SMOOTH BLOW UP STRUCTURES ON PROJECTIVE BUNDLES OVER PROJECTIVE SPACES

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Abstract. Assuming Hartshorne's conjecture on complete intersections, we classify projective bundles over projective spaces that have a smooth blow up structure over another projective space. Under some assumptions, we also classify projective bundles over projective spaces that have a smooth blow up structure over some arbitrary smooth projective variety, not necessarily a projective space. We determine which of the globally generated vector bundles over projective spaces of first Chern class at most five have the property that their projectivisations have smooth blow up structures, with no additional assumption. In the way, we get some new examples of varieties with both projective bundle and smooth blow up structures.

Keywords: Blow up, projective bundle, Bordiga 3-fold, Tango bundle

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1. Introduction

Given a smooth projective variety X, there are two standard ways of constructing another smooth projective variety with Picard number one more than X. One is to construct a projective bundle over X, another is to blow-up a smooth subvariety of codimension at least 2 in X. It is interesting to consider when a smooth projective variety can be constructed by the above procedure in two different ways, in other words, when a smooth projective variety has two different structures of the above kind. We are particularly interested in the case of smooth projective varieties of Picard rank 2 having two different such structures.

There are a few classes of examples of such varieties in literature. In [35],[21] there are examples of smooth projective varieties having two projective bundle structures. In [27],[10], [7], [8] there are examples of smooth projective varieties having two smooth blow up structures. In [6],[33],[13], [26], there are examples of smooth projective varieties having both projective bundle and smooth blow up structures. In §3 of this paper, we give a few examples of projective bundles over projective spaces that have a smooth blow up structure. Some of them have already appeared in literature, some of them are new.

Next, we turn our attention to the classification of such varieties. [28] classifies varieties with two projective bundle structures over projective spaces. [27] gives a satisfactory answer towards the classification of varieties with two smooth blow up structures over projective spaces. In this paper, we tackle the remaining case, of varieties with both projective bundle and smooth blow up structures over projective spaces.

Theorem A. *Let n, r be positive integers, X a smooth projective variety with two contractions $\pi : X \to \mathbb{P}^n$, $\phi : X \to \mathbb{P}^{n+r}$ such that π is a projective bundle structure and ϕ is a smooth blow up along a nonlinear subvariety W of $\mathbb{P}^{n+r}$. Let $d \geq 2$ be the integer such that the rational map $\pi \circ \phi^{-1}$: $\mathbb{P}^{n+r} \dashrightarrow \mathbb{P}^n$ is defined by an $(n+1)$-tuple of homogeneous polynomials of degree d. Then one of the following holds.*

(1) $d = 2$, and X is as in example 3.2 or 3.3 of §3.

(2) $d = 3$ and X is as in example 3.4 of §3.

(3) $d \geq 3$, $r = n/d$ and W has same Betti numbers as $\mathbb{P}^{n-1}$.

If Hartshorne's conjecture on complete intersection holds then (3) *is not possible, so X is always as in one of the examples of §3.*

By Hartshorne's conjecture on complete intersection, we mean the following statement: If W a smooth subvariety of $\mathbb{P}^n$ of dimension $> 2n/3$, then W is a complete intersection.

Next, we want to classify all projective bundles over projective spaces that have a smooth blow up structure. This smooth blow up structure can be over any smooth projective variety, not necessarily a projective space. To the author's best knowledge, there is no known example other than the examples in §3. We prove this in Theorem B under several assumptions. As in [3], for a globally generated vector bundle F on $\mathbb{P}^n$ let P_F be the dual of the kernel of the evaluation morphism $H^0(F) \otimes \mathcal{O}_{\mathbb{P}} \to F$. In [3, Conjecture 0.3], the following conjecture is made for all n and c.

Conjecture $M_{n,c}$: If E is a globally generated vector bundle on $\mathbb{P}^n$ with first Chern class $\leq \min\{c, n-1\}$ and $H^0(E^*) = H^1(E^*) = 0$, then one of the following holds:

(1) $E \cong A \oplus P_B$, where A, B are globally generated split bundles.

(2) $n \geq 3$ and $E \cong \Omega_{\mathbb{P}^n}(2)$.

(3) $n \geq 4$ and $E = \wedge^2(T_{\mathbb{P}^n}(-1))$.

By *split* bundle, we mean a direct sum of line bundles.

By [3],[4], [5], $M_{n,5}$ is true for all n.

Now we can state our result.

Theorem B. *Let n, r be positive integers, X a smooth projective variety which is a $\mathbb{P}^r$-bundle over $\mathbb{P}^n$, and also has a smooth blow up structure over a smooth projective variety Y. Let $\mathcal{E}$ be the unique nef vector bundle of rank $r+1$ over $\mathbb{P}^n$ such that $X \cong \mathbb{P}_{\mathbb{P}^n}(\mathcal{E})$ over $\mathbb{P}^n$ and $\mathcal{E}(-1)$ is not nef. Suppose $\mathcal{E}$ is not ample. Then $c_1(\mathcal{E}) \leq n$ and the following holds:*

(1) Suppose $\mathcal{E}$ is globally generated, equivalently, the ample generator of Y is globally generated. If $M_{n,c_1(\mathcal{E})}$ is true, then either X is one of the examples in §3 or $c_1 = n$, codim $W = 2$, and if $0 \to \mathcal{O}_{\mathbb{P}^n}^k \to F \to \mathcal{E} \to 0$ is a short exact sequence of vector bundles on $\mathbb{P}^n$ with $H^0(F^*) = H^1(F^*) = 0$, then F is indecomposable.

(2) Suppose the ample generator of Y is very ample. Then either X is one of the examples in §3 or rk $\mathcal{E} \leq n$.

The nonampleness of $\mathcal{E}$ is an assumption made in the previous works also, see for example [18, Lemma 2.5], also the equivalent conditions in [17, Proposition 5] in our setup is equivalent to nonampleness of $\mathcal{E}$.

Finally, we classify the projectivizations of globally generated vector bundles over projective spaces of first Chern class at most 5 which has a smooth blow up structure, with no additional assumptions. This is an extension of [6, Theorem E].

Theorem C. *Let n be a positive integer and $\mathcal{E}$ a globally generated vector bundle on $\mathbb{P}^n$ with $c_1(\mathcal{E}) \leq 5$. If $X = \mathbb{P}_{\mathbb{P}^n}(\mathcal{E})$ is a smooth blow up, then X is one of the examples in §3.*

As mentioned in [6], this result is a counterpart of the results in [17] and [18] in the sense that [17] and [18] classify when the blow-up of a variety of small dimensions in projective space has a projective bundle structure, and our result classifies when the projectivization of a globally generated vector bundle of small first Chern class over projective space has a smooth blow up structure.

The same method in this paper also classifies projective bundles over projective spaces that have another projective bundle structure, under some assumptions and assuming M_{n,c_1} (see Remark 4.2). For some previously obtained results regarding these varieties, see [16].

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2. Notation, conventions and lemmas

We work throughout over the field $k = \mathbb{C}$ of complex numbers. A *variety* is an integral, separated scheme of finite type over k. By subvariety we always mean a closed subvariety. By *smooth blow up* we mean blow up of a smooth variety along a smooth subvariety. For positive integers m, n, $Gr(m, n)$ denotes the Grassmanian variety of m-dimensional linear subspaces of k^n. If $\mathcal{E}$ is a vector bundle of rank ≥ 2 on $\mathbb{P}^n$ and the R is the extremal ray of $\mathbb{P}(\mathcal{E})$ which is not contracted by the projective bundle map, then the contraction of R, if it exists, will be called the *other contration* of $\mathbb{P}(\mathcal{E})$.

We shall need the following lemmas.

Lemma 2.1. *Let n, r be positive integers and for $1 \leq i \leq r$, E_i a globally generated vector bundle over $\mathbb{P}^n$. Let $V_i = H^0(\mathbb{P}^n, E_i)$. Let $\phi : \mathbb{P}_{\mathbb{P}^n}(\oplus_i E_i) \to \mathbb{P}(\oplus_i V_i)$ be the morphism given by $|\mathcal{O}_{\mathbb{P}(\oplus_i E_i)}(1)|$, $\pi : \mathbb{P}_{\mathbb{P}^n}(\oplus_i E_i) \to \mathbb{P}^n$ the projection. For $1 \leq i \leq r$, let $\phi_i : \mathbb{P}_{\mathbb{P}^n}(E_i) \to \mathbb{P}(V_i)$ be the morphism given by $|\mathcal{O}_{\mathbb{P}(E_i)}(1)|$, $\pi_i : \mathbb{P}_{\mathbb{P}^n}(E_i) \to \mathbb{P}^n$ the projection. Let $v_i \in V_i^*$ for $1 \leq i \leq r$, $v = (v_i)_i \in (\oplus_i V_i)^*$. Then we have $\phi^{-1}([v]) \cong \pi\phi^{-1}([v]) = \cap_i S_i$, where*

$$S_i = \begin{cases} \pi_i \phi_i^{-1}([v_i]) & \text{if } v_i \neq 0 \\ \mathbb{P}^n & \text{if } v_i = 0. \end{cases}$$

Proof. Same argument as in the proof of [6, Lemma 2.4] shows the result. □

Lemma 2.2. *Let Y be a smooth variety, W a smooth subvariety of Y, $X \xrightarrow{\phi} Y$ the blow of Y along W, $E \hookrightarrow X$ the exceptional divisor. Let W' be a closed subscheme of Y such that $\phi^{-1}(W') = E$ scheme-theoretically. Then $W = W'$.*

Proof. The question is analytic local on Y, so it suffices to prove the statement when $Y = Spec\ k[[X_1, ..., X_n]]$ and W is defined by the ideal $I = (X_1, ..., X_r)$ for some $1 \leq r < n$. Clearly $W'_{red} = W$. Suppose $W' \neq W$. Let W^* be the closed subscheme of Y defined by the ideal $(X_1, ..., X_{r-1}) + \mathfrak{m}I$, where $\mathfrak{m} = (X_1, ..., X_n)$ is the maximal ideal. Upto a change of coordinates, we have $W \hookrightarrow W^* \hookrightarrow W'$. So, $\phi^{-1}(W^*) = E$ scheme-theoretically. But in the blow up chart with coordinates given by $Y_i = X_i/X_r$ for $1 \leq i \leq r-1$ and X_j for $r \leq j \leq n$, the ideal of $\phi^{-1}(W^*)$ has no linear term at the origin. This contradicts $\phi^{-1}(W^*) = E$. So, $W = W'$. □

Now let us consider the following setup.

Setup: Let n, r be positive integers, Y a smooth projective variety of dimension $n+r$, W a smooth subvariety of Y of dimension $m \leq n+r-2$. Let $\phi : X \to Y$ be the blow up of Y along W, $E \subset X$ the exceptional divisor of ϕ. Suppose X has also a structure of a $\mathbb{P}^r$-bundle over $\mathbb{P}^n$, let $\pi : X \to \mathbb{P}^n$ be a

$\mathbb{P}^r$-bundle structure. There is a unique nef vector bundle $\mathcal{E}$ of rank $r+1$ over $\mathbb{P}^n$ such that $X \cong \mathbb{P}_{\mathbb{P}^n}(\mathcal{E})$ over $\mathbb{P}^n$ and $\mathcal{E}(-1)$ is not nef. (Recall that a vector bundle G over $\mathbb{P}^n$ is called nef/ample/big if $\mathcal{O}_{\mathbb{P}(G)}(1)$ is nef/ample/big). Since $\mathbb{Z}^2 \cong \text{Pic } X \cong \mathbb{Z} \oplus \text{Pic } Y$, we see that Pic $Y \cong \mathbb{Z}$. Since Y is also rational, Y is a Fano manifold of Picard number 1. Let $\mathcal{O}_Y(1)$ be the ample generator of Pic Y, and $H_1 = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$, $H_2 = \phi^*\mathcal{O}_Y(1)$, $U = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \in \text{Pic }(X)$. We can regard the Weil divisor E also as an element of Pic(X). Let τ be the index of Y, that is, $-K_Y = \mathcal{O}_Y(\tau)$. Also, let α be the top self-intersection number of $\mathcal{O}_Y(1)$, so $H_2^{n+r} = \alpha$. Let $\mathcal{O}_W(1) = \mathcal{O}_Y(1)|_W$.

Let F_1 be a line in a fiber $\pi^{-1}(z)$ (we have $\pi^{-1}(z) \cong \mathbb{P}^r$), and F_2 be a line in a fiber $\phi|_E^{-1}(w)$ (we have $\phi|_E^{-1}(w) = \mathbb{P}^{n+r-m-1}$). So, $N_1(X)_{\mathbb{Q}}$ has basis $\{F_1, F_2\}$, and $\{H_1, U\}$ and $\{H_2, E\}$ are both bases of Pic$(X) \cong \mathbb{Z}^2$. We have $H_1 \cdot F_1 = H_2 \cdot F_2 = 0$, $U \cdot F_1 = 1$, $E \cdot F_2 = -1$. Let $a = H_1 \cdot F_2$, $d = E \cdot F_1$, $b = U \cdot F_2$. By [17, Lemma 3] and its proof, we have $H_2 \cdot F_1 = a$, $a|1 + bd$, and

$$H_1 = dH_2 - aE$$

$$U = \frac{1+bd}{a} H_2 - bE$$

in Pic(X). Note that if $Y = \mathbb{P}^{n+r}$, then $H_1 = dH_2 - aE$ implies that the rational map $\pi \circ \phi^{-1} : \mathbb{P}^{n+r} \dashrightarrow \mathbb{P}^n$ is defined by an $(n+1)$-tuple of homogeneous polynomials of degree d, so this d is the same d as in the statement of Theorem A. Since U is nef but $U - H_1$ is not nef, we have $0 \le b < a$.

For $0 \le i \le n$ Let $c_i, s_i \in \mathbb{Z}$ denote the i-th Chern and Segre classes of $\mathcal{E}$, respectively. Here we are using the natural identification $H^{2i}(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}$, via the natural generator of $H^{2i}(\mathbb{P}^n, \mathbb{Z})$ given by i-th power of the first Chern class of $\mathcal{O}_{\mathbb{P}^n}(1)$. By definition of Segre classes as in [12, Chapter 3] we have $U^{r+i}H_1^{n-i} = (-1)^i s_i$ for $0 \le i \le n$. Also by[12, Chapter 3] we have

$$\Big(\sum_{i=0}^{n} c_i t^i\Big)\Big(\sum_{i=0}^{n} s_i t^i\Big) \equiv 0 (\text{mod } t^{n+1}). \qquad (1)$$

Lemma 2.3. *In the notation of the setup, we have:*

(i) We have: $\mathcal{E}$ *is not ample* $\Leftrightarrow a = 1, b = 0$.

(ii) We have:

$$\tau = d(n+1-c_1) + (r+1)\frac{1+bd}{a},$$
$$n + r - m - 1 = a(n+1-c_1) + b(r+1).$$

(iii) If $\mathcal{E}$ *is not ample, we have* $d = \frac{\tau - r - 1}{n+r-m-1}, c_1 = m - r + 2.$

(iv) *If $\mathcal{E}$ is not ample, then W is the base locus of $\pi \circ \phi^{-1} : Y \dashrightarrow \mathbb{P}^n$.*

(v) $a^r | \alpha$.

(vi) *If $Y = \mathbb{P}^{n+r}$, then $a = 1, b = 0$, codim $W = n + r - m = \frac{n}{d} + 1, c_1 = \frac{d-1}{d} n + 1$.*

Proof. *(i):* Note that: $\mathcal{E}$ is not ample $\Leftrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)(= U)$ is not ample $\Leftrightarrow U \cdot F_2 \leq 0$ (as $U \cdot F_1 = 1 > 0$) $\Leftrightarrow b \leq 0 \Leftrightarrow b = 0$ (as we assumed $b \geq 0$) $\Leftrightarrow a = 1, b = 0$ (as $a | 1 + bd$).

(ii): By canonical bundle formulae, $\tau H_2 - (n + r - m - 1)E = -K_X = (n+1-c_1)H_1 + (r+1)U = (n+1-c_1)(dH_2 - aE) + (r+1)(\frac{1+bd}{a} H_2 - bE)$. Comparing coefficients of H_2 and E in both sides, we get ii).

(iv): We have $a = 1, b = 0$ by (i). Let W' be the base locus of $\pi \circ \phi^{-1}$. Since $dH_2 = H_1 + E$, we see that $\phi^{-1}(W') = E$ scheme-theoretically. By Lemma 2.2, $W = W'$.

(v): We have $\alpha = H_2^{n+r} = (-bH_1 + aU)^{n+r} = \sum_{i=0}^{n} \binom{n+r}{i} (-b)^i a^{n+r-i} H_1^i U^{n+r-i}$. Clearly each term in this sum is divisible by a^r.

(iii) is immediate from (i) and (ii). For (vi), note that $\alpha = 1$ for $Y = \mathbb{P}^{n+r}$, so $a = 1$ by (v). The rest follows from (i) and (iii).

□

Lemma 2.4. *In the notation of the setup, assume $n \geq 2$.*

(i) *If $1 \leq t \leq min\{r, n + r - m - 1\}$, $j \geq 0$ are integers, such that $a^j | \binom{n+r-t}{n-1}$, then $a^{min\{2, j+1\}} | \binom{n+r-t}{n}$.*

(ii) $a | \binom{n+r-1}{n}$.

(iii) *If $n + r - m \geq 3$, and $r \geq 2$, then $a^2 | \binom{n+r-2}{n}$.*

(iii) $a^3 | \binom{n+r-1}{n} b^2 d + \binom{n+r-1}{n-1} ab(\frac{1+bd}{a} - c_1 d) + \binom{n+r-1}{n-2} a^2 (dU^{r+2} H_1^{n-2} - c_1 \cdot \frac{1+bd}{a})$.

Proof. *(i):* Since $n + r - t > m = \dim W$,we have

$$0 = E^t H_2^{n+r-t} = (-\frac{1+bd}{a} H_1 + dU)^t (-bH_1 + aU)^{n+r-t}.$$

So,

(2)

$$0 = \binom{n+r-t}{n} (-b)^n a^{r-t} d^t$$
$$+ \Big(\sum_{i=0}^{t} \binom{t}{i} \Big(-\frac{1+bd}{a} \Big)^i d^{t-i} H_1^i U^{t-i} \Big) \cdot \Big(\sum_{i=0}^{n-1} \binom{n+r-t}{i} (-b)^i a^{n+r-t-i} H_1^i U^{n+r-t-i} \Big).$$

Note that $a^{r-t+min\{2,j+1\}}|\binom{n+r-t}{i}(-b)^i a^{n+r-t-i}H_1^i U^{n+r-t-i}$ for all $0 \le i \le n-1$. So, right hand side of equation (2) is $\equiv \binom{n+r-t}{n}(-b)^n a^{r-t}d^t (\text{mod } a^{r-t+min\{2,j+1\}})$. So,

$$a^{r-t+min\{2,j+1\}})|\binom{n+r-t}{n}(-b)^n a^{r-t}d^t,$$

hence $a^{min\{2,j+1\}})|\binom{n+r-t}{n}(-b)^n d^t$. As $a|1+bd$, so $\gcd(a,b) = \gcd(a,d) = 1$. So (i) follows.

(ii) follows by putting $t = 1, j = 0$ in (i).

(iii): Put $j = 0$ in (i) to get $a|\binom{n+r-t}{n}$ for $t = 1, 2$. So, $a|\binom{n+r-1}{n} - \binom{n+r-2}{n} = \binom{n+r-2}{n-1}$. Now put $t = 2, j = 1$ in (i) to get (iii).

(iv): Put $t = 1$ in equation (2) and get

$$\begin{aligned}
0 = &\binom{n+r-1}{n}(-b)^n a^{r-1}d \\
&+\sum_{i=0}^{n-1}\binom{n+r-1}{i}(-b)^i a^{n+r-1-i}(dU^{n+r-i}H_1^i - \frac{1+bd}{a}U^{n+r-i-1}H_1^{i+1}) \\
\equiv &\binom{n+r-1}{n}(-b)^n a^{r-1}d + \binom{n+r-1}{n-1}(-b)^{n-1}a^r(c_1 d - \frac{1+bd}{a}) \\
&+\binom{n+r-1}{n-2}(-b)^{n-2}a^{r+1}(dU^{r+2}H_1^{n-2} - \frac{1+bd}{a}c_1)(\text{mod } a^{r+2}).
\end{aligned}$$

Here we used $U^r H_1^n = 1, U^{r+1}H_1^{n-1} = -s_1 = c_1$. Since $\gcd(a,b) = 1$, we can take the common factor $(-b)^{n-2}$ out, and then divide by a^{r-1} to get (iv).

□

Lemma 2.5. *In the notation of the setup, suppose $\mathcal{E}$ is not ample. Then for $c_1 - 1 \le i \le n$, we have $(-1)^i s_i = d^{n-i}\alpha$ and $(-1)^{c_1-2}s_{c_1-2} = d^{n-c_1+2}\alpha - \mathcal{O}_W(1)^{\cdot m} < d^{n-c_1+2}\alpha$.*

Proof. We have $a = 1, b = 0$ by Lemma 2.3. Since $E \to W$ is a $\mathbb{P}^{n+r-m-1}$-bundle, we have $E^{n+r-l}H_2^l = 0$ for $m < l < n+r$ and $E^{n+r-m}H_2^m = (-1)^{n+r-m-1}\mathcal{O}_W(1)^{\cdot m}$ (see also [27, Lemma 2.4]. So, $(-1)^i s_i = U^{r+i}H_1^{n-i} = (dH_2 - E)^{n-i}H_2^{r+i} = d^{n-i}\alpha$, as $H_2^{n+r} = \alpha$ and for $i \ge c_1 - 1$ we have $r+i > c_1+r-2 = m = \dim W$, so $E^{n-i-j}H_2^{r+i+j} = 0$ for $0 \le j < n-i$. Also, $(-1)^{c_1-2}s_{c_1-2} = U^{r+c_1-2}H_1^{n-c_1+2} = (dH_2 - E)^{n-c_1+2}H_2^{r+c_1-2} = (dH_2 - E)^{n+r-m}H_2^m = d^{n+r-m}\alpha + (-1)^{n+r-m}E^{n+r-m}H_2^m = d^{n-c_1+2}\alpha - \mathcal{O}_W(1)^{\cdot m}$. The last inequality in Lemma follows from the fact that $\mathcal{O}_W(1)$ is ample. □

For a compact complex manifold Z and an integer i, denote $\dim_{\mathbb{C}} H^i(Z, \mathbb{C})$ by $h^i(Z)$. Let $a_i = h^{2i}(W)$. Let $P(t) = \sum_i a_i t^i \in \mathbb{Z}[t]$.

Lemma 2.6. *In the notation of the setup, suppose $Y = \mathbb{P}^{n+r}$. We have:*

(i) $a_i \neq 0 \iff 0 \leq i \leq m$

(ii) $P(t) = (\sum_{i=0}^{d-1} t^{in/d})(\sum_{i=0}^{r-1} t^i)$.

(iii) For each k, $H^{2k}(W, \mathbb{Z})$ is free abelian.

(iv) $n/d \leq r$.

Proof. (i) follows from the general fact that for a compact Kahler manifold Y of complex dimension m, we have $H^{2i}(Y, \mathbb{C}) \neq 0$ for $0 \leq i \leq m$.

$\underline{(ii)}$ and $\underline{(iii)}$: By [34, Theorem 7.31], and Lemma 2.3 (vi),

$$H^{2k}(X, \mathbb{Z}) \cong H^{2k}(\mathbb{P}^{n+r}, \mathbb{Z}) \oplus (\oplus_{i=0}^{n/d} H^{2k-2i-2}(W, \mathbb{Z})).$$

Also, we have $H^{2k}(X, \mathbb{Z}) = H^{2k}(\mathbb{P}^n \times \mathbb{P}^r, \mathbb{Z})$, as X is a $\mathbb{P}^r$-bundle over $\mathbb{P}^n$. So, $H^{2k}(W, \mathbb{Z})$ is free abelian, being a subgroup of the free abelian group $H^{2k}(X, \mathbb{Z})$.

By Lemma 2.3 (vi), we have codim $W = n/d + 1$. In the Grothendieck ring of varieties, we have

$$[\mathbb{P}^n] \times [\mathbb{P}^r] = [X] = [\mathbb{P}^{n+r}] + [W]([\mathbb{P}^{n/d}] - 1).$$

This equation, applied to $X =$ blow-up of $\mathbb{P}^{n+r}$ along a linear $\mathbb{P}^{n-1}$, shows

$$[\mathbb{P}^n] \times [\mathbb{P}^r] = [\mathbb{P}^{n+r}] + [\mathbb{P}^{n-1}]([\mathbb{P}^r] - 1).$$

So,

$$[W]([\mathbb{P}^{n/d}] - 1) = [\mathbb{P}^{n-1}]([\mathbb{P}^r] - 1).$$

Now (ii) follows by applying weight characteristic motivic measure (see [19]).

$\underline{\text{(iv)}}$: Suppose $n/d > r$. As $d \geq 2$, Lemma 2.6 (ii) shows $a_{r-1} > 0$, $a_{n/d} > 0$, $a_r = 0$. This is impossible by Lemma 2.6 (i). $\square$

Lemma 2.7. *Let n be a positive integer, F a globally generated vector bundle over $\mathbb{P}^n$ with first Chern class $c_1 \leq n, \mathbb{P}(F) \xrightarrow{\pi} \mathbb{P}^n$ the projection. Let $\phi : \mathbb{P}(F) \to \mathbb{P}(H^0(F))$ be the morphism given by $|\mathcal{O}_{\mathbb{P}(F)}(1)|$. Suppose ϕ is surjective, and is a divisorial contraction. Let $S \subsetneq W_0 \subset \mathbb{P}(H^0(F))$ be subvarieties such that $W_0 \setminus S$ is smooth, ϕ is an isomorphism outside W_0, and for all $w \in W_0 \setminus S$, $\pi(\phi^{-1}(w)_{red})(\cong \phi^{-1}(w)_{red})$ is a linear subvariety of $\mathbb{P}^n$ of dimension $n + 1 - c_1$. Let $0 \leq k \leq$ dim W_0 and $0 \to \mathcal{O}_{\mathbb{P}^n}^k \xrightarrow{s} F \to \mathcal{E} \to 0$ be a short exact sequence of vector bundles, and $L = \mathbb{P}(H^0(\mathcal{E})) \hookrightarrow \mathbb{P}(H^0(F))$ be the codimension k linear subvariety corresponding to s. If $L \cap S = \varnothing$, then $L \cap W_0$ is a smooth subvariety of L of codimension $n + 2 - c_1$, and $\phi|_{\mathbb{P}(\mathcal{E})} : \mathbb{P}(\mathcal{E}) \to L$ is the blow up of L along $L \cap W_0$.*

Proof. Note that $c_1 \leq n$ implies $X = \mathbb{P}(F)$ is Fano. So, ϕ is an extremal contraction and its exceptional locus is an irrducible divisor E.

Since $k \leq \dim W_0$, we have $L \cap W_0 \neq \varnothing$. So, $\psi = \phi|_{\mathbb{P}(\mathcal{E})} : \mathbb{P}(\mathcal{E}) \to L$ is not an isomorphism.

If $L \subset W_0$, then

$$\dim(\mathbb{P}(F)) - k = \dim(\mathbb{P}(\mathcal{E})) = \dim L + n + 1 - c_1$$
$$= \dim(\mathbb{P}(H^0(F))) - k + n + 1 - c_1 = \dim(\mathbb{P}(F)) - k + n + 1 - c_1,$$

the last equality follows as $\dim(\mathbb{P}(F)) = \dim(\mathbb{P}(H^0(F)))$, ϕ being birational. So, $c_1 = n+1$, contradicting $c_1 \leq n$. So, $L \not\subset W_0$.

So, ψ is birational, and $\mathrm{Ex}(\psi) = \mathrm{Ex}(\phi) \cap \phi^{-1}(L)$ is a divisor in $\phi^{-1}(L) = \mathbb{P}(\mathcal{E})$, as $\phi^{-1}(L) \not\subset \mathrm{Ex}(\phi)$. So, if R is the extremal ray of $\overline{NE}(\mathbb{P}(\mathcal{E}))$, which is not contracted by π, then ψ is the contraction of R, and ψ is divisorial. Let $w \in L \cap W_0$ and C a line in $\psi^{-1}(w)_{red} (\cong \mathbb{P}^{n+1-c_1})$. Note that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \cdot C = 0$ as C is contracted by the morphism given by $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$. Also, $\pi^*\mathcal{O}_{\mathbb{P}^n}(1).C = 1$, as $C \xrightarrow{\pi} \mathbb{P}^n$ maps C isomorphically to a line in $\mathbb{P}^n$. Since $-K_{\mathbb{P}(\mathcal{E})} = \pi^*\mathcal{O}_{\mathbb{P}^n}(n+1-c_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mathrm{rk}\ (\mathcal{E}))$, the length of R is given by $l(R) = -K_{\mathbb{P}(\mathcal{E})}.C = n+1-c_1 = \dim\ \psi^{-1}(w)$. By [2, Theorem 5.2], ψ is a smooth blow up. So, $(L \cap W_0)_{red}$ is smooth and ψ is blow up along $(L \cap W_0)_{red}$.

It remains to show that $(L \cap W_0)_{red} = L \cap W_0$. By Lemma 2.2, it suffices to show that $\psi^{-1}(L \cap W_0)$ is reduced. Note that by [10, Theorem 1.1], ϕ is blow up along W_0 away from S. So, $\phi^{-1}(W_0) = E$ away from S. Hence, $\psi^{-1}(L \cap W_0) = \phi^{-1}(L) \cap \phi^{-1}(W_0) = \phi^{-1}(L) \cap E$, an effective Cartier divisor in $\phi^{-1}(L)$. If C is a line in a fibre over a point of $L \cap W_0$, then $(\psi^{-1}(L \cap W_0) \cdot C)_{\psi^{-1}(L)} = (E \cdot C)_X = -1$. This forces $\psi^{-1}(L \cap W_0)$ to be reduced. $\square$

Lemma 2.8. *Let n be a positive integer, F a globally generated vector bundle over $\mathbb{P}^n$ with $c_1(F) \leq n$, $\mathbb{P}(F) \xrightarrow{\phi} \mathbb{P}(H^0(F))$ the morphism given by $|\mathcal{O}_{\mathbb{P}(F)}(1)|$. Let $k \geq 0$, $0 \to \mathcal{O}_{\mathbb{P}^n}^k \to F \to \mathcal{E} \to 0$ be a short exact sequence of vector bundles, and suppose the other contraction of $\mathbb{P}(\mathcal{E})$ is a smooth blow up. Let l be the largest integer such that $c_l(F) \neq 0$, and suppose $S \hookrightarrow \mathbb{P}(H^0(F))$ is a closed subset such that the following holds: for all $s \in S$, $\phi^{-1}(s)_{red}$ has a connected component which is not isomorphic to a projective space of dimension $\leq n - c_1(F) + 1$,*

Then $\dim S + 1 \leq k \leq \mathrm{rk}\ F - l$.

Proof. If $k > \mathrm{rk}\ F - l$, then there is a short exact sequence of vector bundles $0 \to \mathcal{O}_{\mathbb{P}^n}^{\mathrm{rk}\ F - l + 1} \to F \to \mathcal{E}_1 \to 0$. By [3, Lemma 1.4], $c_l(F) = 0$, a contradiction. So $k \leq \mathrm{rk}\ F - l$.

To show $\dim S + 1 \leq k$, it suffices to show $L \cap S = \varnothing$, where $L = \mathbb{P}(H^0(\mathcal{E})) \hookrightarrow \mathbb{P}(H^0(F))$ is the codimension k linear subvariety. Note that $\psi = \phi|_{\mathbb{P}(\mathcal{E})} : \mathbb{P}(\mathcal{E}) \to L$ is the morphism given by $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$, and $\psi^{-1}(y) \cong \phi^{-1}(y)$ for all

$y \in L$. Suppose the contrary, $L \cap S \neq \varnothing$. Let $\mathbb{P}(\mathcal{E}) \xrightarrow{\psi_1} Y \xrightarrow{\eta} L$ be the Stein factorization of ψ. Let $s \in L \cap S$. By assumption, s is in image of ψ, so is in image of η. So, there is $y \in Y$ such that $\eta(y) = s$, and $\psi_1^{-1}(y)$ is not isomorphic to a projective space of dimension $\leq n+1-c_1$.

In particular dim $\psi_1^{-1}(y) > 0$, so ψ_1 is not an isomorphism. So, ψ_1 is the other contraction of $\mathbb{P}(\mathcal{E})$, hence is a smooth blow up along a subvariety W of Y. Since ψ_1 is not an isomorphism, $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ is not ample. By Lemma 2.3, codim $W = n+2-c_1$. So, $\psi_1^{-1}(y) = \mathbb{P}^{\text{codim } W-1} = \mathbb{P}^{n+1-c_1}$, a contradiction. So, $L \cap S = \varnothing$, and we are done. □

Lemma 2.9. *Let n be a positive integer and $\mathcal{E}_1$ a nonzero globally generated vector bundle over $\mathbb{P}^n$. If $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{E}_1)$ is a smooth blow up, then $\mathcal{E}_1$ is direct sum of $\mathcal{O}_{\mathbb{P}^n}(1)$ with a trivial bundle.*

Proof. Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{E}_1$. Suppose $\mathbb{P}(\mathcal{E})$ is blow up of a smooth projective variety Y along a smooth subvariety W. We are in the situation of the setup. Since $\mathcal{E}$ is nonample, we have codim $W = n+2-c_1$ by Lemma 2.3. By Lemma 2.1, the morphism given by $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ has one fiber isomorphic to $\mathbb{P}^n$, so the same is true for contraction of $\mathbb{P}(\mathcal{E})$ given by Stein factorization of this morphism. It follows that the blow up of Y along W has a fiber over Y of dimension n, hence codim $W = n+1$. So, $n+2-c_1 = n+1$, so $c_1 = 1$. So, as $\mathcal{E}_1$ is globally generated, it must be of the form $\mathcal{O}_{\mathbb{P}^n}^{r-1} \oplus \mathcal{O}_{\mathbb{P}^n}(1)$ or $\mathcal{O}_{\mathbb{P}^n}^{r-1} \oplus T_{\mathbb{P}^n}(-1)$. If $n = 1$ both are same, and if $n > 1$ then the 2nd case cannot occur as $s_n(\mathcal{E}_1) = s_n(\mathcal{E}) > 0$ by Lemma 2.5. □

Lemma 2.10. *In the notation of the setup, suppose $Y = \mathbb{P}^n$. Then W is nondegenerate and is not a complete intersection, unless W is linear.*

Proof. Suppose W is nonlinear. The same argument as in [10, Proposition 2.5(a)] shows W is nondegenerate. If W is a complete intersection, then as in the proof of [10, Proposition 2.5(a)], we have $n+1 = h^0(\mathbb{P}^{n+r}, \mathcal{I}_W(d)) \leq n+r-m$, so $m-r+1 \leq 0$. By Lemma 2.3, we get $c_1 = m-r+2 \leq 1$. Now as in the proof of Lemma 2.9 we conclude that $\mathcal{E}$ is direct sum of $\mathcal{O}_{\mathbb{P}^n}(1)$ with a trivial bundle, so W is linear, a contradiction. □

3. Examples

Following are some examples of projective bundles over projective spaces which has a smooth blow up structure.

Example 3.1. *For positive integers m, n, $X = \mathbb{P}_{\mathbb{P}^n}(\mathcal{O}^m \oplus \mathcal{O}(1))$ is blow up of $\mathbb{P}^{n+m}$ along a linear subvariety of dimension $m-1$.*

Proof. This is well-known, see for example [11, Proposition 9.11]. □

Example 3.2. *Let $0 \leq k \leq 2$, $W_0 \hookrightarrow \mathbb{P}^5$ be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$. Let $0 \to \mathcal{O}_{\mathbb{P}^2}^k \to T_{\mathbb{P}^2}(-1)^{\oplus 2} \to E \to 0$ be a short exact sequence of vector bundles. Then $X = \mathbb{P}_{\mathbb{P}^2}(E)$ is blow up of H^{5-k} along $H^{5-k} \cap W_0$, where H^{5-k} is a $(5-k)$-dimensional linear subspace in $\mathbb{P}^5$ such that $H^{5-k} \cap W_0$ is smooth and has codimension 2 in H^{5-k}. For each $0 \leq k \leq 2$, a general map $\mathcal{O}_{\mathbb{P}^2}^k \to T_{\mathbb{P}^2}(-1)^{\oplus 2}$ gives such a short exact sequence.*

Proof. This is the content of [26], also follows from Lemma 2.7. □

Example 3.3. *Let $0 \leq k \leq 3$, $W_0 \hookrightarrow \mathbb{P}^9$ be the Plucker embedding of $Gr(2,5)$. Let $0 \to \mathcal{O}_{\mathbb{P}^4}^k \to \wedge^2 T_{\mathbb{P}^4}(-1) \to E \to 0$ be a short exact sequence of vector bundles. Then $X = \mathbb{P}_{\mathbb{P}^4}(E)$ is blow up of H^{9-k} along $H^{9-k} \cap W_0$, where H^{9-k} is a $(9-k)$-dimensional linear subspace in $\mathbb{P}^9$ such that $H^{9-k} \cap W_0$ is smooth and has codimension 3 in H^{9-k}. For each $0 \leq k \leq 3$, a general map $\mathcal{O}_{\mathbb{P}^4}^k \to \wedge^2 T_{\mathbb{P}^4}(-1)$ gives such a short exact sequence.*

Proof. In Lemma 2.7, we take $F = \wedge^2 T_{\mathbb{P}^4}(-1)$. By [13, Theorem 1.2], and the geometric description of ϕ in its proof, we see that $S = \varnothing$, $W_0 =$ Plücker embedding of $\mathrm{Gr}(2,5)$ in $\mathbb{P}(\mathrm{Alt}(5)) \cong P^9$ satisfies the hypothesis of Lemma 2.7. So Lemma 2.7 completes the proof of everything except the last statement. The last statement follows from [3, Lemma 1.4], as $c_i(F) = 0$ for $i \geq 4$. □

Before stating the next example, we describe it informally. By [13, Theorem 1.1], for any integer $n \geq 2$, there is a birational contraction $\phi : \mathbb{P}_{\mathbb{P}^n}(T_{\mathbb{P}^n}(-1)^{\oplus n}) \to \mathbb{P}^{n^2+n-1}$. This ϕ is not a smooth blow-up in codimension 6, so if we cut restrict ϕ over a general 5-dimensional linear subspace of $\mathbb{P}^{n^2+n-1}$, we get a smooth blow-up along a 3-fold. But after cutting with this linear subspace, the other contraction is a projective bundle only for $n \leq 3$. The $n = 2$ case is in example 3.2. The $n = 3$ case gives the following example.

Example 3.4. *Let $V = M_{3\times 4}(k)$, $W_0 = \{[A] \in \mathbb{P}(V^*) | rkA \leq 2\}$, $S = \{[A] \in \mathbb{P}(V^*) | rkA \leq 1\}$. There is an identification $H^0(\mathbb{P}^3, T_{\mathbb{P}^3}(-1)^{\oplus 3}) = V$, such that the following holds:*

Let $0 \to \mathcal{O}_{\mathbb{P}^3}^6 \xrightarrow{s} T_{\mathbb{P}^3}(-1)^{\oplus 3} \to E \to 0$ be a short exact sequence of vector bundles, such that $L \cap S = \varnothing$, where $L = \mathbb{P}(H^0(E)) \hookrightarrow \mathbb{P}(V)$ is the codimension 6 linear subspace corresponding to s. Then $L \cap W_0$ is smooth and $X = \mathbb{P}_{\mathbb{P}^3}(E)$ is blow up of L along $W := L \cap W_0$. A general map s gives such a short exact sequence with the above property.

Proof. In Lemma 2.7, we take $F = T_{\mathbb{P}^3}(-1)^{\oplus 3}$. Euler exact sequence gives short exact sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^3 \to \mathcal{O}_{\mathbb{P}^3}^{4\times 3} \to T_{\mathbb{P}^3}(-1)^{\oplus 3}$. This gives an identification $H^0(\mathbb{P}^3, T_{\mathbb{P}^3}(-1)^{\oplus 3}) = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}^{4\times 3}) = V$. Note that S is projectively equivalent to the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^3$, so $\dim S = 5$. By [13, Theorem 1.1] and the geometric description of ϕ in its proof, S and W_0 satisfies the hypothesis of Lemma 2.7. So Lemma 2.7 completes the proof of everything except the last statement. The last statement follows from [3, Lemma 1.4], as $c_i(F) = 0$ for $i \geq 4$, and the fact that $\dim S = 5$, so $L \cap S = \varnothing$ for a general linear subvariety L of codimension 6. □

Example 3.5. *i) For a positive integer n, $X = \mathbb{P}_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(1) \oplus T_{\mathbb{P}^n}(-1))$ is the blow up of a smooth quadric in $\mathbb{P}^{2n+1}$ along a linear subvariety of dimension n.*

ii) If

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \xrightarrow{s} \mathcal{O}_{\mathbb{P}^n}(1) \oplus T_{\mathbb{P}^n}(-1) \longrightarrow F \longrightarrow 0$$

is an short exact sequence of vector bundles, where s is a nowhere vanishing section, then $X = \mathbb{P}_{\mathbb{P}^n}(F)$ is the blow up of the smooth quadric in $\mathbb{P}^{2n}$ along a linear subvariety of dimension $n-1$.

Proof. This is [6, Corollary 3.3(1)]. □

Example 3.6. *Let $n \geq 3$. Then:*

(i) $X = \mathbb{P}_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(1) \oplus \Omega_{\mathbb{P}^n}(2))$ is the blow-up of $\mathrm{Gr}(2, n+2)$ along $\mathrm{Gr}(2, n+1)$.

(ii) If s is a nowhere vanishing section of $\mathcal{O}_{\mathbb{P}^n}(1) \oplus \Omega_{\mathbb{P}^n}(2)$ and the vector bundle E is defined by the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \oplus \Omega_{\mathbb{P}^n}(2) \longrightarrow E \longrightarrow 0,$$

then $X = \mathbb{P}_{\mathbb{P}^n}(E)$ is the blow-up of the $H \cap \mathrm{Gr}(2, n+2)$ along $H \cap \mathrm{Gr}(2, n+1)$, where $H \cap \mathrm{Gr}(2, n+2)$ is a smooth hyperplane section of $\mathrm{Gr}(2, n+2)$ under the Plücker embedding, such that $H \cap \mathrm{Gr}(2, n+1)$ is also smooth.

Proof. This is [6, Corollary 3.3(2)]. □

Remark 3.7. *Example 3.1 is well-known, 3.2 has appeared in [26], 3.5 and 3.6 have appeared in [6]. 3.3 with $k = 0$ have appeared in [13, Theorem 2]. The rest of the examples are new, to the author's best knowledge.*

Remark 3.8. *W in example 3.4 is a codimension 2 smooth subvariety of $\mathbb{P}^5$, which is not a complete intersection by Lemma 2.10. Using arguments similar to [9, Example 3.1] and [9, Table 7.3], one can show W is a Bordiga 3-fold.*

Remark 3.9. *The bundle E in example 3.3 with $k = 3$ is a Tango bundle on $\mathbb{P}^4$.*

4. Proofs

We first prove the following result.

Theorem 4.1. *Let $n \geq 3, k \geq 0$ be integers, and*

$$0 \to \mathcal{O}^k \to F \to \mathcal{E} \to 0$$

be a short exact sequence of vector bundles on $\mathbb{P}^n$, where F is one of the following:

(1) $p, q \geq 0$ are integers, a_i, b_j are positive integers for $1 \leq i \leq p, 1 \leq j \leq q$, and $\sum_i a_i + \sum_j b_j \leq n$. $F = \oplus_i \mathcal{O}_{\mathbb{P}^n}(a_i) \oplus (\oplus_j P_{\mathcal{O}(b_j)})$.

(2) $F = \Omega_{\mathbb{P}^n}(2)$.

(3) $F = \wedge^2(T_{\mathbb{P}^n}(-1))$

(4) $F = \mathcal{O}_{\mathbb{P}^n}(1) \oplus \Omega_{\mathbb{P}^n}(2)$.

(5) $F = T_{\mathbb{P}^n}(-1) \oplus \Omega_{\mathbb{P}^n}(2)$.

(6) $F = \mathcal{O}_{\mathbb{P}^n}(1) \oplus \wedge^2(T_{\mathbb{P}^n}(-1))$.

(7) $F = T_{\mathbb{P}^n}(-1) \oplus \wedge^2(T_{\mathbb{P}^n}(-1))$.

Suppose the other contraction of $X = \mathbb{P}(\mathcal{E})$ is a smooth blow up and $\mathcal{E}(-1)$ is not nef. Then X is one of the examples of §3.

Proof. (1) Suppose F is an in (1). Let $V_i = H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(a_i)), W_j = H^0(\mathbb{P}^n, P_{\mathcal{O}(b_j)}), V = \oplus_i V_i, W = \oplus_j W_j$. $|\mathcal{O}_{\mathbb{P}(F)}|$ gives a morphism $\phi : \mathbb{P}(F) \to \mathbb{P}(V \oplus W)$. Let $S = \mathbb{P}(W) \hookrightarrow \mathbb{P}(V \oplus W)$. If $q = 0$, then $\mathcal{E}(-1)$ is globally generated, hence nef, a contradiction. So, $q > 0$, hence $c_n(F) \neq 0$. So, $l = n$ in the notation of Lemma 2.8.

Case 1: $\sum_i a_i + \sum_j b_j > q + 1$.

Using Lemma 2.1, note that for all $s \in S$, dim $\phi^{-1}(s) \geq n - q > n - (\sum_i a_i + \sum_j b_j) + 1 = n - c_1(F) + 1$. So, S satisfies the hypothesis of Lemma 2.8. By Lemma 2.8, $\sum_j$dim $W_j =$ dim $S + 1 \leq rkF - n = p + \sum_j$dim $W_j - q - n$, so $0 \leq p - q - n$. But $n \geq \sum_i a_i + \sum_j b_j \geq p + q$. So, $q \leq 0$, a contradiction.

Case 2: $\sum_i a_i + \sum_j b_j \leq q + 1$.

We have $p + q \leq \sum_i a_i + \sum_j b_j \leq q + 1$. So, $p \leq 1$.

If $p = 1$, then $a_i = 1 = b_j$ for all i, j. If $p = 0$, then either $b_j = 1$ for all j, or $b_j = 2$ for one j and $b_j = 1$ for all other j. So, $F = \mathcal{O}_{\mathbb{P}^n}(1) \oplus T_{\mathbb{P}^n}(-1)^q, T_{\mathbb{P}^n}(-1)^q$ or $T_{\mathbb{P}^n}(-1)^{q-1} \oplus P_{\mathcal{O}(2)}$.

Subcase 1: $F = \mathcal{O}_{\mathbb{P}^n}(1) \oplus T_{\mathbb{P}^n}(-1)^q$.

We can identify $W_j^* = k^{n+1}$ for all j, so $W^* = M_{q,n+1}$. Using Lemma 2.1 one sees that $S = \{[0, A] \in \mathbb{P}(V \oplus W) | A \in W^* = M_{q,n+1},\ \text{rk } A \leq q-1\}$ satisfies the hypothesis of Lemma 2.8. We have dim $S + 1 = (q-1)(n+2)$, by the formula of dimension of determinantal variety, see for example the corresponding wikipedia page. Since $c_n(F) \neq 0$, we have $l = n$ in notation of Lemma 2.8. By Lemma 2.8, $(q-1)(n+2) \leq (q-1)n+1$, so $2(q-1) \leq 1$, So $q = 1$, that is, $F = \mathcal{O}_{\mathbb{P}^n}(1) \oplus T_{\mathbb{P}^n}(-1)$ and $k \leq 1$ by Lemma 2.8. Hence X is as in example 3.5.

Subcase 2: $F = T_{\mathbb{P}^n}(-1)^q$.

We have $q \leq n$ by assumption. If $q < n$ then $s_n(\mathcal{E}) = s_n(F) = 0$, contradiction by Lemma 2.5. So $q = n$. We can identify $W^* = M_{n,n+1}$ as in subcase 1. Using Lemma 2.1 or the geometric description of ϕ in the proof of [13, Theorem 1], one sees that $S = \{[A] \in \mathbb{P}(W) | A \in W^* = M_{n,n+1},\ \text{rk } A \leq n-2\}$ satisfies the hypothesis of Lemma 2.8. We have dim $S + 1 = (n-2)(n+3)$. Since $c_n(F) \neq 0$, we have $l = n$ in notation of Lemma 2.8. By Lemma 2.8, $(n-2)(n+3) \leq n^2 - n$, so $n \leq 3$. So $n = 3 = q$, that is, $F = T_{\mathbb{P}^3}(-1)$. By Lemma 2.8 and its proof, we get $k = 6$ and $L \cap S = \varnothing$, where $L = \mathbb{P}(H^0(\mathcal{E})) \hookrightarrow \mathbb{P}(H^0(F))$ is the codimension k linear subvariety. So, X is as in example 3.4.

Subcase 3: $F = T_{\mathbb{P}^n}(-1)^{q-1} \oplus P_{\mathcal{O}(2)}$.

By assumption, $q + 1 \leq n$. So, $s_n(\mathcal{E}) = s_n(F) = 0$, contradiction by Lemma 2.5.

(2) Suppose F is as in (2). So, $s_n(\mathcal{E}) = s_n(F) = 0$, contradiction by Lemma 2.5.

(3) Suppose F is as in (3). If n is odd, $s_n(\mathcal{E}) = s_n(F) = 0$, contradiction by Lemma 2.5. So n is even.

As in [13], we can identify $H^0(F) = H^0(F)^* = Alt(n+1)$. By the geometric description of ϕ in the proof of [13, Theorem 2], one sees that for all $s \in S := \{[A] \in \mathbb{P}(Alt(n+1)) | \text{ rk } A \leq n-4\}, \phi^{-1}(s)$ is a projective space of dimension ≥ 4. So S satisfies the hypothesis of Lemma 2.8. We have dim $S + 1 = \frac{n^2+n}{2} - 10$. Since $c_{n-1}(F) \neq 0, c_n(F) = 0$, we have $l = n-1$ in notation of Lemma 2.8. By Lemma 2.8, $\frac{n^2+n}{2} - 10 \leq \frac{n^2-n}{2} - n + 1$, so $n \leq 4$. As n is even and ≥ 3, we get $n = 4$, and $k \leq 3$ by Lemma 2.8. So, X is as in example 3.3.

(4) Suppose F is as in (4). By Lemma 2.8, $k \leq 1$. So X is as in example 3.6.

(5) Suppose F is as in (5). By Lemma 2.1, for all $s \in S := \mathbb{P}(H^0(T_{\mathbb{P}^n}(-1)) \hookrightarrow \mathbb{P}(H^0(F))$, we have $\phi^{-1}(s) \cong \mathbb{P}^{n-1}$. So S satisfies the hypothesis of Lemma 2.8, as $n \geq 3$. We have dim $S + 1 = n + 1$. Since $c_n(F) \neq 0$, we have $l = n$ in notation of Lemma 2.8. By Lemma 2.8, $n + 1 \leq n$, which is absurd.

(6) Suppose F is as in (6). Let $F_1 = \wedge^2(T_{\mathbb{P}^n}(-1))$. By the same argument as in [13, Theorem 2], we can identify $H^0(F_1) = H^0(F_1)^* = Alt(n+1)$. Since $c_n(F) \neq 0$, we have $l = n$ in notation of Lemma 2.8.

Case 1: n is odd.

By the same argument as in the geometric description of ϕ in the proof of [13, Theorem 2], one sees that for all $s \in S := \{[0, A] \in \mathbb{P}(H^0(\mathcal{O}(1) \oplus F_1)) | A \in Alt(n+1),\ \text{rk } A \leq n-3\}, \phi^{-1}(s)$ is a projective space of dimension ≥ 3. So S satisfies the hypothesis of Lemma 2.8. We have dim $S + 1 = \frac{n^2+n}{2} - 6$. By Lemma 2.8, $\frac{n^2+n}{2} - 6 \leq \frac{n^2-n}{2} - n + 1$, so $n \leq 3$, hence $n = 3$. But $\wedge^2(T_{\mathbb{P}^3}(-1)) = \Omega_{\mathbb{P}^3}(2)$, so F is an in (5). We have already shown X is as in the examples.

Case 2: n is even.

By the same argument as in the geometric description of ϕ in the proof of [13, Theorem 2], one sees that for all $s \in S := \{[0, A] \in \mathbb{P}(H^0(\mathcal{O}(1) \oplus F_1)) | A \in Alt(n+1),\ \text{rk } A \leq n-2\}, \phi^{-1}(s)$ is a projective space of dimension ≥ 2. So S satisfies the hypothesis of Lemma 2.8. We have dim $S+1 = \frac{n^2+n}{2} - 3$. By Lemma 2.8, $\frac{n^2+n}{2} - 3 \leq \frac{n^2-n}{2} - n + 1$, so $n \leq 2$, a contradiction.

(7) Suppose F is as in (7). Let $F_1 = \wedge^2(T_{\mathbb{P}^n}(-1))$. As in (6) we can identify $H^0(F_1) = H^0(F_1)^* = Alt(n+1)$. Since $c_n(F) \neq 0$, we have $l = n$ in notation of Lemma 2.8.

Case 1: n is even.

As we have seen in (6), for all $s \in S := \{[0, A] \in \mathbb{P}(H^0(T(-1) \oplus F_1)) | A \in Alt(n+1),\ \text{rk } A \leq n-2\}, \phi^{-1}(s)$ is a projective space of dimension ≥ 2. So S satisfies the hypothesis of Lemma 2.8. We have dim $S+1 = \frac{n^2+n}{2} - 3$. By Lemma 2.8, $\frac{n^2+n}{2} - 3 \leq \frac{n^2-n}{2}$, so $n \leq 3$. A contradiction as n is even and $n \geq 3$.

Case 2: n is odd.

By Euler exact sequence, identify $H^0(T(-1)) = k^{n+1}$. Let $S = \{[\underline{\lambda}, A] \in \mathbb{P}(H^0(T(-1) \oplus H^0(F_1)))|\ \text{rk } [A : \underline{\lambda}] \leq n-2\} = \{[\underline{\lambda}, A] \in \mathbb{P}(H^0(T(-1) \oplus H^0(F_1)))|\ \text{rk } A \leq n-3\}$. As in Case 1 of (6), we get $n = 3$. So, F is as in (5), which we already tackled. □

Proof of Theorem B: We are as in the setup. Since $m \leq n + r - 2$, we have $c_1 \leq n$ by Lemma 2.3(2). Assume throughout that $\mathcal{E}$ is not as in example 3.1.

<u>**(1):**</u> Let

$$0 \to \mathcal{O}^k \to F \to \mathcal{E} \to 0$$

be a short exact sequence of vector bundles on $\mathbb{P}^n$ with $H^0(F^*) = H^1(F^*) = 0$. By [3, Lemma 1.2] and Lemma 2.9, there is always such a short exact sequence. Since $H^1(\mathbb{P}^n, \mathcal{O}) = 0$, this short exact sequence induces a short exact sequence on H^0. Since $\mathcal{O}^k$ and $\mathcal{E}$ are globally generated, an application of Snake Lemma shows F is also globally generated. If $c_1(\mathcal{E}) < n$ or if $c_1(\mathcal{E}) = n$ and F is decomposable, then since $c_1(F) = c_1(\mathcal{E})$ and $M_{n,c_1(\mathcal{E})}$ is true, we see that F must be one of (1)-(7) of Theorem 4.1. So by Theorem 4.1, X is one of the examples.

Part **(2):** will be proven in the end.

Proof of Theorem C: If $n \leq 2$ or $r = 1$ then X is one of the examples by [32], [31], [1], [30]. So we assume $n \geq 3, r \geq 2$. So $c_1(\mathcal{E}(-1)) \leq 2$. If $\mathcal{E}(-1)$ is nef, then by[24, Lemma 3], $\mathcal{E}(-1)$ is globally generated. By [6, Theorem E] (or by replacing $\mathcal{E}$ by $\mathcal{E}(-1)$ and proceeding with the proof below), X is one of the examples. So assume $\mathcal{E}(-1)$ is not nef. Hence we are in the situation of the setup. Assume throughout that $\mathcal{E}$ is not as in example 3.1.

Case 1: $\mathcal{E}$ is nonample.

Suppose X is not one of the examples. Since $c_1 \leq 5$, M_{n,c_1} is true. By Theorem B(1), we must have $c_1 = n$, and if $0 \to \mathcal{O}_{\mathbb{P}^n}^k \to F \to \mathcal{E} \to 0$ a short exact sequence of vector bundles on $\mathbb{P}^n$ with $H^0(F^*) = H^1(F^*) = 0$, then F is indecomposable. Looking at the classification of globally generated vector bundles on $\mathbb{P}^n$ with first Chern class ≤ 5 in [3],[5], we see that one of the following cases occur:

Subcase 1: F is as in (xvi) of [3, Theorem 0.2].

We have $s_4(\mathcal{E}) = s_4(F) = 0$ by a computation using exact sequences. Contradiction by Lemma 2.5.

Subcase 2: F is as in (v) of [5, Theorem 0.1].

We have $(-1)^i s_i(\mathcal{E}) = (-1)^i s_i(F) = 21$ for $3 \leq i \leq 5$ by a computation using exact sequences. This is impossible by Lemma 2.5.

Subcase 3: F is as in (vi) of [5, Theorem 0.1].

We have $s_5(\mathcal{E}) = s_5(F) = 0$ by a computation using exact sequences. Contradiction by Lemma 2.5.

Case 2: $\mathcal{E}$ is ample.

Subcase 1: $c_1 \leq n - 1$.

Since $c_1 \leq 5$, M_{n,c_1} is true. So, there is a short exact sequence of vector bundles $0 \to \mathcal{O}_{\mathbb{P}^n}^k \to F \to \mathcal{E} \to 0$ on $\mathbb{P}^n$ with $H^0(F^*) = H^1(F^*) = 0$, and F is one of (1)-(7) as in Theorem 4.1. By Theorem 4.1, X is as in the examples.

Subcase 2: $c_1 = n = 3$.

By [3, Theorem 0.1], there is a short exact sequence of vector bundles $0 \to \mathcal{O}_{\mathbb{P}^n}^k \to F \to \mathcal{E} \to 0$ on $\mathbb{P}^n$ with $H^0(F^*) = H^1(F^*) = 0$, and F is one of (1)-(7) as in Theorem 4.1. By Theorem 4.1, X is as in the examples. But in each such case we have $\mathcal{E}$ nonample, a contradiction.

Subcase 3: $c_1 = n = 4$.

By [3, Theorem 0.2], there is a short exact sequence of vector bundles $0 \to \mathcal{O}_{\mathbb{P}^n}^k \to F \to \mathcal{E} \to 0$ on $\mathbb{P}^n$ with $H^0(F^*) = H^1(F^*) = 0$, and F is one of (1)-(7) as in Theorem 4.1, or F is as in (xvi) of [3, Theorem 0.2]. In the first case, by Theorem 4.1, X is as in the examples. But in each such case we have $\mathcal{E}$ nonample, a contradiction. In the second case, we have We have $s_4(\mathcal{E}) = s_4(F) = 0$, so the top self-intersection number of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is 0, a contradiction as $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is ample.

Subcase 4: $c_1 = n = 5$.

Let $\mathcal{E}_1 = \mathcal{E} \oplus \mathcal{O}_{\mathbb{P}^n}(1)$. So $\mathcal{E}_1$ is ample, $c_1(\mathcal{E}_1) = c_1(\mathbb{P}^n)$, rk $\mathcal{E}_1 \geq 4 = n - 1$. By [25, Theorem 0.1, 0.2], we see that $\mathcal{E}$ must be a split bundle. But then $\mathcal{E}(-1)$ is nef, a contradiction.

Subcase 5: $c_1 = n + 1$.

So $n \leq 4$. We have rk $\mathcal{E} \geq 3 \geq n - 1$. If $n = 4$, then rk $\mathcal{E} = 3$, and by [25, Theorem 7.4], we see that we see that $X = \mathbb{P}(\mathcal{E})$ cannot be a smooth blow up, a contradiction. If $n = 3$, by [25, Theorem 0.1], we see that X cannot be a smooth blow up, a contradiction.

Subcase 6: $c_1 \geq n + 2$.

As $n \geq 3, c_1 \leq 5$, we must have $n = 3, c_1 = 5$. Since $\mathcal{E}$ is ample, restricting $\mathcal{E}$ to a line in $\mathbb{P}^3$ we see that rk $\mathcal{E} \leq c_1(\mathcal{E}) = 5$. If rk $\mathcal{E} = 5$, then restriction of $\mathcal{E}$ at every line must be $\mathcal{O}(1)^5$, hence restriction of $\mathcal{E}(-1)$ at every line is trivial. By [22, Theorem 3.2.1], $\mathcal{E}(-1)$ is trivial, a contradiction as $\mathcal{E}(-1)$ is not nef. So, $3 \leq$ rk $\mathcal{E} \leq 4$, that is, $r = 2$ or 3.

Suppose $r = 3$, so by Lemma 2.4(ii), $a | \binom{5}{3} = 10$, so $a = 2, 5$ or 10. So $r + 1 \nmid a + 1$. So by Lemma 2.3(ii) we must have $n + r - m \geq 3$. By Lemma 2.4(iii), we have $a^2 | \binom{4}{3} = 4$, so $a = 2$. Since $0 < b < a$, we have $b = 1$. Now by Lemma 2.4(iv), we have

$$4 \mid \binom{5}{3} d + \binom{5}{2}(1 + d) - \binom{5}{2} \cdot 10d,$$

hence $4 | 10d + 10(1 + d) = 20d + 10$, which is false.

So, $r = 2$. So, $m \leq n + r - 2 = 3$. If $m \leq 2$, then by Lemma 2.4(iii), $a^2 | \binom{3}{3} = 1$, contradiction to $a > 1$. So, $m = 3$. By Lemma 2.3(ii) and Lemma 2.4(ii), we have $3 | a + 1$ and $a | \binom{4}{3} = 4$. So, $a = 2$. As $0 < b < a$, we have $b = 1$. By Lemma 2.3(v), we have $4 = a^r | \alpha$. So Y is neither a projective space, nor a quadric in a projective space. By [15, Corollaries to Theorem

1.1 and 2.1], we have $\tau \leq n+r-1=4$. So by Lemma 2.3(ii), $\frac{d+3}{2} \leq 4$, so $d \leq 5$. Also, by Lemma 2.4(iv), $8|-50d+6$, so $4|d-3$. $d=3$ is the only possibility. So $d=3, \tau=3$, by Lemma 2.3(ii).

We have $0 = EH_2^4 = (-2H_1+3U)(-H_1+2U)^4 = -48s_3 - 128s_2 + 608$, as $s_1 = -c_1 = -5$. So,

$$3s_3 + 8s_2 = 38. \tag{3}$$

Also, $\alpha = H_2^5 = (-H_1+2U)^5 = -32s_3 - 80s_2 + 360$. So,

$$2s_3 + 5s_2 = \frac{45}{2} - \frac{\alpha}{8}. \tag{4}$$

Solving equations (3) and (4) we get $s_2 = 9 + \frac{3\alpha-8}{16}, s_3 = -\frac{\alpha}{2} - 10$. Using equation (1) we get $c_2 = 16 - \frac{3\alpha-8}{16}, c_3 = 50 - \frac{11\alpha}{8}$.

As $\mathcal{E}$ is ample. we have $c_3 > 0$. So, $\alpha \leq 36$. We also have $16|3\alpha - 8$ as c_2 is an integer. Therefore $\alpha \equiv 8 (\text{mod } 16)$. So, $\alpha = 8$ or 24.

If $\alpha = 24$, then $c_2 = 12, c_3 = 17$. But this is impossible as by [4, Equation 1.7], $c_3 - c_1c_2$ must be even.

If $\alpha = 8$, then $c_2 = 15, c_3 = 39$. $\mathcal{G} := P_{\mathcal{E}}$ is a globally generated vector bundle on $\mathbb{P}^3$ with $c_1(\mathcal{G}) = 5, c_2(\mathcal{G}) = s_2(\mathcal{E}) = 10, c_3(\mathcal{G}) = -s_3(\mathcal{E}) = 14$. But this is also impossible by [4, Theorem 0.1].

So we get a contradiction.

Proof of Theorem A:

If $r = 1$ or $c_1 \leq 5$, then by [1], [32] and Theorem C, we see that either (1) or (2) holds. So we assume $r \geq 2, c_1 \geq 6$. By the already proven part of Theorem B, $n \geq c_1 \geq 6$.

Case 1: Suppose $d = 2$. So n is even, $c_1 = \frac{n}{2} + 1$. As $c_1 \geq 6$ we get $n \geq 10$.

Claim: $n/2 \leq r \leq n/2 + 3$.

Proof. By Lemma 2.6 (iv), $n/2 \leq r$. By Lemma 2.3(iii), W is a quadratic variety in $\mathbb{P}^{n+r}$ in the sense of [14]. By [14] and Lemma 2.10, we have $m \leq \frac{2}{3}(n+r)$, that is, $r \leq n/2 + 3$. □

Let $r = \frac{n}{2} + g, 0 \leq g \leq 3$. So, $m = \frac{n}{2} + g - 1 + \frac{n}{2} = n + g - 1$. By Lemma 2.6 (ii), $P(t) = (1 + t^{n/2})(1 + t + \cdots + t^{n/2+g-1})$. So, $a_i = 1$ if $0 \leq i \leq n/2 - 1$ or $n/2 + g \leq i \leq m$. Since $H^{2i}(W, \mathbb{Z})$ is free abelian by Lemma 2.6(iii), there are positive integers l_i for integers $i \in [0, m] \setminus [n/2, n/2 + g - 1]$, such that $H^{2i}(W, \mathbb{Z})$ is generated by H_2^i / l^i.

Let $S = \{(k,i) \in \mathbb{Z}^2 | 1 \leq k \leq n+r, 0 \leq i \leq min\{n/2-1, k-1\}$ and $k - i - 1 \in [0,m] \setminus [n/2, n/2+g-1]\}$. In notation of [34, Theorem 7.31], for $(k,i) \in S$, we have $H^{2k}(X, \mathbb{Z}) \ni j_* \circ h^i \circ (\phi|_E)^*(H_2^{k-i-1}/l_{k-i-1}) =$

$j_*j^*((-E)^i H_2^{k-i-1}/l_{k-i-1}) = (-1)^i H_2^{k-i-1} E^{i+1}/l_{k-i-1} = (-1)^i U^{k-i-1}(2U - H_1)^{i+1}/l_{k-i-1}$. So,

$$l_{k-i-1} | U^{k-i-1}(2U - H_1)^{i+1} \text{ in } H^{2k}(X, \mathbb{Z}) \text{ for } (k, i) \in S.$$

As $n \geq 10, 0 \leq g \leq 3$, we have $(n/2 + g, g) \in S$. Since $n/2 + g = r$ and $\{U^{r-i}H_1^i | 0 \leq i \leq r\}$ is a basis of the free abelian group $H^{2r}(X, \mathbb{Z})$, we get $l_{n/2-1} = 1$. By Poincare duality, $l_{n/2-1}l_{n/2+g-1} = l_m = e$, where e is the degree of W in $\mathbb{P}^{n+r}$. So, $l_{n/2+g} = e$.

We also have $(n/2 + g + 1, 0) \in S$. So

$$e | U^{n/2+g}(2U - H_1) = 2U^{n/2+g+1} - U^{n/2+g}H_1$$

$$= 2((n/2 + 1)U^r H_1 + \sum_{i=2}^{r+1}(-1)^{i-1}c_i U^{r+1-i}H_1^i) - U^r H_1$$

$$= (n + 1)U^r H_1 + \sum_{i=2}^{r+1}(-1)^{i-1}2c_i U^{r+1-i}H_1^i.$$

As $\{U^{r-i}H_1^{i+1} | 0 \leq i \leq r\}$ is a basis of the free abelian group $H^{2(r+1)}(X, \mathbb{Z})$, we have $e | n + 1$. As W is nondegenerate by Lemma 2.10, we have $e \geq \operatorname{codim}(W) + 1 = n/2 + 1 > (n + 1)/2$. So we must have $e = n + 1$.

As $n \geq 10$, we have $2 \leq n/2 + 1 \leq m - 2$. So $2 \leq$ codim(W) $\leq m - 2$. Also, $e = n + 1 \geq n/2 + 4 =$ codim $(W) + 3$. By [23, Proposition 2.4(2)], we have $n + 1 = e \geq 2$ codim $(W) + 2 = 2(n/2 + 1) + 2 = n + 4$, a contradiction.

Case 2: Suppose $d \geq 3$. By Lemma 2.6 (iv), we have $r \geq n/d$. Suppose $r > n/d$. By Lemma 2.6(ii), $h^{2n/d}(W) = 2$. If $d \geq 4$, or $d = 3$ and $r \geq n/3+2$, we have

$$2n/d \leq n + r - 2(n/d + 1).$$

By Barth-Larsen theorem, we get $h^{2n/d}(W) = 1$, a contradiction. So $d = 3$ and $r = n/d+1$. Since $n \geq 6$, Lemma 2.6(ii) shows $a_{n/3} = 2, a_{n/3+1} = 1$. But by Hard Lefschetz theorem, the map $H^{2n/3}(W, \mathbb{C}) \to H^{2n/3+2}(W, \mathbb{C})$ given by multiplication by a Kahler class in $H^2(W, \mathbb{C})$ is injective, so $a_{n/3} \leq a_{n/3+1}$, a contradiction.

So, we have $r = n/d$. Now Lemma 2.6(ii) shows that W has same Betti numbers as $\mathbb{P}^{n-1}$. So, (3) holds.

Now we prove the last statement. Since we assume Hartshorne's conjecture is true and by Lemma 2.10 W is not a complete intersection, we have $m \leq \frac{2}{3}(n + r)$. Suppose (3) holds, so $r = n/d, m = n - 1$. So, $n - 1 \leq \frac{2(d+1)}{3d}n$, hence $2 \leq r = n/d \leq \frac{3}{d-2}$. This implies $d = 3$ and $r = 2$ or 3. If $r = 2$, then $n = 6$. So by Lemma 2.3, $c_1 = \frac{d-1}{d}n + 1 = 5$, a contradiction to our assumption that $c_1 \geq 6$. So $r = 3, n = 9$.

So, $c_1 = 7 = -s_1$. By Lemma 2.5 we have the following values of the Segre classes: $s_9 = -1, s_8 = 3, s_7 = -9, s_6 = 27$. Now we show that equation (1) cannot hold by a number theoretic argument.

Define integral polynomials $Q(t), R(t)$ by $Q(t) = 1 + 7t + c_2t^2 + c_3t^3 + c_4t^4$, $R(t) = 1 - 7t + s_2t^2 + s_3t^3 + s_4t^4 + s_5t^5$. Equation (1) gives

$$Q(t)(R(t) + 27t^6 - 9t^7 + 3t^8 - t^9) \equiv 1 (\text{mod } t^{10}). \tag{5}$$

Hence,

$$Q(t)(R(t) - t^9) \equiv 1(\text{mod } (3, t^{10})).$$

So, $Q(t)R(t) \equiv t^9 + 1 \equiv (t+1)^9 (\text{mod } (3, t^{10}))$.

As $Q(t)R(t)-(t+1)^9$ has degree at most 9, we get $Q(t)R(t) \equiv (t+1)^9 (\text{mod } 3)$. As $\deg Q \leq 4$ and $\deg R \leq 5$, we must have $Q(t) \equiv (t+1)^4, R(t) \equiv (t+1)^5 (\text{mod } 3)$, since $\mathbb{Z}/3\mathbb{Z}[t]$ is a unique factorization domain. So, there are $A(t), B(t) \in \mathbb{Z}[t]$ such that $Q(t) \equiv (1+t)^4 + 3A(t), R(t) \equiv (1+t)^5 + 3B(t) (\text{mod } 9)$.

By Equation (5), $Q(t)(R(t) + 3t^8 - t^9) \equiv 1(\text{mod } (9, t^{10}))$. Using $c_1 = 7 \equiv -2(\text{mod } 9)$, we get

$$Q(t)R(t) + 3t^8 - 7t^9 \equiv 1(\text{mod } (9, t^{10})).$$

So,

$$Q(t)R(t) \equiv -2t^9 - 3t^8 + 1(\text{mod } (9, t^{10})).$$

As $Q(t)R(t) + 2t^9 + 3t^8 - 1$ has degree at most 9, we get

$$Q(t)R(t) \equiv -2t^9 - 3t^8 + 1(\text{mod } 9).$$

So,

$$((1+t)^4 + 3A(t))((1+t)^5 + 3B(t)) \equiv -2t^9 - 3t^8 + 1(\text{mod } 9).$$

Hence,

$$-2t^9 - 3t^8 + 1 \equiv 0(\text{mod } (9, (1+t)^2)).$$

So, $(1+t)^2 | -2t^9 - 3t^8 + 1 = (1+t)S(t)$ in $\mathbb{Z}/9\mathbb{Z}[t]$, where $S(t) = -2t^8 - t^7 + t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$. As $1+t$ is a nonzerodivisor in $\mathbb{Z}/9\mathbb{Z}[t]$, we get $1 + t | S(t)$ in $\mathbb{Z}/9\mathbb{Z}[t]$. So, $S(-1) \equiv 0(\text{mod } 9)$. But $S(-1) = 6$, a contradiction.

Proof of Theorem B: **(2):** As in the beginning of the proof of Theorem A, it suufices to assume $n \geq c_1 \geq 6$. Suppose rk $\mathcal{E} \geq n+1$, that is, $r \geq n$. So, Y is covered by linear subvarieties $\phi(\pi^{-1}(x))(x \in \mathbb{P}^n)$ of dimension $r \geq \frac{n+r}{2}$. Since Y has Picard rank 1, by [29, Main Theorem] and [20, Corollary 5.3], one of the 3 cases can occur:

Case 1: Y is a projective space.

By Theorem A we are done.

Case 2: $Y = Gr(2, n+2)$.

As in [29, §6], $\mathbb{P}^{n+1}$ gives a family of n-planes in Y with universal family $\mathbb{P}_{\mathbb{P}^{n+1}}(\Omega_{\mathbb{P}^{n+1}}(2))$. Since $\Omega_{\mathbb{P}^{n+1}}(2)$ and $\mathcal{E}$ both has first Chern class n, the family of n-planes in Y given by $\phi(\pi^{-1}(x))(x \in \mathbb{P}^n)$ is induced by a linear map $f : \mathbb{P}^n \to \mathbb{P}^{n+1}$. In other words, $\mathcal{E} = f^*(\Omega_{\mathbb{P}^{n+1}}(2)) = \mathcal{O}_{\mathbb{P}^n}(1) \oplus \Omega_{\mathbb{P}^n}(2)$. So we are as in example 3.6.

Case 2: $Y = Q_{2n}$, the $2n$-dimensional smooth quadric.

Quotienting $\mathcal{E}$ by a general section, we get a nef but nonample vector bundle $\mathcal{E}_1$ of rank n such that $\mathbb{P}(\mathcal{E}_1)$ is blow up of Q_{2n-1} along a smooth codimension 2 subvariety W_1. Since Q_{2n-1} and $\mathbb{P}_{2n-1}$ have the same Betti numbers, the same argument as in Lemma 2.6 (ii) shows $a_1 = 2$. But W_1 is a smooth codimension 3 subvariety of $\mathbb{P}_{2n}$, so by Barth-Larsen theorem we get $2 > 2n - 6$, hence $n \leq 3$, a contradiction.

Remark 4.2. *A similar proof as in Theorem B(1) shows the following:*

Let n, r be positive integers, X a smooth projective variety which is a $\mathbb{P}^r$-bundle over $\mathbb{P}^n$, and has another projective bundle structure over some smooth projective variety Y. Let $\mathcal{E}$ be the unique nef vector bundle of rank $r+1$ over $\mathbb{P}^n$ such that $X \cong \mathbb{P}_{\mathbb{P}^n}(\mathcal{E})$ over $\mathbb{P}^n$ and $\mathcal{E}(-1)$ is not nef. Suppose $\mathcal{E}$ is not ample. Then $c_1(\mathcal{E}) \leq n-1$ and the following holds: Suppose $\mathcal{E}$ is globally generated, equivalently, the ample generator of Y is globally generated. If $M_{n,c_1(\mathcal{E})}$ is true, then $\mathcal{E}$ is either trivial, or $T_{\mathbb{P}^n}(-1)$, or $\Omega_{\mathbb{P}^n}(2)$, or a null-correlation bundle.

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