Generalized principal eigenvalues for parabolic operators in bounded domains

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Abstract

We introduce here new generalized principal eigenvalues for linear parabolic operators with heterogeneous coefficients in space and time. We consider a bounded spatial domain and an unbounded time interval $I : I = \mathbb{R}$, $\mathbb{R}^+$ or $\mathbb{R}^-$, and operators with coefficients having a fairly general dependence on space and time. The notions we introduce rely on the parabolic maximum principle and extend some earlier definitions introduced for elliptic operators [3, 5].

We first show that these eigenvalues hold the key to understanding the large time behavior and entire solutions of heterogeneous Fisher-KPP type equations. We then describe the relation of these quantities with principal Floquet bundles for parabolic operators which provides further characterizations of the principal eigenvalues. These allow us to derive monotonicity properties and comparisons between generalized principal eigenvalues, as well as perturbation results and further properties involving limit operators. We show that the sign of these eigenvalues encodes different versions of the maximum principle for parabolic operators. Lastly, we explicitly compute the generalized principal eigenvalues for several classes of operators such as spatial-independent, periodic, almost periodic, uniquely ergodic or random stationary ergodic coefficients.

Key-words: generalized principal eigenvalues, linear parabolic operator, principal Floquet bundles, semilinear parabolic equation, parabolic maximum principle.

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Contents

1 State of the art

We investigate the notion of the generalized principal eigenvalue for a linear parabolic operator. We consider uniformly parabolic operators of the form

$$P = \partial_t - a_{ij}(t,x)\partial_{ij} - b_i(t,x)\partial_i - c(t,x),$$

defined for $t \in I$ and $x \in \Omega$, where I is an unbounded interval and Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$. The partial derivatives ∂_i refer to the spatial variable x only, and we adopt the summation convention. We focus on the Dirichlet boundary condition. Most of the results derived in this paper should hold true, when properly adapted, for Neumann, Robin or periodic boundary conditions, but we leave such extensions as open problems.

Throughout the paper, we will assume the following hypotheses:

$$\begin{cases} \Omega \text{ is a bounded Lipschitz domain,} \\ a_{ij}, b_i, c \in L^\infty(\mathbb{R}\times\Omega) \text{ and } a_{ij} \in \mathcal{C}^0(\mathbb{R}\times\Omega) \text{ for all } i,j \in \{1,\dots,N\}, \\ \exists \alpha > 0 \mid \forall \xi \in \mathbb{R}^N, \quad (t,x) \in \mathbb{R}\times\Omega, \quad a_{ij}(t,x)\xi_i\xi_j \geq \alpha|\xi|^2. \end{cases} \tag{H}$$

We choose in this paper to consider operators in non-divergence form. Many of our results could easily be extended to operators in divergence form. We will sometimes assume that $a_{ij} \in W^{1,\infty}(\mathbb{R}\times\Omega)$ in order to use previous results for operators in divergence form (see Section 3 for example).

1.1 Floquet and Lyapounov exponents for time-periodic and almost periodic parabolic operators

When the coefficients of the operator are time-periodic (say with period T), the usual notion of eigenvalue is derived from that of Floquet exponents. That is, we consider the principal eigenvalue

$r(P)$ of the Poincaré map $Q_T : u_0(\cdot) \mapsto u(T,\cdot)$, where u is the solution of the Dirichlet problem

$$\begin{cases} Pu = 0 & \text{in } (0,+\infty)\times\Omega \\ u = 0 & \text{on } (0,+\infty)\times\partial\Omega, \end{cases} \tag{1}$$

emerging from the initial datum u_0. It is easily seen that one can write $r(P) = e^{-\lambda(P)T}$, where $\lambda(P)$ is the principal eigenvalue of the operator P, that is, the unique λ associated with some positive function ϕ on $(0,T)\times\Omega$, with $\phi = 0$ on $(0,T)\times\partial\Omega$, $\phi(\cdot+T,\cdot)\equiv\phi$, and $P\phi = \lambda\phi$ on $(0,T)\times\Omega$. Its existence, together with the simplicity, follows from the Krein-Rutman theory.

A comprehensive study of these eigenvalue has been provided by Hess [11]. We refer to [22] for some extensions to space-periodic coefficients.

For almost periodic or random stationary ergodic coefficients with respect to t, the Poincaré map Q_T is not well-defined and the inverse of the operator P is not compact anymore. This prevents the application of the Krein-Rutman theory to obtain the principal eigenvalue. However, one can still prove the existence of a finite quantity λ which plays the role of the Floquet exponent, i.e. such that any solution of problem (1) emerging from a positive initial datum satisfies

$$\lim_{t\to+\infty}\frac{1}{t}\ln\|u(t,\cdot)\|_{L^\infty(\Omega)} = -\lambda.$$

This indeed means that positive solutions behave like $e^{-\lambda t}$ for large t. The above limit holds with some uniformity with respect to translations of the coefficients in almost periodic media [34], and for almost every event in the case of random stationary ergodic coefficients [21]. Indeed, the L^∞ norm could be replaced by other norms or pointwise convergence using classical parabolic estimates. In the almost periodic setting, the quantity λ is called a Lyapounov exponent. Moreover, for sign-changing initial data, one can prove some exponential separation (see [21, 34]), extending earlier spectral gap results that were derived in the periodic framework.

It is easily seen that the above convergence property cannot hold in general when the dependence with respect to t of the coefficients is arbitrary. Among other things, we will show in this paper that for operators with general time/space heterogeneity of the coefficients, the semi limits

$$\limsup_{t\to+\infty}\frac{1}{t}\ln\|u(t,\cdot)\|_{L^\infty(\Omega)} \quad\text{and}\quad \liminf_{t\to+\infty}\frac{1}{t}\ln\|u(t,\cdot)\|_{L^\infty(\Omega)},$$

are characterized by two distinct notions of generalized principal eigenvalue, cf. Theorem 2.5 below.

1.2 Generalized principal eigenvalues for elliptic operators

The notion of generalized principal eigenvalues for elliptic operators has first been introduced by Berestycki, Nirenberg and Varadhan [5] in the case of bounded, non-smooth domains (see also Agmon [1] and Nussbaum and Pinchover [25] for an earlier related notion) and then extended by Berestycki, Hamel and Rossi [3] to unbounded domains. Consider an elliptic operator

$$\mathcal{L} = a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x), \qquad x\in\Omega \tag{2}$$

(with the summation convention on i,j). The notion of generalized principal eigenvalue of elliptic operators of [5] reads

$$\lambda(-\mathcal{L}) := \sup\{\lambda\ :\ \exists\,\phi>0 \text{ in } \Omega,\ -\mathcal{L}\phi\geq\lambda\phi \text{ in } \Omega\}.$$

This notion extends the relevant properties of the classical principal eigenvalue to the case where Ω is non-smooth. This is no longer the case when the domain Ω is unbounded. In order to deal with the unbounded case, an alternative notion was introduced in [2, 3]:

$$\lambda'(-\mathcal{L}) := \inf\{\lambda\ :\ \exists\,\phi>0 \text{ in } \Omega,\ \phi=0 \text{ on } \partial\Omega,\ -\mathcal{L}\phi\leq\lambda\phi \text{ in } \Omega\},$$

and still another one in [6]:

$$\lambda''(-\mathcal{L}) := \sup\{\lambda \ : \ \exists\, \phi, \ \inf_{\Omega} \phi > 0 \text{ in } \Omega, \ -\mathcal{L}\phi \geq \lambda\phi \text{ in } \Omega\}.$$

These notions incorporate in their definition the Dirichlet boundary condition. For their extension to Neumann or oblique derivative boundary conditions we refer to [32].

It has been proved in [3, 6] that, for an arbitrary elliptic operator, there holds

$$\lambda''(-\mathcal{L}) \leq \lambda'(-\mathcal{L}) \leq \lambda(-\mathcal{L}).$$

It may happen that $\lambda'(-\mathcal{L}) < \lambda(-\mathcal{L})$. But equality holds if the operator is self-adjoint. Further relevant cases of equalities between these quantities are established in [6, Theorem 1.9].

These eigenvalues provide both necessary and sufficient conditions for the validity of the maximum principle. They also play a central role in the study of heterogeneous Fisher-KPP type equations, as they allow to characterize existence, positivity, and attractivity of the stationary solutions, see [3, 6] and the discussion in the next section.

In the present paper, we want to find a notion of principal eigenvalue that fits for a parabolic operator P on a bounded, regular, spatial domain Ω. Here we face the same issue of lack of compactness as in the elliptic case when Ω is unbounded, because we consider unbounded intervals of times. As in the elliptic case, this leads us to introduce several notions of generalized principal eigenvalue.

1.3 A first notion of generalized principal eigenvalue

Let us start by introducing a first notion of *generalized principal eigenvalue* for the parabolic operator P on a time interval I that can be one of the following:

$$\mathbb{R}, \qquad \mathbb{R}^+ := [0, +\infty), \qquad \mathbb{R}^- := [-\infty, 0).$$

We consider a fixed, open ball $B \Subset \Omega$, i.e. such that $\bar{B} \subset \Omega$. Then we define

$$\mu_{b,p}(I) := \inf\{\lambda \ : \ \exists\, \phi > 0 \text{ in } I \times \Omega, \ \phi = 0 \text{ on } I \times \partial\Omega,$$
$$\inf_{I\times B} \phi > 0, \ \sup_{I\times\Omega} \phi < +\infty, \ P\phi \leq \lambda\phi \text{ in } I \times \Omega\},$$

The indices "b, p" refer to the requirements that the "test-functions" ϕ are bounded and strictly positive. The functions ϕ are also assumed to belong to the Sobolev space $W^{1,2}_{p,\text{loc}}(I \times \overline{\Omega})$ for some $p > N + 1$, that is, $\phi, \partial_t\phi, \partial_i\phi, \partial_{ij}\phi \in L^p(J \times \Omega)$ for any compact interval $J \subset I$, and the differential inequalities are understood to hold in the strong sense, i.e. almost everywhere. As a consequence, the functions ϕ are continuous up to the boundary of Ω, by Morrey's embedding.

It is a priori not obvious that the the generalized principal eigenvalue $\mu_{b,p}(I)$ is well defined and finite. This is a fact that we show as a byproduct of our main results, cf. Proposition 2.7 below. We will further see that $\mu_{b,p}(I)$ does not depend on the choice of the ball B. Actually, its definition would not change if one requires the stronger condition $\inf_{I\times K} \phi > 0$ for every compact set $K \subset \Omega$, or the weaker one $\inf_{t\in I} \|\phi(t, \cdot)\|_{L^\infty(\Omega)} > 0$ (see Remark 1(*iii*) below).

It is natural to wonder whether there exists a *generalized principal eigenfunction* associated with $\mu_{b,p}(I)$, i.e., a positive eigenfunction fulfilling the requirements $\inf_{I\times B} \phi > 0$, $\sup_{I\times\Omega} \phi < +\infty$. We will see that in general this is not the case (see Remark 3 below).

In Section 2.1, we discuss other possible definitions of generalized eigenvalues playing with the requirements imposed on the test functions or with sup rather than inf. We will see that they are relevant in particular for the study of linear Dirichlet problems.

1.4 Applications to semilinear problems

In order to illustrate the use of the new notion $\mu_{b,p}$ of generalized principal eigenvalue, we now state some applications to the study of the semilinear parabolic problem

$$\begin{cases} \partial_t u - a_{ij}(t,x)\partial_{ij}u - b_i(t,x)\partial_i u = f(t,x,u), & t \in I, \ x \in \Omega, \\ u(t,x) = 0, & t \in I, \ x \in \partial\Omega, \end{cases} \tag{3}$$

where I will be either $\mathbb{R}^\pm$ or $\mathbb{R}$. We will always assume here that the nonlinear term $f = f(t,x,s)$ is of class $\mathcal{C}^1$ with respect to s, uniformly with respect to $(t,x) \in I \times \Omega$, with bounded derivative f'_s. We further assume that

$$\forall (t,x) \in I \times \Omega, \quad f(t,x,0) = 0, \tag{4}$$

and that there exists a constant $M > 0$ such that

$$\forall (t,x) \in I \times \Omega, \ \forall s \geq M, \quad f(t,x,s) \leq 0. \tag{5}$$

Lastly, we will consider the KPP hypothesis:

$$\forall s > 0, \ x \in \Omega, \quad \inf_{t \in I} \big(f'_s(t,x,0)s - f(t,x,s) \big) > 0. \tag{6}$$

For the Fisher KPP equation without temporal dependence on a bounded, non-smooth domain Ω, [5, Theorems 1.1 and 2.2] imply that there exists a positive stationary solution if and only if the eigenvalue $\lambda(-\mathcal{L})$ of the linearized elliptic operator $\mathcal{L}$ is negative. This fact is extended to unbounded smooth domains in [3], using also the notion $\lambda'(-\mathcal{L})$. The eigenvalue $\lambda(-\mathcal{L})$ also allows one to derive conditions for the uniqueness and the stability of this stationary state.

Keeping in mind these results, we aim to derive here analogous properties for the reaction-diffusion problem (3) on a bounded domain Ω, under general space/time dependence of the coefficients, based on criteria involving generalized principal eigenvalues.

A first result concerns the long-time behavior of solutions to the Cauchy problem. It turns out that the quantity $\mu_{b,p}(\mathbb{R}^+)$ encodes the *persistence property*, that is, the fact that solutions preserve a positive mass as $t \to +\infty$. This information is particularly relevant in the context of population dynamics models.

Proposition 1.1. *Assume that $I = \mathbb{R}^+$, that Ω is of class $\mathcal{C}^{2,\alpha}$, for some $0 < \alpha < 1$, and that f satisfies (4)-(6). Let u be the solution of problem (3) with a continuous, nonnegative initial datum $u_0 \not\equiv 0$. Let $\mu_{b,p}(\mathbb{R}^+)$ be the eigenvalue associated with the linearized operator around 0, namely*

$$P := \partial_t - a_{ij}(t,x)\partial_{ij} - b_i(t,x)\partial_i - f'_s(t,x,0). \tag{7}$$

Then the persistence property

$$\liminf_{t \to +\infty} u(t,x) > 0 \qquad \textit{for all } x \in \Omega, \tag{8}$$

holds if and only if $\mu_{b,p}(\mathbb{R}^+) < 0$.

Next, we focus on entire (i.e. time-global) solutions to the semilinear Dirichlet problem. Namely, we consider (3) with $I = \mathbb{R}$. We are concerned with solutions that are uniformly positive away from the boundary of Ω, that is,

$$\forall K \Subset \Omega, \quad \inf_{I \times K} u > 0. \tag{9}$$

The following result shows that the existence of these solutions is completely characterized by the sign of $\mu_{b,p}(\mathbb{R})$, as in the temporal-independent case.

Proposition 1.2. *Assume that $I = \mathbb{R}$, that Ω is of class $\mathcal{C}^{2,\alpha}$, for some $0 < \alpha < 1$, and that f satisfies* (4)-(6)*. Let $\mu_{b,p}(\mathbb{R})$ be the eigenvalue associated with the linearized operator P given by* (7)*. Then the problem* (3) *admits a bounded entire solution satisfying* (9) *if and only if $\mu_{b,p}(\mathbb{R}) < 0$.*

The two results above can be deduced from the results of [20] under the additional hypothesis of random stationarity and ergodicity of the coefficients with respect to time. We will describe this hypothesis in Section 4. In the present article, we prove these results in full generality.

Let us point out that the strict inequality in (6), as well as the $\mathcal{C}^{2,\alpha}$ regularity of Ω, are only required in Propositions 1.1-1.2 to handle the case $\mu_{b,p} = 0$. While the former is really necessary, we believe that the latter could be relaxed.

Next, we show that if one assumes the following stronger version of the KPP hypothesis (6):

$$\forall 0 < s < s', \quad \inf_{\substack{t \in I \\ x \in \Omega}} \left(\frac{f(t,x,s)}{s} - \frac{f(t,x,s')}{s'} \right) > 0, \tag{10}$$

then the bounded solution to problem (3), (9) is unique. This is true not only for entire solutions ($I = \mathbb{R}$), but also for *ancient* solutions ($I = \mathbb{R}^-$).

Theorem 1.3. *Assume that $I = \mathbb{R}^-$, that the a_{ij} are uniformly continuous on $\mathbb{R}^- \times \Omega$ and that f satisfies* (4), (5), (10)*. Then problem* (3), (9) *admits at most a unique bounded ancient solution.*

Theorem 1.3 extends a uniqueness result for ancient solutions obtained by Rodríguez-Bernal and Vidal-López [29] in the particular case where $a_{ij}(t,x) \equiv \delta_{ij}$ and $b_i(t,x) \equiv 0$, and assuming that the solutions are ordered. Namely, [29, Theorem 2] asserts that if u_1 and u_2 are two bounded ancient solutions of (3), (9) such that $u_1 \leq u_2$, then $u_1 \equiv u_2$.

1.5 Aims of this study

The above results highlight the relevance of the generalized principal eigenvalue $\mu_{b,p}$ for discussing the evolution Fisher-KPP type equation (3). The purpose of this paper is to introduce various possible notions of generalized principal eigenvalues and to investigate their properties. We have three main objectives.

Our first task is to extend the notion of generalized principal eigenvalues from the elliptic case to the parabolic framework. Furthermore, following the systematic study carried out for elliptic operators, we would like to derive comparison or equivalence properties between the various notions of generalized principal eigenvalues in the parabolic setting.

Secondly, we establish a connection between the generalized principal eigenvalues and the Floquet theory. In doing so, we derive a new characterization of the exponential growth of the Floquet bundles when the coefficients of the parabolic operator have a general dependence on t. For some classes of operators (such as operators with uniquely ergodic or random stationary ergodic coefficients), we are then able to establish the equivalence between the various notions of generalized principal eigenvalues.

Finally, we show some applications of the generalized principal eigenvalues to the qualitative study of solutions of linear and semilinear parabolic problems.

All these tasks will also give a unified point of view on some results obtained in the literature using distinct tools and techniques.

The main technical tool employed in our proofs is a new notion of average for possibly unbounded functions, that we call *global growth-rate*, see Definition 6.2 below, which extends a previous notion introduced by the last two authors of the present work in [23].

The paper is organized as follows. In Section 2.1 we introduce several definitions of generalized principal eigenvalue. Next, we reclaim the notion of principal Floquet bundle and we state the characterization of the generalized principal eigenvalues in terms of such notion in Section 2.2. These results will allow us to derive some comparisons between the different generalized principal eigenvalues, which are stated in Section 2.3. The two following subsections contain applications to the qualitative study of solutions of linear problems and the maximum principle for ancient solutions. In Sections 3, 4 and 5 we summarize and re-frame some known results in the light of our generalized principal eigenvalues: perturbation results, examples and computation in some particular cases and for limit operators, relation with the notion of exponential type. Section 6 contains some technical results: the Hölder regularity of the Floquet bundle, and the properties of our main tool, the global growth-rate, which is of independent interest. The last two sections contain the proofs of the connection with the Floquet bundle and of the remaining main results.

We hope that this work will inspire future investigations aiming at extending the theory of parabolic generalized principal eigenvalues to unbounded domains. In such domains, the Floquet approach is not well-established. For want of such a theory, most of the results we discuss here are open in general in unbounded domains. Since generalized principal eigenvalues have proved to be successful in the discussion of problems in unbounded domains in the elliptic case, it is natural to expect that the same will be true in the parabolic setting as well.

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2 Definitions and properties of the generalized principal eigenvalues

2.1 Further definitions of generalized principal eigenvalues

We now define further quantities that we also call *generalized principal eigenvalues*, by modifying the requirements on the "test-functions" on which the operator P acts. Such requirements are reflected by the symbol used for their notation, according to the following table:

λ	supersolutions
μ	subsolutions
$(\ )_b$	bounded
$(\ )_p$	locally strictly positive

More precisely, we consider a fixed, open ball $B \Subset \Omega$, i.e. such that $\bar{B} \subset \Omega$, and we define

$$\lambda_{b,p}(I) := \sup\{\lambda \ : \ \exists\, \phi > 0,\ \sup_{I\times\Omega} \phi < +\infty,\ \inf_{I\times B} \phi > 0,\ P\phi \geq \lambda\phi \text{ in } I \times \Omega\}$$

$$\mu_b(I) := \inf\{\lambda \ : \ \exists\, \phi > 0 \text{ in } I \times \Omega,\ \sup_{I\times\Omega} \phi < +\infty,\ \phi = 0 \text{ on } I \times \partial\Omega,\ P\phi \leq \lambda\phi \text{ in } I \times \Omega\}$$

$$\lambda_p(I) := \sup\{\lambda \ : \ \exists\, \phi > 0 \text{ in } I \times \Omega,\ \inf_{I\times B} \phi > 0,\ P\phi \geq \lambda\phi \text{ in } I \times \Omega\}$$

$$\mu_p(I) := \inf\{\lambda,\ \exists\phi > 0,\ \inf_{I\times B} \phi > 0,\ \phi = 0 \text{ on } I \times \partial\Omega,\ P\phi \leq \lambda\phi \text{ in } I \times \Omega\}$$

$$\lambda_b(I) := \sup\{\lambda,\ \exists\phi > 0,\ \sup_{I\times\Omega} \phi < +\infty,\ P\phi \geq \lambda\phi \text{ in } I \times \Omega\}\,.$$

As in the definitions of $\mu_{b,p}$, the test-functions ϕ are assumed to belong to $W^{1,2}_{p,\mathrm{loc}}(I\times\overline{\Omega})$ for some $p>N+1$. The quantities λ,μ are defined as suprema and infima respectively, and doing it the other way around would yield $-\infty$ and $+\infty$ respectively.

It turns out that in the case $I=\mathbb{R}$, all the above quantities are well-defined and finite. We will show it as a byproduct of some characterizations involving the Floquet bundle, see Proposition 2.7.

We will show that the notions of generalized principal eigenvalue introduced so far, including $\mu_{b,p}$, coincide in some cases (see Propositions 4.1, 4.4, 4.6, 4.7 below), but in general they differ, see Remark 3. Some inequalities between them, together with some further properties, are provided by Proposition 2.7 below.

2.2 The relation between generalized principal eigenvalues and the Floquet bundle

We consider here positive entire – i.e., global-in-time – solutions of the Dirichlet problem in a bounded Lipschitz domain Ω, that is,

$$\begin{cases} Pu=0 & \text{in } \mathbb{R}\times\Omega\\ u=0 & \text{on } \mathbb{R}\times\partial\Omega.\end{cases}\tag{11}$$

The existence and uniqueness (up to normalization) of the positive entire solution has been derived in several frameworks [7, 13, 17, 26, 27] and is a first step to extend the Floquet theory to non-periodic settings. In the present paper, we will mostly rely on the work of Húska, Poláčik and Safonov [13][1]

Theorem 2.1 ([13, Theorem 2.6])**.** *There exists a unique time-global, nonnegative solution u_P to* (11) *normalized by $\|u_P(0,\cdot)\|_{L^\infty(\Omega)}=1$.*

The collection of the one-dimensional spaces $X_1(t):=\{ku_P(t,\cdot)\ :\ k\in\mathbb{R}\}$, $t\in\mathbb{R}$, is called the *principal Floquet bundle* of (11).

If the operator P is just defined on the domain $\mathbb{R}^+\times\Omega$ or on $\mathbb{R}^-\times\Omega$, then we extend it by even reflection with respect to t, and we let u_P denote the corresponding function given by Theorem 2.1. As a matter of fact, despite of the fact that the particular choice of extension of the operator P clearly affects the function u_P, our results will be independent of it. They hold true for any arbitrary extension, as long as hypotheses (H) are fulfilled, (see Remark 1(i) below).

The function u_P is continuous, in a certain sense, with respect to L^∞ perturbations of the coefficients of P and of the domain Ω, see [13, Theorems 2.8 and 2.9].

It is also proved in [13, Theorem 2.6] (see also [17, 18, 26, 27]) that, for general solutions, exponential separation occurs, meaning that the long time behavior is essentially determined by the projection on the space generated by u_P, up to a factor converging exponentially to 0 in time, as stated in the next theorem.

Theorem 2.2 ([13])**.** *There exist $C,\gamma>0$ such that, for any continuous initial datum u_0 vanishing on $\partial\Omega$, the solution to the problem $Pu=0$ on $(0,+\infty)\times\Omega$ with Dirichlet boundary condition satisfies*

$$\forall t>0,\quad \|u(t,\cdot)-qu_P(t,\cdot)\|_{L^\infty(\Omega)}\leq C\|u_0-qu_P(0,\cdot)\|_{L^\infty(\Omega)}\|u_P(t,\cdot)\|_{L^\infty(\Omega)}\,e^{-\gamma t},$$

for some $q\in\mathbb{R}$. Moreover, $u(t,\cdot)-qu_P(t,\cdot)$ has a zero in Ω for all $t\geq 0$.

This result is an immediate application of [13, Theorem 2.6]. Indeed, statement [13, Theorem 2.6(ii)] guarantees the existence of a unique $q\in\mathbb{R}$ such that $u_0-qu_P(0,\cdot)$ belongs to the

[1]The existence result in [13] is quoted from [12], where it is derived for operators in divergence form only. However, the strategy – consisting in approximating the operator P by a sequence of time-periodic operators, with large period, and exploiting the existence of the periodic principal eigenfunctions for those – also works in the non-divergence case, without any change.

complementary Floquet bundle associated to the problem, meaning that $u(t,\cdot) - qu_P(t,\cdot)$ vanishes somewhere in Ω, for any $t > 0$. Thus statement [13, Theorem 2.6(iii)] yields Theorem 2.2.

When the operator is in divergence form, that is,

$$Pu := \partial_t u - \partial_i\big(a_{ij}(t,x)\partial_j u\big) - \tilde{b}_i(t,x)\partial_i u - c(t,x)u,$$

the quantity q is actually explicit. Namely, in this case we have $q = \int_\Omega u_0(x)u_{P^\star}(0,x)dx$, where $P^\star$ is the adjoint operator:

$$P^\star u := -\partial_t u - \partial_i\big(a_{ij}(t,x)\partial_j u\big) + \partial_i\big(\tilde{b}_i(t,x)u\big) - c(t,x)u,$$

and $u_{P^\star}$ is the unique entire solution of $P^\star u_{P^\star} = 0$ on $\mathbb{R}\times\Omega$, $u_{P^\star} = 0$ on $\mathbb{R}\times\partial\Omega$, normalized by $\int_\Omega u_P(0,x)u_{P^\star}(0,x)dx = 1$. The function $u_{P^\star}$ is well-defined thanks to Theorem 2.2, with the change of variables $t \mapsto -t$.

We are now in position to state the results providing the link between Floquet bundles and the generalized principal eigenvalues.

Theorem 2.3. *For $I = \mathbb{R}$, $\mathbb{R}^+$ or $\mathbb{R}^-$, there holds*

$$\mu_{b,p}(I) = -\lim_{t\to+\infty}\left(\inf_{s,s+t\in I}\frac{\ln\|u_P(s+t,\cdot)\|_{L^\infty(\Omega)} - \ln\|u_P(s,\cdot)\|_{L^\infty(\Omega)}}{t}\right),$$

$$\lambda_{b,p}(I) = -\lim_{t\to+\infty}\left(\sup_{s,s+t\in I}\frac{\ln\|u_P(s+t,\cdot)\|_{L^\infty(\Omega)} - \ln\|u_P(s,\cdot)\|_{L^\infty(\Omega)}}{t}\right).$$

In the Floquet bundles theory, the closed interval $[-\mu_{b,p}(I), -\lambda_{b,p}(I)]$ is called the *principal spectrum* of the operator P (see for example [14, 18, 19, 21]).

We remark that the L^∞ norm in Theorem 2.3 can be replaced by any L^p norm, $p \geq 1$, owing to the Harnack inequality, cf. [13, Corollary 5.3 (*ii*)]. A first consequence is the following.

Corollary 2.4. *There holds*

$$\mu_{b,p}(\mathbb{R}) = \max\{\mu_{b,p}(\mathbb{R}^-), \mu_{b,p}(\mathbb{R}^+)\},$$

$$\lambda_{b,p}(\mathbb{R}) = \min\{\lambda_{b,p}(\mathbb{R}^-), \lambda_{b,p}(\mathbb{R}^+)\}.$$

We now state further characterizations and comparison results.

Theorem 2.5. *There holds*

$$\mu_p(\mathbb{R}) = \mu_p(\mathbb{R}^+) = -\liminf_{t\to+\infty}\frac{\ln\|u_P(t,\cdot)\|_{L^\infty(\Omega)}}{t},$$

$$\lambda_b(\mathbb{R}) = \lambda_b(\mathbb{R}^+) = -\limsup_{t\to+\infty}\frac{\ln\|u_P(t,\cdot)\|_{L^\infty(\Omega)}}{t}.$$

In particular, one has $\lambda_b(\mathbb{R}) \leq \mu_p(\mathbb{R})$.

Theorem 2.6. *There holds*

$$\mu_b(\mathbb{R}) = \mu_b(\mathbb{R}^-) = -\liminf_{t\to-\infty}\frac{\ln\|u_P(t,\cdot)\|_{L^\infty(\Omega)}}{t},$$

$$\lambda_p(\mathbb{R}) = \lambda_p(\mathbb{R}^-) = -\limsup_{t\to-\infty}\frac{\ln\|u_P(t,\cdot)\|_{L^\infty(\Omega)}}{t}.$$

In particular, one has $\lambda_p(\mathbb{R}) \leq \mu_b(\mathbb{R})$.

Remark 1. (*i*) Since the generalized principal eigenvalues associated with the intervals $\mathbb{R}^-$ or $\mathbb{R}^+$ only depend on the operator P restricted to such intervals, one deduces that the quantities at the the right-hand sides of the formulas in Theorems 2.3, 2.5, 2.6 do not depend on the values of the coefficients of P outside the interval I, or outside $\mathbb{R}^+$ or $\mathbb{R}^-$ respectively. In particular, those theorems hold true independently of the extension of P used in order to define the function u_P when $I \neq \mathbb{R}$, provided it preserves the hypotheses (H).

(*ii*) As a byproduct of Theorems 2.3, 2.5, 2.6 one infers that the quantities $\mu_{b,p}(I)$, $\lambda_{b,p}(I)$, $\mu_p(I)$, $\lambda_b(I)$, $\mu_b(I)$, $\lambda_p(I)$ do not change if one replaces the interval $I = \mathbb{R}^-$ or $I = \mathbb{R}^+$ with any half-line $(-\infty, T)$ or $(T, +\infty)$ respectively.

(*iii*) As a consequence of the above results, it turns out that the definition of the eigenvalues $\mu_{b,p}$, $\lambda_{b,p}$, μ_p, λ_p do not depend on the choice of the ball $B \Subset \Omega$ in the condition $\inf_{I\times B} \phi > 0$ required on the test functions. We will actually show that one could equivalently require the stronger condition $\inf_{I\times K} \phi > 0$ for every compact set $K \subset \Omega$, or the weaker one

$$\inf_{t\in I} \|\phi(t,\cdot)\|_{L^\infty(\Omega)} > 0,$$

(and even a weaker condition) without changing the definitions, see Remarks 5, 6 below.

2.3 Comparison between the different generalized principal eigenvalues

The characterization of the generalized principal eigenvalues via the Floquet bundle yield some relations between the different generalized principal eigenvalues. We summarize them in the following statement. It also includes a monotonicity property with respect to the inclusions of the domains.

Proposition 2.7. *For $I = \mathbb{R}$, $\mathbb{R}^+$ or $\mathbb{R}^-$, the quantities $\lambda_{b,p}(I)$, $\mu_{b,p}(I)$, $\mu_p(I)$, $\lambda_b(I)$, $\mu_b(I)$, $\lambda_p(I)$ are well defined and finite, except for $\lambda_b(\mathbb{R}^-) = \lambda_p(\mathbb{R}^+) = +\infty$, $\mu_p(\mathbb{R}^-) = \mu_b(\mathbb{R}^+) = -\infty$, and satisfy*

$$-\sup_{I\times\Omega} c \;\leq\; \lambda_{b,p}(I) \;\leq\; \min\{\lambda_b(I), \lambda_p(I)\} \;\leq\; \max\{\mu_b(I), \mu_p(I)\} \;\leq\; \mu_{b,p}(I).$$

Moreover, these eigenvalues are nonincreasing with respect to the inclusion of the domain Ω, in the sense that the ones associated with the domain Ω are smaller than or equal to the corresponding ones associated with a smooth domain $\Omega' \subset \Omega$.

2.4 Application to the study of the linear Dirichlet problem

We consider here the Dirichlet problem (11) for $t > 0$, complemented with an initial datum u_0 satisfying

$$u_0 \in \mathcal{C}(\overline{\Omega}), \qquad u_0 \geq 0, \qquad u_0 \not\equiv 0, \qquad u_0 = 0 \text{ on } \partial\Omega. \tag{12}$$

The unique solution of this problem is denoted by $u(t,x;u_0)$. The long-time behavior of this solution can be described combining the exponential separation of [13] with our characterization Theorem 2.5.

Proposition 2.8. *Let u_0 be an initial datum satisfying* (12)*. Then, for any $x \in \Omega$ the following hold:*

$$\liminf_{t\to+\infty} \frac{\ln u(t,x;u_0)}{t} = -\mu_p(\mathbb{R}^+),$$

$$\limsup_{t\to+\infty} \frac{\ln u(t,x;u_0)}{t} = -\lambda_b(\mathbb{R}^+).$$

Next, for all $s \in \mathbb{R}$, we let $u^s = u^s(t,x;u_0)$ denote the solution of $Pu^s = 0$ in $(s,+\infty)\times\Omega$ with initial datum $u^s(s,x;u_0) = u_0(x)$ for all $x \in \Omega$.

Proposition 2.9. *Let u_0 be an initial datum satisfying* (12)*. Then, for any $x \in \Omega$ the following hold:*

$$\liminf_{s\to-\infty} \frac{\ln u^s(0,x;u_0)}{s} = \lambda_p(\mathbb{R}^-),$$

$$\limsup_{s\to-\infty} \frac{\ln u^s(0,x;u_0)}{s} = \mu_b(\mathbb{R}^-).$$

2.5 Application to the maximum principle for ancient solutions

We now turn to the study of ancient solutions to the Dirichlet problem (11).

Definition 2.10. We say that the operator P satisfies the *maximum principle* (*MP* for short) on $\mathbb{R}^- \times \Omega$ if for every ancient subsolution u to (11), i.e. satisfying

$$Pu \le 0 \ \text{ in } \mathbb{R}^- \times \Omega, \qquad \sup_{\mathbb{R}^-\times\Omega} u < +\infty, \qquad u \le 0 \ \text{ on } \mathbb{R}^- \times \partial\Omega,$$

there holds $u \le 0$ in $\mathbb{R}^- \times \Omega$.

Theorem 2.11. *The operator P satisfies the MP on $\mathbb{R}^- \times \Omega$ if $\mu_b(\mathbb{R}^-) > 0$, and only if $\mu_b(\mathbb{R}^-) \ge 0$.*

Remark 2. Since $\lambda_p(\mathbb{R}^-) \le \mu_b(\mathbb{R}^-)$, thanks to Theorem 2.6, a sufficient condition for the *MP* to hold is $\lambda_p(\mathbb{R}^-) > 0$. This is actually easier to check than $\mu_b(\mathbb{R}^-) > 0$, because λ is defined as a supremum rather than an infimum.

We also point out that in the case $\mu_b(\mathbb{R}^-) = 0$, the *MP* may or may not hold. Indeed, on one hand, for $P = \partial_t - \mathcal{L}$ with $-\mathcal{L}$ elliptic operator satisfying $\lambda_D(-\mathcal{L},\Omega) = 0$, Proposition 4.1 yields $\mu_b(\mathbb{R}^-) = \lambda_D(-\mathcal{L},\Omega) = 0$, but the Dirichlet principal eigenfunction ϕ_D violates the *MP*.

On the other hand, adding to the operator a term $\sigma(t) := -(1-t)^{-1/2}$, that is, considering $P = \partial_t - \mathcal{L} + \sigma(t)$, one still has $\mu_b(\mathbb{R}^-) = \lambda_D(-\mathcal{L},\Omega) = 0$, owing to Proposition 4.2, but

$$u_P(t,x) = \phi_D(x) e^{-\lambda_D(-\mathcal{L},\Omega)t + \int_0^t \sigma(s)ds}$$

tends to $+\infty$ as $t \to -\infty$, locally uniformly in $x \in \Omega$. As seen in the proof of Theorem 2.11, such a property entails that the *MP* holds.

3 Perturbation results

In this and the next section, exploiting the connections with the principal Floquet bundle provided by Theorems 2.3, 2.5, 2.6, we derive some other useful properties of the generalized principal eigenvalues.

We first state two results on the dependence of the generalized principal eigenvalues with respect to perturbations of the coefficients of the operator or of the domain Ω. These will be derived from the corresponding perturbation results for u_P obtained in [13] for operators in divergence form.

In order to deal with perturbations of the coefficients, we consider another operator

$$\tilde{P} = \partial_t - \tilde{a}_{ij}(t,x)\partial_{ij} - \tilde{b}_i(t,x)\partial_i - \tilde{c}(t,x).$$

We emphasize the dependence of the generalized principal eigenvalues on the operator P or $\tilde{P}$ by including the operator in the notation. We will assume that a_{ij} and $\tilde{a}_{ij}$ are Lipschitz-continuous in x, and that the operators fulfill the hypotheses of [13] when written in divergence form. We have the following.

Corollary 3.1. *Let I be $\mathbb{R}$, $\mathbb{R}^+$ or $\mathbb{R}^-$. Assume that the coefficients of both operators P and $\tilde{P}$ satisfy (H), together with $a_{ij}, \tilde{a}_{ij} \in W^{1,\infty}(I \times \Omega)$. For all $\varepsilon > 0$, there exists $\delta > 0$ such that if*

$$\|a_{ij} - \tilde{a}_{ij}\|_{L^\infty(I\times\Omega)} < \delta, \quad \|\nabla a_{ij} - \nabla\tilde{a}_{ij}\|_{L^\infty(I\times\Omega)} < \delta, \qquad \|b_i - \tilde{b}_i\|_{L^\infty(I\times\Omega)} < \delta, \quad \|c - \tilde{c}\|_{L^\infty(I\times\Omega)} < \delta,$$

then there holds

$$\begin{aligned} &|\mu_{b,p}(P, I) - \mu_{b,p}(\tilde{P}, I)| < \varepsilon, \quad |\lambda_{b,p}(P, I) - \lambda_{b,p}(\tilde{P}, I)| < \varepsilon, \\ \text{if } I \neq \mathbb{R}^-, \quad &|\mu_p(P, I) - \mu_p(\tilde{P}, I)| < \varepsilon, \quad |\lambda_b(P, I) - \lambda_b(\tilde{P}, I)| < \varepsilon, \\ \text{if } I \neq \mathbb{R}^+, \quad &|\mu_b(P, I) - \mu_b(\tilde{P}, I)| < \varepsilon, \quad |\lambda_p(P, I) - \lambda_p(\tilde{P}, I)| < \varepsilon. \end{aligned}$$

The above result is a consequence of Theorems 2.3, 2.5, 2.6 and of [13, Proposition 8.3], which asserts that, putting the operators in divergence form, for any $\varepsilon > 0$, there are some quantities $C, \delta > 0$ such that if

$$\|a_{ij} - \tilde{a}_{ij}\|_{W^{1,\infty}(\mathbb{R}\times\Omega)} < \delta, \quad \|b_i - \tilde{b}_i\|_{L^\infty(\mathbb{R}\times\Omega)} < \delta, \quad \|c - \tilde{c}\|_{L^\infty(\mathbb{R}\times\Omega)} < \delta,$$

then for all $s \in \mathbb{R}$, $t > 0$, it holds

$$\frac{\|u_P(s+t,\cdot)\|_{L^\infty(\Omega)}}{\|u_P(s,\cdot)\|_{L^\infty(\Omega)}} \leq Ce^{\varepsilon t}\frac{\|u_{\tilde{P}}(s+t,\cdot)\|_{L^\infty(\Omega)}}{\|u_{\tilde{P}}(s,\cdot)\|_{L^\infty(\Omega)}} \tag{13}$$

(in the case $I \neq \mathbb{R}$, this property is applied to the even extension of the coefficients of P).

Next, we consider another bounded Lipschitz domain $\tilde{\Omega}$. We then highlight the dependence of the generalized principal eigenvalues on the domain by specifying in the notation to which one they correspond. The perturbation result is then derived from [13, Proposition 9.5], which gives the estimate (13) provided that Ω, $\tilde{\Omega}$ are close enough, in terms of the Hausdorff distance $d(\partial\Omega, \partial\tilde{\Omega})$ of their boundaries.

Corollary 3.2. *Let I be $\mathbb{R}$, $\mathbb{R}^+$ or $\mathbb{R}^-$. Assume (H) holds in $I\times(\Omega\cup\tilde{\Omega})$ and $a_{ij} \in W^{1,\infty}(\mathbb{R}\times(\Omega\cup\tilde{\Omega}))$. For all $\varepsilon > 0$, there exists $\delta > 0$ such that if*

$$d(\partial\Omega, \partial\tilde{\Omega}) < \delta,$$

then there holds

$$\begin{aligned} &|\mu_{b,p}(P, I\times\tilde{\Omega}) - \mu_{b,p}(P, I\times\Omega)| < \varepsilon, \quad |\lambda_{b,p}(P, I\times\tilde{\Omega}) - \lambda_{b,p}(P, I\times\Omega)| < \varepsilon, \\ \text{if } I \neq \mathbb{R}^-, \quad &|\mu_p(P, I\times\tilde{\Omega}) - \mu_p(P, I\times\Omega)| < \varepsilon, \quad |\lambda_b(P, I\times\tilde{\Omega}) - \lambda_b(P, I\times\Omega)| < \varepsilon, \\ \text{if } I \neq \mathbb{R}^+, \quad &|\mu_b(P, I\times\tilde{\Omega}) - \mu_b(P, I\times\Omega)| < \varepsilon, \quad |\lambda_p(P, I\times\tilde{\Omega}) - \lambda_p(P, I\times\Omega)| < \varepsilon. \end{aligned}$$

4 Computation of the generalized principal eigenvalues for some classes of operators

The properties we have established for the generalized principal eigenvalues allow us to compute them for several important classes of operators. This is the purpose of this section.

Time-independent coefficients

When the coefficients do not depend on t, we recover the classical notion of Dirichlet principal eigenvalue.

Proposition 4.1. *Assume that the coefficients of P do not depend on t, i.e., $P = \partial_t - \mathcal{L}$ with $\mathcal{L}$ given by* (2)*. Then, there holds*

$$\mu_{b,p}(\mathbb{R}) = \mu_{b,p}(\mathbb{R}^\pm) = \lambda_{b,p}(\mathbb{R}) = \lambda_{b,p}(\mathbb{R}^\pm) = \mu_p(\mathbb{R}^+) = \lambda_b(\mathbb{R}^+) = \mu_b(\mathbb{R}^-) = \lambda_p(\mathbb{R}^-) = \lambda_D(-\mathcal{L}, \Omega),$$

where $\lambda_D(-\mathcal{L}, \Omega)$ is the Dirichlet principal eigenvalue of the elliptic operator $-\mathcal{L}$ in Ω.

This result is a particular case of Proposition 4.2 below.

Time-independent principal part

When the operator P is the sum of an elliptic operator plus a time-dependent zero order term σ, then the generalized principal eigenvalues can also be determined through various notions of time average of σ, as the next result asserts.

Proposition 4.2. *Assume that $P = \partial_t - \mathcal{L} - \sigma(t)$, with $\mathcal{L}$ given by (2) and $\sigma \in L^\infty(\mathbb{R})$. Then, for $I = \mathbb{R}$, $\mathbb{R}^+$ or $\mathbb{R}^-$, there holds*

$$\mu_{b,p}(I) = \lambda_D(-\mathcal{L}, \Omega) - \lim_{t\to+\infty} \inf_{s,s+t\in I} \frac{\int_s^{s+t} \sigma(s')ds'}{t},$$

$$\lambda_{b,p}(I) = \lambda_D(-\mathcal{L}, \Omega) - \lim_{t\to+\infty} \sup_{s,s+t\in I} \frac{\int_s^{s+t} \sigma(s')ds'}{t},$$

and

$$\mu_p(\mathbb{R}^+) = \lambda_D(-\mathcal{L}, \Omega) - \liminf_{t\to+\infty} \frac{\int_0^t \sigma(s)ds}{t}, \qquad \lambda_b(\mathbb{R}^+) = \lambda_D(-\mathcal{L}, \Omega) - \limsup_{t\to+\infty} \frac{\int_0^t \sigma(s)ds}{t},$$

$$\mu_b(\mathbb{R}^-) = \lambda_D(-\mathcal{L}, \Omega) - \liminf_{t\to-\infty} \frac{\int_0^t \sigma(s)ds}{t}, \qquad \lambda_p(\mathbb{R}^-) = \lambda_D(-\mathcal{L}, \Omega) - \limsup_{t\to-\infty} \frac{\int_0^t \sigma(s)ds}{t},$$

where $\lambda_D(-\mathcal{L}, \Omega)$ is the Dirichlet principal eigenvalue of the elliptic operator $-\mathcal{L}$ in Ω.

Proof. It is straightforward to check that the function u_P provided by Theorem 2.1 is given by

$$u_P(t,x) = \phi_D(x) e^{-\lambda_D(-\mathcal{L},\Omega)t + \int_0^t \sigma(s)ds}, \tag{14}$$

where ϕ_D is the Dirichlet principal eigenfunction of $-\mathcal{L}$ in Ω, suitably normalized. The equalities then follow from Theorems 2.3, 2.5, 2.6. □

Remark 3. It follows from Proposition 4.2 that all the notions $\mu_{b,p}$, $\lambda_{b,p}$, μ_p, λ_b, μ_b, λ_p differ in general. It is indeed sufficient to choose σ so that the corresponding notions of average are all different. As a consequence of this fact, one infers that in general there does not exist a *principal eigenfunction* associated with the generalized principal eigenvalues $\mu_{b,p}$, $\lambda_{b,p}$, that is, a positive eigenfunction ϕ satisfying the requirements $\sup_{I\times\Omega} \phi < +\infty$ and $\inf_{I\times B} \phi > 0$. Indeed, if it existed, then one would have $\mu_{b,p} \leq \lambda_{b,p}$, hence by Proposition 2.7 the eigenvalues $\mu_{b,p}$ and $\lambda_{b,p}$ would coincide. Likewise, in general there are no principal eigenfunctions associated with the other notions of generalized principal eigenvalues. For if they did exist, one would get $\mu_b(\mathbb{R}) \leq \lambda_b(\mathbb{R})$ and $\mu_p(\mathbb{R}) \leq \lambda_p(\mathbb{R})$ but by Proposition 4.2 one can choose σ in such a way that these inequalities fail.

Coefficients converging when $t \to \pm\infty$

For the next result, we need the Lipschitz regularity of the a_{ij}, that allows one to write the operator in self-adjoint form.

Proposition 4.3. *Assume that $a_{ij} \in W^{1,\infty}(\mathbb{R}^+ \times \Omega)$ (resp. $a_{ij} \in W^{1,\infty}(\mathbb{R}^- \times \Omega)$), and that $a_{ij}(t,x) \to a_{ij}^\infty(x)$, $\nabla a_{ij}(t,x) \to \nabla a_{ij}^\infty(x)$, $b_i(t,x) \to b_i^\infty(x)$, $c(t,x) \to c^\infty(x)$ as $t \to +\infty$ (resp. $t \to -\infty$), uniformly in $x \in \Omega$. Then*

$$\mu_{b,p}(\mathbb{R}^+) = \lambda_{b,p}(\mathbb{R}^+) = \mu_p(\mathbb{R}^+) = \lambda_b(\mathbb{R}^+) = \lambda_D(-\mathcal{L}^\infty, \Omega)$$

$$(\textit{resp. } \mu_{b,p}(\mathbb{R}^-) = \lambda_{b,p}(\mathbb{R}^-) = \mu_b(\mathbb{R}^-) = \lambda_p(\mathbb{R}^-) = \lambda_D(-\mathcal{L}^\infty, \Omega)\ ),$$

where $\lambda_D(-\mathcal{L}^\infty, \Omega)$ is the Dirichlet principal eigenvalue of the elliptic operator

$$-\mathcal{L}^\infty = -a_{ij}^\infty(x)\partial_{ij} - b_i^\infty(x)\partial_i - c^\infty(x)$$

in the domain Ω.

Proof. As noted in Remark 1(*ii*), one has $\mu_{b.p}(\mathbb{R}^+) = \mu_{b,p}((a,+\infty))$ for all $a > 0$. Thus, applying Corollary 3.1 with $I = (a,+\infty)$, one deduces

$$\mu_{b,p}(\mathbb{R}^+) = \mu_{b,p}((a,+\infty)) \xrightarrow{a\to+\infty} \mu_{b,p}(\partial_t - \mathcal{L}^\infty, \mathbb{R}^+),$$

which coincides with $\lambda_D(-\mathcal{L}^\infty, \Omega)$ thanks to Proposition 4.1. The other equivalences follow from the same arguments. □

Periodic coefficients

We assume in this section that the coefficients are periodic in t, with the same period $T > 0$, namely

$$\forall t \in \mathbb{R},\ x \in \Omega,\ i,j = 1,...,N, \quad a_{ij}(t+T,x) = a_{ij}(t,x), \quad b_i(t+T,x) = b_i(t,x), \quad c(t+T,x) = c(t,x).$$

Under this hypothesis, it turns out that all the notions of principal eigenvalues are equivalent, coinciding with the standard periodic principal eigenvalue.

Proposition 4.4. *Assume that the coefficients of P are periodic in t, with period T. Then there holds*

$$\lambda_{b,p}(\mathbb{R}) = \lambda_b(\mathbb{R}) = \lambda_p(\mathbb{R}) = \mu_{b,p}(\mathbb{R}) = \mu_p(\mathbb{R}) = \mu_b(\mathbb{R}) = \lambda_{per}(\Omega),$$

where $\lambda_{per}(\Omega)$ is the unique $\lambda \in \mathbb{R}$ such that there exists a solution ϕ of the problem

$$\begin{cases} P\phi = \lambda\phi & \text{in } \mathbb{R}\times\Omega, \\ \phi \ \text{ is periodic in } t & \text{with period } T, \\ \phi(t,x) = 0 & \text{for } (t,x) \in \mathbb{R}\times\partial\Omega, \\ \phi(t,x) > 0 & \text{for } (t,x) \in \mathbb{R}\times\Omega. \end{cases}$$

Moreover, the principal eigenvalue admits the following characterization:

$$\begin{aligned} \lambda_{per}(\Omega) &= \sup\{\lambda \ : \ \exists\, \phi > 0 \text{ in } \mathbb{R}\times\Omega,\ \phi \text{ is } t\text{-periodic with period } T,\ P\phi \geq \lambda\phi \text{ in } \mathbb{R}\times\Omega\} \\ &= \inf\{\lambda \ : \ \exists\, \phi > 0 \text{ in } \mathbb{R}\times\Omega,\ \phi = 0 \text{ on } \mathbb{R}\times\partial\Omega,\ \phi \text{ is } t\text{-periodic with period } T, \\ &\qquad\qquad P\phi \leq \lambda\phi \text{ in } \mathbb{R}\times\Omega\}. \end{aligned}$$

Proof. The existence and uniqueness of the solution $\lambda_{per}(\Omega)$ of the above eigenproblem (together with its simplicity) follows from the standard Krein-Rutman theory, since the problem is set on a compact domain, due to periodicity. Let ϕ be the eigenfunction associated with $\lambda_{per}(\Omega)$. It is immediately checked that $u_P(t,x) = \phi(t,x)e^{-\lambda_{per}(\Omega)t}$. The equalities stated in the first part of the proposition are then a consequence of Theorems 2.3, 2.5, 2.6. The last equalities then follow too, because

$$\begin{aligned} \lambda_{per}(\Omega) &= \mu_{b,p}(\mathbb{R}) \\ &\leq \inf\{\lambda \ : \ \exists\, \phi > 0 \text{ in } \mathbb{R}\times\Omega,\ \phi = 0 \text{ on } \mathbb{R}\times\partial\Omega,\ \phi \text{ is } t\text{-periodic with period } T, \\ &\qquad\qquad P\phi \leq \lambda\phi \text{ in } \mathbb{R}\times\Omega\} \\ &\leq \lambda_{per}(\Omega) \\ &\leq \sup\{\lambda \ : \ \exists\, \phi > 0 \text{ in } \mathbb{R}\times\Omega,\ \phi \text{ is } t\text{-periodic with period } T,\ P\phi \geq \lambda\phi \text{ in } \mathbb{R}\times\Omega\} \\ &\leq \lambda_{b,p}(\mathbb{R}) = \lambda_{per}(\Omega). \end{aligned}$$

□

Uniquely ergodic coefficients

We now address the case of *uniquely ergodic* coefficients with respect to t.

Definition 4.5. A function $f : I \times \Omega \to \mathbb{R}^m$ is called *uniquely ergodic* with respect to $t \in I$ if it is uniformly continuous and bounded over $I \times \Omega$, and if there exists a unique invariant probability measure on its hull $\mathcal{H}_f := cl\{\tau_a f, a \in I\}$, where cl is the closure with respect to the locally uniform convergence in $I \times \overline{\Omega}$, and where the invariance is understood with respect to the translations $\tau_a f(t,x) := f(t+a,x)$.

This notion is well-known to generalize that of *almost periodic* functions. Indeed, if $f : I \times \Omega \to \mathbb{R}^m$ is almost periodic with respect to t, let $\Psi : \mathcal{H}_f \to \mathbb{R}$. Then we could define $\int_{\mathcal{H}_f} \Psi d\mathbb{P} := \lim_{T\to\infty} \frac{1}{2T} \int_{a-T}^{a+T} \Psi(\tau_t f)dt$ uniformly with respect to $a \in \mathbb{R}$. Moreover, one can check that $\mathbb{P}$ is the unique invariant probability measure on $\mathcal{H}_f := cl\{\tau_a f, a \in I\}$.

When the coefficients of P are uniquely ergodic in t, we are able to prove that all the notions of generalized principal eigenvalues coincide.

Proposition 4.6. *Assume that the coefficients $(a_{ij})_{i,j\in\{1,\dots,N\}}$, $(b_i)_{i\in\{1,\dots,N\}}$ and c are uniquely ergodic with respect to $t \in I$ locally uniformly in $x \in \Omega$. Then there holds*

$$\mu_{b,p}(\mathbb{R}) = \lambda_{b,p}(\mathbb{R}) = \mu_p(\mathbb{R}) = \lambda_p(\mathbb{R}) = \mu_b(\mathbb{R}) = \lambda_p(\mathbb{R}).$$

In the work [14], the authors investigate the principal spectrum when the coefficients are periodic or almost periodic in time. They prove that the principal spectrum is a point, from which one could derive Proposition 4.6 when the coefficients are almost periodic in time. Moreover, some comparisons are derived in [14] between the principal spectrum and the Dirichlet eigenvalues associated with coefficients that are averaged in time.

Proof. This follows from [18, Proposition 2.11 and Theorem 2.15]. Indeed, in that paper, the author defines a principal spectrum $E := [\underline{\lambda}, \overline{\lambda}]$ and characterizes it in Proposition 2.11 as

$$\underline{\lambda} = \lim_{t\to+\infty} \left(\inf_{s\in\mathbb{R}} \frac{\ln \|v(s+t,\cdot)\|_{L^2(\Omega)} - \ln \|v(s,\cdot)\|_{L^2(\Omega)}}{t} \right)$$

$$\overline{\lambda} = \lim_{t\to+\infty} \left(\sup_{s\in\mathbb{R}} \frac{\ln \|v(s+t,\cdot)\|_{L^2(\Omega)} - \ln \|v(s,\cdot)\|_{L^2(\Omega)}}{t} \right)$$

where v is the unique positive time-global solution of $Pv = 0$, with $v = 0$ on $\mathbb{R} \times \partial\Omega$ and $\|v(0,\cdot)\|_{L^2(\Omega)} = 1$ (constructed in [18, Theorem 2.4]). By uniqueness, there exists $\sigma > 0$ such that $v \equiv \sigma u_P$, and we thus get, by Theorem 2.3 and the subsequent remark, that $\mu_{b,p}(\mathbb{R}) = -\underline{\lambda}$ and $\lambda_{b,p}(\mathbb{R}) = -\overline{\lambda}$. Lastly, [18, Theorem 2.15] yields $\underline{\lambda} = \overline{\lambda}$ and the conclusion then follows using also Theorems 2.5, 2.6. □

Remark 4. The identity $\mu_{b,p}(I) = \lambda_{b,p}(I) =: \lambda$ does not imply the existence of a positive function ϕ satisfying $P\phi = \lambda\phi$ on $I \times \Omega$, $\phi = 0$ on $I \times \partial\Omega$, and $\inf_{t\in I} \phi(t,x) > 0$ for $x \in \Omega$, $\sup \phi < +\infty$. Indeed, for instance, in the proof of [31, Proposition 3.5], the third author of the present paper exhibits the existence of an odd, almost periodic function $\sigma : \mathbb{R} \to \mathbb{R}$ such that $\frac{1}{t}\int_0^t \sigma(s)ds \to 0$ as $t \to +\infty$, but $\int_0^t \sigma(s)ds \to +\infty$ as $t \to +\infty$. Then, if a function ϕ with the above properties existed for the operator $P = \partial_t - \Delta - \sigma(t)$ with $I = \mathbb{R}$ or $I = \mathbb{R}^+$, then Theorem 2.2 would imply that $\phi(t,x)e^{-\lambda t} = u_P(t,x)(q + w(t,x))$, for some (necessarily positive) constant q and a function w converging exponentially to 0 as $t \to +\infty$. On the other hand, as seen in the proof of Proposition 4.2, in this case the function u_P is given by (14), whence

$$\phi(t,x) \sim q\phi_D(x)e^{(\lambda-\lambda_D(-\mathcal{L},\Omega))t+\int_0^t \sigma(s)ds} \quad \text{as} \quad t \to +\infty,$$

where $\lambda_D(-\mathcal{L},\Omega)$, ϕ_D are the Dirichlet principal eigenvalue and eigenfunction of $-\mathcal{L}$ over Ω. But then the properties of $\int_0^t \sigma$ prevent ϕ from simultaneously fulfilling the conditions $\inf_{t\in\mathbb{R}^+} \phi(t,x) > 0$ and $\sup \phi < +\infty$.

Random stationary ergodic coefficients

Consider the operator

$$P = \partial_t - a_{ij}(t,x,\omega)\partial_{ij} - b_i(t,x,\omega)\partial_i - c(t,x,\omega).$$

We assume that the coefficients $(a_{ij})_{i,j\in\{1,...,N\}}$, $(b_i)_{i\in\{1,...,N\}}$ and c are random variables, defined for $(t,x,\omega)\in\mathbb{R}\times\Omega\times\mathcal{O}$, where $(\mathcal{O},\mathbb{P},\mathcal{F})$ is a probability space. We suppose that the hypotheses (H) are satisfied almost surely, i.e. for almost every $\omega\in\Omega$ (with respect to the probability measure $\mathbb{P}$).

The functions $(a_{ij})_{i,j\in\{1,...,N\}}$, $(b_i)_{i\in\{1,...,N\}}$ and c are assumed to be random stationary ergodic with respect to $t\in\mathbb{R}$. The stationarity hypothesis means that there exists a group $(\pi_t)_{t\in\mathbb{R}}$ of measure-preserving transformations such that $a_{ij}(t+s,x,\omega)=a_{ij}(t,x,\pi_s\omega)$, $b_i(t+s,x,\omega)=b_i(t,x,\pi_s\omega)$ and $c(t+s,x,\omega)=c(t,x,\pi_s\omega)$ for all $(t,s,x,\omega)\in\mathbb{R}\times\mathbb{R}\times\Omega\times\mathcal{O}$. This hypothesis heuristically means that the statistical properties of the medium do not depend on time at which one observes it. The ergodicity hypothesis means that if $\pi_t A = A$ for all $t\in\mathbb{R}$ and for a given $A\in\mathcal{F}$, then $\mathbb{P}(A)=0$ or 1.

For a given event $\omega\in\mathcal{O}$, we let P^ω denote the operator with coefficients $a_{ij}(\cdot,\omega)$, $b_i(\cdot,\omega)$, $c(\cdot,\omega)$, and stress the dependence on the operator in the notation of the generalized principal eigenvalues.

Proposition 4.7. *Assume that the coefficients of P are random stationary ergodic with respect to $t\in\mathbb{R}$. Then, for almost every event $\omega\in\mathcal{O}$, there holds*

$$\mu_p(P^\omega,\mathbb{R}) = \lambda_p(P^\omega,\mathbb{R}) = \mu_b(P^\omega,\mathbb{R}) = \lambda_p(P^\omega,\mathbb{R}).$$

This case has been addressed in [19] in the framework of Floquet bundles, from which the identities stated in Proposition 4.7 can be derived owing to our results of Section 2.2. We include the short proof here for the sake of completeness.

Proof. Let $u_P(\cdot,\omega)$, $\omega\in\mathcal{O}$, be the entire solution associated with the operator P^ω, provided by Theorem 2.1. It easily follows from the uniqueness of u_P that

$$u_P(t+s,x,\omega) = u_P(t,x,\pi_s\omega)\|u_P(s,\cdot,\omega)\|_{L^\infty(\Omega)}.$$

Hence,

$$\ln\|u_P(t+s,\cdot,\omega)\|_{L^\infty(\Omega)} = \ln\|u_P(t,\cdot,\pi_s\omega)\|_{L^\infty(\Omega)} + \ln\|u_P(s,\cdot,\omega)\|_{L^\infty(\Omega)}.$$

It then follows from the Birkhoff ergodic theorem that the limit

$$\lim_{t\to+\infty}\frac{1}{t}\ln\|u_P(t,\cdot,\omega)\|_{L^\infty(\Omega)}$$

exists almost surely and is deterministic. The conclusion follows from Theorems 2.5 and 2.6. □

Note that there exist situations where $\lambda_{b,p}(P^\omega,\mathbb{R}) > \lambda_p(P^\omega,\mathbb{R}) > \mu_{b,p}(P^\omega,\mathbb{R})$ almost surely. As an example, consider a family of independent, identically distributed random variables $(\tilde{c}_k)_{k\in\mathbb{Z}}$ in $L^\infty(\tilde{\mathcal{O}})$ over a probability space $(\tilde{\mathcal{O}},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$. Let $\mathcal{O} := \tilde{\mathcal{O}}^{\mathbb{Z}}\times\mathbb{T}$, with $\mathbb{T}:=\mathbb{R}/\mathbb{Z}$, and natural Borel space and probability over this space. For all $t\in\mathbb{R}$, consider the transformation $\pi_t:\Omega\to\Omega$ defined by $\pi_t((\omega_k)_k,y) := ((\omega_{k+l})_k, y+r)$ if $t=l+r$, with $l\in\mathbb{Z}$ and $r\in[0,1)$. The ergodicity of $(\pi_t)_t$ is obvious. Consider the operator $P=\partial_t-\Delta-c(t,\omega)$, where

$$c\big(t,((\omega_k)_k,y)\big) := (1-t-y+l)\tilde{c}_l(\omega_l) + (t+y-l)\tilde{c}_{l+1}(\omega_{l+1}) \quad \text{for } t\in[l,l+1),\ l\in\mathbb{Z}.$$

One easily checks that $c(t+s,\omega)=c(t,\pi_s\omega)$ for all $(t,s,\omega)\in\mathbb{R}\times\mathbb{R}\times\mathcal{O}$.

On one hand, we know from Proposition 4.2 and the Birkhoff ergodic theorem that for almost every $\omega\in\mathcal{O}$, one has:

$$\mu_p(P^\omega,\mathbb{R}) = \mu_b(P^\omega,\mathbb{R}) = \lambda_p(P^\omega,\mathbb{R}) = \lambda_b(P^\omega,\mathbb{R}) = \lambda_D(-\Delta,\Omega) - \mathbb{E}[\tilde{c}],$$

where $\mathbb{E}[\tilde{c}]$ is the expectation of $\tilde{c}_k = \tilde{c}$ (which is independent of k by identical distribution). But Proposition 4.2 also yields

$$\mu_{b,p}(P^\omega, \mathbb{R}) = \lambda_D(-\Delta, \Omega) - \lim_{t\to+\infty} \inf_{s\in\mathbb{R}} \frac{\int_s^{s+t} c(s', \omega)ds'}{t},$$

$$\lambda_{b,p}(P^\omega, \mathbb{R}) = \lambda_D(-\Delta, \Omega) - \lim_{t\to+\infty} \sup_{s\in\mathbb{R}} \frac{\int_s^{s+t} c(s', \omega)ds'}{t}.$$

On the other hand, one can show that for all $\varepsilon > 0$ and for almost every $\omega = ((\omega_k)_k, y) \in \mathcal{O}$, one can find arbitrary long consecutive sequences of $k's$ such that $\tilde{c}_k(\omega_k) \leq \operatorname{ess\,inf} \tilde{c} + \varepsilon$ and arbitrary long consecutive sequences of $k's$ such that $\tilde{c}_k(\omega_k) \geq \operatorname{ess\,sup} \tilde{c} - \varepsilon$. One infers that $\mu_{b,p}(P^\omega, \mathbb{R}) = \lambda_D(-\Delta, \Omega) - \operatorname{ess\,inf} \tilde{c}$ and $\lambda_{b,p}(P^\omega, \mathbb{R}) = \lambda_D(-\Delta, \Omega) - \operatorname{ess\,sup} \tilde{c}$.

5 The generalized principal eigenvalues of limit operators

5.1 Main result for limit operators

We assume in this section that the second-order coefficients a_{ij} of P are uniformly continuous over $I \times \overline{\Omega}$. We introduce the notion of *limit operator* associated with the operator P. This is a parabolic operator

$$P^* = \partial_t - a^*_{ij}(t,x)\partial_{ij} - b^*_i(t,x)\partial_i - c^*(t,x),$$

whose coefficients b^*_i, c^* are the weak-$\star$ limits in $L^\infty(I \times \Omega)$ as $n \to +\infty$ of $b_i(\cdot + t_n, \cdot)$, $c(\cdot + t_n, \cdot)$ respectively, and a^*_{ij} is the strong limit in $\mathcal{C}^0_{loc}(I \times \overline{\Omega})$ as $n \to +\infty$ of $a_{ij}(\cdot + t_n, \cdot)$, for some sequence $(t_n)_{n\in\mathbb{N}}$ in I (which does not necessarily diverge). We let $\omega_I(P)$ denote the family of all limit operators associated with P. Notice that, with respect to the usual definition of the omega-limit set of an operator, we also consider bounded sequence of translations $(t_n)_{n\in\mathbb{N}}$, thus $\omega_I(P)$ contains in particular the operator P itself.

In the next result, we stress the operator in the notation of the associated generalized principal eigenvalues.

Theorem 5.1. *Let $I = \mathbb{R}^-$ or $\mathbb{R}^+$. Assume that the coefficients of P satisfy (H), and that the a_{ij} are uniformly continuous over $I \times \Omega$ for all $i, j \in \{1, \ldots, N\}$. Then*

$$\mu_{b,p}(P, I) = \max_{P^*\in\omega_I(P)} \mu_p(P^*, I) = \max_{P^*\in\omega_I(P)} \lambda_b(P^*, I),$$

$$\lambda_{b,p}(P, I) = \min_{P^*\in\omega_I(P)} \mu_p(P^*, I) = \min_{P^*\in\omega_I(P)} \lambda_b(P^*, I).$$

Moreover, if P^* is a limit operator which realizes one of the above maxima/minima, then it realizes the other maximum/minimum of the same chain of equivalences too.

The proof of Theorem 5.1 will be given in Section 8.4. Let us emphasize that the convergence of the coefficients in the definition of the limit operators is not uniform as in Corollary 3.1 above. In fact, the continuity of the generalized principal eigenvalues granted by Corollary 3.1 fails in general under merely local convergence of the coefficients, as one can readily observe from Proposition 4.2. This is why limit operators do not share in general the same principal eigenvalues with the original operator. The proof of Theorem 5.1 relies on the properties of the global growth-rate, rather than on the continuity with respect to the coefficients. We also point out that the minimizers/maximizers P^* appearing in the above expressions are not necessarily unique. Indeed, consider for example the case where $P = \partial_t - \Delta - c(t)$ with c almost periodic in t. Then

$$\mu_p(P^*, \mathbb{R}) = \lambda_D(-\Delta, \Omega) - \liminf_{t\to+\infty} \frac{\int_0^t c^*(s)ds}{t}$$

according to Proposition 4.2 (using the same notations), where $c^*(t) = \lim_{n\to+\infty} c(t+t_n)$ for some sequence $(t_n)_{n\in\mathbb{N}}$ defining the limit operator P^*. But the almost periodicity of c yields that this limit does not depend on the limit operator P^*. This shows that all the limit operators maximize $\mu_p(P^*, \mathbb{R})$.

5.2 Relation with the exponential type

Let us conclude by some comparison with the notion of exponential type. Rodríguez-Bernal and his coauthors studied exponentially stable operators and their links with semilinear parabolic equations under similar KPP-type hypotheses on f. We will not describe their results in details here but just describe some consequences in the case we address in the present paper.

In [30], the notion of *exponential type* is discussed. Such a notion actually coincides with the principal eigenvalue $\lambda_{b,p}$ (up to a sign) introduced in the present paper, as we now show. Note that in [30], Neumann and Robin boundary conditions are also addressed. Let us recall the definition of exponential type. Let P be a linear parabolic operator whose coefficients satisfy our standing assumptions (H). For all $s \in \mathbb{R}$, we let $u^s = u^s(t, x; u_0)$ denote the solution of $Pu^s = 0$ in $(s, +\infty) \times \Omega$ with initial datum $u^s(s, x; u_0) = u_0(x)$ for $x \in \Omega$. The exponential type β_P (resp. exponential type at $-\infty$ β_P^- or at $+\infty$ β_P^+) is the smallest β such that, for any compactly supported and continuous initial datum u_0, one has

$$\limsup_{t\to+\infty} \left(\inf_{s\in\mathbb{R}} \frac{\ln \|u^s(t+s, \cdot; u_0)\|_{L^1(\Omega)} - \ln \|u_0\|_{L^1(\Omega)}}{t} \right) \leq \beta \tag{15}$$

(resp. with $s \in \mathbb{R}^-$ or $s \in \mathbb{R}^+$ in the infimum). One has

$$\beta_P = -\lambda_{b,p}(\mathbb{R}), \quad \beta_P^- = -\lambda_{b,p}(\mathbb{R}^-), \quad \beta_P^+ = -\lambda_{b,p}(\mathbb{R}^+). \tag{16}$$

Indeed, applying Theorem 2.2 with $u_P(t, \cdot)$ replaced by $u_P(t+s, \cdot)/\|u_P(s, \cdot)\|_{L^\infty(\Omega)}$, one gets a constant $q_s \in \mathbb{R}$ such that

$$\forall t > 0, \quad \|u^s(t+s, \cdot; u_0) - q_s u_P(t+s, \cdot)\|_{L^\infty(\Omega)} \leq C\|u_0 - q_s u_P(s, \cdot)\|_{L^\infty(\Omega)} \frac{\|u_P(t+s, \cdot)\|_{L^\infty(\Omega)}}{\|u_P(s, \cdot)\|_{L^\infty(\Omega)}} e^{-\gamma t}$$

where the constants C, γ are independent of s, see [13, Theorem 2.6(iii)]. Moreover, q_s is such that $u^s(t+s, \cdot; u_0) - q_s u_P(t+s, \cdot)$ changes sign in Ω. If u_0 is positive in Ω, then $q_s\|u_P(s, \cdot)\|_{L^\infty(\Omega)}$ can neither be too small nor too large, because otherwise $u^s(t+s, \cdot; u_0) - q_s u_P(t+s, \cdot)$ would not change sign on Ω (as a consequence of the Harnack inequality, cf. Theorem 7.1).

It follows that the left-hand side in (15) is maximized when $u_0 = u_P(s, \cdot)$ and $u^s \equiv u_P$. According to Theorems 2.2 and 2.3, we thus get that $\beta_P = -\lambda_{b,p}(\mathbb{R})$. The proofs for $\beta_P^\pm$ are similar.

One could thus derive estimates on the generalized principal eigenvalue $\lambda_{b,p}$ for a particular class of operators by using the following result of [30].

Proposition 5.2 ([30, Lemma 2.7]). *Consider the operator $P = \partial_t - \Delta - c(t, x)$, with $c \in L^\infty(\mathbb{R} \times \Omega)$. For given $t \in \mathbb{R}$, let $\lambda_D(-\mathcal{L}_t)$ be the Dirichlet principal eigenvalue of the elliptic operator*

$$-\mathcal{L}_t = -\Delta - c(t, x) \quad \text{in } \Omega.$$

Then one has

$$-\beta_P \geq \lim_{t\to+\infty} \left(\inf_{s\in\mathbb{R}} \frac{1}{t} \int_s^{s+t} \lambda_D(-\mathcal{L}_{t'}) dt' \right),$$

and

$$-\beta_P^\pm \geq \lim_{t\to+\infty} \inf_{s\in\mathbb{R}^\pm} \frac{1}{t} \int_s^{s+t} \lambda_D(-\mathcal{L}_{t'}) dt'.$$

In [28], Robinson, Rodríguez-Bernal and Vidal-López studied the following problem with logistic nonlinearity:

$$\begin{cases} \partial_t u - \Delta u = c(t,x)u - n(t,x)u^3, & t \in \mathbb{R},\ x \in \Omega, \\ u(t,x) = 0, & t \in \mathbb{R},\ x \in \partial\Omega, \end{cases} \tag{17}$$

with $c, n \in L^\infty(\mathbb{R} \times \Omega)$ and Hölder continuous in t, and $n \geq 0$. Calling $P := \partial_t - \Delta u - c(t,x)$ the linearized operator, [28, Theorem 8.1] implies that if $\beta_P^- < 0$ then (17) does not admit any positive bounded ancient solution. This result is indeed obtained by considering a nonnegative ancient solution u and applying [28, Theorem 8.1] to $u^{s_n}(t + s_n, \cdot; u(s_n, \cdot))$ on a suitable sequence of times $s_n \to -\infty$ (for which what is called "exponential stability" in [28] holds). We recover this result, and actually extend it in several directions, using ours. Namely, for the more general problem (3) under the hypotheses (4)-(6) on f (which include (17)), a sufficient condition for the non-existence of positive bounded ancient solutions is $\mu_b(\mathbb{R}^-) > 0$. This is an immediate consequence of Theorem 2.11, since solutions to (4) are subsolutions for the linearized operator P thanks to the KPP condition (6). This improves the result of [28] because, by Proposition 2.7, $\mu_b(\mathbb{R}^-) \geq \lambda_{b,p}(\mathbb{R}^-) = -\beta_P^-$.

6 Technical tools

6.1 Hölder-continuity of the Floquet bundles

Let u_P be the time-global, positive solution provided by Theorem 2.1. Recall that if P is just defined on $\mathbb{R}^+ \times \Omega$ or on $\mathbb{R}^- \times \Omega$, then it is extended by even reflection with respect to t. The following function incorporates some crucial information about the dynamical properties of the equation:

$$\beta(t) := \ln(\|u_P(t,\cdot)\|_{L^\infty(\Omega)}). \tag{18}$$

We will apply to the function β some notions of average growth-rate, that require the function to have at most linear growth at infinity, cf. condition (22) below. The latter property is ensured by the uniform continuity, which is granted by the following result.

Lemma 6.1. *The function β is locally Hölder-continuous in $\mathbb{R}$ with some exponent $\alpha > 0$, and it satisfies*

$$\sup_{s\in\mathbb{R},\ t\in(0,1)} \frac{|\beta(s+t) - \beta(s)|}{t^\alpha} < +\infty.$$

Proof. We start with applying parabolic estimates to the function u_P, which, we recall, is defined for all $t \in \mathbb{R}$. Namely, there is a constant $C > 0$ such that

$$\forall s \in \mathbb{R}, \quad \|u_P\|_{W_p^{1,2}((s-1,s+1)\times\Omega)} \leq C\|u_P\|_{L^p((s-2,s+1)\times\Omega)},$$

see e.g. [16, Theorem 7.30]. Then, since $p > N + 1$, we get from Morrey's inequality

$$\forall s \in \mathbb{R}, \quad \|u_P\|_{C^{0,\alpha}((s-1,s+1)\times\Omega)} \leq C\|u_P\|_{L^p((s-2,s+1)\times\Omega)},$$

for some $\alpha > 0$ and another constant independent of s, that we still call C. It follows that

$$\forall s \in \mathbb{R}, \quad \|u_P\|_{C^{0,\alpha}((s-1,s+1)\times\Omega)} \leq C\|u_P\|_{L^\infty((s-2,s+1)\times\Omega)}, \tag{19}$$

for some other $C > 0$, depending also on $|\Omega|$ and p. We make now use of the following two-sided estimate quoted from [13, Corollary 3.10]: there exists $C' > 1$ such that

$$\forall s \in \mathbb{R},\ t \in [0,1], \quad \frac{1}{C'}\|u_P(s,\cdot)\|_{L^\infty(\Omega)} \leq \|u_P(s+t,\cdot)\|_{L^\infty(\Omega)} \leq C'\|u_P(s,\cdot)\|_{L^\infty(\Omega)}. \tag{20}$$

We point out that the second inequality above is a straightforward consequence of the comparison principle (even with $C' \to 1$ as $t \to 0^+$), whereas the first one makes use of a boundary Harnack-type inequality. Gathering together (19), (20) one finds a constant $\hat{C} > 0$ such that

$$\forall s \in \mathbb{R}, \quad \|u_P\|_{\mathcal{C}^{0,\alpha}((s-1,s+1)\times\Omega)} \leq \hat{C}\|u_P(s,\cdot)\|_{L^\infty(\Omega)},$$

whence,

$$\forall s \in \mathbb{R},\ t \in [-1,1], \quad \big|\|u_P(s+t,\cdot)\|_{L^\infty(\Omega)} - \|u_P(s,\cdot)\|_{L^\infty(\Omega)}\big| \leq \hat{C}\|u_P(s,\cdot)\|_{L^\infty(\Omega)}|t|^\alpha.$$

Then, since by the definition of β it holds that

$$\beta(s+t) - \beta(s) = \ln\Big(1 + \frac{\|u_P(s+t,\cdot)\|_{L^\infty(\Omega)} - \|u_P(s,\cdot)\|_{L^\infty(\Omega)}}{\|u_P(s,\cdot)\|_{L^\infty(\Omega)}}\Big),$$

one eventually derives, for $s \in \mathbb{R}$ and $|t|^\alpha < 1/\hat{C}$,

$$\ln(1 - \hat{C}|t|^\alpha) \leq |\beta(s+t) - \beta(s)| \leq \ln(1 + \hat{C}|t|^\alpha).$$

This immediately yields the desired estimate. □

6.2 Global growth-rate

The aim of this section is to derive an equivalent of [23, Lemma 3.2] concerning the notion of the *least mean*, introduced by the last two authors of the present paper. Let us first remind to the reader what that result is. If $g \in L^\infty(\mathbb{R})$, then the least mean of g is defined by

$$\lfloor g \rfloor := \lim_{t\to+\infty}\left(\inf_{s\in\mathbb{R}} \frac{1}{t}\int_s^{s+t} g(\tau)d\tau\right).$$

In [23], after showing that the above limit always exists, the following characterization is derived:

$$\lfloor g \rfloor = \sup_{B\in W^{1,\infty}(\mathbb{R})}\left(\operatorname*{ess\,inf}_{\mathbb{R}}(g + B')\right). \tag{21}$$

This plays a crucial role in the proofs of [23]. In the current paper, we would like to apply the notion of the least mean to the derivative of the function $\beta(t) := \ln(\|u_P(t,\cdot)\|_{L^\infty(\Omega)})$, which is not possible because it is not in L^∞ in general. For this reason, we will need to rewrite the notion of the least mean of a function g in terms of its primitive. This leads us to introduce the notion of the *least global growth-rate*, together with the *greatest global growth-rate*, that we will apply to the function β.

Definition 6.2. Let I be an unbounded open interval and let $G : I \to \mathbb{R}$ be a measurable function such that

$$\exists C > 0, \quad \sup_{t_1,t_2\in I}|G(t_1) - G(t_2)| \leq C(1 + |t_1 - t_2|). \tag{22}$$

Then the following quantities:

$$\llbracket G \rrbracket_I := \limsup_{t\to+\infty}\left(\inf_{s,s+t\in I}\frac{G(s+t)-G(s)}{t}\right), \qquad \lceil\!\lceil G \rceil\!\rceil_I := \liminf_{t\to+\infty}\left(\sup_{s,s+t\in I}\frac{G(s+t)-G(s)}{t}\right)$$

(which exist and are finite) are called the *least global growth-rate* and the *greatest global growth-rate* of G over I respectively.

Notice that the least growth-rate of a function G coincides with the least mean of its derivative G', which is well defined if G is Lipschitz-continuous. Then, in such a case, one could apply the characterization (21) of [23] to G' (see also [33, Lemma 2.2] for some generalizations to functions which are not necessarily bounded) and get an analogous characterization for the least global growth-rate of G. However, in the present paper, the function $G = \beta$ is not Lipschitz-continuous but only Hölder-continuous, and thus we cannot apply (21). This is why we need to reformulate and extend the characterization of the least global growth-rate of a function without passing through its derivative. We will show by the way that the "lim sup" and "lim inf" in Definition 6.2 are actually limits.

Proposition 6.3. *Let $G : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying* (22)*. There holds that*

$$\llbracket G \rrbracket_I = \sup \left\{ \operatorname*{ess\,inf}_I A' \ :\ A - G \in L^\infty(I),\ A' \in L^\infty(I) \right\}$$

and

$$\lceil\!\lceil G \rceil\!\rceil_I = \inf \left\{ \operatorname*{ess\,sup}_I A' \ :\ A - G \in L^\infty(I),\ A' \in L^\infty(I) \right\}.$$

Moreover the "lim sup" *and* "lim inf" *in the definitions of* $\llbracket G \rrbracket_I$ *and* $\lceil\!\lceil G \rceil\!\rceil_I$ *are actually* "lim"*.*

Finally, one has

$$\llbracket G \rrbracket_{\mathbb{R}} = \min \left\{ \llbracket G \rrbracket_{\mathbb{R}^-}, \llbracket G \rrbracket_{\mathbb{R}^+} \right\}, \qquad \lceil\!\lceil G \rceil\!\rceil_{\mathbb{R}} = \max \left\{ \lceil\!\lceil G \rceil\!\rceil_{\mathbb{R}^-}, \lceil\!\lceil G \rceil\!\rceil_{\mathbb{R}^+} \right\}.$$

Proof. We prove the statements about the least global growth-rate. Applying them to the function $-G$ one gets the results for the greatest global growth-rate.

We first assume that $I \neq \mathbb{R}$. Since all the quantities involved in the statement are invariant by reflection of the function G, we can assume that $I = \mathbb{R}^+$.

Take $m < \llbracket G \rrbracket_{\mathbb{R}^+}$. By the definition of the least global growth-rate, there exists $T > 0$ such that

$$m < \inf_{s \geq 0} \frac{1}{T}(G(s+T) - G(s)).$$

We now consider a linear interpolation of the function G. Namely, for $t \geq 0$, we define

$$\forall k \in \mathbb{N},\ t \in [(k-1)T, kT), \qquad \alpha(t) := \frac{G(kT) - G((k-1)T)}{T},$$

then we set

$$A(t) := G(0) + \int_0^t \alpha(s)ds.$$

The function A coincides with G over the set $T\mathbb{N}$. Moreover, outside such a set, it holds $A' = \alpha$, hence by (22) there exists $C > 0$ such that $|A'| = |\alpha| \leq C(1 + 1/T)$. As a consequence, using again (22), one finds, for any $k \in \mathbb{N}$ and $t \in [(k-1)T, kT)$,

$$|A(t) - G(t)| \leq |A(t) - A((k-1)T)| + |G(t) - G((k-1)T)| \leq 2C(T+1),$$

which shows that $A - G$ is bounded over $\mathbb{R}^+$. We further know that $A' = \alpha > m$ outside $T\mathbb{N}$. We have thus proved that the inequality

$$\sup \left\{ \operatorname*{ess\,inf}_{\mathbb{R}^+} A' \ :\ A - G \in L^\infty(\mathbb{R}^+),\ A' \in L^\infty(\mathbb{R}^+) \right\} \geq m$$

holds for any $m < \llbracket G \rrbracket_{\mathbb{R}^+}$, hence it holds true for $m = \llbracket G \rrbracket_{\mathbb{R}^+}$.

In order to show the reverse inequality, consider an arbitrary unbounded interval I and let A be such that $A-G \in L^\infty(I)$ and $A' \in L^\infty(I)$. Then, for all $s \in I$ and $t>0$ such that $s+t \in I$, one has

$$\begin{aligned} G(s+t)-G(s) &= G(s+t)-A(s+t)-G(s)+A(s)+\int_s^{s+t} A' \\ &\geq -2\|A-G\|_{L^\infty(I)} + t \operatorname*{ess\,inf}_I A', \end{aligned}$$

whence

$$[\![G]\!]_I \geq \liminf_{t\to+\infty} \left(\inf_{s,s+t\in I} \frac{G(s+t)-G(s)}{t} \right) \geq \operatorname*{ess\,inf}_I A'. \tag{23}$$

Applying this lower bound with $I=\mathbb{R}^+$ one deduces at once the characterization for $[\![G]\!]_{\mathbb{R}^+}$ and also that the "$\limsup$"in the definitions of $[\![G]\!]_I$ is actually a "$\lim$".

Let us now show the equivalence

$$[\![G]\!]_{\mathbb{R}} = \min\left\{ [\![G]\!]_{\mathbb{R}^-}, [\![G]\!]_{\mathbb{R}^+} \right\}. \tag{24}$$

The inequality "$\leq$" in (24) is a direct consequence of the definition of the involved quantities. For the reverse one, consider any pair of functions A_+, A_- satisfying $A_\pm - G \in L^\infty(\mathbb{R}^\pm)$ and $A'_\pm \in L^\infty(\mathbb{R}^\pm)$. We then define

$$A(t) := \begin{cases} A_-(t) & \text{if } t \leq 0 \\ A_+(t)-A_+(0)+A_-(0) & \text{if } t>0. \end{cases} \tag{25}$$

This is a Lipschitz continuous function satisfying $A-G \in L^\infty(\mathbb{R})$, hence using (23) with $I=\mathbb{R}$ gives

$$[\![G]\!]_{\mathbb{R}} \geq \operatorname*{ess\,inf}_{\mathbb{R}} A' = \min\left\{ \operatorname*{ess\,inf}_{\mathbb{R}^-} A'_-, \operatorname*{ess\,inf}_{\mathbb{R}^+} A'_+ \right\}.$$

Taking the supremum with respect to the functions A_-, A_+ yields $[\![G]\!]_{\mathbb{R}} \geq \min\{[\![G]\!]_{\mathbb{R}^-}, [\![G]\!]_{\mathbb{R}^-}\}$. This proves (24).

It remains to prove the characterization for $[\![G]\!]_{\mathbb{R}}$. We already have one inequality, cf. (23). Let us show the opposite one. By the characterization for $[\![G]\!]_{\mathbb{R}^\pm}$, for any $\varepsilon>0$ there exist two functions $A_\pm$ satisfying $A_\pm - G \in L^\infty(\mathbb{R}^\pm)$ and $A'_\pm \in L^\infty(\mathbb{R}^\pm)$, such that:

$$[\![G]\!]_{\mathbb{R}^\pm} \leq \varepsilon + \operatorname*{ess\,inf}_{\mathbb{R}^\pm} A'_\pm.$$

Defining A as in (25), one then gets from (24)

$$[\![G]\!]_{\mathbb{R}} = \min\{[\![G]\!]_{\mathbb{R}^-}, [\![G]\!]_{\mathbb{R}^-}\} \leq \varepsilon + \operatorname*{ess\,inf}_{\mathbb{R}} A',$$

whence

$$[\![G]\!]_{\mathbb{R}} \leq \sup\left\{ \operatorname*{ess\,inf}_{\mathbb{R}} A' \ : \ A-G \in L^\infty(\mathbb{R}),\ A' \in L^\infty(\mathbb{R}) \right\} + \varepsilon.$$

Since $\varepsilon>0$ was arbitrary, we have obtained the desired upper bound for $[\![G]\!]_{\mathbb{R}}$, which concludes the proof. □

7 Connections with Floquet bundles

All of our results concerning the relation between generalized principal eigenvalues and Floquet bundles can be reformulated in terms of suitable averages of the growth rate of the function β defined by (18). In particular, Theorem 2.3 involves the notion of least and greatest global growth-rates introduced in Section 6.2. Another essential ingredient is the following Harnack-type inequality for quotients of positive solutions, quoted from [13].

Theorem 7.1 ([13, Theorem 2.1])**.** *Let u_1, u_2 be two positive solutions of* (11) *for $t > 0$. Then, for any $s_0 > 0$, it holds*

$$\forall s \geq s_0, \quad \sup_{x\in\Omega} \frac{u_2(s,x)}{u_1(s,x)} \leq C \inf_{x\in\Omega} \frac{u_2(s,x)}{u_1(s,x)},$$

where $C > 0$ only depends on s_0, N, Ω, the ellipticity constant α of (a_{ij}) and the L^∞ bounds of the coefficients of the operator.

We point out that, unlike the standard parabolic Harnack inequality, the sup and inf in the above estimate are taken *at the same time s*.

We are now in a position to prove our main results.

Proof of Theorem 2.3. Lemma 6.1 implies that the function β is uniformly continuous, hence it fulfills (22). We can then consider its least and greatest global growth-rates given by Definition 6.2. The desired equivalences then rewrites as

$$\mu_{b,p}(I) = -\llbracket\beta\rrbracket_I\,, \qquad \lambda_{b,p}(I) = -\lceil\!\lceil\beta\rceil\!\rceil_I\,.$$

The inequalities $\mu_{b,p}(I) \leq -\llbracket\beta\rrbracket_I$ and $\lambda_{b,p}(I) \geq -\lceil\!\lceil\beta\rceil\!\rceil_I$.
Consider an arbitrary $\lambda > -\llbracket\beta\rrbracket_I$. The characterization of $\llbracket\beta\rrbracket_I$ given by Proposition 6.3 provides us with a function A satisfying $A' \in L^\infty(I)$ and $A - \beta \in L^\infty(I)$, such that $A' > -\lambda$ a.e. in I. Define the function

$$\phi(t,x) := u_P(t,x)e^{-A(t)}.$$

This function satisfies

$$P\phi = -A'\phi < \lambda\phi \quad \text{in } I\times\Omega.$$

Observe that

$$\|\phi(t,\cdot)\|_{L^\infty(\Omega)} = e^{\beta(t)-A(t)},$$

which is bounded for $t \in I$. We claim that ϕ is also bounded from below away from 0 on any fixed open ball $B \Subset \Omega$. The boundary Harnack inequality[2] yields the existence of a constant $\hat{C} > 0$ such that

$$\forall t \in I, \quad \|u_P(t-1,\cdot)\|_{L^\infty(\Omega)} \leq \hat{C} \inf_{x\in B} u_P(t,x),$$

and we know from (20) that the left hand-side is bounded from below by $\frac{1}{C'}\|u_P(t,\cdot)\|_{L^\infty(\Omega)}$, for some other constant $C' > 0$. Hence, there exists a positive constant C such that

$$\forall t \in I, \quad \inf_{x\in B} \phi(t,x) = \frac{\inf_{x\in B} u_P(t,x)}{\|u_P(t,\cdot)\|_{L^\infty(\Omega)}} e^{\beta(t)-A(t)} \geq Ce^{\beta(t)-A(t)}.$$

We have shown that the bounded function ϕ is also bounded from below away from 0 on B, hence it can be used in the definition of $\mu_{b,p}(I)$. We deduce that $\mu_{b,p}(I) \leq \lambda$, and this being true for any $\lambda > -\llbracket\beta\rrbracket_I$ shows $\mu_{b,p}(I) \leq -\llbracket\beta\rrbracket_I$.

Proceeding exactly in the same way, but starting from $\lambda < -\lceil\!\lceil\beta\rceil\!\rceil_I$ and then applying Proposition 6.3 to get a function A satisfying a.e. $A' < -\lambda$, one derives the inequality $\lambda_{b,p}(I) \geq -\lceil\!\lceil\beta\rceil\!\rceil_I$.

The inequalities $\mu_{b,p}(I) \geq -\llbracket\beta\rrbracket_I$ and $\lambda_{b,p}(I) \leq -\lceil\!\lceil\beta\rceil\!\rceil_I$.
Take $\lambda > \mu_{b,p}(I)$. Then there exists ϕ such that

$$\phi > 0 \quad \text{and} \quad P\phi \leq \lambda\phi \quad \text{in } I\times\Omega, \qquad \phi = 0 \text{ on } I\times\partial\Omega,$$

and moreover

$$\exists B \Subset \Omega, \quad \inf_{t\in I,\ x\in B} \phi(t,x) > 0, \qquad \sup_{t\in I,\ x\in\Omega} \phi(t,x) < +\infty.$$

[2] Deduced from the localized one of [8] (see also [13, Theorem 3.5]) and extended to the whole Ω using a covering argument and the standard interior Harnack inequality.

Note that ϕ satisfies

$$\liminf_{t\to+\infty}\left(\inf_{s,s+t\in I}\frac{\ln\|\phi(t+s,\cdot)\|_{L^\infty(\Omega)}-\ln\|\phi(s,\cdot)\|_{L^\infty(\Omega)}}{t}\right)\geq 0.$$

This weaker condition on the test-functions is indeed sufficient in order to prove our result.

Fix $s\in I$. Consider the solution u of the problem $Pu=0$ on $(s,+\infty)\times\Omega$ under Dirichlet boundary condition and with initial datum $u(s,\cdot)=\phi(s,\cdot)$ (as usual, in the case $I=\mathbb{R}^-$, P is extended to all times by even reflection). Since $\phi(t,x)e^{-\lambda(t-s)}$ and $\|\phi(s,\cdot)\|_{L^\infty(\Omega)}e^{\|c\|_\infty(t-s)}$ are respectively a subsolution and a supersolution to such a problem, it follows from the comparison principle that

$$\forall t\geq 0,\quad \|\phi(s+t,\cdot)\|_{L^\infty(\Omega)}e^{-\lambda t}\leq\|u(s+t,\cdot)\|_{L^\infty(\Omega)}\leq\|\phi(s,\cdot)\|_{L^\infty(\Omega)}e^{\|c\|_\infty t}. \tag{26}$$

Next, applying the Harnack-type inequality provided by Theorem 7.1, we find a constant $C>0$, independent of s, such that

$$m_s:=\sup_{x\in\Omega}\frac{u(s+1,x)}{u_P(s+1,x)}\leq C\inf_{x\in\Omega}\frac{u(s+1,x)}{u_P(s+1,x)}\leq\frac{C\|\phi(s,\cdot)\|_{L^\infty(\Omega)}e^{\|c\|_\infty}}{\|u_P(s+1,\cdot)\|_{L^\infty(\Omega)}}.$$

Thus, always by comparison, $m_s u_P(s+t,x)\geq u(s+t,x)$ for $t\geq 1$, $x\in\Omega$, whence using (26)

$$\forall t\geq 1,\quad \|u_P(s+t,\cdot)\|_{L^\infty(\Omega)}\geq\frac{1}{m_s}\|u(s+t,\cdot)\|_{L^\infty(\Omega)}\geq\frac{\|\phi(s+t,\cdot)\|_{L^\infty(\Omega)}e^{-\lambda t}}{C\|\phi(s,\cdot)\|_{L^\infty(\Omega)}e^{\|c\|_\infty}}\|u_P(s+1,\cdot)\|_{L^\infty(\Omega)}.$$

This inequality rewrites in terms of β as

$$\beta(s+t)-\beta(s+1)=\ln\frac{\|u_P(s+t,\cdot)\|_{L^\infty(\Omega)}}{\|u_P(s+1,\cdot)\|_{L^\infty(\Omega)}}\geq-\lambda t+\ln\|\phi(t+s,\cdot)\|_{L^\infty(\Omega)}-\ln\|\phi(s,\cdot)\|_{L^\infty(\Omega)}-\ln(Ce^{\|c\|_\infty}),$$

where, we recall, C is independent of s. Owing to Lemma 6.1, there exists C' independent of s such that $|\beta(s+1)-\beta(s)|\leq C'$. As a consequence, we have

$$\liminf_{t\to+\infty}\left(\inf_{s,s+t\in I}\frac{\beta(s+t)-\beta(s)}{t}\right)\geq-\lambda+\liminf_{t\to+\infty}\frac{1}{t}\left(\inf_{s,s+t\in I}\ln\frac{\|\phi(t+s,\cdot)\|_{L^\infty(\Omega)}}{\|\phi(s,\cdot)\|_{L^\infty(\Omega)}}\right).$$

Recalling the definition of $\llbracket\beta\rrbracket_I$, and the condition satisfied by ϕ, one deduces $\llbracket\beta\rrbracket_I\geq-\lambda$, which, being true for any $\lambda>\mu_{b,p}(I)$, implies that $\mu_{b,p}(I)\geq-\llbracket\beta\rrbracket_I$.

The proof of $\lambda_{b,p}(I)\leq-\llbracket\beta\rrbracket_I$ is similar. Take $\lambda<\lambda_{b,p}(I)$. Then there exists ϕ such that

$$\phi>0\quad\text{and}\quad P\phi\geq\lambda\phi\quad\text{in } I\times\Omega,$$

and moreover

$$\exists B\Subset\Omega,\quad\inf_{t\in I,\ x\in B}\phi(t,x)>0,\qquad\sup_{t\in I,\ x\in\Omega}\phi(t,x)<+\infty.$$

Again, we notice that ϕ satisfies the weaker condition

$$\limsup_{t\to+\infty}\left(\sup_{s,s+t\in I}\frac{\ln\|\phi(t+s,\cdot)\|_{L^\infty(\Omega)}-\ln\|\phi(s,\cdot)\|_{L^\infty(\Omega)}}{t}\right)\leq 0$$

which is indeed sufficient to derive our result.

The solution to $Pu=0$ on $(s,+\infty)\times\Omega$ under Dirichlet boundary condition and with initial datum $u(s,\cdot)=\phi(s,\cdot)$ satisfies

$$\forall t\geq 0,\ x\in\Omega,\quad u(s+t,x)\leq\phi(s+t,x)e^{-\lambda t}\leq\|\phi(s+t,\cdot)\|_{L^\infty(\Omega)}e^{-\lambda t}. \tag{27}$$

Moreover, it is easily seen (for instance arguing by contradiction) that there exists another constant $K' > 0$ independent of s such that $u(s+1,x) \geq K'\|u(s,\cdot)\|_{L^\infty(\Omega)} = K'\|\phi(s,\cdot)\|_{L^\infty(\Omega)}$ for $x \in \Omega$. Thus, by Theorem 7.1, there exists a constant $C > 0$, independent of s, such that

$$M_s := \sup_{x\in\Omega} \frac{u_P(s+1,x)}{u(s+1,x)} \leq C \inf_{x\in\Omega} \frac{u_P(s+1,x)}{u(s+1,x)} \leq \frac{C}{K'\|\phi(s,\cdot)\|_{L^\infty(\Omega)}} \|u_P(s+1,\cdot)\|_{L^\infty(\Omega)}.$$

By comparison and using (27) we get for all $t \geq 1,\ x \in \Omega$:

$$u_P(s+t,x) \leq M_s u(s+t,x) \leq M_s\|\phi(s+t,\cdot)\|_{L^\infty(\Omega)}e^{-\lambda t} \leq \frac{C\|\phi(s+t,\cdot)\|_{L^\infty(\Omega)}}{K'\|\phi(s,\cdot)\|_{L^\infty(\Omega)}}\|u_P(s+1,\cdot)\|_{L^\infty(\Omega)}e^{-\lambda t},$$

whence

$$\forall t \geq 1, \quad \beta(s+t) - \beta(s+1) \leq -\lambda t - \ln(C/K') + \ln\|\phi(t+s,\cdot)\|_{L^\infty(\Omega)} - \ln\|\phi(s,\cdot)\|_{L^\infty(\Omega)}.$$

This entails that $\lceil\!\lceil \beta \rceil\!\rceil_I \leq -\lambda$, and finally $\lambda_{b,p}(I) \leq -\lceil\!\lceil \beta \rceil\!\rceil_I$. □

Remark 5. As a matter of fact, we have shown in the above proof that the conclusion of Theorem 2.3 holds true if one requires that the test functions ϕ satisfy $\sup_{I\times\Omega}\phi < \infty$ and $\inf_{I\times B}\phi > 0$, one asks the weaker condition

$$\liminf_{t\to+\infty}\left(\inf_{s,s+t\in I} \frac{\ln\|\phi(t+s,\cdot)\|_{L^\infty(\Omega)} - \ln\|\phi(s,\cdot)\|_{L^\infty(\Omega)}}{t}\right) \geq 0$$

$$\left(\text{resp.} \quad \limsup_{t\to+\infty}\left(\sup_{s,s+t\in I} \frac{\ln\|\phi(t+s,\cdot)\|_{L^\infty(\Omega)} - \ln\|\phi(s,\cdot)\|_{L^\infty(\Omega)}}{t}\right) \leq 0\right)$$

in the definition of $\mu_{b,p}(I)$ (resp. $\lambda_{b,p}(I)$). This means that these changes do not alter the definitions of $\mu_{b,p}(I)$ and $\lambda_{b,p}(I)$.

Proof of Corollary 2.4. By Theorem 2.3, and with the notation of its proof, one has $\mu_{b,p}(I) = -\lfloor\!\lfloor \beta \rfloor\!\rfloor_I$ and $\lambda_{b,p}(I) = -\lceil\!\lceil \beta \rceil\!\rceil_I$. The result then follows from Proposition 6.3. □

Proof of Theorem 2.5. The formula for λ_b.
In terms of the function β defined by (18), we need to show that

$$\lambda_b(\mathbb{R}) = \lambda_b(\mathbb{R}^+) = -\limsup_{t\to+\infty}\frac{\beta(t)}{t}. \tag{28}$$

We start with the lower bound for $\lambda_b(\mathbb{R})$. Take two numbers $\lambda < \lambda'$ satisfying

$$\lambda' > -\liminf_{t\to-\infty}\frac{\beta(t)}{t}, \qquad \lambda < -\limsup_{t\to+\infty}\frac{\beta(t)}{t}.$$

Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a smooth, nonincreasing function satisfying

$$\gamma(t) = \lambda' \text{ if } t \leq -1, \qquad \gamma(t) = \lambda \text{ if } t \geq 1.$$

Define

$$\phi_b(t,x) := u_P(t,x)e^{\int_0^t \gamma(\tau)d\tau}.$$

This function satisfies $P\phi_b = \gamma(t)\phi_b \geq \lambda\phi_b$ in $\mathbb{R}\times\Omega$. Moreover, by our choices of λ and λ' we have, one one hand, for $t > 1$,

$$\|\phi_b(t,\cdot)\|_{L^\infty(\Omega)} = e^{\beta(t)+\int_0^1\gamma(s)ds+\lambda(t-1)} \to 0 \quad \text{as } t \to +\infty,$$

and on the other hand, for $t < -1$,

$$\|\phi_b(t,\cdot)\|_{L^\infty(\Omega)} = e^{\beta(t)+\int_0^{-1}\gamma(s)ds+\lambda'(t+1)} \to 0 \quad \text{as } t \to -\infty.$$

Thus, ϕ_b is bounded. This shows that $\lambda_b(\mathbb{R})$ is well-defined, $\lambda_b(\mathbb{R}) \geq \lambda$ and thus, by the arbitrariness of λ, that

$$\lambda_b(\mathbb{R}) \geq -\limsup_{t\to+\infty} \frac{\beta(t)}{t}.$$

Next, let $\lambda \in \mathbb{R}$ be such that there exists a positive, bounded function ϕ satisfying $P\phi \geq \lambda\phi$ in $\mathbb{R}^+ \times \Omega$. Our aim is to use ϕ to control u_P from above. To do that, we need to show that (a large multiple of) ϕ controls u_P at some time. This would be immediate if we had Hopf's lemma and C^1 regularity of u_P, which is not our case due to the lack of regularity of Ω and of the coefficients of P. To circumvent such difficulty, we consider the solution to $Pu = 0$ on $\mathbb{R}^+ \times \Omega$ under Dirichlet boundary condition and with initial datum $\phi(0,\cdot)$. By comparison we get

$$\forall t \geq 0, \quad \|u(t,\cdot)\|_{L^\infty(\Omega)} \leq \|\phi\|_{L^\infty(\mathbb{R}^+\times\Omega)} e^{-\lambda t}. \tag{29}$$

Moreover, by Theorem 7.1, there is a constant $C > 0$ such that

$$M := \sup_{x\in\Omega} \frac{u_P(1,x)}{u(1,x)} \leq C \inf_{x\in\Omega} \frac{u_P(1,x)}{u(1,x)} < +\infty.$$

Then, again by comparison, and using (29), we infer the desired estimate

$$\forall t \geq 1, \quad \|u_P(t,\cdot)\|_{L^\infty(\Omega)} \leq M\|u(t,\cdot)\|_{L^\infty(\Omega)} \leq M\|\phi\|_{L^\infty(\mathbb{R}^+\times\Omega)} e^{-\lambda t},$$

which in turn yields

$$\limsup_{t\to+\infty} \frac{\beta(t)}{t} \leq -\lambda.$$

Taking the supremum of λ for which a function ϕ as above exists one gets

$$\limsup_{t\to+\infty} \frac{\beta(t)}{t} \leq -\lambda_b(\mathbb{R}^+).$$

Summing up, we have shown that

$$\lambda_b(\mathbb{R}^+) \leq -\limsup_{t\to+\infty} \frac{\beta(t)}{t} \leq \lambda_b(\mathbb{R}).$$

Since the inequality $\lambda_b(\mathbb{R}) \leq \lambda_b(\mathbb{R}^+)$ is an immediate consequence of the definition, (28) follows.

The formula for $\mu_p(\mathbb{R})$.
The inequality $\mu_p(\mathbb{R}) \leq -\liminf_{t\to+\infty} \frac{\beta(t)}{t}$ is obtained in a similar way as the inequality $\lambda_b(\mathbb{R}) \geq -\limsup_{t\to+\infty} \frac{\beta(t)}{t}$. Namely, one considers

$$\phi_p(t,x) := u_P(t,x) e^{\int_0^t \gamma(\tau)d\tau},$$

with γ nondecreasing and coinciding with λ' on $(-\infty,-1]$ and with λ on $[1,+\infty)$, where

$$\lambda' < -\limsup_{t\to-\infty} \frac{\beta(t)}{t}, \qquad \lambda > -\liminf_{t\to+\infty} \frac{\beta(t)}{t}.$$

One checks that this choice of λ, λ' implies $\|\phi_p(t,\cdot)\|_{L^\infty(\Omega)} \to +\infty$ as $t \to \pm\infty$, and the same is true for $\inf_B \phi_p(t,\cdot)$, for any given open ball $B \Subset \Omega$, thanks to the boundary Harnack inequality. One can then use ϕ_p in the definition of $\mu_p(\mathbb{R})$ and infer that $\mu_p(\mathbb{R}) \leq \lambda$. Since this is true for any arbitrary $\lambda > -\liminf_{t\to+\infty} \frac{\beta(t)}{t}$, the desired inequality follows.

Let $\lambda \in \mathbb{R}$ be such that there exists a positive function ϕ vanishing on $\mathbb{R}^+ \times \partial\Omega$ and satisfying $P\phi \leq \lambda\phi$ in $\mathbb{R}^+ \times \Omega$ and $\inf_{\mathbb{R}^+\times B} \phi \geq K > 0$ for some open ball $B \Subset \Omega$. Then, as before, considering the solution to $Pu = 0$ on $\mathbb{R}^+ \times \Omega$ under Dirichlet boundary condition and with initial datum $\phi(0, \cdot)$, one finds

$$\forall t \geq 0,\ x \in \Omega, \quad u(t, x) \geq \phi(t, x)e^{-\lambda t}.$$

Moreover, by the Harnack-type inequality Theorem 7.1, there is $C > 0$ such that

$$m := \sup_{x\in\Omega} \frac{u(1, x)}{u_P(1, x)} \leq C \inf_{x\in\Omega} \frac{u(1, x)}{u_P(1, x)} < +\infty.$$

Then, by comparison we derive

$$\forall t \geq 1, \quad \|u_P(t, \cdot)\|_{L^\infty(\Omega)} \geq \frac{1}{m}\|u(t, \cdot)\|_{L^\infty(\Omega)} \geq \frac{1}{m}\|\phi(t, \cdot)\|_{L^\infty(\Omega)}e^{-\lambda t} \geq \frac{K}{m}e^{-\lambda t},$$

whence

$$\liminf_{t\to+\infty} \frac{\beta(t)}{t} \geq -\lambda.$$

Taking the infimum over λ yields

$$\liminf_{t\to+\infty} \frac{\beta(t)}{t} \geq -\mu_p(\mathbb{R}^+).$$

This concludes the proof, because $\mu_p(\mathbb{R}^+) \leq \mu_p(\mathbb{R})$, by definition. □

Proof of Theorem 2.6. Let $\lambda \in \mathbb{R}$ be such that there exists a positive function ϕ satisfying $P\phi \geq \lambda\phi$ in $\mathbb{R}^- \times \Omega$ and $\inf_{\mathbb{R}^-\times B} \phi \geq K > 0$ for some open ball $B \Subset \Omega$. For $s < 0$, let u^s be the solution to $Pu = 0$ on $(-s, 0] \times \Omega$ under Dirichlet boundary condition and with initial datum $u^s(s, \cdot) = \phi(s, \cdot)$. One gets by comparison

$$\forall t \in (s, 0],\ x \in \Omega, \quad u^s(t, x) \leq \phi(t, x)e^{-\lambda(t-s)}. \tag{30}$$

Moreover, using Theorem 7.1 we infer the existence of a positive constant C such that

$$\forall s \leq -1, \qquad M_s := \sup_{x\in\Omega} \frac{u_P(s + 1, x)}{u^s(s + 1, x)} \leq C \inf_{x\in\Omega} \frac{u_P(s + 1, x)}{u^s(s + 1, x)} \leq \frac{\|u_P(s + 1, \cdot)\|_{L^\infty(\Omega)}}{\inf_B u^s(s + 1, \cdot)}.$$

The fact that $\inf_B u^s(s, \cdot) = \inf_B \phi(s, \cdot) \geq K$ implies that $\inf_B u^s(s + 1, \cdot) \geq K' > 0$ for some K' independent of $s \leq -1$, thanks to the interior Harnack inequality, see e.g. [15, Theorem 1.1]. Hence there exists another constant $K'' > 0$ such that

$$\forall s \leq -1,\ x \in \Omega, \quad u_P(s + 1, x) \leq K''\|u_P(s + 1, \cdot)\|_{L^\infty(\Omega)}u^s(s + 1, x),$$

from which we deduce, by comparison,

$$\forall s \leq -1,\ t \in [s + 1, 0],\ x \in \Omega, \quad u_P(t, x) \leq K''\|u_P(s + 1, \cdot)\|_{L^\infty(\Omega)}u^s(t, x).$$

In particular, reclaiming (30) we get

$$u_P(0, x) \leq K''\|u_P(s + 1, \cdot)\|_{L^\infty(\Omega)}\phi(0, x)e^{\lambda s}.$$

It follows that, for all $s \leq -1$ and for given $x \in \Omega$,

$$\beta(s + 1) \geq -\lambda s + \ln \frac{u_P(0, x)}{K''\phi(0, x)},$$

whence

$$-\limsup_{t\to-\infty}\frac{\beta(t)}{t}\geq\lambda.$$

Taking the supremum in λ we derive the above inequality with λ replaced by $\lambda_p(\mathbb{R}^-)$.

Similarly, one derives the inequality

$$-\liminf_{t\to-\infty}\frac{\beta(t)}{t}\leq\mu_b(\mathbb{R}^-).$$

Namely, starting from $\lambda\in\mathbb{R}$ such that there exists a bounded, positive function ϕ satisfying $P\phi\leq\lambda\phi$ in $\mathbb{R}^-\times\Omega$ and vanishing on $\mathbb{R}^-\times\partial\Omega$, one considers, for $s<0$, the solution to the Dirichlet problem $Pu^s=0$ on $(-s,0]\times\Omega$ with initial datum $u^s(s,\cdot)=\phi(s,\cdot)$. One gets by comparison

$$\forall t\in(s,0],\ x\in\Omega,\quad \phi(t,x)e^{-\lambda(t-s)}\leq u^s(t,x)\leq\|\phi\|_\infty e^{\|c\|_\infty(t-s)}.$$

Moreover, by Theorem 7.1, there exists $C>0$ such that

$$\forall s\leq-1,\qquad m_s:=\sup_{x\in\Omega}\frac{u^s(s+1,x)}{u_P(s+1,x)}\leq C\inf_{x\in\Omega}\frac{u^s(s+1,x)}{u_P(s+1,x)}\leq\frac{C\|\phi\|_\infty e^{\|c\|_\infty}}{\|u_P(s+1,\cdot)\|_{L^\infty(\Omega)}}.$$

Then, by the comparison principle, we deduce that

$$\forall s\leq-1,\ t\in[s+1,0],\ x\in\Omega,\quad u^s(t,x)\leq\frac{C\|\phi\|_\infty e^{\|c\|_\infty}}{\|u_P(s+1,\cdot)\|_{L^\infty(\Omega)}}u_P(t,x),$$

whence in particular

$$\forall s\leq-1,\ x\in\Omega,\quad \phi(0,x)e^{\lambda s}\leq\frac{C\|\phi\|_\infty e^{\|c\|_\infty}}{\|u_P(s+1,\cdot)\|_{L^\infty(\Omega)}}u_P(0,x).$$

Applying this inequality for fixed $x\in\Omega$, and letting $s\to-\infty$, one eventually gets

$$-\liminf_{t\to-\infty}\frac{\beta(t)}{t}\leq\lambda.$$

Taking the infimum in λ we derive the above inequality with λ replaced by $\mu_b(\mathbb{R}^-)$.

Finally, the reverse inequalities for $\lambda_p(\mathbb{R})$, $\mu_b(\mathbb{R})$ are obtained by considering the functions ϕ_b, ϕ_p defined in the proof of Theorem 2.5. They are respectively bounded on $\mathbb{R}\times\Omega$ and strictly positive on $\mathbb{R}\times B$, for $B\Subset\Omega$, and moreover satisfy on $\mathbb{R}\times\Omega$: $P\phi_b\leq\lambda'\phi_b$ with given $\lambda'>-\liminf_{t\to-\infty}\frac{\beta(t)}{t}$ and $P\phi_p\geq\lambda'\phi_p$ with given $\lambda'<-\limsup_{t\to-\infty}\frac{\beta(t)}{t}$, together with Dirichlet boundary conditions. This yields the desired inequalities for $\lambda_p(\mathbb{R})$, $\mu_b(\mathbb{R})$. □

Remark 6. First, the function ϕ_p constructed in the proofs of Theorems 2.5, 2.6 fulfills $\inf_{\mathbb{R}\times K}\phi_p>0$ for every compact set $K\subset\Omega$. This shows that the definitions of $\lambda_p(\mathbb{R})$, $\mu_p(\mathbb{R})$ do not change if one requires such a stronger condition on the test functions. Second, one could check that the proofs of Theorems 2.5, 2.6 still work if the condition $\sup_{\mathbb{R}\times\Omega}\phi<\infty$ is relaxed by

$$\limsup_{t\to+\infty}\frac{1}{t}\ln\|\phi(t,\cdot)\|_{L^\infty(\Omega)}\leq 0,$$

as well as if the condition $\inf_{\mathbb{R}\times B}\phi>0$ is relaxed by

$$\liminf_{t\to+\infty}\frac{1}{t}\ln\|\phi(t,\cdot)\|_{L^\infty(\Omega)}\geq 0.$$

As a consequence, these changes do not alter the definitions of $\lambda_b(\mathbb{R})$, $\mu_b(\mathbb{R})$ and $\lambda_p(\mathbb{R})$, $\mu_p(\mathbb{R})$ respectively.

8 Completing the proofs

8.1 Proof of the comparison results between the generalized principal eigenvalues

Proof of Proposition 2.7. The equality $\lambda_b(\mathbb{R}^-) = +\infty$ simply follows by taking $\phi = e^{\gamma t}$, with γ arbitrarily large, in the definition. The identities $\lambda_p(\mathbb{R}^+) = +\infty$ and $\mu_p(\mathbb{R}^-) = \mu_b(\mathbb{R}^+) = -\infty$ are proved similarly.

Let us prove the inequalities. The lower bound for $\lambda_{b,p}(I) \geq -\sup c$ is simply obtained by taking a test-function ϕ which is constant in the definition of $\lambda_{b,p}(I)$. The inequalities $\lambda_{b,p}(I) \leq \lambda_p(I)$, $\lambda_{b,p}(I) \leq \lambda_b(I)$ and $\mu_b(I) \leq \mu_{b,p}(I)$ and $\mu_p(I) \leq \mu_{b,p}(I)$ are simple consequences of the definitions. Finally, the inequalities $\lambda_b(I) \leq \mu_p(I)$ and $\lambda_p(I) \leq \mu_b(I)$ are derived from Theorems 2.5 and 2.6 respectively. This proves all the above inequalities.

Let us move on to the monotonicity property with respect to the domain. Namely, we need to show that if $\Omega' \subset \Omega$ is another smooth domain, then the associated notions of generalized principal eigenvalues in $I \times \Omega'$ are greater than or equal to the corresponding ones in $I \times \Omega$. This is straightforward in the case of the λ's, since one can use the test functions in $I \times \Omega$ as test functions in $I \times \Omega'$ (up to choosing the ball B inside Ω', which is possible thanks to the Remark 1(iii)). This cannot be done for the μ's because test functions must satisfy the Dirichlet boundary condition. However, one can argue the other way around: starting from a subsolution in $I \times \Omega'$ which vanishes on $I \times \partial\Omega'$, one can consider its extension to 0 outside $I \times \Omega'$, which turns out to be a *generalized* subsolution in $I \times \Omega$ and can be used to derive the desired inequality.

Let us describe it in detail in the case of the notion $\mu_{b,p}$, the other cases being analogous. Let us make the dependence on the domain by writing $\mu_{b,p}(I \times \Omega)$ and $\mu_{b,p}(I \times \Omega')$. One verifies that in the derivation of the inequality $\mu_{b,p}(I \times \Omega) \geq -\llbracket\beta\rrbracket_I$ in the proof of Theorem 2.3, the properties of the test function ϕ associated with $\lambda > \mu_{b,p}(I \times \Omega)$ have only been used to obtain a solution u that satisfies the two-sided estimate (26). As a matter of fact, if ϕ is now associated with $\lambda > \mu_{b,p}(I \times \Omega')$, one may extend it by 0 outside $I \times \Omega'$, getting a generalized subsolution in $I \times \Omega$. Thus one derives (26) exactly as before, using the comparison principle. From that, continuing the proof, one eventually infers $\mu_{b,p}(I \times \Omega') \geq -\llbracket\beta\rrbracket_I$, where β corresponds to the function u_P in the domain Ω (and not in Ω'). But then we conclude by Theorem 2.3 that $\mu_{b,p}(I \times \Omega') \geq \mu_{b,p}(I \times \Omega)$.

It remains to prove the well-posedness of $\mu_{b,p}$, that is, that the set in its definition is nonempty. Since $\mu_{b,p}$ is the greatest among all the notions, thanks to the inequalities showed above, this will imply the finiteness of all the notions. By monotonicity, it is sufficient to prove it in a subdomain of Ω. Consider the ball B in the definition of $\mu_{b,p}(I \times \Omega)$; let us assume without loss of generality that it is centered at the origin, and let r be its radius. Then consider a function $\phi(t, x) := \chi(|x|)$, where χ is an even, smooth function satisfying

$$\chi > 0 \quad \text{in } (-r, r), \qquad \chi(r) = \chi'(r) = 0, \qquad \chi''(r) > 0.$$

On the one hand, by the uniform parabolicity of P and by the boundedness of its coefficients, the function ϕ satisfies $P\phi \geq 0$ in some neighborhood $I \times U$ of $I \times \partial B$. On the other hand, $P\phi$ is bounded and one has

$$\inf_{I \times (B \setminus U)} \phi > 0,$$

hence one can find $\lambda > 0$ such that

$$P\phi \leq \lambda\phi \quad \text{in } I \times (B \setminus U).$$

This gives $\mu_{b,p}(I \times B) \leq \lambda$, which concludes the proof. $\square$

8.2 Proofs of the applications to the semilinear problem

Proof of Proposition 1.1. Let u be as in the statement of the proposition. We start with observing that, as M is a supersolution of (3) by (5), it follows from the comparison principle that $u(t,x) \leq \max\{M, \|u_0\|_\infty\}$ for $t \geq 0$, $x \in \Omega$.

Assume that $\mu_{b,p}(\mathbb{R}^+) < 0$. There exists then $\lambda < 0$ and ϕ such that

$$\phi > 0 \quad \text{and} \quad P\phi \leq \lambda\phi \quad \text{in } \mathbb{R}^+ \times \Omega,$$

and moreover

$$\phi = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega, \qquad \inf_{t\in\mathbb{R}^+,\ x\in B} \phi(t,x) \geq \frac{1}{K} \quad \text{and} \quad \sup_{t\in\mathbb{R}^+,\ x\in\Omega} \phi(t,x) \leq K,$$

for some open ball $B \Subset \Omega$ and some $K > 0$. In order to compare u with a suitable multiple of ϕ, we preliminarily need to show that the comparison holds at some given time, say $t = 1$, that is

$$\forall x \in \Omega, \quad m\phi(1,x) \leq u(1,x). \tag{31}$$

Since, we cannot directly infer this from the Hopf lemma because of the lack of regularity of ϕ, we will rather employ the Harnack inequality. To start with, we write the equation for u in linear form:

$$\partial_t u - a_{ij}(t,x)\partial_{ij}u - b_i(t,x)\partial_i u = c(t,x)u, \tag{32}$$

with $c(t,x) := f(t,x,u(t,x))/u(t,x)$ if $u(t,x) \neq 0$. Then we let v be the solution to (32), under Dirichlet boundary condition and with initial datum $\phi(0,\cdot)$. On the one hand, the Harnack-type inequality Theorem 7.1 yields the existence of a constant $C > 0$ such that

$$m' := \sup_{x\in\Omega} \frac{v(1,x)}{u(1,x)} \leq C \inf_{x\in\Omega} \frac{v(1,x)}{u(1,x)} < +\infty.$$

On the other hand, since u vanishes on $\partial\Omega$ and it is uniformly continuous, by parabolic regularity, there exists a set $K \Subset \Omega$ such that $c(t,x) \geq f_s'(t,x,0) + \lambda$ on $\Omega \setminus K$. It follows that ϕ is a subsolution to (32) on $\Omega \setminus K$. Take $m'' \in (0,1)$ small enough so that $v \geq m''\phi$ on $[0,1] \times K$. Then the comparison principle yields $v \geq m''\phi$ on $[0,1] \times (\Omega \setminus K)$, and thus on $[0,1] \times \Omega$. Gathering together the two inequalities obtained above we get $m'u(1,x) \geq v(1,x) \geq m''\phi(1,x)$ for $x \in \Omega$, i.e. (31)

Now, since $P\phi \leq \lambda\phi$, with $\lambda < 0$, by the regularity of f there exists $\sigma > 0$ such that $f(t,x,s) \geq (f_s(t,x,0) + \lambda)s$ for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$ and $s \in [0,\sigma)$. Therefore, $m\phi$ is a subsolution to the Dirichlet problem (3), up to decreasing m if need be. We can then compare $m\phi$ with u and derive

$$\forall t \geq 1,\ x \in \Omega, \quad m\phi(t,x) \leq u(t,x).$$

In particular, one finds $u(t,x) \geq m/K$ for $t > 1$ and $x \in B$. Then the desired lower bound follows from the standard interior Harnack inequality [15], considering again u as a solution of (32).

Let us show the reverse implication. The KPP hypothesis (6) implies $Pu \leq 0$ on $\mathbb{R}^+ \times \Omega$. Suppose that u fulfills the persistence property (8). Then, thanks to the Harnack inequality (for the linear equation (32)) for any set $K \Subset \Omega$, one can find some $T > 0$ such that $\inf_{[T,+\infty)\times K} u > 0$. This immediately gives $\mu_{b,p}(\mathbb{R}^+) \leq 0$, owing to Remark 1$(ii)$. However, we need to rule out the possibility $\mu_{b,p}(\mathbb{R}^+) = 0$; this is the only point where the strict inequality in (6) is needed. To achieve our goal, we perturb u by a positive sub-barrier, which is not completely standard due to the temporal dependence of the operator. For given $\varepsilon > 0$, we call

$$\phi(t,x) := u(t,x) + \varepsilon\varphi^2(x),$$

where φ is the solution to

$$-\Delta\varphi = 1 \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \partial\Omega.$$

Since Ω is of class $C^{2,\alpha}$, it follows from the standard elliptic theory (see e.g. [10]) that $\varphi \in C^{2,\alpha}(\overline{\Omega})$ and moreover φ is positive in Ω and satisfies $\partial_\nu\varphi > 0$ on $\partial\Omega$ owing to Hopf's maximum principle. Direct inspection reveals

$$P(\varphi^2) = -2\varphi a_{ij}(t,x)\partial_{ij}\varphi - 2a_{ij}(t,x)\partial_i\varphi\partial_j\varphi - 2\varphi b_i(t,x)\partial_i\varphi - f_s'(t,x,0)\varphi^2.$$

Using the ellipticity of (a_{ij}) and the fact that $\varphi = 0$ and $|\nabla\varphi| \neq 0$ on $\partial\Omega$, one can then find some $h > 0$ and $K \Subset \Omega$ such that $P(\varphi^2) < -h$ in $\Omega \setminus K$. For $t > 0$ and $x \in \Omega \setminus K$ we find

$$P\phi = Pu + \varepsilon P(\varphi^2) = f(t,x,u) - f_s'(t,x,0)u + \varepsilon P(\varphi^2) \leq -\varepsilon h,$$

where we have only used the weak inequality in (6). Recalling that u is bounded, we then have $P\phi \leq \lambda\phi$ in $\mathbb{R}^+ \times (\Omega \setminus K)$, for some $\lambda' < 0$ with $|\lambda'|$ sufficiently small. Instead, in K, we use the stronger hypothesis (6). The uniform continuity of u and the persistence property imply that there exist $T, \delta > 0$ such that $u > \delta$ on $[T, +\infty) \times K$. Then, since f is locally continuous in s, uniformly with respect to t, x, property (6) yields

$$\zeta := \inf_{[T,+\infty)\times K} \big(f_s'(t,x,0)u - f(t,x,u)\big) > 0.$$

We then get in $[T, +\infty) \times K$,

$$P\phi \leq -\zeta + \varepsilon P(\varphi^2),$$

which, for ε small enough, gives $P\phi \leq \lambda''\phi$ for a suitable $\lambda'' < 0$. In conclusion, we have shown that for ε small, $P\phi \leq \lambda\phi$ in $[T, +\infty)\times\Omega$ with $\lambda := \max\{\lambda', \lambda''\} < 0$. This implies that $\mu_{b,p}((T, +\infty)) < 0$. Since $\mu_{b,p}((T, +\infty)) = \mu_{b,p}(\mathbb{R}^+)$ (cf. Remark 1(*ii*)), the proof is concluded. □

Proof of Proposition 1.2. Suppose that (3) admits a bounded entire solution u satisfying (9). Then, using exactly the same function $\phi(t,x) := u(t,x) + \varepsilon\varphi^2(x)$ as in the proof of Proposition 1.1 yields $\mu_{b,p}(\mathbb{R}) < 0$.

Suppose now that $\mu_{b,p}(\mathbb{R}) < 0$. There exists then $\lambda < 0$, $\phi > 0$ in $\mathbb{R} \times \Omega$ and $B \Subset \Omega$ satisfying

$$\sup_{\mathbb{R}\times\Omega} \phi < +\infty, \quad \phi = 0 \text{ on } \mathbb{R} \times \partial\Omega, \quad \inf_{\mathbb{R}\times B} \phi > 0, \quad P\phi \leq \lambda\phi \text{ in } \mathbb{R} \times \Omega.$$

The uniform regularity of f implies that $f(t,x,m\phi) \geq (f_s(t,x,0) + \lambda)m\phi$ for all $(t,x) \in \mathbb{R}^{N+1}$, provided that $m > 0$ is sufficiently small. We choose $m > 0$ small enough in such a way that this inequality holds and, in addition, $m\phi \leq M$, where M is from (5). Hence, the function $m\phi$ is a subsolution of (3) which is smaller than the constant function M, the latter being a supersolution of (3). Let u_n be the solution of (3) in $(-n, +\infty) \times \Omega$ with initial condition $u_n(-n, \cdot) = M$. It follows from the standard comparison principle that $m\phi \leq u_n \leq M$ in $(-n, +\infty) \times \Omega$. By parabolic estimates up to the boundary (see e.g. [16, Theorem 7.30]), the sequence $(u_n)_{n\in\mathbb{N}}$ converges (up to subsequences) locally uniformly in $\mathbb{R} \times \overline{\Omega}$ to a solution $m\phi \leq u \leq M$ of (3). One infers in particular that that $\inf_{\mathbb{R}\times B} u \geq m \inf_{\mathbb{R}\times B} \phi > 0$. As usual, the interior Harnack inequality implies that u satisfies also condition (9). □

Remark 7. The reader could easily adapt our proof in order to show that if one drops condition (9), then the relevant notion for the existence result is $\mu_b(\mathbb{R})$ rather than $\mu_{b,p}(\mathbb{R})$. Namely, problem (3) admits a positive bounded entire solution if $\mu_b(\mathbb{R}) < 0$ and only if $\mu_b(\mathbb{R}) \leq 0$.

Proof of Theorem 1.3. Suppose that the problem (3) admits two bounded ancient solutions u and v satisfying (9). Let us write the equation for v in linear form, namely

$$\partial_t v - a_{ij}(t,x)\partial_{ij} v - b_i(t,x)\partial_i v = c(t,x)v,$$

with $c(t,x) := f(t,x,v(t,x))/v(t,x)$. Observe that the function c is bounded, namely,

$$|c| \leq L := \|f'_s\|_\infty.$$

Fix $T \leq -1$. Let $\tilde{u}$ be the solution of the above equation for $t \in (T,0]$, $x \in \Omega$, under Dirichlet boundary condition, with initial datum $\tilde{u}(T,\cdot) = u(T,\cdot)$. It then follows from the Harnack inequality, Theorem 7.1, that there exists a constant $C > 0$ independent of T such that

$$\forall t \in [T+1,0], \quad \sup_{x\in\Omega} \frac{\tilde{u}(t,x)}{v(t,x)} \leq C \inf_{x\in\Omega} \frac{\tilde{u}(t,x)}{v(t,x)} \leq C \frac{\|\tilde{u}(t,\cdot)\|_\infty}{\|v(t,\cdot)\|_\infty}.$$

On the one hand, the term $\|v(t,\cdot)\|_\infty$ is uniformly bounded from below away from zero, because v satisfies (9). On the other hand, the function $\|u\|_\infty e^{L(t-T)}$ is a supersolution of the equation satisfied by $\tilde{u}$ and therefore, one deduces from the comparison principle that $\|\tilde{u}(t,\cdot)\|_\infty \leq \|u\|_\infty e^{L(t-T)}$ for $t \in (T,0]$. As a consequence, there exists another constant C' independent of T such that

$$\sup_{x\in\Omega} \frac{\tilde{u}(T+1,x)}{v(T+1,x)} \leq C'.$$

Next, we have that the function $\hat{u}(t,x) := \tilde{u}(t,x)e^{2L(t-T)}$ satisfies

$$\partial_t \hat{u} - a_{ij}(t,x)\partial_{ij}\hat{u} - b_i(t,x)\partial_i \hat{u} = [c(t,x) + 2L]\hat{u}.$$

Notice that

$$c(t,x) + 2L = \frac{f(t,x,v(t,x))}{v(t,x)} + 2L \geq \frac{f(t,x,u(t,x))}{u(t,x)}.$$

The last term is the zero order coefficient of the equation satisfied by u, written in linear form. Hence, the comparison principle yields $\hat{u} \geq u$ on $[T,0] \times \Omega$. Gathering together the above estimates, one ends up with

$$\sup_{x\in\Omega} \frac{u(T+1,x)}{v(T+1,x)} \leq \sup_{x\in\Omega} \frac{\tilde{u}(T+1,x)}{v(T+1,x)} e^{2L} \leq C' e^{2L}.$$

We emphasize that this estimate holds for all $T \leq -1$, with C' independent of T. We can then define

$$\bar{k} := \inf\{h > 0 \ : \ kv > u \ \text{ in } \mathbb{R}^- \times \Omega\}.$$

There holds that $\bar{k} > 0$ and that the function $w := \bar{k}v - u$ is nonnegative.

We want to show that $\bar{k} \leq 1$. Assume by way of contradiction that this is not the case. Using the hypothesis (10) we get

$$\partial_t w - a_{ij}(t,x)\partial_{ij} w - b_i(t,x)\partial_i w = \bar{k} f(t,x,v) - f(t,x,u) > f(t,x,\bar{k}v) - f(t,x,u).$$

It follows from the strong maximum principle that w is a positive supersolution of the equation

$$\partial_t \tilde{w} - a_{ij}(t,x)\partial_{ij}\tilde{w} - b_i(t,x)\partial_i \tilde{w} = \bar{c}(t,x)\tilde{w},$$

with $\bar{c}(t,x) := (f(t,x,\bar{k}v) - f(t,x,u))/w(t,x)$, which is bounded. We now fix $T \leq -1$ and consider the solutions $\tilde{u}$ and $\tilde{w}$ of the above linear equation on $(T,0] \times \Omega$, under Dirichlet boundary condition, and with initial data $\tilde{u}(T,\cdot) = u(T,\cdot)$ and $\tilde{w}(T,\cdot) = w(T,\cdot)$. Repeating the arguments of the first part of the proof, we end up with the following estimate:

$$\sup_{x\in\Omega} \frac{u(T+1,x)}{\tilde{w}(T+1,x)} \le \sup_{x\in\Omega} \frac{\tilde{u}(T+1,x)}{\tilde{w}(T+1,x)}\, e^{2L} \le C \frac{\|u\|_\infty}{\|\tilde{w}(T+1,\cdot)\|_\infty}\, e^{3L}.$$

Then, as the parabolic comparison principle yields $w \ge \tilde{w}$ on $(T,0] \times \Omega$, one derives

$$\sup_{x\in\Omega} \frac{u(T+1,x)}{w(T+1,x)} \le C \frac{\|u\|_\infty}{\|\tilde{w}(T+1,\cdot)\|_\infty}\, e^{3L}.$$

Suppose for a moment that w satisfies (9). Then, it is easily seen, for instance by contradiction, that for given $K \Subset \Omega$, there exists a constant C' independent of T, such that

$$\min_K \tilde{w}(T+1,\cdot) \ge C'.$$

As a consequence, there exists $C'' > 0$ such that

$$\sup_{\mathbb{R}^-\times\Omega} \frac{u}{w} \le C'',$$

which, recalling the definition of w, rewrites as $\bar{k}v - u \ge u/C''$, that is, $\bar{k}v \ge (1 + 1/C'')u$. This contradicts the definition of $\bar{k}$.

We have therefore shown that w cannot fulfill (9), thus one can find a compact set $K \subset \Omega$ and a sequence $((t_n, x_n))_{n\in\mathbb{N}}$ in $\mathbb{R}^- \times K$ such that

$$w(t_n, x_n) \to 0, \quad x_n \to z \in K \qquad \text{as } n \to +\infty. \tag{33}$$

In order to get a contradiction, we cannot directly pass to the limit in the equation satisfied by $w(\cdot + t_n, \cdot)$, because we do not have enough regularity on the coefficients. To circumvent this difficulty, we apply once again the interior Harnack inequality and infer that, for a given open ball $B \Subset \Omega$ containing the point z, there exists a constant $C > 0$ such that

$$\sup_{[t_n-3,t_n-1]\times B} w \to 0 \quad \text{as } n \to +\infty.$$

Then, we use the interior parabolic estimates (see e.g. [16, Theorem 7.22], this is where the uniform continuity of the a_{ij} is required) and infer that

$$\|w\|_{W^{1,2}_p([t_n-2,t_n-1]\times B')} \to 0 \quad \text{as } n \to +\infty,$$

for any given ball $B' \Subset B$. Thus, integrating the equation satisfied by w yields

$$\lim_{n\to+\infty} \int_{[t_n-2,t_n-1]\times B'} \big(\bar{k}f(t,x,v) - f(t,x,u)\big) = 0.$$

Recalling the definition of the function $\bar{c}$, we rewrite the term inside the above integral as

$$\bar{k}f(t,x,v) - f(t,x,u) = \bar{k}f(t,x,v) - f(t,x,\bar{k}v) + \bar{c}(t,x)w(t,x),$$

and we find

$$\lim_{n\to+\infty} \int_{[t_n-2,t_n-1]\times B'} \big(\bar{k}f(t,x,v) - f(t,x,\bar{k}v)\big) = 0.$$

Recall however that $\inf_{\mathbb{R}^-\times B'} v > 0$ because v fulfills (9), and moreover it holds that

$$\bar{k}f(t,x,v) - f(t,x,\bar{k}v) = \bar{k}v \left(\frac{f(t,x,v)}{v} - \frac{f(t,x,\bar{k}v)}{\bar{k}v} \right),$$

which is then bounded from below by a positive constant for $t \in \mathbb{R}^-$ and $x \in B'$, thanks to the hypothesis (10) and the continuity of $s \mapsto f(t,x,s)$, which is uniform with respect to t,x. We have reached a contradiction.

In conclusion, we have shown that $\bar{k} \le 1$, that is, $u \le v$. Switching the roles of u and v we derive $v \le u$, concluding the proof of the theorem. □

8.3 Proofs of the applications to the linear problem

Proof of Proposition 2.8. Let us call for short $u(t,x) := u(t,x;u_0)$. By Theorem 2.2 we know that there exist $C,\gamma > 0$ and $q \in \mathbb{R}$ such that

$$q - C\|u_0 - qu_P(0,\cdot)\|_\infty \, e^{-\gamma t} \leq \frac{\|u(t,\cdot)\|_\infty}{\|u_P(t,\cdot)\|_\infty} \leq q + C\|u_0 - qu_P(0,\cdot)\|_\infty \, e^{-\gamma t}.$$

We actually know from the last assertion of the theorem that $q > 0$, because $u(t,x) > 0$ for $t > 0$, $x \in \Omega$ due to the strong maximum principle. As a consequence, we derive

$$\lim_{t\to+\infty} \frac{1}{t}\big(\ln \|u(t,\cdot)\|_\infty - \ln \|u_P(t,\cdot)\|_\infty \big) = 0.$$

We then deduce from Theorem 2.5 that

$$\liminf_{t\to+\infty} \frac{\ln \|u(t,\cdot)\|_\infty}{t} = -\mu_p(\mathbb{R}^+), \qquad \limsup_{t\to+\infty} \frac{\ln \|u(t,\cdot)\|_\infty}{t} = -\lambda_b(\mathbb{R}^+).$$

We then conclude using the boundary Harnack inequality, which yields, for given $x \in \Omega$, the existence of a constant $C' > 0$ such that

$$\forall t > 2, \quad \frac{1}{C'}\|u(t-1,\cdot)\|_\infty \leq u(t,x) \leq \|u(t,\cdot)\|_\infty.$$

□

Proof of Proposition 2.9. We write for short $u^s(t,x)$ in place of $u^s(t,x;u_0)$. By Theorem 7.1 there exists a positive constant C such that, for any $s \in \mathbb{R}$ and $x \in \Omega$, there holds

$$\frac{1}{C}\frac{\|u^s(s+1,\cdot)\|_\infty}{\|u_P(s+1,\cdot)\|_\infty} \leq \frac{1}{C}\sup_\Omega \frac{u^s(s+1,\cdot)}{u_P(s+1,\cdot)} \leq \frac{u^s(s+1,x)}{u_P(s+1,x)} \leq C\inf_\Omega \frac{u^s(s+1,\cdot)}{u_P(s+1,\cdot)} \leq C\frac{\|u^s(s+1,\cdot)\|_\infty}{\|u_P(s+1,\cdot)\|_\infty}.$$

One has by comparison that

$$\|u^s(s+1,\cdot)\|_\infty \leq \|u_0\|_\infty e^{\|c\|_\infty}.$$

Furthermore, using that u_0 is continuous and strictly positive somewhere, one easily sees that there exists a constant $K > 0$ independent of s such that $\|u^s(s+1,\cdot)\|_\infty \geq K$. Summing up, there exists $K' > 0$ such that

$$\forall s \in \mathbb{R},\ x \in \Omega, \quad \frac{1}{K'\|u_P(s+1,\cdot)\|_\infty} u_P(s+1,x) \leq u^s(s+1,x) \leq \frac{K'}{\|u_P(s+1,\cdot)\|_\infty} u_P(s+1,x).$$

Therefore, the comparison principle eventually yields

$$\forall s \in \mathbb{R},\ x \in \Omega,\ t \geq s+1, \quad \frac{1}{K'\|u_P(s+1,\cdot)\|_\infty} u_P(t,x) \leq u^s(t,x) \leq \frac{K'}{\|u_P(s+1,\cdot)\|_\infty} u_P(t,x).$$

For $s \leq -1$, we apply these estimates at $t = 0$ and get

$$\forall s \leq -1,\ x \in \Omega, \quad \frac{1}{K'\|u_P(s+1,\cdot)\|_\infty} u_P(0,x) \leq u^s(0,x) \leq \frac{K'}{\|u_P(s+1,\cdot)\|_\infty} u_P(0,x).$$

Finally, we derive

$$\forall s \leq -1,\ x \in \Omega, \quad \big| \ln u^s(0,x) + \ln \|u_P(s+1,\cdot)\|_\infty - \ln u_P(0,x) \big| \leq \ln K'.$$

The results then follow from Theorem 2.6. □

Proof of Theorem 2.11. The fact that that the validity of the *MP* implies $\mu_b(\mathbb{R}^-) \geq 0$ is straightforward, because $\mu_b(\mathbb{R}^-) < 0$ implies the existence of an ancient, bounded, subsolution ϕ which violates the *MP*.

Let us turn to the reverse implication. Suppose that there exists an ancient, bounded, subsolution u to (11) that violates the *MP*, that is, such that $u(\bar{t}, \bar{x}) > 0$ for some $\bar{t} < 0$ and $\bar{x} \in \Omega$. It follows from the comparison principle that $\max_\Omega u(t, \cdot) > 0$ for any $t < \bar{t}$. For $T < \bar{t}$, let v_T be the solution to the problem (11) for $t > T$, with initial datum $v_T(T, \cdot) = \max\{u(T, \cdot), 0\}$. Then, by comparison, we get

$$\forall t \geq T,\ x \in \Omega, \quad u(t,x) \leq v_T(t,x) \leq \|u\|_{L^\infty(\mathbb{R}^- \times \Omega)} e^{\|c\|_\infty (t-T)},$$

and moreover $v_T(t,x) > 0$ for $t > T$ and $x \in \Omega$. Let us call $K := \|u\|_{L^\infty(\mathbb{R}^- \times \Omega)} e^{\|c\|_\infty}$. Then, by Theorem 7.1, there exists a positive constant C such that, for any $T < 0$, it holds

$$\forall x \in \Omega, \quad \frac{v_T(T+1,x)}{u_P(T+1,x)} \leq C \inf_\Omega \frac{v_T(T+1,\cdot)}{u_P(T+1,\cdot)} \leq C \frac{K}{\|u_P(T+1,\cdot)\|_\infty}.$$

It then follows, again by comparison, that, for any $T < 0$,

$$\forall t \in [T+1, 0),\ x \in \Omega, \quad u(t,x) \leq v_T(t,x) \leq C \frac{K}{\|u_P(T+1,\cdot)\|_\infty} u_P(t,x).$$

But, if by contradiction one had $\mu_b(\mathbb{R}^-) > 0$, then Theorem 2.6 would imply the existence of a sequence $(T_n)_{n \in \mathbb{N}}$ diverging to $-\infty$ such that $\|u_P(T_n+1,\cdot)\|_\infty \to +\infty$ as $n \to +\infty$, and therefore the above inequality would imply $u(t,x) \leq 0$ for all $t < 0$, $x \in \Omega$. This is a contradiction. □

8.4 Proofs of the results on limit operators

Proof of Theorem 5.1. We give the proof of the identities stated in the theorem in the case of $\mathbb{R}^+$. The case of $\mathbb{R}^-$ follows from the same arguments, with minor modifications.

Consider an arbitrary limit operator $P^* \in \omega_{\mathbb{R}^+}(P)$ and let $(t_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^+$ be the associated sequence. By parabolic estimates and Harnack's inequality, cf. (20), the functions $u_P(\cdot + t_n, \cdot)/u_P(t_n, \cdot)$ are bounded in $W^{1,2}_{p,loc}(I \times \overline{\Omega})$ for any $p \in (N+1, \infty)$. Hence, up to subsequences, these functions converge weakly-$\star$ in $W^{1,2}_{p,loc}(I \times \overline{\Omega})$ for any $p \in (N+1, \infty)$, and, using the Aubin-Lions lemma, strongly in $W^{0,1}_{p,loc}(I \times \overline{\Omega})$ as $n \to +\infty$, to a function u. We can then pass to the weak limit in the equation and get that u is a positive solution to $P^* u = 0$ in $\mathbb{R} \times \Omega$, vanishing on $\mathbb{R} \times \partial\Omega$ and satisfying $\|u(0,\cdot)\|_{L^\infty(\Omega)} = 1$, namely, $u \equiv u_{P^*}$ given by Theorem 2.1. It follows from Theorem 2.5 that, on the one hand,

$$\begin{aligned}
-\mu_p(P^*, \mathbb{R}^+) &= \liminf_{t \to +\infty} \frac{1}{t} \ln \|u_{P^*}(t,\cdot)\|_{L^\infty(\Omega)} \\
&= \liminf_{t \to +\infty} \Big(\lim_{n \to +\infty} \frac{1}{t} \big(\ln \|u_P(t_n + t, \cdot)\|_{L^\infty(\Omega)} - \ln \|u_P(t_n, \cdot)\|_{L^\infty(\Omega)} \big) \Big),
\end{aligned}$$

and the last term is greater than or equal to $-\mu_{b,p}(P, \mathbb{R}^+)$ by Theorem 2.3. On the other hand, from Theorems 2.5 and 2.3, we infer

$$-\lambda_b(P^*, \mathbb{R}^+) = \limsup_{t \to +\infty} \Big(\lim_{n \to +\infty} \frac{1}{t} \big(\ln \|u_P(t_n + t, \cdot)\|_{L^\infty(\Omega)} - \ln \|u_P(t_n, \cdot)\|_{L^\infty(\Omega)} \big) \Big) \leq -\lambda_{b,p}(P, \mathbb{R}^+).$$

We have thereby shown that

$$\mu_{b,p}(P, \mathbb{R}^+) \geq \sup_{P^* \in \omega_{\mathbb{R}^+}(P)} \mu_p(P^*, \mathbb{R}^+) \quad \text{and} \quad \lambda_{b,p}(P, \mathbb{R}^+) \leq \inf_{P^* \in \omega_{\mathbb{R}^+}(P)} \lambda_b(P^*, \mathbb{R}^+).$$

We know from Theorem 2.5 that $\mu_p(P^*,\mathbb{R}^+) \geq \lambda_b(P^*,\mathbb{R}^+)$, for any operator P^*, therefore to conclude the proof we need to find some limit operators P_1^* and P_2^* for which $\mu_{b,p}(P,\mathbb{R}^+) = \lambda_b(P_1^*,\mathbb{R}^+)$ and $\lambda_{b,p}(P,\mathbb{R}^+) = \mu_p(P_2^*,\mathbb{R}^+)$.

Let $I = \mathbb{R}^-$ or $I = \mathbb{R}^+$. We claim that for all $n \in \mathbb{N}$, there exists $s_n \in I$ satisfying $s_n + n \in I$, with the following property:

$$\forall t \in [1,n], \quad \frac{1}{t}\Big(\ln \|u_P(s_n+t,\cdot)\|_{L^\infty(\Omega)} - \ln \|u_P(s_n,\cdot)\|_{L^\infty(\Omega)}\Big) \leq -\mu_{b,p}(P,I) + \frac{1}{n}. \tag{34}$$

The proof of this claim is similar to that of [24, Proposition 4.4]. Assume that it fails for some $n \in \mathbb{N}$. By Theorem 2.3, for $K \in \mathbb{N}$ large enough, there exists $\tau \in I$, depending on K, such that $\tau + Kn \in I$ and

$$\frac{1}{Kn}\Big(\ln \|u_P(\tau+Kn,\cdot)\|_{L^\infty(\Omega)} - \ln \|u_P(\tau,\cdot)\|_{L^\infty(\Omega)}\Big) < -\mu_{b,p}(P,I) + \frac{1}{2n}. \tag{35}$$

On the other hand, by the contradictory assumption, there exists $t_0 \in [1,n]$ such that $\tau + t_0 \in I$ and

$$\frac{1}{t_0}\Big(\ln \|u_P(\tau+t_0,\cdot)\|_{L^\infty(\Omega)} - \ln \|u_P(\tau,\cdot)\|_{L^\infty(\Omega)}\Big) > -\mu_{b,p}(P,I) + \frac{1}{n}.$$

Then, there is $t_1 \in [1,n]$ such that $\tau + t_0 + t_1 \in I$ and

$$\frac{1}{t_1}\Big(\ln \|u_P(\tau+t_0+t_1,\cdot)\|_{L^\infty(\Omega)} - \ln \|u_P(\tau+t_0,\cdot)\|_{L^\infty(\Omega)}\Big) > -\mu_{b,p}(P,I) + \frac{1}{n}$$

and thus:

$$n\Big(\ln \|u_P(\tau+t_0+t_1,\cdot)\|_{L^\infty(\Omega)} - \ln \|u_P(\tau,\cdot)\|_{L^\infty(\Omega)}\Big) > -n(t_0+t_1)\mu_{b,p}(P,I) + t_0 + t_1.$$

We continue until we find $\bar{t} \in [1,n]$ such that

$$n\Big(\ln \|u_P(\tau+Kn+\bar{t},\cdot)\|_{L^\infty(\Omega)} - \ln \|u_P(\tau,\cdot)\|_{L^\infty(\Omega)}\Big) > -n(Kn+\bar{t})\mu_{b,p}(P,I) + Kn + \bar{t}.$$

It follows that

$$\begin{aligned}&\ln \|u_P(\tau+Kn,\cdot)\|_{L^\infty(\Omega)} - \ln \|u_P(\tau,\cdot)\|_{L^\infty(\Omega)}\\ &> \ln \|u_P(\tau+Kn,\cdot)\|_{L^\infty(\Omega)} - \ln \|u_P(\tau+Kn+\bar{t},\cdot)\|_{L^\infty(\Omega)} - (Kn+\bar{t})\mu_{b,p}(P,I) + K + \bar{t}/n.\end{aligned}$$

Then, using the fact that

$$\|u_P(\tau+Kn+\bar{t},\cdot)\|_{L^\infty(\Omega)} \leq \|u_P(\tau+Kn,\cdot)\|_{L^\infty(\Omega)} e^{\|c\|_\infty \bar{t}},$$

thanks to the comparison principle, and that $\bar{t} \leq n$, one eventually derives

$$\ln \|u_P(\tau+Kn,\cdot)\|_{L^\infty(\Omega)} - \ln \|u_P(\tau,\cdot)\|_{L^\infty(\Omega)} > -n\|c\|_\infty - (Kn+n)\mu_{b,p}(P,I) + K.$$

Choosing K large enough, one obtains a contradiction with (35).

The claim (34) is proved. Consider now $I = \mathbb{R}^+$ and the limit operator P_1^* of P associated with (a subsequence of) $(s_n)_{n\in\mathbb{N}}$. We have already observed that (up to subsequences)

$$\frac{u_P(\cdot+s_n,\cdot)}{\|u_P(s_n,\cdot)\|_{L^\infty(\Omega)}} \to u_{P_1^*} \quad \text{as } n \to +\infty \quad \text{weakly-}\star \text{ in } W^{1,2}_{p,loc}(\mathbb{R}^+ \times \overline{\Omega}).$$

It follows from (34) that

$$\forall t \geq 1, \quad \frac{1}{t} \ln \|u_{P_1^*}(t,\cdot)\|_{L^\infty(\Omega)} \leq -\mu_{b,p}(P,\mathbb{R}^+)$$

and thus, taking the lim sup as $t \to +\infty$, we deduce from Theorem 2.5 that $\mu_{b,p}(P,\mathbb{R}^+) \leq \lambda_b(P_1^*,\mathbb{R}^+)$. In an analogous way one finds another limit operator P_2^* such that $\lambda_{b,p}(P,\mathbb{R}^+) \geq \mu_b(P_2^*,\mathbb{R}^+)$. The conclusion follows by collecting all the inequalities. □

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