DUALS OF LIMITING INTERPOLATION SPACES

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ABSTRACT. The aim of the paper is to establish duals of the limiting real interpolation K- and J-spaces $(X_0, X_1)_{0,q,v;K}$ and $(X_0, X_1)_{0,q,v;J}$, where (X_0, X_1) is a compatible couple of Banach spaces, $1 \le q < \infty$, v is a slowly varying function on the interval $(0, \infty)$, and the symbols K and J stand for the Peetre K- and J-functionals. In the case of the classical real interpolation method $(X_0, X_1)_{\theta,q}$, where $\theta \in (0, 1)$ and $1 \le q < \infty$, this problem was solved by Lions and Peetre.

1. Introduction

Dual spaces play a very important role in numerous parts of mathematics. The same can be said about spaces which appear in the real interpolation (cf., e.g., monographs [9], [4], [60], [47], [3], [8], [50]).

In this paper our goal is to describe duals of the limiting real interpolation K- and J-spaces $(X_0, X_1)_{0,q,v;K}$ and $(X_0, X_1)_{0,q,v;J}$, where (X_0, X_1) is a compatible couple of Banach spaces, $1 \le q < \infty$, v is a slowly varying function on the interval $(0, \infty)$, and the symbols K and J stand for the Peetre K- and J-functionals. Note that we express these duals both in terms of K-spaces and J-spaces.

In the case $\theta \in (0, 1)$, $1 \le q < \infty$, and when the slowly varying function v satisfies $v(t) = 1$ for all $t \in (0, \infty)$, we obtain the classical real interpolation spaces $(X_0, X_1)_{\theta,q}$, which are independent on the fact whether the K- or J-functional is used to define them (see, e.g., [3, Chapter 5, Theorem 2.8 (Equivalence theorem)]). Duals of these spaces have been described by Lions and Peetre in their fundamental paper [49] (see also [48]). It has been proved there that $(X_0, X_1)'_{\theta,q} = (X'_0, X'_1)_{\theta,q'}$ (with equivalent norms), where $1/q + 1/q' = 1$.

However, some problems in mathematical analysis have motivated the investigation of the real interpolation with the limiting values $\theta = 0$ or $\theta = 1$. Nowadays, there is a lot of works, where the limiting real interpolation is studied or applied (see, e.g., [45], [53], [28], [36], [37], [44], [20], [22], [1], [38], [35], [39], [13], [24], [21], [40], [15], [17], [29], [41], [18], [2], [5], [55], [56], [32], [19], [57], and the references given there).

In order that the spaces $(X_0, X_1)_{\theta,q}$ are meaningful also with limiting values $\theta = 0$ or $\theta = 1$ in a general case, one has to extend the given interpolation functor by a convenient weight v. This weight v usually belongs to the class $SV(0, \infty)$ of slowly varying functions on the interval $(0, \infty)$. If $\theta \in (0, 1)$, then the corresponding space $(X_0, X_1)_{\theta,q,v}$ is a particular case of an interpolation space with a function parameter and, by [46, Theorem 2.2], this space is again independent on the fact whether the K- or J-functional is used to define it. However, if $\theta = 0$ or $\theta = 1$, then the corresponding space depends on the fact whether the

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K- or J- functional is used; accordingly, the resulting space is denoted by $(X_0, X_1)_{\theta,q,v;K}$ or by $(X_0, X_1)_{\theta,q,v;J}$.

Now a natural question arises: Given $\theta \in \{0, 1\}, q \in [1, \infty]$ and $v \in SV(0, \infty)$, can we describe the space $(X_0, X_1)_{\theta,q,v;K}$ as a $(X_0, X_1)_{\theta,q,w;J}$ space with a convenient $w \in SV(0, \infty)$?

If the weight function v is of logarithmic form, then the answer is given in [23] for the case that a pair (X_0, X_1) of Banach spaces X_0 and X_1 is ordered, and in [24] and [5] for a general pair (X_0, X_1) of Banach spaces X_0 and X_1. In [56] we have establish conditions under which the limiting K-space $(X_0, X_1)_{0,q,b;K}$, involving a slowly varying function b, can be described by means of the J-space $(X_0, X_1)_{0,q,a;J}$, with a convenient slowly varying function a, and we have also solved the reverse problem. It has been shown that if these conditions are not satisfied that the given problem may not have a solution. Moreover, in [57] it is assumed that these conditions are not fulfilled and then it is proved that

$$(X_0, X_1)_{0,q,b;K} = (X_0, X_0 + X_1)_{0,q,A;J} = X_0 + (X_0, X_1)_{0,q,A;J}$$

and

$$(X_0, X_1)_{0,q,a;J} = (X_0, X_0 \cap X_1)_{0,q,B;K} = X_0 \cap (X_0, X_1)_{0,q,B;K},$$

where A and B are convenient weights. In [57] also equivalent norms in the mentioned spaces have been determined.

These resuls play important role in our calculation of duals of spaces $(X_0, X_1)_{\theta,q,v;K}$ and $(X_0, X_1)_{\theta,q,v;J}$ with $\theta \in \{0, 1\}$.

Note also that it is sufficient to describe duals of these spaces only if $\theta = 0$ since the answer for $\theta = 1$ follows from the solution with $\theta = 0$ and from the equality $(X_0, X_1)_{0,q,v;K} = (X_1, X_0)_{1,q,u;K}$, or $(X_0, X_1)_{0,q,v;J} = (X_1, X_0)_{1,q,u;J}$, where $u(t) = v(1/t)$ for all $t > 0$ (which is a consequence of the fact that $K(f, t; X_0, X_1) = tK(f, t^{-1}; X_1, X_0)$ if $f \in X_0 + X_1$ and $t > 0$, or $J(f, t; X_0, X_1) = tJ(f, t^{-1}; X_1, X_0)$ if $f \in X_0 \cap X_1$ and $t > 0$, and a change of variables).

If $\theta \in (0, 1)$, $q \in [1, \infty]$, and $v \in SV(0, \infty)$, then, as was already mentioned above, the spaces $(X_0, X_1)_{\theta,q,v;K}$, $(X_0, X_1)_{\theta,q,v;J}$ coincides and they are particular cases of interpolation spaces with a function parameter. Consequently, if $q \in [1, \infty)$, then duals of these spaces are described in [59, Theorem 2.4]. Note that it follows from this theorem that if $\theta \in (0, 1)$, $q \in [1, \infty)$, and $v \in SV(0, \infty)$, then $(X_0, X_1)'_{\theta,q,v} = (X'_0, X'_1)_{\theta,q',\tilde{v}}$ (with equivalent norms), where $1/q + 1/q' = 1$ and $\tilde{v}(t) := \frac{1}{v(1/t)}$ for all $t > 0$.

If $\theta \in \{0, 1\}$, $q \in [1, \infty)$, and the weight v is of logarithmic form, then the duals of the spaces $(X_0, X_1)_{\theta,q,v;K}$ have been determined in [24].

The paper is organized as follows: Section 2 contains notation, definitions and preliminaries. In Section 3 we present our main results. In Section 4 we collect some auxiliary assertions. Sections 5–11 are devoted to proofs of the main results. In Section 12 we mention another method which can be used to prove some of our main assertions.

2. Notation, definitions and preliminaries

For two non-negative expressions (i.e. functions or functionals) $\mathcal{A}$, $\mathcal{B}$, the symbol $\mathcal{A} \lesssim \mathcal{B}$ (or $\mathcal{A} \gtrsim \mathcal{B}$) means that $\mathcal{A} \leq c\,\mathcal{B}$ (or $c\,\mathcal{A} \geq \mathcal{B}$), where c is a positive constant independent of significant quantities involved in $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{A} \gtrsim \mathcal{B}$, we write $\mathcal{A} \approx \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are equivalent. Throughout the paper we use the abbreviation LHS($*$) (RHS($*$)) for the left- (right-) hand side of relation ($*$). We adopt the convention that $a/(\infty) = 0$, $a/0 = \infty$ and $(\infty)^a = \infty$ for all $a \in (0, \infty)$. If $q \in [1, \infty]$, the *conjugate number*

q' is given by $1/q + 1/q' = 1$. In the whole paper $\|.\|_{q;(c,d)}$, $q \in [1,\infty]$, denotes the usual L_q-norm on the interval $(c,d) \subseteq \mathbb{R}$.

The symbol $\mathcal{M}^+(0,\infty)$ stands for the class of all (Lebesgue-) measurable functions on the interval $(0,\infty)$, which are non-negative almost everywhere on $(0,\infty)$. We will admit only positive, finite weights, and thus we put

$$\mathcal{W}(0,\infty) := \{w \in \mathcal{M}^+(0,\infty) : w < \infty \text{ a.e. on } (0,\infty)\}.$$

The symbol $AC(0,\infty)$ is used to denote the set of all absolutely continuous functions on the interval $(0,\infty)$.

If f is a monotone function on the interval $(0,\infty)$, then we put

$$f(0) := \lim_{t\to 0_+} f(t) \qquad \textit{and} \qquad f(\infty) := \lim_{t\to\infty} f(t).$$

We say that a positive, finite and Lebesgue-measurable function b is *slowly varying* on $(0,\infty)$, and write $b \in SV(0,\infty)$, if, for each $\varepsilon > 0$, $t^\varepsilon b(t)$ is equivalent to a non-decreasing function on $(0,\infty)$ and $t^{-\varepsilon} b(t)$ is equivalent to a non-increasing function on $(0,\infty)$. Here we follow the definition of $SV(0,\infty)$ given in [44]; for other definitions see, for example, [7], [33], [34], and [54]. The family $SV(0,\infty)$ includes not only powers of iterated logarithms and the broken logarithmic functions of [36] but also such functions as $t \mapsto \exp\left(|\log t|^a\right)$, $a \in (0,1)$. (The last mentioned function has the interesting property that it tends to infinity more quickly than any positive power of the logarithmic function).

We mention some properties of slowly varying functions.

Lemma 2.1. ([56, Lemma 2.1]) *Let $b, b_1, b_2 \in SV(0,\infty)$ and let $d(t) := b(1/t)$ for all $t > 0$.*
(i) *Then $b_1 b_2, d \in SV(0,\infty)$ and $b^r \in SV(0,\infty)$ for each $r \in \mathbb{R}$.*
(ii) *If ε and κ are positive numbers, then there are positive constants c_ε and C_ε such that*

$$c_\varepsilon \min\{\kappa^{-\varepsilon}, \kappa^{\varepsilon}\} b(t) \le b(\kappa t) \le C_\varepsilon \max\{\kappa^{\varepsilon}, \kappa^{-\varepsilon}\} b(t) \quad \textit{for every } t > 0.$$

(iii) *If $\alpha > 0$ and $q \in (0,\infty]$, then, for all $t > 0$,*

$$\|\tau^{\alpha-1/q} b(\tau)\|_{q,(0,t)} \approx t^\alpha b(t) \quad \textit{and} \quad \|\tau^{-\alpha-1/q} b(\tau)\|_{q,(t,\infty)} \approx t^{-\alpha} b(t).$$

(iv) *If $q \in (0,\infty]$, and*

$$B_0(t) := \|\tau^{-1/q} b(\tau)\|_{q,(0,t)}, \quad B_\infty(t) := \|\tau^{-1/q} b(\tau)\|_{q,(t,\infty)}, \quad t > 0,$$

then

$$b(t) \lesssim B_0(t), \quad b(t) \lesssim B_\infty(t), \quad \textit{for all } t > 0. \tag{2.1}$$

Moreover, if $B_i(1) < \infty$, then B_i belongs to $SV(0,\infty)$, $i = 0, \infty$.
(v) *If $q \in (0,\infty)$, then*

$$\limsup_{t\to 0_+} \frac{\|\tau^{-1/q} b(\tau)\|_{q,(0,t)}}{b(t)} = \infty, \qquad \limsup_{t\to\infty} \frac{\|\tau^{-1/q} b(\tau)\|_{q,(0,t)}}{b(t)} = \infty, \tag{2.2}$$

$$\limsup_{t\to 0_+} \frac{\|\tau^{-1/q} b(\tau)\|_{q,(t,\infty)}}{b(t)} = \infty, \qquad \limsup_{t\to\infty} \frac{\|\tau^{-1/q} b(\tau)\|_{q,(t,\infty)}}{b(t)} = \infty. \tag{2.3}$$

Remark 2.2. Note that, by Lemma 2.1 (iii), the function $b \in SV(0,\infty)$ is equivalent to $\bar{b} \in SV(0,\infty) \cap AC(0,\infty)$ given by $\bar{b}(t) := t^{-1}\int_0^t b(s)\,ds$, $t > 0$. Consequently, any $b \in SV(0,\infty)$ is equivalent to a continuous function on the interval $(0,\infty)$.

More properties and examples of slowly varying functions can be found in [62, Chapter V, p. 186], [7], [52], [54], [44] and [42].

Let X and Y be two Banach spaces. We say that X *coincides* with Y (and write $X = Y$) if X and Y are equal in the algebraic and topological sense (their norms are equivalent). Moreover, we say that X and Y *are identical* (and write $X \equiv Y$) provided that X and Y are equal in the algebraic sense and $\|\cdot\|_X = \|\cdot\|_Y$. The symbol $X \hookrightarrow Y$ means that $X \subset Y$ and the natural embedding of X in Y is continuous. The norm of this embedding is denoted by $\|Id\|_{X\to Y}$. By X' we denote the space of all linear and continuous functionals on the space X.

A pair (X_0, X_1) of Banach spaces X_0 and X_1 is called a *compatible couple* if there is a Hausdorff topological vector space $\mathcal{X}$ in which each of X_0 and X_1 is continuously embedded.

If (X_0, X_1) is a compatible couple, then a Banach space X is said to be an *intermediate space* between X_0 and X_1 if $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$.

Definition 2.3. *Let (X_0, X_1) be a compatible couple.*

(i) *The Peetre K-functional is defined for each $f \in X_0 + X_1$ and $t > 0$ by*

$$K(f, t; X_0, X_1) := \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\},$$

where the infimum extends over all representations $f = f_0 + f_1$ of f with $f_0 \in X_0$ and $f_1 \in X_1$. Sometimes, we denote $K(f, t; X_0, X_1)$ simply by $K(f, t)$.

(ii) *The Peetre J-functional is defined for each $f \in X_0 \cap X_1$ and $t > 0$ by*

$$J(f, t; X_0, X_1) := \max\{\|f\|_{X_0}, t\,\|f\|_{X_1}\}.$$

Sometimes, we denote $J(f, t; X_0, X_1)$ simply by $J(f, t)$.

(iii) *For $0 \le \theta \le 1$, $1 \le q \le \infty$, and $v \in \mathcal{W}(0, \infty)$, we put*

$$(X_0, X_1)_{\theta,q,v;K} := \{f \in X_0 + X_1 : \|f\|_{\theta,q,v;K} < \infty\}, \tag{2.4}$$

where

$$\|f\|_{\theta,q,v;K} \equiv \|f\|_{(X_0,X_1)_{\theta,q,v;K}} := \left\| t^{-\theta-1/q}\, v(t)\, K(f, t; X_0, X_1) \right\|_{q,(0,\infty)}. \tag{2.5}$$

(iv) *Let $0 \le \theta \le 1$, $1 \le q \le \infty$, and let $v \in \mathcal{W}(0, \infty)$. The space $(X_0, X_1)_{\theta,q,v;J}$ consists of all $f \in X_0 + X_1$ for which there is a strongly measurable function $u : (0, \infty) \to X_0 \cap X_1$ such that*

$$f = \int_0^\infty u(s)\, \frac{ds}{s} \quad (\textit{convergence in } X_0 + X_1) \tag{2.6}$$

and for which the functional

$$\|f\|_{\theta,q,v;J} \equiv \|f\|_{(X_0,X_1)_{\theta,q,v;J}} := \inf \left\| t^{-\theta-1/q}\, v(t)\, J(u(t), t; X_0, X_1) \right\|_{q,(0,\infty)} \tag{2.7}$$

is finite (the infimum extends over all representations (2.6) of f).

We refer to Lemmas 4.1 and 4.2 mentioned below for properties of the K-functional and the J-functional.

Theorem 2.4. ([56, Theorem 2.3]) *Let (X_0, X_1) be a compatible couple, $0 \le \theta \le 1$, $1 \le q \le \infty$, and let $v \in \mathcal{W}(0, \infty)$.*
A. *If*

$$\left\| t^{-\theta-1/q}\, v(t) \min\{1, t\} \right\|_{q,(0,\infty)} < \infty, \tag{2.8}$$

then:

(i) *The space* $(X_0, X_1)_{\theta,q,v;K}$ *is an intermediate space between* X_0 *and* X_1*, that is,*

$$X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{\theta,q,v;K} \hookrightarrow X_0 + X_1.$$

(ii) *The space* $(X_0, X_1)_{\theta,q,v;K}$ *is a Banach space.*

B. *If condition* (2.8) *is not satisfied, then* $(X_0, X_1)_{\theta,q,v;K} = \{0\}$.

Note that assertion A of Theorem 2.4 also follows from [8, Proposition 3.3.1, p. 338].

Theorem 2.5. ([56, Theorem 2.4]) *Let* (X_0, X_1) *be a compatible couple,* $0 \le \theta \le 1$, $1 \le q \le \infty$, *and let* $v \in \mathcal{W}(0,\infty)$.

A. *If*

$$\left\| t^{\theta-1/q'} \frac{1}{v(t)} \min\left\{1, \frac{1}{t}\right\} \right\|_{q',(0,\infty)} < \infty, \tag{2.9}$$

then:

(i) *The space* $(X_0, X_1)_{\theta,q,v;J}$ *is an intermediate space between* X_0 *and* X_1*, that is,*

$$X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{\theta,q,v;J} \hookrightarrow X_0 + X_1.$$

(ii) *The space* $(X_0, X_1)_{\theta,q,v;J}$ *is a Banach space.*

B. *If condition* (2.9) *is not satisfied, then the functional* $\|.\|_{\theta,q,v;J}$ *vanishes on* $X_0 \cap X_1$ *and thus it is not a norm provided that* $X_0 \cap X_1 \ne \{0\}$.

Sometimes K-spaces or J-spaces coincide with their modifications, which we now introduce: Let (X_0, X_1) be a compatible couple, $0 \le \theta \le 1$, $1 \le q \le \infty$, and let $v \in \mathcal{W}(0,\infty)$. Assuming that

$$\textit{either} \quad (a,b) = (0,1), \qquad \textit{or} \quad (a,b) = (1,\infty),$$

we put

$$(X_0, X_1)_{\theta,q,v;K;(a,b)} := \{f \in X_0 + X_1 : \|f\|_{\theta,q,v;K;(a,b)} < \infty\}, \tag{2.10}$$

where

$$\|f\|_{\theta,q,v;K;(a,b)} \equiv \|f\|_{(X_0,X_1)_{\theta,q,v;K;(a,b)}} := \left\| t^{-\theta-1/q}\, v(t)\, K(f,t;X_0,X_1) \right\|_{q,(a,b)}. \tag{2.11}$$

Similarly, the space $(X_0, X_1)_{\theta,q,v;J;(a,b)}$ consists of all $f \in X_0 + X_1$ for which there is a strongly measurable function $u : (a,b) \to X_0 \cap X_1$ such that

$$f = \int_a^b u(s)\, \frac{ds}{s} \quad (\textit{convergence in } X_0 + X_1) \tag{2.12}$$

and for which the functional

$$\|f\|_{\theta,q,v;J;(a,b)} \equiv \|f\|_{(X_0,X_1)_{\theta,q,v;J;(a,b)}} := \inf \left\| t^{-\theta-1/q}\, v(t)\, J(u(t),t;X_0,X_1) \right\|_{q,(a,b)} \tag{2.13}$$

is finite (the infimum extends over all representations (2.12) of f).

To calculate duals of interpolation spaces, first note that given a compatible couple (X_0, X_1), then the assumption

$$X_0 \cap X_1 \text{ is dense in } \ X_0 \text{ and } \ X_1$$

ensures that the dual couple (X_0', X_1') is compatible as well.

We shall also use the assertions mentioned in the next remark.

Remark 2.6. Let $X_0 \cap X_1$ is dense in X_0 and X_1.

(i) Then (cf. [4, p. 53])

$$K(f', t; X_0', X_1') = \sup_{f \in X_0 \cap X_1} \frac{|\langle f', f\rangle|}{J(f, t^{-1}; X_0, X_1)} \quad \text{for all } f' \in X_0' + X_1' \text{ and } t > 0, \tag{2.14}$$

and

$$J(f', t; X_0', X_1') = \sup_{f \in X_0 + X_1} \frac{|\langle f', f\rangle|}{K(f, t^{-1}; X_0, X_1)} \quad \text{for all } f' \in X_0' \cap X_1' \text{ and } t > 0, \tag{2.15}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $X_0 \cap X_1$ and $X_0' + X_1'$ in (2.14), and between $X_0 + X_1$ and $X_0' \cap X_1'$ in (2.15).

(ii) Since $K(\cdot, 1; X_0, X_1)$ is the norm on $X_0 + X_1$ and $J(\cdot, 1; X_0, X_1)$ is the norm on $X_0 \cap X_1$ it follows from part (i) that

$$(X_0 \cap X_1)' = X_0' + X_1' \qquad \text{and} \qquad (X_0 + X_1)' = X_0' \cap X_1' \tag{2.16}$$

(which is the result mentioned in [4, Theorem 2.7.1, p. 32]).

In the following definition we introduce some weighted variant of the small Lebesgue space $\ell_q(\mathbb{Z})$.

Definition 2.7 (the space $\lambda_{\theta,q,w}$)**.** *Let $0 \le \theta \le 1$, $1 \le q \le \infty$, and let $w \in \mathcal{W}(0,\infty)$. The space $\lambda_{\theta,q,w}$ consists of all sequences of real numbers $\{\alpha_m\}_{m\in\mathbb{Z}}$ satisfying*

$$\|\{\alpha_m\}\|_{\lambda_{\theta,q,w}} = \|\{\alpha_m\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,w}} < \infty,$$

where

$$\|\{\alpha_m\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,w}} := \begin{cases} \left(\sum\limits_{m\in\mathbb{Z}} (2^{-m\theta} w(2^m)|\alpha_m|)^q \right)^{1/q} & \text{if } 1 \le q < \infty, \\ \sup\limits_{m\in\mathbb{Z}} 2^{-m\theta} w(2^m)|\alpha_m| & \text{if } q = \infty. \end{cases}$$

The next remark is a consequence of duality results for $\ell_q(\mathbb{Z})$ spaces.

Remark 2.8. Let $0 \le \theta \le 1$ and $w \in \mathcal{W}(0,\infty)$. If $1 \le q < \infty$, then the space $\lambda_{1-\theta,q',w^{-1}}$ is the dual space of $\lambda_{\theta,q,w}$ via the duality $\sum_{m\in\mathbb{Z}} 2^{-m}\alpha_m\beta_m$.

If $n \in \mathbb{N}$ and $1 \le p \le \infty$, then the symbol $L_p := L_p(\mathbb{R}^n)$ denotes the *Lebesgue space* on $\mathbb{R}^n$ equipped with the usual Lebesgue norm $\|\cdot\|_p$. Moreover, if $k \in \mathbb{N}$, then the *Sobolev space* $W_p^k := W_p^k(\mathbb{R}^n)$ consists of those functions f on $\mathbb{R}^n$ for which all the distributional derivatives of f of order at most k belong to $L_p(\mathbb{R}^n)$ and the functional

$$\|f\|_{W_p^k} := \sum_{|\nu|\le k} \|D^\nu f\|_p, \quad f \in W_p^k,$$

is the norm in this space. We put $W_p^0 = L_p$.

The symbol $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is used to denote the *Schwartz space* of rapidly decreasing C^∞-functions on $\mathbb{R}^n$ endowed with the usual topology. Let $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ be the collection of all tempered distributions on $\mathbb{R}^n$, i.e., the topological dual of $\mathcal{S}$, equipped with the strong topology. Note that if $1 \le p < \infty$, then $\mathcal{S}$ is a dense subspace of L_p.

If $\varphi \in \mathcal{S}$, then

$$\mathcal{F}\varphi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(\xi)\, d\xi, \quad x \in \mathbb{R}^n,$$

denotes the *Fourier transform* $\mathcal{F}$ of φ (here $x\xi = \sum_{j=1}^{n} x_j \xi_j$ is the scalar product in $\mathbb{R}^n$ of $x = [x_1, \cdots, x_n] \in \mathbb{R}^n$ and $\xi = [\xi_1, \cdots, \xi_n] \in \mathbb{R}^n$). The *inverse Fourier transform* $\mathcal{F}^{-1}$ of φ is given by

$$\mathcal{F}^{-1}\varphi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi}\varphi(\xi)\, d\xi, \quad x \in \mathbb{R}^n.$$

Note that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended from $\mathcal{S}$ to $\mathcal{S}'$ in the usual way. (For example, the Fourier transform $\mathcal{F}$ of $f \in \mathcal{S}'$ is defined by $(\mathcal{F}f)(\varphi) = f(\mathcal{F}\varphi)$ for all $\varphi \in \mathcal{S}$.) Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are isomorphic mappings from $\mathcal{S}$ onto itself and from $\mathcal{S}'$ onto itself.

If $s \in \mathbb{R}$ and $1 < p < \infty$, then the *fractional Sobolev* (or *Bessel potential*) *space* $H_p^s := H_p^s(\mathbb{R}^n)$ is the collections of all $f \in \mathcal{S}'$ satisfying

$$\|f\|_{H_p^s} = \|\mathcal{F}^{-1}((1+|x|^2)^{s/2}\mathcal{F}f)\|_p < \infty.$$

The space H_p^s equipped with the norm $\|\cdot\|_{H_p^s}$ is a Banach space (cf. [61, Remark 1, p. 11]) and the sets $C_0^\infty := C_0^\infty(\mathbb{R}^n)$ and $\mathcal{S}$ are dense in H_p^s (this is a consequence of [60, Section 2.3.2, p. 172] and [61, Remark 3, p. 18]). It is well known that if $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $1 < p < \infty$, then

$$H_p^k = W_p^k. \tag{2.17}$$

Let $s \in \mathbb{R}$ and let the operator I_s be given by

$$I_s f = \mathcal{F}^{-1}((1+|x|^2)^{s/2}\mathcal{F}f) \quad \text{if} \quad f \in \mathcal{S}'. \tag{2.18}$$

If $s, \sigma \in \mathbb{R}$ and $1 < p < \infty$, then I_s is a lift which maps H_p^σ isomorphically onto $H_p^{\sigma-s}$ (cf. [61, p. 12]). Consequently,

$$I_s(H_p^\sigma) = H_p^{\sigma-s}. \tag{2.19}$$

Moreover (cf. [60, p. 198]), if $s \in \mathbb{R}$ and $1 < p < \infty$, then

$$(H_p^s)' = H_{p'}^{-s}. \tag{2.20}$$

Let $p \in [1, +\infty)$. The *modulus of continuity* of a function f in $L_p(\mathbb{R}^n)$ is given by

$$\omega_1(f,t)_p := \sup_{|h|\le t} \|\Delta_h f\|_p \quad \text{for all} \quad t \ge 0.$$

Definition 2.9. *Let $n \in \mathbb{N}$, $1 \le p < \infty$, $1 \le q \le \infty$ and let $b \in SV(0,\infty)$ be such that*

$$\|t^{-1/q}\, b(t)\|_{q;(0,1)} = \infty \quad \text{and} \quad \|t^{-1/q}\, b(t)\|_{q;(1,\infty)} < \infty. \tag{2.21}$$

The Besov space $B_{p,q}^{0,b} := B_{p,q}^{0,b}(\mathbb{R}^n)$ consists of all $f \in L_p(\mathbb{R}^n)$ for which the norm

$$\|f\|_{B_{p,q}^{0,b}} := \|f\|_p + \|t^{-1/q} b(t)\omega_1(f,t)_p\|_{q;(0,\infty)} \tag{2.22}$$

is finite.

Note that spaces $B_{p,q}^{0,b}(\mathbb{R}^n)$ are particular cases of Besov spaces with generalized smoothness. Besov spaces $B_{p,q}^{0,b}(\mathbb{R}^n)$ with a logarithmic smoothness b have already appeared in [27] in a connection with the weak-type interpolation. During the recent years spaces $B_{p,q}^{0,b}(\mathbb{R}^n)$ have attracted an increasing attention (see [10], [12], [43], [25], [14], [15], [17], [30], [31], [6], [16], [32], etc.)

Remark 2.10. (i) An equivalent norm on $B^{0,b}_{p,q}(\mathbb{R}^n)$ is given by the functional

$$\|f\|_{B^{0,b}_{p,q}} := \|f\|_p + \|t^{-1/q} b(t)\omega_1(f,t)_p\|_{q;(0,1)}.$$

We refer to [11, Remark 2.5 (iii)] for more details.

(ii) The assumption $\|t^{-1/q}b(t)\|_{q;(1,\infty)} < \infty$ is natural. Otherwise the space $B^{0,b}_{p,q}(\mathbb{R}^n)$ is trivial (that is, it consists only of the zero element). We again refer to [11, Remark 2.5 (iii)] for more details.

(iii) Note also that only the case when $\|t^{-1/q}b(t)\|_{q;(0,1)} = \infty$ is of interest. Otherwise $B^{0,b}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$.

(iv) If the modulus of continuity $\omega_1(f,\cdot)_p$ in (2.22) is replaced by the k-th order modulus of continuity $\omega_k(f,\cdot)_p$, for $k \in \{2,3,4,\ldots\}$, we obtain an equivalent norm on the space $B^{0,b}_{p,q}(\mathbb{R}^n)$, cf. [11, Remark 2.5 (ii)].

3. Main Results

Theorem 3.1 (1. duality theorem for K-spaces and $\theta = 0$)**.** *Let (X_0, X_1) be a compatible couple, $1 \le q < \infty$, and let $X_0 \cap X_1$ be dense in X_0 and X_1. If $b \in SV(0,\infty)$ satisfies*

$$\int_x^\infty t^{-1} b^q(t)\,dt < \infty \quad \textit{for all} \;\; x > 0, \qquad \int_0^\infty t^{-1} b^q(t)\,dt = \infty, \tag{3.1}$$

and $a \in SV(0,\infty)$ is defined by

$$a(x) := b^{-q/q'}(x) \int_x^\infty t^{-1} b^q(t)\,dt \quad \textit{for all } x > 0, \tag{3.2}$$

then

$$(X_0, X_1)'_{0,q,b;K} = (X_0', X_1')_{0,q',\tilde{b};J} = (X_0', X_1')_{0,q',\tilde{a};K}, \tag{3.3}$$

where $\tilde{b}(x) := \frac{1}{b(1/x)}$ and $\tilde{a}(x) := \frac{1}{a(1/x)}$ for all $x > 0$.

(The proof of this theorem is in Section 5.)

There are the following two counterparts of Theorem 3.1.

Theorem 3.2 (1. duality theorem for J-spaces and $\theta = 0$)**.** *Let (X_0, X_1) be a compatible couple, $1 < q < \infty$, and let $X_0 \cap X_1$ be dense in X_0 and X_1. If $a \in SV(0,\infty)$ satisfies*

$$\int_0^x t^{-1} a^{-q'}(t)\,dt < \infty \quad \textit{for all} \;\; x > 0, \qquad \int_0^\infty t^{-1} a^{-q'}(t)\,dt = \infty, \tag{3.4}$$

and $b \in SV(0,\infty)$ is defined by

$$b(x) := a^{-q'/q}(x) \Big(\int_0^x t^{-1} a^{-q'}(t)\,dt \Big)^{-1} \quad \textit{for all} \;\; x > 0, \tag{3.5}$$

then

$$(X_0, X_1)'_{0,q,a;J} = (X_0', X_1')_{0,q',\tilde{a};K} = (X_0', X_1')_{0,q',\tilde{b};J}, \tag{3.6}$$

where $\tilde{a}(x) := \frac{1}{a(1/x)}$ and $\tilde{b}(x) := \frac{1}{b(1/x)}$ for all $x > 0$.

(The proof of this theorem is in Section 6.)

Theorem 3.3 (2. duality theorem for J-spaces and $\theta = 0$)**.** *Let* (X_0, X_1) *be a compatible couple and let* $X_0 \cap X_1$ *be dense in* X_0 *and* X_1*. Assume that* $a \in SV(0,\infty) \cap AC(0,\infty)$ *satisfies*

$$a \text{ is strictly decreasing}, \quad a(0) = \infty, \quad a(\infty) = 0. \tag{3.7}$$

If $b \in SV(0,\infty)$ *and*

$$b(x) := -x\, a'(x) \quad \text{for a.a. } x > 0, \tag{3.8}$$

then

$$(X_0, X_1)'_{0,1,a;J} = (X'_0, X'_1)_{0,\infty,\tilde{a};K} = (X'_0, X'_1)_{0,\infty,\tilde{b};J}, \tag{3.9}$$

where $\tilde{a}(x) := \frac{1}{a(1/x)}$ *and* $\tilde{b}(x) := \frac{1}{b(1/x)}$ *for all* $x > 0$.

(The proof of this theorem is in Section 7.)

There is the following variant of Theorem 3.1.

Theorem 3.4 (2. duality theorem for K-spaces and $\theta = 0$)**.** *Let* (X_0, X_1) *be a compatible couple,* $1 \le q < \infty$*, and let* $X_0 \cap X_1$ *be dense in* X_0 *and* X_1*. If* $b \in SV(0,\infty)$ *satisfies*

$$\int_0^\infty t^{-1} b^q(t)\, dt < \infty, \tag{3.10}$$

the function $B \in SV(0,\infty)$ *is given by*

$$B(x) := b(x) \quad \text{if } x \in [1,\infty) \quad \text{and} \quad B(x) \approx \beta(x) \quad \text{if } x \in (0,1), \tag{3.11}$$

where $\beta \in SV(0,\infty)$ *is such that*

$$\int_x^\infty t^{-1}\beta^q(t)\, dt < \infty \quad \text{for all } x > 0, \qquad \int_0^\infty t^{-1}\beta^q(t)\, dt = \infty, \tag{3.12}$$

and

$$A(x) := B^{-q/q'}(x) \int_x^\infty t^{-1} B^q(t)\, dt \quad \text{for all } x > 0, \tag{3.13}$$

then

$$\begin{aligned}(X_0, X_1)'_{0,q,b;K} &= (X'_0, X'_1)_{0,q',\tilde{b};J} \\ &= (X'_0, X'_0 \cap X'_1)_{0,q',\tilde{B};J} = (X'_0, X'_0 \cap X'_1)_{0,q',\tilde{A};K} \\ &= X'_0 \cap (X'_0, X'_1)_{0,q',\tilde{B};J} = X'_0 \cap (X'_0, X'_1)_{0,q',\tilde{A};K},\end{aligned} \tag{3.14}$$

where $\tilde{b}(x) := \frac{1}{b(1/x)}$, $\tilde{B}(x) := \frac{1}{B(1/x)}$, *and* $\tilde{A}(x) := \frac{1}{A(1/x)}$ *for all* $x > 0$.

Remark 3.5. Under the assumptions of Theorem 3.4,

$$(X_0, X_1)_{0,q,b;K} = (X_0, X_1)_{0,q,b;K;(1,\infty)}, \tag{3.15}$$

$$(X'_0, X'_1)_{0,q',\tilde{b};J} = (X'_0, X'_1)_{0,q',\tilde{b};J;(0,1)}, \tag{3.16}$$

$$(X'_0, X'_0 \cap X'_1)_{0,q',\tilde{B};J} = (X'_0, X'_0 \cap X'_1)_{0,q',\tilde{B};J;(0,1)}, \tag{3.17}$$

$$(X'_0, X'_0 \cap X'_1)_{0,q',\tilde{A};K} = (X'_0, X'_0 \cap X'_1)_{0,q',\tilde{A};K;(0,1)}. \tag{3.18}$$

(The proofs of Theorem 3.4 and Remark 3.5 are in Section 8.)

There is the following variant of Theorem 3.2.

Theorem 3.6 (3. duality theorem for J-spaces and $\theta = 0$)**.** *Let (X_0, X_1) be a compatible couple, $1 < q < \infty$, and let $X_0 \cap X_1$ be dense in X_0 and X_1. If $a \in SV(0,\infty)$ satisfies*

$$\int_0^\infty t^{-1} a^{-q'}(t)\,dt < \infty, \tag{3.19}$$

the function $A \in SV(0,\infty)$ is given by

$$A(x) := a(x) \quad \text{if } \ x \in (0,1], \qquad A(x) \approx \alpha(x) \quad \text{if } \ x \in (1,\infty), \tag{3.20}$$

where the function $\alpha \in SV(0,\infty)$ is such that

$$\int_0^x t^{-1}\alpha^{-q'}(t)\,dt < \infty \quad \text{for all } \ x > 0, \qquad \int_0^\infty t^{-1}\alpha^{-q'}(t)\,dt = \infty, \tag{3.21}$$

and

$$B(x) := A^{-q'/q}(x)\left(\int_0^x t^{-1}A^{-q'}(t)\,dt\right)^{-1} \quad \text{for all } \ x > 0, \tag{3.22}$$

then

$$\begin{aligned}(X_0, X_1)'_{0,q,a;J} =& (X'_0, X'_1)_{0,q',\tilde{a};K}\\ =& (X'_0, X'_0 + X'_1)_{0,q',\tilde{A};K} = (X'_0, X'_0 + X'_1)_{0,q',\tilde{B};J}\\ =& X'_0 + (X'_0, X'_1)_{0,q',\tilde{A};K} = X'_0 + (X'_0, X'_1)_{0,q',\tilde{B};J},\end{aligned} \tag{3.23}$$

where $\tilde{a}(x) := \frac{1}{a(1/x)}$, $\tilde{A}(x) := \frac{1}{A(1/x)}$, and $\tilde{B}(x) := \frac{1}{B(1/x)}$ for all $x > 0$.

Remark 3.7. Under the assumptions of Theorem 3.6,

$$(X_0, X_1)_{0,q,a;J} = (X_0, X_1)_{0,q,a;J;(0,1)}, \tag{3.24}$$

$$(X'_0, X'_1)_{0,q',\tilde{a};K} = (X'_0, X'_1)_{0,q',\tilde{a};K;(1,\infty)}, \tag{3.25}$$

$$(X'_0, X'_0 + X'_1)_{0,q',\tilde{A};K} = (X'_0, X'_0 + X'_1)_{0,q',\tilde{A};K;(1,\infty)}, \tag{3.26}$$

$$(X'_0, X'_0 + X'_1)_{0,q',\tilde{B};J} = (X'_0, X'_0 + X'_1)_{0,q',\tilde{B};;(1,\infty)}. \tag{3.27}$$

(The proofs of Theorem 3.6 and Remark 3.7 are Section 9.)

The next assertion is a variant of Theorem 3.3.

Theorem 3.8 (4. duality theorem for J-spaces and $\theta = 0$)**.** *Let (X_0, X_1) be a compatible couple and let $X_0 \cap X_1$ be dense in X_0 and X_1. Assume that $a \in SV(0,\infty) \cap AC(0,\infty)$ satisfies*

$$a \text{ is strictly decreasing}, \qquad a(0) = \infty, \qquad a(\infty) > 0, \tag{3.28}$$

the function $A \in SV(0,\infty)$ be given by

$$A(x) := a(x) \quad \text{if } \ x \in (0,1], \qquad A(x) := c\,\alpha(x) \quad \text{if } \ x \in (1,\infty), \tag{3.29}$$

where $\alpha \in SV(0,\infty) \cap AC(0,\infty)$ is such that

$$\alpha \text{ is strictly decreasing}, \qquad \alpha(0) = \infty, \qquad \alpha(\infty) = 0, \tag{3.30}$$

and c is a positive constant chosen in such a way that $A \in AC(0,\infty)$. If $B \in SV(0,\infty)$ and

$$B(x) := -x\,A'(x) \quad \text{for a.a. } \ x > 0, \tag{3.31}$$

then

$$(3.32)\qquad \begin{aligned}(X_0,X_1)'_{0,1,a;J} &= (X_0',X_1')_{0,\infty,\tilde a;K}\\ &= (X_0',X_0'+X_1')_{0,\infty,\tilde A;K} = (X_0',X_0'+X_1')_{0,\infty,\tilde B;J}\\ &= X_0' + (X_0',X_1')_{0,\infty,\tilde A;K} = X_0' + (X_0',X_1')_{0,\infty,\tilde B;J},\end{aligned}$$

where $\tilde a(x) := \frac{1}{a(1/x)}$, $\tilde A(x) := \frac{1}{A(1/x)}$, *and* $\tilde B(x) := \frac{1}{B(1/x)}$ *for all* $x > 0$.

Remark 3.9. Under the assumptions of Theorem 3.8,

$$(3.33)\qquad (X_0,X_1)_{0,1,a;J} = (X_0,X_1)_{0,1,a;J;(0,1)},$$

$$(3.34)\qquad (X_0',X_1')_{0,\infty,\tilde a;K} = (X_0',X_1')_{0,\infty,\tilde a;K;(1,\infty)},$$

$$(3.35)\qquad (X_0',X_0'+X_1')_{0,\infty,\tilde A;K} = (X_0',X_0'+X_1')_{0,\infty,\tilde A;K;(1,\infty)}.$$

Moreover, if $\|\tilde B(t)\|_{\infty,(1,e)} < \infty$, then also

$$(3.36)\qquad (X_0',X_0'+X_1')_{0,\infty,\tilde B;J} = (X_0',X_0'+X_1')_{0,\infty,\tilde B;J;(1,\infty)}.$$

(The proofs of Theorem 3.8 and Remark 3.9 are in Section 10.)

Remark 3.10. In Theorem 3.3 we assume that $a \in SV(0,\infty)\cap AC(0,\infty)$. By Remark 2.2, any function $a \in SV(0,\infty)$ is equvivalent to the function $\overline{a} \in SV(0,\infty)\cap AC(0,\infty)$ given by $\overline{a}(x) := x^{-1}\int_0^x a(t)\,dt,\ x > 0$. Moreover, one can prove that if the function a satisfies (3.7), then (3.7) also holds with a replaced by $\overline{a}$. Thus, Theorem 3.3 remains true if the assumption $a \in SV(0,\infty)\cap AC(0,\infty)$ is replaced only by $a \in SV(0,\infty)$ provided that in (3.8) we write $\overline{a}\,'$ instead of a'.

Similar remark can be made about Theorem 3.8.

Duality theorems mentioned above, together with particular choices of compatible couples (X_0,X_1), give duality results for different spaces. For example, we can obtain the following assertion for the Besov space $B^{0,b}_{p,q}(\mathbb{R}^n)$ involving the zero classical smoothness and a slowly varying smoothness b.

Theorem 3.11 (dual of the space $B^{0,b}_{p,q}(\mathbb{R}^n)$)**.** *Let* $n \in \mathbb{N}$, $1 < p < \infty$, $1 \le q < \infty$, *and let* $b \in SV(0,\infty)$ *be such that*

$$(3.37)\qquad \|t^{-1/q}\, b(t)\|_{q;(0,1)} = \infty \quad \text{and} \quad \|t^{-1/q}\, b(t)\|_{q;(1,\infty)} < \infty.$$

If $a \in SV\,(0,\infty)$ *is defined by*

$$a(x) := b^{-q/q'}(x)\int_x^\infty t^{-1}b^q(t)\,dt \quad \text{for all } x > 0,$$

then $(B^{0,b}_{p,q}(\mathbb{R}^n))'$ *is the set of all functions* $f \in H^{-1}_{p'}(\mathbb{R}^n)$ *which satisfy*

$$(3.38)\qquad N(f) := \|I_{-1}f\|_{p'} + \left\|\tau^{-1/q'}\,\frac{\omega_1(I_{-1}f,\tau)_{p'}}{\tau a(\tau)}\right\|_{q',(0,1)} < \infty,$$

where the operator I_{-1} *is defined by* (2.18)*. Moreover,*

$$(3.39)\qquad \|f\|_{(B^{0,b}_{p,q})'} \approx N(f) \quad \text{for all } f \in (B^{0,b}_{p,q})'.$$

(The proof of Theorem 3.11 is in Section 11.)

Remark 3.12. On taking $b(x) = (1 + |\ln x|)^\beta$ if $x > 0$, where $\beta > -1/q$, in Theorem 3.11, one obtains the result corresponding to [14, Theorem 4.3].

4. Auxiliary assertions

In the next two lemmas some basic properties of the K- and J- functionals are summarized.

Lemma 4.1. (cf. [3, Proposition 1.2, p. 294]) *If (X_0, X_1) is a compatible couple, then, for each $f \in X_0 + X_1$, the K-functional $K(f, t; X_0, X_1)$ is a nonnegative concave function of $t > 0$, and*

$$K(f, t; X_0, X_1) = t\, K(f, t^{-1}; X_1, X_0) \quad \text{for all} \;\; t > 0. \tag{4.1}$$

In particular, $K(f, t; X_0, X_1)$ is non-decreasing on $(0, \infty)$ and $K(f, t; X_0, X_1)/t$ is non-increasing on $(0, \infty)$.

Lemma 4.2. (cf. [4, Lemma 3.2.1, p. 42]) *If (X_0, X_1) is a compatible couple, then, for each $f \in X_0 \cap X_1$, the J- functional $J(f, t; X_0, X_1)$ is a nonnegative convex function of $t > 0$, and*

$$J(f, t; X_0, X_1) = t\, J(f, t^{-1}; X_1, X_0) \quad \text{for all} \;\; t > 0, \tag{4.2}$$

$$J(f, t; X_0, X_1) \le \max\{1, t/s\}\, J(f, s; X_0, X_1) \quad \text{for all} \;\; t, s > 0, \tag{4.3}$$

$$K(f, t; X_0, X_1) \le \min\{1, t/s\}\, J(f, s; X_0, X_1) \quad \text{for all} \;\; t, s > 0. \tag{4.4}$$

In particular, $J(f, t; X_0, X_1)$ is non-decreasing on $(0, \infty)$ and $J(f, t; X_0, X_1)/t$ is non-increasing on $(0, \infty)$.

The following result describes when the K-space $(X_0, X_1)_{\theta,q,b;K}$ with the limiting value $\theta = 0$ and $b \in SV(0, \infty)$ coincides with the J-space $(X_0, X_1)_{\theta,q,a;J}$ with a convenient $a \in SV(0, \infty)$.

Theorem 4.3. (cf. [56, Theorem 3.1]) *Let (X_0, X_1) be a compatible couple and $1 \le q < \infty$. If $b \in SV(0, \infty)$ satisfies*

$$\int_x^\infty t^{-1} b^q(t)\, dt < \infty \quad \text{for all} \;\; x > 0, \qquad \int_0^\infty t^{-1} b^q(t)\, dt = \infty, \tag{4.5}$$

and $a \in SV(0, \infty)$ is such that

$$a(x) \approx b^{-q/q'}(x) \int_x^\infty t^{-1} b^q(t)\, dt \quad \text{for a.a. } x > 0, \tag{4.6}$$

then

$$(X_0, X_1)_{0,q,b;K} = (X_0, X_1)_{0,q,a;J}. \tag{4.7}$$

Theorem 4.4. (cf. [56, Theorem 3.2]) *Let (X_0, X_1) be a compatible couple and $1 \le q < \infty$. If $b \in SV(0, \infty)$ satisfies (4.5), then the space $X_0 \cap X_1$ is dense in $(X_0, X_1)_{0,q,b;K}$.*

There are the following counterparts of the previous two assertions.

Theorem 4.5. (cf. [56, Theorem 3.3]) *Let (X_0, X_1) be a compatible couple and $1 < q \le \infty$. If $a \in SV(0, \infty)$ satisfies*

$$\int_0^x t^{-1} a^{-q'}(t)\, dt < \infty \quad \text{for all} \;\; x > 0, \qquad \int_0^\infty t^{-1} a^{-q'}(t)\, dt = \infty, \tag{4.8}$$

and $b \in SV(0, \infty)$ is such that

$$b(x) \approx a^{-q'/q}(x) \Big(\int_0^x t^{-1} a^{-q'}(t)\, dt \Big)^{-1} \quad \text{for a.a. } x > 0, \tag{4.9}$$

then

$$(X_0, X_1)_{0,q,a;J} = (X_0, X_1)_{0,q,b;K}. \tag{4.10}$$

Theorem 4.6. (cf. [56, Theorem 3.4]) *Let (X_0, X_1) be a compatible couple and $1 < q < \infty$. If $a \in SV(0,\infty)$ satisfies (4.8), then the space $X_0 \cap X_1$ is dense in $(X_0, X_1)_{0,q,a;J}$.*

The next two assertions are complements of Theorems 4.3 and 4.5.

Theorem 4.7. ([56, Theorem 9.1]) *Let (X_0, X_1) be a compatible couple. If $b \in SV(0,\infty) \cap AC(0,\infty)$ satisfies*

$$b \text{ is strictly decreasing}, \quad b(0) = \infty, \quad b(\infty) = 0, \tag{4.11}$$

and if

$$a(x) := \frac{b^2(x)}{x(-b'(x))} \quad \text{for a.a. } x > 0, \tag{4.12}$$

then

$$(X_0, X_1)_{0,\infty,b;K} = (X_0, X_1)_{0,\infty,a;J}. \tag{4.13}$$

Theorem 4.8. ([56, Theorem 10.1]) *Let (X_0, X_1) be a compatible couple. If $a \in SV(0,\infty) \cap AC(0,\infty)$ satisfies*

$$a \text{ is strictly decreasing}, \quad a(0) = \infty, \quad a(\infty) = 0, \tag{4.14}$$

and if

$$b(x) := -x\, a'(x) \quad \text{for a.a. } x > 0, \tag{4.15}$$

then

$$(X_0, X_1)_{0,1,a;J} = (X_0, X_1)_{0,1,b;K}. \tag{4.16}$$

Theorem 4.9. ([56, Theorem 10.3]) *Let (X_0, X_1) be a compatible couple. If the function a satisfies the assumptions of* Theorem 4.8*, then $X_0 \cap X_1$ is dense in $(X_0, X_1)_{0,1,a;J}$.*

Another complement of Theorem 4.3 is the next result.

Theorem 4.10. ([57, Theorem 1.11]) *Let (X_0, X_1) be a compatible couple and $1 \le q < \infty$. If $b \in SV\,(0,\infty)$ satisfies*

$$\int_0^\infty t^{-1} b^q(t)\, dt < \infty, \tag{4.17}$$

the function $B \in SV\,(0,\infty)$ is given by

$$B(x) := b(x) \quad \text{if } x \in [1,\infty), \qquad B(x) \approx \beta(x) \quad \text{if } x \in (0,1), \tag{4.18}$$

where $\beta \in SV(0,\infty)$ is such that

$$\int_x^\infty t^{-1}\beta^q(t)\, dt < \infty \quad \text{for all } x > 0, \qquad \int_0^\infty t^{-1}\beta^q(t)\, dt = \infty, \tag{4.19}$$

and

$$A(x) := B^{-q/q'}(x) \int_x^\infty t^{-1} B^q(t)\, dt \quad \text{for all } x > 0, \tag{4.20}$$

then

$$(X_0, X_1)_{0,q,b;K} = (X_0, X_0 + X_1)_{0,q,B;K} = (X_0, X_0 + X_1)_{0,q,A;J}. \tag{4.21}$$

Remark 4.11. ([57, Remark 1.13]) Under the assumptions of Theorem 4.10,

$$(X_0, X_1)_{0,q,b;K} = (X_0, X_1)_{0,q,b;K;(1,\infty)}, \tag{4.22}$$

$$(X_0, X_0 + X_1)_{0,q,B;K} = (X_0, X_0 + X_1)_{0,q,B;K;(1,\infty)}, \tag{4.23}$$

$$(X_0, X_0 + X_1)_{0,q,A;J} = (X_0, X_0 + X_1)_{0,q,A;J;(1,\infty)}. \tag{4.24}$$

Corollary 4.12. ([57, Corollary 1.14]) *Under the assumptions of* Theorem 4.10*,*

$$(X_0, X_1)_{0,q,b;K} = X_0 + (X_0, X_1)_{0,q,B;K} = X_0 + (X_0, X_1)_{0,q,A;J}. \tag{4.25}$$

Further complement of Theorem 4.5 reads as follows.

Theorem 4.13. ([57, Theorem 1.15]) *Let* (X_0, X_1) *be a compatible couple and* $1 < q \le \infty$*. If* $a \in SV(0,\infty)$ *satisfies*

$$\int_0^\infty t^{-1} a^{-q'}(t)\, dt < \infty, \tag{4.26}$$

the function $A \in SV(0,\infty)$ *is given by*

$$A(x) := a(x) \quad \text{if } \ x \in (0,1], \qquad A(x) \approx \alpha(x) \quad \text{if } \ x \in (1,\infty), \tag{4.27}$$

where the function $\alpha \in SV(0,\infty)$ *is such that*

$$\int_0^x t^{-1} \alpha^{-q'}(t)\, dt < \infty \quad \text{for all } \ x > 0, \qquad \int_0^\infty t^{-1} \alpha^{-q'}(t)\, dt = \infty, \tag{4.28}$$

and

$$B(x) := A^{-q'/q}(x) \left(\int_0^x t^{-1} A^{-q'}(t)\, dt \right)^{-1} \quad \text{for all } \ x > 0, \tag{4.29}$$

then

$$(X_0, X_1)_{0,q,a;J} = (X_0, X_0 \cap X_1)_{0,q,A;J} = (X_0, X_0 \cap X_1)_{0,q,B;K}. \tag{4.30}$$

Remark 4.14. ([57, Remark 1.17]) Note that, under the assumptions of Theorem 4.13,

$$(X_0, X_1)_{0,q,a;J} = (X_0, X_1)_{0,q,a;J;(0,1)}, \tag{4.31}$$

$$(X_0, X_0 \cap X_1)_{0,q,A;J} = (X_0, X_0 \cap X_1)_{0,q,A;J;(0,1)}, \tag{4.32}$$

$$(X_0, X_0 \cap X_1)_{0,q,B;K} = (X_0, X_0 \cap X_1)_{0,q,B;K;(0,1)}. \tag{4.33}$$

Corollary 4.15. ([57, Corollary 1.18]) *Under the assumptions of* Theorem 4.13*,*

$$(X_0, X_1)_{0,q,a;J} = X_0 \cap (X_0, X_1)_{0,q,A;J} = X_0 \cap (X_0, X_1)_{0,q,B;K}. \tag{4.34}$$

The following result is a complement of Theorem 4.8.

Theorem 4.16. ([57, Theorem 1.22]) *Let* (X_0, X_1) *be a compatible couple and let* $a \in SV(0,\infty) \cap AC(0,\infty)$ *satisfy*

$$a \text{ is strictly decreasing}, \quad a(0) = \infty, \quad a(\infty) > 0. \tag{4.35}$$

Assume that the function $A \in SV(0,\infty)$ *is given by*

$$A(x) := a(x) \quad \text{if } x \in (0,1], \qquad A(x) := c\,\alpha(x) \quad \text{if } x \in (1,\infty), \tag{4.36}$$

where $\alpha \in SV(0,\infty) \cap AC(0,\infty)$ *is such that*

$$\alpha \text{ is strictly decreasing}, \quad \alpha(0) = \infty, \quad \alpha(\infty) = 0, \tag{4.37}$$

and c is a positive constant chosen in such a way that $A \in AC(0,\infty)$. If

$$B(x) := -x\,A'(x) \quad \textit{for a.a. } x > 0, \tag{4.38}$$

then

$$(X_0, X_1)_{0,1,a;J} = (X_0, X_0 \cap X_1)_{0,1,A;J} = (X_0, X_0 \cap X_1)_{0,1,B;K}. \tag{4.39}$$

Theorem 4.17. ([57, Theorem 1.23]) *Let (X_0, X_1) be a compatible couple. If the function a satisfies the assumptions of* Theorem 4.16*, then $X_0 \cap X_1$ is dense in $(X_0, X_1)_{0,1,a;J}$.*

Remark 4.18. ([57, Remark 1.24]) Note that, under the assumptions of Theorem 4.16,

$$(X_0, X_1)_{0,1,a;J} = (X_0, X_1)_{0,1,a;J;(0,1)}, \tag{4.40}$$

$$(X_0, X_0 \cap X_1)_{0,1,A;J} = (X_0, X_0 \cap X_1)_{0,1,A;J;(0,1)}, \tag{4.41}$$

$$(X_0, X_0 \cap X_1)_{0,1,B;K} = (X_0, X_0 \cap X_1)_{0,1,B;K;(0,1)}. \tag{4.42}$$

Corollary 4.19. ([57, Corollary 1.25]) *Under the assumptions of* Theorem 4.16*,*

$$(X_0, X_1)_{0,1,a;J} = X_0 \cap (X_0, X_1)_{0,1,A;J} = X_0 \cap (X_0, X_1)_{0,1,B;K}. \tag{4.43}$$

There is the following counterpart of Theorem 4.16.

Theorem 4.20 ([57, Theorem 1.19])**.** *Let (X_0, X_1) be a compatible couple and let $b \in SV(0,\infty) \cap AC(0,\infty)$ satisfy*

$$b \textit{ is strictly decreasing}, \quad b(0) < \infty, \quad b(\infty) = 0. \tag{4.44}$$

Assume that the function $B \in SV(0,\infty)$ is given by

$$B(x) := b(x) \quad \textit{if } x \in [1,\infty), \qquad B(x) := c\,\beta(x) \quad \textit{if } x \in (0,1), \tag{4.45}$$

where $\beta \in SV(0,\infty) \cap AC(0,\infty)$ is such that

$$\beta \textit{ is strictly decreasing}, \quad \beta(0) = \infty, \quad \beta(\infty) = 0, \tag{4.46}$$

and c is a positive constant chosen in such a way that $B \in AC(0,\infty)$. If

$$A(x) := \frac{B^2(x)}{x(-B'(x))} \quad \textit{for a.a. } x > 0, \tag{4.47}$$

then

$$(X_0, X_1)_{0,\infty,b;K} = (X_0, X_0 + X_1)_{0,\infty,B;K} = (X_0, X_0 + X_1)_{0,\infty,A;J}. \tag{4.48}$$

Remark 4.21. ([57, Remark 1.20]) Note that, under the assumptions of Theorem 4.20,

$$(X_0, X_1)_{0,\infty,b;K} = (X_0, X_1)_{0,\infty,b;K;(1,\infty)}, \tag{4.49}$$

$$(X_0, X_0 + X_1)_{0,\infty,B;K} = (X_0, X_0 + X_1)_{0,\infty,B;K;(1,\infty)}. \tag{4.50}$$

Moreover, if $\|A(t)\|_{\infty,(1,e)} < \infty$, then also

$$(X_0, X_0 + X_1)_{0,\infty,A;J} = (X_0, X_0 + X_1)_{0,\infty,A;J;(1,\infty)}. \tag{4.51}$$

We shall also need the next two lemmas.

Lemma 4.22. *If* (X_0, X_1) *is a compatible couple,* $\theta \in [0,1]$, $1 \le q \le \infty$, *and* $w \in \mathcal{W}(0,\infty)$, *then*

$$(X_0, X_1)_{\theta,q,w;K} \equiv (X_1, X_0)_{1-\theta,q,w_1;K}, \tag{4.52}$$

where $w_1 \in \mathcal{W}(0,\infty)$ *is given by*

$$w_1(x) := w(1/x) \quad \textit{for all } x > 0. \tag{4.53}$$

Proof. If $f \in X_0 + X_1$, then

$$\|f\|_{(X_0,X_1)_{\theta,q;w,K}} = \|t^{-\theta-1/q} w(t) K(f,t;X_0,X_1)\|_{q,(0,\infty)}. \tag{4.54}$$

Making use of (4.1) and a change of variables, we obtain that

$$\begin{aligned} \text{RHS}(4.54) &= \|t^{1-\theta-1/q} w(t) K(f,t^{-1};X_1,X_0)\|_{q,(0,\infty)} \\ &= \|s^{-(1-\theta)-1/q} w_1(s) K(f,s;X_1,X_0)\|_{q,(0,\infty)} \\ &= \|f\|_{(X_1,X_0)_{1-\theta,q;w_1,K}}, \end{aligned}$$

and the result follows. □

Lemma 4.23. *If* (X_0, X_1) *is a compatible couple,* $\theta \in [0,1]$, $1 \le q \le \infty$, *and* $w \in \mathcal{W}(0,\infty)$, *then*

$$(X_0, X_1)_{\theta,q,w;J} \equiv (X_1, X_0)_{1-\theta,q,w_1;J}, \tag{4.55}$$

where $w_1 \in \mathcal{W}(0,\infty)$ *is given by*

$$w_1(x) := w(1/x) \quad \textit{for all } x > 0. \tag{4.56}$$

Proof. If $f \in X_0 + X_1$ and

$$f = \int_0^\infty u(t)\,\frac{dt}{t} \quad (\textit{convergence in } X_0 + X_1), \tag{4.57}$$

with $u(t) \in X_0 \cap X_1$ for every $t > 0$, then also

$$f = \int_0^\infty u_1(s)\,\frac{ds}{s} \quad \text{with } \ u_1(s) = u(1/s) \text{ for every } s > 0.$$

Furthermore,

$$\|f\|_{(X_0,X_1)_{\theta,q,w;J}} = \inf \|t^{-\theta-1/q} w(t) J(u(t),t;X_0,X_1)\|_{q,(0,\infty)}, \tag{4.58}$$

where the infimum extends over all representation (4.57) of f. Making use of (4.2) and a change of variables, we get

$$\begin{aligned} \|t^{-\theta-1/q} w(t) J(u(t),t;X_0,X_1)\|_{q,(0,\infty)} &= \|t^{1-\theta-1/q} w(t) J(u(t),t^{-1};X_1,X_0)\|_{q,(0,\infty)} \\ &= \|s^{-(1-\theta)-1/q} w_1(s) J(u_1(s),s;X_1,X_0)\|_{q,(0,\infty)}, \end{aligned}$$

which, together with (4.58), implies that

$$\|f\|_{(X_0,X_1)_{\theta,q,w;J}} = \|f\|_{(X_1,X_0)_{1-\theta,q,w_1;J}}.$$

Consequently, (4.55) holds. □

A description when the K-space $(X_0, X_1)_{\theta,q,B;K}$ with the limiting value $\theta = 1$ and $B \in SV(0,\infty)$ coincides with the J-space $(X_0, X_1)_{\theta,q,A;J}$ with a convenient $A \in SV(0,\infty)$ is given in the following assertion.

Theorem 4.24. *Let (X_0, X_1) be a compatible couple and $1 \leq q < \infty$. If $B \in SV(0,\infty)$ satisfies*

$$\int_0^x t^{-1} B^q(t)\, dt < \infty \quad \text{for all} \;\; x > 0, \qquad \int_0^\infty t^{-1} B^q(t)\, dt = \infty, \tag{4.59}$$

and $A \in SV(0,\infty)$ is such that

$$A(x) \approx B^{-q/q'}(x) \int_0^x t^{-1} B^q(t)\, dt \quad \text{for a.a.} \;\; x > 0, \tag{4.60}$$

then

$$(X_0, X_1)_{1,q,B;K} = (X_0, X_1)_{1,q,A;J}. \tag{4.61}$$

Proof. By Lemma 4.22,

$$(X_0, X_1)_{1,q,B;K} \equiv (X_1, X_0)_{0,q,B_1;K}, \tag{4.62}$$

with B_1 given by $B_1(x) := B(1/x)$ for all $x > 0$.

Using assumptions (4.59) and a change of variables, we can see that

$$\int_x^\infty t^{-1} B_1^q(t)\, dt < \infty \quad \text{for all} \;\; x > 0, \qquad \int_0^\infty t^{-1} B_1^q(t)\, dt = \infty.$$

Thus, we can apply Theorem 4.3 to arrive at

$$(X_1, X_0)_{0,q,B_1;K} = (X_1, X_0)_{0,q,\tilde{A};J}\,, \tag{4.63}$$

with $\tilde{A} \in SV(0,\infty)$ satisfying

$$\begin{aligned} \tilde{A}(x) &\approx B_1^{-q/q'}(x) \int_x^\infty t^{-1} B_1^q(t)\, dt \\ &= B^{-q/q'}(1/x) \int_0^{1/x} t^{-1} B^q(t)\, dt \quad \text{for a.a. } x > 0\,. \end{aligned} \tag{4.64}$$

Consequently, by Lemma 4.23,

$$(X_1, X_0)_{0,q,\tilde{A};J} = (X_0, X_1)_{1,q,A;J}, \tag{4.65}$$

with $A \in SV(0,\infty)$ given by $A(x) = \tilde{A}(1/x)$ for all $x > 0$. Together with (4.64), this implies that the function A satisfies (4.60).

On using (4.62), (4.63), and (4.65), we also obtain (4.61). □

There is the following counterpart of the previous result.

Theorem 4.25. *Let (X_0, X_1) be a compatible couple and $1 < q \leq \infty$. If $A \in SV(0,\infty)$ satisfies*

$$\int_x^\infty t^{-1} A^{-q'}(t)\, dt < \infty \;\; \text{for all } x > 0, \qquad \int_0^\infty t^{-1} A^{-q'}(t)\, dt = \infty, \tag{4.66}$$

and $B \in SV(0,\infty)$ is such that

$$B(x) \approx A^{-q'/q}(x) \left(\int_x^\infty t^{-1} A^{-q'}(t)\, dt \right)^{-1} \qquad \text{for a.a. } x > 0, \tag{4.67}$$

then

$$(X_0, X_1)_{1,q,A;J} = (X_0, X_1)_{1,q,B;K}\,. \tag{4.68}$$

Proof. By Lemma 4.23,

$$(X_0, X_1)_{1,q,A;J} \equiv (X_1, X_0)_{0,q,A_1;J}, \tag{4.69}$$

with A_1 given by $A_1(x) := A(1/x)$ for all $x > 0$.

Using assumptions (4.66) and a change of variables, we can see that

$$\int_0^x t^{-1} A_1^{q'}(t)\, dt < \infty \quad \text{for all} \quad x > 0, \qquad \int_0^\infty t^{-1} A_1^{q'}(t)\, dt = \infty.$$

Thus, applying Theorem 4.5, we arrive at

$$(X_1, X_0)_{0,q,A_1;J} = (X_1, X_0)_{0,q,\tilde{B};K}\,, \tag{4.70}$$

with $\tilde{B} \in SV(0,\infty)$ satisfying

$$\begin{aligned} \tilde{B}(x) &:= A_1^{-q'/q}(x) \Big(\int_0^x t^{-1} A_1^{-q'}(t)\, dt \Big)^{-1} \\ &= A^{-q'/q}(1/x) \Big(\int_{1/x}^\infty t^{-1} A^{-q'}(t)\, dt \Big)^{-1} \quad \text{for a.a.} \quad x > 0\,. \end{aligned} \tag{4.71}$$

Consequently, by Lemma 4.22,

$$(X_1, X_0)_{0,q,\tilde{B};K} = (X_0, X_1)_{1,q,B;K}, \tag{4.72}$$

with $B \in SV(0,\infty)$ given by $B(x) = \tilde{B}(1/x)$ for all $x > 0$. Together with (4.71), this implies that the function B satisfies (4.67).

On using (4.69), (4.70), and (4.72), we also obtain (4.68). □

We continue with discrete characterizations of spaces $(X_0, X_1)_{\theta,q,b;K}$ and $(X_0, X_1)_{\theta,q,b;J}$. To this end we shall use the following assertion.

Lemma 4.26. *If $\theta \in [0,1]$, and $b \in SV(0,\infty)$, then there are positive constants c_1, c_2 such that*

$$\frac{c_1}{2^\theta}\, 2^{-m\theta} b(2^m) \le t^{-\theta} b(t) \le c_2\, 2^{-m\theta} b(2^m) \quad \textit{for all} \quad \theta \in [0,1], \ m \in \mathbb{Z} \ \textit{and} \ t \in [2^m, 2^{m+1}).$$

Proof. Since the function $t \mapsto tb(t),\ t \in (0,\infty)$, is equivalent to a non-decreasing function, there is a positive constant k_1 such that

$$2^m b(2^m) \le k_1 t b(t) \quad \text{for all } m \in \mathbb{Z} \text{ and } t \in [2^m, 2^{m+1}),$$

which implies that

$$\frac{b(2^m)}{k_1} \le \frac{tb(t)}{2^m} \le \frac{2^{m+1} b(t)}{2^m} = 2b(t) \quad \text{for all } m \in \mathbb{Z} \text{ and } t \in [2^m, 2^{m+1}). \tag{4.73}$$

Analogously, since the function $t \mapsto \frac{b(t)}{t},\ t \in (0,\infty)$, is equivalent to a non-increasing function, there is a positive constant k_2 such that

$$b(t) = t\frac{b(t)}{t} \le t\, k_2 \frac{b(2^m)}{2^m} \le k_2 \frac{2^{m+1}}{2^m} b(2^m) = 2k_2\, b(2^m) \quad \text{for all } m \in \mathbb{Z} \text{ and } t \in [2^m, 2^{m+1}),$$

which, together with (4.73), implies that

$$\frac{b(2^m)}{2k_1} \le b(t) \le 2k_2\, b(2^m) \quad \text{for all } m \in \mathbb{Z} \text{ and } t \in [2^m, 2^{m+1}). \tag{4.74}$$

Moreover, the function $t \mapsto t^{-\theta},\ t \in (0,\infty)$, with $\theta \in [0,1]$, is non-increasing. Consequently,

$$\big(2^{m+1}\big)^{-\theta} \le t^{-\theta} \le \big(2^m\big)^{-\theta} \quad \text{for all} \quad m \in \mathbb{Z} \text{ and } \quad t \in [2^m, 2^{m+1}). \tag{4.75}$$

Using estimates (4.74) and (4.75), we arrive at

$$\big(2^{m+1}\big)^{-\theta}\,\frac{b(2^m)}{2k_1}\le t^{-\theta}b(t)\le\big(2^m\big)^{-\theta}\,2k_2\,b(2^m)\quad\text{for all }\ \theta\in[0,1],\ m\in\mathbb{Z}\text{ and }\ t\in[2^m,2^{m+1})\,,$$

and the result follows (with $c_1=1/(2k_1)$ and $c_2=2k_2$). □

Lemma 4.27. *If (X_0,X_1) is a compatible couple, $\theta\in[0,1], 1\le q\le\infty$, and $b\in SV(0,\infty)$, then there are positive constants c_1,c_2 such that*

$$\text{(4.76)}\quad \frac{c_1}{2^\theta}(\ln 2)^{1/q}\,\|\{K(f,2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}}\le\|f\|_{\theta,q,b;K}\le 2c_2(\ln 2)^{1/q}\,\|\{K(f,2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}}$$

for all $f\in X_0+X_1$, $\theta\in[0,1]$, and $1\le q\le\infty$.

In particular, given $f\in X_0+X_1$, then $f\in(X_0,X_1)_{\theta,q,b;K}$ if and only if $\{K(f,2^m)\}_{m\in\mathbb{Z}}\in\lambda_{\theta,q,b}$ and

$$\text{(4.77)}\qquad \|f\|_{\theta,q,b;K}\approx\|\{K(f,2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}}$$

for all $f\in(X_0,X_1)_{\theta,q,b;K}$, $\theta\in[0,1]$, and $1\le q\le\infty$.

Proof. Let $f\in X_0+X_1$. Using properties of the K-functional (cf. Lemma 4.1), we obtain

$$K(f,2^m)\le K(f,t)\le 2K(f,2^m)\quad\text{for all }m\in\mathbb{Z}\text{ and }t\in[2^m,2^{m+1}).$$

Together with Lemma 4.26, this implies that

$$\text{(4.78)}\qquad \frac{c_1}{2^\theta}\,2^{-m\theta}b(2^m)K(f,2^m)\le t^{-\theta}b(t)K(f,t)\le 2c_2\,2^{-m\theta}b(2^m)K(f,2^m)$$

for all $\theta\in[0,1]$, $m\in\mathbb{Z}$ and $t\in[2^m,2^{m+1})$.

Hence, if $1\le q<\infty$, then

$$\text{(4.79)}\qquad \Big(\frac{c_1}{2^\theta}\,2^{-m\theta}b(2^m)K(f,2^m)\Big)^q\ln 2\le\int_{2^m}^{2^{m+1}}(t^{-\theta}b(t)K(f,t))^q\frac{dt}{t}$$
$$\le\Big(2c_2\,2^{-m\theta}b(2^m)K(f,2^m)\Big)^q\ln 2,$$

which implies that

$$\frac{c_1}{2^\theta}\,(\ln 2)^{1/q}\Big(\sum_{m\in\mathbb{Z}}\big(2^{-m\theta}b(2^m)K(f,2^m)\big)^q\Big)^{1/q}$$
$$\le\|f\|_{\theta,q,b;K}$$
$$\le 2c_2\,(\ln 2)^{1/q}\Big(\sum_{m\in\mathbb{Z}}\big(2^{-m\theta}b(2^m)K(f,2^m)\big)^q\Big)^{1/q},$$

i.e,

$$\frac{c_1}{2^\theta}\,(\ln 2)^{1/q}\,\|\{K(f,2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}}\le\|f\|_{\theta,q,b;K}\le 2c_2\,(\ln 2)^{1/q}\,\|\{K(f,2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}}$$

for all $f\in X_0+X_1$, $\theta\in[0,1]$ and $1\le q<\infty$, which is (4.76) with $1\le q<\infty$.

Furthermore, if $q=\infty$, then using (4.78), we obtain

$$\frac{c_1}{2^\theta}\,2^{-m\theta}b(2^m)K(f,2^m)\le\operatorname*{ess\,sup}_{t\in[2^m,2^{m+1})}t^{-\theta}b(t)K(f,t)\le 2c_2\;2^{-m\theta}b(2^m)K(f,2^m)\quad\text{for all }m\in\mathbb{Z},$$

which immediately gives

$$\frac{c_1}{2^\theta}\|\{K(f,2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}}\le\|f\|_{\theta,q,b;K}\le 2c_2\|\{K(f,2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}}$$

for all $f\in X_0+X_1$, $\theta\in[0,1]$, and $q=\infty$, which is (4.76) with $q=\infty$. □

Lemma 4.28. *Let (X_0, X_1) be a compatible couple, $\theta \in [0,1], 1 \le q \le \infty$, and $b \in SV(0,\infty)$. Then $f \in (X_0, X_1)_{\theta,q,b;J}$ if and only if there exist $u_m \in X_0 \cap X_1$ for all $m \in \mathbb{Z}$ such that*

$$f = \sum_{m\in\mathbb{Z}} u_m \qquad (\textit{convergence in } X_0 + X_1) \tag{4.80}$$

and

$$\{J(u_m, 2^m)\}_{m\in\mathbb{Z}} \in \lambda_{\theta,q,b}. \tag{4.81}$$

Moreover, there are positive constants c_1, c_2 such that

$$\frac{1}{2c_2}(\ln 2)^{1/q'} \, \|f\|_{\theta,q,b;J} \le \inf \|\{J(u_m, 2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}} \le \frac{2^\theta}{c_1}(\ln 2)^{1/q'} \|f\|_{\theta,q,b;J} \tag{4.82}$$

for all $f \in (X_0, X_1)_{\theta,q,b;J}$, $\theta \in [0,1]$, and $1 \le q \le \infty$, where the infimum is extended over all sequences $\{u_m\}_{m\in\mathbb{Z}}$ satisfying (4.80) and (4.81).

Proof. Let $f \in (X_0, X_1)_{\theta,q,b;J}$ and

$$f = \int_0^\infty u(t)\frac{dt}{t} \qquad \text{(convergence in } X_0 + X_1), \tag{4.83}$$

where $u : (0,\infty) \to X_0 \cap X_1$ is a strongly measurable function. Putting $u_m = \int_{2^m}^{2^{m+1}} u(t)\, dt/t$ for all $m \in \mathbb{Z}$, we see that (4.80) holds. Since

$$J(u_m, 2^m) = J\Big(\int_{2^m}^{2^{m+1}} u(t)\frac{dt}{t}, 2^m\Big) \le \int_{2^m}^{2^{m+1}} J(u(t), 2^m)\frac{dt}{t} \le \int_{2^m}^{2^{m+1}} J(u(t), t)\frac{dt}{t} \tag{4.84}$$

for all $m \in \mathbb{Z}$ and $t \in [2^m, 2^{m+1})$, we get

$$\|\{J(u_m, 2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}} \le \left\| \left\{ \int_{2^m}^{2^{m+1}} J(u(t), t)\frac{dt}{t} \right\}_{m\in\mathbb{Z}} \right\|_{\lambda_{\theta,q,b}}. \tag{4.85}$$

If $1 \le q < \infty$, then, by Lemma 4.26,

$$\begin{aligned} \text{RHS}(4.85) &= \left(\sum_{m\in\mathbb{Z}} \left(2^{-m\theta} b(2^m) \int_{2^m}^{2^{m+1}} J(u(t), t)\frac{dt}{t} \right)^q \right)^{1/q} \\ &\le \frac{2^\theta}{c_1} \left(\sum_{m\in\mathbb{Z}} \left(\int_{2^m}^{2^{m+1}} t^{-\theta} b(t) J(u(t), t)\frac{dt}{t} \right)^q \right)^{1/q}. \end{aligned} \tag{4.86}$$

Moreover, by the Jensen inequality,

$$\left(\int_{2^m}^{2^{m+1}} t^{-\theta} b(t) J(u(t), t)\frac{dt}{t\ln 2} \right)^q \le \int_{2^m}^{2^{m+1}} (t^{-\theta} b(t) J(u(t), t))^q \frac{dt}{t\ln 2} \quad \text{for all } m \in \mathbb{Z}.$$

Together with (4.86) and (4.85), this gives

$$\|\{J(u_m, 2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}} \le \frac{2^\theta}{c_1}(\ln 2)^{1/q'} \|t^{-\theta-1/q} b(t) J(u(t), t)\|_{q,(0,\infty)},$$

which implies (4.81). Thus, taking the infimum over all sequences $\{u_m\}_{m\in\mathbb{Z}}$ satisfying (4.80) and (4.81), we get the second inequality in (4.82) for all $f \in (X_0, X_1)_{\theta,q,b;J}$, $\theta \in [0,1]$, and $1 \le q < \infty$.

If $q = \infty$, then, using (4.85) and Lemma 4.26, we obtain

$$\mathrm{RHS}(4.85) = \sup_{m\in\mathbb{Z}} 2^{-m\theta} b(2^m) \int_{2^m}^{2^{m+1}} J(u(t),t)\frac{dt}{t} \le \frac{2^\theta}{c_1} \sup_{m\in\mathbb{Z}} \int_{2^m}^{2^{m+1}} t^{-\theta} b(t) J(u(t),t)\frac{dt}{t}.$$

Since (cf. [58, Lemma 5.5, p. 47])

$$t^{-\theta} b(t) J(u(t),t) \le \operatorname*{ess\,sup}_{s\in[2^m,2^{m+1})} s^{-\theta} b(s) J(u(s),s) \quad \text{for all } m\in\mathbb{Z} \text{ and for a.a. } t\in[2^m,2^{m+1}),$$

we arrive at

$$\begin{aligned} \mathrm{RHS}(4.85) &= \sup_{m\in\mathbb{Z}} 2^{-m\theta} b(2^m) \int_{2^m}^{2^{m+1}} J(u(t),t)\frac{dt}{t} \\ &\le \frac{2^\theta}{c_1}(\ln 2) \sup_{m\in\mathbb{Z}} \operatorname*{ess\,sup}_{s\in<2^m,2^{m+1})} s^{-\theta} b(s) J(u(s),s) \\ &= \frac{2^\theta}{c_1}(\ln 2)\, \|s^{-\theta-1/q} b(s) J(u(s),s)\|_{q,(0,\infty)}. \end{aligned} \tag{4.87}$$

Together with (4.85), this gives

$$\|\{J(u_m,2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}} \le \frac{2^\theta}{c_1}(\ln 2)\, \|s^{-\theta-1/q} b(s) J(u(s),s)\|_{q,(0,\infty)},$$

which implies that the second inequality in (4.82) holds for all $f \in X_0, X_1)_{\theta,q,b;J}$, $\theta\in[0,1]$, and $q=\infty$.

Conversely, assume that

$$f = \sum_{m\in\mathbb{Z}} u_m \quad (\text{convergence in } X_0+X_1), \tag{4.88}$$

where $u_m \in X_0\cap X_1$ for all $m\in\mathbb{Z}$, and

$$\{J(u_m,2^m)\}_{m\in\mathbb{Z}} \in \lambda_{\theta,q,b}. \tag{4.89}$$

Putting

$$u(t) = \frac{u_m}{\ln 2} \quad \text{if } \ 2^m \le t < 2^{m+1} \ \text{ and } \ m\in\mathbb{Z}, \tag{4.90}$$

we get $u:(0,\infty)\to X_0\cap X_1$ and

$$f = \sum_{m\in\mathbb{Z}} u_m = \sum_{m\in\mathbb{Z}} \int_{2^m}^{2^{m+1}} \frac{u_m}{\log 2}\frac{dt}{t} = \int_0^\infty u(t)\frac{dt}{t}.$$

Thus,

$$\|f\|_{\theta,q,b;J} \le \|t^{-\theta-1/q} b(t) J(u(t),t)\|_{q,(0,\infty)}. \tag{4.91}$$

If $1\le q<\infty$, then using (4.90), we obtain

$$\begin{aligned} \mathrm{RHS}(4.91) &= \left(\sum_{m\in\mathbb{Z}} \int_{2^m}^{2^{m+1}} \left(t^{-\theta} b(t) J\left(\frac{u_m}{\ln 2},t\right)\right)^q \frac{dt}{t}\right)^{1/q} \\ &= \frac{1}{\ln 2}\left(\sum_{m\in\mathbb{Z}} \int_{2^m}^{2^{m+1}} \left(t^{-\theta} b(t) J(u_m,t)\right)^q \frac{dt}{t}\right)^{1/q}. \end{aligned}$$

Together with the estimate

$$J(u_m,t) \le 2\, J(u_m,2^m) \quad \text{for all } m\in\mathbb{Z} \text{ and } t\in[2^m,2^{m+1}), \tag{4.92}$$

and Lemma 4.26, this implies that

$$\text{RHS}(4.91) \le \frac{2}{\ln 2}\, c_2 \left(\sum_{m\in\mathbb{Z}} \int_{2^m}^{2^{m+1}} \left(2^{-m\theta} b(2^m) J\left(u_m, 2^m\right)\right)^q \frac{dt}{t} \right)^{1/q} \tag{4.93}$$

$$= \frac{2}{\ln 2}\, c_2 \,(\ln 2)^{1/q} \left(\sum_{m\in\mathbb{Z}} \left(2^{-m\theta} b(2^m) J\left(u_m, 2^m\right)\right)^q \right)^{1/q}$$

$$= 2c_2 (\ln 2)^{-1/q'} \|\{J(u_m, 2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}}.$$

Combining estimates (4.91) and (4.93), we arrive at

$$\frac{1}{2c_2} (\ln 2)^{1/q'} \, \|f\|_{\theta,q,b;J} \ \le \|\{J(u_m, 2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}},$$

and taking the infimum over all sequences $\{u_m\}_{m\in\mathbb{Z}}$ satisfying (4.88) and (4.89), we get the first inequality in (4.82).

If $q = \infty$, then, by (4.90),

$$\text{RHS}(4.91) = \sup_{m\in\mathbb{Z}} \operatorname*{ess\,sup}_{s\in[2^m,2^{m+1})} s^{-\theta} b(s) J(u(s), s)$$

$$= \frac{1}{\ln 2} \sup_{m\in\mathbb{Z}} \operatorname*{ess\,sup}_{s\in[2^m,2^{m+1})} s^{-\theta} b(s) J(u_m, s).$$

Using estimate (4.92) and Lemma 4.26, we obtain

$$\text{RHS}(4.91) \le \frac{2}{\ln 2} c_2 \sup_{m\in\mathbb{Z}} \operatorname*{ess\,sup}_{s\in[2^m,2^{m+1})} 2^{-m\theta} b(2^m) J(u_m, 2^m) \tag{4.94}$$

$$= \frac{2}{\ln 2} c_2 \sup_{m\in\mathbb{Z}} 2^{-m\theta} b(2^m) J(u_m, 2^m)$$

$$= \frac{2}{\ln 2} c_2 \,\|\{J(u_m, 2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}}.$$

Combining estimates (4.91) and (4.94), we arrive at

$$\frac{1}{2c_2} (\ln 2)\, \|f\|_{\theta,q,b;J} \ \le \|\{J(u_m, 2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}},$$

and taking the infimum over all sequences $\{u_m\}_{m\in\mathbb{Z}}$ satisfying (4.88) and (4.89), we get the first inequality in (4.82). ☐

Remark 4.29. Note that the constants c_1 and c_2 in Lemmas 4.27 and 4.28 are the same as those in Lemma 4.26 and thus they depend only on the function $b \in SV(0,\infty)$.

We shall also need the following assertion.

Lemma 4.30. *Let $\theta \in [0,1]$, $1 \le q \le \infty$, and $a \in SV(0,\infty)$. Then*

$$\left\| t^{\theta - 1/q'} a^{-1}(t) \min\left\{1, t^{-1}\right\} \right\|_{q',(0,\infty)} < \infty, \tag{4.95}$$

if and only if

$$\{\min\{1, 2^m\}\}_{m\in\mathbb{Z}} \in \lambda_{1-\theta,q',a^{-1}}. \tag{4.96}$$

PROOF. Note that

$$\frac{1}{2}\min\{1,t\} \le \frac{1}{2}\min\{1,2^{m+1}\} \le \min\{1,2^m\} \le \min\{1,t\} \tag{4.97}$$

for all $m \in \mathbb{Z}$ and $t \in [2^m, 2^{m+1})$.

If $1 < q \le \infty$, then, using Lemma 4.26 and (4.97), we obtain

$$\begin{aligned}
\Big\|\{\min\{1,2^m\}\}_{m\in\mathbb{Z}}\Big\|_{\lambda_{1-\theta,q',a^{-1}}} &= \Big(\sum_{m\in\mathbb{Z}} \big(2^{-m(1-\theta)}a^{-1}(2^m)\min\{1,2^m\}\big)^{q'}\Big)^{1/q'} \\
&\approx \Big(\sum_{m\in\mathbb{Z}} \big(2^{-m(1-\theta)}a^{-1}(2^m)\min\{1,2^m\}\big)^{q'} \int_{2^m}^{2^{m+1}} \frac{dt}{t}\Big)^{1/q'} \\
&\approx \Big(\sum_{m\in\mathbb{Z}} \int_{2^m}^{2^{m+1}} \big(t^{-(1-\theta)}a^{-1}(t)\min\{1,t\}\big)^{q'} \frac{dt}{t}\Big)^{1/q'} \\
&= \Big(\int_0^\infty \big(t^{\theta}a^{-1}(t)\min\{1,t^{-1}\}\big)^{q'} \frac{dt}{t}\Big)^{1/q'} \\
&= \big\|t^{\theta-1/q'}a^{-1}(t)\min\{1,t^{-1}\}\big\|_{q',(0,\infty)},
\end{aligned}$$

and the result follows.

Assume now that $q = 1$. Then, by Lemma 4.26 and (4.97),

$$\begin{aligned}
\Big\|\{\min\{1,2^m\}\}_{m\in\mathbb{Z}}\Big\|_{\lambda_{1-\theta,q',a^{-1}}} &= \sup_{m\in\mathbb{Z}} 2^{-m(1-\theta)}a^{-1}(2^m)\min\{1,2^m\} \\
&\approx \sup_{m\in\mathbb{Z}} \operatorname*{ess\,sup}_{s\in[2^m,2^{m+1})} t^{-(1-\theta)}a^{-1}(t)\min\{1,t\} \\
&= \big\|t^{-(1-\theta)}a^{-1}(t)\min\{1,t\}\big\|_{q',(0,\infty)} \\
&= \big\|t^{\theta-1/q'}a^{-1}(t)\min\{1,t^{-1}\}\big\|_{q',(0,\infty)},
\end{aligned}$$

and the result follows. □

Theorem 4.31. *Let (X_0, X_1) be a compatible couple, $\theta \in [0,1]$, $1 \le q < \infty$, and let $X_0 \cap X_1$ be dense in X_0 and X_1. If $a \in SV(0,\infty)$ satisfies condition (4.95), then*

$$(X_0, X_1)'_{\theta,q,a;J} \hookrightarrow (X_1', X_0')_{1-\theta,q',a^{-1};K}. \tag{4.98}$$

PROOF. By Theorem 2.5, condition (4.95) guarantees that the space $(X_0, X_1)_{\theta,q,a;J}$ is a Banach space, which is an intermediate space between X_0 and X_1. In particular,

$$X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{\theta,q,a;J}. \tag{4.99}$$

Moreover, by (2.16),

$$(X_0 \cap X_1)' = X_0' + X_1'. \tag{4.100}$$

Let $0 \ne f' \in (X_0, X_1)'_{\theta,q,a;J}$. By (4.99) and (4.100),

$$f' \in X_0' + X_1'. \tag{4.101}$$

Since $K(f', 2^m; X_0', X_1')$ is the equivalent norm on $X_0' + X_1'$ for every $m \in \mathbb{Z}$, from (4.101) we see that

$$0 < K(f', 2^m; X_0', X_1') < \infty. \tag{4.102}$$

Given $\varepsilon > 0$ and $m \in \mathbb{Z}$. Then (2.14) implies that there is $w_m \in X_0 \cap X_1$ such that

$$K(f', 2^{-m}; X_0', X_1') - \varepsilon \min\{1, 2^{-m}\} \leq \frac{\langle f', w_m \rangle}{J(w_m, 2^m)}\,. \tag{4.103}$$

Choose a sequence of numbers $\tau := \{\tau_m\}_{m\in\mathbb{Z}} \in \lambda_{\theta,q,a}$, and put

$$f_\tau := \sum_{m\in\mathbb{Z}} u_m \qquad \text{(convergence in } X_0 + X_1)\,, \tag{4.104}$$

where

$$u_m := \frac{\tau_m w_m}{J(w_m, 2^m)} \in X_0 \cap X_1, \qquad \text{for all } m \in \mathbb{Z}\,. \tag{4.105}$$

Then

$$\begin{aligned}\|\{J(u_m, 2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,a}} &= \left(\sum_{m\in\mathbb{Z}} \left(2^{-m\theta} a(2^m) J\left(\frac{\tau_m w_m}{J(w_m, 2^m)}, 2^m\right)\right)^q\right)^{1/q} \\ &= \left(\sum_{m\in\mathbb{Z}} \left(2^{-m\theta} a(2^m) \frac{\tau_m}{J(w_m, 2^m)} J\left(w_m, 2^m\right)\right)^q\right)^{1/q} \\ &= \|\{\tau_m\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,a}} < \infty.\end{aligned}$$

Thus, by Lemma 4.28,

$$f_\tau \in (X_0, X_1)_{\theta,q,a;J} \qquad \text{and} \qquad \|f_\tau\|_{\theta,q,a;J} \lesssim \|\{\tau_m\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,a}}\,. \tag{4.106}$$

Consequently,

$$\begin{aligned}|\,\langle f', f_\tau \rangle\,| &\leq \|f'\|_{(X_0,X_1)'_{\theta,q,a;J}} \cdot \|f_\tau\|_{\theta,q,a;J} \\ &\lesssim \|f'\|_{(X_0,X_1)'_{\theta,q,a;J}} \cdot \|\{\tau_m\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,a}}.\end{aligned} \tag{4.107}$$

Moreover, by (4.104), (4.105), and (4.103),

$$\begin{aligned}\langle f', f_\tau \rangle &= \sum_{m\in\mathbb{Z}} \tau_m \frac{\langle f', w_m \rangle}{J(w_m, 2^m)} \\ &\geq \sum_{m\in\mathbb{Z}} \tau_m \left(K(f', 2^{-m}; X_0', X_1') - \varepsilon \min\{1, 2^{-m}\}\right).\end{aligned} \tag{4.108}$$

Together with (4.107), (4.106), and the fact that

$$K(f', 2^{-m}; X_0', X_1') = 2^{-m} K(f', 2^m; X_1', X_0'),$$

this implies that

$$\sum_{m\in\mathbb{Z}} 2^{-m}\tau_m (K(f', 2^m; X_1', X_0') - \varepsilon \min\{1, 2^m\}) \lesssim \|f'\|_{(X_0,X_1)'_{\theta,q,a;J}} \|\{\tau_m\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,a}},$$

i.e.,

$$\begin{aligned}\sum_{m\in\mathbb{Z}} 2^{-m}\tau_m K(f', 2^m; X_1', X_0') &\lesssim \|f'\|_{(X_0,X_1)'_{\theta,q,a;J}} \|\{\tau_m\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,a}} \\ &\quad + \varepsilon \sum_{m\in\mathbb{Z}} 2^{-m}\tau_m \min\{1, 2^m\}.\end{aligned} \tag{4.109}$$

By Remark 2.8,

$$\sum_{m\in\mathbb{Z}} 2^{-m}\tau_m \min\{1, 2^m\} \leq \|\{\tau_m\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,a}} \|\{\min\{1, 2^m\}\}_{m\in\mathbb{Z}}\|_{\lambda_{1-\theta,q',a^{-1}}}.$$

Since, by our assumption (4.95) and Lemma 4.30,

$$c := \|\{\min\{1, 2^m\}\}_{m\in\mathbb{Z}}\|_{\lambda_{1-\theta,q',a^{-1}}} < \infty.$$

we get from (4.109) that

$$\sum_{m\in\mathbb{Z}} 2^{-m}\tau_m K(f', 2^m; X_1', X_0') \lesssim (\|f'\|_{(X_0,X_1)'_{\theta,q,a;J}} + c\,\varepsilon)\,\|\{\tau_m\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,a}}\,,$$

for all positive ε, all sequences $\tau := \{\tau_m\}_{m\in\mathbb{Z}} \in \lambda_{\theta,q,a}$ and all $f' \in (X_0, X_1)'_{\theta,q,a;J}$. Together with Remark 2.8, this implies that

$$\|\{K(f', 2^m; X_1', X_0')\}\|_{\lambda_{1-\theta,q',a^{-1}}} \lesssim \|f'\|_{(X_0,X_1)'_{\theta,q,a;J}} \quad \text{all} \;\; f' \in (X_0, X_1)'_{\theta,q,a;J}. \tag{4.110}$$

By Lemma 4.27,

$$\mathrm{LHS}(4.110) \approx \|f'\|_{(X_1',X_0')_{1-\theta,q',a^{-1};K}}\,.$$

This estimate and (4.110) show that embedding (4.98) holds. □

Theorem 4.32. *Let (X_0, X_1) be a compatible couple, $\theta \in [0, 1]$, $1 \le q < \infty$, and let $X_0\cap X_1$ be dense in X_0 and X_1. If $b \in SV(0,\infty)$ satisfies*

$$\left\|t^{1-\theta-1/q} b(t) \min\{1, t^{-1}\}\right\|_{q,(0,\infty)} < \infty, \tag{4.111}$$

then

$$(X_0', X_1')_{1-\theta,q',b^{-1};J} \hookrightarrow (X_1, X_0)'_{\theta,q,b;K}. \tag{4.112}$$

Proof. By Theorem 2.5, condition (4.111) guarantees that the space $(X_0', X_1')_{1-\theta,q',b^{-1};J}$ is a Banach space (which is intermediate between spaces X_0' and X_1').

Since

$$t \min\{1, t^{-1}\} = \min\{1, t\} \quad \text{for all} \;\; t > 0,$$

it is clear that condition (4.111) is equivalent to

$$\left\|t^{-\theta-1/q} b(t) \min\{1, t\}\right\|_{q,(0,\infty)} < \infty.$$

Consequently, by Theorem 2.4, the space $(X_1, X_0)_{\theta,q,b;K}$ is also a Banach, space (which is intermediate between spaces X_0 and X_1).

Let $f' \in (X_0', X_1')_{1-\theta,q',b^{-1};J}$. By Lemma 4.28, there are $u_m' \in X_0' \cap X_1'$ for all $m \in \mathbb{Z}$ such that

$$f' = \sum_{m\in\mathbb{Z}} u_m' \qquad \text{(convergence in } X_0' + X_1'), \tag{4.113}$$

$$\|\{J(u_m', 2^m, X_0', X_1')\}_{m\in\mathbb{Z}}\|_{\lambda_{1-\theta,q',b^{-1}}} < \infty, \tag{4.114}$$

and

$$\|f'\|_{(X_0',X_1')_{1-\theta,q',b^{-1};J}} \approx \inf \|\{J(u_m', 2^m)\}_{m\in\mathbb{Z}}\|_{\lambda_{1-\theta,q',b^{-1}}}\,, \tag{4.115}$$

where the infimum extends over all sequences $\{u_m'\}_{m\in\mathbb{Z}}$ satisfying (4.113) and (4.114).

By (2.15), for any $f \in (X_1, X_0)_{\theta,q,b;K}$,

$$\begin{aligned} |\langle f', f\rangle| &\le \sum_{m\in\mathbb{Z}} J(u_m', 2^{-m}; X_1', X_0') K(f, 2^m; X_1, X_0) \\ &= \sum_{m\in\mathbb{Z}} 2^{-m} J(u_m', 2^m; X_0', X_1') K(f, 2^m; X_1, X_0). \end{aligned} \tag{4.116}$$

Since, cf. Remark 2.8, $\lambda_{\theta,q,b}$ and $\lambda_{1-\theta,q',b^{-1}}$ are dual spaces via the duality $\sum_{m\in\mathbb{Z}} 2^{-m}\alpha_m\beta_m$, we get

$$\mathrm{RHS}(4.116) \le \|\{J(u'_m, 2^m; X'_0, X'_1)\}_{m\in\mathbb{Z}}\|_{\lambda_{1-\theta,q',b^{-1}}} \, \|\{K(f, 2^m; X_1, X_0)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}}.$$

Now, by Lemma 4.27,

$$\|\{K(f, 2^m; X_1, X_0)\}_{m\in\mathbb{Z}}\|_{\lambda_{\theta,q,b}} \approx \|f\|_{(X_1,X_0)_{\theta,q,b;K}} \quad \text{for all} \;\; f \in (X_1, X_0)_{\theta,q,b;K}.$$

Together with (4.116), this implies that

$$\|f'\|_{(X_1,X_0)'_{\theta,q,b;K}} \lesssim \|\{J(u'_m, 2^m; X'_0, X'_1)\}_{m\in\mathbb{Z}}\|_{\lambda_{1-\theta,q',b^{-1}}} \quad \text{for all} \;\; f' \in (X'_0, X'_1)_{1-\theta,q',b^{-1};J}.$$

Thus, using (4.115) and taking the infimum over all sequences $\{u_m\}_{m\in\mathbb{Z}}$ satisfying (4.113) and (4.114), we arrive at

$$\|f'\|_{(X_1,X_0)'_{\theta,q,b;K}} \lesssim \|f'\|_{(X'_0,X'_1)_{1-\theta,q',b^{-1};J}} \quad \text{for all} \;\; f' \in (X'_0, X'_1)_{1-\theta,q',b^{-1};J},$$

which means that embedding (4.112) holds. □

Lemma 4.33. ([56, Lemma 11.2]) *Assume that* $1 < q < \infty$.
(i) *Let* $b \in SV(0,\infty)$ *be such that*

$$\int_x^\infty t^{-1} b^q(t)\, dt < \infty \quad \text{for all} \;\; x > 0 \qquad \text{and} \quad \int_0^\infty t^{-1} b^q(t)\, dt = \infty. \tag{4.117}$$

If

$$a(x) := b^{-q/q'}(x) \int_x^\infty t^{-1} b^q(t)\, dt \quad \text{for all} \;\; x > 0, \tag{4.118}$$

then $a \in SV(0,\infty)$,

$$\int_0^x t^{-1} a^{-q'}(t)\, dt < \infty \quad \text{for all} \;\; x > 0 \qquad \text{and} \quad \int_0^\infty t^{-1} a^{-q'}(t)\, dt = \infty. \tag{4.119}$$

Moreover,

$$b(x) \approx a^{-q'/q}(x) \Big(\int_0^x t^{-1} a^{-q'}(t)\, dt \Big)^{-1} \quad \text{for a.a.} \;\; x > 0 \tag{4.120}$$

and

$$\Big(\int_0^x t^{-1} a^{-q'}(t)\, dt \Big)^{1/q'} \Big(\int_x^\infty t^{-1} b^q(t)\, dt \Big)^{1/q} = \Big(\frac{1}{q'-1} \Big)^{1/q'} \quad \text{for all} \;\; x > 0. \tag{4.121}$$

(ii) *Let* $a \in SV(0,\infty)$ *be such that* (4.119) *holds. If*

$$b(x) := a^{-q'/q}(x) \Big(\int_0^x t^{-1} a^{-q'}(t)\, dt \Big)^{-1} \quad \text{for all} \;\; x > 0, \tag{4.122}$$

then $b \in SV(0,\infty)$ *and* (4.117) *is satisfied. Moreover,*

$$a(x) \approx b^{-q/q'}(x) \int_x^\infty t^{-1} b^q(t)\, dt \quad \text{for a.a.} \;\; x > 0, \tag{4.123}$$

and

$$\Big(\int_0^x t^{-1} a^{-q'}(t)\, dt \Big)^{1/q'} \Big(\int_x^\infty t^{-1} b^q(t)\, dt \Big)^{1/q} = \Big(\frac{1}{q-1} \Big)^{1/q} \quad \text{for all} \;\; x > 0. \tag{4.124}$$

Lemma 4.34. *If* (X_0, X_1) *is a compatible couple and* $X_0 \cap X_1$ *is dense in* X_0 *and* X_1*, then* $X_0 \cap X_1$ *is also dense in* $X_0 + X_1$.

Proof. Let $f \in X_0 + X_1$ and $f = f_0 + f_1$, where $f_i \in X_i, i = 1, 2$, and let $\varepsilon > 0$. Since $X_0 \cap X_1$ is dense in X_0 and X_1, there are $g_i \in X_0 \cap X_1$ satisfying

$$\|f_i - g_i\| < \varepsilon/2, \quad i = 1, 2.$$

Putting $g := g_0 + g_1$, we see that $g \in X_0 \cap X_1$, and

$$\|f - g\|_{X_0+X_1} = \inf\{\|h_0\|_{X_0} + \|h_1\|_{X_1} : f - g = h_0 + h_1\},$$

where the infimum extends over all representation $f - g = h_0 + h_1$ of $f - g$ with $h_i \in X_i$, $i = 1, 2$.

Since $f - g = (f_0 - g_0) + (f_1 - g_1)$, $f_i - g_i \in X_i$, i=1,2, we get

$$\|f - g\|_{X_0+X_1} \leq \|f_0 - g_0\|_{X_0} + \|f_1 - g_1\|_{X_1} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Consequently, $X_0 \cap X_1$ is dense in $X_0 + X_1$. □

Theorem 4.35 (cf. [3, Chap. V, Theorem 4.12, p. 339])**.** *If* $1 \leq p \leq \infty$, *then*

$$K(f, t; L_p(\mathbb{R}^n), W_p^1(\mathbb{R}^n)) \approx \min\{1, t\}\|f\|_p + \omega_1(f, t)_p$$

for all $f \in L_p(\mathbb{R}^n)$ *and all* $t \in (0, \infty)$.

5. Proof of Theorem 3.1

We claim that condition (4.95) of Lemma 4.30 is satisfied with $\theta = 0$. Indeed,

$$\|t^{-1/q'}a^{-1}(t)\min\{1, t^{-1}\}\|_{q',(0,\infty)} \leq \|t^{-1/q'}a^{-1}(t)\|_{q',(0,1)} + \|t^{-1-1/q'}a^{-1}(t)\|_{q',(1,\infty)} =: I_1 + I_2.$$

Note that $I_2 < \infty$ by Lemma 2.1 (i) and (iii). If $1 < q < \infty$, then $I_1 < \infty$ by Lemma 4.33 (i). If $q = 1$, then

$$a(x) = \int_x^\infty t^{-1}b(t)\,dt \quad \text{for all} \;\; x > 0. \tag{5.1}$$

Thus,

$$I_1 = \|a^{-1}(t)\|_{\infty,(0,1)} = \Big(\int_1^\infty t^{-1}b(t)\,dt\Big)^{-1} < \infty,$$

and our claim follows.

Applying Theorem 4.31 with $\theta = 0$, we obtain

$$(X_0, X_1)'_{0,q,a;J} \hookrightarrow (X'_1, X'_0)_{1,q',a^{-1};K}. \tag{5.2}$$

Now we claim that also condition (4.111) of Theorem 4.32 with $\theta = 0$ is satisfied. Indeed,

$$\|t^{1-1/q}b(t)\min\{1, t^{-1}\}\|_{q,(0,\infty)} \leq \|t^{1-1/q}b(t)\|_{q,(0,1)} + \|t^{-1/q}b(t)\|_{q,(1,\infty)}. \tag{5.3}$$

The first term on RHS(5.3) is finite by Lemma 2.1 (iii), while the second one by (3.1).

Using Theorem 4.32 with $\theta = 0$, we arrive at

$$(X'_1, X'_0)_{1,q',b^{-1};J} \hookrightarrow (X_0, X_1)'_{0,q,b;K}. \tag{5.4}$$

Further, by Theorem 4.3,

$$\text{RHS}(5.4) = (X_0, X_1)'_{0,q,b;K} = (X_0, X_1)'_{0,q,a;J} = \text{LHS}(5.2). \tag{5.5}$$

Now we are going to show that RHS(5.2)=LHS(5.4), i.e.,

$$(X'_1, X'_0)_{1,q',a^{-1};K} = (X'_1, X'_0)_{1,q',b^{-1};J}. \tag{5.6}$$

To verify (5.6), we consider two cases:

Let $1 < q < \infty$. Then we apply Theorem 4.24 with $B := a^{-1}, A := b^{-1}$, with q replaced by q', and with (X_0, X_1) replaced by (X_1', X_0'). Note that then (4.59) reads as (4.119), and (4.60) reads as(4.120). But (4.119) and (4.120) hold by Lemma 4.33 (i). Thus, (5.6) is true by Theorem 4.24.

If $q = 1$, then $q' = \infty$ and the function a is given by (5.1). Now we make use of Theorem 4.25 with $A := b^{-1}, B := a^{-1}$, with $q = \infty$ and with (X_0, X_1) replaced by (X_1', X_0'). Note that then (4.66) holds by our assumption (3.1), while (4.67) follows from (5.1). Thus, (5.6) holds by Theorem 4.25.

Using (5.5), (5.2), (5.6), and (5.4), we obtain

$$\begin{aligned}(X_0, X_1)'_{0,q,b;K} &= (X_0, X_1)'_{0,q,a;J} \hookrightarrow (X_1', X_0')_{1,q',a^{-1};K} \\ &= (X_1', X_0')_{1,q',b^{-1};J} \hookrightarrow (X_0, X_1)'_{0,q,b;K}.\end{aligned}$$

Consequently,

$$(X_0, X_1)'_{0,q,b;K} = (X_1', X_0')_{1,q',b^{-1};J} = (X_1', X_0')_{1,q',a^{-1};K}. \tag{5.7}$$

Furthermore, by Lemmas 4.23 and 4.22,

$$(X_1', X_0')_{1,q',b^{-1};J} = (X_0', X_1')_{0,q',\tilde{b};J} \quad \text{and} \quad (X_1', X_0')_{1,q',a^{-1};K} = (X_0', X_1')_{0,q',\tilde{a};K}.$$

Together with (5.7), this gives (3.3). □

6. Proof of Theorem 3.2

Since $1 < q < \infty$ and $a \in SV(0, \infty)$ satisfies

$$\int_0^x t^{-1} a^{-q'}(t)\, dt < \infty \quad \text{for all} \;\; x > 0, \qquad \int_0^\infty t^{-1} a^{-q'}(t)\, dt = \infty, \tag{6.1}$$

we can apply Theorem 4.5 to get

$$(X_0, X_1)'_{0,q,a;J} = (X_0, X_1)'_{0,q,b;K} \tag{6.2}$$

where

$$b(x) = a^{-q'/q}(x) \Big(\int_0^x t^{-1} a^{-q'}(t)\, dt \Big)^{-1} \quad \text{for all } x > 0. \tag{6.3}$$

Moreover, by Lemma 4.33 (ii),

$$\int_x^\infty t^{-1} b^q(t)\, dt < \infty \quad \text{for all} \;\; x > 0, \qquad \int_0^\infty t^{-1} b^q(t)\, dt = \infty, \tag{6.4}$$

and

$$a(x) \approx b^{-q/q'}(x) \int_x^\infty t^{-1} b^q(t)\, dt, \quad \text{for a.a.} \;\; x > 0.$$

Thus, by Theorem 3.1,

$$(X_0, X_1)'_{0,q,b;K} = (X_0', X_1')_{0,q',\tilde{b};J} = (X_0', X_1')_{0,\, q',\tilde{a};K}, \tag{6.5}$$

where where $\tilde{a}(x) := \frac{1}{a(1/x)}$ and $\tilde{b}(x) := \frac{1}{b(1/x)}$ for all $x > 0$. Now (6.2) and (6.5) imply that

$$(X_0, X_1)'_{0,q,a;J} = (X_0', X_1')_{0,q',\tilde{a};K} = (X_0', X_1')_{0,q',\tilde{b};J}.$$

□

7. Proof of Theorem 3.3

Let the function a satisfy the assumption of Theorem 3.3 and let $b \in SV(0,\infty)$ be given by (3.8). Then, by Theorem 4.8,

$$(X_0, X_1)_{0,1,a;J} = (X_0, X_1)_{0,1,b;K}. \tag{7.1}$$

Solving differential equation (3.8) and using the condition $a(\infty) = 0$, we arrive at

$$a(x) := \int_x^\infty t^{-1} b(t)\, dt \quad \text{for all } x > 0. \tag{7.2}$$

Moreover, condition (3.7) implies that the function b satisfies condition (3.1) with $q = 1$ of Theorem 3.1. Thus, on using this theorem, we arrive at

$$(X_0, X_1)'_{0,1,b;K} = (X'_0, X'_1)_{0,\infty,\tilde{b};J} = (X'_0, X'_1)_{0,\infty,\tilde{a};K}, \tag{7.3}$$

where $\tilde{b}(x) := \frac{1}{b(1/x)}$ and $\tilde{a}(x) := \frac{1}{a(1/x)}$ for all $x > 0$. Now, by (7.1) and (7.3),

$$(X_0, X_1)'_{0,1,a;J} = (X'_0, X'_1)_{0,\infty,\tilde{a};K} = (X'_0, X'_1)_{0,\infty,\tilde{b};J},$$

and the proof is complete. □

Remark 7.1. In the proof of Theorem 3.3 one can apply Theorem 4.3 instead of Theorem 4.8. Indeed, making use of (7.2) and the fact that condition (3.7) implies that the function b satisfies condition (4.5) with $q = 1$ of Theorem 4.3, one can get (7.1) from Theorem 4.3.

8. Proofs of Theorem 3.4 and Remark 3.5

I. Proof of Theorem 3.4. By Theorem 4.10,

$$(X_0, X_1)_{0,q,b;K} = (X_0, X_0 + X_1)_{0,q,B;K}. \tag{8.1}$$

The function B satisfies (cf. (3.11), (3.10), and (3.12))

$$\int_x^\infty t^{-1} B^q(t)\, dt < \infty \quad \text{for all} \;\; x > 0, \qquad \int_0^\infty t^{-1} B^q(t)\, dt = \infty. \tag{8.2}$$

Thus, using (8.1), Lemma 4.34, Theorem 3.1 (with (X_0, X_1) replaced by $(X_0, X_0 + X_1)$ and with B, A instead of b, a), and (2.16), we arrive at

$$\begin{aligned} (X_0, X_1)'_{0,q,b;K} &= (X_0, X_0 + X_1)'_{0,q,B;K} \\ &= (X'_0, (X_0 + X_1)')_{0,q',\tilde{B};J} = (X'_0, (X_0 + X_1)')_{0,q',\tilde{A};K} \\ &= (X'_0, X'_0 \cap X'_1)_{0,q',\tilde{B};J} = (X'_0, X'_0 \cap X'_1)_{0,q',\tilde{A};K}. \end{aligned} \tag{8.3}$$

Moreover, by Corollary 4.12,

$$(X_0, X_1)_{0,q,b;K} = X_0 + (X_0, X_1)_{0,q,B;K} = X_0 + (X_0, X_1)_{0,q,A;J}. \tag{8.4}$$

By Theorem 2.4 (i),

$$X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{0,q,B;K}.$$

Thus,

$$X_0 \cap X_1 \hookrightarrow X_0 \cap (X_0, X_1)_{0,q,B;K}. \tag{8.5}$$

By our assumption, $X_0 \cap X_1$ is dense in X_0, and, by Theorem 4.4 (with b replaced by B), $X_0 \cap X_1$ is dense in $(X_0, X_1)_{0,q,B;K}$. Together with embedding (8.5), this shows that

$X_0 \cap (X_0, X_1)_{0,q,B;K}$ is dense in X_0 and in $(X_0, X_1)_{0,q,B;K}$. Therefore, we can apply (2.16) (with X_0, X_1 replaced by $X_0, (X_0, X_1)_{0,q,B;K}$) to get that

$$(X_0 + (X_0, X_1)_{0,q,B;K})' = X_0' \cap (X_0, X_1)'_{0,q,B;K}. \tag{8.6}$$

Moreover, by Theorem 3.1 (with B and A instead of b and a),

$$(X_0, X_1)'_{0,q,B;K} = (X_0', X_1')_{0,q',\tilde{B};J}. \tag{8.7}$$

Using (8.4), (8.6), and (8.7), we arrive at

$$(X_0, X_1)'_{0,q,b;K} = X_0' \cap (X_0', X_1')_{0,q',\tilde{B};J}. \tag{8.8}$$

Let $1 < q < \infty$. Then the function A satisfies

$$\int_0^x t^{-1} A^{-q'}(t)\, dt < \infty \quad \text{for all} \;\; x > 0, \qquad \int_0^\infty t^{-1} A^{-q'}(t)\, dt = \infty, \tag{8.9}$$

which follows from Lemma 4.33 (i) (with b and a replaced by B and A). By Theorem 2.5 (i),

$$X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{0,q,A;J}.$$

Thus,

$$X_0 \cap X_1 \hookrightarrow X_0 \cap (X_0, X_1)_{0,q,A;J}. \tag{8.10}$$

By our assumption, $X_0 \cap X_1$ is dense in X_0, and, by Theorem 4.6 (with A instead of a), $X_0 \cap X_1$ is dense in $(X_0, X_1)_{0,q,A;J}$. Together with embedding (8.10), this shows that $X_0 \cap (X_0, X_1)_{0,q,A;J}$ is dense both in X_0 and in $(X_0, X_1)_{0,q,A;J}$. Therefore, we can apply (2.16) (with X_0, X_1 replaced by $X_0, (X_0, X_1)_{0,q,A;J}$) to get that

$$(X_0 + (X_0, X_1)_{0,q,A;J})' = X_0' \cap (X_0, X_1)'_{0,q,A;J}. \tag{8.11}$$

Moreover, by Theorem 3.2 (with A and B instead of a and b),

$$(X_0, X_1)'_{0,q,A;J} = (X_0', X_1')_{0,q',\tilde{A};K}. \tag{8.12}$$

Using (8.4), (8.11), and (8.12), we arrive at

$$(X_0, X_1)'_{0,q,b;K} = X_0' \cap (X_0', X_1')_{0,q',\tilde{A};K}. \tag{8.13}$$

Let $q = 1$. Then $q' = \infty$ and

$$A(x) = \int_x^\infty t^{-1} B(t)\, dt \quad \text{for all} \;\; x > 0.$$

Hence, $A \in SV(0,\infty) \cap AC(0,\infty)$ satisfies

$$A \text{ is strictly decreasing}, \quad A(0) = \infty, \quad A(\infty) = 0,$$

and

$$B(x) := -x\, A'(x) \quad \text{for a.a. } x > 0\,.$$

Consequently, by Theorem 4.9 (with A instead of a), $X_0 \cap X_1$ is dense in $(X_0, X_1)_{0,1,A;J}$. Therefore, as in the case that $1 < q < \infty$, we obtain that (8.11) holds with $q = 1$. Moreover, Theorem 3.3 (with A, and B instead of a, and b) implies that (8.12) holds with $q = 1$. Thus, on using (8.4), (8.11), and (8.12), we see that (8.13) remains true if $q = 1$.

Applying Theorem 4.13 (with X_0, X_1 replaced by X_0', X_1' and with q, a, α, and A replaced by $q', \tilde{b}, \tilde{\beta}$, and $\tilde{B}$, respectively [1]), we obtain

$$(X_0', X_1')_{0,q',\tilde{b};J} = (X_0', X_0' \cap X_1')_{0,q',\tilde{B};J} = (X_0', X_0' \cap X_1')_{0,q',\tilde{A};K},$$

which, together with (8.3), (8.8) and (8.13), implies that (3.14) holds. □

II. Proof of Remark 3.5. Equality (3.15) holds by (4.22), while (3.16)-(3.18) follow from Remark 4.14 (used with X_0', X_1' instead of X_0, X_1 and with $q', \tilde{b}, \tilde{B}$, and $\tilde{A}$ instead of q, a, A, and B, respectively). □

9. Proofs of Theorem 3.6 and Remark 3.7

I. Proof of Theorem 3.6. By Theorem 4.13,

$$(X_0, X_1)_{0,q,a;J} = (X_0, X_0 \cap X_1)_{0,q,A;J}. \tag{9.1}$$

The function A satisfies (cf. (3.20), (3.19), and (3.21))

$$\int_0^x t^{-1} A^{-q'}(t)\, dt < \infty \quad \text{for all} \quad x > 0, \qquad \int_0^\infty t^{-1} A^{-q'}(t)\, dt = \infty. \tag{9.2}$$

Thus, using (9.1), Theorem 3.2 (with (X_0, X_1) replaced by $(X_0, X_0 \cap X_1)$ and with A, B instead of a, b), and (2.16), we arrive at

$$\begin{aligned} (X_0, X_1)'_{0,q,a;J} &= (X_0, X_0 \cap X_1)'_{0,q,A;J} \\ &= (X_0', (X_0 \cap X_1)')_{0,q',\tilde{A};K} = (X_0', (X_0 \cap X_1)')_{0,q',\tilde{B};J} \\ &= (X_0', X_0' + X_1')_{0,q',\tilde{A};K} = (X_0', X_0' + X_1')_{0,q',\tilde{B};J}\,. \end{aligned} \tag{9.3}$$

Moreover, by Corollary 4.15,

$$(X_0, X_1)_{0,q,a;J} = X_0 \cap (X_0, X_1)_{0,q,A;J} = X_0 \cap (X_0, X_1)_{0,q,B;K}. \tag{9.4}$$

As in the proof of Theorem 3.4, one can show that $X_0 \cap (X_0, X_1)_{0,q,A;J}$ is dense both in X_0 and in $(X_0, X_1)_{0,q,A;J}$. Therefore, applying (2.16) (with X_0, X_1 replaced by $X_0, (X_0, X_1)_{0,q,A;J}$), we arrive at

$$(X_0 \cap (X_0, X_1)_{0,q,A;J})' = X_0' + (X_0, X_1)'_{0,q,A;J}. \tag{9.5}$$

Moreover, by Theorem 3.2 (with A and B instead of a and b), (8.12) holds. Using (9.4), (9.5), and (8.12), we arrive at

$$(X_0, X_1)'_{0,q,a;J} = X_0' + (X_0', X_1')_{0,q',\tilde{A};K}. \tag{9.6}$$

Making use of (9.2), (3.22), and Lemma 4.33 (ii) (with a and b replaced by A and B), we obtain that (8.2) holds. As in the proof of Theorem 3.4, one can show that $X_0 \cap (X_0, X_1)_{0,q,B;K}$ is dense both in X_0 and in $(X_0, X_1)_{0,q,B;K}$. Therefore, applying (2.16) (with X_0, X_1 replaced by $X_0, (X_0, X_1)_{0,q,B;K}$), we get

$$(X_0 \cap (X_0, X_1)_{0,q,B;K})' = X_0' + (X_0, X_1)'_{0,q,B;K}. \tag{9.7}$$

Moreover, by Theorem 3.1 (with B and A instead of b and a), (8.7) holds. Relations (9.4), (9.7), and (8.7) imply that

$$(X_0, X_1)'_{0,q,a;J} = X_0' + (X_0', X_1')_{0,q',\tilde{B};J}\,. \tag{9.8}$$

[1] Note that (3.13) implies

$$\tilde{A}(x) := (\tilde{B})^{-q/q'}(x) \left(\int_0^x t^{-1} (\tilde{B})^{-q}(t)\, dt \right)^{-1} \quad \text{for all} \quad x > 0,$$

which is (4.29) with the above mentioned replacement in Theorem 4.13.

Applying Theorem 4.10 (with X_0, X_1 replaced by X_0', X_1' and with q, b, β, and B replaced by $q', \tilde{a}, \tilde{\alpha}$, and $\tilde{A}$, respectively [2]), we obtain

$$(X_0', X_1')_{0,q',\tilde{a};K} = (X_0', X_0' + X_1')_{0,q',\tilde{A};K} = (X_0', X_0' + X_1')_{0,q',\tilde{B};J},$$

which, together with (9.3), (9.6), and (9.8) shows that (3.23) holds. □

II. Proof of Remark 3.7. Equality (3.24) holds by (4.31), while (3.25)-(3.27) follow from Remark 4.11 (used with X_0', X_1' instead of X_0, X_1 and with $q', \tilde{a}, \tilde{A}$, and $\tilde{B}$ instead of q, b, B, and A, respectively). □

10. Proof of Theorem 3.8 and and Remark 3.9

I. Proof of Theorem 3.8. By Theorem 4.16,

$$(X_0, X_1)_{0,1,a;J} = (X_0, X_0 \cap X_1)_{0,1,A;J}. \tag{10.1}$$

The function $A \in SV(0,\infty) \cap AC(0,\infty)$ satisfies

$$A \text{ is strictly decreasing}, \qquad A(0) = \infty, \qquad A(\infty) = 0. \tag{10.2}$$

Thus, using (10.1), Theorem 3.3 with (X_0, X_1) replaced by $(X_0, X_0 \cap X_1)$ and with A, B instead of a, b), and (2.16), we arrive at

$$\begin{aligned}(X_0, X_1)'_{0,1,a;J} &= (X_0, X_0 \cap X_1)'_{0,1,A;J} \\ &= (X_0', (X_0 \cap X_1)')_{0,\infty,\tilde{A};K} = (X_0', (X_0 \cap X_1)')_{0,\infty,\tilde{B};J} \\ &= (X_0', X_0' + X_1')_{0,\infty,\tilde{A};K} = (X_0', X_0' + X_1')_{0,\infty,\tilde{B};J}\,.\end{aligned} \tag{10.3}$$

Moreover, by Corollary 4.19,

$$(X_0, X_1)_{0,1,a;J} = X_0 \cap (X_0, X_1)_{0,1,A;J} = X_0 \cap (X_0, X_1)_{0,1,B;K}. \tag{10.4}$$

By Theorem 2.5 (i),

$$X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{0,1,A;J}.$$

Thus,

$$X_0 \cap X_1 \hookrightarrow X_0 \cap (X_0, X_1)_{0,1,A;J}. \tag{10.5}$$

By our assumption, $X_0 \cap X_1$ is dense in X_0, and, by Theorem 4.9 (with A instead of a), $X_0 \cap X_1$ is dense in $(X_0, X_1)_{0,q,A;J}$. Together with embedding (10.5), this shows that $X_0 \cap (X_0, X_1)_{0,1,A;J}$ is dense both in X_0 and in $(X_0, X_1)_{0,1,A;J}$. Therefore, we can apply (2.16) (with X_0, X_1 replaced by $X_0, (X_0, X_1)_{0,q,A;J}$) to get that

$$(X_0 \cap (X_0, X_1)_{0,1,A;J})' = X_0' + (X_0, X_1)'_{0,1,A;J}. \tag{10.6}$$

Moreover, by Theorem 3.3 (with A and B instead of a and b),

$$(X_0, X_1)'_{0,1,A;J} = (X_0', X_1')_{0,\infty,\tilde{A};K}. \tag{10.7}$$

Using (10.4), (10.6), and (10.7), we arrive at

$$(X_0, X_1)'_{0,1,A;J} = X_0' + (X_0', X_1')_{0,\infty,\tilde{A};K}. \tag{10.8}$$

---

[2] Note that (3.22) implies

$$\tilde{B}(x) := (\tilde{A})^{-q'/q}(x) \int_x^\infty t^{-1} (\tilde{A})^{q'}(t)\, dt \quad \text{for all } x > 0,$$

which is (4.20) with the above mentioned replacement in Theorem 4.10.

Solving differential equation (3.31) and using the condition $A(\infty)=0$, we obtain that

$$A(x) := \int_x^\infty t^{-1}B(t)\,dt \quad \text{for all } x>0, \tag{10.9}$$

which, together with the condition $A(0)=\infty$, implies that

$$\int_x^\infty t^{-1}B(t)\,dt < \infty \quad \text{for all } \ x>0, \qquad \int_0^\infty t^{-1}B(t)\,dt = \infty. \tag{10.10}$$

By Theorem 2.4 (i),

$$X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{0,1,B;K}.$$

Thus,

$$X_0 \cap X_1 \hookrightarrow X_0 \cap (X_0, X_1)_{0,1,B;K}. \tag{10.11}$$

By our assumption, $X_0 \cap X_1$ is dense in X_0, and, by Theorem 4.4 (with B instead of b), $X_0 \cap X_1$ is dense in $(X_0, X_1)_{0,1,B;K}$. Together with embedding (10.11), this shows that $X_0 \cap (X_0, X_1)_{0,1,B;K}$ is dense both in X_0 and in $(X_0, X_1)_{0,1,B;K}$. Therefore, we can apply (2.16) (with X_0, X_1 replaced by $X_0, (X_0, X_1)_{0,1,B;K}$) to get that

$$(X_0 \cap (X_0, X_1)_{0,1,B;K})' = X_0' + (X_0, X_1)'_{0,1,B;K}. \tag{10.12}$$

Moreover, by Theorem 3.1 (with B and A instead of b and a),

$$(X_0, X_1)'_{0,1,B;K} = (X_0', X_1')_{0,\infty,\tilde{B};J}. \tag{10.13}$$

Using (10.4), (10.12), and (10.13), we arrive at

$$(X_0, X_1)'_{0,1,a;J} = X_0' + (X_0', X_1')_{0,\infty,\tilde{B};J}. \tag{10.14}$$

Applying Theorem 4.20 (with X_0, X_1 replaced by X_0', X_1' and with b, β, B, and A replaced by $\tilde{a}, \tilde{\alpha}, \tilde{A}$, and $\tilde{B}$, respectively [3]),

$$(X_0', X_1')_{0,\infty,\tilde{a};K} = (X_0', X_0' + X_1')_{0,\infty,\tilde{A};K} = (X_0', X_0' + X_1')_{0,\infty,\tilde{B};J},$$

which, together with (10.3), (10.8), and (10.14), shows that (3.32) holds. ☐

II. Proof of Remark 3.9. Equality (3.33) holds by (4.40), while (3.34)-(3.36) follow from Remark 4.21 (with X_0, X_1 replaced by X_0', X_1' and with b, β, B, and A replaced by $\tilde{a}, \tilde{\alpha}, \tilde{A}$, and $\tilde{B}$, respectively). ☐

11. Proofs of Theorem 3.11

Assume that all the assumptions of Theorem 3.11 are satisfied. Let

$$X := (L_p, W_p^1)_{0,q,b;K}. \tag{11.1}$$

By Theorem 4.35,

$$K(f,t;L_p,W_p^1) \approx \min\{1,t\}\|f\|_p + \omega_1(f,t)_p$$

for all $f \in L_p$ and all $t \in (0,\infty)$. Thus, for all $f \in L_p$,

$$\begin{aligned} \|f\|_X &\approx \|t^{-1/q}b(t)\min\{1,t\}\|f\|_p\|_{q;(0,\infty)} + \|t^{-1/q}b(t)\,\omega_1(f,t)_p\|_{q;(0,\infty)} \\ &=: N_1 + N_2. \end{aligned} \tag{11.2}$$

---

[3] Note that (3.31) implies

$$\tilde{B}(x) := \frac{(\tilde{A})^2(x)}{x(-(\tilde{A})'(x))} \quad \text{for a.a. } x>0,$$

which is (4.47) with the above mentioned replacement in Theorem 4.20.

Moreover, $N_1 \approx N_{11} + N_{12}$, where, by Lemma 2.1 (iii),

$$N_{11} := \|f\|_p \cdot \|t^{1-1/q} b(t)\|_{q;(0,1)} \approx \|f\|_p \,,$$

and, by (3.37),

$$N_{12} := \|f\|_p \cdot \|t^{-1/q} b(t)\|_{q;(1,\infty)} \approx \|f\|_p.$$

Therefore,

$$N_1 \approx \|f\|_p. \tag{11.3}$$

Together with (11.2) and (11.3), this shows that the norm of the space X is equivalent to

$$\|\cdot\|_p + \|t^{-1/q} b(t)\, \omega_1(\cdot, t)_p\|_{q;(0,\infty)}.$$

This and (2.22) imply that $X = B^{0,b}_{p,q}$, which, together with (11.1), shows that

$$(L_p, W^1_p)_{0,q,b;K} = B^{0,b}_{p,q}.$$

Consequently,

$$(B^{0,b}_{p,q})' = (L_p, W^1_p)'_{0,q,b;K}. \tag{11.4}$$

Note that conditions (3.37) and (3.1) are equivalent. Thus, on using Theorem 3.1, (2.17), (2.20), we arrive at

$$(B^{0,b}_{p,q})' = (L_{p'}, H^{-1}_{p'})_{0,q',\tilde{a};K},$$

where the function $\tilde{a}(x) = \frac{1}{a(1/x)}$ for all $x > 0$ and a is given by (3.2).

Since the lift operator I_s with $s = -1$ satisfies (cf. (2.17), (2.19))

$$I_{-1} L_{p'} = I_{-1} H^0_{p'} = H^1_{p'} = W^1_{p'} \quad \text{and} \quad I_{-1} H^{-1}_{p'} = H^0_{p'} = L_{p'},$$

and, together with its inverse, it is bounded, we get that

$$K(f, t; L_{p'}, H^{-1}_{p'}) \approx K(I_{-1} f, t; W^1_{p'}, L_{p'})$$

for all $f \in H^{-1}_{p'}$ and all $t > 0$. Moreover, by (4.1) and Theorem 4.35, we obtain

$$\begin{aligned}
K(I_{-1} f, t; W^1_{p'}, L_{p'}) &= t\, K(I_{-1} f, 1/t; L_{p'}, W^1_{p'}) \\
&\approx t\,(\min\{1, 1/t\} \|I_{-1} f\|_{p'} + \omega_1(I_{-1} f, 1/t)_{p'}) \\
&= \min\{t, 1\} \|I_{-1} f\|_{p'} + t\, \omega_1(I_{-1} f, 1/t)_{p'}
\end{aligned}$$

for all $f \in H^{-1}_{p'}$ and all $t > 0$. Therefore,

$$\begin{aligned}
\|f\|_{(B^{0,b}_{p,q})'} &= \|f\|_{(L_{p'}, H^{-1}_{p'})_{0,q',\tilde{a};K}} = \|t^{-1/q'} \tilde{a}(t) K(f, t; L_{p'}, H^{-1}_{p'})\|_{q',(0,\infty)} \\
&\approx \|t^{-1/q'} \tilde{a}(t) \min\{t, 1\} \|I_{-1} f\|_{p'}\|_{q',(0,\infty)} + \|t^{-1/q'} \tilde{a}(t)\, t\, \omega_1(I_{-1} f, 1/t)_{p'}\|_{q',(0,\infty)} \\
&=: V_1 + V_2.
\end{aligned}$$

Let $1 < q < \infty$. By Lemma 4.33,

$$\int_0^x t^{-1} a^{-q'}(t)\, dt < \infty \quad \text{for all} \;\; x > 0 \qquad \text{and} \quad \int_0^\infty t^{-1} a^{-q'}(t)\, dt = \infty. \tag{11.5}$$

Making use of (11.5), one can easily verify that

$$\int_0^1 t^{-1} \tilde{a}^{q'}(t)\, dt = \int_1^\infty \tau^{-1} a^{-q'}(\tau)\, d\tau = \infty \tag{11.6}$$

and

$$\int_1^\infty t^{-1}\tilde{a}^{q'}(t)\,dt = \int_0^1 \tau^{-1}a^{-q'}(\tau)\,d\tau < \infty. \tag{11.7}$$

Applying properties of slowly varying functions and (11.7), we get

$$V_1 \approx \|t^{1-1/q'}\tilde{a}(t)\|_{q',(0,1)}\|I_{-1}f\|_{p'} + \|t^{-1/q'}\tilde{a}(t)\|_{q',(1,\infty)}\|I_{-1}f\|_{p'} \approx \|I_{-1}f\|_{p'}, \tag{11.8}$$

$$\begin{aligned} V_2 &\approx \|t^{1-1/q'}\tilde{a}(t)\omega_1(I_{-1}f,1/t)_{p'}\|_{q',(0,1)} + \|t^{1-1/q'}\tilde{a}(t)\omega_1(I_{-1}f,1/t)_{p'}\|_{q',(1,\infty)} \\ &=: V_{21}+V_{22}, \end{aligned} \tag{11.9}$$

$$V_{21} \lesssim \|t^{1-1/q'}\tilde{a}(t)\|_{q',(0,1)}\|I_{-1}f\|_{p'} \approx \|I_{-1}f\|_{p'}. \tag{11.10}$$

Using the change of variable $t = 1/\tau$ and the fact that $\tilde{a}(t) = \frac{1}{a(1/t)}$ for all $t > 0$, we obtain

$$V_{22} = \left\| \tau^{-1/q'} \frac{\omega_1(I_{-1}f,\tau)_{p'}}{\tau\, a(\tau)} \right\|_{q',(0,1)}. \tag{11.11}$$

If $q = 1$, then $q' = \infty$. Thus, by (3.2), $a(x) = \int_x^\infty t^{-1}b(t)\,dt$ for all $x > 0$. This implies that the function a is decreasing on $(0,\infty)$. Hence, the function $\tilde{a}$ is decreasing on $(0,\infty)$ as well. Therefore,

$$\|t^{-1/q'}\tilde{a}(t)\|_{q',(1,\infty)} = \|\tilde{a}(t)\|_{\infty,(1,\infty)} \approx \tilde{a}(1) < \infty,$$

and one can easily verify that (11.8)-(11.11) remain true.

Consequently, for all $q \in [1,\infty)$,

$$\|f\|_{(B^{0,b}_{p,q})'} \approx \|I_{-1}f\|_{p'} + \left\| \tau^{-1/q'} \frac{\omega_1(I_{-1}f,\tau)_{p'}}{\tau\, a(\tau)} \right\|_{q',(0,1)}$$

and the result follows. □

12. Concluding remarks

Note that in [26] discrete versions of K- and J-spaces are defined and then in [26, Theorems 3.1 and 3.2] duals of these spaces are described. Thus, there is another way how to prove some results mentioned in Section 3 of our paper. For example, to prove the first equality mentioned in (3.3) of Theorem 3.1, one can proceed as follows:

1) Use Lemma 4.27 to rewrite the given space $(X_0, X_1)_{0,q,b;K}$ in the discrete form.
2) Apply results [26, Theorems 3.1] to determine the dual of the given discrete K-space. (Note that the dual is expressed as a discrete J-space.)
3) Use Lemma 4.28 to rewrite this discrete J-space in the continuous form.

Note that such a method was used in [24, Theorem 5.6], where the particular case of Theorem 3.1, with b of a logarithmic form, was proved.

References

[1] I. Ahmed, D. E. Edmunds, W.D. Evans, and G. E. Karadzhov, *Reiteration theorems for the K-interpolation method in limiting cases*, Math. Nachr. 284(4) (2011), 421–442.

[2] I. Ahmed, A. Fiorenza, and A. Hafeez, *Some interpolation formulae for grand and small Lorentz spaces*, Mediterr. J. Math. 17(2) (2020), Art. 57, 21 pp.

[3] C. Bennett and R. Sharpley, “Interpolation of Operators”, Academic Press, New York, 1988.

[4] J. Bergh and J. Löfström, “Interpolation spaces, An introduction”, Springer-Verlag, Berlin-New York, 1976.

[5] B. F. Besoy, F. Cobos, and L. M. Fernández-Cabrera, *On the Structure of a Limit Class of Logarithmic Interpolation Spaces*, Mediterr. J. Math. 17(5) (2020), Art. 168, 17 pp.

[6] B. F. Besoy and F. Cobos, *Duality for logarithmic interpolation spaces when* $0 < q < 1$ *and applications*, J. Math. Anal. Appl. 466(1) (2018), 373–399.

[7] N. H. Bingham, C. M. Goldie, and J. L. Teugels, "Regular Variation", Cambridge University Press, Cambridge, 1987.

[8] Yu. A. Brudnyi and N. Ya. Krugljak, "Interpolation Functors and Interpolation Spaces", North-Holland Publishing Co., Amsterdam, 1991.

[9] P. L. Butzer and H. Berens, "Semi-Groups of Operators and Approximation", Springer-Verlag New York Inc., New York, 1967.

[10] A. M. Caetano, A. Gogatishvili, and B. Opic, *Sharp embeddings of Besov spaces involving only logarithmic smoothness*, J. Approx. Theory 152(2) (2008), 188–214.

[11] A. M. Caetano, A. Gogatishvili, and B. Opic, *Embeddings and the growth envelope of Besov spaces involving only slowly varying smoothness*, J. Approx. Theory 163(10) (2011), 1373–1399.

[12] F. Cobos and Ó. Domínguez, *Embeddings of Besov spaces of logarithmic smoothness*, Studia Math. 223(3) (2014), 193–204.

[13] F. Cobos and Ó. Domínguez, *Approximation spaces, limiting interpolation and Besov spaces*, J. Approx. Theory 189 (2015), 43–66.

[14] F. Cobos and Ó. Domínguez, *On Besov spaces of logarithmic smoothness and Lipschitz spaces*, J. Math. Anal. Appl. 425(1) (2015), 71–84.

[15] F. Cobos and Ó. Domínguez, *On the relationship between two kinds of Besov spaces with smoothness near zero and some other applications of limiting interpolation*, J. Fourier Anal. Appl. 22(5) (2016), 1174–1191.

[16] F. Cobos, Ó. Domínguez, and T. Kühn, *Approximation and entropy numbers of embeddings between approximation spaces*, Constr. Approx. 47(3) (2018), 453–486.

[17] F. Cobos, Ó. Domínguez, and H. Triebel, *Characterizations of logarithmic Besov spaces in terms of differences, Fourier-analytical decompositions, wavelets and semi-groups*, J. Funct. Anal. 270, (2016), 4386–4425.

[18] F. Cobos and L. M. Fernández-Cabrera, *Duality for logarithmic interpolation spaces and applications to Besov spaces*, Banach Center Publ. 119 (2019), 109–122.

[19] F. Cobos, L. M. Fernández-Cabrera, and M. Grover, *Measure of non-compactness and limiting interpolation with slowly varying functions*, Banach J. Math. Anal. (2024), Paper No. 25, 28 pp.

[20] F. Cobos, L. M. Fernández-Cabrera, T. Kühn, and T. Ullrich, *On an extreme class of real interpolation spaces*, J. Funct. Anal. 256(7) (2009), 2321–2366.

[21] F. Cobos, L. M. Fernández-Cabrera, and A. Martínez, *On a paper of Edmunds and Opic on limiting interpolation of compact operators between* L_p *spaces*, Math. Nachr. 288(2–3) (2015), 167–175.

[22] F. Cobos, L. M. Fernández-Cabrera, and M. Mastyło, *Abstract limit* J*-spaces*, J. Lond. Math. Soc. 82(2) (2010), 501–525.

[23] F. Cobos, T. Kühn, *Equivalence of* K*- and* J*-methods for limiting real interpolation spaces*, J. Funct. Anal. 261(12) (2011), 3696–3722.

[24] F. Cobos, A. Segurado, *Description of logarithmic interpolation spaces by means of the* J*-functional and applications*, J. Funct. Anal. 268(10) (2015), 2906–2945.

[25] F. Colombini, D. Del Santo, F. Fanelli, and G. Métivier, *The well-posedness issue in Sobolev spaces for hyperbolic systems with Zygmund-type coefficients*, Comm. Partial Differential Equations 40 (2015), 2082–2121.

[26] M. Cwikel, J. Peetre, *Abstract* K *and* J *spaces*, J. Math. Pures Appl. 60 (1981), 1–50.

[27] R. A. DeVore, S. D. Riemenschneider, and R. C. Sharpley, *Weak interpolation in Banach spaces*, J. Funct. Anal. 33(1) (1979), 58–94.

[28] R. Ya. Doktorskiĭ, *Reiterative relations of the real interpolation method*, Dokl. Akad. Nauk SSSR 321(2) (1991), 241–245; translation in Soviet Math. Dokl. 44 (1992), no. 3, 665–669.

[29] R. Ya. Doktorskiĭ, *An application of limiting interpolation to Fourier series theory*, In: "The diversity and beauty of applied operator theory", A. Böttcher, D. Potts, P. Stollmann, and D. Wenzel (eds.), Oper. Theory Adv. Appl. 268, Birkhäuser/Springer, Cham, 2018, 179–191.

[30] Ó. Domínguez, *Tractable embeddings of Besov spaces into small Lebesgue spaces*, Math. Nachr. 289(14-15) (2016), 1739–1759.

[31] Ó. Domínguez, *Ul'yanov-type inequalities and embeddings between Besov spaces: the case of parameters with limit values*, Math. Inequal. Appl. 20(3) (2017), 755–772.
[32] Ó. Domínguez and S. Tikhonov, *Function spaces of logarithmic smoothness: embeddings and characterizations*, Mem. Amer. Math. Soc. 282 (2023), no. 1393, vii+166 pp.
[33] D. E. Edmunds and W. D. Evans, "Hardy Operators, Function Spaces and Embeddings", Springer-Verlag, Berlin, Heidelberg, 2004.
[34] D. E. Edmunds, R. Kerman, and L. Pick, *Optimal Sobolev Imbeddings Involving Rearrangement-Invariant Quasinorms*, J. Funct. Anal. 170 (2000), 307–355.
[35] D. E. Edmunds and B. Opic *Limiting variants of Krasnosel'skiĭ's compact interpolation theorem*, J. Funct. Anal. 266(5) (2014), 3265–3285.
[36] W. D. Evans and B. Opic, *Real interpolation with logarithmic functors and reiteration*, Canad. J. Math. 52 (2000), 920–960.
[37] W. D. Evans, B. Opic, and L. Pick, *Real Interpolation with Logarithmic Functors*, J. Inequal. Appl. 7(2) (2002), 187–269.
[38] P. Fernández-Martínez and T. Signes, *Real interpolation with symmetric spaces and slowly varying functions*, Q. J. Math. 63(1) (2012), 133–164.
[39] P. Fernández-Martínez and T. Signes, *Reiteration theorems with extreme values of parameters*, Ark. Mat. 52(2) (2014), 227–256.
[40] P. Fernández-Martínez and T. Signes, *Limit cases of reiteration theorems*, Math. Nachr. 288(1) (2015), 25–47.
[41] A. Fiorenza, M. R. Formica, A. Gogatishvili, T. Kopaliani, and J. M. Rakotoson, *Characterization of interpolation between grand, small or classical Lebesgue spaces*, Nonlinear Anal. 177(B) (2018), 422–453.
[42] A. Gogatishvili, J. S. Neves, and B. Opic, *Optimal embeddings of Bessel-potential-type spaces into generalized Hölder spaces involving k-modulus of smoothness*, Potential Anal. 32(3) (2010), 201–228.
[43] A. Gogatishvili, B. Opic, S. Tikhonov, and W. Trebels, *Ulyanov-type inequalities between Lorentz-Zygmund spaces*, J. Fourier Anal. Appl. 20(5) (2014), 1020–1049.
[44] A. Gogatishvili, B. Opic, and W. Trebels, *Limiting reiteration for real interpolation with slowly varying functions*, Math. Nachr. 278 (2005), 86–10.
[45] M. E. Gomez, M. Milman, *Extrapolation spaces and almost-everywhere convergence of singular integrals*, J. London Math. Soc. 34(2) (1986), 305–316.
[46] J. Gustavsson, *A function parameter in connection with interpolation of Banach spaces*, Math. Scand. 42(2) (1978), 289–305.
[47] S. G. Kreĭn, Ju. I. Petunin and E. M. Semënov, "Interpolation of linear operators", (Russian), Nauka, Moscow, 1978.
[48] J.-L. Lions, *Sur les espaces d'interpolation: dualité*, Math. Scand. 9 (1961), 147–177.
[49] J.-L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math. 19 (1964), 5–68.
[50] A. Lunardi, "Interpolation theory", Edizioni della Normale, Pisa, 2018.
[51] L. Maligranda, *The K-functional for symmetric spaces*, In: "Interpolation spaces and allied topics in analysis", M. Cwikel and J. Peetre (eds.), Springer-Verlag, Berlin, 1984, 169–182.
[52] V. Marić, "Regular Variation and Differential Equations", Springer-Verlag, Berlin, 2000.
[53] M. Milman, "Extrapolation and optimal decompositions with applications to analysis", Springer-Verlag, Berlin, 1994.
[54] J. S. Neves, *Lorentz-Karamata spaces, Bessel and Riesz potentials and embeddings*, Dissertationes Math. 405 (2002), 46 pp.
[55] J. S. Neves and B. Opic, *Optimal local embeddings of Besov spaces involving only slowly varying smoothness*, J. Approx. Theory 254 (2020), (105393), 25 pp.
[56] B. Opic and M. Grover, *Description of K-spaces by means of J-spaces and the reverse problem in the limiting interpolation*, Math. Nachr. 296 (2023), 4002–4031.
[57] B. Opic and M. Grover, *Relationship between limiting K-spaces and J-spaces in the real interpolation*, arXiv:2501.08965 [math.FA].
https://doi.org/10.48550/arXiv.2501.08965
[58] B. Opic and A. Kufner, "Hardy-type inequalities", Pitman Research Notes in Math., Series 219, Longman Sci. & Tech., Harlow, 1990.
[59] L.E. Persson, *Interpolation with a parameter function*, Math. Scand. 59 (1986), 199–222.

[60] H. Triebel, "Interpolation theory, function spaces, differential operators", North-Holland Publishing Co., Amsterdam-New York, 1978.
[61] H. Triebel, "Theory of function spaces II", Birkhäuser Verlag, Basel, 1992.
[62] A. Zygmund, "Trigonometric Series", volume I, Cambridge University Press, Cambridge, 1957.

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