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The equivariant Euler characteristic of $\mathcal{A}_3[2]$

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Abstract. We compute the weighted Euler characteristic, equivariant with respect to the action of the symplectic group of degree six over the field of two elements, of the moduli space of principally polarized Abelian threefolds together with a level two structure.

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1. Introduction

Let $\mathcal{A}_g[2]$ denote the moduli space of principally polarized Abelian varieties of dimension g together with a full level 2 structure. Recall that, if we avoid characteristic 2, a full level 2 structure on an Abelian variety A is a choice of an identification of the Weil pairing on $A[2]$ with the standard symplectic form on $(\mathbb{Z}/2\mathbb{Z})^{2g}$. Similarly, let $\mathcal{M}_g[2]$ denote the moduli space of smooth curves of genus g together with a full level 2 structure on their Jacobians. Note that we are considering these spaces as *coarse* moduli spaces. The two types of moduli spaces are connected through the Torelli morphism $t_g : \mathcal{M}_g[2] \to \mathcal{A}_g[2]$ sending a smooth curve to its Jacobian. There is an action of $\mathrm{Sp}_2(2g)$, the symplectic group of degree $2g$ over the field of two elements, on $\mathcal{A}_g[2]$ and $\mathcal{M}_g[2]$ via its action on the level 2 structure (for some more details on this see for instance [2]).

The locus $\mathcal{H}_g[2]$ inside $\mathcal{M}_g[2]$ consisting of hyperelliptic curves can, for any $g \geq 2$, be described as a disjoint union of copies of $\mathcal{M}_{0,2g+2}$ (see [3,7,11] and [12]), the moduli space of smooth genus 0 curves together with $2g+2$ marked points. Moreover, the moduli space $\mathcal{M}_{1,1}[2]$ of elliptic curves is isomorphic to $\mathcal{M}_{0,4}$. The cohomology of $\mathcal{M}_{0,2g+2}$, together with the action of the symmetric group S_{2g+2}, can (because of purity) be computed using counts of points over finite fields (see for instance [3, 5] and [10]). In Section 5.1 respectively Section 5.2 below, we compute in this way the $\mathrm{Sp}_2(2)$-action and Hodge structure of the cohomology of $\mathcal{M}_{1,1}[2] \cong \mathcal{A}_1[2]$ respectively the $\mathrm{Sp}_2(4)$-action and Hodge structure of the coho-

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mology of $\mathcal{M}_2[2]$. By adding the complement of $t_2(\mathcal{M}_2[2])$ inside $\mathcal{A}_2[2]$ consisting of products of elliptic curves we also compute the $\mathrm{Sp}_2(4)$-equivariant weighted Euler characteristic of $\mathcal{A}_2[2]$. For a definition of this type of Euler characteristic see Section 2.

The main result of this article is Table 4.1 which contains the $\mathrm{Sp}_2(6)$-equivariant weighted Euler characteristic of $\mathcal{A}_3[2]$. This is based upon the work of the second author in [2] in which the $\mathrm{Sp}_2(6)$-equivariant cohomology of $\mathcal{M}_3[2]$ is computed, see Section 3 below. There are two other loci consisting of either products of an indecomposable Abelian surface and an elliptic curve, or products of three elliptic curves. The cohomology of these loci are computed in Section 5 and Section 6 respectively.

We note in Section 4 that the weighted Euler characteristic of $\mathcal{A}_3[2]$ contains much fewer classes than the weighted Euler characteristics of its different loci. This cancellation property was noted also in [1] for the integer valued Euler characteristic of local systems upon the corresponding strata inside $\mathcal{A}_3$, the moduli space of principally polarized Abelian threefolds with no level structure. The Hodge structure of the cohomology of $\mathcal{A}_3$ was previously known, see [8].

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2. Euler characteristics

For a quasi-projective variety X defined over $\mathbb{C}$, let $\mathrm{Gr}_k^W H^i(X)$ denote the kth subquotient of the weight filtration on $H^i(X)$, the ith Betti cohomology group with complex coefficients. Say that a finite group G acts on X. This will make $\mathrm{Gr}_k^W H^i(X)$ into a representation of G. Define the G-equivariant weighted Euler characteristic of X to be

$$e_X(v) = \sum_{i,k\geq 0} (-1)^i \, \mathrm{Gr}_k^W H^i(X)\, v^k,$$

which is a polynomial in the formal variable v whose coefficients are virtual representations of G. This Euler characteristic is additive in the sense that if $X = Y \sqcup Z$, where Y and Z are preserved by G, then $e_X(v) = v^{2\mathrm{codim}_X(Y)} e_Y(v) + v^{2\mathrm{codim}_X(Z)} e_Z(v)$. If X fullfils purity, in the sense of Dimca and Lehrer in [5], then one can from this Euler characteristic determine the individual cohomology groups as representations of G.

Now let X be a quasi-projective variety defined over $\mathcal{O}[\frac{1}{N}]$, where $\mathcal{O}$ is a ring of integers of an algebraic number field, together with an action of a finite group G. For any prime power $q = p^r$ (where $p \nmid N$), let F_q denote the geometric Frobenius morphism acting on $X \otimes \bar{\mathbb{F}}_q$. For any $g \in G$ we will by $X^{F_q \circ g}$ denote the set of fixed points of $F_q \circ g$ acting on $X \otimes \bar{\mathbb{F}}_q$. Note that $X^{F_q \circ \mathrm{id}} = X(\mathbb{F}_q)$. Determining the numbers $|X^{F_q \circ g}|$, for all $g \in G$ and almost all prime powers $q = p^r$ (meaning for all but a finite set of primes p and all $r \geq 1$), will be called a twisted point count of X.

Say that there, for all $g \in G$, are polynomials $P_g(t)$, with complex coefficients and of degree $2 \dim X$, such that $P_g(q) = |X^{F_q \circ g}|$ for almost all prime powers q. Using the Lefschetz fixed point theorem, we can then from this information determine the G-equivariant weighted Euler characteristic of $X(\mathbb{C})$, see [6, Appendix A].

All spaces we consider here will be quasi-projective varieties defined over $\mathbb{Z}[\frac{1}{2}]$.

3. Decomposable and indecomposable Abelian threefolds

We say that a principally polarized Abelian threefold is indecomposable if it is not isomorphic to a product of Abelian varieties of lower dimension. We denote the corresponding locus in $\mathcal{A}_3[2]$ by $\mathcal{A}_3^{\text{in}}[2]$.

The Torelli morphism t_3 gives an isomorphism $\mathcal{M}_3[2] \cong \mathcal{A}_3^{\text{in}}[2]$ (on the level of coarse moduli spaces). The moduli space $\mathcal{M}_3[2]$ can be decomposed as a disjoint union

$$\mathcal{M}_3[2] = \mathcal{Q}[2] \sqcup \mathcal{H}_3[2],$$

where $\mathcal{Q}[2]$ denotes the locus consisting of curves whose canonical model is a plane quartic curve and where $\mathcal{H}_3[2]$ denotes the hyperelliptic locus.

The cohomology groups of $\mathcal{Q}[2]$ and $\mathcal{H}_3[2]$ were determined as representations of $\mathrm{Sp}_2(6)$ by the second author in [2]. For completeness, we recall the results in Table 7.1 and Table 7.2.

There are two types of decomposable Abelian threefolds. The threefold can either be isomorphic to a product of an indecomposable Abelian surface and an elliptic curve or to a product of three elliptic curves. We denote the corresponding loci in $\mathcal{A}_3[2]$ by $\mathcal{A}_{2,1}[2]$ and $\mathcal{A}_{1,1,1}[2]$ respectively.

4. The main result

We have the decomposition

$$\mathcal{A}_3[2] = t_3(\mathcal{Q}[2]) \sqcup t_3(\mathcal{H}_3[2]) \sqcup \mathcal{A}_{2,1}[2] \sqcup \mathcal{A}_{1,1,1}[2]$$

and below we compute the cohomology groups of each of the spaces on the right hand side as representations of $\mathrm{Sp}_2(6)$. Moreover, we will see that each cohomology group H^i of a space on the right hand side is pure of weight $2i$ and Tate type (i, i).

By the additivity of the weighted Euler characteristic,

$$e_{\mathcal{A}_3[2]}(v) = e_{t_3(\mathcal{Q}[2])}(v) + v^2\, e_{t_3(\mathcal{H}_3[2])}(v) + v^4\, e_{\mathcal{A}_{2,1}[2]}(v) + v^6\, e_{\mathcal{A}_{1,1,1}}(v).$$

Putting the results together for the different strata we get the $\mathrm{Sp}_2(6)$-equivariant weighted Euler characteristic of $\mathcal{A}_3[2]$, see Table 4.1. Each column in this table corresponds to an irreducible representation of $\mathrm{Sp}_2(6)$. The irreducible representations are denoted ϕ_{dn} where d is the dimension of the representation and n is letter used to distinguish different representations of the same dimension, see [4].

Table 4.1. The $\mathrm{Sp}_2(6)$-equivariant weighted Euler characteristic of $\mathcal{A}_3[2]$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}
$e_{\mathcal{A}_3[2]}(v)$	$1+v^2+v^4+v^6+v^{12}$	0	v^{12}	0	0
	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
$e_{\mathcal{A}_3[2]}(v)$	$-v^6-v^8$	0	$-v^6-v^8+v^{12}$	0	0
	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}
$e_{\mathcal{A}_3[2]}(v)$	v^{12}	0	v^4	0	v^{10}
	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
$e_{\mathcal{A}_3[2]}(v)$	v^{10}	0	0	0	$-v^6$
	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}
$e_{\mathcal{A}_3[2]}(v)$	$-v^6$	0	0	v^{10}	0
	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
$e_{\mathcal{A}_3[2]}(v)$	0	0	0	v^8	$-v^{12}$

Note that the results for ϕ_{1a} agree with the computation of the cohomology groups of $\mathcal{A}_3$ together with their Hodge structure in [8]. Note also that only 13 of the 30 irreducible representations of $\mathrm{Sp}_2(6)$ occur in $e_{\mathcal{A}_3[2]}(v)$ and that for each irreducible representation the coefficients of v^i are all either zero or ± 1. This is in sharp contrast to the cohomology of the individual pieces - all irreducible representations except ϕ_{7a} occur in some cohomology group of some piece and they occur with multiplicities up to 14.

5. An indecomposable Abelian surface and an elliptic curve

As in the genus 3 case, t_2 gives an isomorphism $\mathcal{M}_2[2] \cong \mathcal{A}_2^{\mathrm{in}}[2]$, where $\mathcal{A}_2^{\mathrm{in}}[2]$ denotes the indecomposable locus inside $\mathcal{A}_2[2]$.

There is a close relationship between $\mathcal{A}_{2,1}[2]$ and the product space $\mathcal{M}_2[2] \times \mathcal{A}_1[2]$. Let C be a genus 2 curve with level 2 structure represented by the symplectic basis (e_1, e_2, f_1, f_2) of $\mathrm{Jac}(C)[2]$ and let E be an elliptic curve with level 2 structure (e_3, f_3). Then $t_2(C) \times E$ is an Abelian threefold and $t_2(C)[2] \times E[2]$ is a six dimensional vector space over $\mathbb{F}_2$ with a symplectic pairing given by

$$e_i \cdot e_j = f_i \cdot f_j = 0$$

and

$$e_i \cdot f_j = \delta_{i,j}$$

for all i and j, where we identify e_i with $t_2(e_i)$ and f_i with $t_2(f_i)$. Clearly, not all level 2 structures on $t_2(C) \times E$ arise in this way but those that do are permuted by the group $\mathrm{Sp}_2(4) \times \mathrm{Sp}_2(2)$. Let $\mathscr{C}$ be the quotient set $\mathrm{Sp}_2(6)/(\mathrm{Sp}_2(4) \times \mathrm{Sp}_2(2))$. We may then describe the locus $\mathcal{A}_{2,1}[2]$ as

$$\mathcal{A}_{2,1}[2] \cong \coprod_{c \in \mathscr{C}} (\mathcal{M}_2[2] \times \mathcal{A}_1[2])_c,$$

where $(\mathcal{M}_2[2] \times \mathcal{A}_1[2])_c$ is an isomorphic copy of $\mathcal{M}_2[2] \times \mathcal{A}_1[2]$ indexed by c and the components are permuted as

$$g(\mathcal{M}_2[2] \times \mathcal{A}_1[2])_c = (\mathcal{M}_2[2] \times \mathcal{A}_1[2])_{gc}$$

for $g \in \mathrm{Sp}_2(6)$. In terms of cohomology groups this means that

$$H^i(\mathcal{A}_{2,1}[2]) = \mathrm{Ind}_{\mathrm{Sp}_2(4)\times\mathrm{Sp}_2(2)}^{\mathrm{Sp}_2(6)} H^i(\mathcal{M}_2[2] \times \mathcal{A}_1[2]).$$

By the Künneth theorem we have that

$$H^i(\mathcal{M}_2[2] \times \mathcal{A}_1[2]) \cong \bigoplus_{p+q=i} H^p(\mathcal{M}_2[2]) \otimes H^q(\mathcal{A}_1[2]).$$

Thus, in order to understand the action of $\mathrm{Sp}_2(4) \times \mathrm{Sp}_2(2)$ on $H^i(\mathcal{M}_2[2] \times \mathcal{A}_1[2])$ it is enough to understand the action of $\mathrm{Sp}_2(4)$ on $H^i(\mathcal{M}_2[2])$ and the action of $\mathrm{Sp}_2(2)$ on $H^i(\mathcal{A}_1[2])$ for all i.

5.1. The moduli space of elliptic curves with level two structure

In order to understand the action of $\mathrm{Sp}_2(2)$ on $H^i(\mathcal{A}_1[2])$ we note that $\mathrm{Sp}_2(2)$ is isomorphic to the symmetric group S_3 and that $\mathcal{A}_1[2]$ is isomorphic (over $\mathbb{Z}[\frac{1}{2}]$) to $\mathcal{M}_{0,4}$, the moduli space of four ordered points on $\mathbb{P}^1$. Under these identifications, the action of $\mathrm{Sp}_2(2)$ is given by permuting the first three points. Since $\mathcal{M}_{0,4}$ is pure in the sense of Dimca and Lehrer [5] we can deduce the action of $\mathrm{Sp}_2(2)$ on its cohomology groups by a twisted point count, see Section 2.

Say that $\sigma = (1, 2, \ldots, 2g+2) \in S_{2g+2}$. Then $(p_1, \ldots, p_n) \in \mathcal{M}_{0,2g+2}^{F_q \circ \sigma}$ if and only if p_1 is defined over $\mathbb{F}_{q^n}$, but not over any strict subfield of $\mathbb{F}_{q^n}$, and $p_i = F^{i-1} p_1$ for all i. Using this fact we easily find that for all prime powers q,

$$\begin{aligned}
\left|\mathcal{M}_{0,4}^{F_q \circ \mathrm{id}}\right| &= \frac{(q+1)q(q-1)(q-2)}{|\mathrm{PGL}_2(\mathbb{F}_q)|} = q-2 \\
\left|\mathcal{M}_{0,4}^{F_q \circ (12)}\right| &= \frac{(q+1)q(q^2-q)}{|\mathrm{PGL}_2(\mathbb{F}_q)|} = q \\
\left|\mathcal{M}_{0,4}^{F_q \circ (123)}\right| &= \frac{(q+1)(q^3-q)}{|\mathrm{PGL}_2(\mathbb{F}_q)|} = q+1.
\end{aligned}$$

Thus, the traces of (id, (12), (123)) on $H^0(\mathcal{A}_1[2])$ and $H^1(\mathcal{A}_1[2])$ are (1, 1, 1) and $(2, 0, -1)$, respectively. In other words, $H^0(\mathcal{A}_1[2])$ is the trivial representation of $\mathrm{Sp}_2(2)$ while $H^1(\mathcal{A}_1[2])$ is the standard representation.

5.2. The moduli space of genus two curves with level two structure

In order to understand the action of $\mathrm{Sp}_2(4)$ on $H^i(\mathcal{M}_2[2])$ we note that $\mathrm{Sp}_2(4)$ is isomorphic to the symmetric group S_6 and that $\mathcal{M}_2[2]$ is isomorphic (over $\mathbb{Z}[\frac{1}{2}]$) to $\mathcal{M}_{0,6}$, the moduli space of six ordered points on $\mathbb{P}^1$. Under these identifications, the action of $\mathrm{Sp}_2(4)$ on $\mathcal{M}_2[2]$ is given by permuting the points. Also, $\mathcal{M}_{0,6}$ is pure so we can again deduce the action of $\mathrm{Sp}_2(4)$ on the cohomology via twisted point counts. As in the above, we easily find that for all prime powers q,

$$\begin{aligned}
\left|\mathcal{M}_{0,6}^{F_q\circ \mathrm{id}}\right| &= \frac{(q+1)q(q-1)(q-2)(q-3)(q-4)}{|\mathrm{PGL}_2(\mathbb{F}_q)|} &&= q^3-9q^2+26q-24\\
\left|\mathcal{M}_{0,6}^{F_q\circ(12)}\right| &= \frac{(q+1)q(q-1)(q-2)(q^2-q)}{|\mathrm{PGL}_2(\mathbb{F}_q)|} &&= q^3-3q^2+2q\\
\left|\mathcal{M}_{0,6}^{F_q\circ(12)(34)}\right| &= \frac{(q+1)q(q^2-q)(q^2-q-2)}{|\mathrm{PGL}_2(\mathbb{F}_q)|} &&= q^3-q^2-2q\\
\left|\mathcal{M}_{0,6}^{F_q\circ(12)(34)(56)}\right| &= \frac{(q^2-q)(q^2-q-2)(q^2-q-4)}{|\mathrm{PGL}_2(\mathbb{F}_q)|} &&= q^3-3q^2-2q+8\\
\left|\mathcal{M}_{0,6}^{F_q\circ(123)}\right| &= \frac{(q+1)q(q-1)(q^3-q)}{|\mathrm{PGL}_2(\mathbb{F}_q)|} &&= q^3-q\\
\left|\mathcal{M}_{0,6}^{F_q\circ(123)(45)}\right| &= \frac{(q+1)(q^2-q)(q^3-q)}{|\mathrm{PGL}_2(\mathbb{F}_q)|} &&= q^3-q\\
\left|\mathcal{M}_{0,6}^{F_q\circ(123)(456)}\right| &= \frac{(q^3-q)(q^3-q-3)}{|\mathrm{PGL}_2(\mathbb{F}_q)|} &&= q^3-q-3\\
\left|\mathcal{M}_{0,6}^{F_q\circ(1234)}\right| &= \frac{(q+1)q(q^4-q^2)}{|\mathrm{PGL}_2(\mathbb{F}_q)|} &&= q^3+q^2\\
\left|\mathcal{M}_{0,6}^{F_q\circ(1234)(56)}\right| &= \frac{(q^2-q)q(q^4-q^2)}{|\mathrm{PGL}_2(\mathbb{F}_q)|} &&= q^3-q^2\\
\left|\mathcal{M}_{0,6}^{F_q\circ(12345)}\right| &= \frac{(q+1)(q^5-q)}{|\mathrm{PGL}_2(\mathbb{F}_q)|} &&= q^3+q^2+q+1\\
\left|\mathcal{M}_{0,6}^{F_q\circ(123456)}\right| &= \frac{q^6-q^3-q^2+q}{|\mathrm{PGL}_2(\mathbb{F}_q)|} &&= q^3+q-1.
\end{aligned}$$

We may read off $\mathrm{Tr}(\sigma, H^i(\mathcal{M}_{0,6}))$ as the coefficient of $(-1)^i q^{3-i}$ in $|\mathcal{M}_{0,6}^{F_q\circ\sigma'}|$, where σ' is the element in the above list which is conjugate to σ.

We now know $H^i(\mathcal{M}_2[2])$ as a representation of $\mathrm{Sp}_2(4)$ for all i and we know $H^i(\mathcal{A}_1[2])$ as a representation of $\mathrm{Sp}_2(2)$ for all i so we therefore know $H^i(\mathcal{M}_2[2]\times \mathcal{A}_1[2])$ as a representation of $\mathrm{Sp}_2(4)\times\mathrm{Sp}_2(2)$. Inducing this representation to $\mathrm{Sp}_2(6)$ gives us $H^i(\mathcal{A}_{2,1}[2])$ as a representation of $\mathrm{Sp}_2(6)$. We give the result in Table 7.3.

As an aside we note that the complement of $t_2(\mathcal{M}_2[2])$ inside $\mathcal{A}_2[2]$ consists of products of elliptic curves. The cohomology of this locus can be computed using the same techniques as in Section 6, but in this case we will omit the details. Let us denote the irreducible representations of $\mathrm{Sp}_2(4)\cong S_6$ by s_λ, which are indexed in the standard way by λ, a partition of 6. Adding the contributions from the two loci we find that,

$$e_{\mathcal{A}_2[2]}(v) = \big(1+v^2\big)s_6 - v^4\big(s_{5,1}+s_{4,2}\big) + v^6 s_{3,2,1}.$$

6. Products of three elliptic curves

There is a close relationship between the product space $(\mathcal{A}_1[2])^3$ and the locus $\mathcal{A}_{1,1,1}[2]$. Let E_1, E_2 and E_3 be three elliptic curves with level 2 structures (e_1, f_1), (e_2, f_2) and (e_3, f_3), respectively. Then $E_1\times E_2\times E_3$ is an Abelian threefold and $E_1[2]\times E_2[2]\times E_3[2]$ is a six dimensional vector space over $\mathbb{F}_2$ with a symplectic pairing given by

$$e_i\cdot e_j = f_i\cdot f_j = 0$$

for all i and j and

$$e_i\cdot f_j = \delta_{i,j}.$$

Clearly, not all level 2 structures on $E_1\times E_2\times E_3$ arise in this way but those that do are permuted by the group $(\mathrm{Sp}_2(2))^3$ while the three curves themselves are permuted by the symmetric group S_3. Let $\mathscr{C}$ be the quotient set $\mathrm{Sp}_2(6)/(S_3\ltimes(\mathrm{Sp}_2(2))^3)$. We may describe the locus $\mathcal{A}_{1,1,1}[2]$ as

$$\mathcal{A}_{1,1,1}[2]\cong\coprod_{c\in\mathscr{C}}\big(\mathcal{A}_1[2]\big)^3_c,$$

where $(\mathcal{A}_1[2])^3_c$ is an isomorphic copy of $(\mathcal{A}_1[2])^3$ indexed by c and the components are permuted as

$$g\big(\mathcal{A}_1[2]\big)^3_c = \big(\mathcal{A}_1[2]\big)^3_{gc}$$

for $g \in \mathrm{Sp}_2(6)$. In terms of cohomology groups this means that

$$H^i\big(\mathcal{A}_{1,1,1}[2]\big) = \mathrm{Ind}_{S_3 \ltimes (\mathrm{Sp}_2(2))^3}^{\mathrm{Sp}_2(6)} H^i\left(\big(\mathcal{A}_1[2]\big)^3\right).$$

By the Künneth theorem we have that

$$H^i\left(\big(\mathcal{A}_1[2]\big)^3\right) \cong \bigoplus_{p+q+r=i} H^p\big(\mathcal{A}_1[2]\big) \otimes H^q\big(\mathcal{A}_1[2]\big) \otimes H^r\big(\mathcal{A}_1[2]\big).$$

Thus, in order to understand the action of $S_3 \ltimes (\mathrm{Sp}_2(2))^3$ on $H^i((\mathcal{A}_1[2])^3$ it is enough to understand the action of $\mathrm{Sp}_2(2)$ on $H^i(\mathcal{A}_1[2])$ for all i and the action of S_3 on the factors.

Since the action of $\mathrm{Sp}_2(2)$ was described in Section 5.1 we only consider the action of S_3 on the factors. Let $\alpha \otimes \beta \otimes \gamma \in H^p(\mathcal{A}_1[2]) \otimes H^q(\mathcal{A}_1[2]) \otimes H^r(\mathcal{A}_1[2]) \subseteq H^{p+q+r}((\mathcal{A}_1[2])^3)$. We have

$$\begin{aligned}(12).\alpha \otimes \beta \otimes \gamma &= (-1)^{pq} \beta \otimes \alpha \otimes \gamma,\\ (13).\alpha \otimes \beta \otimes \gamma &= (-1)^{pr} \gamma \otimes \beta \otimes \alpha,\\ (23).\alpha \otimes \beta \otimes \gamma &= (-1)^{qr} \alpha \otimes \gamma \otimes \beta,\end{aligned}$$

where the signs are a consequence of the Künneth isomorphism. Since S_3 is generated by transpositions and $H^{p+q+r}((\mathcal{A}_1[2])^3)$ is generated by elements of the form $\alpha \otimes \beta \otimes \gamma$ for all possible choices of p, q and r, this determines the action of S_3 on $H^{p+q+r}((\mathcal{A}_1[2])^3)$.

We now have all the information we need in order to understand the action of $S_3 \ltimes (\mathrm{Sp}_2(2))^3$ on $H^i((\mathcal{A}_1[2])^3)$.

Example 6.1. Let u be a basis vector for the trivial representation of $\mathrm{Sp}_2(2)$ and let v_1 and v_2 be basis vectors for the standard representation of $\mathrm{Sp}_2(2)$. Let $\sigma \in \mathrm{Sp}_2(2)$ be an element of order 3 acting as

$$\begin{aligned}\sigma.u &= u,\\ \sigma.v_1 &= v_2,\\ \sigma.v_2 &= -v_1 - v_2,\end{aligned}$$

and let $g \in S_3 \ltimes (\mathrm{Sp}_2(2))^3$ on $H^i((\mathcal{A}_1[2])^3$ be the element $g = ((23), (\sigma, \sigma, \mathrm{id}))$. We of course have $g.u \otimes u \otimes u = u \otimes u \otimes u$, so $\mathrm{Tr}(g, H^0((\mathcal{A}_1[2])^3) = 1$. In cohomological degree 1 we have

$$g.v_2 \otimes u \otimes u = -(v_1 + v_2) \otimes u \otimes u$$

while $g.\alpha$ has no component in the direction of α for all other choices of $\alpha \in H^1((\mathcal{A}_1[2])^3)$. Thus, $\mathrm{Tr}(g, H^1((\mathcal{A}_1[2])^3)) = -1$. In degree 2 we have

$$g.u \otimes v_2 \otimes v_2 = u \otimes v_2 \otimes (v_1 + v_2)$$

while $g.\alpha$ has no component in the direction of α for all other choices of $\alpha \in H^2((\mathcal{A}_1[2])^3)$. We conclude that $\mathrm{Tr}(g, H^2((\mathcal{A}_1[2])^3)) = 1$. Finally, in degree 3 we have

$$g.v_2 \otimes v_2 \otimes v_2 = -(v_1 + v_2) \otimes v_2 \otimes (v_1 + v_2)$$

while $g.\alpha$ has no component in the direction of α for all other choices of $\alpha \in H^3((\mathcal{A}_1[2])^3)$. Hence, $\mathrm{Tr}(g, H^3((\mathcal{A}_1[2])^3) = -1$. We thus have

$$\sum_{i=0}^{3} \mathrm{Tr}\left(g, H^i\left((\mathcal{A}_1[2])^3\right)\right) t^i = 1 - t + t^2 - t^3.$$

Similar computations for the other conjugacy classes of $S_3 \ltimes (\mathrm{Sp}_2(2))^3$ give the results in Table 6.1, where

$$P_g\left((\mathcal{A}_1[2])^3, t\right) := \sum_{i=0}^{3} \mathrm{Tr}\left(g, H^i\left((\mathcal{A}_1[2])^3\right)\right) t^i,$$

is called the equivariant Poincaré polynomial. See [9, Chapter 4] for a beautiful description of how to compute representatives of $S_3 \ltimes (\mathrm{Sp}_2(2))^3$. In Table 6.1, σ is the element of $\mathrm{Sp}_2(2)$ described in Example 6.1 while τ is the element of order 2 acting as

$$\tau.v_1 = -v_1, \quad \tau.v_2 = v_1 + v_2$$

where v_1 and v_2 are the same basis vectors of the standard representation considered in Example 6.1.

Table 6.1. Equivariant Poincaré polynomials of $(\mathcal{A}_1[2])^3$ for a representative g of every conjugacy class of $S_3 \ltimes (\mathrm{Sp}_2(2))^3$.

g	$P_g((\mathcal{A}_1[2])^3, t)$	g	$P_g((\mathcal{A}_1[2])^3, t)$	g	$P_g((\mathcal{A}_1[2])^3, t)$
(id, (id, id, id))	$1 + 6t + 12t^2 + 8t^3$	((23), (id, id, id))	$1 + 2t - 2t^2 - 4t^3$	((123), (id, id, id))	$1 + 2t^3$
(id, (id, id, τ))	$1 + 4t + 4t^2$	((23), (id, τ, id))	$1 + 2t$	((123), (τ, id, id))	1
(id, (id, id, σ))	$1 + 3t - 4t^3$	((23), (id, σ, id))	$1 + 2t + t^2 + 2t^3$	((123), (σ, id, id))	$1 - t^3$
(id, (id, τ, τ))	$1 + 2t$	((23), (τ, id, id))	$1 - 2t^2$		
(id, (id, τ, σ))	$1 + t - 2t^2$	((23), (τ, τ, id))	1		
(id, (id, σ, σ))	$1 - 3t^2 + 2t^3$	((23), (τ, σ, id))	$1 + t^2$		
(id, (τ, τ, τ))	1	((23), (σ, id, id))	$1 - t - 2t^2 + 2t^3$		
(id, (τ, τ, σ))	$1 - t$	((23), (σ, τ, id))	$1 - t$		
(id, (τ, σ, σ))	$1 - 2t + t^2$	((23), (σ, σ, id))	$1 - t + t^2 - t^3$		
(id, (σ, σ, σ))	$1 - 3t + 3t^2 - t^3$				

By inducing the corresponding representations from $S_3 \ltimes (\mathrm{Sp}_2(2))^3$ to $\mathrm{Sp}_2(6)$ we obtain the cohomology of $\mathcal{A}_{1,1,1}[2]$ as a representation of $\mathrm{Sp}_2(6)$. We give the result in Table 7.4.

7. Cohomology groups of strata

In this section we give the cohomology groups of $\mathcal{Q}[2]$, $\mathcal{H}_3[2]$, $\mathcal{A}_{2,1}[2]$ and $\mathcal{A}_{1,1,1}[2]$ as representations of $\mathrm{Sp}_2(6)$. The results are presented in Table 7.1-7.4. Each column in these tables corresponds to an irreducible representation of $\mathrm{Sp}_2(6)$. The irreducible representations are denoted ϕ_{dn} where d is the dimension of the representation and n is a letter used to distinguish different representations of the same dimension, see [4].

Table 7.1. The cohomology groups of $\mathcal{Q}[2]$ as a representation of $\mathrm{Sp}_2(6)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	1	0	0
H^2	0	0	0	0	0	0	0	0	0	0
H^3	0	0	0	1	0	0	0	0	0	0
H^4	0	0	0	0	0	0	0	0	0	1
H^5	0	0	0	0	0	1	1	1	0	0
H^6	1	0	2	0	1	1	1	3	0	0

	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	0	0	0	0	0	0	0	0	0	1
H^3	0	0	1	0	0	0	1	0	0	2
H^4	0	0	2	0	2	1	2	1	0	3
H^5	1	2	2	1	2	4	3	3	3	4
H^6	5	1	1	4	0	3	2	2	5	3

	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	0	0	0	1	0	0	0	0	0	0
H^3	1	0	0	0	0	0	1	2	2	1
H^4	4	0	3	1	3	2	3	6	5	4
H^5	4	4	4	6	5	6	6	6	8	9
H^6	1	6	3	6	1	6	4	2	6	6

Table 7.2. The cohomology groups of $\mathcal{H}_3[2]$ as a representation of $\mathrm{Sp}_2(6)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	0	0	1	0	0
H^1	0	0	0	0	0	1	0	1	0	0
H^2	0	0	0	1	0	0	0	0	0	0
H^3	0	0	0	1	0	0	0	0	0	1
H^4	0	0	0	0	0	1	1	1	0	1
H^5	0	0	1	0	1	1	1	2	0	0

	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	1	0	0	0	1
H^2	0	0	1	0	2	1	1	0	0	3
H^3	0	0	3	1	3	2	4	1	0	5
H^4	2	2	3	2	3	6	5	4	4	6
H^5	4	2	1	4	1	4	3	3	6	4

	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	1	0	0	0	0	0	0
H^2	1	0	0	2	0	0	1	3	2	2
H^3	5	1	3	3	4	3	5	10	7	7
H^4	6	5	7	8	7	9	9	10	12	14
H^5	2	7	4	8	3	8	6	4	8	9

Table 7.3. The cohomology groups of $\mathcal{A}_{2,1}[2]$ as representations of $\mathrm{Sp}_2(6)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	1	0	1	0	0
H^1	0	0	0	0	0	2	0	2	0	0
H^2	0	0	0	1	0	0	0	1	0	0
H^3	0	0	0	1	0	0	0	0	0	2
H^4	0	0	0	0	0	0	0	0	1	1

	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	1	0	0	1	0	0	0	0
H^1	1	0	1	0	2	2	0	0	0	2
H^2	1	0	2	2	3	3	3	0	0	5
H^3	1	0	3	2	2	3	5	2	1	6
H^4	0	1	1	0	2	1	2	2	1	2

	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	1	0	0	3	0	0	0	1	1	1
H^2	2	2	0	6	1	2	3	6	5	5
H^3	4	2	4	4	4	5	6	10	8	10
H^4	4	1	3	2	5	3	5	6	5	5

Table 7.4. The cohomology groups of $\mathcal{A}_{1,1,1}[2]$ as representations of $\mathrm{Sp}_2(6)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	1	0	1	0	0
H^1	0	0	0	0	0	1	0	2	0	0
H^2	0	0	0	1	0	0	0	0	0	1
H^3	0	0	0	0	0	0	0	0	1	1

	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	1	0	1	0	0	1	0	0	0	0
H^1	1	0	1	1	2	2	1	0	0	3
H^2	0	0	2	1	2	2	3	1	0	4
H^3	0	0	1	0	1	0	1	1	0	1

	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	1	0	0	0	0	1	0
H^1	1	1	0	4	0	1	1	2	2	2
H^2	2	1	1	3	2	2	3	6	4	5
H^3	3	0	2	0	3	1	3	4	3	3

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