

# Singular solutions of the Yamabe problem in the Heisenberg group and their bifurcation

Claudio Afeltra\*

## Abstract

We prove the existence of a homogeneous singular solution of the critical equation

$$-\Delta_{\mathbf{H}^n} u = u^{\frac{Q+2}{Q-2}}$$

on the Heisenberg group  $H^n$ , where  $Q$  is the *homogeneous dimension*. In order to do this, we introduce a suitable concept of normal curvature for hypersurfaces. Furthermore we study the bifurcation of non-homogeneous solutions from the homogeneous one.

## 1 Introduction

The Yamabe problem has drawn a large interest in Riemannian geometry. Its solution in the compact case (due to the works of Yamabe, Trudinger, Aubin and Schoen) has constituted a major advance in the fields of geometric analysis and partial differential equations, and it has been drawing attention until today (on this topic see, for example, [3] for a general treatment).

In the field of CR geometry, the analogous problem of finding a contact form with constant Webster curvature (the analogous of scalar curvature) arises even more naturally, due to the fact that the choice of a contact form on a given CR structure is determined up to the multiplication by a nowhere zero scalar function.

On the Heisenberg group, which is the “model” CR manifold, the Yamabe problem is equivalent to finding the positive solutions of the equation

$$-\Delta_{\mathbf{H}^n} u = u^{\frac{Q+2}{Q-2}}, \tag{1}$$

where  $\Delta_{\mathbf{H}^n}$  is the sublaplacian and  $Q = 2n + 2$  is the homogeneous dimension (precise definitions are given later).

The positive solutions of this equation satisfying some integrability hypotheses were classified by Jerison and Lee [13]; geometrically they correspond to conformal factors that transform the standard pseudohermitian structure of  $\mathbf{H}^n$  into

---

\*Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa (Italy) - claudio.afeltra@sns.it

the push-forward of the pseudohermitian structure of the sphere  $\mathbf{S}^{2n+1} \subset \mathbf{C}^{n+1}$  with respect to the Cayley transform, up to translations and dilations. This classification plays an important role in the solution of the CR Yamabe problem, see [14], [12], [11] and [7].

Additionally to this, it is interesting to study the problem on  $\mathbf{H}^n \setminus \{0\}$ . In the Riemannian case all the solutions singular at a point were classified by Caffarelli, Gidas and Spruck (see [4]). These form a continuous one-parameter family of radially periodic metrics depending on a parameter  $\tau \in (0, 1]$ , called Delaunay metrics: for  $\tau = 1$  the metric is homogeneous and corresponds to the cylindrical metric, and for  $\tau \rightarrow 0$  tends to a superposition of regular solutions. This classification has been useful in the study of the profiles of general singular solutions (see [15]), as well as in the study of blow-ups in the problem of prescribed curvature (see [17], [18], [5]).

The author of this article proved (see [1]) the existence of analogues of the Delaunay metrics for small values of  $\tau$ , constructed by perturbing an approximated solution consisting of a series of regular solutions suitable dilated. In this article, it is proved the existence of a solution analogous to the homogeneous Euclidean one, that is, homogeneous with respect to the Heisenberg dilations  $\{\delta_\lambda\}_{\lambda>0}$  defined in Section 2.

**Theorem 1.1.** *There exists a smooth solution  $\Psi$  of the equation*

$$-\Delta_{\mathbf{H}^n} \Psi = \Psi^{\frac{Q+2}{Q-2}}$$

*on  $\mathbf{H}^n \setminus \{0\}$  such that  $\Psi \circ \delta_\lambda = \lambda^{\frac{Q-2}{2}} \Psi$  and  $\Psi(z, t) = \Psi(|z|, t)$ .*

The above result is proved by posing the problem in a variational form, and then performing a conformal change that transforms  $\mathbf{H}^n \setminus \{0\}$  in a pseudohermitian cylinder, and imposing symmetries in order to reduce the problem to an ODE with variational structure.

The main difficulty is that, because of the non compactness of  $\mathbf{H}^n \setminus \{0\}$ , the problem has to be formulated on a closed annulus  $\{x \in \mathbf{H}^n \mid 1 \leq |x| \leq r\}$  (where  $|\cdot|$  is the homogeneous norm), and so one has to put boundary conditions that, under a conformal change, behave in a treatable way. It is known that the mean curvature behaves in such a way, indeed the prescription of the mean curvature of the boundary is considered the most natural boundary condition in the prescribed curvature problem for manifolds with boundary (see, for example, [9]). In our case there is not such a concept, except in dimension three (see [6]). So we introduce, in arbitrary dimension, the notion of canonical pseudohermitian normal curvature. In such a way we can formulate variationally the problem of the prescription of the Webster curvature with boundary conditions, with a functional that is conformally invariant.

In the second part of the article, we study the problem of the bifurcation of radially periodic solutions from the homogeneous one we found (for an introduction to bifurcation theory, we refer to [2]). That is, considering a parameter  $T$ , we want to prove that solutions to equation (1) such that  $u \circ \delta_T = T^{-\frac{Q-2}{2}} u$  bifurcate from the homogeneous solution for infinitely many values of  $T$ . This

problem has a variational structure: the radially periodic solutions are critical points of the functional

$$\mathcal{J}_T(u) = \int_{\Omega_T} \left( |\nabla_{\mathbf{H}^n} u|^2 - \frac{1}{2^*} |u|^{2^*} \right)$$

defined on the space

$$X_T = \left\{ u \in S_{\text{loc}}^1(\mathbf{H}^n) \mid u \circ \delta_T = T^{-\frac{Q-2}{2}} u \right\},$$

where  $S_{\text{loc}}^1(\mathbf{H}^n)$  is the Stein-Folland space (see [10]). We prove the following result.

**Theorem 1.2.** *There exists arbitrarily large values of  $T$  for which  $d^2 \mathcal{J}_T$  is singular, and every such value is a bifurcation value.*

The article is structured as follows. After the preliminaries of Section 2, in Section 3 we introduce a notion of curvature of a hypersurface in a pseudohermitian manifold that serves our purposes. In Section 4 we prove Theorem 1.1 performing a conformal change of contact form, and through the imposition of natural symmetries. In Section 5 we study the problem of bifurcation.

## 2 Preliminaries and notation

For a general introduction to CR manifolds we refer to [8], but we recall here some basic concepts.

A *CR manifold* is a real smooth manifold  $M$  endowed with a subbundle  $\mathcal{H}$  of the complexified tangent bundle of  $M$ ,  $T^{\mathbf{C}}M$ , such that  $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$  and  $[\mathcal{H}, \mathcal{H}] \subseteq \mathcal{H}$ . We will assume  $M$  to be of hypersurface type, that is that  $\dim M = 2n + 1$  and that  $\dim \mathcal{H} = n$ .

There exists a non-zero real differential form  $\theta$  that is zero on  $\Re(\mathcal{H} \oplus \overline{\mathcal{H}})$ ; it is unique up to scalar multiple by a nowhere zero function. Such a form is called *pseudohermitian* or *contact form*. On a *pseudohermitian manifold*, the *Levi form* on  $\mathcal{H}$  is defined as the 2-form  $L_\theta(V, W) = -id\theta(V, \overline{W}) = i\theta([V, \overline{W}])$ . A CR manifold is said to be *pseudoconvex* if it admits a positive definite Levi form (this implies every Levi form to be definite), it is said *nondegenerate* if it admits a nondegenerate Levi form.

An almost complex structure  $J$  can be defined on  $H(M) = \Re(\mathcal{H} + \overline{\mathcal{H}})$  in a natural way, by  $J(V + \overline{V}) = i(V - \overline{V})$ .

There exists a unique vector field  $T$  such that  $\theta(T) = 1$  and  $i_T d\theta = 0$ . This permits to define a natural Riemannian metric  $g_\theta$ , which coincides with the metric  $G_\theta$  associated to  $L_\theta$  on  $H(M)$ , and such that  $g_\theta(T, T) = 1$  and  $T$  is orthogonal to  $H(M)$ .

Given a function  $u \in C^1(M)$ , the subriemannian gradient is the section  $\nabla_b u$  of  $H(M)$  defined by  $du(X) = G_\theta(\nabla_b u, X)$  for every section  $X$  of  $H(M)$ , and the sublaplacian  $\Delta_b u$  is defined as the divergence of  $\nabla_b u$  with respect to the volume form  $\theta \wedge (d\theta)^n$ , which is the natural one in such context.

On a nondegenerate pseudohermitian manifold one can define a connection, the Tanaka-Webster connection. This allows to define curvature operators in an analogous manner as in Riemannian geometry: the pseudohermitian curvature tensor is the curvature of the Tanaka-Webster connection, the Ricci tensor is

$$\text{Ric}(X, Y) = \text{trace}(Z \mapsto R(Z, X)Y),$$

and the Webster scalar curvature is the trace of the Ricci tensor with respect to the Levi form.

If  $\tilde{\theta} = u^{2/n}\theta$ , the transformation law of the Webster curvature is

$$\tilde{W} = u^{-1-2/n} \left( -\frac{2n+2}{n} \Delta_b u + Wu \right), \quad (2)$$

(see [8]). So the CR Yamabe problem leads to the equation

$$-\frac{2n+2}{n} \Delta_b u + Wu = \lambda u^{1+2/n}.$$

The model pseudohermitian manifold, and the one we will study, is the Heisenberg group  $\mathbf{H}^n$ , that is the Lie group  $\mathbf{C}^n \times \mathbf{R}$  with the product

$$(z_1, t_1) \cdot (z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\Im(z_1 \cdot \bar{z}_2)),$$

endowed with the subbundle spanned by the standard left invariant vector fields

$$Z_\alpha = \frac{\partial}{\partial z_\alpha} + i\bar{z}_\alpha \frac{\partial}{\partial t},$$

and with the left invariant contact form

$$\theta = dt + i \sum_{\alpha=1}^n (z^\alpha d\bar{z}^\alpha - \bar{z}^\alpha dz^\alpha).$$

We will also use the real vector fields

$$X_\alpha = \frac{\partial}{\partial x_\alpha} + 2y_\alpha \frac{\partial}{\partial t}, \quad Y_\alpha = \frac{\partial}{\partial x_\alpha} - 2x_\alpha \frac{\partial}{\partial t},$$

which form a left invariant frame for  $H(M)$ .

On  $\mathbf{H}^n$  with this CR structure the CR subriemannian gradient coincides with the standard subriemannian gradient given by the formula

$$\nabla_{\mathbf{H}^n} u = \sum_{\alpha=1}^n (X_\alpha u X_\alpha + Y_\alpha u Y_\alpha).$$

Furthermore in this case the sublaplacian coincides with the group sublaplacian given by

$$\Delta_{\mathbf{H}^n} = \sum_{\alpha=1}^n (X_\alpha^2 + Y_\alpha^2) = 2 \sum_{\alpha=1}^n (Z_\alpha \bar{Z}_\alpha + \bar{Z}_\alpha Z_\alpha).$$

We will use also the Koranyi norm, given by

$$|(z, t)| = (|z|^4 + t^2)^{1/4},$$

and the dilations given by

$$\delta_\lambda(z, t) = (\lambda z, \lambda^2 t)$$

for  $\lambda > 0$ . Notice that, in the above notation,  $|\delta_\lambda x| = \lambda|x|$ .

It turns out that  $\mathbf{H}^n$  has zero Webster curvature, and so, up to an inessential constant, the Yamabe problem is equivalent to find positive solution to equation

$$-\Delta_{\mathbf{H}^n} u = u^{\frac{Q+2}{Q-2}},$$

(where  $Q = 2n + 2$  is the *homogeneous dimension*).

An important transformation in the Heisenberg group is the Kelvin inversion

$$\mathcal{K} : \mathbf{H} \setminus \{0\} \rightarrow \mathbf{H} \setminus \{0\}$$

given by

$$\mathcal{K}(z, t) = \left( \frac{-iz}{t + i|z|^2}, -\frac{t}{\rho^4} \right).$$

where  $\rho = |(z, t)|$ .  $\mathcal{K}$  leaves the unit sphere invariant, but, unlike its analogous on the Euclidean space, it does not fix the unit sphere pointwise.

We recall the following formulas from [16] for conformal changes of pseudohermitian metric, to which we refer also for the notation.

**Proposition 2.1.** *If  $Z_1, \dots, Z_n$  is a frame for  $\mathcal{H}$ ,  $\theta^1, \dots, \theta^n$  is the corresponding dual frame, and the connection forms  $\omega_\alpha^\beta$  are defined by  $\nabla Z_\alpha = \omega_\alpha^\beta \otimes Z_\beta$ , then, under the conformal change  $\theta \mapsto \tilde{\theta} = e^{2f}\theta$ , the Tanaka-Webster connection transforms as*

$$\tilde{\omega}_\alpha^\beta = \omega_\alpha^\beta + 2(f_\beta \theta^\alpha - f_\alpha \theta^\beta) + \delta_\alpha^\beta (f_\gamma \theta^\gamma - f^\gamma \theta_\gamma) + F \cdot \theta$$

(where  $F$  is a function of  $f$  explicitly known, but whose expression is irrelevant for our purposes).

### 3 The canonical pseudohermitian normal curvature

Let  $M$  be a CR manifold endowed with a contact form  $\theta$ . Let  $\Sigma$  be a two-sided hypersurface in  $M$  such that  $V = \dim(T\Sigma \cap H(M)) = 2n - 1$  at every point. If  $N$  is a normal vector field to  $\Sigma$  with respect to  $g_\theta$ , the normalization of its orthogonal projection on  $H(M)$ ,

$$\nu = \frac{\pi_{H(M)}(N)}{|\pi_{H(M)}(N)|},$$

is orthogonal to  $V$ . Let  $\xi = -J\nu$ . This is tangent to  $\Sigma$ , and it is canonical (given an orientation on  $\Sigma$ ). So we define the *canonical pseudohermitian normal curvature* of  $\Sigma$  as

$$\kappa = g_\theta(\nabla_\xi \xi, \nu),$$

where  $\nabla$  is the Tanaka-Webster connection.

**Proposition 3.1.** *Under the conformal change  $\theta \mapsto \tilde{\theta} = u^{2/n}\theta$ ,  $\kappa$  the canonical pseudohermitian normal curvature of the new pseudohermitian metric is given by the formula*

$$\kappa u - \frac{3}{n}\nu(u) = u^{1+\frac{1}{n}}\tilde{\kappa}.$$

*Proof.* Since  $\xi \in \mathcal{H}$  and is a unit vector with respect to  $g_\theta$ , there exists an orthonormal frame  $Z_1, \dots, Z_n$  for  $\mathcal{H}$  such that  $Z_1 + \bar{Z}_1 = \sqrt{2}\xi$ . Then  $\nu = \frac{i}{\sqrt{2}}(Z_1 - \bar{Z}_1)$ . Because of formula (4.2) in [16],  $\omega_1^1 = -\omega_1^1$ , and so

$$\begin{aligned} \kappa &= g_\theta(\nabla_\xi \xi, \nu) = \frac{1}{2}g_\theta(\nabla_\xi Z_1 + \nabla_\xi \bar{Z}_1, i(Z_1 - \bar{Z}_1)) = \\ &= -\frac{1}{2}d\theta(\omega_1^1(\xi)Z_1 + \omega_1^{\bar{1}}(\xi)\bar{Z}_1, Z_1 + \bar{Z}_1) = \\ &= -\frac{i}{2}\omega_1^1(\xi)L_\theta(Z_1, \bar{Z}_1) - \frac{i}{2}\omega_1^1(\xi)L_\theta(Z_1, \bar{Z}_1) = -i\omega_1^1(\xi). \end{aligned}$$

Applying Proposition 2.1, since  $h_{1\bar{1}} = 1$  we obtain that

$$\begin{aligned} \tilde{\omega}_1^1 &= \omega_1^1 + 2(f_1\theta^1 - f_1\theta^1) + \delta_1^1(f_1\theta^1 - f^1\theta_1) + F \cdot \theta = \\ &= \omega_1^1 + 3(Z_1 f \theta^1 - \bar{Z}_1 f \bar{\theta}^1) \quad \text{mod } \theta. \end{aligned}$$

Considering that after the conformal change the Levi form is multiplied by  $e^{2f}$ , and so the canonical tangent vector becomes  $\tilde{\xi} = e^{-f}\xi$ , we obtain that

$$\begin{aligned} \tilde{\kappa} &= -i\tilde{\omega}_1^1(\tilde{\xi}) = -ie^{-f}(\omega_1^1 + 3(Z_1 f \theta^1 - \bar{Z}_1 f \bar{\theta}^1))(\xi) = \\ &= e^{-f}\kappa - \frac{3i}{\sqrt{2}}(Z_1 f \theta^1 - \bar{Z}_1 f \bar{\theta}^1)(Z_1 + \bar{Z}_1) = e^{-f}\kappa - \frac{3i}{\sqrt{2}}(Z_1 - \bar{Z}_1)f = \\ &= e^{-f}\kappa - 3\nu(f). \end{aligned}$$

This concludes the proof.  $\square$

Recall that the Webster curvature transforms by the formula

$$-b_n \Delta_b u + Wu = \tilde{W}u^{1+\frac{2}{n}},$$

where  $b_n = 2 + \frac{2}{n}$ .

Now let us pick a local orthonormal frame  $Z_1, \dots, Z_n$  of  $\mathcal{H}$  such that  $\xi = \frac{1}{\sqrt{2}}(Z_n + \bar{Z}_n)$  and  $\nu = \frac{i}{\sqrt{2}}(Z_n - \bar{Z}_n)$ , and let  $\theta^1, \dots, \theta^n$  be the associated coframe. Then if  $e_{2\alpha-1} = \frac{1}{\sqrt{2}}(Z_\alpha + \bar{Z}_\alpha)$  and  $e_{2\alpha} = \frac{i}{\sqrt{2}}(Z_\alpha - \bar{Z}_\alpha)$ ,  $e_1, \dots, e_{2n}$  is

an orthonormal frame with respect to  $G_\theta$ . Let  $e^1, \dots, e^{2n}$  be the associated coframe. Then  $\theta^\alpha = \frac{1}{\sqrt{2}}(e^{2\alpha-1} + ie^{2\alpha})$ .

So, by definition of the Levi form, we have

$$d\theta = i \sum_{\alpha=1}^n \theta^\alpha \wedge \theta^{\bar{\alpha}} = \frac{i}{2} \sum_{\alpha=1}^n (e^{2\alpha-1} + ie^{2\alpha}) \wedge (e^{2\alpha-1} - ie^{2\alpha}) = \sum_{\alpha=1}^n e^{2\alpha-1} \wedge e^{2\alpha},$$

therefore

$$\theta \wedge (d\theta)^n = \theta \wedge \left( \sum_{\alpha=1}^n e^{2\alpha-1} \wedge e^{2\alpha} \right)^n = n! \theta \wedge e^1 \wedge \dots \wedge e^{2n} = n! \text{vol}_{g_\theta}.$$

We want to give a variational formulation to the problem of the prescription of the Webster curvature and the prescription of the canonical pseudohermitian normal curvature on the boundary.

In order to state the next proposition, we define the differential  $2n - 1$ -form on  $\partial M$

$$\sigma = e^1 \wedge e^2 \wedge \dots \wedge e^{2n-1} \quad (3)$$

which is independent by the chosen frame. This can be verified by a change of frame, or alternatively it can be noted that it is the pull-back with respect to inclusion in  $M$  of  $i_\nu i_T \text{vol}_{g_\theta}$ .

**Proposition 3.2.** *The functional*

$$Q(v) = \int_M (b_n |\nabla_b v|^2 + Wv^2) \theta \wedge (d\theta)^n - c_n \int_{\partial M} \kappa v^2 \sigma \wedge \theta,$$

where  $c_n = \frac{b_n}{3} nn!$  and  $\sigma$  is defined in formula (3), is invariant by the transformation

$$\theta \mapsto \tilde{\theta} = u^{2/n} \theta, \quad v \mapsto \tilde{v} = vu^{-1}. \quad (4)$$

*Proof.* Under this conformal change the subriemannian metric transforms as  $G_\theta \mapsto u^{2/n} G_\theta$ , and so  $\tilde{\nabla}_b = u^{-2/n} \nabla_b$ . Therefore

$$\begin{aligned} \int_M |\tilde{\nabla}_b \tilde{v}|^2 \tilde{\theta} \wedge (d\tilde{\theta})^n &= \int_M u^{-2/n} |\nabla_b (u^{-1}v)|^2 u^{2(n+1)/n} \theta \wedge (d\theta)^n = \\ &= u^2 \int_M |u^{-1} \nabla_b v - u^{-2} v \nabla_b u|^2 \theta \wedge (d\theta)^n = \\ &= \int_M (|\nabla_b v|^2 + u^{-2} v^2 |\nabla_b u|^2 - 2u^{-1} v \nabla_b u \cdot \nabla_b v) \theta \wedge (d\theta)^n = \\ &= \int_M |\nabla_b v|^2 + \int_M (v^2 |\nabla_b \log u|^2 - \nabla_b \log u \cdot \nabla_b (v^2)) \theta \wedge (d\theta)^n = \\ &= \int_M |\nabla_b v|^2 + \int_M v^2 (|\nabla_b \log u|^2 + \Delta_b \log u) \theta \wedge (d\theta)^n - n! \int_{\partial M} v^2 g_\theta(\nabla_b \log u, \xi) \mathcal{V}, \end{aligned}$$

where  $\mathcal{V}$  is the volume form associated to the restriction of  $g_\theta$ . It is easy to verify that for every  $X$  in  $H(M)$ , the restriction of  $g_\theta(\xi, X)\mathcal{V}$  is equal to the restriction of  $e^{2n}(X)\sigma \wedge \theta$ . So

$$\begin{aligned} & \int_M |\widetilde{\nabla}_b \widetilde{v}|^2 \widetilde{\theta} \wedge (d\widetilde{\theta})^n = \\ &= \int_M |\nabla_b v|^2 + \int_M v^2 (|\nabla_b \log u|^2 + \Delta_b \log u) \theta \wedge (d\theta)^n - n! \int_{\partial M} v^2 \nu(\log u) \sigma \wedge \theta. \end{aligned}$$

Thanks to the conformal change formula,

$$\begin{aligned} \int_M \widetilde{W} \widetilde{v}^2 \widetilde{\theta} \wedge (d\widetilde{\theta})^n &= \int_M (-b_n u^{-1-2/n} \Delta_b u + W u^{-2/n}) v^2 u^{-2} u^{2+2/n} \theta \wedge (d\theta)^n = \\ &= \int_M (-b_n u^{-1} \Delta_b u + W) v^2 \theta \wedge (d\theta)^n. \end{aligned}$$

It holds that

$$\Delta_b \log u = \operatorname{div}(\nabla_b \log u) = \operatorname{div} \left( \frac{\nabla_b u}{u} \right) = \frac{\Delta_b u}{u} - \frac{|\nabla_b u|^2}{u^2} = \frac{\Delta_b u}{u} - |\nabla_b \log u|^2,$$

and so

$$\int_M \widetilde{W} \widetilde{v}^2 \widetilde{\theta} \wedge (d\widetilde{\theta})^n = \int_M W v^2 \theta \wedge (d\theta)^n - b_n \int_M (\Delta_b \log u + |\nabla_b \log u|^2) v^2 \theta \wedge (d\theta)^n.$$

Finally

$$\begin{aligned} \int_{\partial M} \widetilde{\kappa} \widetilde{v}^2 \widetilde{\sigma} \wedge \widetilde{\theta} &= \int_{\partial M} \left( u^{-1/n} \kappa - \frac{3}{n} u^{-1-1/n} \nu(u) \right) v^2 u^{-2} u^{2+1/n} \sigma \wedge \theta = \\ &= \int_{\partial M} \kappa v^2 \sigma \wedge \theta - \frac{3}{n} \int_{\partial M} \nu(\log u) v^2 \sigma \wedge \theta. \end{aligned}$$

By summing the above identities we get the desired result.  $\square$

Thanks to the above result, and through standard calculus of variations methods, one can easily check the following Proposition.

**Proposition 3.3.** *A conformal change has Webster curvature  $W_1$  and canonical pseudohermitian normal curvature  $\kappa_1$  if and only if it is a stationary point of the functional*

$$I_{W_1, \kappa_1}(v) = Q(v) - \frac{n}{n+1} \int_M W_1 v^{2+2/n} \theta \wedge (d\theta)^n + \frac{n c_n}{2n+1} \int_{\partial M} \kappa_1 v^{2+1/n} \sigma \wedge \theta,$$

that is invariant by the transformation in formula (4).

## 4 Proof of Theorem 1.1

Now that we have a variational and conformally covariant formulation of the problem of prescribed curvature with boundary conditions, thanks to Proposition 3.3, we study this problem on suitable annuli, imposing that the boundary has zero curvature, a natural condition because of the symmetry given by the Cayley transform. So let us study the problem

$$\begin{cases} -b_n \Delta_b u = u^{1+2/n} & \text{on } A_r \\ -\frac{3}{n} \nu(u) + \kappa_{A_r} u = 0 & \text{on } \partial A_r \end{cases}$$

where  $A_r = B_r \setminus \overline{B}_1$ , and  $B_r = B_r(0)$  with respect to the Koranyi norm. The latter problem is equivalent to find the critical points of

$$I(v) = b_n \int_M |\nabla v|^2 \theta \wedge (d\theta)^n - c_n \int_{\partial M} \kappa v^2 \sigma \wedge \theta - \frac{n}{n+1} \int_M v^{2+2/n} \theta \wedge (d\theta)^n.$$

We restrict the functional to functions such that  $u(x, t) = u(|x|, t)$ .

Let us consider the conformal change

$$\theta \mapsto \tilde{\theta} = \rho^{-2} \theta,$$

where  $\rho = |x|$ .

**Lemma 4.1.** *The Webster curvature of  $\tilde{\theta}$  is*

$$\tilde{W} = -b_n u^{-1-\frac{2}{n}} \Delta_b u = -b_n \rho^{n+2} \Delta_b (\rho^{-n}) = b_n n^2 \frac{|x|^2}{\rho^2},$$

and the canonical pseudohermitian normal curvature of the boundary of  $A_r$  is zero.

*Proof.* We have

$$\begin{aligned} X_\alpha(\rho^4) &= \left( \frac{\partial}{\partial x^\alpha} + 2y^\alpha \frac{\partial}{\partial t} \right) (|x|^4 + t^2) = \\ &= 4(x_\alpha^3 + (|x|^2 - x_\alpha^2)x_\alpha + y_\alpha t) = 4(|x|^2 x_\alpha + y_\alpha t); \\ Y_\alpha(\rho^4) &= \left( \frac{\partial}{\partial y^\alpha} - 2x^\alpha \frac{\partial}{\partial t} \right) (|x|^4 + t^2) = 4(|x|^2 y_\alpha - x_\alpha t); \\ X_\alpha^2(\rho^4) &= 4 \left( \frac{\partial}{\partial x^\alpha} + 2y^\alpha \frac{\partial}{\partial t} \right) (x_\alpha^3 + |y_\alpha|^2 x_\alpha + y_\alpha t) = 4(|x|^2 + 2|x_\alpha|^2 + 2|y_\alpha|^2); \\ Y_\alpha^2(\rho^4) &= 4 \left( \frac{\partial}{\partial y^\alpha} - 2x^\alpha \frac{\partial}{\partial t} \right) (y_\alpha^3 + |x_\alpha|^2 y_\alpha - x_\alpha t) = 4(|x|^2 + 2|x_\alpha|^2 + 2|y_\alpha|^2). \end{aligned}$$

$$\begin{aligned}
X_\alpha^2(\rho^{-n}) &= X_\alpha(X_\alpha((\rho^4)^{-n/4})) = -\frac{n}{4}X_\alpha(\rho^{-n-4}X_\alpha(\rho^4)) = \\
&= \frac{n(n+4)}{16}\rho^{-n-8}|X_\alpha(\rho^4)|^2 - \frac{n}{4}\rho^{-n-4}X_\alpha^2(\rho^4) = \\
&= \frac{n(n+4)}{16}\rho^{-n-8}16(|x|^2x_\alpha + y_\alpha t)^2 - \frac{n}{4}\rho^{-n-4}4(|x|^2 + 2|x_\alpha|^2 + 2|y_\alpha|^2) = \\
&= n(n+4)\rho^{-n-8}(|x|^2x_\alpha + y_\alpha t)^2 - n\rho^{-n-4}(|x|^2 + 2|x_\alpha|^2 + 2|y_\alpha|^2),
\end{aligned}$$

and analogously

$$Y_\alpha^2(\rho^{-n}) = n(n+4)\rho^{-n-8}(|x|^2y_\alpha - x_\alpha t)^2 - n\rho^{-n-4}(|x|^2 + 2|x_\alpha|^2 + 2|y_\alpha|^2),$$

so

$$\begin{aligned}
\Delta_b(\rho^{-n}) &= \sum_{\alpha=1}^n (X_\alpha^2 + Y_\alpha^2)(\rho^{-n}) = \\
&= n(n+4)\rho^{-n-8}(|x|^6 + |x|^2t^2) - 2n(n+2)\rho^{-n-4}|x|^2 = -n^2\rho^{-n-4}|x|^2.
\end{aligned}$$

Since  $u = \rho^{-n}$ , by formula (2) we get the desired result.

It can be readily verified that the Kelvin transform is isopseudohermitian with respect to  $\tilde{\theta}$  (that is, it preserves the pseudohermitian structure). Also the transformations of  $\mathbf{H}^n$  of the form  $(z, t) \mapsto (Az, t)$  with  $A$  unitary, and the dilations, are isopseudohermitian. Furthermore, the function  $(z, t) \mapsto (\bar{z}, -t)$  is isopseudohermitian from  $(\mathbf{H}^n, \mathcal{H}, \theta)$  to  $(\mathbf{H}^n, \overline{\mathcal{H}}, \theta)$ . So, for every point  $x$  of  $\partial A_r$ , there is a isopseudohermitian transformation from  $(\mathbf{H}^n, \mathcal{H}, \theta)$  to  $(\mathbf{H}^n, \overline{\mathcal{H}}, \theta)$  that fixes  $x$ , leaves its component of  $\partial A_r$  invariant, but reverses the orientation. Since conjugating the CR structure leaves  $\tilde{\kappa}$  invariant and reversing the orientation changes its sign, it follows that  $\tilde{\kappa} = 0$ .

Since the Cayley transform is isopseudohermitian with respect to  $\tilde{\theta}$  (that is, it preserves the pseudohermitian structure), and that the transformations of  $\mathbf{H}^n$  of the form  $(z, t) \mapsto (Az, t)$  with  $A$  unitary also are, this shows that by symmetry that  $\tilde{\kappa} = 0$ .  $\square$

Therefore, thanks to Proposition 3.3,

$$\tilde{I}(v) = b_n \int_{A_r} \left( |\tilde{\nabla}_b v|_{\tilde{\theta}}^2 + n^2 \frac{|x|^2}{\rho^2} v^2 \right) \tilde{\theta} \wedge (d\tilde{\theta})^n - \frac{n}{n+1} \int_{A_r} v^{2+2/n} \tilde{\theta} \wedge (d\tilde{\theta})^n.$$

We want to impose that the solution is homogeneous and symmetric, in the sense that  $u \circ \delta_\lambda = \lambda^{\frac{Q-2}{2}} u$  and  $u(x, t) = u(|x|, t)$ .

Furthermore we want to express this functional in more convenient coordinates.

**Lemma 4.2.** *If  $v = v(|x|, t)$ , in the coordinates  $l = \frac{1}{n} \log \rho \in \mathbf{R}$ ,  $\tau = t/\rho^2 \in [-1, 1]$  and  $\gamma = x/|x| \in \mathbf{S}^{2n-1}$ , it holds that*

$$|\tilde{\nabla}_b v|_{\tilde{\theta}}^2 = (1 - \tau^2)^{3/2} \left| \frac{\partial v}{\partial \tau} \right|^2 + \frac{1}{4n^2} (1 - \tau^2)^{1/2} \left| \frac{\partial v}{\partial l} \right|^2.$$

*Proof.* A transformation of  $\mathbf{H}^n$  of the form  $(x, t) \mapsto (Ax, t)$  with  $A$  linear is an isomorphism of the pseudohermitian structure if and only if  $A$  is unitary. Since this kind of transformations preserves the sphere of unit radius, and since the action of the unitary group is transitive between vectors of the same length, we can calculate  $|\tilde{\nabla}_b v|_{\theta}^2$  in the points of the curve

$$(\sqrt[4]{1-t^2}, 0, \dots, 0, t).$$

At such points

$$X_{\alpha} = \frac{\partial}{\partial x_{\alpha}}$$

for every  $\alpha = 1, \dots, n$ ,

$$Y_{\alpha} = \frac{\partial}{\partial y_{\alpha}}$$

for every  $\alpha \neq 1$ , and

$$Y_1 = \frac{\partial}{\partial y_1} - 2\sqrt[4]{1-t^2} \frac{\partial}{\partial t}.$$

Using the symmetry of  $v$  in  $x$ , we get

$$\begin{aligned} |\tilde{\nabla}_b v|_{\theta}^2 &= v^{-2/n} |\nabla_b v|_{\theta}^2 = \frac{1}{4} |X_1 u|^2 + \frac{1}{4} |Y_1 u|^2 = \\ &= \frac{1}{4} \left| \frac{\partial v}{\partial x_1} \right|^2 + (1-t)^{1/2} \left| \frac{\partial v}{\partial t} \right|^2. \end{aligned}$$

Since

$$\begin{aligned} \frac{\partial \tau}{\partial x_1} &= -\frac{1}{2} \tau^3 \frac{\partial(\tau^{-2})}{\partial x_1} = -\frac{1}{2} \frac{\tau^3}{t^2} \frac{\partial}{\partial x_1} (x_1^4 + t^2) = -2tx_1^3; \\ \frac{\partial l}{\partial x_1} &= \frac{1}{n} \frac{\partial \log \rho}{\partial x_1} = \frac{1}{4n} \frac{\partial \rho^4}{\partial x_1} = \frac{1}{n} x_1^3; \\ \frac{\partial \tau}{\partial t} &= \frac{1}{2} \tau^{-1} \frac{\partial(\tau^2)}{\partial t} = \frac{1}{2} t^{-1} (2t - 2t^3) = (1-t^2); \\ \frac{\partial l}{\partial t} &= \frac{1}{4n} \frac{\partial \rho^4}{\partial t} = \frac{1}{2n} t, \end{aligned}$$

we obtain

$$\begin{aligned} |\tilde{\nabla}_b v|_{\theta}^2 &= \frac{1}{4} \left| \frac{\partial v}{\partial x_1} \right|^2 + (1-t^2)^{1/2} \left| \frac{\partial v}{\partial t} \right|^2 = \\ &= \frac{1}{4} \left| -2tx_1^3 \frac{\partial v}{\partial \tau} + \frac{x_1^3}{n} \frac{\partial v}{\partial l} \right|^2 + (1-t^2)^{1/2} \left| |x|^4 \frac{\partial v}{\partial \tau} + \frac{t}{2n} \frac{\partial v}{\partial l} \right|^2 = \\ &= t^2 (1-t^2)^{3/2} \left| \frac{\partial v}{\partial \tau} \right|^2 + \frac{1}{4n^2} (1-t^2)^{3/2} \left| \frac{\partial v}{\partial l} \right|^2 - \frac{1}{n} t (1-t^2)^{3/2} \frac{\partial v}{\partial \tau} \frac{\partial v}{\partial l} + \\ &+ (1-t^2)^{5/2} \left| \frac{\partial v}{\partial \tau} \right|^2 + \frac{1}{4n^2} t^2 (1-t^2)^{1/2} \left| \frac{\partial v}{\partial l} \right|^2 + \frac{1}{n} (1-t^2)^{3/2} t \frac{\partial v}{\partial \tau} \frac{\partial v}{\partial l} = \end{aligned}$$

$$= (1-t^2)^{3/2} \left| \frac{\partial v}{\partial \tau} \right|^2 + \frac{1}{4n^2} (1-t^2)^{1/2} \left| \frac{\partial v}{\partial l} \right|^2.$$

By the dilation invariance of  $\tilde{\theta}$  we obtain the formula in the general case  $\square$

Now we compute the volume form.

**Lemma 4.3.** *In the coordinates of Lemma 4.2*

$$\tilde{\theta} \wedge (d\tilde{\theta})^n = n!(1-\tau^2)^{(n-2)/2} dl \wedge d\gamma \wedge d\tau.$$

*Proof.* The volume form becomes

$$\tilde{\theta} \wedge (d\tilde{\theta})^n = \rho^{-2(n+1)} \theta \wedge (d\theta)^n = \frac{n!}{\rho^{2(n+1)}} \text{vol}_{g_\theta} = \frac{n!}{\rho^{2(n+1)}} |x|^{2n-1} d|x| \wedge d\gamma \wedge dt.$$

By an easy computation

$$\begin{aligned} d|x| &= \frac{1}{4|x|^3} d(\rho^4 - t^2) = \frac{1}{4|x|^3} d(e^{4ln}(1-\tau^2)) = \\ &= \frac{1}{4(1-\tau^2)^{3/4} e^{3nl}} e^{4ln} (4n(1-\tau^2)dl - 2\tau d\tau) = \\ &= e^{ln} \left( n(1-\tau^2)^{1/4} dl - \frac{\tau}{2(1-\tau^2)^{3/4}} d\tau \right); \\ dt &= d(e^{2nl}\tau) = e^{2nl} (2n\tau dl + d\tau); \\ d|x| \wedge dt &= e^{3nl} \frac{1}{(1-\tau^2)^{3/4}} dl \wedge d\tau, \end{aligned}$$

so

$$\begin{aligned} \tilde{\theta} \wedge (d\tilde{\theta})^n &= -\frac{n!}{\rho^{2(n+1)}} (1-\tau^2)^{(2n-1)/4} \rho^{2n-1} d|x| \wedge dt \wedge d\gamma = \\ &= -\frac{n!}{e^{3nl}} (1-\tau^2)^{(2n-1)/4} e^{3nl} \frac{1}{(1-\tau^2)^{3/4}} dl \wedge d\tau \wedge d\gamma = \\ &= n!(1-\tau^2)^{(n-2)/2} dl \wedge d\gamma \wedge d\tau, \end{aligned}$$

as desired.  $\square$

Using Lemmas 4.2 and 4.3 we have that

$$\begin{aligned} \tilde{I}(v) &= b_n \int_{A_r} \left( (1-\tau^2)^{3/2} \left| \frac{\partial v}{\partial \tau} \right|^2 + \frac{1}{4n^2} (1-\tau^2)^{1/2} \left| \frac{\partial v}{\partial l} \right|^2 v^2 + \right. \\ &\quad \left. + n^2 (1-\tau^2)^{1/2} \right) n!(1-\tau^2)^{(n-2)/2} dl \wedge d\gamma \wedge d\tau + \\ &\quad - \frac{n}{n+1} \int_{A_r} v^{2+2/n} n!(1-\tau^2)^{(n-2)/2} dl \wedge d\gamma \wedge d\tau = \end{aligned}$$

$$\begin{aligned}
&= b_n n! \int_0^{\frac{\log r}{n}} \int_{-1}^1 \left( (1 - \tau^2)^{(n+1)/2} \left| \frac{\partial v}{\partial \tau} \right|^2 + \frac{1}{4n^2} (1 - \tau^2)^{(n-1)/2} \left| \frac{\partial v}{\partial l} \right|^2 v^2 + \right. \\
&\left. + n^2 (1 - \tau^2)^{(n-1)/2} \right) dl \wedge d\tau - \frac{nn!}{n+1} \int_0^{\frac{\log r}{n}} \int_{-1}^1 v^{2+2/n} (1 - \tau^2)^{(n-2)/2} dl \wedge d\tau.
\end{aligned}$$

If  $\tau = \sin s$ , then

$$\begin{aligned}
\tilde{I}(v) &= b_n n! \int_0^{\frac{\log r}{n}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{(\cos s)^{n+1}}{(\cos s)^2} \left| \frac{\partial v}{\partial s} \right|^2 + \frac{1}{4n^2} (\cos s)^{n-1} \left| \frac{\partial v}{\partial l} \right|^2 + \right. \\
&\left. + n^2 (\cos s)^{n-1} v^2 \right) dl (\cos s) ds - \frac{nn!}{n+1} \int_0^{\frac{\log r}{n}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v^{2+2/n} (\cos s)^{n-2} dl (\cos s) ds = \\
&= b_n n! \int_0^{\frac{\log r}{n}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos s)^n \left( \left| \frac{\partial v}{\partial s} \right|^2 + \frac{1}{4n^2} \left| \frac{\partial v}{\partial l} \right|^2 + n^2 v^2 \right) dl ds + \\
&\quad - \frac{nn!}{n+1} \int_0^{\frac{\log r}{n}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v^{2+2/n} (\cos s)^{n-1} dl ds.
\end{aligned}$$

Now let us look for homogeneous solutions. Homogeneous solutions in the original setting correspond to solutions invariant by translation (in the  $l$  direction), and so let us set  $\frac{\partial v}{\partial l} = 0$ , and  $v = v(s)$ . In this special case we have

$$\begin{aligned}
\tilde{I}(v) &= b_n n! \frac{\log r}{n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos s)^n ((v')^2 + n^2 v^2) ds + \\
&\quad - \frac{nn!}{n+1} \frac{\log r}{n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v^{2+2/n} (\cos s)^{n-1} ds.
\end{aligned}$$

The Euler-Lagrange equation for this functional is

$$-\frac{d}{ds} ((\cos s)^n v'(s)) + n^2 (\cos s)^n v(s) = \frac{n}{2(n+1)} (\cos s)^{n-1} v(s)^{1+2/n},$$

or equivalently

$$-\cos s v''(s) + n \sin s v'(s) + n^2 \cos s v(s) = \frac{n}{2(n+1)} v(s)^{1+2/n},$$

on the interval  $(-\frac{\pi}{2}, \frac{\pi}{2})$ , with Neumann boundary conditions, that is also the Euler-Lagrange equation (up to rescaling, thanks to homogeneity) of

$$J(v) = \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos s)^n ((v')^2 + n^2 v^2) ds}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v^{2+2/n} (\cos s)^{n-1} ds}.$$

Let us define the weighted Sobolev and Lebesgue spaces

$$X = \left\{ u \in H_{\text{loc}}^1 \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \mid \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos s)^n ((v')^2 + v^2) ds < \infty \right\},$$

$$Y = \left\{ u \in L_{\text{loc}}^1 \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \mid \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos s)^{n-1} v^{2+2/n} ds < \infty \right\}.$$

**Proposition 4.4.** *X embeds compactly in Y.*

*Proof.* Let  $Z$  be the subspace of  $H^1(S^{n+1})$  formed by functions invariant by rotation around the last coordinate axis. Every such function is of the form  $v(x) = u(\cos x^{n+2})$ , and it is easy to verify that under such an identification

$$\|v\|_Z = \|u\|_X.$$

So this is an isometric isomorphism between  $X$  and  $Z$ . By Rellich-Kondrachov's theorem,  $Z$  embeds compactly into  $L^p(S^n)$  for every  $p \in [1, 2\frac{n+1}{n-1})$ , which by similar arguments is isometrically isomorphic to  $L^p((-\frac{\pi}{2}, \frac{\pi}{2}), (\cos s)^n ds)$ . Given  $\alpha > 0, q > 1$ ,

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} (\cos s)^{n-1} v^{2+2/n} ds &= \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{(\cos s)^{n-1+\alpha}}{(\cos s)^\alpha} ds \leq \\ &\leq \left( \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} v^{(2+2/n)q} (\cos s)^{(n-1+\alpha)q} ds \right)^{1/q} \left( \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} (\cos s)^{-\alpha q'} ds \right)^{1/q'}. \end{aligned}$$

If we impose that  $(n-1+\alpha)q = n$ , then taking  $\alpha$  small enough, we can find that  $p = (2 + \frac{2}{n}q) < 2\frac{n+1}{n-1}$  and  $\alpha q' < n$ , getting that

$$\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} (\cos s)^{n-1} v^{2+2/n} ds \leq C \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} (v^p (\cos s)^n ds)^{1/q},$$

that is

$$\|v\|_Y \leq C \|v\|_{L^p((\cos s)^n ds)},$$

and so  $L^p((\frac{\pi}{2}, \frac{\pi}{2}) (\cos s)^n ds)$  embeds into  $Y$ . So we get the thesis.  $\square$

Now Theorem 1.1 can be proved, in a standard way, by the direct methods of the calculus of variation. Since the solution does not depend on  $r$ , by homogeneity this defines a solution on the whole  $\mathbf{H}^n \setminus \{0\}$ .

## 5 Proof of Theorem 1.2

Let  $\Omega_T = \{1 \leq |x| \leq T\}$  be a cylinder in the Heisenberg group. In the following, all integrals are meant with respect to the Haar measure and volume elements will be omitted.

Let

$$\mathcal{J}_T(u) = \int_{\Omega_T} \left( |\nabla_{\mathbf{H}^n} u|^2 - \frac{1}{2^*} |u|^{2^*} \right)$$

be defined on the space

$$X_T = \left\{ u \in S_{\text{loc}}^1(\mathbf{H}^n) \mid u \circ \delta_T = T^{-\frac{Q-2}{2}} u \right\},$$

where  $S_{\text{loc}}^1(\mathbf{H}^n)$  is the Stein-Folland space (see [10]). Let  $\Psi$  be the homogeneous solution of the PDE

$$-\Delta_{\mathbf{H}^n} u = u^{2^*-1}$$

found in the previous section, that is, a stationary point of  $\mathcal{J}_T$ .

**Proposition 5.1.** *The Morse index of  $\mathcal{J}_T$  at  $\Psi$  is finite and tends to infinity as  $T \rightarrow \infty$ .*

*Proof.* The Morse index is finite because the operator on  $X_T$  associated to the bilinear form

$$d^2 \mathcal{J}_T(\Psi)[u, v] = \int_{\Omega_T} \left( \nabla_{\mathbf{H}^n} u \nabla_{\mathbf{H}^n} v - (2^* - 1) \Psi^{2^*-2} uv \right)$$

is of the sum of the identity and a compact operator (thanks to the Rellich-Kondrachov theorem for the Stein-Folland space).

The signature of a symmetric bilinear form remains invariant passing to the complexification and extending it to a hermitian form. So let us take

$$u(x) = \exp\left(i \frac{1}{M} \log |x|\right) \Psi(x),$$

where

$$\frac{\log T}{2\pi M} \in \mathbf{Z}.$$

Then

$$\begin{aligned} d^2 \mathcal{J}_T(\Psi)[u, u] &= \int_{\Omega_T} \left( \left| \nabla_{\mathbf{H}^n} \left( \exp\left(i \frac{1}{M} \log |x|\right) \Psi \right) \right|^2 - (2^* - 1) \Psi^{2^*-2} \Psi^2 \right) = \\ &= \int_{\Omega_T} \left( \frac{1}{M^2 |x|^2} \Psi^2 + |\nabla_{\mathbf{H}^n} \Psi|^2 + 2 \frac{1}{M |x|} \Psi \nabla_{\mathbf{H}^n} \Psi \cdot \nabla_{\mathbf{H}^n} |x| - (2^* - 1) \Psi^{2^*} \right) = \\ &= \int_{\Omega_T} \left( \frac{1}{M^2 |x|^2} \Psi^2 + 2 \frac{1}{M |x|} \Psi \nabla_{\mathbf{H}^n} \Psi \cdot \nabla_{\mathbf{H}^n} |x| - (2^* - 2) \Psi^{2^*} \right) = \end{aligned}$$

$$= -(2^* - 2) \int_{\Omega_T} \Psi^{2^*} + \int_{\Omega_T} \left( \frac{1}{M^2|x|^2} \Psi^2 + 2 \frac{1}{M|x|} \Psi \nabla_{\mathbf{H}^n} \Psi \cdot \nabla_{\mathbf{H}^n} |x| \right).$$

By homogeneity the three integrals

$$\int_{\Omega_T} \Psi^{2^*}, \int_{\Omega_T} \frac{1}{|x|^2} \Psi^2, \int_{\Omega_T} \frac{1}{|x|} \Psi \nabla_{\mathbf{H}^n} \Psi \cdot \nabla_{\mathbf{H}^n} |x|$$

are constant multiples of  $\log T$ , so there exists a constant  $C$  such that if  $M \geq C$  then  $d^2 \mathcal{J}_T(\Psi)[u, u]$  is negative. Given  $k \in \mathbf{N}$ , let  $T$  be big enough so that

$$\frac{2\pi k}{\log T} \leq \frac{1}{C}.$$

Then the functions

$$u_m(x) = \exp\left(i \frac{2\pi m}{\log T} \log |x|\right) \Psi(x)$$

with  $m = 1, \dots, k$ , are such that  $d^2 \mathcal{J}_T(\Psi)[u_m, u_m] \leq -\varepsilon \log T$  is negative. If  $f$  is a homogeneous function of degree zero and  $m \neq 0$  then

$$\int_{\Omega_T} \exp\left(i \frac{2\pi m}{\log T} \log |x|\right) \frac{f(x)}{|x|^Q} = \int_{S_1} d\sigma f \int_1^T dr \frac{\exp\left(i \frac{2\pi m}{\log T} \log r\right)}{r} = 0.$$

When calculating

$$d^2 \mathcal{J}_T(\Psi)[u_m, u_j]$$

with  $m \neq j$ , the result is a sum of terms of this kind, so it is zero. So the functions  $u_m$  span a vector space of dimension  $k$  on which  $d^2 \mathcal{J}_T(\Psi)$  is negative definite.  $\square$

In order to apply bifurcation theory, let us rewrite the functional with respect to the pseudohermitian form given by the conformal change corresponding to  $\Psi$ . So we get the functional

$$\widetilde{\mathcal{J}}_T(u) = \int_{\Omega_T} \left( |\widetilde{\nabla}_b u|^2 + \frac{1}{2} u^2 - \frac{1}{2^*} |u|^{2^*} \right),$$

defined on the space

$$Y_T = \left\{ u \in S_{\text{loc}}^1(\mathbf{H}^n) \mid u \circ \delta_T = u, \int_{\Omega_T} u = 0 \right\}.$$

Let  $\Sigma$  be the sphere with respect to the Euclidean metric <sup>1</sup>. Let  $\phi_k$  be a complete set in  $L^2(\Sigma)$  consisting of analytic functions. So

$$\gamma_{k,m,T}(x) = \phi_k \left( \frac{x}{|x|_{\text{eucl}}} \right) \sin \left( i \frac{2\pi m}{\log T} \log |x|_{\text{eucl}} \right)$$

<sup>1</sup>this is necessary to perform the next steps of the proofs because the sphere with respect to the Heisenberg metric is not smooth

is a complete set of functions in  $H^1(\Omega_T)$ , analytic with respect to the couple  $(x, T)$ . SO it is complete also in  $S^1(\Omega_T)$ . With the Gram-Schmidt algorithm, we can obtain a family of Hilbert bases  $\psi_{k,T}$  of  $S^1(\Omega_T)$ , and preserve the analyticity property. Let us define the isometry  $\Psi_T$  between  $Y_T$  and  $Y_2$  obtained sending  $\psi_{k,T}$  into  $\psi_{k,2}$ . Let us call

$$L_T = \Psi_T \circ \widetilde{\mathcal{F}}_T''(1) \circ \Psi_T^{-1}.$$

Then, for every  $l, k$ ,

$$\begin{aligned} \langle L_T u_{2,k}, u_{2,k} \rangle_{Y_2} &= \left\langle \widetilde{\mathcal{F}}_T''(1) u_{k,T}, u_{l,T} \right\rangle_{Y_T} = \\ &= \int_{\Omega_T} \widetilde{\nabla}_b u_{k,T} \widetilde{\nabla}_b u_{l,T} + u_{k,T} u_{l,T} - (2^* - 1) u_{k,T} u_{l,T}. \end{aligned}$$

It is an analytic function by the following lemma of immediate proof.

**Lemma 5.2.** *If  $S : I \rightarrow X$ ,  $T : I \rightarrow Y$  are two analytic vector valued functions and  $L : X \times Y \rightarrow Z$  is a bilinear continuous form, then  $t \mapsto B(S(t), T(t))$  is analytic.*

So  $L_T$  is an analytic operator-valued function.

It holds that

$$\begin{aligned} \left\langle \widetilde{\mathcal{F}}_T''(1) u, v \right\rangle &= \int_{\Omega_T} \widetilde{\nabla}_b u \widetilde{\nabla}_b v + uv - (2^* - 1) uv = \int_{\Omega_T} \widetilde{\nabla}_b u \widetilde{\nabla}_b v - (2^* - 2) \Delta(G_T u) v = \\ &= \int_{\Omega_T} \widetilde{\nabla}_b u \widetilde{\nabla}_b v + (2^* - 2) \widetilde{\nabla}_b(G_T u) \widetilde{\nabla}_b v, \end{aligned}$$

where  $G_T : Y_T \rightarrow Y_T$  is the Green's operator, so

$$\widetilde{\mathcal{F}}_T''(1) = I + (2^* - 2) G_T.$$

Since  $L_T$  is, by definition, conjugated to  $\widetilde{\mathcal{F}}_T''(1)$ , it is of the form  $I - K(T)$ , where  $K(T)$  is an analytic operator-valued function of compact operators.

Now, by means of known results in bifurcation theory, we can prove Theorem 1.2.

*Proof of Theorem 1.2.* It suffices to apply Theorem 8.9 in [19]. In our case the hypotheses of that Theorem are all either trivial or standardly verifiable, with exception of hypothesis  $\gamma$ , that is consequence of Corollary 8.3 in the same book.  $\square$

## References

- [1] C. Afeltra, *Singular periodic solutions to a critical equation in the Heisenberg group*, Pacific J. Math. 305 (2020), no. 2, 385–406.

- [2] A. Ambrosetti and A. Malchiodi, “Nonlinear analysis and semilinear elliptic problems”, Cambridge Studies in Advanced Mathematics, 104. Cambridge University Press, Cambridge, 2007.
- [3] T. Aubin, “Some nonlinear problems in Riemannian geometry”, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.
- [4] L. A. Caffarelli, B. Gidas and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*. Comm. Pure Appl. Math. 42 (1989), no. 3, 271–297.
- [5] C.-C. Chen and C.-S. Lin, *Estimate of the conformal scalar curvature equation via the method of moving planes. II*. J. Differential Geom. 49 (1998), no. 1, 115–178.
- [6] J.-H. Cheng, J.-F. Hwang, A. Malchiodi and P. Yang, *Minimal surfaces in pseudohermitian geometry*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 4 (2005), no. 1, 129–177.
- [7] J.-H. Cheng, A. Malchiodi and P. Yang, *A positive mass theorem in three dimensional Cauchy-Riemann geometry*. Adv. Math. 308 (2017), 276–347.
- [8] S. Dragomir and G. Tomassini, “Differential geometry and analysis on CR manifolds, Progress in Mathematics, 246. Birkhäuser Boston, Inc., Boston, MA, 2006.
- [9] J. F. Escobar, *The Yamabe problem on manifolds with boundary*, J. Differential Geom. 35 (1992), no. 1, 21–84.
- [10] G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat. 13 (1975), no. 2, 161–207.
- [11] N. Gamara, *The CR Yamabe conjecture—the case  $n=1$* . J. Eur. Math. Soc. (JEMS) 3 (2001), no. 2, 105–137.
- [12] N. Gamara and R. Yacoub, *CR Yamabe conjecture—the conformally flat case*. Pacific J. Math. 201 (2001), no. 1, 121–175.
- [13] D. Jerison and J. M. Lee, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*. J. Amer. Math. Soc. 1 (1988), no. 1, 1–13.
- [14] D. Jerison and J. M. Lee, *The Yamabe problem on CR manifolds.*, J. Differential Geom. 25 (1987), no. 2, 167–197.
- [15] N. Korevaar, R. Mazzeo, F. Pacard and R. Schoen, *Refined asymptotics for constant scalar curvature metrics with isolated singularities*. Invent. Math. 135 (1999), no. 2, 233–272.
- [16] J. M. Lee, *The Fefferman metric and pseudo-Hermitian invariants*, Trans. Amer. Math. Soc. 296 (1986), no. 1, 411–429.

- [17] Y. Y. Li, *Prescribing scalar curvature on  $S_n$  and related problems. I.* J. Differential Equations 120 (1995), no. 2, 319–410.
- [18] Y. Li, *Prescribing scalar curvature on  $S_n$  and related problems. II. Existence and compactness.* Comm. Pure Appl. Math. 49 (1996), no. 6, 541–597.
- [19] J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems*, Applied Mathematical Sciences, 74. Springer-Verlag, New York, 1989. xiv+277 pp.