

The integral Chow ring of \mathcal{R}_2

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Abstract

In this paper we compute the integral Chow ring of the moduli stack \mathcal{R}_2 of Prym pairs of genus 2.

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1 Introduction

Prym curves have been studied for more than 100 years due to their connections to the theory of curves, admissible covers and abelian varieties [Mum74, SJ09, FR70, Bea77a, Bea77b, Do92, DS81]. They are connected, étale double covers of a smooth, projective curve or, equivalently, non-trivial line bundles on the base curve that are 2-torsion in the Picard group.

After fixing the genus g of the base curve, the moduli stack of Prym curves has a coarse moduli space, which is denoted by R_g , and it is a finite cover of the moduli space of genus g smooth curves M_g , of degree $2^{2g} - 1$. Just as for M_g , it is natural to study the geometric properties of R_g .

The study of the Chow rings of M_g started with Mumford [Mum93]. With rational coefficients, they are known for genus up to 9 [Fab90, Iza95, PV15, CL24], but the

torsion elements in the Chow ring of the moduli stack \mathcal{M}_g have drawn attention only recently. The full Chow group of \mathcal{M}_2 was computed in [Vis98], relying on the fact that the curves in question are hyperelliptic. Following this direction, the Chow groups of the moduli stack of (pointed) hyperelliptic curves of any genus have been studied in [EF09, Per22, DL21].

The integral Chow groups of the Deligne-Mumford compactification $\overline{\mathcal{M}}_2$ were independently computed in [Lar21, DLV21] and the ones of $\overline{\mathcal{M}}_3$ after inverting 6 by [Per24].

On the other hand, there are fewer results for the moduli stack of Prym curves, although the Picard groups with rational coefficients can be obtained from the results in [Put12]. This paper aims to begin filling this gap by computing the integral Chow ring of the moduli stack of genus 2 Prym pairs. It also initiates a series of works computing the integral Chow rings of hyperelliptic Prym pairs [CL25a, CL25b], with another paper forthcoming.

We should remark that, even though there is no ambiguity on what the stack \mathcal{M}_g should parametrize, there are two natural choices of stacks that have R_g as a coarse moduli space (see §2.1.1 below). We will denote them by $\tilde{\mathcal{R}}_g$ and \mathcal{R}_g , and the second one is the rigidification (in the sense of [ACV03]) of the first one along the generic $\mathbb{Z}/2\mathbb{Z}$ automorphism coming from the étale double cover. This distinction had already appeared in [Cor91, BCF03].

1.1 Results and methods

In this paper, we compute the integral Chow ring of the stack \mathcal{R}_2 (defined over an algebraically closed field k of characteristic different from 2 and 3), using the techniques of equivariant intersection theory developed in [EG98]. The first step is to find an explicit presentation of \mathcal{R}_2 .

To state our results, we require the following definition.

Definition 1. *Let*

$$G = (\mathbb{G}_m \times \mathbb{G}_m) \rtimes \mathbb{Z}/2\mathbb{Z}$$

where the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{G}_m \times \mathbb{G}_m$ permutes the factors.

We denote by Γ the representation of G arising from the sign representation of the $\mathbb{Z}/2\mathbb{Z}$ quotient of G .

Furthermore, we use V to denote the standard representation of G coming from the inclusion $G \subseteq \mathrm{GL}_2$ as the subgroup preserving the set of lines $\{k(1, 0), k(0, 1)\} \subseteq k^2$.

Theorem 2. *We have an isomorphism of algebraic stacks*

$$\left[\frac{\mathrm{Sym}^4(V^\vee) \otimes \det(V) \otimes \Gamma \setminus \Delta}{G} \right] \xrightarrow{\sim} \mathcal{R}_2.$$

where Δ is the locus of polynomials having either a root at 0 or ∞ , or having a double root.

The proof will be given in §2.4. Here, we content ourselves to explain the main idea. The starting point is [Ver11, Lemma 4.3], which we recall here in the case $g = 2$.

Let C be a smooth genus 2 curve defined over an algebraically closed field k of characteristic different from 2¹ and let $q : C \rightarrow \mathbb{P}^1$ be its associated double cover (well-defined up to an isomorphism of \mathbb{P}^1). Then the ramification divisor of q is the set

$$W = \{w_1, \dots, w_6\}$$

of the Weierstrass points of C . Let $H = q^*\mathcal{O}_{\mathbb{P}^1}(1)$ be the hyperelliptic line bundle of C and E be the family of effective divisors of cardinality 2, which are supported on W . For $e \in E$ the line bundle $H \otimes \mathcal{O}_C(-e)$ is a non-trivial square root of \mathcal{O}_C , i.e. an element of $\text{Pic}^0(C)[2] \setminus \{\mathcal{O}_C\}$. The following lemma is proven in [Ver11].

Lemma 3. [Ver11, Lemma 2.3] *The map*

$$\begin{aligned} E &\rightarrow \text{Pic}^0(C)[2] \setminus \{\mathcal{O}_C\} \\ e &\mapsto H \otimes \mathcal{O}_C(-e) \end{aligned}$$

is a bijection.

Thus, for $\eta \in \text{Pic}^0(C)[2] \setminus \{\mathcal{O}_C\}$, we can uniquely write $\eta = H \otimes \mathcal{O}_C(-e)$ for $e \in E$ and, up to composing q with an isomorphism of \mathbb{P}^1 , assume that $q(e) = \{0, \infty\} \subseteq \mathbb{P}^1$. Therefore, we have an isomorphism

$$\eta^{\otimes 2} = q^*(\mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(-[0] - [\infty])) \xrightarrow{\sim} \mathcal{O}_C$$

given by the polynomial $XY \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(-[0] - [\infty]))$.

This provides a map

$$\text{Sym}^4(V^\vee) \setminus \Delta \rightarrow \mathcal{R}_2 \tag{1}$$

which Theorem 2 states is a G -torsor. We refer to (9), (11), (12), (14) and Lemma 23 for the actual definition of the map (1).

Next, in §3.4, we use the machinery of equivariant intersection theory and the presentation of \mathcal{R}_2 in Theorem 2 to compute the Chow ring of \mathcal{R}_2 . Finally, in §4, we identify the generators as naturally defined geometric classes on \mathcal{R}_2 . The final result is the following.

Theorem 4. *Over any algebraically closed base field of characteristic distinct from 2 and 3, the Chow ring of the moduli space of Prym pairs of genus 2 is given by*

$$\text{CH}^*(\mathcal{R}_2) = \frac{\mathbb{Z}[\lambda_1, \lambda_2, \gamma]}{(2\lambda_1, 2\gamma, 8\lambda_2, \gamma^2 + \lambda_1\gamma, \lambda_1^2 + \lambda_1\gamma)}$$

where λ_1 and λ_2 denote respectively the first and second Chern classes of the Hodge bundle, and γ denotes the first class of the pushforward of the structure sheaf of a natural degree 2 cover of \mathcal{R}_2 (see Lemma 43).

Assumptions on Characteristic: for the remainder of the paper, we work over an algebraically closed field k of characteristic distinct from 2 and 3.

¹In [Ver11] the field k is assumed to be the field of complex numbers, but the result we are interested in remains true (with the same proof) over every algebraically closed field of characteristic different from 2.

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2 Presentation of the stack of genus 2 Prym pairs

2.1 Definitions

We start with recalling the definition of \mathcal{R}_g .

Definition 5. A Prym curve of genus g is the datum of (C, η, β) where C is a smooth geometrically connected genus g curve, $\eta \in \text{Pic}(C)$ is non-trivial and $\beta : \eta^{\otimes 2} \rightarrow \mathcal{O}_C$ is an isomorphism of invertible sheaves on C .

Definition 6. A family of Prym curves of genus g is a smooth family of genus g curves $f : C \rightarrow S$ with an invertible sheaf η on C and an isomorphism $\beta : \eta^{\otimes 2} \rightarrow \mathcal{O}_C$ such that the restriction of these data to any geometric fiber of f gives rise to a Prym curve.

Definition 7. The prestack $\mathcal{R}_g^{\text{pre}}$ is defined over the Big étale site $\text{Sch}_{\text{ét}}$ as the category whose objects over a scheme S are families $(C \rightarrow S, \eta, \beta)$ of genus g Prym curves over S and a morphism $(C \rightarrow S, \eta, \beta) \rightarrow (C' \rightarrow S', \eta', \beta')$ is the data of a cartesian diagram

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & C' \\ \downarrow & \square & \downarrow \\ S & \xrightarrow{f} & S' \end{array} \quad (2)$$

such that there exists an isomorphism $\tau : \varphi^* \eta' \rightarrow \eta^2$ such that the diagram

$$\begin{array}{ccc} \varphi^* \eta'^{\otimes 2} & \xrightarrow{\tau^{\otimes 2}} & \eta^{\otimes 2} \\ \downarrow \varphi^*(\beta'^{\otimes 2}) & & \downarrow \beta^{\otimes 2} \\ \varphi^* \mathcal{O}_{C'} & \longrightarrow & \mathcal{O}_C \end{array} \quad (3)$$

commutes.

The moduli stack \mathcal{R}_g is the stackification of $\mathcal{R}_g^{\text{pre}}$.

²Observe that we are adopting the convention that the datum of τ is not included in the definition of a morphism in $\mathcal{R}_g^{\text{pre}}$.

We will discuss some properties of \mathcal{R}_g is §2.1.1.

Remark 8. *Stackification is a two step process (see [Alp24, Section 2.5.6] for the details): given a prestack \mathcal{Y} , one first defines another prestack $\mathcal{Y}^{\text{st}_1}$ satisfying the condition that morphisms of $\mathcal{Y}^{\text{st}_1}$ glue and a morphism $\mathcal{Y} \rightarrow \mathcal{Y}^{\text{st}_1}$ such that*

$$\text{Mor}(\mathcal{Y}^{\text{st}_1}, \mathcal{Z}) \rightarrow \text{Mor}(\mathcal{Y}, \mathcal{Z})$$

is an equivalence of categories for all prestacks \mathcal{Z} in which morphisms glue. Then, one define the stackification \mathcal{Y}^{st} with a morphism $\mathcal{Y}^{\text{st}_1} \rightarrow \mathcal{Y}^{\text{st}}$ having the same universal property as above but where \mathcal{Z} is required to be a stack.

In our paper, we will use the notation $\mathcal{R}_g^{\text{st}_1}$ to denote the resulting prestack after the first step. Unravelling the definitions, we see that, its objects are the same as the objects of $\mathcal{R}_g^{\text{pre}}$, but a morphism $(C \rightarrow S, \eta, \beta) \rightarrow (C' \rightarrow S', \eta', \beta')$ is the data of a cartesian diagram as in (2) such that there exists an étale cover $\tilde{S} \rightarrow S$ and a isomorphism $\tau : h^ \varphi^* \eta' \rightarrow h^* \eta$ such that the diagram (3) commutes after pullback under $\tilde{S} \times_S C \xrightarrow{h} C$.*

Finally, we note that, after stackification, an object of \mathcal{R}_g still includes the data of a genus g curve, and, given objects $(C/S, \eta, \beta)$ and $(C'/S', \eta', \beta')$ of \mathcal{R}_g , one has an inclusion

$$\text{Mor}_{\mathcal{R}_g}((C/S, \eta, \beta), (C'/S', \eta', \beta')) \subseteq \text{Mor}_{\mathcal{M}_g}(C/S, C'/S'). \quad (4)$$

2.1.1 Background facts about \mathcal{R}_g

As we pointed out in the introduction, there is another natural stack with the same coarse moduli space as \mathcal{R}_g . The aim of this section is to shortly clarify the differences and the relations between these two stacks.

Definition 9. *Let $\tilde{\mathcal{R}}_g$ be the stack over the big étale site whose objects are families of genus g Prym curves, and a morphism $(C \rightarrow S, \eta, \beta) \rightarrow (C' \rightarrow S', \eta', \beta')$ is a Cartesian diagram of curves as in (2) and an isomorphism $\tau : \varphi^* \eta' \rightarrow h^* \eta$ such that the diagram (3) commutes.*

It is possible to show that $\tilde{\mathcal{R}}_g$ is a smooth DM-stack. We will use this throughout.

Remark 10. *The stack $\tilde{\mathcal{R}}_g$ has a $\mathbb{Z}/2\mathbb{Z}$ -2-structure (see [AGV06, Appendix C] for the definition) given by the multiplication by -1 on the line bundle η , and the corresponding rigidification is exactly \mathcal{R}_g .*

Let $\tilde{\mathcal{J}}_g$ be the stack of smooth curves of genus g and line bundles of degree 0 with morphisms given by maps between the curves and the line bundles. Then, $\tilde{\mathcal{J}}_g$ has a \mathbb{G}_m -2-structure, whose rigidification we denote by \mathcal{J}_g . The resulting stack is projective over \mathcal{M}_g by [Del85, Theorem 4.3].

Note that there is a natural morphism $\tilde{\iota} : \tilde{\mathcal{R}}_g \rightarrow \tilde{\mathcal{J}}_g$.

Proposition 11. *With notation as above, \mathcal{R}_g is a smooth DM stack, with a representable, finite and étale map to \mathcal{M}_g of degree $2^{2g} - 1$. Moreover, there is a commutative diagram*

$$\begin{array}{ccc} \tilde{\mathcal{R}}_g & \xrightarrow{\tilde{\iota}} & \tilde{\mathcal{J}}_g \\ \downarrow \pi & & \downarrow \\ \mathcal{R}_g & \xrightarrow{\iota} & \mathcal{J}_g \end{array}$$

where the vertical arrows are the rigidification morphisms, and the lower horizontal arrow is a closed immersion.

The previous proposition might be well-known to experts. However, we included a sketch of the proof due to our lack of knowledge of a precise reference.

Sketch of proof. Since $\tilde{\mathcal{R}}_g$ is a smooth DM-stack, so is \mathcal{R}_g , by [ACV03, Theorem 5.1.5.]. Moreover, the morphism $\iota : \mathcal{R}_g \rightarrow \mathcal{J}_g$ is obtained from $\tilde{\iota}$ and the universal property of rigidification [ACV03, Theorem 5.1.5.].

Next we show that ι is a closed embedding. Equation (4) shows that $\mathcal{R}_g \rightarrow \mathcal{M}_g$ is representable (by algebraic spaces), so ι is also representable.

Now, we show that it is proper. Recall that, from Remark 8, an S -object of \mathcal{R}_g is a curve $C \rightarrow S$ of genus g , and the data (η, β) of a line bundle η and trivialization of $\eta^{\otimes 2}$ defined only after an étale base change $\tilde{S} \rightarrow S$. Let $S = \text{Spec}(R)$ be the spectrum of a DVR, and consider a smooth curve $C \rightarrow S$ of genus g with the data of two Prym curves (η_i, β_i) for $i = 1, 2$, defined only up to an étale cover $\tilde{S} \rightarrow S$, that agree on the generic fiber. To check the uniqueness part of the valuative criterion, we have to show that the identity $C \rightarrow C$ is a morphism between (C, η_1, β_1) and (C, η_2, β_2) . Because $\mathcal{J}_g \rightarrow \mathcal{M}_g$ is separated, up to a further étale cover, there is an isomorphism $\tau : \eta_1 \rightarrow \eta_2$. Finally, perhaps after a further étale cover, we can also arrange τ in a way that makes the diagram (2) commute (see the proof of Proposition 26 below for more details on how to arrange τ in this way).

To check the existence part of the valuative criterion, let $C \rightarrow S = \text{Spec}(R)$ be as before, and let (C, η, β) be a $\text{Frac}(R)$ -point of \mathcal{R}_g . Since \mathcal{J}_g is proper, there is an extension of DVR's $R \rightarrow R_1$ such that η is in fact a line bundle on the generic point of C_{R_1} which can be extended to the whole C_{R_1} . Since the Picard group of a curve over a DVR and over its generic point are isomorphic, the extension is necessarily 2-torsion too, and this shows the existence of an extension of β .

It is clear that $\kappa : \mathcal{R}_g \rightarrow \mathcal{M}_g$ is quasi-finite. We conclude that it is representable by schemes by [Sta18, 03XX], and finite by the Zariski Main Theorem.

Moreover, the geometric fibers of κ have constant cardinality equal to $2^{2g} - 1$ and, since the two stacks are smooth, this is also the degree of κ , and this implies that all fibers are reduced. Therefore κ is finite étale.

We also know that ι is representable by algebraic spaces, quasi-finite, and it is proper because κ is, so it is representable by schemes and finite by the Zariski Main Theorem. In fact, it is injective on geometric points. Since κ is unramified, so is ι , and so ι is a closed embedding by [Sta18, 04DG]. \square

Remark 12. *The stack $\tilde{\mathcal{R}}_g$ is also proper over \mathcal{M}_g (as it has the same coarse moduli space as \mathcal{R}_g), but it is not representable over \mathcal{M}_g and does not embed into $\tilde{\mathcal{J}}_g$. In fact, $\tilde{\mathcal{R}}_g$ is DM but $\tilde{\mathcal{J}}_g$ has 1-dimensional automorphism groups.*

From this perspective, the stack \mathcal{R}_g is better behaved than $\tilde{\mathcal{R}}_g$. Both variants already appeared in the literature. The paper [Cor91] discusses $\tilde{\mathcal{R}}_g$, while [Cor89, BCF03] focus on \mathcal{R}_g .

Remark 13. *There is nothing special about 2-torsion in this argument. For any n that is invertible over the base field, if one studies curves with a line bundle of order n , there*

are two stacks $\mathcal{M}_g(n)$ and $\widetilde{\mathcal{M}}_g(n)$. The first one is the $\mathbb{Z}/n\mathbb{Z}$ -rigidification of the latter, and embeds into \mathcal{J}_g .

Remark 14. As A. Landi pointed out to us, there is a canonical map $\mathcal{H}_{g,2}^w \rightarrow \mathcal{R}_g$ from the moduli stack of hyperelliptic curves with 2-Weierstrass sections (see [EH21]), which is $\mathbb{Z}/2\mathbb{Z}$ -torsor. This provides a further presentation of \mathcal{R}_g .

2.2 Preliminaries on G

We start with some notation. Recall that $G = (\mathbb{G}_m \times \mathbb{G}_m) \rtimes \mathbb{Z}/2\mathbb{Z}$.

Notation 15. We will often think of G inside GL_2 embedded as the subgroup of matrices of the form

$$(a, b; 0) := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ or } (a, b; 1) := \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

for $a, b \in k^*$.

The next proposition explains how the group G arises in our computation.

Proposition 16. The group scheme G is isomorphic to the group scheme

$$\underline{\mathrm{Aut}}_{\{0,\infty\}}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-3))$$

of automorphisms of $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-3))$ that preserve the set $\{0, \infty\} \subseteq \mathbb{P}^1$.

Proof. From the exact sequence of group schemes

$$1 \rightarrow \mu_3 \rightarrow \underline{\mathrm{Aut}}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \rightarrow \underline{\mathrm{Aut}}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-3)) \rightarrow 1$$

and identifying $\underline{\mathrm{Aut}}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = \mathrm{GL}_2$, we obtain an isomorphism

$$\underline{\mathrm{Aut}}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-3)) \xrightarrow{\sim} \mathrm{GL}_2/\mu_3. \tag{5}$$

where $\mu_3 \subseteq \mathrm{GL}_2$ is the group of diagonal matrices of the form $\mathrm{diag}(\zeta, \zeta)$ where $\zeta \in \mathbb{G}_m$ is a 3-rd root of unity. Moreover, we have an isomorphism of group schemes

$$\mathrm{GL}_2/\mu_3 \rightarrow \mathrm{GL}_2 \tag{6}$$

$$[A] \mapsto \det(A)A \tag{7}$$

with inverse given by

$$B \mapsto \left[\frac{B}{\det(B)^{\frac{1}{3}}} \right].$$

Note that cubic roots are well-defined mod μ_3 . By a direct computation,

$$\underline{\mathrm{Aut}}_{\{0,\infty\}}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \subset \underline{\mathrm{Aut}}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = \mathrm{GL}_2$$

is identified with $G \subseteq \mathrm{GL}_2$ and the restriction of the isomorphism (5), yields

$$\underline{\mathrm{Aut}}_{\{0,\infty\}}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-3)) \xrightarrow{\sim} G/\mu_3$$

while then the restriction of (6)

$$G/\mu_3 \xrightarrow{\sim} G.$$

The composition of the two morphisms is the desired isomorphism. \square

We will need to know the Chow ring of BG , which was computed by E. Larson [Lar21].

Notation 17. *Following [Lar21], we set*

$$\begin{aligned}\beta_i &= c_i(V_G) \in \mathrm{CH}^*(BG), \\ \gamma &= c_1(\Gamma_G) \in \mathrm{CH}^*(BG),\end{aligned}$$

where V_G, Γ_G are the vector bundles over BG associated to the representations V and Γ defined in Definition 1.

Theorem 18. [Lar21, Theorem 5.2.] *The Chow ring of BG is given by*

$$\mathrm{CH}^*(BG) \cong \frac{\mathbb{Z}[\beta_1, \beta_2, \gamma]}{(2\gamma, \gamma(\gamma + \beta_1))},$$

2.3 Construction of the morphism

We start with defining a morphism from the quotient prestack

$$\alpha^{\mathrm{pre}} : \mathcal{X}^{\mathrm{pre}} := \left[\frac{\mathrm{Sym}^4(V^\vee) \otimes \det(V) \otimes \Gamma \setminus \Delta}{G} \right]^{\mathrm{pre}} \rightarrow \mathcal{R}_2. \quad (8)$$

as follows.

The map (8) is defined at the level of objects as follows. Given a morphism $F : S \rightarrow \mathrm{Sym}^4(V^\vee) \setminus \Delta$, we map F to the triple $(C/S, \eta, \beta)$ where

$$C = \underline{\mathrm{Spec}}_{\mathbb{P}_S^1}(\mathcal{O}_{\mathbb{P}_S^1} \oplus \mathcal{O}_{\mathbb{P}_S^1}(-3)) \quad (9)$$

where the $\mathcal{O}_{\mathbb{P}_S^1}$ -algebra structure on $\mathcal{O}_{\mathbb{P}_S^1} \oplus \mathcal{O}_{\mathbb{P}_S^1}(-3)$ is given by

$$\mathcal{O}_{\mathbb{P}_S^1}(-3) \otimes \mathcal{O}_{\mathbb{P}_S^1}(-3) \xrightarrow{XY.F} \mathcal{O}_{\mathbb{P}_S^1}.$$

where $XY : S \rightarrow \mathrm{Sym}^2(V^\vee)$ is the constant function equal to XY and $XY.F$ denotes its product with the polynomial F . Clearly the curve C comes with a map

$$q : C \rightarrow \mathbb{P}_S^1. \quad (10)$$

We have two sections $\sigma_0, \sigma_\infty : S \rightarrow C$, corresponding to the 0 and ∞ section of $\mathbb{P}_S^1 \rightarrow S$. As explained in [Per24, Section 1], given a section $s : S \rightarrow \mathbb{P}_S^1$ and an $\mathcal{O}_{\mathbb{P}_S^1}$ -algebra \mathcal{A} , one has the functorial bijective map

$$\mathrm{Hom}_{\mathbb{P}_S^1}(S, \underline{\mathrm{Spec}}_{\mathbb{P}_S^1}(\mathcal{A})) \rightarrow \mathrm{Hom}_{\mathcal{O}_S\text{-alg}}(s^*(\mathcal{A}), \mathcal{O}_S).$$

Note now that

$$\mathrm{Hom}_{\mathcal{O}_S\text{-alg}}(0^*(\mathcal{O}_{\mathbb{P}_S^1} \oplus \mathcal{O}_{\mathbb{P}_S^1}(-3)), \mathcal{O}_S) \subseteq \mathrm{Hom}_{\mathcal{O}_S\text{-mod}}(0^*\mathcal{O}_{\mathbb{P}_S^1}(-3), \mathcal{O}_S)$$

is the subset of \mathcal{O}_S -module homomorphisms j_0 such that the two maps $j_0^{\otimes 2} : 0^*\mathcal{O}_{\mathbb{P}_S^1}(-6) \rightarrow \mathcal{O}_S$ and $XY.F : 0^*\mathcal{O}_{\mathbb{P}_S^1}(-6) \rightarrow \mathcal{O}_S$ coincide. But the last morphism is clearly 0, thus we

can take $j_0 = 0$. In conclusion, the section of σ_0 is no further data. Similarly, the section σ_∞ is no further data.

We set

$$\eta = q^* \mathcal{O}_{\mathbb{P}_S^1}(1) \otimes \mathcal{O}_C(-\sigma_0 - \sigma_\infty) \quad (11)$$

and finally the isomorphism

$$\eta^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_C \quad (12)$$

is given by XY .

Remark 19. As explained in [Vis98, proof of Proposition 3.1], in our situation, an isomorphism $q^* \mathcal{O}_{\mathbb{P}_S^1}(1) \cong \omega_{C/S}$ is functorially the same data as an isomorphism

$$\omega_{\mathbb{P}_S^1/S}(-1) \otimes \mathcal{O}_{\mathbb{P}_S^1}(3) \cong \mathcal{O}_{\mathbb{P}_S^1}.$$

We have such an isomorphism, up to fixing once and for all an isomorphism $\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$. We will fix such an isomorphism.

The next lemma will be useful later.

Lemma 20. The union $\sigma_0(S) \cup \sigma_\infty(S) \subseteq C$ is the unique effective relative Cartier divisor of $\omega_{C/S} \otimes \eta^\vee$.

Proof. The restriction of $\omega_{C/S} \otimes \eta^\vee$ to each geometric fiber C_s is a degree 2 line bundle not isomorphic to ω_{C_s} , thus $h^0(C_s, \omega_{C/S} \otimes \eta^\vee) = 1$ for all $s \in S$. \square

Now, we define the morphism (8) at the level of morphisms. A morphism

$$(F : S \rightarrow \mathrm{Sym}^4(V^\vee) \otimes \det(V) \otimes \Gamma \setminus \Delta) \rightarrow (F' : S' \rightarrow \mathrm{Sym}^4(V^\vee) \otimes \det(V) \otimes \Gamma \setminus \Delta)$$

in $\mathcal{X}^{\mathrm{pre}}$ is by definition the data of morphisms $f : S \rightarrow S'$ and $g : S \rightarrow G$ such that $F' \circ f = g.F$, where $g.F$ denotes the action of g on F .

Remark 21. Write $g(s) = (a(s), b(s), \sigma^i(s))$ for $s \in S$. Then, spelling out the action of G on F , this is equivalent to

$$F'(f(s)) = a(s)b(s) \left(F \circ (a(s), b(s), \sigma^i(s))^{-1} \right) \quad (13)$$

for $s \in S$.

Let $(C/S, \eta, \beta)$ and $(C'/S', \eta', \beta')$ be the objects in \mathcal{R}_2 associated to F and F' respectively. Call also

$$\phi = f \times g : \mathbb{P}_S^1 \rightarrow \mathbb{P}_{S'}^1.$$

Here, we view g as an automorphism of \mathbb{P}^1 via the isomorphism in Proposition 16. We have a morphism

$$\varphi : C \rightarrow C' \quad (14)$$

given by $\varphi^\# : \phi^* \mathcal{O}_{\mathbb{P}_{S'}^1}(-3) \rightarrow \mathcal{O}_{\mathbb{P}_S^1}(-3)$ defined at $(s, \ell) \in \mathbb{P}_S^1 = S \times \mathbb{P}^1$ by

$$(\varphi^\#(\phi^*(v)))(x, \ell) = g(s)^{-1} \cdot v(f(s), g(s) \cdot \ell)$$

Again, G acts on $(\mathbb{P}^1, \mathcal{O}(-3))$ via the isomorphism in Proposition 16.

Lemma 22. *The morphism*

$$\phi^*(\mathcal{O}_{\mathbb{P}_S^1} \oplus \mathcal{O}_{\mathbb{P}_S^1}(-3)) \rightarrow \mathcal{O}_{\mathbb{P}_S^1} \oplus \mathcal{O}_{\mathbb{P}_S^1}(-3)$$

induced by $\varphi^\#$ is an homomorphism of $\mathcal{O}_{\mathbb{P}_S^1}$ -algebras.

Proof. Here it is crucial to take into account how G acts on the space $\text{Sym}^4(V^\vee) \setminus \Delta$. Since the algebra structure on $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ is determined by $XY.F$, the statement is equivalent to the identity

$$((\varphi^\#)^\vee)^{\otimes 2} \left(\phi^*(XYF') \right) = XYF \quad (15)$$

as sections of $\mathcal{O}_{\mathbb{P}_S^1}(6)$. We check this at a point $(s, \ell) \in \mathbb{P}_S^1 = S \times \mathbb{P}^1$:

$$\begin{aligned} ((\varphi^\#)^\vee)^{\otimes 2} \left(\phi^*(XYF') \right) (s, \ell) &= g(s)^{-1} \cdot \left((XYF')(f(s), g(s) \cdot \ell) \right) \\ &= (XYF') \circ \left(\frac{(a(s), b(s); \sigma^i(s))}{\det(a(s), b(s); \sigma^i(s))^{1/3}} \right) (s, \ell) \\ &= \left(a(s)^{-1} b(s)^{-1} (XY)(F' \circ (a(s), b(s); \sigma^i(s))) \right) (s, \ell) \\ &= (XYF)(s, \ell) \end{aligned}$$

where we wrote $g(s) = (a(s), b(s); \sigma^i(s)) \in G(S)$ with $\sigma^i(s) = \delta_0^i$ as explained in Notation 15. Note that in the last equality we used Equation (13). \square

This shows that the morphism (14) is well-defined.

To conclude the definition of α^{pre} on morphisms we need to provide, étale locally on S , a morphism τ between $\varphi^*\eta'$ and η making the diagram (3) commute. Note that, since g preserves $\{0, \infty\}$, we have

$$\varphi^{-1}(\sigma'_0(S') \cup \sigma'_\infty(S')) = \sigma_0(S) \cup \sigma_\infty(S)$$

and thus

$$\varphi^*(-\sigma'_0(S') - \sigma'_\infty(S')) = -\sigma_0(S) - \sigma_\infty(S) \quad (16)$$

as Weil divisors. Furthermore, étale locally on S , we have an isomorphism

$$\phi^*\mathcal{O}_{\mathbb{P}_S^1}(1) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}_S^1}(1) \quad (17)$$

that over a point $(s, \ell) \in \mathbb{P}_S^1$ maps $\varphi \in (g(s) \cdot \ell)^\vee$ to $\varphi \circ \frac{g(s)}{\sqrt{a(s)b(s)}} \in \ell^\vee$. A square root of ab is defined étale locally on S . We claim that we can take τ to be the product of the morphisms in (16) and (17).

Lemma 23. *Let τ be the isomorphism $\varphi^*\eta' \rightarrow \eta$ obtained as the product of the morphisms in (16) and (17). Then, the diagram (3) commutes.*

Proof. This is an easy check left to the reader. \square

This concludes the definition of the morphism (8).

2.4 Proof of Theorem 2

By the universal property of stackification, the morphism (8) yields a morphism

$$\alpha : \mathcal{X} := \left[\frac{\mathrm{Sym}^4(V^\vee) \otimes \det(V) \otimes \Gamma \setminus \Delta}{G} \right] \rightarrow \mathcal{R}_2 \quad (18)$$

which we aim to prove is an isomorphism of stacks.

We will use the following well-known lemma.

Lemma 24. *Let $b : \mathcal{X}^{\mathrm{pre}} \rightarrow \mathcal{X}$ be a stackification. Let $\alpha^{\mathrm{pre}} : \mathcal{X}^{\mathrm{pre}} \rightarrow \mathcal{Y}$ be a morphism to a stack \mathcal{Y} , and let $\alpha : \mathcal{X} \rightarrow \mathcal{Y}$ be the induced morphism. If α^{pre} satisfies the following two conditions:*

(i) *it is fully faithful,*

(ii) *for every object $Y \in \mathcal{Y}(S)$ over S there is a covering $\tilde{S} \rightarrow S$ and an object $\tilde{X} \in \mathcal{X}^{\mathrm{pre}}(\tilde{S})$ such that $\alpha^{\mathrm{pre}}(\tilde{X}) = Y|_{\tilde{S}}$,*

then α is an isomorphism.

We start with showing that α^{pre} is fully faithful.

Proposition 25. *The morphism α^{pre} satisfies condition (i) of Lemma 24.*

Proof. First, we notice that α^{pre} factors as

$$\alpha^{\mathrm{pre}} : \mathcal{X}^{\mathrm{pre}} \rightarrow \mathcal{R}_2^{\mathrm{st}_1} \rightarrow \mathcal{R}_2.$$

Since the second map is fully faithful by construction, we only need to prove that the first map is faithful. We will refer to again by the symbol α^{pre} . It is enough to prove that $\alpha^{\mathrm{pre}}(S) : \mathcal{X}^{\mathrm{pre}}(S) \rightarrow \mathcal{R}_2^{\mathrm{st}_1}(S)$ is fully faithful for all schemes S . Let $u, u' : S \rightarrow \mathcal{X}^{\mathrm{pre}}$ be objects in \mathcal{X} and let $\alpha^{\mathrm{pre}}(u) = (\pi : C \rightarrow S, \eta, \beta)$ and $\alpha^{\mathrm{pre}}(u') = (\pi' : C' \rightarrow S, \eta', \beta')$ be their images under α^{pre} . We wish to show that the induced map

$$\mathrm{Mor}_{\mathcal{X}(S)}(u, u') \rightarrow \mathrm{Mor}_{\mathcal{R}_2^{\mathrm{st}_1}(S)}((C/S, \eta, \beta), (C'/S, \eta', \beta')) \quad (19)$$

is bijective.

Let $\varphi : C \rightarrow C'$ be an isomorphism for which there exists an isomorphism $\tau : \varphi^* \eta' \rightarrow \eta$ such that the diagram (3) commutes. First of all, we note that there exists a unique isomorphism $\phi : \mathbb{P}_S^1 \rightarrow \mathbb{P}_S^1$ such that the following diagram

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & C' \\ \downarrow q & & \downarrow q' \\ \mathbb{P}_S^1 & \xrightarrow{\phi} & \mathbb{P}_S^1 \end{array} \quad (20)$$

commutes. The uniqueness of ϕ is clear. To show the existence of ϕ , we consider the corresponding diagram where the vertical arrows are the canonical maps and the lower horizontal arrow is given by the pullback of forms via φ^*

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & C' \\ \downarrow p & & \downarrow p' \\ \mathbb{P}(\pi_* \omega_{C/S})^\vee & \xrightarrow{\varphi^*} & \mathbb{P}(\pi'_* \omega_{C'/S})^\vee. \end{array}$$

This diagram clearly commutes.

By Remark 19, the maps q and q' are given by two sections of $\omega_{C/S}$ and $\omega_{C'/S}$ respectively. These two sections yields commutative diagrams

$$\begin{array}{ccc} & C & \\ p \swarrow & & \searrow q \\ \mathbb{P}(\pi_*\omega_{C/S})^\vee & \xrightarrow{h} & \mathbb{P}_S^1 \end{array} \quad \begin{array}{ccc} & C' & \\ p' \swarrow & & \searrow q' \\ \mathbb{P}(\pi'_*\omega_{C'/S})^\vee & \xrightarrow{h'} & \mathbb{P}_S^1 \end{array}$$

where h and h' are isomorphisms. In these terms, the morphism ϕ in diagram (20) is the composition $h' \circ \varphi^* \circ h^{-1} : \mathbb{P}_S^1 \rightarrow \mathbb{P}_S^1$.

At this point, the morphism φ induces an isomorphism

$$\varphi^\# : \phi^* \mathcal{O}_{\mathbb{P}_S^1}(-3) \rightarrow \mathcal{O}_{\mathbb{P}_S^1}(-3).$$

This shows that φ can be the image of a unique element $g \in \underline{\text{Aut}}(\mathbb{P}_S^1, \mathcal{O}_{\mathbb{P}_S^1}(-3))$ and injectivity of (19) follows.

In order to prove surjectivity of (19), it is enough to check that $g \in \underline{\text{Aut}}_{\{0, \infty\}}(\mathbb{P}_S^1, \mathcal{O}_{\mathbb{P}_S^1}(-3))$. The existence of τ implies that there is an isomorphism

$$\varphi^* \mathcal{O}_{C'}(\sigma'_0 + \sigma'_\infty) \cong \mathcal{O}_C(\sigma_0 + \sigma_\infty).$$

and, from Lemma 20, we obtain that

$$\varphi^{-1}(\sigma'_0(S) \cup \sigma'_\infty(S)) = \sigma_0(S) \cup \sigma_\infty(S).$$

Finally, the commutativity of (20) implies that $\phi(0(S) \cup \infty(S)) = 0'(S) \cup \infty'(S)$ and thus that $g \in \underline{\text{Aut}}_{\{0, \infty\}}(\mathbb{P}_S^1, \mathcal{O}_{\mathbb{P}_S^1}(-3))$. This concludes the proof. \square

Proposition 26. *The morphism α^{pre} satisfies condition (ii) of Lemma 24.*

Proof of Proposition 26. Let X be an object of \mathcal{R}_2 over a scheme S . Up to an étale cover of S , we may assume that X is of the form $X = (C'/S, \eta', \beta' : \eta'^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_C)$. We need to show that, up to an étale cover of S , there exists $F : S \rightarrow \text{Sym}^4(V^\vee) \otimes \det(V) \otimes \Gamma \setminus \Delta$ and:

- (a) an isomorphism φ between the curve C in (9) and C' ;
- (b) an isomorphism $\tau : \varphi^* \eta' \rightarrow \eta$ where η is the line bundle defined in (11) such that the diagram (3) commutes.

Let \bar{s} be a geometric point of S . We will prove that (a) and (b) holds in an étale neighborhood of \bar{s} . By [Vis98], we know that, perhaps after base-change by an étale cover of S , there exists F in \mathcal{X}^{pre} over S such that there is an isomorphism $\varphi : C \rightarrow C'$ over S . By Lemma 3, we have $\eta'_{\bar{s}} = q'_{\bar{s}} \mathcal{O}(1) \otimes \mathcal{O}_{C'_{\bar{s}}}(-w_i - w_j)$ for exactly one subset $\{w_i, w_j\}$ of cardinality 2 of the set of Weierstrass points of $C'_{\bar{s}}$. Here $q'_{\bar{s}} : C'_{\bar{s}} \rightarrow \mathbb{P}^1$ is any degree 2 map such that $q'_{\bar{s}}(\{w_i, w_j\}) = \{0, \infty\} \subseteq \mathbb{P}^1$. Up to composing the map (10) with an automorphism of \mathbb{P}^1 fixing $\{0, \infty\}$ (which amounts to changing F – but not C), we may assume that

$$q'_{\bar{s}} \circ \varphi_{\bar{s}} = q_{\bar{s}}$$

where q is the map in (10). It follows then $\varphi^*\eta'_s = \eta_{\bar{s}}$. The locus

$$\{s \in S \mid \varphi^*\eta'_s \cong q_s^*\mathcal{O}_{\mathbb{P}^1_S}(1) \otimes \mathcal{O}_{C_s}(-\sigma_0(s) - \sigma_\infty(s))\}$$

is, by Lemma 3, open and closed in S and contains \bar{s} . We may thus replace S with it. It follows that

$$\varphi^*\eta' = \eta \otimes L$$

where L is a line bundle pulled back to C from S . Zariski-locally on S , the line bundle L is trivial. Up to replacing S by an open subset containing \bar{s} we may then assume that we have an isomorphism $\varphi^*\eta' \xrightarrow{\tau} \eta$. The diagram

$$\begin{array}{ccc} \varphi^*\eta'^{\otimes 2} & \xrightarrow{\tau^{\otimes 2}} & \eta^{\otimes 2} \\ \downarrow \varphi^*(\beta'^{\otimes 2}) & & \downarrow \beta^{\otimes 2} \\ \varphi^*\mathcal{O}_{C'} & \longrightarrow & \mathcal{O}_C \end{array}$$

commutative up to some $\lambda \in \mathbb{G}_m(S)$. Here β is defined in (12). However, étale locally around \bar{s} , we can find a square root $\mu \in \mathbb{G}_m(S)$ of λ and, replacing τ with $\frac{\tau}{\mu}$, we obtain the commutativity of the above diagram. This concludes the proof of the proposition. \square

Proof of Theorem 2. Theorem 2 now follows immediately from Lemma 24 and Propositions 25 and 26. \square

3 Computation of the Chow ring of \mathcal{R}_2

In this section we use Theorem 2 and the tools in [EG98] to compute the Chow ring of \mathcal{R}_2 . In the following, we will repeatedly use the isomorphism $\mathrm{CH}^*([X/G]) = \mathrm{CH}_G^*(X)$ without explicitly mentioning it.

The first observation is that we have a G -equivariant projection

$$p : \mathrm{Sym}^4(V^\vee) \otimes \det(V) \otimes \Gamma \setminus \Delta \rightarrow \mathbb{P}\mathrm{Sym}^4(V^\vee) \setminus \underline{\Delta} \quad (21)$$

where $\underline{\Delta} \subseteq \mathbb{P}\mathrm{Sym}^4(V^\vee)$ is the locus of polynomials (up to non-zero scalars) having either a root at 0 or ∞ , or a double root in any field extension of the base field k .

Lemma 27. *The pullback p^* induces an isomorphism*

$$\frac{\mathrm{CH}_G^*(\mathbb{P}(\mathrm{Sym}^4(V^\vee)) \setminus \underline{\Delta})}{(-h + \beta_1 + \gamma)} \xrightarrow{\sim} \mathrm{CH}_G^*(\mathrm{Sym}^4(V^\vee) \otimes \det(V) \otimes \Gamma \setminus \Delta)$$

where $h = c_1^G(\mathcal{O}_{\mathbb{P}\mathrm{Sym}^4(V^\vee)}(1))$, and β_1 and γ are defined in Notation 17.

Proof. The morphism p is a \mathbb{G}_m -torsor with associated line bundle $\mathcal{O}_{\mathbb{P}\mathrm{Sym}^4(V^\vee)}(-1) \otimes \det(V) \otimes \Gamma$. \square

Next, we compute $\mathrm{CH}_G^*(\mathbb{P}\mathrm{Sym}^4(V^\vee) \setminus \underline{\Delta})$ using the excision sequence:

$$\mathrm{CH}_*^G(\underline{\Delta}) \rightarrow \mathrm{CH}_*^G(\mathbb{P}\mathrm{Sym}^4(V^\vee)) \rightarrow \mathrm{CH}_*^G(\mathbb{P}\mathrm{Sym}^4(V^\vee) \setminus \underline{\Delta}) \rightarrow 0 \quad (22)$$

In order to do so, we will need a G -equivariant envelope of $\underline{\Delta}$.

3.1 A G -equivariant envelope for $\underline{\Delta}$

We first recall the definition.

Definition 28. [Ful84, Definition 18.3] and [EG98, page 9] An envelope of a scheme X is a proper morphism $p : X' \rightarrow X$ such that for every subvariety³ Y of X , there is a subvariety Y' of X' such that p maps Y' birationally onto Y .

Suppose that G is a linear algebraic group acting on X and X' . We will say that $p : X' \rightarrow X$ is a G -equivariant envelope, if p is G -equivariant, proper and if we can take Y' to be G -invariant for G -invariant Y .

The importance of G -envelope is explained by the next lemma.

Lemma 29. Let $p : X' \rightarrow X$ be a G -equivariant envelope. Then the pushforward

$$p_* : \mathrm{CH}_*^G(X') \rightarrow \mathrm{CH}_*^G(X)$$

is surjective.

Proof. This follows from [EG98, Lemma 3] and [Ful84, Lemma 18.3.(6)]. \square

Finally, the next lemma explains how to construct a G -equivariant envelope.

Lemma 30. Let X be a scheme and G a linear algebraic group acting on X . Given G -equivariant proper and surjective morphisms $p_i : X'_i \rightarrow X_i$ for $i = 1, \dots, d$ such that:

- (i) $X_1 = X$;
- (ii) $X_{i+1} \subseteq X_i$ for $i = 1, \dots, d-1$;
- (iii) if $K \supseteq k$ is any extension and x is a K -valued point of $X_i \setminus X_{i+1}$, then there is a unique K -valued point y of X'_i mapping to x .

then the map $p = \sqcup p_i : \sqcup_{i=1}^d X'_i \rightarrow X$ is a G -equivariant envelope.

Proof. See [EF09, Proof of Proposition 4.1]. \square

In our situation, we take p to be the disjoint union of the following maps p_i for $i = 1, \dots, 4$.

We let

$$p_{1,1} : \mathbb{P}V^\vee \times \mathbb{P}\mathrm{Sym}^2(V^\vee) \rightarrow \underline{\Delta}$$

defined by $(F, G) \mapsto F^2G$ and

$$p_{1,2} : \mathbb{P}\mathrm{Sym}^3(V^\vee) \times \{X, Y\} \subset \mathbb{P}\mathrm{Sym}^3(V^\vee) \times \mathbb{P}V^\vee \rightarrow \underline{\Delta}$$

defined by $(F, G) \mapsto FG$, where $G \in \{X, Y\}$. Then, we set $p_1 = p_{1,1} \sqcup p_{1,2}$ and $\underline{\Delta}_1 = \underline{\Delta}$ to be the image of p_1 .

Let

$$p_{2,1} : \mathbb{P}\mathrm{Sym}^2(V^\vee) \rightarrow \underline{\Delta}$$

³By variety we mean an integral scheme. A subvariety of a scheme is a closed subscheme which is a variety.

be defined by $F \mapsto F^2$,

$$p_{2,2} : \mathbb{P}V^\vee \times \mathbb{P}V^\vee \times \{X, Y\} \subset (\mathbb{P}V^\vee)^{\times 3} \rightarrow \underline{\Delta}$$

defined by $(F_1, F_2, G) \mapsto F_1^2 F_2 G$, where $G \in \{X, Y\}$ and

$$p_{2,3} : \mathbb{P}\text{Sym}^2(V^\vee) \times \{X^2, Y^2, XY\} \subset (\mathbb{P}\text{Sym}^2(V^\vee))^{\times 2} \rightarrow \underline{\Delta}$$

be defined by $(F, G) \mapsto FG$, where $G \in \{X^2, Y^2, XY\}$. We set $p_2 = p_{2,1} \sqcup p_{2,2} \sqcup p_{2,3}$ and $\underline{\Delta}_2 \subseteq \underline{\Delta}_1$ to be the image of p_2 .

Define

$$p_{3,1} : \mathbb{P}V^\vee \times \{X^3, Y^3, X^2Y, XY^2\} \subset \mathbb{P}V^\vee \times \mathbb{P}\text{Sym}^3(V^\vee) \rightarrow \underline{\Delta}$$

by $(F, G) \mapsto FG$, where $G \in \{X^3, Y^3, X^2Y, XY^2\}$, and

$$p_{3,2} : \mathbb{P}V^\vee \times \{X^2, Y^2, XY\} \subset \mathbb{P}V^\vee \times \mathbb{P}\text{Sym}^2(V^\vee) \rightarrow \underline{\Delta}$$

by $(F, G) \mapsto F^2G$, where $G \in \{X^2, Y^2, XY\}$. We set $p_3 = p_{3,1} \sqcup p_{3,2}$ and $\underline{\Delta}_3 \subseteq \underline{\Delta}_2$ to be the image of p_3 .

Finally, we set

$$p_4 : \{X^4, Y^4, X^3Y, XY^3, X^2Y^2\} \rightarrow \underline{\Delta}$$

to be the inclusion, and $\underline{\Delta}_4$ will be its image.

Let $\underline{\Delta}'$ be the disjoint union of the domains of p_1, p_2, p_3 and p_4 .

Proposition 31. *The morphism*

$$p = p_1 \sqcup p_2 \sqcup p_3 \sqcup p_4 : \underline{\Delta}' \longrightarrow \underline{\Delta}$$

is a G -equivariant envelope.

Proof. We will check the conditions of Lemma 30. Condition (i) and (ii) are obvious. We will check that condition (iii) holds for $i = 1$, the other cases are similar. Let $f \in K[X, Y]$ be a K valued point of $\underline{\Delta}_1 \setminus \underline{\Delta}_2$ and let n_0 and n_∞ be respectively the multiplicities of 0 and ∞ as roots of $f \in K[X, Y]$. Clearly $n_0 + n_\infty \leq 1$ otherwise f would be in $\underline{\Delta}_2$.

If $n_0 + n_\infty = 1$, then f cannot be in the image of $p_{1,1}$ and must be in the image of $p_{1,2}$. Suppose $n_0 = 1$ and $n_\infty = 0$. Then, we can write $f = XF$ for a unique $F \in K[X, Y]_3$ (up to K^*) and $f = p_{1,2}((F, X))$.

If $n_0 + n_\infty = 0$, then clearly f is not in the image of $p_{1,2}$ and must be in the image of $p_{1,1}$. In fact, by definition, f must have a double root in some extension of K , and thus also in K as by assumption $\text{char}(k)$ is bigger than the degree of f . Write $f = F^2G$ where $F \in K[X, Y]_1$ and $G \in K[X, Y]_2$. Then, since $f \notin \underline{\Delta}_2$, the polynomial G is not a square (in any extension of K). Thus (F, G) is uniquely determined. \square

3.2 Review of multiplication maps

In order to compute the pushforward along the morphism p we will make use of the machinery developed in [Lar21, Section 4]. We recall here the basic definitions and results.

Let E be a vector bundle of rank 2 over an algebraic stack \mathcal{X} with Chern classes $c_1, c_2 \in \text{CH}^*(\mathcal{X})$. Denote by s_r^j the pushforward of

$$\underbrace{c_1(\mathcal{O}_{\mathbb{P}E}(1)) \otimes \dots \otimes c_1(\mathcal{O}_{\mathbb{P}E}(1))}_{j \text{ times}} \otimes 1$$

along the multiplication map

$$(\mathbb{P}E)^j \times_{\mathcal{X}} \mathbb{P}\text{Sym}^{r-j}(E) \rightarrow \mathbb{P}\text{Sym}^r(E).$$

If $h = c_1(\mathcal{O}_{\mathbb{P}\text{Sym}^r(E)}(1))$, then $s_r^0 = 1$, $s_r^1 = h$ and, more generally, for all $j \geq 0$ the following relation holds:

$$s_r^{j+1} = (h + jc_1)s_r^j + j(r+1-j)c_2s_r^{j-1}.$$

So one obtains:

$$h^2 = s_r^2 - c_1s_r^1 - rc_2 \quad (23)$$

$$h^3 = s_r^3 - 3c_1s_r^2 + (c_1^2 + (2-3r)c_2)s_r^1 + rc_1c_2 \quad (24)$$

$$h^4 = s_r^4 - 6c_1s_r^3 + (7c_1^2 - (6r-8)c_2)s_r^2 + ((10r-8)c_1c_2 - c_1^3)s_r^1 - (rc_1^2c_2 - (3r^2-2r)c_2^2)s_r^0 \quad (25)$$

where $r \geq 2, 3$ and 4 in (23), (24) and (25) respectively. Moreover, the pushforward along multiplication maps

$$\text{mult} : \mathbb{P}\text{Sym}^a(E) \times_{\mathcal{X}} \mathbb{P}\text{Sym}^b(E) \longrightarrow \mathbb{P}\text{Sym}^{a+b}(E)$$

is obtained from the bilinear map

$$\begin{aligned} \text{mult}_* : \text{CH}_*(\mathbb{P}\text{Sym}^a(E)) \times \text{CH}_*(\mathbb{P}\text{Sym}^b(E)) &\rightarrow \text{CH}_*(\mathbb{P}\text{Sym}^{a+b}(E)) \\ (s_a^\alpha, s_b^\beta) &\mapsto \begin{pmatrix} a+b-\alpha-\beta \\ a-\alpha \end{pmatrix} s_{a+b}^{\alpha+\beta} \end{aligned} \quad (26)$$

The maps $p_{1,1}$ and $p_{2,2}$ involve squaring, so we will also need the class of the diagonal.

Lemma 32. *The pushforward along the squaring map*

$$\text{sq} : \mathbb{P}E \longrightarrow \mathbb{P}\text{Sym}^2(E)$$

is given by

$$\begin{aligned} s_1^0 &\mapsto 2s_2^1 + 2c_1 \\ s_1^1 &\mapsto s_2^2 - 2c_2. \end{aligned}$$

The pushforward along the squaring map

$$\text{sq} : \mathbb{P}\text{Sym}^2(E) \longrightarrow \mathbb{P}\text{Sym}^4(E)$$

is given by

$$\begin{aligned} s_2^0 &\mapsto 4s_4^2 + 12c_1s_4^1 + 12c_1^2 \\ s_2^1 &\mapsto 2s_4^3 + 2c_1s_4^2 - 12c_2s_4^1 - 24c_1c_2 \\ s_2^2 &\mapsto s_4^4 - 4c_2s_4^2 + 24c_2^2. \end{aligned}$$

Proof. Let W denote the vector bundle E or $\text{Sym}^2(E)$, and consider the tautological sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow W \longrightarrow Q \longrightarrow 0$$

on $\mathbb{P}W$. The diagonal $\Delta_{\mathbb{P}W} \subset \mathbb{P}W \times_{\mathcal{X}} \mathbb{P}W$ is the vanishing locus of the map

$$p_1^* \mathcal{O}(-1) \longrightarrow p_1^* W = p_2^* W \longrightarrow p_2^* Q,$$

where $p_1, p_2 : \mathbb{P}W \times_{\mathcal{X}} \mathbb{P}W \rightarrow \mathbb{P}W$ are the projections. Therefore, its class is given by

$$[\Delta_{\mathbb{P}W}] = \left[\frac{c(W)}{(1-h_1)(1-h_2)} \right]_{\text{rank}(W)-1},$$

where $h_i = p_i^* h$ for $i = 1, 2$, $h = c_1(\mathcal{O}_{\mathbb{P}W}(1))$. Moreover, $\Delta_{\mathbb{P}W}^*(h_1)$ is again the hyperplane class, so by the projection formula,

$$\Delta_{\mathbb{P}W,*}(h^k) = h_1^k \cdot [\Delta_{\mathbb{P}W}].$$

If $W = E$, then $c(W) = 1 + c_1 + c_2$, so $[\Delta_{\mathbb{P}E}] = h_1 + h_2 + c_1$ and

$$\begin{aligned} \Delta_{\mathbb{P}E,*}(1) &= h_1 + h_2 + c_1 \\ \Delta_{\mathbb{P}E,*}(h) &= h_1^2 + h_1 \otimes h_2 + c_1 h_1 = h_1 \otimes h_2 - c_2 \end{aligned}$$

where in the last equality we have used the projective bundle formula. If $W = \text{Sym}^2(E)$, $c(W) = 1 + 3c_1 + 2c_1^2 + 4c_2 + 4c_1c_2$, so

$$\begin{aligned} \Delta_{\mathbb{P}\text{Sym}^2(E),*}(1) &= h_1^2 + h_1 \otimes h_2 + h_2^2 + 3c_1(h_1 + h_2) + 2c_1^2 + 4c_2 \\ \Delta_{\mathbb{P}\text{Sym}^2(E),*}(h) &= h_1^3 + h_1^2 \otimes h_2 + h_1 \otimes h_2^2 + 3c_1 h_1 \otimes h_2 + 3c_1 h_1^2 + (2c_1^2 + 4c_2)h_1 \\ &= h_1^2 \otimes h_2 + h_1 \otimes h_2^2 + 3c_1 h_1 \otimes h_2 - 4c_1 c_2 \\ \Delta_{\mathbb{P}\text{Sym}^2(E),*}(h^2) &= h_1^3 \otimes h_2 + h_1^2 \otimes h_2^2 + 3c_1 h_1^2 \otimes h_2 - 4c_1 c_2 h_1 \\ &= h_1^2 \otimes h_2^2 - (2c_1^2 + 4c_2)h_1 \otimes h_2 - 4c_1 c_2 (h_1 + h_2). \end{aligned}$$

where again we have used the projective bundle formula. The squaring maps are the composition $\text{mult} \circ \Delta_{\mathbb{P}W}$, so the result follows from formula (26), after using (23). \square

For us, E will be V_G^\vee , and X will be BG , so $c_1 = -\beta_1$ and $c_2 = \beta_2$.

3.3 The classes of the finite subsets

Almost all the maps p_{ij} in §3.1 are obtained as a composition

$$\mathbb{P}\text{Sym}^a(V_G^\vee) \times Z \hookrightarrow \mathbb{P}\text{Sym}^a(V_G^\vee) \times \mathbb{P}\text{Sym}^b(V_G^\vee) \xrightarrow{\text{mult}} \mathbb{P}\text{Sym}^{a+b}(V_G^\vee)$$

for some substack $Z \subseteq \text{Sym}^b(V_G^\vee)$ corresponding to a finite set of points in V^\vee . Here we calculate the equivariant fundamental classes of such finite subsets.

Lemma 33. *The class of $\{X, Y\} \subset \mathbb{P}V_G^\vee$ is $2s_1^1 - (\beta_1 + \gamma)s_1^0$.*

Proof. Let $\xi : B\mathbb{G}_m^2 \rightarrow BG$ the map induced by $\mathbb{G}_m^2 \rightarrow G$, and let $t_1, t_2 \in \text{CH}^*(B\mathbb{G}_m^2)$ be the Chern roots of $\xi^*V_G^\vee = V_{\mathbb{G}_m^2}$. There is a Cartesian diagram

$$\begin{array}{ccc} \mathbb{P}V_{\mathbb{G}_m^2}^\vee & \xrightarrow{\xi} & \mathbb{P}V_G^\vee \\ \downarrow & & \downarrow \\ B\mathbb{G}_m^2 & \xrightarrow{\xi} & BG \end{array}$$

where $\xi^*\mathcal{O}(1) = \mathcal{O}(1)$ and $\xi_*[\{X\}] = [\{X, Y\}]$. Using [EF09, Lemma 2.4.], one sees that

$$[X]^{\mathbb{G}_m^2} = c_1^{\mathbb{G}_m^2}(\mathcal{O}_{\mathbb{P}V^\vee}(1)) - t_1$$

and so by the projection formula,

$$[\{X, Y\}] = \xi_*(c_1^{\mathbb{G}_m^2}(\mathcal{O}_{\mathbb{P}V^\vee}(1)) - t_1) = \xi_*(1)c_1^G(\mathcal{O}_{\mathbb{P}V^\vee}(1)) - \xi_*(t_1) = 2s_1^1 - (\beta_1 + \gamma),$$

where we are using the formulas in [Lar21, Lemma 7.3.] to pushforward along $\xi : B\mathbb{G}_m^2 \rightarrow BG$. \square

Lemma 34. *The class of $\{XY\} \subset \mathbb{P}\text{Sym}^2(V_G^\vee)$ is $s_2^2 + (\gamma - \beta_1)s_2^1 + 2\beta_2s_2^0$*

Proof. Let $\pi : \mathbb{P}\text{Sym}^2(V_G^\vee) \rightarrow BG$ be the natural map. Note that XY is a section of the bundle $\text{Sym}^2(V_G^\vee) \otimes \Gamma_G \otimes \det(V_G)$ over BG . Define Q as the quotient

$$0 \longrightarrow \mathcal{O}_{BG} \xrightarrow{XY} \text{Sym}^2(V_G^\vee) \otimes \Gamma_G \otimes \det(V_G) \longrightarrow Q \longrightarrow 0. \quad (27)$$

There is a natural diagram on $\mathbb{P}\text{Sym}^2(V_G^\vee)$

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & \mathcal{O}_{\mathbb{P}\text{Sym}^2(V_G^\vee)} & & \\ & & & & \downarrow XY & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}\text{Sym}^2(V_G^\vee)}(-1) \otimes \Gamma_G \otimes \det(V_G) & \longrightarrow & \pi^*\text{Sym}^2(V_G^\vee) \otimes \Gamma_G \otimes \det(V_G) & & \\ & & & \searrow \psi & \downarrow & & \\ & & & & \pi^*Q & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

and $\{XY\} \subset \mathbb{P}\text{Sym}^2(V^\vee)$ is precisely the locus where

$$\psi \in H^0(\mathbb{P}\text{Sym}^2(V_G^\vee), \pi^*Q \otimes \mathcal{O}_{\mathbb{P}\text{Sym}^2(V_G^\vee)}(1) \otimes \Gamma_G^\vee \otimes \det(V_G^\vee))$$

vanishes. Since the codimension of $\{XY\} \subseteq \mathbb{P}\text{Sym}^2(V_G^\vee)$ is 2, the expected one, we have by [Ful84, Theorem 14.4] that

$$[\{XY\}] = c_2(\pi^*Q \otimes \mathcal{O}_{\mathbb{P}\text{Sym}^2(V_G^\vee)}(1) \otimes \Gamma_G^\vee \otimes \det(V_G^\vee)).$$

If r_1, r_2 are the Chern roots of V_G^\vee (so $r_1 + r_2 = -\beta_1$ and $r_1 r_2 = \beta_2$) then, using the relations in $\text{CH}^*(BG)$, we see that the total Chern class of $\text{Sym}^2(V_G^\vee) \otimes \Gamma \otimes \det(V_G)$ is

$$\begin{aligned}
c(\text{Sym}^2(V_G^\vee) \otimes \Gamma \otimes \det(V_G)) &= (1 + 2r_1 + \gamma + \beta_1)(1 + r_1 + r_2 + \gamma + \beta_1)(1 + 2r_2 + \gamma + \beta_1) \\
&= (1 + \gamma)(1 + \gamma + r_1 - r_2)(1 + \gamma + r_2 - r_1) \\
&= (1 + \gamma)((1 + \gamma)^2 - (r_1 - r_2)^2) \\
&= (1 + \gamma)(1 + 2\gamma + \gamma^2 - \beta_1^2 + 4\beta_2) \\
&= (1 + \gamma)(1 - \beta_1(\gamma + \beta_1) + 4\beta_2) \\
&= 1 + \gamma - \beta_1(\gamma + \beta_1) + 4\beta_2,
\end{aligned}$$

which is the same as the total Chern class of Q by the sequence in (27). Therefore,

$$\begin{aligned}
[\{XY\}] &= c_2(\pi^*Q \otimes \mathcal{O}_{\mathbb{P}\text{Sym}^2(V_G^\vee)}(1) \otimes \Gamma_G^\vee \otimes \det(V_G^\vee)) \\
&= c_2(\pi^*Q) + c_1(\pi^*Q)(c_1(\mathcal{O}_{\mathbb{P}\text{Sym}^2(V_G^\vee)}(1)) - \gamma - \beta_1) + (c_1(\mathcal{O}_{\mathbb{P}\text{Sym}^2(V_G^\vee)}(1)) - \gamma - \beta_1)^2 \\
&= 4\beta_2 - \beta_1(\gamma + \beta_1) + \gamma s_2^1 - \gamma(\gamma + \beta_1) + s_2^2 + \beta_1 s_2^1 - 2\beta_2 - 2(\gamma + \beta_1)s_2^1 + (\gamma + \beta_1)^2 \\
&= s_2^2 + (\gamma - \beta_1)s_2^1 + 2\beta_2,
\end{aligned}$$

where we have used the identity (23) to write $c_1(\mathcal{O}_{\mathbb{P}\text{Sym}^2(V_G^\vee)}(1))^2$ in terms of the s_2^i . \square

Lemma 35. *The following relations hold in $\text{CH}_0(\text{Sym}^k(V_G^\vee))$ for various k :*

- $[\{X^2, Y^2\}] = 2s_2^2 - 2\beta_1 s_2^1 + 2(\beta_1^2 - 2\beta_2)s_2^0.$
- $[\{X^2Y, XY^2\}] = 2s_3^3 - (3\beta_1 + \gamma)s_3^2 + 2(2\beta_2 + \beta_1^2)s_3^1 - 6\beta_2\beta_1 s_3^0.$
- $[\{X^3, Y^3\}] = 2s_3^3 + (\gamma - 3\beta_1)s_3^2 + 2(3\beta_1^2 - 6\beta_2)s_3^1 + 6\beta_1(3\beta_2 - \beta_1^2)s_3^0.$
- $[\{X^2Y^2\}] = s_4^4 - 2\beta_1 s_4^3 + 2(\beta_1^2 + 2\beta_2)s_4^2 - 12\beta_2\beta_1 s_4^1 + 24\beta_2^2 s_4^0.$
- $[\{X^3Y, XY^3\}] = 2s_4^4 - 4\beta_1 s_4^3 + 6\beta_1^2 s_4^2 - 6\beta_1^3 s_4^1 + 24\beta_2(\beta_1^2 - 2\beta_2).$
- $[\{X^4, Y^4\}] = 2s_4^4 - 4\beta_1 s_4^3 + 12(\beta_1^2 - 2\beta_2)s_4^2 + 24\beta_1(\beta_2 - \beta_1^2)s_4^1 + 24(\beta_1^4 + 2\beta_2^2 - 4\beta_2\beta_1^2)s_4^0.$

Proof. In this proof we will repeatedly apply formula (26) to compute the pushforwards along mult . In $\text{CH}_0(\mathbb{P}\text{Sym}^2(V_G^\vee))$ we have

$$\begin{aligned}
[\{X^2, Y^2\}] &= \text{mult}_*([\{X, Y\}], [\{X, Y\}]) - 2[\{XY\}] \\
&= \text{mult}_*(2s_1^1 - (\beta_1 + \gamma)s_1^0, 2s_1^1 - (\beta_1 + \gamma)s_1^0) - 2(s_2^2 + (\gamma - \beta_1)s_2^1 + 2\beta_2 s_2^0) \\
&= 2s_2^2 + (-4(\beta_1 + \gamma) - 2(\gamma - \beta_1))s_2^1 + \left(\binom{2}{1}(\beta_1 + \gamma)^2 - 4\beta_2\right)s_2^0 \\
&= 2s_2^2 - 2\beta_1 s_2^1 + 2(\beta_1^2 - 2\beta_2)s_2^0.
\end{aligned}$$

In $\text{CH}_0(\mathbb{P}\text{Sym}^3(V_G^\vee))$ we have

$$\begin{aligned}
[\{X^2Y, XY^2\}] &= \text{mult}_*([\{X, Y\}], [\{XY\}]) \\
&= \text{mult}_*(2s_1^1 - (\beta_1 + \gamma)s_1^0, s_2^2 + (\gamma - \beta_1)s_2^1 + 2\beta_2 s_2^0) \\
&= s_3^3 + (2(\gamma - \beta_1) - (\beta_1 + \gamma))s_3^2 + \left(4\beta_2 - \binom{2}{1}(\beta_1 + \gamma)(\gamma - \beta_1)\right)s_3^1 - 2\binom{3}{1}(\beta_1 + \gamma)\beta_2 s_3^0 \\
&= 2s_3^3 - (3\beta_1 + \gamma)s_3^2 + 2(2\beta_2 + \beta_1^2)s_3^1 - 6\beta_2\beta_1 s_3^0
\end{aligned}$$

and

$$\begin{aligned}
[\{X^3, Y^3\}] &= \text{mult}_*([\{X, Y\}], [\{X^2, Y^2\}]) - [\{X^2Y, XY^2\}] \\
&= \text{mult}_*(2s_1^1 - (\beta_1 + \gamma)s_1^0, 2s_2^2 - 2\beta_1s_2^1 + 2(\beta_1^2 - 2\beta_2)s_2^0) \\
&\quad - (2s_3^3 - (3\beta_1 + \gamma)s_3^2 + 2(2\beta_2 + \beta_1^2)s_3^1 - 6\beta_2\beta_1s_3^0) \\
&= 4s_3^3 - (2(\beta_1 + \gamma) + 4\beta_1)s_3^2 + \left(2\beta_1 \binom{2}{1}(\beta_1 + \gamma) + 4(\beta_1^2 - 2\beta_2)\right) s_3^1 \\
&\quad - \binom{3}{1}2(\beta_1 + \gamma)(\beta_1^2 - 2\beta_2)s_3^0 - 2s_3^3 + (3\beta_1 + \gamma)s_3^2 - 2(2\beta_2 + \beta_1^2)s_3^1 + 6\beta_2\beta_1s_3^0 \\
&= 2s_3^3 + (\gamma - 3\beta_1)s_3^2 + 6(\beta_1^2 - 2\beta_2)s_3^1 + 6\beta_1(3\beta_2 - \beta_1^2)s_3^0.
\end{aligned}$$

In $\text{CH}_0(\mathbb{P}\text{Sym}^4(V_G^\vee))$ we have

$$\begin{aligned}
[\{X^2Y^2\}] &= \text{mult}_*([\{XY\}], \{XY\}) \\
&= \text{mult}_*(s_2^2 + (\gamma - \beta_1)s_2^1 + 2\beta_2s_2^0, s_2^2 + (\gamma - \beta_1)s_2^1 + 2\beta_2s_2^0) \\
&= s_4^4 + 2(\gamma - \beta_1)s_4^3 + \left(\binom{2}{1}(\gamma - \beta_1)^2 + 4\beta_2\right) s_4^2 + 4\binom{3}{1}\beta_2(\gamma - \beta_1)s_4^1 + 4\binom{4}{2}\beta_2^2s_4^0 \\
&= s_4^4 - 2\beta_1s_4^3 + 2(\beta_1^2 + 2\beta_2)s_4^2 - 12\beta_2\beta_1s_4^1 + 24\beta_2^2s_4^0
\end{aligned}$$

and

$$\begin{aligned}
[\{X^3Y, XY^3\}] &= \text{mult}_*([\{XY\}], \{X^2, Y^2\}) \\
&= \text{mult}_*(s_2^2 + (\gamma - \beta_1)s_2^1 + 2\beta_2s_2^0, 2s_2^2 - 2\beta_1s_2^1 + 2(\beta_1^2 - 2\beta_2)s_2^0) \\
&= 2s_4^4 + (2(\gamma - \beta_1) - 2\beta_1)s_4^3 + \left(4\beta_2 + 2(\beta_1^2 - 2\beta_2) - \binom{2}{1}2\beta_1(\gamma - \beta_1)\right) s_4^2 \\
&\quad + \binom{3}{1}(-4\beta_2\beta_1 + 2(\gamma - \beta_1)(\beta_1^2 - 2\beta_2))s_4^1 + \binom{4}{2}4(\beta_1^2 - 2\beta_2)\beta_2s_4^0 \\
&= 2s_4^4 - 4\beta_1s_4^3 + 6\beta_1^2s_4^2 - 6\beta_1^3s_4^1 + 24\beta_2(\beta_1^2 - 2\beta_2)s_4^0,
\end{aligned}$$

and

$$\begin{aligned}
[\{X^4, Y^4\}] &= \text{mult}_*([\{X^2, Y^2\}], \{X^2, Y^2\}) - 2[\{X^2Y^2\}] \\
&= \text{mult}_*(2s_2^2 - 2\beta_1s_2^1 + 2(\beta_1^2 - 2\beta_2)s_2^0, 2s_2^2 - 2\beta_1s_2^1 + 2(\beta_1^2 - 2\beta_2)s_2^0) \\
&\quad - 2(s_4^4 - 2\beta_1s_4^3 + 2(\beta_1^2 + 2\beta_2)s_4^2 - 12\beta_2\beta_1s_4^1 + 24\beta_2^2s_4^0) \\
&= 4s_4^4 - 8\beta_1s_4^3 + \left(\binom{2}{1}4\beta_1^2 + 8(\beta_1^2 - 2\beta_2)\right) s_4^2 - 8\binom{3}{1}\beta_1(\beta_1^2 - 2\beta_2)s_4^1 + 4\binom{4}{2}(\beta_1^2 - 2\beta_2)^2s_4^0 \\
&\quad - 2s_4^4 + 4\beta_1s_4^3 - 4(\beta_1^2 + 2\beta_2)s_4^2 + 24\beta_2\beta_1s_4^1 - 48\beta_2^2s_4^0 \\
&= 2s_4^4 - 4\beta_1s_4^3 + 12(\beta_1^2 - 2\beta_2)s_4^2 + 24\beta_1(\beta_2 - \beta_1^2)s_4^1 + 24(\beta_1^4 + 2\beta_2^2 - 4\beta_2\beta_1^2)s_4^0.
\end{aligned}$$

□

3.4 Proof of Theorem 4

Recall that, by the projective bundle formula,

$$\mathrm{CH}^*(\mathbb{P}\mathrm{Sym}^4(V_G^\vee)) = \frac{\mathrm{CH}^*(BG)[h]}{(P(h))} = \frac{\mathbb{Z}[\beta_1, \beta_2, \gamma, h]}{(2\gamma, \gamma(\gamma + \beta_1), P(h))},$$

where

$$P(h) = h^5 + h^4 c_1(\mathrm{Sym}^4(V_G^\vee)) + \dots + c_5(\mathrm{Sym}^4(V_G^\vee)).$$

Therefore, by Lemma 27, and Proposition 31

$$\mathrm{CH}^*(\mathrm{Sym}^4(V_G^\vee) \otimes \det(V_G) \otimes \Gamma \setminus \Delta) = \frac{Z[\beta_1, \beta_2, \gamma]}{(2\gamma, \gamma(\gamma + \beta_1), P(h), \mathrm{im}(p'_*))}, \quad (28)$$

where

$$p' : \underline{\Delta}' \longrightarrow \mathbb{P}(\mathrm{Sym}^4(V_G^\vee))$$

is the composition of p and the inclusion $\underline{\Delta} \subseteq \mathbb{P}\mathrm{Sym}^4(V_G^\vee)$ and we are substituting $h = \beta_1 + \gamma$ in P and in $\mathrm{im}(p'_*)$. Let $I \subset \mathbb{Z}[\beta_1, \beta_2, \gamma]$ be the ideal in the denominator of (28). We will show that

$$I = (2\gamma, 2\beta_1, 8\beta_2, \gamma^2 + \beta_1\gamma, \beta_1^2 + \beta_1\gamma) \quad (29)$$

and this will prove Theorem 4 (up to identification of the classes β_i and γ done in §4).

Lemma 36. $\{2\beta_1, \beta_1^2 + \beta_1\gamma, 8\beta_2\}$ are in I .

Proof. From Lemmas 34 and 35, we have

$$\begin{aligned} (p_{1,2})_*(s_3^0 \otimes [\{X, Y\}]) &= 2\mathrm{mult}(s_3^0, s_1^1) - (\beta_1 + \gamma)\mathrm{mult}(s_3^0, s_1^0) = 2s_4^1 - 4\beta_1 \\ (p_{1,2})_*(s_3^1 \otimes [\{X, Y\}]) &= 2\mathrm{mult}(s_3^1, s_1^1) - (\beta_1 + \gamma)\mathrm{mult}(s_3^1, s_1^0) = 2s_4^2 - 3(\beta_1 + \gamma)s_4^1 \\ (p_{1,1})_*(s_1^1 \otimes s_2^0) &= \mathrm{mult}(\mathrm{sq}(s_1^1), s_2^0) = \mathrm{mult}(s_2^2, s_2^0) - 2\beta_2\mathrm{mult}(s_2^0, s_2^0) \\ &= s_4^2 - 12\beta_2. \end{aligned}$$

Which, after using (23), the substitution $h = \beta_1 + \gamma$, and the relations coming from $\mathrm{CH}^*(BG)$, yield $-2\beta_1, 8\beta_2 - 3\beta_1(\beta_1 + \gamma)$ and $-8\beta_2$, respectively, and linear combinations of these give the desired elements. \square

From now on, and until the rest of the section, we work modulo the relations

$$2\beta_1 = 2\gamma = 8\beta_2 = 0, \beta_1^2 = \beta_1\gamma = \gamma^2, \quad (30)$$

and without mentioning that we are substituting $h = \beta_1 + \gamma$. Note that $h^2 = 0$ modulo the relations (30).

Also, note that, by (23), (24) and (25)

$$\begin{aligned} s_4^2 &= h^2 - \beta_1 h + 4\beta_2 = 4\beta_2, \\ s_4^3 &= h^3 - 3\beta_1 s_4^2 - (\beta_1^2 - 10\beta_2)s_4^1 + 4\beta_1\beta_2 = 0, \\ s_4^4 &= h^4 - 6\beta_1 s_4^3 - (7\beta_1^2 - 16\beta_2)s_4^2 + (-\beta_1^3 + 32\beta_1\beta_2)s_4^1 + 4\beta_1^2\beta_2 - 40\beta_2^2 = 0. \end{aligned}$$

An immediate but useful consequence is the following.

Remark 37. We have $\alpha \cdot \text{mult}(s_a^{a'}, s_{4-a}^{b'}) = 0$ whenever $(\beta_1 + \gamma) \mid \alpha$, or $a' + b' \geq 1$ and $2 \mid \alpha$, or $a' + b' \geq 3$.

Lemma 38. Modulo the relations (30), we have

$$\text{im}(p'_*) = 0.$$

Proof. We have to pushforward all the s_r^i classes along maps $p_{i,j}$. Note that $p_{2,2}$ and $p_{3,2}$ factor through $p_{1,1}$, and so there is no need to consider them. Remark 37 will be used repeatedly and without further mention.

Pushforward along $p_{1,1} = \text{mult}(\text{sq}(\cdot), \cdot)$:

$$\begin{aligned} (p_{1,1})_*(s_1^0 \otimes s_2^0) &= 2\text{mult}(s_2^1, s_2^0) - 2\beta_1 \text{mult}(s_2^0, s_2^0) = 0 \\ (p_{1,1})_*(s_1^0 \otimes s_2^1) &= 2\text{mult}(s_2^1, s_2^1) - 2\beta_1 \text{mult}(s_2^0, s_2^1) = 0 \\ (p_{1,1})_*(s_1^0 \otimes s_2^2) &= 2\text{mult}(s_2^1, s_2^2) - 2\beta_1 \text{mult}(s_2^0, s_2^2) = 0 \\ (p_{1,1})_*(s_1^1 \otimes s_2^0) &= \text{mult}(s_2^2, s_2^0) - 2\beta_2 \text{mult}(s_2^0, s_2^0) = s_4^2 - 12\beta_2 = 0 \\ (p_{1,1})_*(s_1^1 \otimes s_2^1) &= \text{mult}(s_2^2, s_2^1) - 2\beta_2 \text{mult}(s_2^0, s_2^1) = 0 \\ (p_{1,1})_*(s_1^1 \otimes s_2^2) &= \text{mult}(s_2^2, s_2^2) - 2\beta_2 \text{mult}(s_2^0, s_2^2) = 0 \end{aligned}$$

Pushforward along $p_{1,2} = \text{mult}(\cdot, \{X, Y\})$:

$$\begin{aligned} (p_{1,2})_*(s_3^0) &= 2\text{mult}(s_3^0, s_1^1) - (\beta_1 + \gamma) \text{mult}(s_3^0, s_1^0) = 0 \\ (p_{1,2})_*(s_3^1) &= 2\text{mult}(s_3^1, s_1^1) - (\beta_1 + \gamma) \text{mult}(s_3^1, s_1^0) = 0 \\ (p_{1,2})_*(s_3^2) &= 2\text{mult}(s_3^2, s_1^1) - (\beta_1 + \gamma) \text{mult}(s_3^2, s_1^0) = 0 \\ (p_{1,2})_*(s_3^3) &= 2\text{mult}(s_3^3, s_1^1) - (\beta_1 + \gamma) \text{mult}(s_3^3, s_1^0) = 0 \end{aligned}$$

Pushforward along $p_{2,1} = \text{sq}(\cdot)$:

$$\begin{aligned} (p_{2,1})_*(s_2^0) &= 4s_4^2 - 12\beta_1 s_4^1 + 12\beta_1^2 = 0 \\ (p_{2,1})_*(s_2^1) &= 2s_4^3 - 2\beta_1 s_4^2 - 12\beta_2 s_4^1 + 24\beta_1 \beta_2 = 0 \\ (p_{2,1})_*(s_2^2) &= s_4^4 - 4\beta_2 s_4^2 + 24\beta_2^2 = 0, \end{aligned}$$

see Lemma 32.

Pushforward along $p_{2,3} = \text{mult}(\cdot, \{X^2, Y^2, XY\})$

$$\begin{aligned} (p_{2,3})_*(s_2^0 \otimes [\{X^2, Y^2\}]) &= 2\text{mult}(s_2^0, s_2^2) - 2\beta_1 \text{mult}(s_2^0, s_2^1) + 2(\beta_1^2 - 2\beta_2) \text{mult}(s_2^0, s_2^0) = 0 \\ (p_{2,3})_*(s_2^1 \otimes [\{X^2, Y^2\}]) &= 2\text{mult}(s_2^1, s_2^2) - 2\beta_1 \text{mult}(s_2^1, s_2^1) + 2(\beta_1^2 - 2\beta_2) \text{mult}(s_2^1, s_2^0) = 0 \\ (p_{2,3})_*(s_2^2 \otimes [\{X^2, Y^2\}]) &= 2\text{mult}(s_2^2, s_2^2) - 2\beta_1 \text{mult}(s_2^2, s_2^1) + 2(\beta_1^2 - 2\beta_2) \text{mult}(s_2^2, s_2^0) = 0 \\ (p_{2,3})_*(s_2^0 \otimes [\{XY\}]) &= \text{mult}(s_2^0, s_2^2) + (\gamma - \beta_1) \text{mult}(s_2^0, s_2^1) + 2\beta_2 \text{mult}(s_2^0, s_2^0) \\ &= s_4^2 + 12\beta_2 = 0 \\ (p_{2,3})_*(s_2^1 \otimes [\{XY\}]) &= \text{mult}(s_2^1, s_2^2) + (\gamma - \beta_1) \text{mult}(s_2^1, s_2^1) + 2\beta_2 \text{mult}(s_2^1, s_2^0) = 0 \\ (p_{2,3})_*(s_2^2 \otimes [\{XY\}]) &= \text{mult}(s_2^2, s_2^2) + (\gamma - \beta_1) \text{mult}(s_2^2, s_2^1) + 2\beta_2 \text{mult}(s_2^2, s_2^0) = 0 \end{aligned}$$

Pushforward along $p_{3,1} = \text{mult}(\cdot, \{X^3, Y^3, X^2Y, XY^2\})$:

$$\begin{aligned}
(p_{3,1})_*(s_1^0, [\{X^3, Y^3\}]) &= 2\text{mult}(s_1^0, s_3^3) + (\gamma - 3\beta_1)\text{mult}(s_1^0, s_3^2) \\
&\quad + 2(3\beta_1^2 - 6\beta_2)\text{mult}(s_1^0, s_3^1) + 6\beta_1(3\beta_2 - \beta_1^2)\text{mult}(s_1^0, s_3^0) = 0 \\
(p_{3,1})_*(s_1^1, [\{X^3, Y^3\}]) &= 2\text{mult}(s_1^1, s_3^3) + (\gamma - 3\beta_1)\text{mult}(s_1^1, s_3^2) \\
&\quad + 2(3\beta_1^2 - 6\beta_2)\text{mult}(s_1^1, s_3^1) + 6\beta_1(3\beta_2 - \beta_1^2)\text{mult}(s_1^1, s_3^0) = 0 \\
(p_{3,1})_*(s_1^0, [\{X^2Y, XY^2\}]) &= 2\text{mult}(s_1^0, s_3^3) - (3\beta_1 + \gamma)\text{mult}(s_1^0, s_3^2) \\
&\quad + 2(2\beta_2 + \beta_1^2)\text{mult}(s_1^0, s_3^1) - 6\beta_2\beta_1\text{mult}(s_1^0, s_3^0) = 0 \\
(p_{3,1})_*(s_1^1, [\{X^2Y, XY^2\}]) &= 2\text{mult}(s_1^1, s_3^3) - (3\beta_1 + \gamma)\text{mult}(s_1^1, s_3^2) \\
&\quad + 2(2\beta_2 + \beta_1^2)\text{mult}(s_1^1, s_3^1) - 6\beta_2\beta_1\text{mult}(s_1^1, s_3^0) = 0
\end{aligned}$$

Pushforward along p_4 : This is just the fundamental classes of $\{X^4, Y^4\}$, $\{X^3Y, XY^3\}$ and $\{X^2Y^2\}$, which we calculated in Lemma 35, and are easily seen to be 0. \square

Lemma 39. *Modulo the relations (30), we have*

$$P(h) = 0.$$

Proof. If r_1, r_2 are the Chern roots of V_G ($r_1 + r_2 = \beta_1$, $r_1r_2 = \beta_2$) then

$$c_5(\text{Sym}^4(V_G^\vee)) = -\prod_{i=0}^4(ir_1 + (4-i)r_2) \text{ and } c_4(\text{Sym}^4(V_G^\vee)) = \sum_{j=0}^4 \prod_{i \neq j} (ir_1 + (4-i)r_2).$$

From this it follows that $2\beta_1 \mid c_5(\text{Sym}^4(V_G^\vee))$ and $2h \mid hc_4(\text{Sym}^4(V_G^\vee))$. Also, $h^2 = 0$, so $P(h) = 0$. \square

4 Interpretation of the generators

In this section we will give a geometric interpretation of the generators β_1, β_2 and γ .

We start with identifying the β 's with the Chern classes of the Hodge bundle. This is a natural vector bundle \mathbb{E} of rank 2 on \mathcal{R}_2 : it is pulled back from \mathcal{M}_2 and if $\pi : C \rightarrow S$ is a family of curves of genus 2 corresponding to a morphism $S \rightarrow \mathcal{M}_2$, and ω_π is the relative dualizing sheaf of π , then the pullback of \mathbb{E} to S is $\pi_*(\omega_\pi)$. The Chern classes $\lambda_i = c_i(\mathbb{E})$ are among the tautological classes introduced by Mumford.

Proposition 40. *The pullback of V_G to \mathcal{X} yields the dual of the Hodge bundle, i.e.*

$$\alpha^*\mathbb{E} = V_G^\vee$$

on \mathcal{X} .

Remark 41. *In the proof of the above proposition we will refer to [Vis98, Proposition 3.1]. However, with the notation of that paper, there is an error in the description of the GL_2 -action on the space X . It is incorrectly stated that this action, induced by the*

isomorphism of X with the GL_2 -equivariant variety Y , identifies X as an open subset of $\mathrm{Sym}^6(V^\vee) \otimes \det(V)^{\otimes 2}$. Instead, X is identified with the open subset of $\mathrm{Sym}^6(V) \otimes \det(V^\vee)^{\otimes 2}$ consisting of polynomials with no repeated roots.

Base changing over the morphism

$$B\mathrm{GL}_2 \rightarrow B\mathrm{GL}_2$$

given by $A \mapsto (A^{-1})^t$, we get an isomorphism

$$\mathcal{M}_2 \cong [X'/\mathrm{GL}_2], \quad (31)$$

where $X' \subset \mathrm{Sym}^6(V^\vee) \otimes \det(V)^{\otimes 2}$ is the locus with no repeated roots, and under this new isomorphism, the Hodge bundle is the pullback of $V_{\mathrm{GL}_2}^\vee$ from $B\mathrm{GL}_2$.

Proof of Proposition 40. The morphism

$$\begin{aligned} \chi : \mathrm{Sym}^4(V^\vee) \otimes \det(V) \otimes \Gamma &\longrightarrow \mathrm{Sym}^6(V^\vee) \otimes \det(V)^{\otimes 2} \\ F &\mapsto XYF \end{aligned}$$

is equivariant with respect to the inclusion $\iota : G \rightarrow \mathrm{GL}_2$, and so we have a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\tilde{\chi}} & [X'/\mathrm{GL}_2] \\ \downarrow \alpha & & \downarrow (31) \\ \mathcal{R}_2 & \longrightarrow & \mathcal{M}_2, \end{array}$$

where the bottom arrow is the forgetful morphism. Therefore, the proposition follows because the pullback of the Hodge bundle under (31) is $V_{\mathrm{GL}_2}^\vee$, and $\iota^*V_{\mathrm{GL}_2} = V_G$. \square

We obtain the following immediate corollary:

Corollary 42. *We have*

$$\beta_i = (-1)^i \lambda_i$$

for $i = 1, 2$.

Finally we identify the class $\gamma \in \mathrm{CH}^*(\mathcal{R}_2)$. There is a natural double cover of \mathcal{R}_2 which we now describe. Denote by U the affine scheme $\mathrm{Sym}^4(V^\vee) \otimes \det(V) \otimes \Gamma \setminus \Delta$. Over U we have the following commutative diagram of G equivariant maps

$$\begin{array}{ccc} D = \sigma_0(U) \sqcup \sigma_\infty(U) & \hookrightarrow & C = \underline{\mathrm{Spec}}_{\mathbb{P}_U^1}(\mathcal{O}_{\mathbb{P}_U^1} \oplus \mathcal{O}_{\mathbb{P}_U^1}(-3)) \xrightarrow{q} \mathbb{P}_U^1 = U \times \mathbb{P}(V) \\ & \searrow \delta & \downarrow \pi \\ & & U \xleftarrow{\rho} \end{array}$$

The action of G on C is given by Proposition 16 and we see that D is G -equivariant. Then δ descends to

$$\left[\frac{D}{G} \right] \rightarrow \mathcal{R}_2,$$

and we have the following lemma.

Lemma 43. *We have $c_1^G(\delta_*\mathcal{O}_D) = \gamma$.*

Proof. The pushforward of the structure sheaf \mathcal{O}_D under δ is clearly $\mathcal{O}_U \otimes A$ where A is the two dimensional representation of G arising from the representation of $\mathbb{Z}/(2)$ where -1 switches the two entries of the vector. Clearly, $\det(A) = \Gamma$ and the lemma follows. \square

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