Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XX (2020), 1295-1324

Trudinger-Moser inequalities on a closed Riemannian surface with the action of a finite isometric group

YU FANG AND YUNYAN YANG

Abstract. Let (Σ, g) be a closed Riemannian surface, $W^{1,2}(\Sigma, g)$ be the usual Sobolev space, $\mathbf{G}$ be a finite isometric group acting on (Σ, g), and $\mathscr{H}_{\mathbf{G}}$ be the function space including all functions $u \in W^{1,2}(\Sigma, g)$ with $\int_\Sigma u dv_g = 0$ and $u(\sigma(x)) = u(x)$ for all $\sigma \in \mathbf{G}$ and all $x \in \Sigma$. Denote the number of distinct points of the set $\{\sigma(x) : \sigma \in \mathbf{G}\}$ by $I(x)$ and $\ell = \min_{x\in\Sigma} I(x)$. Let $\lambda_1^{\mathbf{G}}$ be the first eigenvalue of the Laplace-Beltrami operator on the space $\mathscr{H}_{\mathbf{G}}$. Using blow-up analysis, we prove that if $\alpha < \lambda_1^{\mathbf{G}}$ and $\beta \le 4\pi\ell$, then there holds

$$\sup_{u\in\mathscr{H}_{\mathbf{G}},\, \int_\Sigma |\nabla_g u|^2 dv_g - \alpha\int_\Sigma u^2 dv_g \le 1} \int_\Sigma e^{\beta u^2} dv_g < \infty;$$

if $\alpha < \lambda_1^{\mathbf{G}}$ and $\beta > 4\pi\ell$, or $\alpha \ge \lambda_1^{\mathbf{G}}$ and $\beta > 0$, then the above supremum is infinity; if $\alpha < \lambda_1^{\mathbf{G}}$ and $\beta \le 4\pi\ell$, then the above supremum can be attained. Moreover, similar inequalities involving higher order eigenvalues are obtained. Our results partially improve original inequalities of J. Moser [17], L. Fontana [9] and W. Chen [4].

Mathematics Subject Classification (2010): 58J05 (primary).

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $W_0^{1,n}(\Omega)$ be the usual Sobolev space, and ω_{n-1} be the area of the unit sphere in $\mathbb{R}^n$. It was proved by Moser [17] that for any $\alpha \le \alpha_n = n\omega_{n-1}^{1/(n-1)}$, there holds

$$\sup_{u\in W_0^{1,n}(\Omega),\, \int_\Omega |\nabla u|^n dx \le 1} \int_\Omega e^{\alpha |u|^{n/(n-1)}} dx < \infty. \tag{1.1}$$

This work is partly supported by the National Science Foundation of China (Grant Nos. 11171347 and 11471014).

Received November 16, 2017; accepted in revised form October 31, 2018.
Published online December 2020.

Moreover, α_n is the best constant in the sense that if $\alpha > \alpha_n$, the integrals in the above inequality are still finite, but the supremum is infinity. Such kind of inequalities are known as Trudinger-Moser inequalities in literature. Earlier contributions are due to Yudovich [34], Pohozaev [21], Peetre [20] and Trudinger [24]. Let $\lambda_1(\Omega)$ be the first eigenvalue of the Laplace operator with respect to the Dirichlet boundary condition. Adimurthi-Druet [1] proved that for any $\alpha < \lambda_1(\Omega)$, there holds

$$\sup_{u\in W_0^{1,2}(\Omega),\, \int_\Omega |\nabla u|^2 dx\leq 1} \int_\Omega e^{4\pi u^2(1+\alpha\|u\|_2^2)}dx < \infty; \tag{1.2}$$

moreover, if $\alpha \geq \lambda_1(\Omega)$, then the above supremum is infinity, where $\|u\|_2^2 = \int_\Omega u^2 dx$. The inequality (1.2) is stronger than (1.1) and was extended by the second named author [26] to the higher dimensional case. Later, Tintarev [23] proved among other results that for any $\alpha < \lambda_1(\mathbb{B}_R(0))$, there holds

$$\sup_{u\in W_0^{1,2}(\Omega),\, \int_\Omega |\nabla u|^2 dx-\alpha\int_\Omega u^2dx\leq 1} \int_\Omega e^{4\pi u^2}dx < \infty, \tag{1.3}$$

where $\mathbb{B}_R(0)$ denotes the ball centered at 0 with radius R and its measure is equal to that of Ω. As one expected, $\lambda_1(\mathbb{B}_R(0))$ can be replaced by $\lambda_1(\Omega)$, which is a consequence of [28, Theorem 1].

One can ask whether the supremum in (1.1) can be attained or not. Existence of extremal functions was proved first by Carleson-Chang [3] in the case that Ω is the unit ball, then by Struwe [22] in the case that Ω is close to a ball in the sense of measure, later by Flucher [8] when Ω is a planar domain, and finally by Lin [13] when Ω is a domain in $\mathbb{R}^n$. In [25], the second named author claimed that the supremum in (1.2) can be attained for all $0 \leq \alpha < \lambda_1(\Omega)$. We remark that there is a mistake during that test function computation ([25, page 338, line 8]). In fact, in two dimensions, extremal function for (1.2) exists only for sufficiently small α, see for example [27]. Concerning extremal functions for inequalities of the type (1.2), we refer the reader to [6, 10, 14, 15, 19, 29, 30, 32, 33, 35]. As a comparison, it was proved in [28] that the supremum in (1.3) can be attained for all $\alpha < \lambda_1(\Omega)$. It is remarkable that (1.3) is stronger than (1.2), however, there is no relation on existence of extremal functions between (1.2) and (1.3).

Let $(\mathbb{S}^2, g_0)$ be the 2-dimensional sphere $x_1^2+x_2^2+x_3^2 = 1$ with the metric $g_0 = dx_1^2 + dx_2^2 + dx_3^2$ and the corresponding volume element dv_{g_0}. According to Moser [17], one can find a constant C such that for all functions u with $\int_{\mathbb{S}^2} |\nabla_{g_0} u|^2 dv_{g_0} \leq 1$ and $\int_{\mathbb{S}^2} u dv_{g_0} = 0$,

$$\int_{\mathbb{S}^2} e^{4\pi u^2} dv_{g_0} \leq C. \tag{1.4}$$

Concerning all even functions u, it was indicated by Moser [18] that the best constant $\alpha_2 = 4\pi$ would double. Namely, there exists a constant C such that for all functions u satisfying $u(-x) = u(x)$, $\forall x \in \mathbb{S}^2$, $\int_{\mathbb{S}^2} |\nabla_{g_0} u|^2 dv_{g_0} \leq 1$, and $\int_{\mathbb{S}^2} u dv_{g_0} = 0$, there holds

$$\int_{\mathbb{S}^2} e^{8\pi u^2} dv_{g_0} \leq C. \tag{1.5}$$

Later, by using an isoperimetric inequality on closed Riemannian surfaces with conical singularities, Chen [4] proved a Trudinger-Moser inequality for a class of "symmetric" functions, which particularly generalized (1.4) and (1.5).

Let (M, g) be a closed n-dimensional Riemannian manifold. Among other results, it was proved by Fontana [9] that there exists a constant C, depending only on (M, g), such that if $u \in W^{1,n}(M, g)$ satisfies $\int_M |\nabla_g u|^n dv_g \leq 1$ and $\int_M u dv_g = 0$, then

$$\int_M e^{\alpha_n |u|^{n/(n-1)}} dv_g \leq C. \tag{1.6}$$

The existence of extremal functions for (1.6) was obtained by Li [11, 12]. Precisely, there exists some $u_0 \in W^{1,n}(M) \cap C^1(M)$ with $\int_M |\nabla_g u_0|^n dv_g = 1$ and $\int_M u_0 dv_g = 0$ such that

$$\int_M e^{\alpha_n |u_0|^{n/(n-1)}} dv_g = \sup_{u \in W^{1,n}(M),\, \int_M |\nabla_g u|^n dv_g \leq 1,\, \int_M u dv_g = 0} \int_M e^{\alpha_n |u|^{n/(n-1)}} dv_g. \tag{1.7}$$

Obviously (1.7) implies (1.6). In [27], the inequality (1.2) was generalized to a closed Riemannian surface version, namely for any α with $0 \leq \alpha < \lambda_1(\Sigma) = \inf_{\|u\|_2 = 1,\, \int_\Sigma u dv_g = 0} \|\nabla_g u\|_2^2$,

$$\sup_{u \in W^{1,2}(\Sigma, g),\, \int_\Sigma |\nabla_g u|^2 dv_g \leq 1,\, \int_\Sigma u dv_g = 0} \int_\Sigma e^{4\pi u^2 (1 + \alpha \|u\|_2^2)} dv_g < \infty; \tag{1.8}$$

moreover, the supremum in (1.8) can be attained for sufficiently small α. However, in a recent work [28], an analog of (1.3) was also established on a closed Riemannian surface, say for any $\alpha < \lambda_1(\Sigma)$,

$$\sup_{u \in W^{1,2}(\Sigma, g),\, \int_\Sigma |\nabla_g u|^2 dv_g - \alpha \int_\Sigma u^2 dv_g \leq 1} \int_\Sigma e^{4\pi u^2} dv_g < \infty. \tag{1.9}$$

Moreover, the above supremum can be attained for any $\alpha < \lambda_1(\Sigma)$. Further, this kind of inequalities involving higher order eigenvalues of the Laplace-Beltrami operator has been studied.

In this paper, our aim is to establish Trudinger-Moser inequalities for "symmetric" functions and prove the existence of their extremal functions on a closed Riemannian surface with the action of a finite isometric group. They can be viewed as a "combination" of (1.5) and (1.9). We believe that such inequalities would play an important role in the study of prescribing Gaussian curvature problem and mean field equations. Before ending this introduction, we mention Mancini-Martinazzi [16], who studied the classical Trudinger-Moser inequality by estimating the energy of extremals for subcritical functionals.

ACKNOWLEDGEMENTS. The authors thank the referee for very careful reading and constructive comments, which greatly improve the previous version of this paper.

2. Notation and main results

Let (Σ, g) be a closed Riemannian surface and $\mathbf{G} = \{\sigma_1, \cdots, \sigma_N\}$ be an isometric group acting on it, where N is some positive integer. By definition, $\mathbf{G}$ is a group and each $\sigma_i : \Sigma \to \Sigma$ is an isometric map, particularly $\sigma_i^* g_x = g_{\sigma_i(x)}$ for all $x \in \Sigma$. Let $u : \Sigma \to \mathbb{R}$ be a measurable function, we say that $u \in \mathscr{I}_{\mathbf{G}}$ if u is $\mathbf{G}$-invariant, namely $u(\sigma_i(x)) = u(x)$ for any $1 \le i \le N$ and almost every $x \in \Sigma$. We denote $W^{1,2}(\Sigma, g)$ the closure of $C^\infty(\Sigma)$ under the norm

$$\|u\|_{W^{1,2}(\Sigma,g)} = \left(\int_\Sigma \left(|\nabla_g u|^2 + u^2\right) dv_g\right)^{1/2},$$

where ∇_g and dv_g stand for the gradient operator and the Riemannian volume element respectively. Define a Hilbert space

$$\mathscr{H}_{\mathbf{G}} = \left\{u \in W^{1,2}(\Sigma, g) \cap \mathscr{I}_{\mathbf{G}} : \int_\Sigma u dv_g = 0\right\} \tag{2.1}$$

with an inner product

$$\langle u, v\rangle_{\mathscr{H}_{\mathbf{G}}} = \int_\Sigma \langle\nabla_g u, \nabla_g v\rangle dv_g,$$

where $\langle\nabla_g u, \nabla_g v\rangle$ stands for the Riemannian inner product of $\nabla_g u$ and $\nabla_g v$. Let $\Delta_g = -\mathrm{div}_g \nabla_g$ be the Laplace-Beltrami operator, and

$$\lambda_1^{\mathbf{G}} = \inf_{u\in\mathscr{H}_{\mathbf{G}},\, u\not\equiv 0} \frac{\int_\Sigma |\nabla_g u|^2 dv_g}{\int_\Sigma u^2 dv_g} \tag{2.2}$$

be the first eigenvalue of Δ_g on the space $\mathscr{H}_{\mathbf{G}}$. For any $x \in \Sigma$, we set $I(x) = \sharp\mathbf{G}(x)$, where $\sharp A$ stands for the number of all distinct points in the set A, and $\mathbf{G}(x) = \{\sigma_1(x), \cdots, \sigma_N(x)\}$. Let

$$\ell = \min_{x\in\Sigma} I(x). \tag{2.3}$$

Clearly we have $1 \le \ell \le N$ since $1 \le I(x) \le N$ for all $x \in \Sigma$. As one will see, the best constant in the Trudinger-Moser inequality for "symmetric" functions would be $4\pi\ell$. Precisely we state the following theorem.

Theorem 2.1. *Let (Σ, g) be a closed Riemannian surface and $\mathbf{G} = \{\sigma_1, \cdots, \sigma_N\}$ be an isometric group acting on it. Assume $\mathscr{H}_{\mathbf{G}}$, $\lambda_1^{\mathbf{G}}$ and ℓ are defined by* (2.1), (2.2) *and* (2.3) *respectively. Then we have the following assertions:*

(i) *For any $\alpha < \lambda_1^{\mathbf{G}}$ and $\beta \leq 4\pi\ell$, there holds*

$$\sup_{u\in\mathscr{H}_{\mathbf{G}},\,\int_\Sigma |\nabla_g u|^2 dv_g - \alpha\int_\Sigma u^2 dv_g \leq 1} \int_\Sigma e^{\beta u^2} dv_g < \infty; \tag{2.4}$$

(ii) *If $\alpha < \lambda_1^{\mathbf{G}}$ and $\beta > 4\pi\ell$, or $\alpha \geq \lambda_1^{\mathbf{G}}$ and $\beta > 0$, then the supremum in* (2.4) *is infinity;*

(iii) *If $\alpha < \lambda_1^{\mathbf{G}}$ and $\beta \leq 4\pi\ell$, then the supremum in* (2.4) *can be attained by some function $u_0 \in \mathscr{H}_{\mathbf{G}} \cap C^1(\Sigma, g)$ with $\int_\Sigma |\nabla_g u_0|^2 dv_g - \alpha \int_\Sigma u_0^2 dv_g = 1$.*

As in [28], we may consider the effect of higher order eigenvalues on the Trudinger-Moser inequality. For this purpose, we define the first eigenfunction space with respect to $\lambda_1^{\mathbf{G}}$ by

$$E_{\lambda_1^{\mathbf{G}}} = \left\{u \in \mathscr{H}_{\mathbf{G}} : \Delta_g u = \lambda_1^{\mathbf{G}} u\right\}.$$

By an induction, the j-th ($j \geq 2$) eigenvalue and eigenfunction space will be defined as

$$\lambda_j^{\mathbf{G}} = \inf_{u\in\mathscr{H}_{\mathbf{G}},\, u\in E_{j-1}^{\perp},\, u\not\equiv 0} \frac{\int_\Sigma |\nabla_g u|^2 dv_g}{\int_\Sigma u^2 dv_g} \tag{2.5}$$

and

$$E_{\lambda_j^{\mathbf{G}}} = \left\{u \in E_{j-1}^{\perp} : \Delta_g u = \lambda_j^{\mathbf{G}} u\right\}$$

respectively, where $E_{j-1} = E_{\lambda_1^{\mathbf{G}}} \oplus \cdots \oplus E_{\lambda_{j-1}^{\mathbf{G}}}$ and

$$E_{j-1}^{\perp} = \left\{u \in \mathscr{H}_{\mathbf{G}} : \int_\Sigma uv\, dv_g = 0,\ \forall v \in E_{j-1}\right\}. \tag{2.6}$$

Then higher order eigenvalues of Δ_g affect the Trudinger-Moser inequality in the following way:

Theorem 2.2. *Let (Σ, g) be a closed Riemannian surface and $\mathbf{G} = \{\sigma_1, \cdots, \sigma_N\}$ be an isometric group acting on it. Assume $\mathscr{H}_{\mathbf{G}}$, ℓ, $\lambda_j^{\mathbf{G}}$ and $E_{j-1}^{\perp}$ are defined by* (2.1), (2.3), (2.5) *and* (2.6) *respectively, $j \geq 2$.*

(i) *For any $\alpha < \lambda_j^{\mathbf{G}}$ and $\beta \leq 4\pi\ell$, there holds*

$$\sup_{u\in E_{j-1}^{\perp},\,\int_\Sigma |\nabla_g u|^2 dv_g - \alpha\int_\Sigma u^2 dv_g \leq 1} \int_\Sigma e^{\beta u^2} dv_g < \infty; \tag{2.7}$$

(ii) *If* $\alpha < \lambda_j^{\mathbf{G}}$ *and* $\beta > 4\pi\ell$, *or* $\alpha \geq \lambda_j^{\mathbf{G}}$ *and* $\beta > 0$, *then the supremum in* (2.7) *is infinity;*

(iii) *For any* $\alpha < \lambda_j^{\mathbf{G}}$ *and* $\beta \leq 4\pi\ell$, *the supremum in* (2.7) *can be attained by some function* $u_0 \in E_{j-1}^{\perp} \cap C^1(\Sigma, g)$ *with* $\int_\Sigma |\nabla_g u_0|^2 dv_g - \alpha \int_\Sigma u_0^2 dv_g = 1$.

Let us give several examples for the finite isometric group $\mathbf{G}$ acting on a closed Riemannian surface (Σ, g). (a) If $\mathbf{G} = \{Id\}$, where Id denotes the identity map, then $\mathbf{G}$ is a trivial isometric group action, and Theorems 2.1 and 2.2 are reduced to ([28, Theorems 3 and 4]). (b) Let $(\mathbb{S}^2, g_0)$ be the standard 2-sphere given as in the introduction, and $\mathbf{G} = \{Id, \sigma_0\}$, where $\sigma_0(x) = -x$ for any $x \in \mathbb{S}^2$. Then we have $\sharp\mathbf{G}(x) = \sharp\{x, -x\} = 2$ for any $x \in \mathbb{S}^2$, and thus $\ell = 2$. Hence Moser's inequality (1.5) for even functions is a special case of our theorems. (c) If $\mathbf{G}$ has a fixed point, namely there exists some point $p \in \Sigma$ such that $\sigma(p) = p$ for all $\sigma \in \mathbf{G}$, then we have $\ell = \sharp\mathbf{G}(p) = 1$, and whence both of the best constants in (2.4) and (2.7) are 4π.

From now on, to simplify notations, we write

$$\|u\|_{1,\alpha} = \left(\int_\Sigma |\nabla_g u|^2 dv_g - \alpha \int_\Sigma u^2 dv_g \right)^{1/2}, \tag{2.8}$$

provided that the right hand side of the above equality makes sense, say, if $\alpha < \lambda_1^{\mathbf{G}}$ and $u \in \mathscr{H}_{\mathbf{G}}$, then $\|u\|_{1,\alpha}$ is well defined. For the proof of Theorems 2.1 and 2.2, we follow the lines of [28] and thereby follow closely [11]. Pioneer works are due to Carleson-Chang [3], Ding-Jost-Li-Wang [7], and Adimurthi-Struwe [2]. Since both of them are similar, we only give the outline of the proof of Theorem 2.1. Firstly, we prove that the best constant in (2.4) is $4\pi\ell$, which is based on Moser's original inequality and test function computations; Secondly, a direct method of variation shows that every subcritical Trudinger-Moser functional has a maximizer, namely for any $\epsilon > 0$, there exists some $u_\epsilon \in \mathscr{H}_{\mathbf{G}}$ with $\|u_\epsilon\|_{1,\alpha} = 1$ satisfying

$$\int_\Sigma e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g = \sup_{u \in \mathscr{H}_{\mathbf{G}},\ \|u\|_{1,\alpha} \leq 1} \int_\Sigma e^{(4\pi\ell-\epsilon)u^2} dv_g,$$

where $\alpha < \lambda_1^{\mathbf{G}}$ and $\|u\|_{1,\alpha}$ is defined as in (2.8); Thirdly, we use blow-up analysis to show that if $\sup_{x\in\Sigma} |u_\epsilon| \to \infty$ as $\epsilon \to 0$, then

$$\sup_{u \in \mathscr{H}_{\mathbf{G}},\ \|u\|_{1,\alpha} \leq 1} \int_\Sigma e^{4\pi\ell u^2} dv_g \leq \mathrm{Vol}_g(\Sigma) + \pi\ell e^{1+4\pi\ell A_{x_0}},$$

where A_{x_0} is a constant related to a certain Green function (see (3.42) below); Finally, we construct a sequence of functions $\phi_\epsilon \in \mathscr{H}_{\mathbf{G}}$ with $\|\phi_\epsilon\|_{1,\alpha} \leq 1$ such that

$$\int_\Sigma e^{4\pi\ell\phi_\epsilon^2} dv_g > \mathrm{Vol}_g(\Sigma) + \pi\ell e^{1+4\pi\ell A_{x_0}},$$

provided that $\epsilon > 0$ is chosen sufficiently small. Combining the above two estimates, we get a contradiction, which implies that u_ϵ must be uniformly bounded. Then applying elliptic estimates to the equation of u_ϵ, we get a desired extremal function.

In the remaining part of this paper, we shall prove Theorems 2.1 and 2.2. Throughout this paper, we do not distinguish sequence and subsequence. Moreover we often denote various constants by the same C, but the dependence of C will be given only if necessary. Also we use symbols $|O(R\epsilon)| \le CR\epsilon$, $o_\epsilon(1) \to 0$ as $\epsilon \to 0$, $o_\delta(1) \to 0$ as $\delta \to 0$, and so on.

3. Proof of Theorem 2.1

In this section, we shall prove Theorem 2.1. In the first subsection, we show that the best constant in (2.4) is equal to $4\pi\ell$. The essential tools we use are subcritical Trudinger-Moser inequality and Moser's sequence of functions. Also we prove (ii) of Theorem 2.1. In the second subsection, we consider the existence of maximizers for subcritical Trudinger-Moser functionals and study their energy concentration phenomenon. In the third subsection, assuming blow-up occurs, we derive an upper bound of the supremum in (2.4), which obviously leads to (i) of Theorem 2.1. In the final subsection, we construct a sequence of test functions to show that the upper bound we obtained in the third subsection is not really an upper bound. Therefore blow-up can not occur and elliptic estimates lead to existence of extremal function. This concludes (iii) of Theorem 2.1.

3.1. The best constant

In view of (2.2), one can see that $\lambda_1^{\mathbf{G}} > 0$ by using a direct method of variation. For any fixed $\alpha < \lambda_1^{\mathbf{G}}$, if $u \in \mathscr{H}_{\mathbf{G}}$ satisfies $\|u\|_{1,\alpha} \le 1$, then $\|\nabla_g u\|_2^2 \le \lambda_1^{\mathbf{G}}/(\lambda_1^{\mathbf{G}} - \alpha)$. By Fontana's inequality (1.6), there exists a positive constant β_0 depending only on $\lambda_1^{\mathbf{G}}$ and α such that

$$\sup_{u\in\mathscr{H}_{\mathbf{G}},\, \|u\|_{1,\alpha}\le 1} \int_\Sigma e^{\beta_0 u^2} dv_g < \infty.$$

Now we define

$$\beta^* = \sup\left\{\beta : \sup_{u\in\mathscr{H}_{\mathbf{G}},\, \|u\|_{1,\alpha}\le 1} \int_\Sigma e^{\beta u^2} dv_g < \infty\right\}. \tag{3.1}$$

Lemma 3.1. *Let ℓ and β^* be defined as in* (2.3) *and* (3.1) *respectively. Then $\beta^* = 4\pi\ell$.*

Proof. We divide the proof into two steps.

Step 1. *There holds* $\beta^* \leq 4\pi\ell$. In view of (2.3), there exists some point $x_0 \in \Sigma$ satisfying $\ell = \sharp\mathbf{G}(x_0) = \sharp\{\sigma_1(x_0), \cdots, \sigma_N(x_0)\}$. Without loss of generality, we assume that $\sigma_1 = Id$ is the identity map, and that $\mathbf{G}(x_0) = \{\sigma_i(x_0)\}_{i=1}^{\ell}$. Take

$$r_0 = \frac{1}{4} \min_{1\leq i<j\leq\ell} d_g\big(\sigma_i(x_0), \sigma_j(x_0)\big),$$

where $d_g(\sigma_i(x_0), \sigma_j(x_0))$ denotes the Riemannian distance between $\sigma_i(x_0)$ and $\sigma_j(x_0)$. Since every $\sigma_i : \Sigma \to \Sigma$ is an isometric map, we can see that for all $0 < r \leq r_0$,

$$B_r(\sigma_i(x_0)) = \sigma_i(B_r(x_0)), \quad 1 \leq i \leq \ell, \tag{3.2}$$

where $B_r(x)$ stands for the geodesic ball centered at $x \in \Sigma$ with radius r.

Fixing $p \in \Sigma$, $k \in \mathbb{N}$ and $0 < r \leq r_0$, we take a sequence of Moser functions by

$$M_{p,k} = M_{p,k}(x, r) = \begin{cases} \log k & \text{when} \rho \leq rk^{-1/4} \\ 4\log\dfrac{r}{\rho} & \text{when} rk^{-1/4} < \rho \leq r \\ 0 & \text{when} \rho > r, \end{cases} \tag{3.3}$$

where ρ denotes the Riemannian distance between x and p. Define

$$\widetilde{M}_k = \widetilde{M}_k(x, r) = \begin{cases} M_{\sigma_i(x_0),k}(x, r) & x \in B_{r_0}(\sigma_i(x_0)),\ 1 \leq i \leq \ell \\ 0, & x \in \Sigma \setminus \cup_{i=1}^{\ell} B_{r_0}(\sigma_i(x_0)). \end{cases} \tag{3.4}$$

If $x \in B_{r_0}(\sigma_i(x_0))$ for some i, then it follows from (3.2) that for any $j = 1, \cdots, N$, $\sigma_j(x) \in B_r(\sigma_j(\sigma_i(x_0)))$ and $d_g(\sigma_j(x), \sigma_j(\sigma_i(x_0))) = d_g(x, \sigma_i(x_0))$. In view of (3.3) and (3.4), one can easily check that

$$\widetilde{M}_k(\sigma_j(x), r) = \widetilde{M}_k(x, r),\ \forall x \in B_{r_0}(\sigma_i(x_0)),\ 1 \leq i \leq \ell,\ 1 \leq j \leq N. \tag{3.5}$$

If $x \in \Sigma \setminus \cup_{i=1}^{\ell} B_{r_0}(\sigma_i(x_0))$, then $\sigma_j(x) \in \Sigma \setminus \cup_{i=1}^{\ell} B_{r_0}(\sigma_i(x_0))$, and thus $\widetilde{M}_k(\sigma_j(x), r) = 0$ for $j = 1, \cdots, N$. This together with (3.5) leads to

$$\widetilde{M}_k\big(\sigma_j(x), r\big) = \widetilde{M}_k(x, r),\ \forall x \in \Sigma,\ 1 \leq j \leq N. \tag{3.6}$$

A straightforward calculation shows

$$\int_\Sigma |\nabla_g \widetilde{M}_k|^2 dv_g = (1 + O(r))8\pi\ell \log k, \tag{3.7}$$

$$\int_\Sigma \widetilde{M}_k^m dv_g = O(1),\ m = 1, 2. \tag{3.8}$$

Denote $\overline{\widetilde{M}_k} = \frac{1}{\mathrm{Vol}_g(\Sigma)} \int_\Sigma \widetilde{M}_k dv_g$ and define

$$M_k^* = M_k^*(x,r) = \frac{\widetilde{M}_k(x,r) - \overline{\widetilde{M}_k}}{\|\widetilde{M}_k - \overline{\widetilde{M}_k}\|_{1,\alpha}}.$$

In view of (3.6), we have $M_k^* \in \mathscr{H}_{\mathbf{G}}$. Note that $\|M_k^*\|_{1,\alpha} = 1$. By (3.7) and (3.8),

$$\|\widetilde{M}_k - \overline{\widetilde{M}_k}\|_{1,\alpha} = (1 + O(r))8\pi\ell \log k + O(1).$$

Hence we have for any $\beta_1 > 4\pi\ell$,

$$\begin{aligned}\int_{B_{r_0}(x_0)} e^{\beta_1 M_k^{*2}} dv_g &\geq \int_{B_{rk^{-1/4}}(x_0)} e^{\beta_1 \frac{(\log k + O(1))^2}{(1+O(r))8\pi\ell \log k + O(1)}} dv_g \\ &= e^{\frac{\beta_1(1+o_k(1))\log k}{(1+O(r))8\pi\ell}} \pi r^2 k^{-1/2}(1 + o_k(1)).\end{aligned}$$

Choosing $r > 0$ sufficiently small and then passing to the limit $k \to \infty$ in the above estimate, we conclude

$$\int_{B_{r_0}(x_0)} e^{\beta_1 M_k^{*2}} dv_g \to \infty \quad \text{as} \quad k \to \infty.$$

Therefore $\beta^* \leq 4\pi\ell$.

Step 2. *There holds* $\beta^* \geq 4\pi\ell$. Suppose $\beta^* < 4\pi\ell$. Then for any $k \in \mathbb{N}$, there is a $u_k \in \mathscr{H}_{\mathbf{G}}$ with $\|u_k\|_{1,\alpha} \leq 1$ such that

$$\int_\Sigma e^{(\beta^* + k^{-1})u_k^2} dv_g \to \infty \quad \text{as} \quad k \to \infty. \tag{3.9}$$

Since $\alpha < \lambda_1^{\mathbf{G}}$, we can see that u_k is bounded in $W^{1,2}(\Sigma, g)$. Up to a subsequence, we can assume that u_k converges to some function u_0 weakly in $W^{1,2}(\Sigma,g)$, strongly in $L^q(\Sigma, g), \forall q > 1$, and for almost every $x \in \Sigma$. Clearly $u_0 \in \mathscr{H}_{\mathbf{G}}$ and $\|u_0\|_{1,\alpha} \leq 1$. We now *claim* that $u_0 \equiv 0$. For otherwise, we have

$$\|u_k - u_0\|_{1,\alpha}^2 \leq 1 - \|u_0\|_{1,\alpha}^2 + o_k(1) \leq 1 - \frac{1}{2}\|u_0\|_{1,\alpha}^2 < 1 \tag{3.10}$$

for sufficiently large k. Given any $\epsilon > 0$. We calculate

$$\begin{aligned}\int_\Sigma e^{(\beta^* + k^{-1})u_k^2} dv_g &\leq \int_\Sigma e^{(\beta^* + k^{-1})(1+\epsilon)(u_k - u_0)^2 + Cu_0^2} dv_g \\ &\leq C\left(\int_\Sigma e^{(\beta^* + k^{-1})(1+2\epsilon)(u_k - u_0)^2} dv_g\right)^{\frac{1+\epsilon}{1+2\epsilon}},\end{aligned} \tag{3.11}$$

where C is a constant depending only on u_0, β^* and ϵ. In view of (3.10), one can find a small $\epsilon > 0$ and a large integer k_0 such that when $k \geq k_0$, there holds

$$\left(\beta^* + k^{-1}\right)(1+2\epsilon)\|u_k - u_0\|_{1,\alpha}^2 \leq \beta^* \left(1 - 8^{-1}\|u_0\|_{1,\alpha}^2\right).$$

This together with (3.11) leads to

$$\int_\Sigma e^{(\beta^*+k^{-1})u_k^2} dv_g \leq C,$$

contradicting (3.9). This confirms our claim $u_0 \equiv 0$.

For any fixed $x \in \Sigma$, we let $I = I(x) = \sharp\mathbf{G}(x)$. Without loss of generality, we assume that $\sigma_1 = Id$ and that $\mathbf{G}(x) = \{\sigma_1(x), \cdots, \sigma_I(x)\}$. There exists sufficiently small $r_1 > 0$ such that $\cap_{i=1}^I B_{r_1}(\sigma_i(x)) = \varnothing$. Since σ_i's are all isometric maps, if $0 < r \leq r_1$, then we have

$$\int_{B_r(\sigma_i(x))} |\nabla_g u_k|^2 dv_g = \int_{B_r(x)} |\nabla_g u_k|^2 dv_g, \quad \forall 1 \leq i \leq I.$$

Noting that $I \geq \ell$, $\|u_k\|_{1,\alpha} \leq 1$ and $u_0 \equiv 0$, we have for $0 < r \leq r_1$,

$$\int_{B_r(x)} |\nabla_g u_k|^2 dv_g \leq \frac{1}{\ell} + o_k(1). \tag{3.12}$$

Let $\zeta \in C_0^1(B_r(x))$, $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on $B_{r/2}(x)$ and $|\nabla_g \zeta| \leq \frac{2}{r}$. This together with (3.12) and $u_0 \equiv 0$ implies that $\zeta u_k \in W_0^{1,2}(B_r(x))$ and

$$\int_{B_r(x)} \left|\nabla_g(\zeta u_k)\right|^2 dv_g \leq \frac{1}{\ell} + o_k(1). \tag{3.13}$$

Take a normal coordinate system $(B_r(x), \exp_x^{-1}; \{y\})$, where $y = (y_1, y_2) \in \mathbb{B}_r(0) \subset \mathbb{R}^2$, and $\exp_x : \mathbb{B}_r(0) \to B_r(x)$ denotes the exponential map. Let $\psi_k(y) = (\zeta u_k)(\exp_x(y))$, $y \in \mathbb{B}_r(0)$. In view of (3.13), one easily gets

$$\begin{aligned}\int_{\mathbb{B}_r(0)} \left|\nabla_{\mathbb{R}^2}\psi_k(y)\right|^2 dy &= (1+O(r))\int_{B_r(x)} \left|\nabla_g(\zeta u_k)\right|^2 dv_g \\ &\leq (1+O(r))\left(\frac{1}{\ell} + o_k(1)\right),\end{aligned} \tag{3.14}$$

where $\nabla_{\mathbb{R}^2}$ denotes the usual gradient operator in $\mathbb{R}^2$. Also there holds $\psi_k \in W_0^{1,2}(\mathbb{B}_r(0))$ since $\zeta u_k \in W_0^{1,2}(B_r(x))$. Hence, if $K \in \mathbb{N}$ is chosen sufficiently large and $r > 0$ is chosen sufficiently small, it then follows from (3.14) and Moser's inequality (1.1) that

$$\begin{aligned}\int_{B_{r/2}(x)} e^{(\beta^*+k^{-1})u_k^2} dv_g &\leq \int_{B_r(x)} e^{(\beta^*+k^{-1})(\zeta u_k)^2} dv_g \\ &= (1+O(r)) \int_{\mathbb{B}_r(0)} e^{(\beta^*+k^{-1})\psi_k^2} dy \\ &\leq C\end{aligned}$$

for some constant C and all $k \geq K$. Since (Σ, g) is compact, there exists some constant C such that for all $k \geq K$,

$$\int_\Sigma e^{(\beta^*+k^{-1})u_k^2} dv_g \leq C.$$

This contradicts (3.9) again. Hence $\beta^* \geq 4\pi\ell$.

We finish the proof of the lemma by combining Steps 1 and 2. □

We now clarify the proof of (ii) of Theorem 2.1, which is partially implied by Lemma 3.1.

Proof of (ii) *of Theorem* 2.1. If $\alpha < \lambda_1^{\mathbf{G}}$ and $\beta > 4\pi\ell$, then Step 1 of the proof of Lemma 3.1 gives the desired result. In the following, we assume $\alpha \geq \lambda_1^{\mathbf{G}}$ and $\beta > 0$. By a direct method of variation, one can find a function $u_0 \not\equiv 0$ satisfying $u_0 \in \mathscr{H}_{\mathbf{G}} \cap C^1(\Sigma)$ and

$$\int_\Sigma |\nabla_g u_0|^2 dv_g = \lambda_1^{\mathbf{G}} \int_\Sigma u_0^2 dv_g.$$

For any $t \in \mathbb{R}$, we have $tu_0 \in \mathscr{H}_{\mathbf{G}}$ and

$$\int_\Sigma |\nabla_g (tu_0)|^2 dv_g - \alpha \int_\Sigma (tu_0)^2 dv_g \leq 0.$$

Moreover, there holds

$$\int_\Sigma e^{\beta(tu_0)^2} dv_g \to \infty \quad \text{as} \quad t \to \infty.$$

Again this gives the desired result. □

3.2. Maximizers for subcritical functionals

Let $\alpha < \lambda_1^{\mathbf{G}}$. As in ([28, page 3183]), by Lemma 3.1 and a direct method of variation, we can prove that for any $0 < \epsilon < 4\pi\ell$, there exists some $u_\epsilon \in \mathscr{H}_{\mathbf{G}}$ with $\|u_\epsilon\|_{1,\alpha} = 1$ such that

$$\int_\Sigma e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g = \sup_{u\in\mathscr{H}_{\mathbf{G}},\ \|u\|_{1,\alpha}\leq 1} \int_\Sigma e^{(4\pi\ell-\epsilon)u^2} dv_g. \tag{3.15}$$

The Euler-Lagrange equation for the maximizer u_ϵ reads

$$\begin{cases} \Delta_g u_\epsilon - \alpha u_\epsilon = \dfrac{1}{\lambda_\epsilon} u_\epsilon e^{(4\pi\ell-\epsilon)u_\epsilon^2} - \dfrac{\mu_\epsilon}{\lambda_\epsilon} \\ u_\epsilon \in \mathscr{H}_{\mathbf{G}},\ \ \|u_\epsilon\|_{1,\alpha} = 1 \\ \lambda_\epsilon = \displaystyle\int_\Sigma u_\epsilon^2 e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g \\ \mu_\epsilon = \dfrac{1}{\mathrm{Vol}_g(\Sigma)} \displaystyle\int_\Sigma u_\epsilon e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g. \end{cases} \tag{3.16}$$

Regularity theory implies that $u_\epsilon \in C^1(\Sigma, g)$. Using an argument of ([28, page 3184]), one has

$$\liminf_{\epsilon\to 0} \lambda_\epsilon > 0, \quad |\mu_\epsilon|/\lambda_\epsilon \leq C. \tag{3.17}$$

By (3.15), one can easily see that

$$\lim_{\epsilon\to 0} \int_\Sigma e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g = \sup_{u\in\mathscr{H}_G,\ \|u\|_{1,\alpha}\leq 1} \int_\Sigma e^{4\pi\ell u^2} dv_g. \tag{3.18}$$

Note that we do *not* assume the supremum on the right hand side of (3.18) is finite. If $|u_\epsilon| \leq C$, in view of (3.17), applying elliptic estimates to (3.16), we obtain $u_\epsilon \to u^*$ in $C^1(\Sigma, g)$, which implies that $u^* \in \mathscr{H}_{\mathbf{G}}$ and $\|u^*\|_{1,\alpha} = 1$. In view of (3.18), we know that u^* is a desired extremal function. From now on, we assume $c_\epsilon = \max_\Sigma |u_\epsilon| \to +\infty$ as $\epsilon \to 0$. Noting that $-u_\epsilon$ also satisfies (3.15) and (3.16), we may assume with no loss of generality that

$$c_\epsilon = \max_\Sigma |u_\epsilon| = \max_\Sigma u_\epsilon = u_\epsilon(x_\epsilon) \to +\infty \tag{3.19}$$

and that

$$x_\epsilon \to x_0 \in \Sigma \quad as \quad \epsilon \to 0. \tag{3.20}$$

To proceed, we need the following energy concentration phenomenon of u_ϵ.

Lemma 3.2. *Under the assumptions* (3.19) *and* (3.20)*, we have*

(i) u_ϵ *converges to* 0 *weakly in* $W^{1,2}(\Sigma, g)$*, strongly in* $L^2(\Sigma, g)$*, and almost everywhere in* Σ*;*

(ii) $I(x_0) = \sharp \boldsymbol{G}(x_0) = \ell$;
(iii) $\lim_{r\to 0}\lim_{\epsilon\to 0} \int_{B_r(x_0)} |\nabla_g u_\epsilon|^2 dv_g = 1/\ell$.

Proof. (i) Since $\alpha < \lambda_1^{\mathbf{G}}$ and $\|u_\epsilon\|_{1,\alpha} = 1$, u_ϵ is bounded in $W^{1,2}(\Sigma, g)$. Hence we may assume u_ϵ converges to u_0 weakly in $W^{1,2}(\Sigma, g)$, strongly in $L^2(\Sigma, g)$, and almost everywhere in Σ. If $u_0 \not\equiv 0$, then

$$\|u_\epsilon - u_0\|_{1,\alpha}^2 = 1 - \|u_0\|_{1,\alpha}^2 + o_\epsilon(1) \leq 1 - \frac{1}{2}\|u_0\|_{1,\alpha}^2,$$

provided that ϵ is sufficiently small. It follows from Lemma 3.1 that $e^{(4\pi\ell-\epsilon)u_\epsilon^2}$ is bounded in $L^q(\Sigma, g)$ for some $q > 1$. Then applying elliptic estimates to (3.16), we have that $\|u_\epsilon\|_{L^\infty(\Sigma)} \leq C$, which contradicts (3.19). Therefore $u_0 \equiv 0$.

(ii) Since $\ell = \min_{x\in\Sigma} I(x)$, we have $I(x_0) \geq \ell$. Suppose $I = I(x_0) > \ell$. Using the same argument as we derived (3.12), we have

$$\int_{B_r(x_0)} |\nabla_g u_\epsilon|^2 dv_g \leq \frac{1}{I} + o_\epsilon(1), \tag{3.21}$$

provided that $r > 0$ is chosen sufficiently small. Similar to (3.15), it follows from (3.21) and Moser's inequality (1.1) that

$$\int_{B_{r/2}(x_0)} e^{4\pi\ell p u_\epsilon^2} dv_g \leq C$$

for some sufficiently small $r > 0$ and some $p > 1$, where C is a constant depending only on r, p, I and ℓ. Applying elliptic estimates to (3.16), we have that u_ϵ is uniformly bounded in $B_{r/4}(x_0)$. This contradicts (3.19). Therefore $I(x_0) = \ell$.

(iii) By (ii), there exists some $r_0 > 0$ such that $\|\nabla_g u_\epsilon\|_{L^2(B_{r_0}(x_0))}^2 \leq \frac{1}{\ell} + o_\epsilon(1)$. It follows that

$$\lim_{r\to 0}\lim_{\epsilon\to 0} \int_{B_r(x_0)} \left|\nabla_g u_\epsilon\right|^2 dv_g \leq \frac{1}{\ell}. \tag{3.22}$$

We *claim* that the equality of (3.22) holds. For otherwise, there exist two positive constants ν and r_1 with $0 < r_1 < r_0$ such that

$$\int_{B_{r_1}(x_0)} \left|\nabla_g u_\epsilon\right|^2 dv_g < \frac{1}{\ell} - \nu.$$

Similarly as we did in the proof of (ii), we have that $e^{(4\pi\ell-\epsilon)u_\epsilon^2}$ is bounded in $L^q(B_{r_1/2}(x_0))$ for some $q > 1$. Then applying elliptic estimates to (3.16), we obtain that u_ϵ is uniformly bounded in $B_{r_1/4}(x_0)$, which contradicts (3.19). This concludes our claim and (iii) holds. □

3.3. Blow-up analysis

Set

$$r_\epsilon = \frac{\sqrt{\lambda_\epsilon}}{c_\epsilon} e^{-(2\pi\ell - \epsilon/2)c_\epsilon^2}. \tag{3.23}$$

For any $0 < a < 4\pi\ell$, by Lemma 3.1, the Hölder inequality and (i) of Lemma 3.2, one has

$$\lambda_\epsilon = \int_\Sigma u_\epsilon^2 e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g = e^{ac_\epsilon^2} \int_\Sigma u_\epsilon^2 e^{(4\pi\ell-\epsilon-a)u_\epsilon^2} dv_g \le e^{ac_\epsilon^2} o_\epsilon(1).$$

It then follows that

$$r_\epsilon^2 c_\epsilon^2 e^{(4\pi\ell-\epsilon-a)c_\epsilon^2} = o_\epsilon(1). \tag{3.24}$$

In particular, $r_\epsilon \to 0$ as $\epsilon \to 0$. Let $0 < \delta < \frac{1}{2} i_g(\Sigma)$ be fixed, where $i_g(\Sigma)$ is the injectivity radius of (Σ, g). For $y \in \mathbb{B}_{\delta r_\epsilon^{-1}}(0) \subset \mathbb{R}^2$, we define $\psi_\epsilon(y) = c_\epsilon^{-1} u_\epsilon(\exp_{x_\epsilon}(r_\epsilon y))$, $\varphi_\epsilon(y) = c_\epsilon(u_\epsilon(\exp_{x_\epsilon}(r_\epsilon y)) - c_\epsilon)$ and $g_\epsilon(y) = (\exp_{x_\epsilon}^* g)(r_\epsilon y)$, where $\mathbb{B}_{\delta r_\epsilon^{-1}}(0)$ is the Euclidean ball of radius δr_ϵ^{-1} centered at 0, and $\exp_{x_\epsilon}$ is the exponential map at x_ϵ. Note that g_ϵ converges to g_0 in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ as $\epsilon \to 0$, where g_0 denotes the standard Euclidean metric. By (3.16), we have on $\mathbb{B}_{\delta r_\epsilon^{-1}}(0)$,

$$\Delta_{g_\epsilon} \psi_\epsilon(y) = \alpha r_\epsilon^2 \psi_\epsilon(y) + c_\epsilon^{-2} \psi_\epsilon(y) e^{(4\pi\ell-\epsilon)(u_\epsilon^2(\exp_{x_\epsilon}(r_\epsilon y)) - c_\epsilon^2)} - r_\epsilon^2 c_\epsilon^{-1} \frac{\mu_\epsilon}{\lambda_\epsilon} \tag{3.25}$$

$$\Delta_{g_\epsilon} \varphi_\epsilon(y) = \alpha r_\epsilon^2 c_\epsilon^2 \psi_\epsilon(y) + \psi_\epsilon(y) e^{(4\pi\ell-\epsilon)(u_\epsilon^2(\exp_{x_\epsilon}(r_\epsilon y)) - c_\epsilon^2)} - r_\epsilon^2 c_\epsilon \frac{\mu_\epsilon}{\lambda_\epsilon}. \tag{3.26}$$

In view of (3.24), applying elliptic estimates to (3.25) and (3.26) respectively, we have

$$\psi_\epsilon \to 1 \quad \text{in} \quad C^1_{\mathrm{loc}}(\mathbb{R}^2), \tag{3.27}$$

and

$$\varphi_\epsilon \to \varphi \quad \text{in} \quad C^1_{\mathrm{loc}}(\mathbb{R}^2), \tag{3.28}$$

where φ satisfies

$$\begin{cases} -\Delta_{\mathbb{R}^2} \varphi = e^{8\pi\ell\varphi} & \text{in } \mathbb{R}^2 \\ \varphi(0) = 0 = \sup_{\mathbb{R}^2} \varphi \\ \int_{\mathbb{R}^2} e^{8\pi\ell\varphi(y)} dy < \infty. \end{cases}$$

By a result of Chen-Li [5], we have

$$\varphi(y) = -\frac{1}{4\pi\ell} \log\left(1 + \pi\ell|y|^2\right),$$

which leads to

$$\int_{\mathbb{R}^2} e^{8\pi\ell\varphi(y)} dy = \frac{1}{\ell}. \tag{3.29}$$

By (3.23), (3.27) and (3.28), there holds for any $R > 0$,

$$\begin{aligned}\int_{\mathbb{B}_R(0)} e^{4\pi\ell\varphi(y)}dy &= \lim_{\epsilon\to 0}\int_{\mathbb{B}_R(0)} e^{(4\pi\ell-\epsilon)(u_\epsilon^2(\exp_{x_\epsilon}(r_\epsilon y))-c_\epsilon^2)}dy \\ &= \lim_{\epsilon\to 0}\frac{c_\epsilon^2}{\lambda_\epsilon}\int_{B_{Rr_\epsilon}(x_\epsilon)} e^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g \\ &= \lim_{\epsilon\to 0}\frac{1}{\lambda_\epsilon}\int_{B_{Rr_\epsilon}(x_\epsilon)} u_\epsilon^2 e^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g.\end{aligned}$$

This together with (3.29) gives

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\frac{1}{\lambda_\epsilon}\int_{B_{Rr_\epsilon}(x_\epsilon)} u_\epsilon^2 e^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g = \frac{1}{\ell}. \tag{3.30}$$

By (ii) of Lemma 3.2 and (3.20), one has for all sufficiently small $\epsilon > 0$,

$$\cap_{i=1}^{\ell} B_{Rr_\epsilon}(\sigma_i(x_\epsilon)) = \varnothing. \tag{3.31}$$

Noting that $u_\epsilon \in \mathscr{H}_{\mathbf{G}}$, we have

$$\int_{B_{Rr_\epsilon}(\sigma_i(x_\epsilon))} u_\epsilon^2 e^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g = \int_{B_{Rr_\epsilon}(x_\epsilon)} u_\epsilon^2 e^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g, \quad 1 \le i \le \ell.$$

This together with (3.30) and (3.31) leads to

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\frac{1}{\lambda_\epsilon}\int_{B_{Rr_\epsilon}(\sigma_i(x_\epsilon))} u_\epsilon^2 e^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g = \frac{1}{\ell}, \quad 1 \le i \le \ell. \tag{3.32}$$

By definition of λ_ϵ in (3.16), we conclude from (3.32) that

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\frac{1}{\lambda_\epsilon}\int_{\Sigma\setminus\cup_{i=1}^{\ell} B_{Rr_\epsilon}(\sigma_i(x_\epsilon))} u_\epsilon^2 e^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g = 0. \tag{3.33}$$

Similar to [1,11], $\forall 0 < \beta < 1$, we let $u_{\epsilon,\beta} = \min\{u_\epsilon, \beta c_\epsilon\}$.

Lemma 3.3. $\forall\, 0 < \beta < 1$, *there holds*

$$\lim_{\epsilon\to 0}\int_\Sigma |\nabla_g u_{\epsilon,\beta}|^2 dv_g = \beta.$$

Proof. Multiplying (3.16) by $u_{\epsilon,\beta}$, we have

$$\begin{aligned}\int_\Sigma |\nabla_g u_{\epsilon,\beta}|^2 dv_g &= \int_\Sigma \nabla_g u_{\epsilon,\beta} \nabla u_\epsilon dv_g \\ &= \frac{1}{\lambda_\epsilon}\int_\Sigma u_{\epsilon,\beta} u_\epsilon e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g \\ &\quad + \alpha \int_\Sigma u_{\epsilon,\beta} u_\epsilon dv_g - \frac{\mu_\epsilon}{\lambda_\epsilon}\int_\Sigma u_{\epsilon,\beta} dv_g \\ &= \frac{1}{\lambda_\epsilon}\sum_{i=1}^{\ell}\int_{B_{Rr_\epsilon}(\sigma_i(x_\epsilon))} u_{\epsilon,\beta} u_\epsilon e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g \\ &\quad + \frac{1}{\lambda_\epsilon}\int_{\Sigma\setminus\cup_{i=1}^{\ell} B_{Rr_\epsilon}(\sigma_i(x_\epsilon))} u_{\epsilon,\beta} u_\epsilon e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g + o_\epsilon(1).\end{aligned} \tag{3.34}$$

Note that $0 \le u_{\epsilon,\beta} u_\epsilon \le u_\epsilon^2$ on Σ, and $u_{\epsilon,\beta} = \beta(1+o_\epsilon(1))u_\epsilon$ on $B_{Rr_\epsilon}(\sigma_i(x_\epsilon))$ for $1 \le i \le \ell$. In view of (3.30), (3.33) and (3.34), letting $\epsilon \to 0$ first and then $R \to \infty$, we conclude the lemma. □

Lemma 3.4. *There holds* $\liminf_{\epsilon\to 0} \lambda_\epsilon / c_\epsilon^2 > 0$.

Proof. Let $0 < \beta < 1$. In view of Lemma 3.3, we have by using the Hölder inequality

$$\int_{u_\epsilon \le \beta c_\epsilon} u_\epsilon^2 e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g \le \int_\Sigma u_\epsilon^2 e^{(4\pi\ell-\epsilon)u_{\epsilon,\beta}^2} dv_g = o_\epsilon(1).$$

Similarly

$$\begin{aligned}\frac{\lambda_\epsilon}{c_\epsilon^2} &\ge \beta^2 \int_{u_\epsilon > \beta c_\epsilon} e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g + o_\epsilon(1) \\ &\ge \beta^2 \left(\int_\Sigma e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g - \int_\Sigma e^{(4\pi\ell-\epsilon)u_{\epsilon,\beta}^2} dv_g\right) + o_\epsilon(1) \\ &= \beta^2 \int_\Sigma (e^{(4\pi\ell-\epsilon)u_\epsilon^2} - 1) dv_g + o_\epsilon(1).\end{aligned} \tag{3.35}$$

This together with (3.18) ends the proof of the lemma. □

Lemma 3.5. *For any* $1<q<2$, *we have* $c_\epsilon u_\epsilon$ *converges to* G *weakly in* $W^{1,q}(\Sigma, g)$, *strongly in* $L^{2q/(2-q)}(\Sigma)$, *and almost everywhere in* Σ, *where* G *is a Green function satisfying*

$$\begin{cases} \Delta_g G - \alpha G = \dfrac{1}{\ell}\displaystyle\sum_{i=1}^{\ell} \delta_{\sigma_i(x_0)} - \dfrac{1}{\mathrm{Vol}_g(\Sigma)} \\ \displaystyle\int_\Sigma G dv_g = 0 \\ G(\sigma_i(x)) = G(x),\ x \in \Sigma \setminus \left\{\sigma_j(x_0)\right\}_{j=1}^{\ell},\ 1 \le i \le \ell. \end{cases} \tag{3.36}$$

Proof. By (3.16),

$$\Delta_g(c_\epsilon u_\epsilon) - \alpha(c_\epsilon u_\epsilon) = h_\epsilon = \frac{1}{\lambda_\epsilon} c_\epsilon u_\epsilon e^{(4\pi\ell-\epsilon)u_\epsilon^2} - \frac{c_\epsilon \mu_\epsilon}{\lambda_\epsilon}. \tag{3.37}$$

It follows from Lemmas 3.3 and 3.4 that for any $0 < \beta < 1$,

$$\begin{aligned}\int_\Sigma \frac{c_\epsilon}{\lambda_\epsilon}|u_\epsilon| e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g &= \frac{c_\epsilon}{\lambda_\epsilon}\int_{u_\epsilon \le \beta c_\epsilon} |u_\epsilon| e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g \\ &\quad + \frac{c_\epsilon}{\lambda_\epsilon}\int_{u_\epsilon > \beta c_\epsilon} u_\epsilon e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g \\ &\le \frac{c_\epsilon}{\lambda_\epsilon}\int_\Sigma |u_\epsilon| e^{(4\pi\ell-\epsilon)u_{\epsilon,\beta}^2} dv_g + \frac{1}{\beta} \\ &\le \frac{1}{\beta} + o_\epsilon(1),\end{aligned}$$

and that

$$\begin{aligned}\frac{c_\epsilon|\mu_\epsilon|}{\lambda_\epsilon} &\le \frac{1}{\mathrm{Vol}_g(\Sigma)}\frac{c_\epsilon}{\lambda_\epsilon}\int_{u_\epsilon \le \beta c_\epsilon} |u_\epsilon| e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g \\ &\quad + \frac{1}{\mathrm{Vol}_g(\Sigma)}\frac{c_\epsilon}{\lambda_\epsilon}\int_{u_\epsilon > \beta c_\epsilon} u_\epsilon e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g \\ &\le \frac{1}{\mathrm{Vol}_g(\Sigma)}\frac{1}{\beta} + o_\epsilon(1).\end{aligned}$$

Hence h_ϵ is bounded in $L^1(\Sigma, g)$. Then by ([31, Lemma 2.11]), we have $c_\epsilon u_\epsilon$ is bounded in $W^{1,q}(\Sigma, g)$ for any $1 < q < 2$. Up to a subsequence, for any $1 < q < 2$ and $1 < s \le 2q/(2-q)$, $c_\epsilon u_\epsilon$ converges to G weakly in $W^{1,q}(\Sigma)$, strongly in $L^s(\Sigma, g)$, and almost everywhere in Σ.

We calculate

$$\int_{u_\epsilon \le \beta c_\epsilon} \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g = o_\epsilon(1), \tag{3.38}$$

$$\begin{aligned}&\int_{\{u_\epsilon > \beta c_\epsilon\}\setminus \cup_{i=1}^{\ell} B_{Rr_\epsilon}(\sigma_i(x_\epsilon))} \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g \\ \le& \frac{1}{\beta}\frac{1}{\lambda_\epsilon}\int_{\Sigma\setminus \cup_{i=1}^{\ell} B_{Rr_\epsilon}(\sigma_i(x_\epsilon))} u_\epsilon^2 e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g = o(1),\end{aligned} \tag{3.39}$$

$$\begin{aligned}&\int_{B_{Rr_\epsilon}(\sigma_i(x_\epsilon))} \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g \\ =& \frac{1+o_\epsilon(1)}{\lambda_\epsilon}\int_{B_{Rr_\epsilon}(\sigma_i(x_\epsilon))} u_\epsilon^2 e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g = \frac{1}{\ell} + o(1),\ 1 \le i \le \ell,\end{aligned} \tag{3.40}$$

where $o(1) \to 0$ as $\epsilon \to 0$ first and then $R \to \infty$. Integrating the equation (3.37), we have by combining (3.38)-(3.40),

$$\frac{c_\epsilon \mu_\epsilon}{\lambda_\epsilon}\mathrm{Vol}_g(\Sigma) = \int_\Sigma \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g = 1 + o_\epsilon(1).$$

In view of (3.38)-(3.40) again, testing the equation (3.37) by $\phi \in C^2(\Sigma)$ and passing to the limit $\epsilon \to 0$, we have

$$\int_\Sigma G\Delta_g \phi dv_g - \alpha \int_\Sigma G\phi dv_g = \frac{1}{\ell}\sum_{i=1}^{\ell} \phi(\sigma_i(x_0)) - \frac{1}{\mathrm{Vol}_g(\Sigma)}\int_\Sigma \phi dv_g.$$

Since $c_\epsilon u_\epsilon \in \mathscr{H}_{\mathbf{G}}$, we have $\int_\Sigma G dv_g = 0$ and $G(\sigma_i(x)) = G(x)$ for all $x \in \Sigma \setminus \{\sigma_1(x_0), \cdots, \sigma_\ell(x_0)\}$ and all $1 \le i \le \ell$. □

Let

$$\psi(x) = G(x) + \frac{1}{2\pi\ell}\sum_{i=1}^{\ell} \log d_g(\sigma_i(x_0), x).$$

It follows from (3.36) that the distributional Laplacian of ψ belongs to $L^s(\Sigma, g)$ for some $s > 2$. Then we have by elliptic estimates that $\psi \in C^1(\Sigma, g)$. Let $r_0 = \frac{1}{4}\min_{1\le i<j\le \ell} d_g(\sigma_i(x_0), \sigma_j(x_0))$. For $x \in B_{r_0}(x_0)$, the Green function G can be decomposed as

$$G(x) = -\frac{1}{2\pi\ell}\log d_g(x, x_0) + A_{x_0} + \widetilde{\psi}(x), \tag{3.41}$$

where $\widetilde{\psi} \in C^1(\overline{B_{r_0}(x_0)})$, $\widetilde{\psi}(x_0) = 0$ and

$$\begin{aligned} A_{x_0} &= \lim_{x\to x_0}\left(G(x) + \frac{1}{2\pi\ell}\log d_g(x, x_0)\right) \\ &= \lim_{x\to x_0}\left(\psi(x) - \frac{1}{2\pi\ell}\sum_{i=2}^{\ell}\log d_g(\sigma_i(x_0), x)\right). \end{aligned} \tag{3.42}$$

By (3.36), we have

$$\begin{aligned} \int_{\Sigma\setminus\cup_{i=1}^{\ell} B_\delta(\sigma_i(x_0))} |\nabla_g G|^2 dv_g &= \alpha \int_{\Sigma\setminus\cup_{i=1}^{\ell} B_\delta(\sigma_i(x_0))} G^2 dv_g - \int_{\cup_{i=1}^{\ell}\partial B_\delta(\sigma_i(x_0))} G\frac{\partial G}{\partial \nu} d\sigma \\ &\quad - \frac{1}{\mathrm{Vol}_g(\Sigma)}\int_{\Sigma\setminus\cup_{i=1}^{\ell} B_\delta(\sigma_i(x_0))} G dv_g \\ &= -\frac{1}{2\pi\ell}\log\delta + A_{x_0} + \alpha\int_\Sigma G^2 dv_g + o_\delta(1). \end{aligned}$$

Hence

$$\int_{\Sigma\setminus\cup_{i=1}^{\ell}B_\delta(\sigma_i(x_0))}|\nabla_g u_\epsilon|^2dv_g$$
$$=\frac{1}{c_\epsilon^2}\left(-\frac{1}{2\pi\ell}\log\delta+A_{x_0}+\alpha\int_\Sigma G^2dv_g+o_\delta(1)+o_\epsilon(1)\right).$$

It follows that

$$\int_{\cup_{i=1}^{\ell}B_\delta(\sigma_i(x_0))}|\nabla_g u_\epsilon|^2dv_g=1+\alpha\int_\Sigma u_\epsilon^2dv_g-\int_{\Sigma\setminus\cup_{i=1}^{\ell}B_\delta(\sigma_i(x_0))}|\nabla_g u_\epsilon|^2dv_g$$
$$=1-\frac{1}{c_\epsilon^2}\left(-\frac{1}{2\pi\ell}\log\delta+A_{x_0}+o_\delta(1)+o_\epsilon(1)\right).$$

Let $s_\epsilon=\sup_{\partial B_\delta(x_0)}u_\epsilon$ and $\widetilde{u}_\epsilon=(u_\epsilon-s_\epsilon)^+$. Then $\widetilde{u}_\epsilon\in W_0^{1,2}(B_\delta(x_0))$, and satisfies

$$\int_{\cup_{i=1}^{\ell}B_\delta(\sigma_i(x_0))}|\nabla_g\widetilde{u}_\epsilon|^2dv_g\le\tau_\epsilon=1-\frac{1}{c_\epsilon^2}\left(-\frac{1}{2\pi\ell}\log\delta+A_{x_0}+o_\delta(1)+o_\epsilon(1)\right).$$

Now we choose an isothermal coordinate system $(U,\phi;\{x^1,x^2\})$ near x_0 such that $B_{2\delta}(x_0)\subset U$, $\phi(x_0)=0$, and the metric $g=e^h({dx^1}^2+{dx^2}^2)$ for some function $h\in C^1(\phi(U))$ with $h(0)=0$. Clearly, for any $\delta>0$, there exists some $c(\delta)>0$ with $c(\delta)\to 0$ as $\delta\to 0$ such that $\sqrt{g}\le 1+c(\delta)$ and $\phi(B_\delta(p))\subset\mathbb{B}_{\delta(1+c(\delta))}(0)\subset\mathbb{R}^2$. Noting that $\widetilde{u}_\epsilon=0$ outside $B_\delta(p)$ for sufficiently small δ, we have

$$\int_{\mathbb{B}_{\delta(1+c(\delta))}(0)}|\nabla_{\mathbb{R}^2}(\widetilde{u}_\epsilon\circ\phi^{-1})|^2dx=\int_{\phi^{-1}(\mathbb{B}_{\delta(1+c(\delta))}(0))}|\nabla_g\widetilde{u}_\epsilon|^2dv_g$$
$$=\int_{B_\delta(x_0)}|\nabla_g\widetilde{u}_\epsilon|dv_g\le\frac{\tau_\epsilon}{\ell}.$$

This together with a result of Carleson-Chang [3] leads to

$$\begin{aligned}&\limsup_{\epsilon\to 0}\int_{B_\delta(p)}(e^{4\pi\ell\widetilde{u}_\epsilon^2/\tau_\epsilon}-1)dv_g\\&\le\limsup_{\epsilon\to 0}(1+c(\delta))\int_{\mathbb{B}_{\delta(1+c(\delta))}(0)}(e^{4\pi\ell(\widetilde{u}_\epsilon\circ\phi^{-1})^2/\tau_\epsilon}-1)dx\\&\le\pi\delta^2(1+c(\delta))^3e.\end{aligned}\tag{3.43}$$

Note that $|u_\epsilon|\le c_\epsilon$ and $u_\epsilon/c_\epsilon=1+o_\epsilon(1)$ on the geodesic ball $B_{Rr_\epsilon}(x_\epsilon)\subset\Sigma$. We estimate on $B_{Rr_\epsilon}(x_\epsilon)$,

$$\begin{aligned}(4\pi\ell-\epsilon)u_\epsilon^2&\le 4\pi\ell(\widetilde{u}_\epsilon+s_\epsilon)^2\\&\le 4\pi\ell\widetilde{u}_\epsilon^2+8\pi\ell s_\epsilon\widetilde{u}_\epsilon+o_\epsilon(1)\\&\le 4\pi\ell\widetilde{u}_\epsilon^2-4\log\delta+8\pi\ell A_{x_0}+o(1)\\&\le 4\pi\ell\widetilde{u}_\epsilon^2/\tau_\epsilon-2\log\delta+4\pi\ell A_{x_0}+o(1).\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{B_{Rr_\epsilon}(x_\epsilon)} e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g \\
&\le \delta^{-2} e^{4\pi\ell A_{x_0}+o(1)} \int_{B_{Rr_\epsilon}(x_\epsilon)} e^{4\pi\ell\widetilde{u}_\epsilon^2/\tau_\epsilon} dv_g \\
&= \delta^{-2} e^{4\pi\ell A_{x_0}+o(1)} \int_{B_{Rr_\epsilon}(x_\epsilon)} (e^{4\pi\ell\widetilde{u}_\epsilon^2/\tau_\epsilon} - 1) dv_g + o(1) \\
&\le \delta^{-2} e^{4\pi\ell A_{x_0}+o(1)} \int_{B_\delta(x_0)} (e^{4\pi\ell\widetilde{u}_\epsilon^2/\tau_\epsilon} - 1) dv_g + o(1),
\end{aligned} \tag{3.44}$$

where $o(1) \to 0$ as $\epsilon \to 0$ first and then $\delta \to 0$. Combining (3.43) with (3.44), letting $\epsilon \to 0$ first, and then letting $\delta \to 0$, we conclude

$$\limsup_{\epsilon\to 0} \int_{B_{Rr_\epsilon}(x_\epsilon)} e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g \le \pi e^{1+4\pi\ell A_{x_0}}.$$

Therefore

$$\limsup_{\epsilon\to 0} \int_{\cup_{i=1}^{\ell} B_{Rr_\epsilon}(\sigma_i(x_\epsilon))} e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g \le \pi\ell e^{1+4\pi\ell A_{x_0}}. \tag{3.45}$$

Proposition 3.6. *Under the assumptions* (3.19) *and* (3.20)*, there holds*

$$\sup_{u\in\mathscr{H}_G,\, \|u\|_{1,\alpha}\le 1} \int_\Sigma e^{4\pi\ell u^2} dv_g = \lim_{\epsilon\to 0} \int_\Sigma e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g \le \mathrm{Vol}_g(\Sigma) + \pi\ell e^{1+4\pi\ell A_{x_0}}.$$

Proof. We calculate

$$\begin{aligned}
\int_{B_{Rr_\epsilon}(x_\epsilon)} e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g &= (1+o_\epsilon(1)) \int_{\mathbb{B}_R(0)} e^{(4\pi\ell-\epsilon)u_\epsilon^2(\exp_{x_\epsilon}(r_\epsilon y))} r_\epsilon^2 dy \\
&= (1+o_\epsilon(1)) \frac{\lambda_\epsilon}{c_\epsilon^2} \left(\int_{\mathbb{B}_R(0)} e^{8\pi\ell\varphi(y)} dy + o_\epsilon(1) \right).
\end{aligned}$$

In view of (3.29) and (3.45),

$$\lim_{R\to\infty} \lim_{\epsilon\to 0} \int_{B_{Rr_\epsilon}(x_\epsilon)} e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g = \frac{1}{\ell} \lim_{\epsilon\to 0} \frac{\lambda_\epsilon}{c_\epsilon^2}.$$

Hence

$$\lim_{R\to\infty} \lim_{\epsilon\to 0} \int_{\cup_{i=1}^{\ell} B_{Rr_\epsilon}(\sigma_i(x_\epsilon))} e^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g = \lim_{\epsilon\to 0} \frac{\lambda_\epsilon}{c_\epsilon^2}. \tag{3.46}$$

By (3.35), we have

$$\lim_{\epsilon\to 0} \int_\Sigma (e^{(4\pi\ell-\epsilon)u_\epsilon^2} - 1) dv_g \le \frac{1}{\beta^2} \lim_{\epsilon\to 0} \frac{\lambda_\epsilon}{c_\epsilon^2}, \quad \forall 0 < \beta < 1.$$

Letting $\beta \to 1$, we obtain

$$\lim_{\epsilon\to 0}\int_\Sigma (e^{(4\pi\ell-\epsilon)u_\epsilon^2}-1)dv_g \le \lim_{\epsilon\to 0}\frac{\lambda_\epsilon}{c_\epsilon^2}.$$

This together with (3.45) and (3.46) completes the proof of the proposition. □

3.4. Test function computation

In this subsection, we shall complete the proof of (iii) of Theorem 2.1. Let $\alpha < \lambda_1^{\mathbf{G}}$ be fixed and ℓ be an integer defined as in (2.3). In particular, we shall construct a function sequence ϕ_ϵ satisfying $\phi_\epsilon \in \mathscr{H}_{\mathbf{G}}$,

$$\int_\Sigma |\nabla_g\phi_\epsilon|^2 dv_g - \alpha\int_\Sigma \phi_\epsilon^2 dv_g = 1 \tag{3.47}$$

and

$$\int_\Sigma e^{4\pi\ell\phi_\epsilon^2}dv_g > \mathrm{vol}_g(\Sigma) + \pi\ell e^{1+4\pi\ell A_{x_0}} \tag{3.48}$$

for sufficiently small $\epsilon > 0$, where x_0 and A_{x_0} are defined as in (3.20) and (3.42) respectively. If there exists such a sequence ϕ_ϵ, then we have by Proposition 3.6 that c_ϵ must be bounded. Applying elliptic estimates to (3.16), we conclude the existence of the desired extremal function.

To do this, we define a sequence of functions by

$$\mathsf{b}_\epsilon(x) = \begin{cases} c + \dfrac{-\frac{1}{4\pi\ell}\log(1+\pi\ell\frac{r^2}{\epsilon^2}) + B}{c} & x \in B_{R\epsilon}(x_0) \\ \dfrac{G - \zeta\widetilde{\psi}}{c} & x \in B_{2R\epsilon}(x_0)\setminus B_{R\epsilon}(x_0), \end{cases} \tag{3.49}$$

where $\widetilde{\psi}$ is defined as in (3.41), $\zeta \in C_0^\infty(B_{2R\epsilon}(x_0))$ satisfies that $\zeta \equiv 1$ on $B_{R\epsilon}(x_0)$ and $\|\nabla_g\zeta\|_{L^\infty} = O(1/(R\epsilon))$, $r = r(x) = \mathrm{dist}_g(x, x_0)$, $R = -\log\epsilon$, B and c are constants depending only on ϵ to be determined later. Define another sequence of functions

$$\eta_\epsilon(x) = \begin{cases} \mathsf{b}_\epsilon(x) & x \in B_{2R\epsilon}(x_0) \\ \mathsf{b}_\epsilon(\sigma_i^{-1}(x)) & x \in B_{2R\epsilon}(\sigma_i(x_0)),\ 2\le i\le \ell \\ \dfrac{G}{c} & x \in \Sigma \setminus \cup_{i=1}^{\ell} B_{2R\epsilon}(\sigma_i(x_0)). \end{cases} \tag{3.50}$$

Noting that $G(\sigma_i(x)) = G(x)$ for all $x \in \Sigma \setminus \{\sigma_1(x_0),\cdots,\sigma_\ell(x_0)\}$, one can easily check that

$$\eta_\epsilon(\sigma_i(x)) = \eta_\epsilon(x),\quad \forall x\in\Sigma,\ \forall 1\le i\le \ell. \tag{3.51}$$

In view of (3.49) and (3.50), in order to ensure that $\eta_\epsilon \in W^{1,2}(\Sigma, g)$, we set

$$c + \frac{1}{c}\left(-\frac{1}{4\pi\ell}\log(1+\pi\ell R^2) + B\right) = \frac{1}{c}\left(-\frac{1}{2\pi\ell}\log(R\epsilon) + A_{x_0}\right),$$

which gives

$$2\pi\ell c^2 = -\log\epsilon - 2\pi\ell B + 2\pi\ell A_{x_0} + \frac{1}{2}\log(\pi\ell) + O\left(\frac{1}{R^2}\right). \tag{3.52}$$

Noting that $\int_\Sigma G dv_g = 0$, we have

$$\begin{aligned}
&\int_{\Sigma\setminus\cup_{i=1}^{\ell} B_{R\epsilon}(\sigma_i(x_0))} |\nabla_g G|^2 dv_g \\
&= \int_{\Sigma\setminus\cup_{i=1}^{\ell} B_{R\epsilon}(\sigma_i(x_0))} G\Delta_g G dv_g - \int_{\cup_{i=1}^{\ell}\partial B_{R\epsilon}(\sigma_i(x_0))} G\frac{\partial G}{\partial \nu} d\sigma \\
&= \alpha\int_{\Sigma\setminus\cup_{i=1}^{\ell} B_{R\epsilon}(\sigma_i(x_0))} G^2 dv_g - \frac{1}{\mathrm{Vol}_g(\Sigma)}\int_{\Sigma\setminus\cup_{i=1}^{\ell} B_{R\epsilon}(\sigma_i(x_0))} G dv_g \\
&\quad - \sum_{i=1}^{\ell}\int_{\partial B_{R\epsilon}(\sigma_i(x_0))} G\frac{\partial G}{\partial\nu} d\sigma \\
&= -\frac{1}{2\pi\ell}\log(R\epsilon) + \alpha\int_\Sigma G^2 dv_g + A_{x_0} + O(R\epsilon\log(R\epsilon)).
\end{aligned} \tag{3.53}$$

Since $\widetilde{\psi} \in C^1(\Sigma, g)$ and $\widetilde{\psi}(x_0) = 0$, we have

$$\int_{B_{2R\epsilon}(x_0)\setminus B_{R\epsilon}(x_0)} |\nabla_g\zeta|^2\widetilde{\psi}^2 dv_g = O((R\epsilon)^2), \tag{3.54}$$

$$\int_{B_{2R\epsilon}(x_0)\setminus B_{R\epsilon}(x_0)} \nabla_g G\nabla_g\zeta\widetilde{\psi} dv_g = O(R\epsilon), \tag{3.55}$$

$$\int_{B_{R\epsilon}(x_0)} |\nabla_g\eta_\epsilon|^2 dv_g = \frac{1}{\ell^2 c^2}\left(\frac{1}{2\pi}\log R + \frac{\log(\pi\ell)}{4\pi} - \frac{1}{4\pi} + O\left(\frac{1}{R^2}\right)\right). \tag{3.56}$$

Combining (3.53)-(3.56) and noting that

$$\int_{\cup_{i=1}^{\ell} B_{R\epsilon}(\sigma_i(x_0))} |\nabla_g\eta_\epsilon|^2 dv_g = \ell\int_{B_{R\epsilon}(x_0)} |\nabla_g\eta_\epsilon|^2 dv_g,$$

we obtain

$$\begin{aligned}
\int_\Sigma |\nabla_g\eta_\epsilon|^2 dv_g = \frac{1}{4\pi\ell c^2}\Big(&2\log\frac{1}{\epsilon} + \log(\pi\ell) - 1 + 4\pi\ell A_{x_0} + 4\pi\ell\alpha\int_\Sigma G^2 dv_g \\
&+ O\left(\frac{1}{R^2}\right) + O(R\epsilon\log(R\epsilon))\Big).
\end{aligned} \tag{3.57}$$

Observing

$$\begin{aligned}\int_\Sigma \eta_\epsilon dv_g &= \frac{1}{c}\left(\int_{\Sigma\setminus\cup_{i=1}^{\ell}B_{2R\epsilon}(\sigma_i(x_0))} G dv_g + O(R\epsilon\log(R\epsilon))\right)\\ &= \frac{1}{c}\left(-\int_{\cup_{i=1}^{\ell}B_{2R\epsilon}(\sigma_i(x_0))} G dv_g + O(R\epsilon\log(R\epsilon))\right)\\ &= \frac{1}{c}O(R\epsilon\log(R\epsilon)),\end{aligned} \tag{3.58}$$

we have

$$\overline{\eta}_\epsilon = \frac{1}{\mathrm{Vol}_g(\Sigma)}\int_\Sigma \eta_\epsilon dv_g = \frac{1}{c}O(R\epsilon\log(R\epsilon)). \tag{3.59}$$

Hence

$$\begin{aligned}\int_\Sigma (\eta_\epsilon-\overline{\eta}_\epsilon)^2 dv_g &= \int_\Sigma \eta_\epsilon^2 dv_g - 2\overline{\eta}_\epsilon\int_\Sigma \eta_\epsilon dv_g + \overline{\eta}_\epsilon^2\mathrm{Vol}_g(\Sigma)\\ &= \frac{1}{c^2}\left(\int_\Sigma G^2 dv_g + O(R\epsilon\log(R\epsilon))\right).\end{aligned}$$

This together with (3.57) yields

$$\begin{aligned}\|\eta_\epsilon-\overline{\eta}_\epsilon\|_{1,\alpha}^2 &= \int_\Sigma |\nabla_g\eta_\epsilon|^2 dv_g - \alpha\int_\Sigma(\eta_\epsilon-\overline{\eta}_\epsilon)^2 dv_g\\ &= \frac{1}{4\pi\ell c^2}\left(2\log\frac{1}{\epsilon}+\log(\pi\ell)-1+4\pi\ell A_{x_0}\right.\\ &\qquad\left.+O\left(\frac{1}{R^2}\right)+O(R\epsilon\log(R\epsilon))\right).\end{aligned} \tag{3.60}$$

Now we choose B in (3.52) such that

$$\|\eta_\epsilon-\overline{\eta}_\epsilon\|_{1,\alpha} = 1. \tag{3.61}$$

Combining (3.60) and (3.61), we have

$$c^2 = -\frac{\log\epsilon}{2\pi\ell}+\frac{\log(\pi\ell)}{4\pi\ell}-\frac{1}{4\pi\ell}+A_{x_0}+O\left(\frac{1}{R^2}\right)+O(R\epsilon\log(R\epsilon)). \tag{3.62}$$

It then follows from (3.52) and (3.62) that

$$B = \frac{1}{4\pi\ell}+O\left(\frac{1}{R^2}\right)+O(R\epsilon\log(R\epsilon)). \tag{3.63}$$

Let

$$\phi_\epsilon = \eta_\epsilon-\overline{\eta}_\epsilon. \tag{3.64}$$

In view of (3.51), (3.64) and the fact that $\eta_\epsilon \in W^{1,2}(\Sigma, g)$, we have $\phi_\epsilon \in \mathscr{H}_{\mathbf{G}}$. Moreover, the equality (3.61) is exactly $\|\phi_\epsilon\|_{1,\alpha} = 1$, and thus (3.47). A straightforward calculation shows on $B_{R\epsilon}(x_0)$,

$$4\pi\ell\phi_\epsilon^2 \geq 4\pi\ell c^2 - 2\log\left(1 + \pi\ell\frac{r^2}{\epsilon^2}\right) + 8\pi\ell B + O(R\epsilon\log(R\epsilon)).$$

This together with (3.62) and (3.63) yields

$$\int_{B_{R\epsilon}(x_0)} e^{4\pi\ell\phi_\epsilon^2} dv_g \geq \pi e^{1+4\pi\ell A_{x_0}} + O\left(\frac{1}{(\log\epsilon)^2}\right),$$

which immediately leads to

$$\int_{\cup_{i=1}^{\ell} B_{R\epsilon}(\sigma_i(x_0))} e^{4\pi\ell\phi_\epsilon^2} dv_g \geq \pi\ell e^{1+4\pi\ell A_{x_0}} + O\left(\frac{1}{(\log\epsilon)^2}\right). \tag{3.65}$$

Now we shall calculate the integral $\int_{\Sigma\setminus\cup_{i=1}^{\ell} B_{2R\epsilon}(\sigma_i(x_0))} e^{4\pi\ell\phi_\epsilon^2} dv_g$. By our choices of $R = -\log\epsilon$ and $c^2 = O(\log\epsilon)$ (see (3.62)), one can easily see that

$$R\epsilon\log(R\epsilon) = o\left(\frac{1}{c^2}\right). \tag{3.66}$$

Recalling the representation of the Green function G, namely (3.41), one has

$$\begin{aligned}\int_{\cup_{i=1}^{\ell} B_{2R\epsilon}(\sigma_i(x_0))} G^2 dv_g &= \sum_{i=1}^{\ell}\int_{B_{2R\epsilon}(\sigma_i(x_0))} G^2 dv_g \\ &= O\left((R\epsilon)^2(\log(R\epsilon))^2\right).\end{aligned}$$

This together with (3.66) gives

$$\begin{aligned}\int_{\Sigma\setminus\cup_{i=1}^{\ell} B_{2R\epsilon}(\sigma_i(x_0))} G^2 dv_g &= \int_\Sigma G^2 dv_g - \int_{\cup_{i=1}^{\ell} B_{2R\epsilon}(\sigma_i(x_0))} G^2 dv_g \\ &= \|G\|_2^2 + o\left(\frac{1}{c^2}\right).\end{aligned} \tag{3.67}$$

Moreover, in view of (3.58), (3.59), (3.64) and (3.66), there holds

$$\begin{aligned}\int_{\Sigma\setminus\cup_{i=1}^{\ell} B_{2R\epsilon}(\sigma_i(x_0))} \phi_\epsilon^2 dv_g &= \int_{\Sigma\setminus\cup_{i=1}^{\ell} B_{2R\epsilon}(\sigma_i(x_0))} \eta_\epsilon^2 dv_g + o\left(\frac{1}{c^2}\right) \\ &= \int_{\Sigma\setminus\cup_{i=1}^{\ell} B_{2R\epsilon}(\sigma_i(x_0))} \frac{G^2}{c^2} dv_g + o\left(\frac{1}{c^2}\right).\end{aligned} \tag{3.68}$$

Obviously it follows from $R = -\log \epsilon$ and (3.62) that

$$\int_{\Sigma \setminus \cup_{i=1}^{\ell} B_{2R\epsilon}(\sigma_i(x_0))} dv_g = \mathrm{vol}_g(\Sigma) + o\left(\frac{1}{c^2}\right). \tag{3.69}$$

Combining (3.67)-(3.69) and using the inequality $e^t \geq 1 + t$ for $t \geq 0$, we obtain

$$\begin{aligned}\int_{\Sigma \setminus \cup_{i=1}^{\ell} B_{R\epsilon}(\sigma_i(x_0))} e^{4\pi\ell\phi_\epsilon^2} dv_g &\geq \int_{\Sigma \setminus \cup_{i=1}^{\ell} B_{2R\epsilon}(\sigma_i(x_0))} \left(1 + 4\pi\ell\phi_\epsilon^2\right) dv_g \\ &\geq \mathrm{vol}_g(\Sigma) + 4\pi\ell\frac{\|G\|_2^2}{c^2} + o\left(\frac{1}{c^2}\right).\end{aligned}$$

Noting that $O(\frac{1}{(\log \epsilon)^2}) = o(\frac{1}{c^2})$ and combining (3.65) and (3.70), we conclude (3.48) for sufficiently small $\epsilon > 0$. This completes the proof of Theorem 2.1. □

4. Proof of Theorem 2.2

In this section, we shall prove Theorem 2.2. Since the proof is very similar to that of Theorem 2.1, we only give its outline.

Let $j \geq 2$, $\lambda_j^{\mathbf{G}}$ and $E_{j-1}^{\perp}$ be defined as in (2.5) and (2.6) respectively. For $\alpha < \lambda_j^{\mathbf{G}}$, we define

$$\beta_j^* = \sup\left\{\beta : \sup_{u \in E_{j-1}^{\perp},\, \|u\|_{1,\alpha} \leq 1} \int_\Sigma e^{\beta u^2} dv_g < \infty\right\}. \tag{4.1}$$

Comparing (3.1) with (4.1), similar to Lemma 3.1, we have $\beta_j^* = 4\pi\ell$, where ℓ is defined as in (2.3).

We now prove (ii) of Theorem 2.2. If $\alpha \geq \lambda_j^{\mathbf{G}}$ and $\beta > 0$, we take $u_j \in \mathscr{H}_{\mathbf{G}} \cap C^1(\Sigma, g)$ satisfies $\Delta_g u_j = \lambda_j^{\mathbf{G}} u_j$ and $u_j \not\equiv 0$. It follows that

$$\int_\Sigma |\nabla_g(tu_j)|^2 dv_g - \alpha \int_\Sigma (tu_j)^2 dv_g \leq 0, \quad \forall t \in \mathbb{R} \tag{4.2}$$

and that

$$\int_\Sigma e^{\beta(tu_j)^2} dv_g \to \infty \quad \text{as} \quad t \to \infty. \tag{4.3}$$

Then (4.2) and (4.3) imply that the supremum in (2.7) is infinity.

If $\alpha < \lambda_j^{\mathbf{G}}$ and $\beta > 4\pi\ell$, then we shall prove that the supremum in (2.7) is infinity. To do this, we let $\{e_i\}_{i=1}^{m_{j-1}} \subset \mathscr{H}_{\mathbf{G}} \cap C^1(\Sigma, g)$ be an orthonormal basis of

$E_{j-1} = E_{\lambda_1^{\mathbf{G}}} \oplus \cdots \oplus E_{\lambda_{j-1}^{\mathbf{G}}}$ with respect to the inner product on $L^2(\Sigma, g)$, namely, $E_{j-1} = \mathrm{span}\{e_1, \cdots, e_{m_{j-1}}\}$ and

$$(e_i, e_k) = \int_\Sigma e_i e_k dv_g = \delta_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

for all $i, k = 1, \cdots, m_{j-1}$. Let $\widetilde{M}_k$ be defined as in (3.4). Set

$$Q_k = \widetilde{M}_k - \frac{1}{\mathrm{Vol}_g(\Sigma)} \int_\Sigma \widetilde{M}_k dv_g - \sum_{i=1}^{m_{j-1}} (\widetilde{M}_k, e_i) e_i.$$

Then $Q_k \in E_{j-1}^{\perp}$. By a straightforward calculation, $\|Q_k\|_{1,\alpha}^2 = (1+O(r))8\pi\ell \log k + O(1)$. Denote $Q_k^* = Q_k / \|Q_k\|_{1,\alpha}$. It follows that for any fixed $\beta > 4\pi\ell$,

$$\begin{aligned} \int_\Sigma e^{\beta Q_k^{*2}} dv_g &\geq \int_{B_{rk^{-1/4}}(x_0)} e^{\frac{\beta(\log k + O(1))^2}{(1+O(r))8\pi\ell \log k + O(1)}} dv_g \\ &= e^{\frac{\beta(1+o_k(1))\log k}{(1+O(r))8\pi\ell}} \pi r^2 k^{-1/2}(1 + o_k(1)). \end{aligned}$$

Choosing $r > 0$ sufficiently small and then passing to the limit $k \to \infty$ in the above estimate, we conclude

$$\int_\Sigma e^{\beta Q_k^{*2}} dv_g \to \infty \quad \text{as} \quad k \to \infty.$$

Hence the supremum in (2.7) is infinity.

In the following, we sketch the proof of (i) and (iii) of Theorem 2.2.

Let $\alpha < \lambda_j^{\mathbf{G}}$. By a direct method of variation, one can see that for any $0 < \epsilon < 4\pi\ell$, there exists some $u_\epsilon \in E_{j-1}^{\perp}$ with $\|u_\epsilon\|_{1,\alpha} = 1$ such that

$$\int_\Sigma e^{(4\pi\ell - \epsilon)u_\epsilon^2} dv_g = \sup_{u \in E_{j-1}^{\perp}, \|u\|_{1,\alpha} \leq 1} \int_\Sigma e^{(4\pi\ell - \epsilon)u^2} dv_g.$$

Clearly u_ϵ satisfies the Euler-Lagrange equation

$$\begin{cases} \Delta_g u_\epsilon - \alpha u_\epsilon = \dfrac{1}{\lambda_\epsilon} u_\epsilon e^{(4\pi\ell - \epsilon)u_\epsilon^2} - \dfrac{\mu_\epsilon}{\lambda_\epsilon} - \displaystyle\sum_{k=1}^{m_{j-1}} \gamma_k e_k \\ u_\epsilon \in E_{j-1}^{\perp}, \ \|u_\epsilon\|_{1,\alpha} = 1 \\ \lambda_\epsilon = \displaystyle\int_\Sigma u_\epsilon^2 e^{(4\pi\ell - \epsilon)u_\epsilon^2} dv_g \\ \mu_\epsilon = \dfrac{1}{\mathrm{Vol}_g(\Sigma)} \displaystyle\int_\Sigma u_\epsilon e^{(4\pi\ell - \epsilon)u_\epsilon^2} dv_g \\ \gamma_k = \displaystyle\int_\Sigma \frac{1}{\lambda_\epsilon} e_k u_\epsilon e^{(4\pi\ell - \epsilon)u_\epsilon^2} dv_g. \end{cases} \tag{4.4}$$

Without loss of generality, we assume $c_\epsilon = u_\epsilon(x_\epsilon) = \sup_\Sigma |u_\epsilon| \to +\infty$ and $x_\epsilon \to x_0$ as $\epsilon \to 0$. Let r_ϵ be the blow-up scale defined as in (3.23) and $\varphi_\epsilon(y) = c_\epsilon(u_\epsilon(\exp_{x_\epsilon}(r_\epsilon y)) - c_\epsilon)$ for $y \in \mathbb{B}_{\delta r_\epsilon^{-1}}(0)$, where $0 < \delta < \frac{1}{2} i_g(\Sigma)$. As before, we can derive

$$\varphi_\epsilon(y) \to \varphi(y) = -\frac{1}{4\pi\ell}\log\left(1+\pi\ell|y|^2\right) \quad \text{in} \quad C^1_{\mathrm{loc}}(\mathbb{R}^2).$$

Moreover, we can prove that $\forall 1 < q < 2$, $c_\epsilon u_\epsilon$ converges to a Green function G weakly in $W^{1,q}(\Sigma, g)$, strongly in $L^{\frac{2q}{2-q}}(\Sigma, g)$, and almost everywhere in Σ. In this case, G satisfies

$$\begin{cases} \Delta_g G - \alpha G = \dfrac{1}{\ell}\displaystyle\sum_{i=1}^{\ell} \delta_{\sigma_i(x_0)} - \dfrac{1}{\mathrm{Vol}_g(\Sigma)} - \displaystyle\sum_{j=1}^{m_{j-1}} e_k(x_0)e_k \\ \displaystyle\int_\Sigma G\phi dv_g = 0, \ \ \forall \phi \in E_{j-1} \\ G(\sigma(x)) = G(x), \ \forall x \in \Sigma \setminus \mathbf{G}(x_0), \ \forall \sigma \in \mathbf{G}. \end{cases} \tag{4.5}$$

Similarly, G has a decomposition (3.41) near x_0 and A_{x_0} is defined as in (3.42). Analogous to Proposition 3.6, we arrive at

$$\sup_{u \in E_{j-1}^\perp,\ \|u\|_{1,\alpha} \le 1} \int_\Sigma e^{4\pi\ell u^2} dv_g \le \mathrm{Vol}_g(\Sigma) + \pi\ell e^{1+4\pi\ell A_{x_0}}. \tag{4.6}$$

This particularly leads to (i) of Theorem 2.2.

Finally we construct a sequence of functions to show that the estimate (4.6) is not true. This implies that blow-up can not occur and elliptic estimates on (4.4) give a desired extremal function. To do this, we let η_ϵ, ϕ_ϵ be defined respectively as in (3.50) and (3.64) satisfying $\eta_\epsilon \in W^{1,2}(\Sigma, g)$ and $\|\phi_\epsilon\|_{1,\alpha} = 1$. Note that the constants c and B in definitions of η_ϵ and ϕ_ϵ are given by (3.62) and (3.63) respectively. It then follows that

$$\int_\Sigma e^{4\pi\ell\phi_\epsilon^2} dv_g \ge \mathrm{Vol}_g(\Sigma) + \pi\ell e^{1+4\pi\ell A_{x_0}} + \frac{4\pi\ell\|G\|^2_{L^2(\Sigma,g)}}{-\log\epsilon} + o\left(\frac{1}{-\log\epsilon}\right). \tag{4.7}$$

Let

$$\widetilde{\phi}_\epsilon = \phi_\epsilon - \sum_{i=1}^{m_{j-1}} (\phi_\epsilon, e_i)e_i. \tag{4.8}$$

Obviously $\widetilde{\phi}_\epsilon \in E_{j-1}^\perp$. Since G satisfies (4.5) and

$$\int_\Sigma G e_i dv_g = \lim_{\epsilon\to 0}\int_\Sigma c_\epsilon u_\epsilon e_i dv_g = 0, \quad \forall 1 \le i \le m_{j-1},$$

we calculate

$$(\phi_\epsilon, e_i) = \int_{\cup_{i=1}^{\ell} B_{2R\epsilon}(\sigma_i(x_0))} (\eta_\epsilon - \overline{\eta}_\epsilon) e_i dv_g + \int_{\Sigma\setminus\cup_{i=1}^{\ell} B_{2R\epsilon}(\sigma_i(x_0))} \left(\frac{G}{c} - \overline{\eta}_\epsilon\right) e_i dv_g = o\left(\frac{1}{\log^2\epsilon}\right).$$

This together with (4.8) leads to

$$\widetilde{\phi}_\epsilon = \phi_\epsilon + o\left(\frac{1}{\log^2\epsilon}\right), \quad \|\widetilde{\phi}_\epsilon\|_{1,\alpha}^2 = 1 + o\left(\frac{1}{\log^2\epsilon}\right). \tag{4.9}$$

It follows from (4.7) and (4.9) that

$$\begin{aligned}\int_\Sigma e^{4\pi\ell\frac{\widetilde{\phi}_\epsilon^2}{\|\widetilde{\phi}_\epsilon\|_{1,\alpha}^2}} dv_g &= \int_\Sigma e^{4\pi\ell\phi_\epsilon^2 + o(\frac{1}{-\log\epsilon})} dv_g \\ &\geq \left(1 + o\left(\frac{1}{-\log\epsilon}\right)\right) \\ &\quad\times\left(\mathrm{Vol}_g(\Sigma) + \pi\ell e^{1+4\pi\ell A_{x_0}} + \frac{4\pi\ell\|G\|_{L^2(\Sigma,g)}^2}{c^2} + o\left(\frac{1}{c^2}\right)\right) \\ &\geq \mathrm{Vol}_g(\Sigma) + \pi\ell e^{1+4\pi\ell A_{x_0}} + \frac{4\pi\ell\|G\|_{L^2(\Sigma,g)}^2}{-\log\epsilon} + o\left(\frac{1}{-\log\epsilon}\right),\end{aligned}$$

which implies that (4.6) does not hold. This completes the proof of (iii) of Theorem 2.2.

References

[1] A. ADIMURTHI and O. DRUET, *Blow-up analysis in dimension 2 and a sharp form of Moser-Trudinger inequality*, Comm. Partial Differential Equations **29** (2004), 295–322.

[2] A. ADIMURTHI and M. STRUWE, *Global compactness properties of semilinear elliptic equation with critical exponential growth*, J. Funct. Anal. **175** (2000), 125–167.

[3] L. CARLESON and A. CHANG, *On the existence of an extremal function for an inequality of J. Moser*, Bull. Sci. Math. **110** (1986), 113–127.

[4] W. CHEN, *A Trudinger inequality on surfaces with conical singularities*, Proc. Amer. Math. Soc. **108** (1990), 821–832.

[5] W. CHEN and C. LI, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. **63** (1991), 615–622.

[6] M. DE SOUZA and J. M. DO Ó, *A sharp Trudinger-Moser type inequality in* $\mathbb{R}^2$, Trans. Amer. Math. Soc. **366** (2014), 4513–4549.

[7] W. DING, J. JOST, J. LI and G. WANG, *The differential equation* $-\mathbb{D}u = 8\pi - 8\pi he^u$ *on a compact Riemann Surface*, Asian J. Math. **1** (1997), 230–248.

[8] M. FLUCHER, *Extremal functions for Trudinger-Moser inequality in* 2 *dimensions*, Comment. Math. Helv. **67** (1992), 471–497.

[9] L. FONTANA, *Sharp borderline Sobolev inequalities on compact Riemannian manifolds*, Comment. Math. Helv. **68** (1993), 415–454.

[10] S. IULA and G. MANCINI, *Extremal functions for singular Moser-Trudinger embeddings*, Nonlinear Anal. **156** (2017), 215–248.

[11] Y. LI, *Moser-Trudinger inequality on compact Riemannian manifolds of dimension two*, J. Partial Differential Equations **14** (2001), 163–192.

[12] Y. LI, *The existence of the extremal function of Moser-Trudinger inequality on compact Riemannian manifolds*, Sci. China Math. (Series A) **48** (2005), 618–648.

[13] K. LIN, *Extremal functions for Moser's inequality*, Trans. Amer. Math. Soc. **348** (1996), 2663–2671.

[14] G. LU and Y. YANG, *The sharp constant and extremal functions for Moser-Trudinger inequalities involving L^p norms*, Discrete Cont. Dyn. Syst. **25** (2009), 963–979.

[15] G. LU and Y. YANG, *Adams' inequalities for bi-Laplacian and extremal functions in dimension four*, Adv. Math. **220** (2009), 1135–1170.

[16] G. MANCINI and L. MARTINAZZI, *The Moser-Trudinger inequality and its extremals on a disk via energy estimates*, Calc. Var. **56** (2017), 94, https://doi.org/10.1007/s00526-017-1184-y.

[17] J. MOSER, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1091.

[18] J. MOSER, *On a nonlinear problem in differential geometry*, In: "Dynamical Systems" (Proc. Sympos., Univ. Bahia, Salvador, 1971), Academic Press, New York, 1973, 273–280.

[19] V. H. NGUYEN, *Extremal functions for the Moser-Trudinger inequality of Adimurthi-Druet type in $W^{1,N}(\mathbb{R}^N)$*, arXiv:1702.07970.

[20] J. PEETRE, *Espaces d'interpolation et theoreme de Soboleff*, Ann. Inst. Fourier (Grenoble) **16** (1966), 279–317.

[21] S. POHOZAEV, *The Sobolev embedding in the special case $pl = n$*, In: "Proceedings of the Technical Scientific Conference on Advances of Scientific Research 1964-1965", Mathematics sections, Moscov. Energet. Inst., Moscow, 1965, 158–170.

[22] M. STRUWE, *Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **5** (1988), 425–464.

[23] C. TINTAREV, *Trudinger-Moser inequality with remainder terms*, J. Funct. Anal. **266** (2014), 55–66.

[24] N. TRUDINGER, *On embeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–484.

[25] Y. YANG, *Extremal functions for a sharp Moser-Trudinger inequality*, Internat. J. Math. **17** (2006), 331–338.

[26] Y. YANG, *A sharp form of Moser-Trudinger inequality in high dimension*, J. Funct. Anal. **239** (2006), 100–126.

[27] Y. YANG, *A sharp form of the Moser-Trudinger inequality on a compact Riemannian surface*, Trans. Amer. Math. Soc. **359** (2007), 5761–5776.

[28] Y. YANG, *Extremal functions for Trudinger-Moser inequalities of Adimurthi-Druet type in dimension two*, J. Differential Equations **258** (2015), 3161–3193.

[29] Y. YANG, *A Trudinger-Moser inequality on a compact Riemannian surface involving Gaussian curvature*, J. Geom. Anal. **26** (2016), 2893–2913.

[30] Y. YANG and X. ZHU, *Blow-up analysis concerning singular Trudinger-Moser inequalities in dimension two*, J. Funct. Anal. **272** (2017), 3347–3374.

[31] Y. YANG and X. ZHU, *Existence of solutions to a class of Kazdan-Warner equations on compact Riemannian surface*, Sci. China Math. **61** (2018), 1109–1128.

[32] A. YUAN and Z. HUANG, *An improved singular Trudinger-Moser inequality in dimension two*, Turkish J. Math. **40** (2016), 874–883.

[33] A. YUAN and X. ZHU, *An improved singular Trudinger-Moser inequality in unit ball*, J. Math. Anal. Appl. **435** (2016), 244–252.

[34] V. YUDOVICH, *Some estimates connected with integral operators and with solutions of elliptic equations*, Sov. Math. Dokl. **2** (1961), 746–749.
[35] J. ZHU, *Improved Moser-Trudinger inequality involving L^p norm in n dimensions*, Adv. Nonlinear Stud. **14** (2014), 273–293.

Department of Mathematics
Renmin University of China
Beijing 100872, P. R. China
2011202402@ruc.edu.cn
yunyanyang@ruc.edu.cn