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Boundary regularity for Monge–Ampère equations with unbounded right hand side

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Abstract. We consider Monge–Ampère equations with right hand side f that degenerate to ∞ near the boundary of a convex domain Ω, which are of the type

$$\det D^2u = f \quad \text{in } \Omega, \qquad f \sim d_{\partial\Omega}^{-\alpha} \quad \text{near } \partial\Omega,$$

where $d_{\partial\Omega}$ represents the distance to $\partial\Omega$ and $-\alpha$ is a negative power with $\alpha \in (0, 2)$. We study the boundary regularity of the solutions and establish a localization theorem for boundary sections.

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1. Introduction

In this paper we consider degenerate Monge–Ampère equations of the type

$$\det D^2u = f \quad \text{in } \Omega, \qquad f \sim d_{\partial\Omega}^{-\alpha} \quad \text{near } \partial\Omega, \tag{1.1}$$

where $d_{\partial\Omega}$ represents the distance to the boundary of the domain Ω and $-\alpha$ is a negative power with $\alpha \in (0, 2)$.

Boundary estimates for the Monge–Ampère equation in the nondegenerate case $f \in C^2(\overline{\Omega})$, $f > 0$, were obtained by Ivočkina [8], Krylov [9], Caffarelli–Nirenberg–Spruck [3] (see also [1,15]).

In [12], a localization theorem at boundary points was proved when the right hand side f is only bounded away from 0 and ∞. It states that under natural local assumptions on the domain and boundary data, the sections $S_h(x_0)$ with $x_0 \in \partial\Omega$ are "equivalent" to half-ellipsoids centered at x_0. This extends up to the boundary a result that is valid for sections compactly included in Ω, which is a consequence of John's lemma from convex geometry. These localization theorems are the key ingredients in establishing optimal $C^{2,\alpha}$ and $W^{2,p}$ estimates for solutions under

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further regularity properties of the right-hand side f and boundary data (see [2,12, 13]).

In [14], the first author studied degenerate Monge–Ampère equations of the type

$$\det D^2 u = f \quad \text{in } \Omega, \qquad f \sim d_{\partial\Omega}^{\alpha} \quad \text{near } \partial\Omega, \tag{1.2}$$

where $\alpha > 0$ is a positive power. A localization theorem and pointwise C^2 estimate were established in [14] and they were later used in [10] to prove the global smoothness for the eigenfunctions of the Monge–Ampère operator $(\det D^2 u)^{1/n}$.

In this paper, we consider the case of the Monge-Ampère equation with right hand side which degenerates to ∞ near the boundary of Ω. This type of equations appear for example in the study of affine spheres in gemetry [4,5], the p-Minkowski problem [11], or in optimal transportation problems involving two densities with only one of them having compact support.

We study the case when f is "comparable" with a negative power $d_{\partial\Omega}^{-\alpha}$ of the distance function to $\partial\Omega$. It can be checked from a simple 1D example that the Dirichlet problem for equation (1.1) is well posed only for $\alpha \in (0, 2)$. Moreover, when $\alpha \in (0, 1)$ solutions are expected to have bounded gradients, and when $\alpha \in [1, 2)$ the gradient should tend to ∞ as we approach the boundary. We study the geometry of boundary sections of solutions to (1.1) and prove two localization theorems Theorems 1.1 and 1.4 depending whether α is smaller or larger than 1.

We first give the localization theorem for the case $\alpha \in (0, 1)$. It states that under appropriate assumptions on the domain and boundary data, the sections

$$S_h(x_0) := \left\{x \in \bar{\Omega} |\ u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + h\right\}$$

with $x_0 \in \partial\Omega$ have the shape of half-ellipsoids centered at x_0.

Theorem 1.1. *Assume $\Omega \subset \mathbb{R}^n$ is a bounded convex set, $\partial\Omega \in C^2$. Let $u : \overline{\Omega} \to \mathbb{R}$ be continuous, convex, satisfying*

$$\det D^2 u = f, \qquad \lambda_0 d_{\partial\Omega}^{-\alpha} \le f \le \Lambda_0 d_{\partial\Omega}^{-\alpha} \quad \text{in } \Omega \tag{1.3}$$

for some $\alpha \in (0, 1)$, and on $\partial\Omega$, u separates quadratically from its tangent plane, namely

$$\mu|x-x_0|^2 \le u(x) - u(x_0) - \nabla u(x_0)\cdot(x-x_0) \le \mu^{-1}|x-x_0|^2, \quad \forall x, x_0 \in \partial\Omega, \tag{1.4}$$

for some $\mu > 0$. Then there is a constant $c > 0$ depending only on n, λ_0, Λ_0, α, μ, $\operatorname{diam}(\Omega)$ *and $\|\partial\Omega\|_{C^2}$ such that for each $x_0 \in \partial\Omega$ and $h \le c$ we have*

$$\mathcal{E}_{ch}(x_0) \cap \overline{\Omega} \subset S_h(x_0) \subset \mathcal{E}_{c^{-1}h}(x_0),$$

where

$$\mathcal{E}_h(x_0) := \left\{|(x - x_0)_\tau|^2 + |(x - x_0) \cdot \nu_{x_0}|^{2-\alpha} < h\right\}, \quad \forall h > 0,$$

ν_{x_0} denotes the unit inner normal to $\partial\Omega$ at x_0 and

$$(x-x_0)_\tau := (x-x_0) - [(x-x_0)\cdot\nu_{x_0}]\nu_{x_0}$$

is the projection of $x-x_0$ onto the tangent plane of $\partial\Omega$ at x_0.

Theorem 1.1 states that a boundary section S_h is equivalent to an ellipsoid of axes $h^{\frac{1}{2}}$ in the tangential direction to $\partial\Omega$ and $h^{\frac{1}{2-\alpha}}$ in the normal. As a corollary, it can be proved that the maximal interior sections have the same geometry as boundary sections. Namely, for any $y_0 \in \Omega$, let $S_{\bar h}(y_0)$ denote the maximal interior section centered at y_0 which becomes tangent to $\partial\Omega$ at some point x_0. Then $S_{\bar h}(y_0)$ is equivalent to an ellipsoid of axes $\bar h^{\frac{1}{2}}$ in the tangential direction to $\partial\Omega$ at x_0 and $\bar h^{\frac{1}{2-\alpha}}$ in the normal ν_{x_0}.

We remark that if $u|_{\partial\Omega} = \varphi$ and $\partial\Omega \in C^3$, $\varphi \in C^3(\partial\Omega)$, and Ω is uniformly convex, then the quadratic separation condition (1.4) is satisfied. The proof is given in [12, Proposition 3.2], where only the lower bound of det D^2u is used. Since in our degenerate case, det D^2u is also bounded below by a constant, the estimate still applies.

Theorem 1.1 implies global $W^{2,p}$ estimates of solutions if we assume further that $f = g\, d_{\partial\Omega}^{-\alpha}$ for some function $g \in C(\overline{\Omega})$ which is strictly positive. In a subsequent work we will show that $u \in W^{2,p}(\Omega)$ for any $p < \frac{1}{\alpha}$.

For the case $\alpha \in (0,1)$, we establish the following Liouville type theorem for global solutions to (1.1).

Theorem 1.2. *Let $u \in C(\overline{\mathbb{R}^n_+})$ be a convex function that satisfies*

$$c_0\left(|x'|^2 + x_n^{2-\alpha}\right) \le u(x) \le c_0^{-1}\left(|x'|^2 + x_n^{2-\alpha}\right) \tag{1.5}$$

for some $c_0 > 0$ and

$$\det D^2u = x_n^{-\alpha}, \qquad u(x',0) = \frac{1}{2}|x'|^2. \tag{1.6}$$

Then

$$u(x) = \frac{1}{2}|x'|^2 + \frac{x_n^{2-\alpha}}{(2-\alpha)(1-\alpha)}.$$

Theorem 1.1 and the Liouville theorem imply a pointwise C^2 tangential estimate at the boundary.

Theorem 1.3. *Assume that $\Omega \subset \{x_n > 0\}$ is a bounded convex set, $0 \in \partial\Omega$, $\partial\Omega \in C^2$ near the origin, and the principal curvatures of $\partial\Omega$ at 0 are strictly positive. Let $u \in C(\overline{\Omega})$ be a convex solution to the equation*

$$\det D^2u = f(x)d_{\partial\Omega}^{-\alpha} \quad \text{in } \Omega, \qquad u = \varphi \quad \text{on } \partial\Omega.$$

for some $\alpha \in (0, 1)$, where f is a nonnegative function that is continuous at the origin and $f(0) > 0$, the boundary data φ is C^2 at 0, and it separates quadratically away from 0. Assume further that

$$u(0) = 0, \quad \nabla u(0) = 0.$$

Then there exists a constant $a > 0$ such that

$$u(x) = Q(x') + a x_n^{2-\alpha} + o\left(|x'|^2 + x_n^{2-\alpha}\right),$$

where Q represents the quadratic part of the boundary data φ at the origin.

Next we give the localization theorem when $\alpha \in (1, 2)$. In this case we consider the maximal sections included in Ω which become tangent to $\partial\Omega$ at boundary points.

Theorem 1.4. *Assume $\Omega \subset \mathbb{R}^n$ is uniformly convex, $\partial\Omega \in C^2$. Assume further that $0 \in \partial\Omega$ and the x_n coordinate axis lies in the direction ν_0 (ν_0 is the unit inner normal to $\partial\Omega$ at 0).*

Let $u : \overline{\Omega} \to \mathbb{R}$ be continuous, convex, satisfying

$$\det D^2 u = f, \qquad \lambda_0 d_{\partial\Omega}^{-\alpha} \le f \le \Lambda_0 d_{\partial\Omega}^{-\alpha} \quad \text{in } \Omega,$$

for some $\alpha \in [1, 2)$, and assume $u|_{\partial\Omega} = \varphi \in C^2$. Suppose that $S_{\bar h}(y_0)$ is the maximal section included in Ω which becomes tangent to $\partial\Omega$ at 0. Then

$$\nabla_{x'} u(y_0) = \nabla_{x'} \varphi(0), \qquad M = -u_n(y_0) \ge -C,$$

and the following hold:

(i) *If $\alpha \in (1, 2)$, denote $\beta := \frac{n+\alpha-1}{n}$, then we have*

$$c\bar h^{\frac{1-\beta}{2-\beta}} \le \max\{M, 1\} \le C\bar h^{\frac{1-\beta}{2-\beta}}, \qquad c\bar h^{\frac{1}{2-\beta}} \le d_{\partial\Omega}(y_0) \le C\bar h^{\frac{1}{2-\beta}},$$
$$\left\{|x'|^2 + |x_n| \le c\bar h^{\frac{1}{2-\beta}}\right\} \subset S_{\bar h}(y_0) - y_0 \subset \left\{|x'|^2 + |x_n| \le C\bar h^{\frac{1}{2-\beta}}\right\};$$

(ii) *If $\alpha = 1$, denote $\bar h_* := \min\{\bar h, 1\}$, then we have*

$$-c\log(C\bar h) \le |M|^n \le -C\log(c\bar h), \qquad c\bar h_*^C \le d_{\partial\Omega}(y_0) \le C\bar h_*^c,$$
$$B_{c\bar h_*^C} \subset S_{\bar h}(y_0) - y_0 \subset B_{C\bar h_*^c}.$$

Here the constants c, C depend only on $n, \lambda_0, \Lambda_0, \alpha, \operatorname{diam}(\Omega)$, and $\varphi, \partial\Omega$ up to their second derivatives.

In the case $\alpha \in (1, 2)$, Theorem 1.4 states that for any $y_0 \in \Omega$, the maximal interior section $S_{\bar h}(y_0)$ which becomes tangent to $\partial\Omega$ at some point x_0 is equivalent to an ellipsoid of axes $\bar h^{\frac{1}{2(2-\beta)}}$ in the tangential direction to $\partial\Omega$ at x_0 and $\bar h^{\frac{1}{2-\beta}}$ in the normal ν_{x_0}. For the border line case $\alpha = 1$, it cannot be concluded from (ii) that $S_{\bar{h}}(y_0)$ is equivalent to an ellipsoid whose shape depends only on $\bar{h}$, y_0 and Ω. Probably more precise information is needed on the ratio between f and $d_{\partial\Omega}^{-1}$ in order to reach a similar conclusion as in the case $\alpha \in (1, 2)$.

The proofs of Theorems 1.1 and 1.4 are quite different. Theorem 1.4 follows directly from comparison with explicit barriers. Theorem 1.1 is much more involved and most of the paper will be devoted towards its proof. We will follow similar ideas as in the nondegenerate case treated in [12].

The paper is organized as follows. In Section 2 we introduce some notation, then we reduce Theorem 1.1 to its local version Theorem 2.1. This is further reduced to Theorem 2.2, where the distance function is replaced by x_n. We also give a more precise quantitative version of Theorem 1.3 (see Theorem 2.3). Sections 3-4 are devoted to the proof of Theorem 2.2. In Section 5, the proof of Theorem 1.2 is given. In Section 6, we give the proof of Theorem 2.3 and then finish the proof of Theorem 1.3. In the last section, we give the proof of Theorem 1.4.

2. Statement of main results

We introduce some notation. We denote points in $\mathbb{R}^n$ as

$$x = (x_1, \dots, x_n) = (x', x_n), \quad x' \in \mathbb{R}^{n-1}.$$

Let u be a convex function defined on a convex set $\overline{\Omega}$, we denote by $S_h(x_0)$ the section centered at x_0 and at height $h > 0$,

$$S_h(x_0) := \left\{x \in \overline{\Omega} \,|\, u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + h\right\}.$$

When $x_0 \in \partial\Omega$, the term $\nabla u(x_0)$ is understood in the sense that

$$x_{n+1} = u(x_0) + \nabla u(x_0) \cdot (x - x_0)$$

is a supporting hyperplane for the graph of u at x_0 but for any $\epsilon > 0$,

$$x_{n+1} = u(x_0) + (\nabla u(x_0) + \epsilon \nu_{x_0}) \cdot (x - x_0)$$

is not a supporting hyperplane, where ν_{x_0} denotes the unit inner normal to $\partial\Omega$ at x_0. We denote for simplicity $S_h = S_h(0)$, and sometimes when we specify the dependence on the function u we use the notation $S_h(u) = S_h$.

For a set $E \subset \mathbb{R}^n$, we always denote $\pi(E)$ the projection of E into $\mathbb{R}^{n-1}$, *i.e.*,

$$\pi(E) := \left\{x' \in \mathbb{R}^{n-1} : \exists\, t \in \mathbb{R} \; s.t. \; (x', t) \in E\right\}.$$

In the case $\alpha \in (0, 1)$, for any $h > 0$ we often use the particular sets

$$\mathcal{E}_h := \left\{|x'|^2 + x_n^{2-\alpha} < h\right\}, \qquad \mathcal{E}_h^+ := \mathcal{E}_h \cap \{x_n > 0\},$$

and the diagonal matrix

$$F_h := \operatorname{diag}\left(h^{\frac{1}{2}}, h^{\frac{1}{2}}, \ldots, h^{\frac{1}{2}}, h^{\frac{1}{2-\alpha}}\right)$$

in our estimates.

Next we give a local version of Theorem 1.1. Our assumptions are the following.

Let $\Omega \subset \mathbb{R}^n$ be a open convex set. Assume that for some fixed small $\rho > 0$,

$$B_\rho(\rho e_n) \subset \Omega \subset \{x_n > 0\} \cap B_{\frac{1}{\rho}}, \tag{2.1}$$

and

$$\begin{gathered}\Omega \text{ contains an interior ball of radius } \rho \text{ tangent to } \partial\Omega \\ \text{at each point on } \partial\Omega \cap \{x_n \le \rho\}.\end{gathered} \tag{2.2}$$

The part $\partial\Omega \cap \{x_n \le \rho\}$ is then given by $x_n = g(x')$ for some convex function g, where

$$g \in C^2\left(\pi(\partial\Omega \cap \{x_n < \rho\})\right), \quad g(0) = 0, \quad \nabla g(0) = 0. \tag{2.3}$$

Let $u : \overline{\Omega} \to \mathbb{R}$ be a convex solution to

$$\det D^2 u = f, \quad 0 < \lambda(x_n - g)^{-\alpha} \le f \le \Lambda(x_n - g)^{-\alpha} \quad \text{in } \Omega \cap \{x_n < \rho/2\} \tag{2.4}$$

for some $\alpha \in (0, 1)$. Moreover,

$$x_{n+1} = 0 \text{ is the tangent plane to } u \text{ at } 0, \tag{2.5}$$

that is, $u \ge 0$, $u(0) = 0$, $\nabla u(0) = 0$ in the sense that $x_{n+1} = \epsilon x_n$ is not a supporting plane for the graph of u at 0 for any $\epsilon > 0$.

We also assume that u separates quadratically on $\partial\Omega$ (in a neighborhood of $\{x_n = 0\}$) from the tangent plane at 0, *i.e.*,

$$\mu|x|^2 \le u(x) \le \mu^{-1}|x|^2 \quad \text{on } \partial\Omega \cap \{x_n \le \rho\}. \tag{2.6}$$

Theorem 2.1. *Assume Ω and u satisfy* (2.1)-(2.6). *Then there is a constant $c > 0$ depending only on $n, \lambda, \Lambda, \alpha, \mu$ and ρ such that for each $h \le c$ we have*

$$\mathcal{E}_{ch} \cap \overline{\Omega} \subset S_h \subset \mathcal{E}_{c^{-1}h}.$$

Assume Ω and u satisfy the hypotheses in Theorem 1.1. Fix a point $x_0 \in \partial\Omega$, by a translation and a rotation of coordinates we can assume that $x_0 = 0$, and the x_n coordinate axis lies in the direction ν_{x_0}. Since $\partial\Omega \in C^2$, there exists $\rho > 0$ such that (2.1)-(2.3) hold, and after subtracting a linear function we have (2.5) and (2.6). By (2.1)-(2.3), it is easy to see that

$$\|D^2 g\|_{C(\pi(\partial\Omega \cap \{x_n \le \rho/2\}))} \le C(n, \rho) \tag{2.7}$$

and therefore

$$d_{\partial\Omega}(x) \le x_n - g(x') \le C'(n, \rho) d_{\partial\Omega}(x),$$

where $C(n, \rho)$ and $C'(n, \rho)$ are constants depending only on n and ρ. It follows that u satisfies (2.4) with $\lambda := \lambda_0$, $\Lambda := C'(n, \rho)\Lambda_0$. Therefore we reduce the proof of Theorem 1.1 to that of Theorem 2.1 above.

Let Ω and u satisfy the hypotheses in Theorem 2.1. By constructing some lower barrier for u, we will prove in Section 3 that in some domain $\Omega_0 \subset \Omega$ we have $x_n - g \sim x_n$, and u still satisfies the quadratic separation (2.6) on $\partial\Omega_0$ in a neighborhood of $\{x_n = 0\}$. Therefore we reduce the proof of Theorem 2.1 to that of Theorem 2.2 below.

We assume (2.1), (2.5), (2.6) hold while replacing the equation (2.4) by

$$\det D^2 u = f, \quad 0 < \lambda x_n^{-\alpha} \le f \le \Lambda x_n^{-\alpha} \quad \text{in } \Omega \cap \{x_n < \rho\}. \tag{2.8}$$

Note that we do not assume (2.2) and (2.3) hold here.

Theorem 2.2. *Assume Ω and u satisfy* (2.1), (2.5), (2.6) *and* (2.8). *Then there is a constant $c > 0$ depending only on n, λ, Λ, α, μ and ρ such that for each $h \le c$ we have*

$$\mathcal{E}_{ch} \cap \overline{\Omega} \subset S_h \subset \mathcal{E}_{c^{-1}h}.$$

We prove Theorem 2.2 using the compactness methods in [12]. We first obtain some preliminary estimates about u. Next we consider the rescaling v of u. Then we reduce the proof of the theorem to that of a statement about v. We reduce this to the proof of a statement (Proposition 4.2) about the limiting function (still denoted by u) of such v. Different from the case that $\alpha = 0$ (in this case the estimate of the volume of $S_t(v)$ is $|S_t(v)|^2 \sim t^n$), the estimate of the volume of $S_t(v)$ becomes

$$(x_t^*(v) \cdot e_n)^{-\alpha} |S_t(v)|^2 \sim t^n, \tag{2.9}$$

where $x_t^*(v)$ is the center of mass of $S_t(v)$. The limiting function u also satisfies this estimate. To prove Proposition 4.2, we construct some lower barrier for the limiting function u and use (2.9). Since we do not have the estimate of $|S_h(u)|$, we also use the convexity of the original solution to estimate the quantity $x_t^*(v) \cdot e_n$ from below. The estimate (2.9) brings another difficulty when we prove Proposition 4.2. We use John's lemma and find an ellipsoid E_h equivalent to the section $S_h(u)$ of the limiting solution u. In the case $\alpha = 0$, we use the estimate $|E_h|^2 \sim h^n$ to estimate the shape of $S_h(u)$, but in our degenerate case, we do not have the estimate of the volume of E_h. For this, we use the estimate $(x_h^*(u) \cdot e_n)^{-\alpha} |E_h|^2 \sim h^n$ to obtain an estimate of the shape of $S_h(u)$ in terms of the quantity $x_h^*(u) \cdot e_n$. Using this estimate, we rescale u and reduce Proposition 4.2 to the lower-dimensional case. Again, since we do not have the estimate of $|E_h|$, we perform a different rescaling (which corresponds to our estimate (2.9)) from the $\alpha = 0$ case.

At the end of this section we give a more precise quantitative version of Theorem 1.3 as follows.

Theorem 2.3. *For any $\eta > 0$ there exists $\epsilon_0 > 0$ depending only on η, n, α such that if* (2.1)-(2.5) *hold with $\lambda = 1 - \epsilon_0$, $\Lambda = 1 + \epsilon_0$ and*

$$\left(\frac{1}{2} - \epsilon_0\right)|x'|^2 \le u(x) \le \left(\frac{1}{2} + \epsilon_0\right)|x'|^2 \quad \text{on } \partial\Omega \cap \{x_n \le \rho\}, \tag{2.10}$$

then for all $h \le c$, we have

$$(1-\eta)S_h(U_0) \cap \overline{\Omega} \subset S_h(u) \subset (1+\eta)S_h(U_0),$$

where

$$U_0(x) := \frac{1}{2}|x'|^2 + \frac{x_n^{2-\alpha}}{(2-\alpha)(1-\alpha)}, \qquad S_h(U_0) := \{x \in \mathbb{R}^n : U_0(x) < h\},$$

and the constant $c > 0$ depends only on η, n, α, ρ.

3. Proof of Theorem 2.2 (I)

As mentioned in Section 2, we first show that we can reduce the proof of Theorem 2.1 to that of Theorem 2.2.

Proposition 3.1. *Theorem* 2.2 *implies Theorem* 2.1.

Proof. In this proof we always denote by $c, C, c_i, C_i (i = 0, 1, 2, \dots)$ constants depending only on $n, \lambda, \Lambda, \mu, \alpha$ and ρ. For simplicity of notation, their values may change from line to line whenever there is no possibility of confusion.

Let

$$v_0 := \mu|x'|^2 + \frac{\Lambda}{(2-\alpha)(1-\alpha)\,\mu^{n-1}}(x_n - g)^{2-\alpha}.$$

Then by straightforward computation and using (2.7), we obtain that

$$\begin{aligned} \det D^2 v_0 &= \frac{\Lambda}{\mu^{n-1}}(x_n - g)^{-\alpha}\det\left(2\mu I_{n-1} - \frac{\Lambda}{(1-\alpha)\,\mu^{n-1}}(x_n - g)^{1-\alpha} D^2 g\right) \\ &\ge \Lambda(x_n - g)^{-\alpha} \quad \text{in } \Omega \cap \{x_n < c_*\}, \end{aligned} \tag{3.1}$$

where $c_* \le \rho/2$ is small depending only on n, Λ, μ, α and ρ.

Denote $D := \pi\,(\Omega \cap \{x_n = c_*\})$. For $x' \in D$, define

$$g^*(x') := \sup\left\{ l(x') : l \le g \text{ in } D,\ l \text{ is affine, and } |\nabla l| \le \frac{c_*\rho}{2} \right\}.$$

Then g^* is convex in D since it is the supremum of a family of convex functions.

We claim that for any $x \in \Omega \cap \{x_n = c_*\}$, we have

$$x_n - g^*(x') \ge \frac{c_*}{2}. \tag{3.2}$$

Indeed, if l is affine, $l \le g$ in D and $|\nabla l| \le \frac{c_*\rho}{2}$, then

$$0 = g(0) \ge l(0) = l(x') - \nabla l \cdot x',$$

it follows that

$$l(x') \le \nabla l \cdot x' \le \frac{c_*\rho}{2} \cdot \frac{1}{\rho} = \frac{c_*}{2},$$

where we use the fact that $\Omega \subset B^+_{1/\rho}$. Thus the claim follows.

We also claim that

$$\pi(\Omega \cap \{x_n \le c_0\rho\}) \subset D \cap \left\{|\nabla g| \le \frac{c_*\rho}{2}\right\} \subset \{g^* = g\} \tag{3.3}$$

for some small constant c_0.

Indeed, the second inclusion in (3.3) follows easily from the convexity of g and the definition of g^*. Therefore we only need to prove the first inclusion. Let $c_0 > 0$ be a small constant to be chosen. For any $x_0 \in \partial\Omega \cap \{x_n \le c_0\rho\}$, we have $B_\rho(y_0) \subset \Omega \subset \{x_n \ge 0\}$ by (2.2), where $y_0 := x_0 + \rho\nu_{x_0}$. Let

$$t = \inf_{x \in B_\rho(y_0)} x_n,$$

then $(y_0', t) \in \partial B_\rho(y_0)$ and

$$\rho\nu_{x_0} \cdot e_n = (y_0 \cdot e_n - t) - (x_0 \cdot e_n - t) = \rho - (x_0 \cdot e_n - t) \ge (1 - c_0)\rho,$$

which gives

$$\frac{1}{\sqrt{1 + |\nabla g(x_0')|^2}} = \nu_{x_0} \cdot e_n \ge 1 - c_0.$$

Hence,

$$|\nabla g(x_0')| \le \sqrt{\left(\frac{1}{1 - c_0}\right)^2 - 1} \le \frac{c_*\rho}{2} \tag{3.4}$$

if c_0 is small. The desired conclusion (3.3) follows.

Let

$$v^* := \mu|x'|^2 + \frac{\Lambda}{(2-\alpha)(1-\alpha)\,\mu^{n-1}}(x_n - g)^{2-\alpha} - C^*(x_n - g^*(x')).$$

Then v^* is a lower barrier for u in $\Omega \cap \{x_n \le c_*\}$ if C^* is large depending only on n, Λ, μ, α and ρ.

Indeed, since g^* is convex, we find from (3.1) that v^* is a subsolution of the equation

$$\det D^2 w = \Lambda(x_n - g)^{-\alpha}.$$

On $\partial\Omega \cap \{x_n \le c_*\}$, we have $x_n - g^* = g - g^* \ge 0$, which implies

$$v^* \le \mu|x'|^2 \le u.$$

On $\Omega \cap \{x_n = c_*\}$, we obtain from (3.2) that

$$v^* \le \frac{\mu}{\rho^2} + \frac{\Lambda}{(2-\alpha)(1-\alpha)\,\mu^{n-1}} c_*^{2-\alpha} - C^* \frac{c_*}{2} \le 0 \le u$$

if C^* is large.

Thus,

$$v^* \le u \quad \text{in } \Omega \cap \{x_n \le c_*\}.$$

This together with (3.3) implies that

$$u \ge \mu|x'|^2 - C^*(x_n - g(x')) \quad \text{in } \Omega \cap \{x_n \le c_0\rho\}. \tag{3.5}$$

Therefore, if δ is small, we have

$$u \ge \frac{\mu|x'|^2}{2} \quad \text{in } \Omega \cap \{x_n \le c_0\rho\} \cap \{x_n \le g(x') + \delta|x'|^2\}. \tag{3.6}$$

On the other hand, the convexity of u and the quadratic separation of u on $\partial\Omega \cap \{x_n \le \rho\}$ (see (2.6)) implies that

$$u \le C|x'|^2 \quad \text{in } \Omega \cap \{x_n \le c_0\rho\} \cap \{x_n \le g(x') + \delta|x'|^2\}. \tag{3.7}$$

In particular, if we denote $\Omega_0 := \Omega \cap \{x_n < c_0\rho\} \cap \{x_n > g(x') + \delta|x'|^2\}$, then the above two estimates hold on $\partial\Omega_0 \cap \{x_n \le c_0\rho\}$.

We have

$$\lambda x_n^{-\alpha} \le \det D^2 u \le C x_n^{-\alpha} \quad \text{in } \Omega_0.$$

We apply Theorem 2.2 to u in Ω_0 and obtain that

$$\mathcal{E}_{ch} \cap \overline{\Omega_0} \subset S_h \cap \overline{\Omega_0} \subset \mathcal{E}_{Ch}, \quad \forall h \le c.$$

We claim that the last estimate also holds for S_h (instead of $S_h \cap \overline{\Omega_0}$). Indeed, we have by (3.7)

$$(\overline{\Omega} \setminus \Omega_0) \cap \mathcal{E}_{ch} \subset S_h$$

and therefore

$$\mathcal{E}_{ch} \cap \overline{\Omega} \subset S_h.$$

On the other hand, we obtain from (3.6) that

$$(S_h \cap \{x_n \le c_0\rho\}) \setminus \Omega_0 \subset \left\{|x'| \le C h^{\frac{1}{2}}\right\}.$$

Since

$$(\overline{\Omega} \setminus \Omega_0) \cap \{x_n \le c_0\rho\} \subset \left\{x_n \le g(x') + \delta|x'|^2\right\},$$

we obtain

$$S_h \setminus \Omega_0 \subset \{|x'| \le C h^{\frac{1}{2}}, x_n \le Ch\} \subset \mathcal{E}_{Ch}. \qquad \square$$

In the following we give the first part of the proof of Theorem 2.2. In the remaining part of this section we denote by $c, C, c_i, C_i (i = 0, 1, 2, \dots)$ positive constants depending on n, λ, Λ, μ and α. The dependence of various constants also on ρ will be denoted by $c(\rho), C(\rho), c_i(\rho), C_i(\rho) (i = 0, 1, 2, \dots)$.

Proposition 3.2. *Assume that Ω and u satisfy the hypotheses of Theorem 2.2. Then, for each $h \leq c(\rho)$ there exists a linear transformation (sliding along $x_n = 0$)*

$$A_h x = x - \nu x_n, \quad \nu_n = 0, \quad |\nu| \leq C(\rho) h^{-\frac{n}{2(n+1-\alpha)}},$$

such that the rescaled function

$$\tilde{u}(A_h x) = u(x)$$

satisfies in

$$\tilde{S}_h := A_h S_h = \{\tilde{u} < h\}$$

the following:

(i) *the center of mass $\tilde{x}_h^*$ of $\tilde{S}_h$ lies on the x_n axis,* i.e. $\tilde{x}_h^* = d_h e_n$*;*

(ii)
$$c h^n \leq |S_h|^2 d_h^{-\alpha} \leq C h^n.$$

And after a rotation of the $x_1, \dots, x_{n-1}$ variables we have

$$\tilde{x}_h^* + c D_h B_1 \subset \tilde{S}_h \subset C D_h B_1,$$

where $D_h :=$ diag$(d_1, d_2, \dots, d_{n-1}, d_n)$ is a diagonal matrix that satisfies

$$\left(\prod_1^{n-1} d_i^2\right) d_n^{2-\alpha} = h^n \tag{3.8}$$

and

$$c d_h \leq d_n \leq C d_h;$$

(iii) *Denote $\tilde{\Omega}_h := A_h \Omega$ and $\tilde{G}_h := \partial \tilde{S}_h \cap \{\tilde{u} < h\}$, then $\tilde{G}_h$ is a graph* i.e.

$$\tilde{G}_h = (x', \tilde{g}_h(x')) \qquad \text{with} \qquad \tilde{g}_h(x') \leq \frac{2}{\rho} |x'|^2$$

and the function $\tilde{u}$ satisfies on $\tilde{G}_h$

$$\frac{\mu}{2} |x'|^2 \leq \tilde{u}(x) \leq 2\mu^{-1} |x'|^2.$$

Proof. Let

$$v := \mu|x'|^2 + \frac{\Lambda}{(2-\alpha)(1-\alpha)\,\mu^{n-1}} x_n^{2-\alpha} - C(\rho)x_n,$$

where $C(\rho)$ is large such that

$$\frac{\Lambda}{(2-\alpha)(1-\alpha)\,\mu^{n-1}} x_n^{2-\alpha} - \frac{C(\rho)}{2} x_n \le 0 \quad \text{in } \Omega \cap \{x_n \le \rho\},$$

then it is straightforward to check that v is a lower barrier for u in $\Omega \cap \{x_n \le \rho\}$. It follows that

$$S_h \cap \{x_n \le \rho\} \subset \{v < h\} \subset \{x_n > c(\rho)(\mu|x'|^2 - h)\}. \tag{3.9}$$

Let x_h^* be the center of mass of S_h and $d_h := x_h^* \cdot e_n$. We claim that

$$d_h \ge c_0(\rho) h^{\frac{n}{n+1-\alpha}} \tag{3.10}$$

for some small $c_0(\rho) > 0$.

Indeed, if

$$d_h \ge c(n)\rho$$

with $c(n)$ depending only on n, then (3.10) holds clearly. On the other hand, if

$$d_h \le c(n)\rho,$$

then by John's lemma, for some constant $C(n)$ depending only on n we have

$$S_h \subset \left\{ x_n \le C(n) d_h \le \frac{\rho}{2} \right\}$$

if $c(n)$ is small. If (3.10) does not hold, then from the last estimate, (3.9) and John's lemma that

$$S_h \subset \left\{ x_n \le C(n)c_0(\rho) h^{\frac{n}{n+1-\alpha}} \le h^{\frac{n}{n+1-\alpha}} \right\} \cap \left\{ |x'| \le C_1(\rho) h^{\frac{n}{2(n+1-\alpha)}} \right\}.$$

Define

$$w = \epsilon x_n + \frac{h}{2} \left(\frac{|x'|}{C_1(\rho) h^{\frac{n}{2(n+1-\alpha)}}} \right)^2 + \frac{\Lambda [C_1(\rho)]^{2(n-1)} h}{(2-\alpha)(1-\alpha)} \left(\frac{x_n}{h^{\frac{n}{n+1-\alpha}}} \right)^{2-\alpha}.$$

Then we have in S_h,

$$w \le \epsilon + \frac{h}{2} + \frac{\Lambda [C_1(\rho)]^{2(n-1)} h}{(2-\alpha)(1-\alpha)} [C(n)c_0(\rho)]^{2-\alpha} \le h$$

if $c_0(\rho)$ is small. On $S_h \cap \partial\Omega$,

$$w \le \frac{\epsilon}{\rho}|x'|^2 + \frac{h^{\frac{1-\alpha}{n+1-\alpha}}}{2C_1(\rho)^2}|x'|^2 + \frac{\Lambda[C_1(\rho)]^{2(n-1)}h^{\frac{1-\alpha}{n+1-\alpha}}}{(2-\alpha)(1-\alpha)} \cdot \frac{|x'|^2}{\rho} \le \mu|x'|^2$$

if $h \le c(\rho)$. In conclusion,

$$w \le u \quad \text{in } S_h,$$

which contradicts that $\nabla u(0) = 0$. Thus (3.10) holds.

Now we prove that for all small h we have

$$d_h \le C_0 h^{\frac{1}{2-\alpha}} \tag{3.11}$$

for some large constant C_0.

Assume by contradiction that $d_h \ge C_0 h^{\frac{1}{2-\alpha}}$. Then (3.9) implies that

$$|(x_h^*)'| \le C(\rho) d_h^{\frac{1}{2}}. \tag{3.12}$$

From (2.1) and (2.6) we know that if $h \le c(\rho)$, then S_h contains the set $\partial\Omega \cap \{x_n \le \rho\} \cap \{x : |x'| \le (ch)^{\frac{1}{2}}\}$ for some small c depending only on μ. Therefore S_h contains the convex set generated by $\partial\Omega \cap \{x_n \le \rho\} \cap \{x : |x'| \le (ch)^{\frac{1}{2}}\}$ and the point x_h^*. Let $x_n = b$ be a hyperplane in $\mathbb{R}^n$, where $b \le \rho$ is chosen such that

$$ch + (b-\rho)^2 = \rho^2.$$

For each $x_0 \in \partial\Omega \cap \{x_n \le \rho\} \cap \{x : |x'| = (ch)^{\frac{1}{2}}\}$, let y_0 be the intersection of the segment $\overline{x_0 x_h^*}$ (which is the segment joining x_0 and x_h^*) and the hyperplane $x_n = b$. We can write

$$y_0 = (1-\theta)x_0 + \theta x_h^*$$

for some $\theta = \theta(x_0) \in (0,1)$. Since

$$(1-\theta)x_0 \cdot e_n + \theta d_h = y_0 \cdot e_n = b \le \frac{ch}{\rho},$$

we obtain

$$\theta \le \frac{ch}{\rho d_h}.$$

Recall that $d_h \ge C_0 h^{\frac{1}{2-\alpha}}$, then by (3.12) we obtain that for all small h

$$\begin{aligned} |y_0'| = |(1-\theta)x_0' + \theta(x_h^*)'| &\ge |x_0'| - \theta\left(|(x_h^*)'| + |x_0'|\right) \\ &\ge (ch)^{\frac{1}{2}} - \frac{ch}{\rho d_h}\left(C(\rho)d_h^{\frac{1}{2}} + (ch)^{\frac{1}{2}}\right) \\ &\ge \frac{(ch)^{\frac{1}{2}}}{2}. \end{aligned}$$

Since S_h contains the convex set generated by all such y_0 and x_h^*, this means that S_h contains a convex set of measure $c(n)\left(\frac{(ch)^{\frac{1}{2}}}{2}\right)^{n-1} d_h$, and therefore

$$|S_h| \geq c(n)\left(\frac{(ch)^{\frac{1}{2}}}{2}\right)^{n-1} d_h. \tag{3.13}$$

Let v solves

$$\det D^2 v = \lambda(C(n)d_h)^{-\alpha} \leq \det D^2 u \quad \text{in } S_h, \qquad v = h \quad \text{on } \partial S_h.$$

Then

$$v \geq u \geq 0 \quad \text{in } S_h.$$

It follows

$$h^n \geq |h - \min_{S_h} v|^n \geq c(n,\alpha)\lambda d_h^{-\alpha}|S_h|^2.$$

Namely,

$$d_h^{-\alpha}|S_h|^2 \leq C(n,\lambda,\alpha)h^n. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$d_h \leq C h^{\frac{1}{2-\alpha}}.$$

We reach a contradiction if C_0 is sufficiently large, hence (3.11) is proved.

Define

$$A_h x = x - \nu x_n, \quad \nu = \frac{(x_h^*)'}{d_h}$$

and

$$\tilde{u}(A_h x) = u(x).$$

Then the center of mass of $\tilde{S}_h = A_h S_h$ is

$$\tilde{x}_h^* = A_h x_h^*$$

and lies on the x_n-axis from the definition of A_h. We obtain from (3.9) and (3.10) that

$$|\nu| = \frac{|(x_h^*)'|}{d_h} \leq C(\rho) d_h^{-\frac{1}{2}} \leq C(\rho) h^{-\frac{n}{2(n+1-\alpha)}}. \tag{3.15}$$

Part (i) of Proposition 3.2 follows.

Let $\tilde{\Omega}_h := A_h\Omega$ and $\tilde{G}_h := \partial\tilde{S}_h \cap \partial\tilde{\Omega}_h = \partial\tilde{S}_h \cap \{\tilde{u} < h\}$.

On $\partial\Omega \cap \{x_n \leq \rho\} \cap \{|x'| \leq (\mu^{-1}h)^{\frac{1}{2}}\}$, we have

$$|A_h x - x| = |\nu| x_n \leq C(\rho) h^{-\frac{n}{2(n+1-\alpha)}} |x'|^2 \leq C(\rho) h^{\frac{1-\alpha}{2(n+1-\alpha)}} |x'|.$$

Note that

$$\partial S_h \cap \partial\Omega \subset \{x_n \le \rho\} \cap \left\{|x'| \le (\mu^{-1}h)^{\frac{1}{2}}\right\},$$

thus on $\tilde{G}_h = \partial\tilde{S}_h \cap \partial\tilde{\Omega}_h$,

$$x_n \le \frac{1}{\rho}|(A_h^{-1}x)'|^2 \le \frac{2}{\rho}|x'|^2$$

and

$$\frac{\mu}{2}|x'|^2 \le \tilde{u}(x) = u(A_h^{-1}x) \le \mu^{-1}|(A_h^{-1}x)|^2 \le 2\mu^{-1}|x'|^2.$$

It remains to prove (ii). After a rotation of $x_1, \dots, x_{n-1}$ variables, we can assume that $\tilde{S}_h \cap \{x_n = d_h\}$ is equivalent to an ellipsoid of axes $d_1 \le d_2 \le \dots \le d_{n-1}$ *i.e.*

$$\left\{\sum_1^{n-1}\left(\frac{x_i}{d_i}\right)^2 \le 1\right\} \cap \{x_n = d_h\} \subset \tilde{S}_h \cap \{x_n = d_h\} \subset \left\{\sum_1^{n-1}\left(\frac{x_i}{d_i}\right)^2 \le C(n)\right\}.$$

Thus,

$$\tilde{S}_h \subset \left\{\sum_1^{n-1}\left(\frac{x_i}{d_i}\right)^2 \le C(n)\right\} \cap \{0 \le x_n \le C(n)d_h\}.$$

Since $\tilde{u} \le 2\mu^{-1}|x'|^2$ on $\tilde{G}_h$, we see that the domain of definition of $\tilde{G}_h$ contains a ball in $\mathbb{R}^{n-1}$ of radius $(\mu h/2)^{\frac{1}{2}}$. This implies that

$$d_i \ge c_1 h^{\frac{1}{2}}, \quad i = 1, \dots, n-1. \tag{3.16}$$

Now we prove that

$$d_h^{2-\alpha}\prod_1^{n-1} d_i^2 \ge c_2 h^n. \tag{3.17}$$

Indeed, if the last estimate does not hold, then we construct

$$w := \epsilon x_n + \left[\sum_1^{n-1}\left(\frac{x_i}{d_i}\right)^2 + \left(\frac{x_n}{d_h}\right)^{2-\alpha}\right] \cdot ch.$$

If c_2 is small, then we have

$$\det D^2 w \ge \frac{c^n 2^{n-1}(2-\alpha)(1-\alpha)x_n^{-\alpha}}{c_2} > \Lambda x_n^{-\alpha}.$$

On $\partial\tilde{S}_h \setminus \tilde{G}_h$,

$$w \le \epsilon + C(n,\alpha)ch \le h,$$

and on $\tilde{G}_h$, we use (3.16) and (3.10) to obtain

$$w \le \frac{2\epsilon}{\rho}|x'|^2 + ch\sum_1^{n-1}\left(\frac{x_i}{d_i}\right)^2 + chC(n)^{1-\alpha}\frac{2|x'|^2}{\rho d_h} \le \frac{\mu}{2}|x'|^2$$

if c is small. We conclude that $w \le \tilde{u}$ in $\tilde{S}_h$. This contradicts $\nabla\tilde{u}(0) = 0$ and therefore (3.17) holds.

Since $\tilde{S}_h$ contains the convex set generated by $\left\{\sum_1^{n-1}(\frac{x_i}{d_i})^2 \le 1\right\} \cap \{x_n = d_h\}$ and the point 0, we have

$$|\tilde{S}_h| \ge c(n)\left(\prod_1^{n-1} d_i\right)\cdot d_h.$$

This together with (3.17) and (3.14) implies that

$$Ch^n \ge d_h^{-\alpha}|\tilde{S}_h|^2 \ge c(n)d_h^{2-\alpha}\prod_1^{n-1} d_i^2 \ge ch^n. \tag{3.18}$$

Define d_n from $d_1, \dots, d_{n-1}$ by (3.8), and (3.18) gives

$$cd_h \le d_n \le Cd_h.$$

This proves (ii). □

Theorem 2.2 follows from Proposition 3.2 and the following result.

Lemma 3.3. *Assume that Ω and u satisfy the hypotheses of Theorem 2.2. Then for any $h \le c(\rho)$, we have*

$$d_n \ge ch^{\frac{1}{2-\alpha}}. \tag{3.19}$$

Lemma 3.3 implies Theorem 2.2.

From Lemma 3.3 and Proposition 3.2 we obtain

$$ch^{\frac{1}{2}} \le d_i \le Ch^{\frac{1}{2}},\ i = 1, \dots, n-1, \qquad ch^{\frac{1}{2-\alpha}} \le d_n \le Ch^{\frac{1}{2-\alpha}}.$$

It follows that

$$\tilde{x}_h^* + cF_hB_1 \subset A_hS_h \subset CF_hB_1, \tag{3.20}$$

where we recall from Section 2 that

$$F_hx = \left(h^{\frac{1}{2}}x', h^{\frac{1}{2-\alpha}}x_n\right).$$

Since the domain of definition of $\tilde{G}_h$ contains a ball of radius $(\mu h/2)^{\frac{1}{2}}$, we have

$$cF_hB_1 \cap A_h\overline{\Omega} \subset A_hS_h \subset CF_hB_1. \tag{3.21}$$

It follows that

$$c\mathcal{E}_h \cap A_h\overline{\Omega} \subset A_h S_h \subset C\mathcal{E}_h. \tag{3.22}$$

Denote $A_h x = x - \nu_h x_n$. Using in (3.20) that $S_{h/2} \subset S_h$ we find

$$|\nu_{h/2} - \nu_h| \le C h^{\frac{1}{2} - \frac{1}{2-\alpha}}, \quad \forall h \le c(\rho),$$

which gives

$$|\nu_h| \le C(\rho) h^{\frac{1}{2} - \frac{1}{2-\alpha}}, \quad \forall h \le c(\rho). \tag{3.23}$$

This easily implies that

$$\mathcal{E}_{c_1(\rho)h} \subset A_h^{-1}\mathcal{E}_h \subset \mathcal{E}_{C_1(\rho)h} \tag{3.24}$$

for some constants $c_1(\rho), C_1(\rho) > 0$.

The conclusion of Theorem 2.2 follows from (3.22) and (3.24).

In order to prove Lemma 3.3, we modify the definition of the quantity $b_u(h)$ in [12].

Fix $\alpha \in (0, 1)$. Given a convex function u we define

$$b_u(h) = h^{-\frac{1}{2-\alpha}} \sup_{S_h} x_n. \tag{3.25}$$

Whenever there is no possibility of confusion we drop the subindex u and write for simplicity $b(h)$.

$b(h)$ satisfies the following properties which are slightly different from those in [12]:

1) If $h_1 \le h_2$, then
$$\left(\frac{h_1}{h_2}\right)^{\frac{1-\alpha}{2-\alpha}} \le \frac{b(h_1)}{b(h_2)} \le \left(\frac{h_2}{h_1}\right)^{\frac{1}{2-\alpha}};$$

2) If A is a linear transformation which leaves the x_n-coordinate invariant and
$$\tilde{u}(Ax) = u(x),$$
then
$$b_{\tilde{u}}(h) = b_u(h);$$

3) If A is a linear transformation which leaves the plane $\{x_n = 0\}$ invariant, then
$$\frac{b_{\tilde{u}}(h_1)}{b_{\tilde{u}}(h_2)} = \frac{b_u(h_1)}{b_u(h_2)};$$

4) If

$$\tilde{u}(x) = \beta u(x)$$

with β a positive constant, then

$$b_{\tilde{u}}(\beta h) = \beta^{-\frac{1}{2-\alpha}} b_u(h)$$

and therefore

$$\frac{b_{\tilde{u}}(\beta h_1)}{b_{\tilde{u}}(\beta h_2)} = \frac{b_u(h_1)}{b_u(h_2)}.$$

From part (ii) of Proposition 3.2 we know that

$$cd_n \le d_h = x_h^* \cdot e_n \le Cd_n,$$

and it follows that

$$cd_n \le b_u(h) h^{\frac{1}{2-\alpha}} = \sup_{S_h} x_n \le Cd_n.$$

Thus Lemma 3.3 will follow if we show that $b_u(h)$ is bounded below. This will follow from property 1) above and the following lemma.

Lemma 3.4. *If $h \le c(\rho)$ and $b_u(h) \le c_0$, then*

$$\frac{b_u(th)}{b_u(h)} > 2$$

for some $t \in [c_0, 1]$.

In order to prove Lemma 3.4, we recall the function $\tilde{u}$, the section $\tilde{S}_h$ and the matrix D_h in Proposition 3.2. Define

$$v(x) = \frac{1}{h}\tilde{u}(D_h x) = \frac{1}{h} u(A_h^{-1} D_h x).$$

The section $S_1(v) = \{v < 1\} = D_h^{-1} A_h S_h$ satisfies

$$x^* + cB_1 \subset S_1(v) \subset CB_1$$

with x^* the center of mass of $S_1(v)$. The function v satisfies in $S_1(v)$

$$\lambda x_n^{-\alpha} \le \det D^2 v(x) = d_n^{\alpha} \det D^2 u(A_h^{-1} D_h x) \le \Lambda x_n^{-\alpha}$$

and

$$v(0) = 0, \quad 0 \le v \le 1.$$

Moreover, let $0 < t \le 1$, $x_t^*(v)$ and x_{th}^* be the centers of mass of $S_t(v)$ and $S_{th}(u)$ respectively, and $d_{th} = x_{th}^* \cdot e_n$. Then

$$(x_t^*(v) \cdot e_n)^{-\alpha} |S_t(v)|^2 = \frac{d_{th}^{-\alpha} |S_{th}(u)|^2}{h^n}.$$

Since Proposition 3.2 implies that $c(th)^n \le d_{th}^{-\alpha} |S_{th}(u)|^2 \le C(th)^n$, we obtain

$$ct^n \le (x_t^*(v)\cdot e_n)^{-\alpha} |S_t(v)|^2 \le Ct^n.$$

From the convexity of u we have

$$x_t^*(v)\cdot e_n = \frac{d_{th}}{d_n} \ge c\frac{d_{th}}{d_h} \ge c\cdot \frac{\sup_{S_{th}(u)} x_n}{\sup_{S_h(u)} x_n} \ge ct.$$

Let $G_v := \partial S_1(v) \cap \{v < 1\}$. We claim that

$$G_v \subset \{x_n \le \sigma\}, \qquad \sigma = C(\rho) h^{\frac{1-\alpha}{n+1-\alpha}}.$$

Indeed, for $x \in G_v = D_h^{-1}\tilde{G}_h$,

$$d_n x_n \le \frac{2}{\rho} |D_h' x'|^2 \le C(\rho) h,$$

which gives

$$x_n \le C(\rho) h^{1-\frac{n}{n+1-\alpha}} = \sigma$$

by (3.10). Thus the claim follows.

We also have

$$v = 1 \quad \text{on } \partial S_1(v) \setminus G_v.$$

On G_v,

$$\mu \sum_1^{n-1} a_i^2 x_i^2 \le v(x) \le \mu^{-1} \sum_1^{n-1} a_i^2 x_i^2,$$

where

$$a_i = \frac{d_i}{h^{\frac{1}{2}}} \ge c_1, \quad i = 1, \ldots, n-1$$

by (3.16).

In order to prove Lemma 3.4, we only need to show that there exist constants $c(\rho)$, c_0 small and C sufficiently large such that if $h \le c(\rho)$ and $\max_{1\le i\le n-1} a_i \ge C$, then the rescaled function v satisfies

$$b_v(t) \ge 2b_v(1) \tag{3.26}$$

for some $t \in [c_0, 1]$.

4. Proof of Theorem 2.2 (II)

We consider the class of solutions v that satisfy the properties above. After relabeling the constants μ and a_i, and by abuse of notation writing u instead of v, we may assume we are in the following case.

Fix μ, λ, Λ and $\alpha \in (0, 1)$. For an increasing sequence

$$a_1 \le a_2 \le \cdots \le a_{n-1}$$

with

$$a_1 \ge \mu,$$

we consider the family of solutions

$$u \in \mathcal{D}^{\mu}_{\sigma}(a_1, a_2, \ldots, a_{n-1})$$

of convex functions $u : \Omega \to \mathbb{R}$ that satisfy

$$\lambda x_n^{-\alpha} \le \det D^2 u \le \Lambda x_n^{-\alpha}, \qquad 0 \le v \le 1 \quad \text{in } \Omega; \tag{4.1}$$

$$0 \in \partial\Omega, \qquad B_\mu(x_0) \subset \Omega \subset B^+_{1/\mu}; \tag{4.2}$$

$$\mu h^n \le (x_h^* \cdot e_n)^{-\alpha} |S_h|^2 \le \mu^{-1} h^n, \qquad x_h^* \cdot e_n \ge \mu h \tag{4.3}$$

with x_h^* the center of mass of S_h.

Moreover, there exists a closed set $G \subset \partial\Omega$ such that

$$G \subset \partial\Omega \cap \{x_n \le \sigma\}, \tag{4.4}$$

and G is a graph in the e_n direction with projection $\pi(G)$ along e_n,

$$\left\{\mu^{-1} \sum_1^{n-1} a_i^2 x_i^2 \le 1\right\} \subset \pi(G) \subset \left\{\mu \sum_1^{n-1} a_i^2 x_i^2 \le 1\right\}. \tag{4.5}$$

The boundary values of $u = \varphi$ on $\partial\Omega$ satisfy

$$\varphi = 1 \quad \text{on } \partial\Omega \setminus G, \tag{4.6}$$

and

$$\mu \sum_1^{n-1} a_i^2 x_i^2 \le \varphi(x) \le \min\left\{1, \mu^{-1} \sum_1^{n-1} a_i^2 x_i^2\right\} \quad \text{on } G. \tag{4.7}$$

As explained in [12, see page 79], Property (3.26) is a corollary of the following proposition.

Proposition 4.1. *For any $M > 0$ there exists C_* depending only on $M, n, \mu, \lambda, \Lambda$ and α such that if $u \in \mathcal{D}_\sigma^\mu(a_1, a_2, \dots, a_{n-1})$ with*

$$a_{n-1} \geq C_*, \qquad \sigma \leq C_*^{-1}$$

then

$$b(h) = (\sup_{S_h} x_n) h^{-\frac{1}{2-\alpha}} \geq M$$

for some $h \in [C_*^{-1}, 1]$.

We prove Proposition 4.1 by compactness as in [12]. We introduce the limiting solutions from the class $\mathcal{D}_\sigma^\mu(a_1, \dots, a_{n-1})$ when $a_{k+1} \to \infty$ and $\sigma \to 0$.

For an increasing sequence

$$a_1 \leq a_2 \leq \cdots \leq a_k$$

with

$$a_1 \geq \mu,$$

we denote by

$$\mathcal{D}_0^\mu(a_1, \dots, a_k, \infty, \dots, \infty), \quad 0 \leq k \leq n-2,$$

the class of functions u that satisfy

$$\lambda x_n^{-\alpha} \leq \det D^2 u \leq \Lambda x_n^{-\alpha}, \qquad 0 \leq u \leq 1 \quad \text{in } \Omega; \tag{4.8}$$

$$0 \in \partial\Omega, \qquad B_\mu(x_0) \subset \Omega \subset B_{1/\mu}^+; \tag{4.9}$$

$$\mu h^n \leq (x_h^* \cdot e_n)^{-\alpha} |S_h|^2 \leq \mu^{-1} h^n, \qquad x_h^* \cdot e_n \geq \mu h, \tag{4.10}$$

where x_h^* is the center of mass of S_h. There exists a closed set G such that

$$G \subset \partial\Omega \cap \{x_i = 0, i > k\}. \tag{4.11}$$

If we restrict to the space generated by the first k coordinates, then

$$\left\{\mu^{-1} \sum_1^k a_i^2 x_i^2 \leq 1\right\} \subset G \subset \left\{\mu \sum_1^k a_i^2 x_i^2 \leq 1\right\}. \tag{4.12}$$

The boundary values of $u = \varphi$ on $\partial\Omega$ satisfy

$$\varphi = 1 \quad \text{on } \partial\Omega \setminus G, \tag{4.13}$$

and

$$\mu \sum_1^k a_i^2 x_i^2 \leq \varphi(x) \leq \min\left\{1, \mu^{-1} \sum_1^k a_i^2 x_i^2\right\} \quad \text{on } G. \tag{4.14}$$

As in [12], Proposition 4.1 will follow from the proposition below.

Proposition 4.2. *For any $M > 0$ and $0 \le k \le n-2$ there exists c_k depending only on $M, k, n, \mu, \lambda, \Lambda$ and α such that if*

$$u \in \mathcal{D}_0^\mu(a_1, \dots, a_k, \infty, \dots, \infty),$$

then

$$b(h) = (\sup_{S_h} x_n) h^{-\frac{1}{2-\alpha}} \ge M$$

for some $h \in [c_k, 1]$.

To prove the above proposition, we use the notation introduced in [12]. Denote

$$x = (y, z, x_n), \quad y = (x_1, \dots, x_k) \in \mathbb{R}^k, \quad z = (x_{k+1}, \dots, x_{n-1}) \in \mathbb{R}^{n-1-k}.$$

A sliding along the y direction is defined as follows:

$$Tx := x + \nu_1 z_1 + \nu_2 z_2 + \dots + \nu_{n-k-1} z_{n-k-1} + \nu_{n-k} x_n$$

with

$$\nu_1, \nu_2, \dots, \nu_{n-k} \in \operatorname{span}\{e_1, \dots, e_k\}.$$

Lemma 4.3. *Assume that*

$$u \ge p(|z| - q x_n)$$

for some $p, q > 0, q \le q_0$ and assume that for each section S_h of u, $h \in (0, 1)$, there exists T_h, a sliding along the y direction, such that

$$T_h S_h \subset C_0 F_h B_1^+$$

for some constant C_0. Then

$$u \notin \mathcal{D}_0^\mu(1, \dots, 1, \infty, \dots, \infty).$$

Proof. Assume by contradiction that $u \in \mathcal{D}_0^\mu(1, \dots, 1, \infty, \dots, \infty)$. We will show that

$$u \ge p'(|z| - q' x_n), \qquad q' = q - \eta, \tag{4.15}$$

where $\eta > 0$ depends only on $q_0, C_0, \Lambda, \mu, n, \alpha$ and $0 < p' \ll p$.

Apply this result a finite number of times we obtain

$$u \ge \epsilon(|z| + x_n)$$

for some $\epsilon > 0$ small. Thus we obtain $S_h \subset \{x_n \le \epsilon^{-1} h\}$ and it follows that

$$T_h S_h \subset \left\{ x_n \le \epsilon^{-1} h \right\}.$$

This together with the hypothesis of the lemma and (4.10) in the definition of the class $\mathcal{D}_0^{\mu}$ implies that

$$\mu h^n \le (x_h^* \cdot e_n)^{-\alpha}|S_h|^2 = (x_h^* \cdot e_n)^{-\alpha}|T_h S_h|^2 \le C h^{n+1-\alpha},$$

where C is a constant depending only on ϵ, C_0, n, μ and α. This is a contradiction as $h \to 0$.

It remains to prove (4.15). Since $u \in \mathcal{D}_0^{\mu}(1, \dots, 1, \infty, \dots, \infty)$, there is a closed set

$$G_h \subset \partial S_h \cap \{(z, x_n) = 0\}$$

such that when we restrict to the subspace $\{(z, x_n) = 0\}$,

$$\left\{\mu^{-1}|y|^2 \le h\right\} \subset G_h \subset \left\{\mu|y|^2 \le h\right\},$$

and the boundary values φ_h of u on ∂S_h satisfy

$$\varphi_h = h \quad \text{on } \partial S_h \setminus G_h;$$

$$\mu|y|^2 \le \varphi_h \le \min\left\{h, \mu^{-1}|y|^2\right\} \quad \text{on } G_h.$$

Define

$$w(x) = \frac{1}{h}u(T_h^{-1}F_h x).$$

Then

$$S_1(w) = F_h^{-1}T_h S_h \subset C_0 B_1^+,$$

and

$$\lambda x_n^{-\alpha} \le \det D^2 w \le \Lambda x_n^{-\alpha} \quad \text{in } S_1(w).$$

Also,

$$w(x) \ge \frac{1}{h}p\left(|h^{\frac{1}{2}}z| - qh^{\frac{1}{2-\alpha}}x_n\right) = \frac{p}{h^{\frac{1}{2}}}\left(|z| - qh^{\frac{\alpha}{2(2-\alpha)}}x_n\right). \tag{4.16}$$

Moreover, the boundary values φ_w of w on $\partial S_1(w)$ satisfy

$$\varphi_w = 1 \quad \text{on } \partial S_1(w) \setminus G_w;$$

$$\mu|y|^2 \le \varphi_w \le \min\left\{1, \mu^{-1}|y|^2\right\} \quad \text{on } G_w = F_h^{-1}G_h.$$

Define

$$v := \delta\left(|x'|^2 + \frac{x_n^{2-\alpha}}{(2-\alpha)(1-\alpha)}\right) + \frac{\Lambda}{\delta^{n-1}}\left(z_1 - qh^{\frac{\alpha}{2(2-\alpha)}}x_n\right)^2$$
$$+ N\left(z_1 - qh^{\frac{\alpha}{2(2-\alpha)}}x_n\right) + \delta h^{\frac{\alpha}{2(2-\alpha)}}x_n,$$

where δ is small depending only on μ, C_0, α and N is large such that

$$\frac{\Lambda}{\delta^{n-1}} t^2 + Nt$$

is increasing in the interval $|t| \le (1+q_0)C_0$.

By a straightforward computation and similar arguments to the proof of [12, Lemma 5.4], we find that v is a lower barrier for w in $S_1(w)$, which implies

$$w \ge N\left(z_1 - qh^{\frac{\alpha}{2(2-\alpha)}} x_n\right) + \delta h^{\frac{\alpha}{2(2-\alpha)}} x_n \quad \text{in } S_1(w).$$

Since this inequality holds for all directions in the z-plane, we obtain

$$w \ge N\left[|z| - \left(q - \frac{\delta}{N}\right) h^{\frac{\alpha}{2(2-\alpha)}} x_n\right].$$

Back to u we have

$$u(x) = hw(F_h^{-1} T_h x) \ge h^{\frac{1}{2}} N\left[|z| - \left(q - \frac{\delta}{N}\right) x_n\right] \quad \text{in } S_h.$$

From the convexity of u and $u(0) = 0$, we know that this inequality holds in Ω and therefore (4.15) is proved. □

Now we give the proof of Proposition 4.2.

$\mathbf{k = 0}$: Assume Proposition 4.2 is not true, then by compactness, there exist $M > 0$ and $u \in \mathcal{D}_0^{\mu}(\infty, \dots, \infty)$ such that $b(h) \le M$ for any $0 < h \le 1$. Let

$$v := \delta\left(|x'| + \frac{1}{2}|x'|^2\right) + \frac{\Lambda}{\delta^{n-1}(2-\alpha)(1-\alpha)} x_n^{2-\alpha} - Nx_n,$$

where δ is small depending only on μ and N is large such that

$$\frac{\Lambda}{\delta^{n-1}(2-\alpha)(1-\alpha)} x_n^{2-\alpha} - Nx_n \le 0$$

in $B_{1/\mu}^+$. It is easily seen that

$$v \le u \qquad \text{in } \Omega.$$

It follows that

$$u \ge \delta |x'| - Nx_n$$

and then

$$S_h \subset \left\{ |x'| \le \delta^{-1}(Nx_n + h) \right\}.$$

Since $b(h) \leq M$ implies that $x_n \leq Mh^{\frac{1}{2-\alpha}} \leq Mh^{\frac{1}{2}}$, we obtain

$$S_h \subset \left\{|x'| \leq Ch^{\frac{1}{2}}, x_n \leq Mh^{\frac{1}{2-\alpha}}\right\},$$

where C is a constant depending only on M, μ, Λ and α. This contradicts Lemma 4.3 and therefore Proposition 4.2 is true for $k = 0$.

Assume Proposition 4.2 holds for $0, 1, \ldots, k-1$, $1 \leq k \leq n-2$, and now we prove it for k.

By the induction hypothesis, it suffices to consider the case $a_k \leq C_k$, where C_k is a constant depending only on $M, k, n, \mu, \lambda, \Lambda$ and α. Assume in contradiction that no c_k exists, then we can find a limiting solution u such that

$$u \in \mathcal{D}_0^{\tilde{\mu}}(1, \ldots, 1, \infty, \ldots, \infty) \tag{4.17}$$

with

$$b(h) \leq M, \quad \forall h > 0, \tag{4.18}$$

where $\tilde{\mu}$ depends only on μ and C_k.

Denote as before

$$x = (y, z, x_n), \quad y = (x_1, \ldots, x_k) \in \mathbb{R}^k, \quad z = (x_{k+1}, \ldots, x_{n-1}) \in \mathbb{R}^{n-1-k}.$$

Similar to the case $k = 0$, the function

$$v := \delta\left(|z| + \frac{1}{2}|x'|^2\right) + \frac{\Lambda}{\delta^{n-1}(2-\alpha)(1-\alpha)} x_n^{2-\alpha} - Nx_n$$

is a lower barrier for u, where δ is small depending only on $\tilde{\mu}$ and N is large. Therefore,

$$u \geq \delta|z| - Nx_n. \tag{4.19}$$

This together with (4.18) implies that

$$S_h \subset \left\{|z| \leq \delta^{-1}(Nx_n + h)\right\} \cap \left\{x_n \leq Mh^{\frac{1}{2-\alpha}}\right\}. \tag{4.20}$$

From John's lemma, there is an ellipsoid E_h such that

$$E_h \subset S_h - x_h^* \subset C(n)E_h \tag{4.21}$$

with x_h^* the center of mass of S_h. By a fact in linear algebra (see the arguments in [12, page 83]), there is T_h, a sliding along the y direction, such that

$$T_h E_h = |E_h|^{\frac{1}{n}} AB_1, \tag{4.22}$$

where, after rotating coordinates in the $(y, 0, 0)$ and $(0, z, 0)$ subspaces, the matrix A satisfies

$$A(y, z, x_n) = (A_1 y, A_2(z, x_n)),$$

$$A_1 = \begin{pmatrix} \beta_1 & 0 & \dots & 0 \\ 0 & \beta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \beta_k \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} \gamma_{k+1} & 0 & \dots & 0 & \theta_{k+1} \\ 0 & \gamma_{k+2} & \dots & 0 & \theta_{k+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \gamma_{n-1} & \theta_{n-1} \\ 0 & 0 & \dots & 0 & \theta_n \end{pmatrix}$$

with

$$0 < \beta_1 \le \cdots \le \beta_k, \qquad \gamma_j > 0, \quad \theta_n > 0, \qquad \left(\prod_1^k \beta_i\right)\left(\prod_{k+1}^{n-1} \gamma_j\right)\theta_n = 1.$$

Let

$$\tilde{u}(x) = u(T_h^{-1}x), \qquad \tilde{S}_h = T_h S_h,$$

then (4.21) implies that

$$\tilde{x}_h^* + |E_h|^{\frac{1}{n}} AB_1 \subset \tilde{S}_h \subset C(n)|E_h|^{\frac{1}{n}} AB_1, \tag{4.23}$$

where $\tilde{x}_h^*$ is the center of mass of $\tilde{S}_h$.

Since $u \in \mathcal{D}_0^{\tilde{\mu}}(1, \dots, 1, \infty, \dots, \infty)$, there exists $\tilde{G}_h = G_h$,

$$\tilde{G}_h \subset \{(z, x_n) = 0\} \cap \partial \tilde{S}_h$$

such that on the subspace $\{(z, x_n) = 0\}$,

$$\left\{\tilde{\mu}^{-1}|y|^2 \le h\right\} \subset \tilde{G}_h \subset \left\{\tilde{\mu}|y|^2 \le h\right\},$$

and the boundary values $\tilde{\varphi}_h$ of $\tilde{u}$ on $\partial\tilde{S}_h$ satisfy

$$\tilde{\varphi}_h = h \quad \text{on } \partial\tilde{S}_h \setminus \tilde{G}_h;$$

$$\tilde{\mu}|y|^2 \le \tilde{\varphi}_h \le \min\left\{h, \tilde{\mu}^{-1}|y|^2\right\} \quad \text{on } \tilde{G}_h.$$

For any $h > 0$, denote $d_h := x_h^* \cdot e_n$, then

$$\tilde{\mu}h^n \le (\tilde{x}_h^* \cdot e_n)^{-\alpha}|\tilde{S}_h|^2 = d_h^{-\alpha}|S_h|^2 \le \tilde{\mu}^{-1}h^n, \qquad \tilde{x}_h^* \cdot e_n = d_h \ge \tilde{\mu}h. \tag{4.24}$$

We will show that

$$|E_h|^{\frac{1}{n}} AB_1 \subset C diag\left(h^{\frac{1}{2}}, \dots, h^{\frac{1}{2}}, h^{\frac{1}{2}} d_h^{\frac{\alpha}{2}}\right) B_1, \tag{4.25}$$

where C is a constant depending only on μ, M, k, λ, Λ, n and α.

This together with (4.23), (4.18) gives

$$T_h S_h \subset C\mathrm{diag}\left(h^{\frac{1}{2}}, \dots, h^{\frac{1}{2}}, h^{\frac{1}{2-\alpha}}\right) B_1. \tag{4.26}$$

Now we prove (4.25). Let

$$\begin{aligned}\bar{A} &= |E_h|^{\frac{1}{n}}\mathrm{diag}\left(h^{-\frac{1}{2}}, \dots, h^{-\frac{1}{2}}, h^{-\frac{1}{2}} d_h^{-\frac{\alpha}{2}}\right) A \\ &= |E_h|^{\frac{1}{n}} h^{-\frac{1}{2}} \begin{pmatrix} A_1 & \\ & \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & d_h^{-\frac{\alpha}{2}} \end{pmatrix} A_2 \end{pmatrix} = \begin{pmatrix} \bar{A}_1 & \\ & \bar{A}_2 \end{pmatrix}.\end{aligned}$$

Since $\tilde{G}_h \subset \partial \tilde{S}_h \cap \{(z, x_n) = 0\}$ contains a ball in $\mathbb{R}^k$ of radius $(\tilde{\mu} h)^{1/2}$, then from the second inclusion in (4.23) we obtain

$$\bar{\beta}_i := h^{-\frac{1}{2}} |E_h|^{\frac{1}{n}} \beta_i \ge c, \quad i = 1, \dots, k, \tag{4.27}$$

where c is a constant depending only on n and $\tilde{\mu}$.

From (4.18) we know that for any $x = (y, z, x_n) \in \tilde{S}_h$ we have

$$x_n \le C(n) d_h \le C(n)\left(M h^{\frac{1}{2-\alpha}}\right)^{\frac{2-\alpha}{2}} d_h^{\frac{\alpha}{2}} \le C(n, M, \alpha) h^{\frac{1}{2}} d_h^{\frac{\alpha}{2}},$$

combining this, (4.24) and (4.20) we obtain that

$$\tilde{S}_h \subset \left\{ |(z, x_n)| \le C h^{\frac{1}{2}} d_h^{\frac{\alpha}{2}} \right\}.$$

This together with the first inclusion in (4.23) implies that $\|\bar{A}_2\| \le C$ and if follows that

$$\begin{aligned} &\bar{\gamma}_j := h^{-\frac{1}{2}} |E_h|^{\frac{1}{n}} \gamma_j, \le C, \\ &h^{-\frac{1}{2}} |E_h|^{\frac{1}{n}} |\theta_n \nu| \le C, \\ &\bar{\theta}_n := h^{-\frac{1}{2}} |E_h|^{\frac{1}{n}} d_h^{-\frac{\alpha}{2}} \theta_n \le C, \end{aligned} \tag{4.28}$$

where C is a constant depending only on n, $\tilde{\mu}$, Λ, α and M.

Also, we have by (4.23)

$$|E_h|^{\frac{1}{n}} \theta_n \le \tilde{x}_h^* \cdot e_n = d_h \le C(n) |E_h|^{\frac{1}{n}} \theta_n. \tag{4.29}$$

We define

$$w(x) := \frac{1}{h} \tilde{u}\left(|E_h|^{\frac{1}{n}} A x\right),$$

then from (4.23) we know that

$$B_1(x_0) \subset S_1(w) = |E_h|^{-\frac{1}{n}} A^{-1} \tilde{S}_h \subset C(n) B_1$$

for some x_0, and from (4.24) and (4.29) we find that

$$\bar{\lambda} x_n^{-\alpha} \le \det D^2 w \le \bar{\Lambda} x_n^{-\alpha}$$

with $\bar{\lambda}$, $\bar{\Lambda}$ depending only on λ, Λ, n, α, $\tilde{\mu}$.

Moreover, for $t > 0$ let $x_t^*(w)$ be the center of mass of the section $S_t(w)$, then

$$S_t(w) = |E_h|^{-\frac{1}{n}} A^{-1} T_h S_{th}(u),$$

and we have by (4.29)

$$\frac{d_{th}}{d_h} \le x_t^*(w) \cdot e_n = |E_h|^{-\frac{1}{n}} \theta_n^{-1} d_{th} \le C(n) \frac{d_{th}}{d_h}.$$

Then (4.24) implies that

$$c t^n \le (x_t^*(w) \cdot e_n)^{-\alpha} |S_t(w)|^2 \le C t^n$$

for some constants c, C depending only on n, α and $\tilde{\mu}$.

Let $G_w = \partial S_1(w) \cap \{w < 1\} = |E_h|^{-\frac{1}{n}} A^{-1} \tilde{G}_h$, then the boundary values φ_w of w satisfy

$$\varphi_w = 1 \quad \text{on } \partial S_1(w) \setminus G_w,$$

and from the definition of $\bar{\beta}_i$ we find that

$$\tilde{\mu} \sum_1^k \bar{\beta}_i^2 y_i^2 \le \varphi_w \le \tilde{\mu}^{-1} \sum_1^k \bar{\beta}_i^2 y_i^2.$$

This implies that

$$w \in \mathcal{D}_0^{\bar{\mu}}(\bar{\beta}_1, \ldots, \bar{\beta}_k, \infty, \ldots, \infty)$$

for some $\bar{\mu}$ depending only on μ, M, k, λ, Λ, n and α.

Note that (4.24) implies that

$$c \le \left(\prod_1^k \bar{\beta}_i \right) \left(\prod_{k+1}^{n-1} \bar{\gamma}_j \right) \bar{\theta}_n = h^{-\frac{n}{2}} |E_h| d_h^{-\frac{\alpha}{2}} \le C \tag{4.30}$$

with c, C depending only on n and $\tilde{\mu}$.

We claim

$$\bar{\theta}_n \ge c_* \tag{4.31}$$

for some small c_* to be chosen.

Indeed, if we c_* is small, then (4.28) and (4.30) imply that

$$\bar{\beta}_k \geq C_k(\bar{\mu}, \bar{M}, \bar{\lambda}, \bar{\Lambda}, n, \alpha)$$

with $\bar{M} := 2\bar{\mu}^{-1}$. Then by the induction hypothesis,

$$b_w(\bar{h}) \geq \bar{M} \geq 2b_w(1)$$

for some $\bar{h} > C_k^{-1}$. It follows that

$$\frac{b_u(h\bar{h})}{b_u(h)} = \frac{b_w(\bar{h})}{b_w(1)} \geq 2,$$

which implies $b_u(h\bar{h}) \geq 2b_u(h)$ for any $h > 0$. This contradicts (4.18) and therefore the claim holds.

Similarly, we obtain that

$$\bar{\gamma}_j \geq \tilde{c}_* \tag{4.32}$$

for some small $\tilde{c}_*$.

We obtain from (4.30), (4.31), (4.32) that

$$\bar{\beta}_i \leq C, \quad i = 1, \dots, k, \tag{4.33}$$

where C is a constant depending only on μ, M, k, λ, Λ, n and α. This implies that $\|\bar{A}_1\| \leq C$ and therefore $\|\bar{A}\| \leq C$.

Thus, the estimate (4.25) holds. Then the proof is finished because (4.19), (4.26) and (4.17) contradict Lemma 4.3.

5. Proof of Theorem 1.2

In this section we always denote by $c, C, c_i, C_i, i \in \mathbb{N}$ constants depending only on n, c_0 and α (c_0 is the constant in (1.5)). Their values may change from line to line whenever there is no possibility of confusion.

Lemma 5.1. *Assume the hypotheses in Theorem* 1.2 *hold, then for* $i = 1, \dots, n-1$ *we have* $u_i \in C(\overline{\mathbb{R}^n_+})$.

Proof. We first claim that for some constant c_1 small, we have

$$|\nabla u| \leq c_1^{-1}, \quad \text{in } B^+_{c_1}. \tag{5.1}$$

Indeed, we note that (1.5) implies that

$$B^+_k \subset S_1(u) \subset B^+_{k^{-1}}$$

for some constant k depending only on c_0 and α. We can use the convexity of u and obtain an upper bound for u_n and all $|u_i|$, $1 \le i \le n-1$, in $B^+_{k/4}$. On the other hand, for any $x_0 \in B^+_{c_1}$, the function

$$w_{(x_0',0)}(x) := \frac{1}{2}|x_0'|^2 + x_0' \cdot (x' - x_0') + \delta |x' - x_0'|^2 + \frac{\delta^{1-n}}{(2-\alpha)(1-\alpha)}\left(x_n^{2-\alpha} - k^{-1} x_n\right)$$

is a lower barrier u in $S_1(u)$, where δ is small depending only on n, c_0 and α. This together with the convexity of u gives a lower bound for $u_n(x_0)$.

Next we prove that for any $1 \le i \le n-1$, u_i is continuous at any point $x_0 \in \{|x'| \le c_1/2, x_n = 0\}$.

Indeed, fix $1 \le i \le n-1$ and $x_0 \in \{|x'| \le c_1/2, x_n = 0\}$, define

$$u_{x_0}(x) = u(x_0 + x) - u(x_0) - \nabla u(x_0) \cdot x.$$

We only need to prove that $\partial_i u_{x_0}$ is continuous at 0. Assume there is a sequence $x^{(m)} \to 0$, $m \to \infty$ with

$$\partial_i u_{x_0}(x^{(m)}) \ge \epsilon$$

for some $\epsilon > 0$. We have

$$u_{x_0} \ge u_{x_0}(x^{(m)}) + \nabla u_{x_0}(x^{(m)}) \cdot (x - x^{(m)}).$$

Note that $|\nabla u_{x_0}(x^{(m)})|$ is bounded by (5.1). Let $m \to \infty$, we obtain

$$u_{x_0} \ge a \cdot x$$

for some $a = (a', a_n) \in \mathbb{R}^n$ with $a_i \ge \epsilon$. From the value of u_{x_0} on the boundary $\{x_n = 0\}$ we find that $a' = 0$. This is a contradiction.

For any $\lambda > 0$, we define

$$u_\lambda(y) := \frac{1}{\lambda} u(F_\lambda y),$$

then u_λ satisfies (1.5) and (1.6). The results above show that for any $1 \le i \le n-1$, $\partial_i u_\lambda$ is continuous on $\{|x'| \le c_1/2, x_n = 0\}$. Therefore, u_i is continuous on $F_\lambda\{|x'| \le c_1/2, x_n = 0\}$. Let $\lambda \to \infty$ and we conclude that u_i is continuous on $\{x_n = 0\}$. □

Proof of Theorem 1.2. As before we have

$$B^+_k \subset S_1(u) \subset B^+_{k^{-1}} \tag{5.2}$$

and

$$|u_i| \le C \quad \text{in } B^+_{k/4}, \quad i = 1, \dots, n-1, \tag{5.3}$$

where k is a constant depending only on c_0 and α.

Let

$$L\varphi := \mathrm{tr}\big[(D^2u)^{-1}D^2\varphi\big]$$

be the linearized Monge–Ampère operator for u. Then for $i = 1, \dots, n-1$ we have

$$\begin{aligned} Lu_i &= 0, \qquad u_i = x_i \quad \text{on } \{x_n = 0\},\\ Lu &= n, \end{aligned}$$

and if we define $P(x) = \delta|x'|^2 + \delta^{1-n}\frac{x_n^{2-\alpha}}{(2-\alpha)(1-\alpha)}$ with $\delta > 0$ a small constant to be chosen, then

$$LP = \mathrm{tr}[(D^2u)^{-1}D^2P] \geq n[\det(D^2u)^{-1}\det D^2P]^{\frac{1}{n}} > n.$$

Let γ_1, γ_2 be large constants to be chosen and define

$$v^{\mp}(x) := x_i \pm \gamma_1\left[\delta|x'|^2 + \delta^{1-n}\left(\frac{x_n^{2-\alpha}}{(2-\alpha)(1-\alpha)} - \gamma_2 x_n\right) - u(x)\right].$$

We have

$$Lv^- = \gamma_1[LP - Lu] > 0.$$

On $\partial B^+_{k/4} \cap \{x_n = 0\}$, we choose $\delta \leq 1/2$ and obtain

$$v^- = x_i + \gamma_1\left[\delta|x'|^2 - \frac{1}{2}|x'|^2\right] \leq x_i.$$

We choose γ_2 large such that

$$\frac{x_n^{2-\alpha}}{(2-\alpha)(1-\alpha)} - \gamma_2 x_n \leq 0 \quad \text{in } B^+_{k/4}.$$

Then on $\partial B^+_{k/4} \cap \{x_n > 0\}$, we use (1.5) and obtain

$$v^- \leq x_i + \gamma_1\left[\delta|x'|^2 - c_0|x'|^2 - c_0 x_n^{2-\alpha}\right] \leq x_i - \frac{\gamma_1 c_0}{2}(|x'|^2 + x_n^{2-\alpha}) \leq -C,$$

where C is the constant in (5.3), and we choose $\delta \leq c_0/2$ and γ_1 large.

By Lemma 5.1, $u_i \in C(\overline{B^+_{k/4}})$ and therefore the maximum principle for linear elliptic equations implies that

$$v^- \leq u_i \quad \text{in } B^+_{k/4}.$$

Similarly,

$$v^+ \geq u_i \quad \text{in } B^+_{k/4}.$$

Therefore,

$$|u_i - x_i| \le \gamma_1\big[\delta^{1-n}\gamma_2 x_n + u\big] \quad \text{in } B^+_{k/4}. \tag{5.4}$$

For any $\lambda > 0$, we define

$$u_\lambda(y) := \frac{1}{\lambda} u(F_\lambda y),$$

then u_λ satisfies (1.5) and (1.6).

Apply (5.4) with $u \rightsquigarrow u_\lambda$ and we obtain

$$|\partial_i u_\lambda(y) - y_i| \le \gamma_1[\delta^{1-n}\gamma_2 y_n + u_\lambda(y)] \qquad \text{in } B^+_{k/4}.$$

Back to u we have

$$|u_i(x) - x_i| \le \gamma_1 \left[\delta^{1-n}\gamma_2 \lambda^{\frac{1}{2}-\frac{1}{2-\alpha}} x_n + \lambda^{-\frac{1}{2}} u(x)\right] \quad \text{in } F_\lambda B^+_{k/4}. \tag{5.5}$$

Let $\lambda \to \infty$, we obtain

$$u_i = x_i, \quad \forall x \in \mathbb{R}^n_+. \tag{5.6}$$

For any $x = (x', x_n) \in \mathbb{R}^n_+$,

$$u(x', x_n) = u(0, x_n) + \int_0^1 \nabla_{x'} u(\theta x', x_n) \cdot x' d\theta = u(0, x_n) + \frac{1}{2}|x'|^2. \tag{5.7}$$

Thus,

$$\det D^2 u = u_{nn}(0, x_n) = x_n^{-\alpha},$$

it follows that

$$u(0, x_n) = \frac{x_n^{2-\alpha}}{(2-\alpha)(1-\alpha)} + A x_n + B$$

for some constants $A, B \in \mathbb{R}$.

Since $u \in C(\overline{\mathbb{R}^n_+})$ and satisfies (1.5), we obtain $A = B = 0$. The conclusion of the theorem follows from this and (5.7). □

6. Proof of Theorem 1.3

Proof of Theorem 2.3. By the localization theorem (Theorem 2.1),

$$c U_0(x) \le u(x) \le c^{-1} U_0(x) \quad \text{in } \Omega \cap S_c(U_0),$$

where c is a constant depending only on n, α and ρ. Let $\Omega_h = F_h^{-1}\Omega$, then $\Omega_h \cap S_1(U_0)$ can be denoted by $x_n = g_h(x')$, where

$$g_h(x') = h^{-\frac{1}{2-\alpha}} g(h^{\frac{1}{2}} x') \le C h^{\frac{1-\alpha}{2-\alpha}} |x'|^2 \tag{6.1}$$

for some constant $C = C(n, \rho)$. Define

$$u_h(x) = \frac{1}{h}u(F_h x), \quad x \in \Omega_h.$$

Then we have

$$cU_0(x) \le u_h(x) \le c^{-1}U_0(x) \quad \text{in } \Omega_h \cap S_1(U_0). \tag{6.2}$$

The assumptions of Theorem 2.3 imply that

$$\begin{aligned}(1-\epsilon_0)(x_n - g_h(x'))^{-\alpha} &\le \det D^2 u_h(x) \\ &\le (1+\epsilon_0)(x_n - g_h(x'))^{-\alpha} \quad \text{in } \Omega_h \cap S_1(U_0)\end{aligned} \tag{6.3}$$

and

$$\left(\frac{1}{2} - \epsilon_0\right)|x'|^2 \le u_h(x) \le \left(\frac{1}{2} + \epsilon_0\right)|x'|^2 \quad \text{on } \partial\Omega_h \cap S_1(U_0). \tag{6.4}$$

Assume by contradiction that Theorem 2.3 does not hold. Then there is a constant $\eta > 0$ such that for any $m \in \mathbb{N}, m \ge 1$, there exist Ω^m, g^m, u^m that satisfy the hypotheses of Theorem 2.3 with $\epsilon_0 \rightsquigarrow 1/m$, and some $0 < h_m \le \min\{1/m, c\}$ such that if we denote $\Omega^m_{h_m} = F^{-1}_{h_m}\Omega^m$ and

$$u^m_{h_m}(x) := \frac{1}{h_m}u^m(F_{h_m}x),$$

then the part $\partial\Omega^m_{h_m} \cap S_1(U_0)$ is given by $x_n = g^m_{h_m}(x')$ for some convex function $g^m_{h_m}$ satisfying (6.1) with $h \rightsquigarrow h_m$, and the function $u^m_{h_m}$ satisfies (6.2)-(6.4) with $\epsilon_0 \rightsquigarrow 1/m$, while the inclusion in Theorem 2.3 does not hold for η and $S_{h_m}(u^m)$.

Let $m \to \infty$, we can extract a subsequence $u^m_{h_m}$ that converges uniformly on compact sets to a global solution u_0 defined in $\mathbb{R}^n_+$ such that

$$cU_0(x) \le u_0(x) \le c^{-1}U_0(x) \quad \text{in } \mathbb{R}^n_+ \tag{6.5}$$

and

$$\det D^2 u_0(x) = x_n^{-\alpha} \quad \text{in } \mathbb{R}^n_+, \qquad u_0(x', 0) = \frac{1}{2}|x'|^2. \tag{6.6}$$

Theorem 2 implies that

$$u_0 = U_0 = \frac{1}{2}|x'|^2 + \frac{x_n^{2-\alpha}}{(2-\alpha)(1-\alpha)}.$$

We reach a contradiction. □

Proof of Theorem 1.3. Assume the hypotheses in Theorem 1.3 hold, then we can assume that

$$\varphi = \frac{1}{2}\langle Mx', x'\rangle + o(|x'|^2),$$

for some positive definite matrix $M \in \mathbb{R}^{(n-1)\times(n-1)}$.

It suffices to prove the theorem for the case $f(0) = 1$ and $M = I_{n-1}$. Indeed, let $D' \in \mathbb{R}^{(n-1)\times(n-1)}$ be a positive definite matrix such that

$$D'MD' = I_{n-1}.$$

Let $\lambda > 0$ be a constant to be chosen. Define

$$D := \begin{pmatrix} D' & \\ & \lambda \end{pmatrix}.$$

For any $y \in \tilde{\Omega} = D^{-1}\Omega$, define

$$\tilde{u}(y) = u(Dy)$$

and $\tilde{\varphi} := \tilde{u}|_{\partial\tilde{\Omega}}$. Then we have

$$\det D^2\tilde{u}(y) = \tilde{f}(y) d_{\partial\tilde{\Omega}}^{-\alpha}(y), \qquad \tilde{f}(y) := (\det M)^{-1}\lambda^2 f(Dy)\frac{d_{\partial\Omega}^{-\alpha}(Dy)}{d_{\partial\tilde{\Omega}}^{-\alpha}(y)}.$$

It is easy to see that

$$\lim_{y\to 0}\frac{d_{\partial\Omega}(Dy)}{d_{\partial\tilde{\Omega}}(y)} = \lambda.$$

Thus we can choose $\lambda > 0$ such that

$$\lim_{y\to 0}\tilde{f}(y) = (\det M)^{-1}\lambda^2 f(0)\lambda^{-\alpha} = 1.$$

Now we assume $f(0) = 1$ and $M = I_{n-1}$, and we will prove that

$$u(x) = \frac{1}{2}|x'|^2 + \frac{x_n^{2-\alpha}}{(2-\alpha)(1-\alpha)} + o\left(|x'|^2 + x_n^{2-\alpha}\right). \tag{6.7}$$

For any $\epsilon_1 > 0$ small, we can choose $R = R(\epsilon_1) > 0$ such that $\partial\Omega \cap B_R$ is given by $x_n = g(x')$ for some convex function g, where

$$g \in C^2(\pi(\partial\Omega \cap B_R)), \quad g(0)=0, \quad \nabla g(0)=0, \quad D^2 g(0) \ge k_0 I_{n-1} > 0, \tag{6.8}$$

$$\det D^2 u = f(x) d_{\partial\Omega}^{-\alpha}, \quad 1-\epsilon_1 \le f \le 1+\epsilon_1 \quad \text{in } \Omega\cap B_R, \tag{6.9}$$

$$\left(\frac{1}{2}-\epsilon_1\right)|x'|^2 \le u(x) = \varphi(x') \le \left(\frac{1}{2}+\epsilon_1\right)|x'|^2 \quad \text{on } \partial\Omega\cap B_R, \tag{6.10}$$

where k_0 depends only on the principal curvatures of $\partial\Omega$ at 0.

It is obvious that

$$\lim_{x \to 0} \frac{x_n - g(x')}{d_{\partial\Omega}(x)} = 1.$$

Therefore we can choose $R = R(\epsilon_1)$ smaller such that

$$(1-4\epsilon_1)(x_n - g(x'))^{-\alpha} \le \det D^2 u \le (1+4\epsilon_1)(x_n - g(x'))^{-\alpha} \quad \text{in } \Omega \cap B_R, \tag{6.11}$$

For any $\eta > 0$, let ϵ_0 be the constant given by Theorem 2.3 and $\epsilon_1 := \epsilon_0/4$. Using (6.8), (6.10) and (6.11), we can choose $\rho > 0$ such that the hypotheses of Theorem 2.3 hold. Then Theorem 2.3 implies that

$$|u(x) - U_0(x)| \le C\eta U_0(x) \quad \text{in } \Omega \cap S_c(U_0),$$

where U_0 is defined as in Theorem 2.3, c is a constant depending only on η, n, α, ρ and $C = C(n, \alpha)$ depends only on n, α. This proves (6.7) and therefore the proof of Theorem 1.3 is complete. □

7. Proof of Theorem 1.4

In this section we always denote by $c, C, c_i, C_i (i = 0, 1, \dots)$ constants depending only on $n, \lambda_0, \Lambda_0, \alpha, \operatorname{diam}(\Omega)$, and $\varphi, \partial\Omega$ up to their second derivatives. For any $A, B \in \mathbb{R}$, we write $A \sim B$ if

$$c \le \frac{A}{B} \le C$$

for some constants c, C depending only on $n, \lambda_0, \Lambda_0, \alpha, \operatorname{diam}(\Omega)$, and $\varphi, \partial\Omega$ up to their second derivatives.

Suppose the assumptions of Theorem 1.4 hold. First we can use barriers to obtain that

$$\|u\|_{C(\overline{\Omega})} \le C. \tag{7.1}$$

Now we restrict to a neighborhood of $0 \in \partial\Omega$. As in Section 2, we can assume that for some fixed small $\rho > 0$, the part $\partial\Omega \cap \{x_n \le \rho\}$ is given by $x_n = g(x')$ for some convex function g, where

$$g \in C^2(\pi(\partial\Omega \cap \{x_n < \rho\})), \quad g(0) = 0, \quad \nabla g(0) = 0. \tag{7.2}$$

The function $u : \overline{\Omega} \to \mathbb{R}$ satisfies $u = \varphi(x')$ on $\partial\Omega \cap \{x_n \le \rho\}$, and

$$\det D^2 u = f, \quad 0 < \lambda(x_n - g)^{-\alpha} \le f \le \Lambda(x_n - g)^{-\alpha} \quad \text{in } \Omega \cap \{x_n < \rho/2\}, \tag{7.3}$$

where $\alpha \in [1, 2)$.

Case 1 : $\alpha \in (1, 2)$.
Denote $\beta := \frac{n+\alpha-1}{n} > 1$. We claim that for any $x \in \partial\Omega \cap \{x_n \le \rho/2\}$,

$$-\delta^{-1} x_n - \delta^{-1}(x_n - g)^{2-\beta} \le \tilde{u} \le \delta^{-1} x_n - \delta(x_n - g)^{2-\beta}, \tag{7.4}$$

where $\delta > 0$ is a small constant, and

$$\tilde{u} := u - u(0) - \nabla_{x'}\varphi(0) \cdot x'.$$

Indeed, let $C_0, C_1 > 0$ be two constants and define

$$v^- := \varphi(0) + \nabla_{x'}\varphi(0) \cdot x' - \frac{C_0}{(2-\beta)}(x_n - g)^{2-\beta} - C_1 x_n.$$

Since Ω is uniformly convex, $\partial\Omega, \varphi \in C^2$, and u is bounded below by (7.1), by straightforward computation we obtain that v^- is a lower barrier for u in $\Omega \cap \{x_n \leq \rho/2\}$ if C_0, C_1 are sufficiently large. Similarly, the function

$$v^+ := \varphi(0) + \nabla_{x'}\varphi(0) \cdot x' - \frac{c_1}{(2-\beta)}(x_n - g)^{2-\beta} + C x_n$$

is an upper barrier for u in $\Omega \cap \{x_n \leq \rho/2\}$ if c_1 is small and C is sufficiently large. Hence the claim follows.

The estimate (7.4) implies that

$$|u - u(x_0)| \leq C|x - x_0|^{2-\beta} \quad \forall\, x_0 \in \partial\Omega,\ x \in \Omega.$$

This together with the convexity of u implies that u is Hölder continuous in Ω and

$$\|u\|_{C^{0,2-\beta}(\overline{\Omega})} \leq C. \tag{7.5}$$

Let $y_0 \in \Omega$ and assume $S_{\bar{h}}(y_0)$ is the maximal section included in Ω which becomes tangent to $\partial\Omega$ at 0 with $\bar{h} \leq c$. Then we obtain that

$$\nabla_{x'}u(y_0) = \nabla_{x'}\varphi(0), \qquad S_{\bar{h}}(y_0) = \{x \in \Omega : \tilde{u}(x) < u_n(y_0)x_n\}.$$

Note that $u_n(y_0)$ is bounded above since $\varphi \in C^2$ and Ω is uniformly convex. Thus

$$\bar{h} = -\tilde{u}(y_0) + u_n(y_0)y_0 \cdot e_n$$

is bounded above.

Denote $M := -u_n(y_0)$. We only need to consider two cases: $-C < M < 2\delta^{-1}$ and $M \geq 2\delta^{-1}$, where δ is the constant in (7.4).

If $-C < M < 2\delta^{-1}$, then at the point $x = (0, c_0)$ with c_0 a small constant, we have by (7.4)

$$\tilde{u} + Mx_n \leq 3\delta^{-1}x_n - \delta x_n^{2-\beta} \leq -\frac{\delta}{2}x_n^{2-\beta} = -\frac{\delta}{2}c_0^{2-\beta}.$$

It follows that $\bar{h}$ is bounded below. Hence by (7.5),

$$S_{\bar{h}}(y_0) \supset B_c(y_0).$$

It remains to consider the case $M \geq 2\delta^{-1}$. For some c_1 small, the second inequality in (7.4) implies that the point $x = (0, c_1 M^{\frac{1}{1-\beta}}) \in S_{\bar{h}}(y_0)$ and

$$\tilde{u} + Mx_n \leq -\frac{\delta}{2}x_n^{2-\beta} + Mx_n = c_1 M^{\frac{2-\beta}{1-\beta}}\left[1 - \frac{\delta}{2}c_1^{1-\beta}\right] \leq -c_1 M^{\frac{2-\beta}{1-\beta}}.$$

Therefore,

$$\bar{h} \geq c_1 M^{\frac{2-\beta}{1-\beta}}. \tag{7.6}$$

On the other hand, the first inequality in (7.4) and the uniform convexity of Ω imply that

$$S_{\bar{h}}(y_0) \subset \left\{x_n \leq CM^{\frac{1}{1-\beta}}, |x'| \leq CM^{\frac{1}{2(1-\beta)}}\right\}. \tag{7.7}$$

Using this and the first inequality in (7.4) again, we obtain that for any $x \in S_{\bar{h}}(y_0)$

$$\tilde{u} + Mx_n \geq \frac{M}{2}x_n - \delta^{-1}x_n^{2-\beta} \geq -\delta^{-1}x_n^{2-\beta} \geq -CM^{\frac{2-\beta}{1-\beta}}$$

and therefore

$$\bar{h} \leq CM^{\frac{2-\beta}{1-\beta}}. \tag{7.8}$$

By (7.4), (7.6) and (7.8), we have

$$S_{\bar{h}/2}(y_0) \subset \left\{x_n \geq cM^{\frac{1}{1-\beta}}\right\}. \tag{7.9}$$

Using the first inequality in (7.4), (7.7) and (7.9),we obtain that

$$x_n - g \sim x_n \sim M^{\frac{1}{1-\beta}} \qquad \forall\, x \in S_{\bar{h}/2}(y_0). \tag{7.10}$$

The volume estimate for interior sections in [6, Corollary 3.2.4] and the definition of β imply that

$$|S_{\bar{h}/2}(y_0)| \sim M^{\frac{n(2-\beta)+\alpha}{2(1-\beta)}} = M^{\frac{n+1}{2(1-\beta)}}. \tag{7.11}$$

Define $Ty := (M^{\frac{1}{2(1-\beta)}}y', M^{\frac{1}{1-\beta}}y_n)$, and

$$v(y) := \frac{1}{M^{\frac{2-\beta}{1-\beta}}}\left[\tilde{u}(Ty) + M(Ty)\cdot e_n + \bar{h}/2\right] \quad \text{in } T^{-1}S_{\bar{h}/2}(y_0).$$

We have

$$\det D^2 v(y) = M^{\frac{\alpha}{1-\beta}} \det D^2 u(Ty) \sim 1 \quad \text{in } T^{-1}S_{\bar{h}/2}(y_0)$$

and

$$|T^{-1}S_{\bar{h}/2}(y_0)| \sim 1.$$

Since (7.7) implies that

$$T^{-1}S_{\bar{h}/2}(y_0) \subset B_C,$$

hence the Aleksandrov's maximum principle [6, Theorem 1.4.2] implies that

$$T^{-1}S_{\bar{h}/2}(y_0) \supset B_c(T^{-1}y_0).$$

Part (i) of Theorem 1.4 is proved.

Case 2 : $\alpha = 1$.
By straightforward computation, the functions

$$v^- := \varphi(0) + \nabla_{x'}\varphi(0) \cdot x' - C_0(x_n - g)(-\log(x_n - g))^{\frac{1}{n}} - C_1 x_n$$

and

$$v^+ := \varphi(0) + \nabla_{x'}\varphi(0) \cdot x' - c_1(x_n - g)(-\log(x_n - g))^{\frac{1}{n}} + C x_n,$$

are barriers for u in $\Omega \cap \{x_n \leq \rho/2\}$ if C_0, C_1, C are large constants and c_1 is small. Hence, we have in $\Omega \cap \{x_n \leq \rho/2\}$

$$\begin{aligned} &-\delta^{-1}x_n - \delta^{-1}(x_n - g)\,(-\log(x_n - g))^{\frac{1}{n}} \\ &\leq \tilde{u} \leq \delta^{-1}x_n - \delta(x_n - g)\,(-\log(x_n - g))^{\frac{1}{n}}, \end{aligned} \tag{7.12}$$

where $\delta > 0$ is a small constant, and $\tilde{u}$ is defined as in the case $\alpha \in (1, 2)$.

Using (7.12) and similar arguments to the previous case, part (ii) of the theorem is proved. The proof of Theorem 1.4 is complete.

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