ON STABILITY AND SCALAR CURVATURE RIGIDITY OF QUATERNION-KÄHLER MANIFOLDS

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Abstract. We show that every quaternion-Kähler manifold of negative scalar curvature is stable as an Einstein manifold and therefore scalar curvature rigid. In particular, this implies that every irreducible nonpositive Einstein manifold of special holonomy is stable. In contrast, we demonstrate that there exist quaternion-Kähler manifolds of positive scalar curvature which are not scalar curvature rigid even though they are semi-stable.

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1. Introduction

1.1. **Stability of Einstein manifolds.** Let g be an Einstein metric, meaning that g satisfies $\mathrm{Ric}_g = \lambda \cdot g$ for some $\lambda \in \mathbb{R}$. On a compact manifold, Einstein metrics are critical points of the Einstein-Hilbert functional

$$S(g) = \int_M \mathrm{scal}_g \, dV_g.$$

on the space $\mathcal{M}_1$ of Riemannian metrics of volume 1. It is well-known that Einstein metrics are always saddle points of the functional, and that index and coindex of $D^2_g S$ are both infinite (see e.g. [2]). Recall that a trace-free and divergence-free symmetric 2-tensor is named a tt-tensor and denote the space of such tensors by TT. If $h \in TT$,

$$D^2_g S(h,h) = -\frac{1}{2}\int_M \langle \nabla^*\nabla h - 2\mathring{R}h, h\rangle \, dV, \qquad (\mathring{R}h)(X,Y) = \sum_i h(R_{X,e_i}Y, e_i)$$

Thus, $D^2_g S$ has finite coindex on TT. We call $\Delta_E = \nabla^*\nabla - 2\mathring{R}$ the Einstein operator.

Definition 1.1. A compact Einstein manifold is called *stable* if all eigenvalues of Δ_E on TT are positive and *semi-stable* if all eigenvalues of Δ_E on TT are nonnegative.

Let g_t be a curve of Einstein metrics through $g_0 = g$ and suppose that $h = \dot{g}_0$ is orthogonal to constant rescalings and the orbit of the diffeomorphism group acting on g. Then h is a tt-tensor and $\Delta_E h = 0$. This motivates the following definition:

Definition 1.2. An element $h \in \epsilon(g) := \ker(\Delta_E|_{TT})$ is called an *infinitesimal Einstein deformation*. We call it *integrable*, if it is tangent to a curve of Einstein metrics. We call an Einstein metric *integrable*, if all of its infinitesimal deformations are integrable.

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An important problem is to decide whether g is *non-deformable*, i.e. whether the equivalence class $[g]$ under homothetic scaling and pullback by diffeomorphisms is isolated in the moduli space of Einstein structures. If $\epsilon(g)$ is trivial, g is obviously non-deformable. If $\epsilon(g)$ is nontrivial, g can still be non-deformable. For example, the product metric on $S^2 \times \mathbb{C}P^{2n}$ admits infinitesimal deformations but none are integrable [16]. In contrast, all known Ricci-flat metrics on compact manifolds are integrable, see Subsection 1.3.1 below

Define $E_g := \mathrm{Ric}_g - \frac{1}{n}(\int_M \mathrm{scal}_g \, dV_g)g$ and let $D_g^k E(h, \ldots, h) = \frac{d^k}{dt^k}|_{t=0} E_{g+th}$ be its k'th Frechet derivative at g. If $h \in \epsilon(g)$, one computes that $D_g E(h) = 0$. Suppose that h is integrable and let g_t be a curve of Einstein metrics such that $g_0 = g$ and $g_0' = h$. We may assume that $g_t \in \mathcal{M}_1$, so that $E_{g_t} = 0$. By differentiating twice, we obtain

$$\frac{d}{dt}|_{t=0} E_{g_t} = D_g E(h) = 0, \qquad \frac{d^2}{dt^2}|_{t=0} E_{g_t} = D_g E(k) + D_g^2 E(h, h) = 0,$$

where $k = g_0''$. The second equation is not automatic and motivates the following definition:

Definition 1.3. An infinitesimal Einstein deformation $h \in \epsilon(g)$ is *integrable to second order* if and only if there is a symmetric 2-tensor k field satisfying

$$D_g E(k) + D_g^2 E(h, h) = 0.$$

If h is integrable, it is necessarily integrable to second order. Furthermore, h is integrable to second order if and only if $D_g^2 E(h, h) \perp \epsilon(g)$ [16, Lem. 4.8]. Hence, the vanishing of the so-called Koiso obstruction $\Psi(h) := (D_g^2 E(h, h), h)_{L^2}$ is a necessary condition for the integrability of second order for an infinitesimal deformation $h \in \epsilon(g)$. Integrability of higher order can be defined in a similar way, see e.g. [1, Def. 2.4] or [2, 12.38] for details.

1.2. **Scalar curvature rigidity.** In a recent work [7], Dahl and the first author discovered a deep relation between stability and scalar curvature rigidity. The latter notion is defined in [7] as follows:

Definition 1.4. Let $\hat{g}$ be an Einstein metric on a manifold M. For each compact set $K \subset M$ consider the metrics g with $g \equiv \hat{g}$ on $M \setminus K$, $\mathrm{Vol}(K, g) = \mathrm{Vol}(K, \hat{g})$ and $\mathrm{scal}_g \geq \mathrm{scal}_{\hat{g}}$. If close to $\hat{g}$, there is no such g with $\mathrm{scal}_g \not\equiv \mathrm{scal}_{\hat{g}}$, then we say that $(M, \hat{g})$ *scalar curvature rigid.*

Note that in the definition, we only consider metrics *near* an Einstein metric. Many rigidity phenomena involving scalar curvature are global on the space of metrics and do not require a volume constraint. For example, as a consequence of the positive mass theorem (see [27]), every metric g on $\mathbb{R}^n$ with $\mathrm{scal}_g \geq 0$ which agrees with the Euclidean metric outside a compact set is isometric to the Euclidean metric. An analogous statement, holds for hyperbolic space, see [4].

A consequence of Theorem 1.4 and Remark 1.6 in [7] is the following result:

Theorem 1.5 ([7])**.** *Let (M, g) be a compact Einstein manifold. If it is semi-stable and integrable, it is also scalar curvature rigid. If (M, g) is unstable, it is not scalar curvature rigid.*

Remark 1.6. Because the statement is not explicitly proven in this form in [7], we briefly sketch here why it holds: For a parameter $\alpha > 0$, let λ_α be the smallest eigenvalue of the operator $4\alpha\Delta + \mathrm{scal}$. Because the smallest eigenvalue is simple, the functionals $g \mapsto \lambda_\alpha(g)$ depend smoothly on the metric. Einstein metrics are critical points of the functionals λ_α on $\mathcal{M}_1$, see [7, Prop. 5.3]. Let μ be the smallest nonzero eigenvalue of the Laplace-Beltrami operator, if (M, g) is not isometric to the round sphere, and the second smallest nonzero eigenvalue of the Laplace-Beltrami operator if (M, g) is isometric to the round sphere. By the Obata-Lichnerowicz eigenvalue estimate, we can choose $\alpha > 0$ so small that

$$\left(1 - \frac{n-2}{n-1}\alpha\right)\mu > \frac{\mathrm{scal}_g}{n-1}.$$

If (M, g) is semi-stable and the parameter α satisfies the above inequality, $D^2_g\lambda_\alpha$ is nonpositive on $T_g\mathcal{M}_1$ and its kernel is spanned by the Lie-derivatives of g and the space $\epsilon(g)$, which can be concluded from the formulas in [7, Sec. 5.4]. Integrability implies that there is a manifold $\mathcal{E}_1 \subset \mathcal{M}_1$ of Einstein metrics which is tangent to $\ker(D^2_g\lambda_\alpha)$ and along which λ_α is constant. Orthogonal to its kernel, $D^2_g\lambda_\alpha$ has a uniform negative upper bound. By a Taylor expansion argument along the lines of [7, Thm. 7.1] (where the case of manifolds with boundary is done), one proves that g is a local maximum of λ_α on $\mathcal{M}_1$, and therefore, it is scalar curvature rigid by [7, Lem. 5.2].

On the other hand, if (M, g) is unstable, we consider the set

$$\mathcal{C}_1 = \{g \in \mathcal{M}_1 \mid \mathrm{scal}_g \equiv \mathit{const}\}.$$

In [15], Koiso has shown that if (M, g) is a unit-volume Einstein manifold which is not isometric to a round sphere, $\mathcal{C}_1$ is a manifold near g with tangent space

$$T_g\mathcal{C}_1 = \{\mathcal{L}_X g \mid X \in \Gamma(TM)\} \oplus TT.$$

Therefore, if h is a TT-tensor such that $(\Delta_E h, h)_{L^2} < 0$, we find a curve $g_t \in \mathcal{C}_1$ such that

$$\frac{d}{dt}S(g_t)|_{t=0} = 0, \qquad \frac{d^2}{dt^2}S(g_t)|_{t=0} = -\frac{1}{2}(\Delta_E h, h)_{L^2} > 0,$$

so that for small $t \neq 0$, $\mathrm{scal}_{g_t} = S(g_t) > S(g) = \mathrm{scal}_g$. Because all g_t have the same volume, this implies that g is not scalar curvature rigid.

Remark 1.7. Stability is a definition of an *infinitesimal* nature and often referred to as linear stability in the literature. Theorem 1.5 demonstrates that scalar curvature rigidity is the corresponding *local* definition. Furthermore, under some additional assumptions, scalar curvature rigidity is equivalent to dynamical stability under the Ricci flow, see [7] and references therein.

Note that Definition 1.4 assumes neither compactness nor completeness. To formulate an analouge of Theorem 1.5 for open manifolds, we extend the definition of stability by saying that a (possibly open) Einstein manifold is stable if there exists a constant $C > 0$ such that $\Delta_E h|_{TT} \geq C$ holds in the L^2-sense, that is, $(\Delta_E h, h)_{L^2} \geq C\,\|h\|^2_{L^2}$ for all tt-tensors h with compact support. It is semi-stable, if $(\Delta_E h, h)_{L^2} \geq 0$ for all tt-tensors h with compact support. The analogue of Theorem 1.5 for open manifolds now reads as follows:

Theorem 1.8 ([7])**.** *Let (M, g) be an open Einstein manifold of nonpositive scalar curvature which is semi-stable. Then it is scalar curvature rigid.*

This result is a consequence of Theorem 1.9 and Remark 1.10 in [7]. The full form of [7, Thm. 1.9] states an equivalence of linear stability and scalar curvature rigidity without any restriction to the scalar curvature, provided that some additional conditions hold. We formulated Theorem 1.8 in this form in order to keep the statement simple. Note that in constrast to the compact case, we do not need to assume an integrability condition. The essential reason is that according to Definition 1.4, we only need to consider perturbations of g which are supported on compact subsets $K \subset M$. Because M is open, we find for each compact $K \subset M$ another open subset N with compact closure such that $K \subsetneq N \subsetneq M$. By domain monotonicity of the smallest Dirichlet eigenvalue, N is stable and thus, $\Delta_E h|_{TT}$ does not have a Dirichlet kernel.

In this paper, we show that the integrability condition in Theorem 1.5 is essential:

Theorem 1.9. *Let (M, g) be a compact Einstein manifold and suppose that it admits infinitesimal deformations which are not integrable to second order. Then (M, g) is not scalar curvature rigid.*

The failure of integrability of second order allows us to construct a curve g_t of metrics of volume 1 and constant scalar curvature along which we can increase $S(g_t)$, which is by the assumptions on g_t equal to the constant function scal_{g_t}. This will be shown in Section 3.

1.3. **Special holonomy metrics.** An oriented Riemannian manifold (M, g) is said to be of special holonomy if its restricted holonomy group is a proper subgroup of $SO(n)$. If (M, g) is simply-connected, irreducible and nonsymmetric, there are six possible types of holonomy groups. For five of them (with the Kähler case as the only exception), (M, g) is necessarily Einstein. Conversely, many Einstein metrics are of special holonomy. It is thus natural to consider the questions of stability and scalar curvature rigidity specifically for special holonomy metrics. For all holonomy types except one, answers are already given in the literature and we give a brief overview over these results in the following. In this article, we study stability and scalar curvature rigidity of the remaining holonomy type, which are the quaternion-Kähler manifolds.

1.3.1. *Ricci flat metrics.* In the Ricci-flat case, every special holonomy metric has a universal cover which admits a parallel spinor. Thus, they are semi-stable by [9]. This follows essentially from [25], although it is not written there explicitly. The integrability for compact manifolds of special holonomy is also a well-known result, see [25]. Therefore, all these metrics are scalar curvature rigid by Theorem 1.5. All known examples of compact Ricci-flat manifolds are of special holonomy and it is a long-standing open question whether other examples exist.

1.3.2. *Kähler-Einstein metrics of nonzero scalar curvature.* Using *spin*^{c}-geometry, Dai, Wang and Wei showed that Kähler-Einstein metrics of negative scalar curvature are semi-stable [10]. This already follows from work by Koiso ([18], see also the discussion in [2, p. 361–364]), but is not stated there explicitly.

It is still open whether Kähler-Einstein metrics of negative scalar curvature are integrable. It has however been shown by Nagy and the second author in [20] that all infinitesimal Einstein deformations of such metrics are integrable to second order. In dimension four, this also follows from combining results of LeBrun, Dahl and the first author with Theorem 1.9:

By [19], compact four-dimensional Kähler-Einstein manifolds of negative scalar curvature maximise the Yamabe invariant. Therefore by [7, Theorem 1.1], they are scalar curvature rigid. On the other hand, if there were infitesimal Einstein deformations which are not integrable to second order, Theorem 1.9 would imply that the metric is not scalar curvature rigid, which leads to a contradiction.

Because the above mentioned result by Nagy and the second author [20] holds in any dimension, this raises the following question.

Question 1.10. Are all compact Kähler-Einstein manifolds with negative Einstein constant scalar curvature rigid?

On the other hand, Kähler-Einstein manifolds of positive scalar curvature and $h^{1,1}(M) > 1$ are unstable and therefore not scalar curvature rigid (see [3]). The easiest way to construct such an example is by taking a product of two Kähler-Einstein manifolds. The above mentioned example $S^2 \times \mathbb{C}P^{2n}$ is a positive Kähler-Einstein manifold whose infinitesimal deformations are all non-integrable.

1.3.3. *Symmetric spaces.* Let $M = G/K$ be an irreducible symmetric space with the metric g induced by the Killing form of G. The holonomy of the Levi-Civita connection can be identified with K, i.e. we may consider symmetric spaces as manifolds with special holonomy. Then g is Einstein with non-negative sectional curvature if M is of compact type and with non-positive sectional curvature if M is of non-compact type. In the later case the metric g is stable (see [2, 12.73]). If M is of compact type then the stability type was decided by Koiso in [17], see also [22] for the treatment of a few remaining cases. It turns out that all symmetric spaces of compact type are semi-stable, with the exception of the following unstable cases: $\mathrm{Sp}(r), \mathrm{Sp}(n)/\mathrm{U}(n)$ and the Grassmannians $\mathrm{Gr}_2(\mathbb{R}^5)$ and $\mathrm{Gr}_r(\mathbb{H}^{r+s})$ for $r, s \geq 2$. Infinitesimal Einstein deformations exist on the following spaces:

$$\mathrm{SU}(n),\ \ \mathrm{SU}(n)/\mathrm{SO}(n),\ \ \mathrm{SU}(2n)/\mathrm{Sp}(n),\ \ \mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q)),\ \ \mathrm{E}_6/\mathrm{F}_4$$

where $n \geq 3$ and $p, q \geq 2$. For the four $\mathrm{SU}(m)$ quotients in this list the space of infinitesimal Einstein deformations integrable to second order can be explicitly described and turns out to be a proper subspace of $\epsilon(g)$ (see [1] for $\mathrm{SU}(n)$, [12] for $\mathrm{SU}(n)/\mathrm{SO}(n)$ and $\mathrm{SU}(2n)/\mathrm{Sp}(n)$ and [13, 20] for $\mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$). It follows from Theorem 1.9 that none of these spaces are scalar curvature rigid. For the space $\mathrm{E}_6/\mathrm{F}_4$ it is known (see [16]) that all infinitesimal Einstein deformations are integrable to second order. Hence, this is the only symmetric space for which it is still not known whether it is scalar curvature rigid or not.

Note that reducible symmetric spaces are unstable, as is always the case for products of Einstein manifolds.

1.3.4. *Quaternion-Kähler manifolds.* Recall that quaternion-Kähler manifolds are defined by the condition that the holonomy is a subgroup of $\mathrm{Sp}(1)\cdot\mathrm{Sp}(n) \subset \mathrm{SO}(4n)$. For $n \geq 2$, they are automatically Einstein. Furthermore, they are deRham irreducible if the scalar curvature is different from zero. If the scalar curvature vanishes the restricted holonomy reduces further to $\mathrm{Sp}(n)$, i.e. the manifold is then locally hyper-Kähler (see [2, Chap. 14] for further details). Our main result in the case of negative scalar curvature is as follows.

Theorem 1.11. *Every quaternion-Kähler manifold (M, g) of negative scalar curvature is stable. In particular, it is scalar curvature rigid and, if M is compact, the metric g is non-deformable as an Einstein metric.*

Using representations of the Holonomy group, we construct a parallel bundle embedding $\Phi : \mathrm{Sym}^2 T^*M \to \Lambda^4 T^*M$ via which $\Delta_E + 2\frac{\mathrm{scal}_g}{\dim M}$ corresponds to the (nonnegative) Hodge-Laplace operator. This construction will be explained in Section 2.

The first examples of quaternion-Kähler manifolds of negative scalar are the noncompact Wolf spaces. These are symmetric spaces and there is one such space for each simple Lie algebra, e.g. the quaternionic hyperbolic space $\mathbb{H}H^n$. There are many more examples with negative scalar curvature, homogeneous and non-homogeneous, see e.g. [5, 6] and references therein. However, the only known compact examples are compact quotients of the non-compact Wolf spaces.

In the case of positive scalar curvature the compact Wolf spaces are the only known examples and it is an open question whether there are other examples. All these symmetric examples are stable, with the exception of the complex 2-plane Grassmannian $\mathrm{Gr}_2(\mathbb{C}^{n+2})$ which is semi-stable, i.e. it admits infinitesimal Einstein deformations (see [22]) and, as already mentioned in Subsection 1.3.3, some of them are not integrable to second order. Thus we have

Corollary 1.12. *There are quaternion-Kähler manifolds of positive scalar curvature which are not scalar curvature rigid.*

We see a pattern for quaternion-Kähler manifolds which is similar to other special holonomy Einstein manifolds and with our results, we can summarize the discussion in Subsection 1.3 as follows: All Einstein metrics of special holonomy and negative scalar curvature are semi-stable. Among Einstein manifolds of positive scalar curvature, we find for each type of special holonomy type (Kähler, quaternion-Kähler, symmetric spaces) exmples which are not scalar curvature rigid.

2. Stability of quaternionic Kähler manifolds

2.1. **The standard Laplace operator.** Let (M^n, g) be an oriented Riemannian manifold and let $\mathrm{Hol} := \mathrm{Hol}(g) \subset \mathrm{SO}(n)$ be the holonomy group of the Levi-Civita connection, with holonomy reduction $P_{\mathrm{Hol}} \subset P_{\mathrm{SO}(n)}$, where $P_{\mathrm{SO}(n)}$ is the frame bundle of (M^n, g). Then any geometric vector bundle EM on M is associated to P_{Hol} via some representation $\rho : \mathrm{Hol} \to \mathrm{Aut}(E)$, e.g. the tangent bundle TM is associated to the standard representation on $T = \mathbb{R}^n$.

Let $\mathfrak{hol}(M) \subset \Lambda^2 TM$ be the holonomy algebra bundle defined as associated vector bundle via the adjoint representation of Hol on its Lie algebra $\mathfrak{hol}$. The differential of the representation ρ defines fibrewise a parallel bundle map $* : \mathfrak{hol}(M) \otimes EM \to EM$. We denote with ∇ the connection on sections of EM induced by the Levi–Civita connection of g. Its curvature is defined as $R^\nabla_{X,Y} = \nabla^2_{X,Y} - \nabla^2_{Y,X}$ for any tangent vectors X, Y. It is well-known that $R^\nabla_{X,Y} \in \mathfrak{hol}(M)$. In particular, the curvature can be written as $R^\nabla_{X,Y} = R_{X,Y} \circ *$, where R is the curvature of the Levi-Civita connection of g.

The parallel orthogonal projection map $\mathrm{pr}_{\mathfrak{hol}} : \Lambda^2 TM \to \mathfrak{hol}(M) \subset \Lambda^2 TM$ allows one to define the standard curvature endomorphism for every geometric vector bundle:

$$q(R) := \frac{1}{2} \sum_{i,j} \mathrm{pr}_{\mathfrak{hol}}(e_i \wedge e_j) * R_{e_i,e_j} * \; \in \mathrm{End}(EM) \ .$$

The definition of $q(R)$ is independent of the choice of local orthonormal frame $\{e_i\}$. The main example comes from the classical Weitzenböck formula for the Hodge-Laplace operator

$$\Delta = dd^* + d^*d = \nabla^*\nabla + q(R) \ .$$

In particular, $q(R)$ is the Ricci tensor on 1-forms. On symmetric 2-tensors we have

$$q(R) = 2\mathring{R} + 2\mathrm{Ric} \ ,$$

where Ric acts as a derivation, e.g. as 2λ on an Einstein manifold with Einstein constant λ.

The standard Laplace operator $\Delta = \Delta_\rho$ acting on sections of a geometric vector bundle EM is defined as the sum $\Delta_\rho = \nabla^*\nabla + q(R)$. Then Δ_ρ coincides with the Hodge-Laplace operator Δ if ρ is the restriction to Hol of the standard representation of $\mathrm{SO}(n)$ on forms and similarly with the Lichnerowicz Laplacian Δ_L on symmetric tensors if ρ is the restriction of the standard representation of $\mathrm{SO}(n)$ on symmetric tensors. The notion of standard Laplace operator was introduced in [21]. A similar construction can be found in [2, Ch. 1.I]

The most important property of the standard Laplacian Δ_ρ is that it commutes with parallel bundle maps, i.e. with maps induced by Hol-equivariant maps between Hol-representations. In particular, if $EM \subset \Lambda^k T^*M$ is a parallel subbundle, then the restriction of the Hodge-Laplace operator to sections of EM coincides with the standard Laplace operator Δ_ρ of the bundle EM. The same is true for any parallel subbundle of the bundle of symmetric tensors and the Lichnerowicz Laplacian. As a consequence we have $\Delta_L \geq 0$ on all parallel subbundles $EM \subset \mathrm{Sym}^*TM$ which are also parallel subbundles of the form bundle Λ^*T^*M. Note, that EM is a parallel subbundle of $\Lambda^k T^*M$ if and only if $E \subset \Lambda^k T$ is an Hol-invariant subspace with respect to the standard representation of $\mathrm{SO}(n)$ on $\Lambda^k T$ restricted to Hol.

2.2. **Quaternion-Kähler manifolds of negative scalar curvature.** Let (M^{4n}, g) be a quaternion-Kähler manifold. Then $\mathrm{Hol}(g) \subset \mathrm{Sp}(1) \cdot \mathrm{Sp}(n) \subset \mathrm{SO}(4n)$ and equivalently there are locally defined almost complex structures I, J, K compatible with the metric, satisfying the quaternionic relation $IJ = K$ and spanning a globally defined parallel subbundle of $\mathrm{End}(\mathrm{T}M)$. Such a triple $\{I, J, K\}$ is called a local quaternionic frame.

The standard representations of $\mathrm{Sp}(1)$ on $\mathrm{H} := \mathbb{H}^1$ and of $\mathrm{Sp}(n)$ on $\mathrm{E} := \mathbb{H}^n$ give rise to locally defined vector bundles again denoted by H and E. Any tensor product of these bundles with an even number of factors leads to globally defined bundles, e.g. the complexified tangent bundle can be written as $\mathrm{T}M^{\mathbb{C}} = \mathrm{H} \otimes \mathrm{E}$. Especially important will be the following decomposition.

Lemma 2.1. *The vector bundle of symmetric 2-tensors decomposes into the direct sum of three globally defined parallel subbundles:*

$$\mathrm{Sym}^2\mathrm{T}^*M^{\mathbb{C}} \;\cong\; (\mathrm{Sym}^2\mathrm{H}^* \otimes \mathrm{Sym}^2\mathrm{E}^*) \,\oplus\, \Lambda^2_0\mathrm{E}^* \,\oplus\, \mathbb{C} \ . \tag{2.1}$$

Here $\Lambda^2_0\mathrm{E}^*$ denotes the space of primitive 2-forms on E, i.e. 2-forms orthogonal to the symplectic form σ_{E}. The trivial bundle $\mathbb{C}$ is spanned by the metric $g = \sigma_H \otimes \sigma_E$.

Proof. Decomposition (2.1) corresponds to a decomposition of the $\mathrm{Sp}(1)\cdot\mathrm{Sp}(n)$ representations $\mathrm{Sym}^2(\mathrm{H}^*\otimes\mathrm{E}^*)$ into irreducible summands. Hence all three summands in the decomposition define parallel subbundles of the bundle of symmetric 2-tensors.

The decomposition of $\mathrm{Sym}^2(\mathrm{H}^*\otimes\mathrm{E}^*)$ is well-known and follows from general formulas for Schur functors (see [11]). It can also be proved by first giving explicit embeddings of the three summands and then comparing the dimensions. The embedding of the summands can be described as follows. Consider $\alpha_H\in\mathrm{Sym}^2\mathrm{H}^*$ and $\alpha_E\in\mathrm{Sym}^2\mathrm{E}^*$, then $\alpha=\alpha_H\otimes\alpha_E$ defined by $\alpha(a\otimes e,\tilde a\otimes\tilde e)=\alpha_H(a,\tilde a)\alpha_E(e,\tilde e)$ is obviously in $\mathrm{Sym}^2\mathrm{T}M^{\mathbb{C}}$. For any $\eta\in\mathrm{E}^*\otimes\mathrm{E}^*$ we define $\eta_T=\sigma_H\otimes\eta$ by $\eta_T(a\otimes e,\tilde a\otimes\tilde e)=\sigma_H(a,\tilde a)\eta(e,\tilde e)$. If η is skew-symmetric, i.e. in Λ^2E, then η_T is symmetric. This defines the embedding of the second and third summand. Recall that the Riemannian metric is given as $g=\sigma_H\otimes\sigma_E=(\sigma_E)_T$. For later use we also note, that η_T is skew-symmetric if η is symmetric. □

The proof of Theorem 1.11 is based on the following simple observation.

Lemma 2.2. *The three bundles* $\mathrm{Sym}^2\mathrm{H}^*\otimes\mathrm{Sym}^2\mathrm{E}^*$, $\Lambda^2_0\mathrm{E}^*$ *and the trivial bundle* $\mathbb{C}$ *of the decomposition* (2.1) *all appear as parallel subbundles of the bundle of* 4*-forms* $\Lambda^4\mathrm{T}M^*$*.*

Proof. Again it follows directly from properties of $\mathrm{Sp}(1)\cdot\mathrm{Sp}(n)$ representations. The decomposition of the space of k forms on $\mathrm{H}\otimes\mathrm{E}$ is given in [26]. It turns out that the three irreducible summands of the $\mathrm{Sp}(1)\cdot\mathrm{Sp}(n)$ representation $\mathrm{Sym}^2(\mathrm{H}^*\otimes\mathrm{E}^*)$ given in (2.1) also appear in the decomposition of the representation $\Lambda^4(\mathrm{H}^*\otimes\mathrm{E}^*)$. Hence, there are corresponding parallel subbundles in $\Lambda^4\mathrm{T}^*M^{\mathbb{C}}$ and parallel bundle maps identifying the three subbundles in $\mathrm{Sym}^2\mathrm{T}^*M^{\mathbb{C}}$ with the corresponding subbundles in $\Lambda^4\mathrm{T}^*M^{\mathbb{C}}$.

The embeddings can also be described explicitly. In the notation from above the map $\alpha\otimes\beta\mapsto\alpha_T\wedge\beta_T$ defines an embedding of $\mathrm{Sym}^2\mathrm{H}^*\otimes\mathrm{Sym}^2\mathrm{E}^*$ into the space of 4-forms $\Lambda^4\mathrm{T}^*M^{\mathbb{C}}$. In order to describe the embedding of $\Lambda^2\mathrm{E}^*$ we fix a local quaternionic frame $\{I,J,K\}$ with corresponding Kähler forms $\omega_I,\omega_J,\omega_K$, i.e. $\omega_I(\cdot,\cdot)=g(I\cdot,\cdot)$ and similarly for ω_J and ω_K. For any $\eta\in\Lambda^2\mathrm{E}^*$ we define a 4-form $\hat\eta$ on M by the following formula

$$\hat\eta\ :=\ \omega_I\wedge(\eta_T(I\cdot,\cdot))+\omega_J\wedge(\eta_T(J\cdot,\cdot))+\omega_K\wedge(\eta_T(K\cdot,\cdot))\ .$$

That $\eta_T(I\cdot,\cdot)$ is indeed a 2-form can be seen by the following argument. The almost complex structures of a local quaternionic frame are elements of Sym^2H, where $I=f^2\in\mathrm{Sym}^2H$ acts on $TM^{\mathbb{C}}=H\otimes E$ as $f^2(h\otimes e)=\sigma_H(f,h)f\otimes e$. Then $\eta_T(I\cdot,\cdot)$ is skew-symmetric because of

$$\eta_T(f^2(h\otimes e),\tilde h\otimes\tilde e)=\sigma_H(f,h)\sigma_H(f,\tilde h)\eta(e,\tilde e)\ .$$

The map $\eta\mapsto\hat\eta$ then defines the embedding $\Lambda^2\mathrm{E}^*\to\Lambda^4\mathrm{T}^*M^{\mathbb{C}}$. In particular, the symplectic form $\sigma_E\in\Lambda^2\mathrm{E}^*$ is mapped to the Kraines form $\Omega:=\omega_I\wedge\omega_I+\omega_J\wedge\omega_J+\omega_K\wedge\omega_K$. The parallel 4-form Ω spans the trivial subbundle in $\Lambda^4\mathrm{T}^*M^{\mathbb{C}}$. □

Proof of Theorem 1.11. Since the standard Laplace operator Δ_L commutes with parallel bundle maps we see that there is an isometric embedding $\Phi:\mathrm{Sym}^2\mathrm{T}^*M\to\Lambda^4\mathrm{T}^*M$ of bundles such that $\Phi\circ\Delta_L=\Delta_H\circ\Phi$. Since $\Delta_E=\Delta_L-2\frac{\mathrm{scal}}{4n}$ it follows that Δ_E is strictly positive in the L^2-sense if the Einstein constant is negative. Indeed, if h is a compactly supported tt-tensor and $\omega = \Phi(h)$, we have

$$\begin{aligned}(\Delta_E h, h)_{L^2} &= (\Delta_L h, h)_{L^2} - 2\frac{\mathrm{scal}}{4n}\,\|h\|^2_{L^2} \\ &= (\Delta_H \omega, \omega)_{L^2} - 2\frac{\mathrm{scal}}{4n}\,\|h\|^2_{L^2} \geq -2\frac{\mathrm{scal}}{4n}\,\|h\|^2_{L^2}\,,\end{aligned}$$

where we used that the Hodge-Laplace operator is non-negative. It follows that quaternion-Kähler manifolds of negative scalar curvature are stable, thus proving Theorem 1.11. □

Remark 2.3. For quaternion-Kähler manifolds of positive scalar curvature the situation is more complicated. Here the stability question is still open. As already mentioned, the complex 2-plane Grassmannian $\mathrm{Gr}_2(\mathbb{C}^{n+2})$ is only semi-stable, as it has a non-trivial space of infinitesimal Einstein deformations. On a quaternion-Kähler manifold of positive scalar curvature one can show that the Lichnerowicz Laplacian on trace-free symmetric 2-tensors is bounded from below by $2\,\frac{\mathrm{scal}}{4n}\,\frac{n+1}{n+2}$ (see [14]). It is perhaps interesting to note that $\frac{\mathrm{scal}}{4n}\,\frac{n+1}{n+2}$ is exactly the Einstein constant of the Kähler Einstein metric of the twistor space associated to the quaternion-Kähler manifold.

3. Non-integrable deformations and scalar curvature rigidity

Throughout this section, let us assume that (M, g) is an Einstein manifold of volume 1 and with Einstein constant λ. Recall that g is a critical point of S on $\mathcal{M}_1$. In order to avoid technical complications, we want to avoid the volume constraint. For this reason, we work instead with the Einstein-Hilbert functional with cosmological constant λ, given by

$$S_\lambda : g \mapsto S_\lambda(g) = \int_M (\mathrm{scal}_g + (2-n)\lambda)\; dV_g.$$

A standard computation shows that

$$D_g S_\lambda(h) = -\left(F_g, h\right)_{L^2},$$

where

$$F_g = \mathrm{Ric}_g - \frac{1}{2}\mathrm{scal}_g \cdot g - \frac{1}{2}(2-n)\lambda g.$$

Note that the Euler-Lagrange equation $F_g = 0$ is equivalent to $\mathrm{Ric}_g = \lambda g$.

3.1. A symmetric trilinear map.

Recall the definitions $E_g = \mathrm{Ric}_g - \frac{1}{n}\left(\int_M \mathrm{scal}_g\, dV_g\right) g$ and $\epsilon(g) = \ker(\Delta_E|_{TT})$ from Subsection 1.1. For a map $\Phi : V \to W$ between Banach spaces, we denote the k'th Frechet derivative at g by $D^k_g \Phi : V \times \ldots \times V \to W$. Our goal in this section is to show that the expression $(D^2_g E(h, k), l)_{L^2}$ defines a totally symmetric trilinear form on $\epsilon(g)$.

Lemma 3.1. *We have*

$$(D^2_g E(h, h), k)_{L^2} = (D^2_g F(h, h), k)_{L^2}$$

for all $h, k \in \epsilon(g)$.

Proof. Let $g_t = g + th$ and compute

$$\frac{d^2}{dt^2}E_{g_t} = \frac{d^2}{dt^2}\mathrm{Ric}_{g_t} - \frac{1}{n}\left(\int_M \left(\frac{d^2}{dt^2}\mathrm{scal}_{g_t}\right)\, dV_{g_t}\right) g_t - \frac{1}{n}\left(\int_M \mathrm{scal}_{g_t}\left(\frac{d^2}{dt^2}\, dV_{g_t}\right)\right) g_t$$
$$- \frac{2}{n}\left(\int_M \left(\frac{d}{dt}\mathrm{scal}_{g_t}\right)\left(\frac{d}{dt}\, dV_{g_t}\right)\right) g_t - \frac{2}{n}\left(\int_M \frac{d}{dt}(\mathrm{scal}_{g_t}\, dV_{g_t})\right) h.$$

By the first variation of the scalar curvature and the volume element (see e.g. [2, Thm. 1.174 and Prop. 1.186], we have $D_g\mathrm{scal}(h) = 0$ and $D_g dV(h) = 0$ for every tt-tensor h. In particular, these identities hold for $h \in \epsilon(g)$, so that $\frac{d}{dt}|_{t=0}\mathrm{scal}_{g_t} = 0$ and $\frac{d}{dt}|_{t=0}\, dV_{g_t} = 0$. Therefore, evaluating the above identity at $t = 0$ yields

$$D_g^2 E(h,h) = D_g^2\mathrm{Ric}(h,h) - \frac{1}{n}\left(\int_M D_g^2\mathrm{scal}(h,h)\, dV_g + \mathrm{scal}_g \cdot D_g^2 dV(h,h)\right) g.$$

A similar computation along the same curve g_t yields

$$\begin{aligned} D_g^2 F(h,h) &= D_g^2\mathrm{Ric}(h,h) - \frac{1}{2}D_g^2\mathrm{scal}\cdot g - D_g\mathrm{scal}(h)\cdot h \\ &= D_g^2\mathrm{Ric}(h,h) - \frac{1}{2}D_g^2\mathrm{scal}(h,h)\cdot g. \end{aligned}$$

By taking the scalar product of the two right hand sides with a tensor $k \in \epsilon(g)$ and using that $\langle g, k\rangle = \mathrm{tr}_g h = 0$, we see that

$$\langle D_g^2 E(h,h), k\rangle_g = \langle D_g^2\mathrm{Ric}(h,h), k\rangle_g = \langle D_g^2 F(h,h), k\rangle_g$$

and integrating over M yields the desired result. □

This lemma allows us to identify $(D_g^2 E(h,k), l)_{L^2}$ as the third variation of S_λ which is key for the following assertion.

Proposition 3.2. *The trilinear form*

$$\Phi : \epsilon(g)\times\epsilon(g)\times\epsilon(g) \to \mathbb{R}, \qquad (h,k,l) \mapsto (D^2E(h,k), l)_{L^2}$$

is totally symmetric. In particular, if we find $h, k \in \epsilon(g)$ such that $(D^2E(h,h),k)_{L^2} \neq 0$, we also find $l \in \epsilon(g)$ such that $(D^2E(l,l),l)_{L^2} \neq 0$.

Remark 3.3. The symmetry of Φ was already shown in [20]. There the authors obtain as a result of a direct but lengthy calculation an explicit formula for the full obstruction Φ in terms of the Frölicher-Nijenhuis bracket. It turns out to be symmetric in all three arguments and in particular it recovers the Koiso obstruction, i.e. $\Psi(h) = \Phi(h,h,h)$. Hence, Proposition 3.2 gives a simple way to check the explicit formula for Φ by first reformulating the Koiso obstruction and then polarising it. In particular, we see that the vanishing of the Koiso obstruction is a necessary and sufficient condition for integrability of second order.

Proof. We are going to show that the trilinear form of the lemma equals the third variation of $S_\lambda(g)$. Let $h, k, l \in \epsilon(g)$ and compute

$$\begin{aligned}\frac{d}{dt}\frac{d}{ds}\frac{d}{dr}S_\lambda(g+th+sk+rl) &= \frac{d}{dt}\frac{d}{ds}D_{g+th+sk+rl}S_\lambda(l)\\ &= -\frac{d}{dt}\frac{d}{ds}(F_{g+th+sk+rl}, l)_{L^2}\\ &= -\frac{d}{dt}[(D_{g+th+sk+rl}F(k), l)_{L^2} + (F_{g+th+sk+rl}, k * l)_{L^2}].\end{aligned}$$

Here, the second term on the right hand side comes from differentiating the scalar product and the volume element. We use the $*$-notation to denote a linear combination of tensor products and contractions. The two terms on the right hand side are computed to be

$$\begin{aligned}\frac{d}{dt}(D_{g+th+sk+rl}F(k), l)_{L^2} &= (D^2_{g+th+sk+rl}F(h,k), l)_{L^2} + (D_{g+th+sk+rl}F(k), h * l)_{L^2},\\ \frac{d}{dt}(F_{g+th+sk+rl}, k * l)_{L^2} &= (D_{g+th+sk+rl}F(h), k * l)_{L^2} + (F_{g+th+sk+rl}, h * k * l)_{L^2}.\end{aligned}$$

Also here, the second terms on the right hand sides comes from differentiating the scalar product and the volume element. Now, we evaluate at $t, s, r = 0$. Because g is Einstein, $F_g = 0$. Because $h, k \in \epsilon(g)$, we have $DF_g(k) = DF_g(h) = 0$ by the first variation of the Ricci tensor and the scalar curvature, see [2, Thm. 1.174]. Therefore by the above,

$$\frac{d}{dt}\frac{d}{ds}\frac{d}{dr}|_{t,s,r=0}S_\lambda(g+th+sk+rl) = -(D^2_gF(h,k), l)_{L^2} = -(D^2_gE(h,k), l)_{L^2},$$

where the second equation follows from Lemma 3.1. The left hand side is totally symmetric by Schwarz's theorem, and so is the right hand side. The second statement of the Lemma follows from the identity

$$6\Phi(h,h,k) = \Phi(k+h, k+h, k+h) + \Phi(k-h, k-h, k-h) - 2\Phi(k,k,k),$$

which holds for any totally symmetric triliear form. □

3.2. **Proof of Theorem 1.9.** Before we prove Theorem 1.9, we compute the first three derivatives of S_λ along a curve g_t which is tangent to $h \in \epsilon(g)$ at $g_0 = g$. In particular, we show that these derivatives depend on $g_0' = h$ but not on higher derivatives of g_t.

Lemma 3.4. *Let $h \in \epsilon(g)$. Then for every smooth family g_t of metrics with $g_0 = g$ and $\frac{d}{dt}|_{t=0}g_t = h$, we have*

$$\frac{d}{dt}|_{t=0}S_\lambda(g_t) = 0, \qquad \frac{d^2}{dt^2}|_{t=0}S_\lambda(g_t) = 0, \qquad \frac{d^3}{dt^3}|_{t=0}S_\lambda(g_t) = -(D^2_gE(h,h), h)_{L^2}.$$

Proof. Let $f(t) = S_\lambda(g_t)$. Then we compute

$$\begin{aligned}f'(t) &= D_{g_t}S_\lambda(g_t') = -(F_{g_t}, g_t')_{L^2},\\ f''(t) &= -(D_{g_t}F(g_t'), g_t')_{L^2} - (F_{g_t}, g_t'')_{L^2} + (F_{g_t}, g_t' * g_t')_{L^2},\\ f'''(t) &= -(D^2_{g_t}F(g_t', g_t'), g_t')_{L^2} - 2(D_{g_t}F(g_t'), g_t'')_{L^2} - (D_{g_t}F(g_t''), g_t')_{L^2} - (F_{g_t}, g_t''')_{L^2}\\ &\quad + (D_{g_t}F(g_t'), g_t' * g_t')_{L^2} + (F_{g_t}, g_t' * g_t'')_{L^2} + (F_{g_t}, g_t' * g_t' * g_t')_{L^2}.\end{aligned}$$

The second term for $f''(t)$ and the terms in the last line come from differentiating the scalar product and the volume element. Since $F_g = 0$, we have $f'(0) = 0$. Because $h \in \epsilon(g)$, the first variation of the Ricci tensor and the scalar curvature (see [2, Thm. 1.174]) yield

$$f''(0) = -(D_g F(h), h)_{L^2} = -\frac{1}{2}(\Delta_E h, h)_{L^2} = 0.$$

Before we are going to evaluate the third variation, we compute

$$\begin{aligned}\frac{d}{dt}\frac{d}{ds}S_\lambda(g+th+sk) &= \frac{d}{dt}D_g S_\lambda(k)\\ &= -\frac{d}{dt}(F_{g+th+sk}, k)_{L^2} = -(D_{g+th+sk}F(h), k)_{L^2} + (F_{g+th+sk}, h * k)_{L^2}.\end{aligned}$$

Evaluating at $t = s = 0$ yields

$$\frac{d}{dt}\frac{d}{ds}|_{t,s=0}S_\lambda(g+th+sk) = -(D_g F(h), k)_{L^2},$$

which in particular shows that DF_g is symmetric. Thus, $(D_g F(g_0''), g_0')_{L^2} = (D_g F(g_0'), g_0'')_{L^2}$. In combination with $F_g = 0$ and $D_g F(g_0') = DF_g(h) = 0$, we obtain

$$f'''(0) = -(D_g^2 F(h,h), h)_{L^2} = -(D_g^2 E(h,h), h)_{L^2},$$

which finishes the proof. □

Now we have all ingredients together to prove Theorem 1.9.

Proof of Theorem 1.9. We may assume that $\mathrm{vol}(M, g) = 1$. If $l \in \epsilon(g)$ is not integrable to second order, we find $k \in \epsilon(g)$ such that

$$(D_g^2 E(l,l), k)_{L^2} \neq 0.$$

By Proposition 3.2, we also find $h \in \epsilon(g)$ such that

$$(D_g^2 E(h,h), h)_{L^2} \neq 0.$$

Because $h \in TT$, it is tangent to the manifold $\mathcal{C}_1$, c.f. Remark 1.6. Therefore, we find a curve $g_t \in \mathcal{C}_1$ with $g_0 = g$ and $g_0' = h$. By Lemma 3.4, we have

$$\frac{d}{dt}|_{t=0}S(g_t) = 0, \qquad \frac{d^2}{dt^2}|_{t=0}S(g_t) = 0, \qquad \frac{d^3}{dt^3}|_{t=0}S(g_t) = -(D_g^2 E(h,h), h)_{L^2} \neq 0.$$

Depending on the sign of the third derivative, we get $S(g_t) > S(g)$ either for $t \in (0, \epsilon)$ or $t \in (-\epsilon, 0)$. By definition of S and $\mathcal{C}_1$, we get

$$\mathrm{scal}_{g_t} > \mathrm{scal}_g, \qquad \mathrm{vol}(M, g_t) = 1 = \mathrm{vol}(M, g),$$

which finishes the proof of the theorem. □

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