ON THE GEOMETRY OF SINGULAR EPW CUBES

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Abstract. EPW cubes form a locally complete family of smooth projective hyper-Kähler varieties of dimension 6, constructed by Iliev–Kapustka–Kapustka–Ranestad. Their construction and behavior share a lot of similarities with the double EPW sextics constructed by O'Grady. Adapting the methods of O'Grady, we construct a projective smooth small resolution of singular EPW cubes.

Contents

1. Introduction

Very few geometric constructions of locally complete families of smooth projective hyper-Kähler varieties are known. One of these families is that of so-called EPW cubes, constructed by Iliev–Kapustka–Kapustka–Ranestad in [?]. Their construction goes as follows.

Let V_6 be a 6-dimensional complex vector space. We endow $\bigwedge^3 V_6$ with the conformal symplectic structure given by wedge product and we denote by $\mathrm{LG}(\bigwedge^3 V_6)$ the Lagrangian Grassmannian that parametrizes Lagrangian subspaces $A \subset \bigwedge^3 V_6$. For any $[A] \in \mathrm{LG}(\bigwedge^3 V_6)$ and nonnegative integer k, the authors of [?] define the degeneracy loci

$$\mathsf{Z}_A^{\geq k} := \{[U_3] \in \mathrm{Gr}(3, V_6) \mid \dim(A \cap (\textstyle\bigwedge^2 U_3 \wedge V_6)) \geq k\}, \quad \mathsf{Z}_A^k := \mathsf{Z}_A^{\geq k} \smallsetminus \mathsf{Z}_A^{\geq k+1}$$

in the Grassmannian $\mathrm{Gr}(3, V_6)$.

When A has no decomposable vectors (this means that the intersection $\mathbf{P}(A) \cap \mathrm{Gr}(3, V_6)$, inside $\mathbf{P}(\bigwedge^3 V_6)$, is empty, and happens exactly when $[A]$ is outside an irreducible divisor $\Sigma \subset \mathrm{LG}(\bigwedge^3 V_6)$), the scheme $\mathsf{Z}_A^{\geq 2}$ is an integral normal sixfold whose singular locus is the threefold $\mathsf{Z}_A^{\geq 3}$. Moreover, the locus $\mathsf{Z}_A^{\geq 4}$ is finite, and it is empty for $[A]$ outside another irreducible divisor $\Gamma \subset \mathrm{LG}(\bigwedge^3 V_6)$.

In [?, Theorem 5.7], when $[A] \notin \Sigma$ (so that A has no decomposable vectors), the authors constructed a double cover

$$g\colon \widetilde{\mathsf{Z}}_A^{\geq 2} \longrightarrow \mathsf{Z}_A^{\geq 2} \tag{1}$$

branched over $\mathsf{Z}_A^{\geq 3}$. When moreover $[A] \notin \Gamma$ (namely when $\mathsf{Z}_A^{\geq 4} = \varnothing$), the authors of [?] prove that *the scheme* $\widetilde{\mathsf{Z}}_A^{\geq 2}$ *is a smooth hyper-Kähler variety of* $\mathrm{K3}^{[3]}$*-type with a polarization of square* 4 *and divisibility* 2, called an EPW cube. This is an important result because, as explained above, there are very few explicit constructions of this type. It justifies a more extensive study of this construction.

Date: July 14, 2025.

2020 *Mathematics Subject Classification.* 14J10, 14J42.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Project HyperK — grant agreement 854361).

The aim of this paper is to study the singular sixfold $\widetilde{\mathsf{Z}}_A^{\geq 2}$ when $[A] \in \Gamma \setminus \Sigma$. Adapting the methods of [?] (where the author worked with double EPW sextics), we show the following:

(a) The scheme $\mathsf{Z}_A^{\geq 4} = \{z_1, \dots, z_r\}$ is finite and smooth, and equals Z_A^4; moreover, if $[A]$ is general in Γ, then Z_A^4 is a singleton ($r = 1$).
(b) The singular set of the sixfold $\widetilde{\mathsf{Z}}_A^{\geq 2}$ is equal to $g^{-1}(\mathsf{Z}_A^4) = \{g^{-1}(z_1), \dots, g^{-1}(z_r)\}$. For each $i \in \{1, \dots, r\}$, the tangent cone to $\widetilde{\mathsf{Z}}_A^{\geq 2}$ at the point $g^{-1}(z_i)$ is isomorphic to the (affine) cone over the incidence variety $I \subset \mathbf{P}^3 \times (\mathbf{P}^3)^\vee$. Therefore, the exceptional divisor of the blowup $\widetilde{X}_A \to \widetilde{\mathsf{Z}}_A^{\geq 2}$ along $g^{-1}(\mathsf{Z}_A^4)$ is the disjoint union of smooth divisors $E_1, \dots, E_r$, all isomorphic to I and with normal bundles $\mathscr{O}_{E_i}(-1, -1)$. For any choice $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r)$ of contractions of each E_i onto either $\mathbf{P}^3$ or $(\mathbf{P}^3)^\vee$, we obtain an (analytic) small resolution $\mathsf{X}_A^\varepsilon \to \widetilde{\mathsf{Z}}_A^{\geq 2}$ with exceptional locus a disjoint union of r copies of $\mathbf{P}^3$.
(c) There exists a choice of ε such that X_A^ε is a projective smooth quasi-polarized hyper-Kähler sixfold with a projective contraction $\mathsf{X}_A^\varepsilon \to \widetilde{\mathsf{Z}}_A^{\geq 2}$ of r copies of $\mathbf{P}^3$.

Concerning Item (a), the finiteness and smoothness of $\mathsf{Z}_A^{\geq 4}$ are proved in Theorem **??** by generalizing the argument of [?, Lemma 2.8]. In Section **??**, following [?, Section 2.1], we show that $\mathsf{Z}_A^{\geq 4}$ consists of a single point for $[A]$ general in Γ. Regarding Item (b), we show in Proposition **??** and Lemma **??** that, analytically, the singularity of $\mathsf{Z}_A^{\geq 2}$ at a point of Z_A^4 is the cone over a general hyperplane section, in the space of (4×4)-matrices, of the subscheme $\mathscr{S}_{\leq 2}$ of symmetric matrices of rank ≤ 2. By the uniqueness of the double covers of degeneracy loci constructed in [?, Theorem 3.1], the singularity of $\widetilde{\mathsf{Z}}_A^{\geq 2}$ at a point of $g^{-1}(\mathsf{Z}_A^4)$ is the singularity of the "universal double cover" of this general hyperplane section of $\mathscr{S}_{\leq 2}$, studied in details in Section **??** (in other words, it is the cone over the incidence variety I mentioned above).

The construction of the small analytic resolutions $\mathsf{X}_A^\varepsilon \to \widetilde{\mathsf{Z}}_A^{\geq 2}$ is analogous to the one in [?, Sections 3.1 and 3.2]. We do it in Section **??**.

The most subtle part of our results is Item (c), that is, proving that there exists a small *projective* resolution of $\widetilde{\mathsf{Z}}_A^{\geq 2}$ (as in the double EPW sextic case, if Z_A^4 has more than one point, we do not expect the small resolutions $\mathsf{X}_A^\varepsilon \to \widetilde{\mathsf{Z}}_A^{\geq 2}$ to be all projective – that is, Kähler). For double EPW sextics, this was done by O'Grady by explicitly constructing a projective K3 surface S_A and (under a mild generality hypothesis) an isomorphism between the Hilbert square $S_A^{[2]}$ and one of these analytic resolutions. We cannot hope for an analogous construction in our case, because there are no associated K3 surfaces, and Y_A cannot be realized, even birationally, as a moduli space of sheaves over a K3 surface (we prove this in Proposition **??**).

However, using the properties of the period map for EPW cubes and the surjectivity of the period map for hyper-Kähler sixfolds, we produce in Proposition **??** a smooth projective hyper-Kähler sixfold Y_A with a big and nef line bundle H that induces a small contraction $\mathsf{Y}_A \to \widetilde{\mathsf{Z}}_A^{\geq 2}$; the pair (Y_A, H) is a deformation of standard (smooth) EPW cubes. Finally, using results from [?], we show in Theorem **??** that Y_A is actually a divisorial contraction of the blowup $\widetilde{\mathsf{X}}_A$, hence one of the small analytic resolutions X_A^ε.

In Appendix **??**, we describe the Heegner divisors in the period domain of hyper-Kähler varieties of $\mathrm{K3}^{[3]}$-type with polarization of square 4 and divisibility 2, and show that the divisors Γ and Σ of $\mathrm{LG}(\bigwedge^3 V_6)$ map onto distinct Heegner divisors.

Acknowledgment. I would like to express my deep gratitude to my advisor, Olivier Debarre, for suggesting this problem to me and for his guidance. I thank Alexander Kuznetsov and Claire Voisin for their help with Theorem **??**. I also thank Pietro Beri, Francesco Denisi, Franco Giovenzana, Alexander Kuznetsov, and Rita Pardini for useful discussions.

2. The degeneracy loci $\mathsf{Z}_A^{\geq k}$

The following description of the degeneracy loci $\mathsf{Z}_A^{\geq k} \subset \mathrm{Gr}(3, V_6)$ associated with a Lagrangian subspace $A \subset \bigwedge^3 V_6$ with no decomposable vectors, namely vectors of the form $v_1 \wedge v_2 \wedge v_3$, is stated in [**?**, Theorem 5.6].

Theorem 2.1. *If a Lagrangian subspace $A \subset \bigwedge^3 V_6$ has no decomposable vectors, the following properties hold.*

(a) $\mathsf{Z}_A^{\geq 1}$ *is an integral normal quartic hypersurface in* $\mathrm{Gr}(3, V_6)$.
(b) $\mathsf{Z}_A^{\geq 2}$ *is the singular locus of* $\mathsf{Z}_A^{\geq 1}$*; it is an integral normal Cohen–Macaulay sixfold of degree* 480.
(c) $\mathsf{Z}_A^{\geq 3}$ *is the singular locus of* $\mathsf{Z}_A^{\geq 2}$*; it is an integral normal Cohen–Macaulay threefold of degree* 4944.
(d) $\mathsf{Z}_A^{\geq 4}$ *is the singular locus of* $\mathsf{Z}_A^{\geq 3}$*; it is finite and smooth, and is empty for A general.*
(e) $\mathsf{Z}_A^{\geq 5}$ *is empty.*

However, in the reference given in [**?**] for this result ([**?**, Proposition 2.6 and Corollary 2.10]), the smoothness of Z_A^{k} is only studied for $k \in \{1, 2, 3\}$, so (d) and (e) need to be proven. This is the aim of this section.

We fix some notation. Let $[U_0]$ be a point of $\mathrm{Gr}(3, V_6)$. We set

$$T_{U_0} := \bigwedge^2 U_0 \wedge V_6. \tag{2}$$

This is a (10-dimensional) Lagrangian subspace of $\bigwedge^3 V_6$ and the projective tangent space to $\mathrm{Gr}(3, V_6)$ at $[U_0]$ is $\mathbf{P}(T_{U_0}) \subset \mathbf{P}(\bigwedge^3 V_6)$. If we choose a subspace $U_\infty \subset V_6$ complementary to U_0, then T_{U_0} decomposes as $\bigwedge^3 U_0 \oplus \mathrm{Hom}(U_0, U_\infty)$ and the Zariski tangent space to $\mathrm{Gr}(3, V_6)$ at $[U_0]$ is isomorphic to $\mathrm{Hom}(U_0, U_\infty)$.

We also set

$$\mathscr{C}_{U_0} := \mathbf{P}(T_{U_0}) \cap \mathrm{Gr}(3, V_6),$$

the intersection of $\mathbf{P}(T_{U_0})$ with the space of decomposable vectors of $\mathrm{LG}(\bigwedge^3 V_6)$. This is a 5-dimensional cone with vertex $[\bigwedge^3 U_0]$ over the smooth Segre fourfold

$$\mathscr{M}_1 \simeq \mathbf{P}(U_0^\vee) \times \mathbf{P}(U_\infty) \subset \mathbf{P}(\mathrm{Hom}(U_0, U_\infty))$$

of morphisms of rank 1.

Proof (of Theorem **??***(e)).* This is very easy to prove. Assume $[U_0]$ is in $\mathsf{Z}_A^{\geq 5}$, so that $A \cap T_{U_0}$ has dimension at least 5. Then, $\mathbf{P}(A \cap T_{U_0})$ has dimension at least 4, hence must meet the fivefold $\mathscr{C}_{U_0}$ in $\mathbf{P}(T_{U_0}) = \mathbf{P}^9$, which contradicts the fact that A has no decomposable vectors. Therefore, $\mathsf{Z}_A^{\geq 5}$ is empty.

It remains to prove Theorem **??**(d). This will be done in Section **??**. Before that, we prove general preliminary results that will also be used later on in this paper. ☐

2.1. **Tangent space to** $\mathrm{Gr}(3,V_6)$**.** In this section, we study the map $T\colon \mathrm{Gr}(3,V_6)\to \mathrm{LG}(\bigwedge^3 V_6)$ that takes a point $[U]$ to the Lagrangian subspace $[T_U]$.

Proposition 2.2. *The tangent map to T at a point $[U_0]$ gives an identification*

$$\Theta\colon T_{\mathrm{Gr}(3,V_6),[U_0]} \xrightarrow{\sim} H^0(\mathbf{P}(T_{U_0}), I_{\mathscr{C}_{U_0}}(2))$$

of 9-dimensional vector spaces, where $I_{\mathscr{C}_{U_0}}$ is the ideal sheaf of the subscheme $\mathscr{C}_{U_0}\subset \mathbf{P}(T_{U_0})$.

Proof. Consider the affine open subsets

$$\mathscr{U} := \{[U]\in \mathrm{Gr}(3,V_6)\mid U\cap U_\infty = 0\}\subset \mathrm{Gr}(3,V_6),$$

neighborhood of $[U_0]$, and

$$\mathscr{V} := \{[A]\in \mathrm{LG}(\textstyle\bigwedge^3 V_6)\mid A\cap T_{U_\infty} = 0\}\subset \mathrm{LG}(\textstyle\bigwedge^3 V_6).$$

We get identifications $\mathrm{Hom}(U_0,U_\infty)\simeq \mathscr{U}$ and $\mathrm{Sym}^2 T_{U_0}^\vee \simeq \mathscr{V}$ by sending a morphism to its graph and a quadratic form on T_{U_0}, viewed as a symmetric morphism $T_{U_0}\to T_{U_0}^\vee$, to the graph of the composition of this morphism with the inverse of the isomorphism $T_{U_\infty}\xrightarrow{\sim} T_{U_0}^\vee$ given by the symplectic form.

The morphism $T|_{\mathscr{U}}\colon \mathscr{U}\to\mathscr{V}$ then corresponds to a morphism

$$\begin{aligned}\mathrm{Hom}(U_0,U_\infty) &\longrightarrow \mathrm{Sym}^2(T_{U_0}^\vee)\\ [U] &\longmapsto [q_U],\end{aligned}$$

where q_U is described in [**?**, Lemma 2.7] as a polynomial in the entries of the matrix $(b_{ij})_{1\le i,j\le 3}$ representing U in fixed bases of U_0 and U_∞. In particular, if we write an element of $T_{U_0} = \bigwedge^3 U_0\oplus \mathrm{Hom}(U_0,U_\infty)$ as (m,M), the linear part in the b_{ij} of $q_U(m,M)$ is given by $\sum_{ij} b_{i,j}M^{i,j}$, where $M^{i,j}$ is the determinant of the 2×2 submatrix of M in which we remove the column i and the row j.

Therefore, the image of the tangent space $T_{\mathscr{U},[U_0]}$ by $T_{\Theta,[U_0]}$ is generated by the cofactors $M^{i,j}$, namely, this image is, in $\mathrm{Sym}^2(T_{U_0}^\vee) = H^0(\mathbf{P}(T_{U_0}),\mathscr{O}(2))$, the linear subsystem of quadrics in $\mathbf{P}(T_{U_0})$ that vanish on the set of matrices of rank 1. ☐

2.2. **The restriction map** r_K**.** Let $A\subset \bigwedge^3 V_6$ be a Lagrangian subspace and let $[U_0]$ be a point of Z_A^k, so that $K := A\cap T_{U_0}$ has dimension k. The restriction map

$$r_K\colon H^0(\mathbf{P}(T_{U_0}), I_{\mathscr{C}_{U_0}}(2)) \longrightarrow H^0(\mathbf{P}(K),\mathscr{O}(2))\simeq \mathrm{Sym}^2 K^\vee \tag{3}$$

will play a very important role for us. The following proposition is crucial for our argument.

Proposition 2.3. *Let $A\subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and let $[U_0]$ be a point of Z_A^k, with $k\in\{1,2,3,4\}$. With the notation above,*

(a) *when $k\in\{1,2,3\}$, the map r_K is surjective with kernel of dimension $9-k(k+1)/2$,*
(b) *when $k=4$, the map r_K is injective with image a general hyperplane in $H^0(\mathbf{P}(K),\mathscr{O}(2))$.*

The meaning of "general" in Item (b) is the following: a hyperplane in $H^0(\mathbf{P}(K),\mathscr{O}(2))\simeq \mathrm{Sym}^2K^\vee$ is defined by a nonzero element of $\mathrm{Sym}^2 K$ and we want the corresponding tensor to be of maximal rank $k=4$.

The proof will show that the proposition is true more generally for any linear subspace $K\subset T_{U_0}$ of dimension $k\le 4$ with no decomposable vectors.

Proof. Recall that $\mathscr{C}_{U_0}$ is the cone with vertex $\mathbf{v} := [\bigwedge^3 U_0]$ over the smooth fourfold $\mathscr{M}_1 \subset \mathbf{P}(\mathrm{Hom}(U_0, U_\infty))$ of morphisms of rank 1. Since the latter is not contained in any hyperplane, so is $\mathscr{C}_{U_0}$, hence its projective Zariski tangent space at $\mathbf{v}$ is the whole space $\mathbf{P}^9$. Any quadric containing $\mathscr{C}_{U_0}$ is therefore a cone with vertex $\mathbf{v}$.

Since A contains no decomposable vectors, $\mathbf{P}(K)$ is disjoint from $\mathscr{C}_{U_0}$. In particular, $\mathbf{v} \notin \mathbf{P}(K)$, so we may, upon projecting from $\mathbf{v}$, consider $\mathbf{P}(K)$ as a linear subspace of $\mathbf{P}(\mathrm{Hom}(U_0, U_\infty))$ (as above, $U_\infty \subset V_6$ is a subspace complementary to U_0). All in all, this proves that r_K identifies with the restriction morphism

$$r'_K \colon H^0(\mathbf{P}(\mathrm{Hom}(U_0, U_\infty)), I_{\mathscr{M}_1}(2)) \longrightarrow H^0(\mathbf{P}(K), \mathscr{O}(2)),$$

where $\mathbf{P}(K) \cap \mathscr{M}_1 = \varnothing$.

That being said, Item (a) is actually [**?**, Lemma 2.8], so we may assume $k = 4$ and we need to prove that r'_K is injective with general image. Suppose by contradiction that there exists a nonzero element in the kernel of r'_K. Its zero-locus $Q \subset \mathbf{P}(\mathrm{Hom}(U_0, U_\infty))$ is a quadric that contains the disjoint $\mathbf{P}(K) = \mathbf{P}^3$ and $\mathscr{M}_1$. This contradicts the following lemma and proves the injectivity of r'_K.

Lemma 2.4. *Let $\mathscr{M}_1 \subset \mathbf{P}(\mathrm{Hom}(U_0, U_\infty)) = \mathbf{P}^8$ be the smooth Segre fourfold of morphisms of rank 1 and let $Q_0 \subset \mathbf{P}^8$ be a quadric containing $\mathscr{M}_1$. Any $\mathbf{P}^3 \subset Q_0$ meets $\mathscr{M}_1$.*

Proof. The space of quadrics in $\mathbf{P}^8$ containing $\mathscr{M}_1$ is the $\mathbf{P}^8 = \mathbf{P}(\bigwedge^2 U_0 \otimes \bigwedge^2 U_\infty^\vee)$ spanned by the 9 maximal minors of elements of $\mathrm{Hom}(U_0, U_\infty)$. The group $G = \mathrm{GL}(U_0) \times \mathrm{GL}(U_\infty)$ acts on it, and under the isomorphisms $\bigwedge^2 U_0 \simeq U_0^\vee$ and $\bigwedge^2 U_\infty \simeq U_\infty^\vee$, the action of G on $\mathbf{P}^8$ corresponds to the action of $\mathrm{GL}(U_0) \times \mathrm{GL}(U_\infty)$ on $\mathbf{P}(\mathrm{Hom}(U_0, U_\infty))$. Therefore, the are only three orbits, given by:

(a) quadrics of (maximal) rank 9, represented for example by $M^{1,1} + M^{2,2} + M^{3,3}$;
(b) quadrics of rank 6, represented for example by $M^{2,2} + M^{3,3}$;
(c) quadrics of rank 4, represented for example by $M^{3,3}$,

where $M^{i,j}$ is the (i,j)-minor of an element $M = (x_{ij})_{1 \le i,j \le 3}$ of $\mathrm{Hom}(U_0, U_\infty)$ (in each case, it is enough to find one quadric containing $\mathscr{M}_1$ of the correct rank).

Since the action of $\mathrm{GL}(U_0) \times \mathrm{GL}(U_\infty)$ fixes $\mathscr{M}_1$, we can suppose that the quadric Q_0 is of one in the previous list.

In case (a), the Lefschetz hyperplane theorem shows that $H^{2i}(Q_0, \mathbf{Z}) \simeq H^{2i}(\mathbf{P}^8, \mathbf{Z}) = \mathbf{Z}h^i$, where $h \subset \mathbf{P}^8$ is a hyperplane section, for each $i \le 3$. Therefore, $[\mathscr{M}_1] = mh^3|_{Q_0} \in H^6(Q_0, \mathbf{Z})$ for some $m \in \mathbf{Z}_{>0}$. Thus, for each $\mathbf{P}^3 \subset Q_0$, we have

$$[\mathbf{P}^3] \cdot [\mathscr{M}_1] = m(h|_{\mathbf{P}^3})^3 = m \neq 0.$$

Hence, each $\mathbf{P}^3 \subset Q_0$ meets $\mathscr{M}_1$.

In case (b), the quadric

$$Q_0 = M^{2,2} + M^{3,3} = x_{11}(x_{22} + x_{33}) - x_{12}x_{21} - x_{13}x_{31}$$

has rank 6, hence is a cone over a smooth quadric

$$Q_1 \subset \mathbf{P}^5 = \mathbf{P}\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & 0 \end{pmatrix}$$

with vertex

$$\mathbf{P}^2 = (x_{11} = x_{22} + x_{33} = x_{12} = x_{21} = x_{13} = x_{31} = 0).$$

One sees that $\mathscr{M}_1 \cap \mathbf{P}^2$ is the smooth conic $x_{22}^2 + x_{23}x_{32} = 0$ and that $\mathscr{M}_1 \cap Q_1$ contains the planes

$$P_1 = \mathbf{P}(x_{11}, x_{12}, x_{13}) \quad \text{and} \quad P_2 = \mathbf{P}(x_{11}, x_{21}, x_{31})$$

that intersect at one point.

The Hilbert scheme $F_2(Q_1)$ of planes contained in Q_1 has 2 connected components, and two planes are in different components if and only if they intersect at one point (see [**?**, p. 35]). Therefore, P_1 and P_2 are in two different components of $F_2(Q_1)$.

Consider now our $\mathbf{P}^3 \subset Q_0$ disjoint from $\mathscr{M}_1$. Its intersection with the vertex $\mathbf{P}^2$ does not meet the conic $\mathscr{M}_1 \cap \mathbf{P}^2$ hence is finite. Since the smooth quadric Q_1 contains no 3 dimensional linear spaces, the projection of $\mathbf{P}^3$ from the vertex $\mathbf{P}^2$ to $\mathbf{P}^5$, which is contained in Q_1, is a plane contained in Q_1. Therefore, it intersects nontrivially the P_i that is not in the same component of $F_2(Q_1)$. Since P_i is contained in $\mathscr{M}_1$, we obtain that $\mathbf{P}^3$ meets $\mathscr{M}_1$.

In case (c), the quadric

$$Q_0 = M^{3,3} = x_{11}x_{22} - x_{12}x_{21}$$

is a cone over a smooth quadric $Q_1 \subset \mathbf{P}^3(x_{11}, x_{12}, x_{21}, x_{22})$ with vertex $\mathbf{P}^4$. Hence, the projection from $\mathbf{P}^4$ to $\mathbf{P}^2$ of any linear subspace $\mathbf{P}^3 \subset Q_0$ is contained in a line on Q_1, say defined by $x_{12} = x_{22} = 0$. The preimage in $\mathbf{P}^8$ of this line is

$$\mathbf{P}^6 = \mathbf{P}\begin{pmatrix} x_{11} & 0 & x_{13} \\ x_{21} & 0 & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},$$

whose intersection with $\mathscr{M}_1$ has dimension 3. Therefore, $\mathbf{P}^3 \subset \mathbf{P}^6 \subset Q_0$ must meet $\mathscr{M}_1$. □

To finish the proof of the proposition, we need to prove that (still in the case $k = 4$), the image of r'_K is a general hyperplane. This follows from the next lemma. □

Lemma 2.5. *In the above setting, the point* $\mathbf{p} \in \mathbf{P}(\mathrm{Sym}^2 K)$ *defined by the hyperplane* $\mathrm{Im}(r'_K) \subset H^0(\mathbf{P}(K), \mathcal{O}(2))$ *has maximal rank.*

Proof. We follow the proof of [**?**, Proposition 2.5] and consider the rational morphism

$$\begin{array}{rccc} \Phi\colon \mathbf{P}^8 \simeq \mathbf{P}(\mathrm{Hom}(U_0, U_\infty)) & \dashrightarrow & |I_{\mathscr{M}_1}(2)|^\vee \simeq \mathbf{P}^8 \\ M = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} & \longmapsto & \begin{pmatrix} M^{1,1} & M^{1,2} & M^{1,3} \\ M^{2,1} & M^{2,2} & M^{2,3} \\ M^{3,1} & M^{3,2} & M^{3,3} \end{pmatrix} \end{array} \tag{4}$$

induced by the linear system $|I_{\mathscr{M}_1}(2)| \subset H^0(\mathbf{P}^8, \mathscr{O}(2))$.

Using that $\det(M)$ can be expressed as a linear combination of minors of M, one sees that the (i,j)-minor of $\Phi(M)$ is equal to

$$\Phi(M)^{i,j} = \det(M) \cdot x_{i,j}.$$

It follows that Φ is a birational involution which is regular on $\mathbf{P}^8 \smallsetminus \mathscr{M}_1$ and an isomorphism on $\mathbf{P}^8 \smallsetminus \mathscr{M}_2$, while $\mathscr{M}_2 \smallsetminus \mathscr{M}_1$ is contracted onto $\mathscr{M}_1$.

We show that Φ restricted to $\mathbf{P}(K) \subset \mathbf{P}^8 \smallsetminus \mathscr{M}_1$ is an embedding.

First, we note that the fibers of $\Phi\colon \mathscr{M}_2\smallsetminus\mathscr{M}_1 \to \mathscr{M}_1$ are complements of quadric surfaces in their span $\mathbf{P}^3$. Indeed, the group $\mathrm{GL}(U_0)\times\mathrm{GL}(U_\infty)$ acts on $I_{\mathscr{M}_1}(2)$ (see the proof of Theorem **??**) and transitively on $\mathscr{M}_1$. Therefore, all fibers of Φ are conjugate under this action. Now, the preimage of $\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$ via $\Phi\colon \mathscr{M}_2\smallsetminus\mathscr{M}_1 \to \mathscr{M}_1$ is, in the set of matrices M all of whose minors but $M^{1,1}$ are 0, which is equal to

$$\mathbf{P}^3 = \mathbf{P}\begin{pmatrix}0&0&0\\0&x_{22}&x_{23}\\0&x_{32}&x_{33}\end{pmatrix},$$

the set of those of rank exactly 2. It is defined in this $\mathbf{P}^3$ by $x_{22}x_{33}-x_{23}x_{32}\neq 0$.

The morphism $\Phi|_{\mathbf{P}(K)}$ is an embedding outside $\mathscr{M}_2$ and, at points of $\mathscr{M}_2\smallsetminus\mathscr{M}_1$, its fiber is the linear space $\mathbf{P}(K)\cap\mathbf{P}^3$, which must be contained in the complement of a quadric surface, so it is also one reduced point. This proves our claim that $\Phi|_{\mathbf{P}(K)}$ is an embedding. But this restriction can be identified with the composition

$$\mathbf{P}(K) \xhookrightarrow{\nu_2} \mathbf{P}(\mathrm{Sym}^2K) \dashrightarrow \mathbf{P}(\mathrm{Im}(r'_K)^\vee),$$

where ν_2 is the second Veronese embedding and the second map is the projection from the point $\mathbf{p}$ of $\mathbf{P}(\mathrm{Sym}^2K)$ defined by the hyperplane $\mathrm{Im}(r'_K)\subset\mathrm{Sym}^2K^\vee$. The fact that it is an embedding means that $\mathbf{p}$ is not in the second secant variety of $\nu_2(\mathbf{P}(K))$ and leaves only two possibilities:

(a) $\mathrm{rank}(\mathbf{p})=3$;
(b) $\mathrm{rank}(\mathbf{p})=4$ (the hyperplane $\mathrm{Im}(r'_K)$ is general).

We prove that case (a) cannot happen. Since any rank 3 symmetric matrix can be written as the sum of three rank 1 symmetric matrices, this case happens if and only if there exist three distinct points $x,y,z\in\mathbf{P}(K)$ such that $\mathbf{p}=\nu_2(x)+\nu_2(y)+\nu_2(z)$, or equivalently, such that the points $\Phi(x)$, $\Phi(y)$, and $\Phi(z)$ are distinct and lie on a line ℓ.

We show that in this case, we have $\ell=\Phi(C)$ for some curve C contained in $\mathbf{P}(K)$. This leads to a contradiction, since Φ is a regular embedding on $\mathbf{P}(K)$ which is defined by quadrics, hence the image of any curve in $\mathbf{P}(K)$ has even degree.

We first consider the case $\ell\subset\mathscr{M}_1$. Note that every line in $\mathscr{M}_1$ is contained in a $\mathbf{P}^2\subset\mathscr{M}_1$ that is given by matrices with the same 2-dimensional kernel or the same 1-dimensional image. Up to change of basis and transposition, we can suppose $\mathbf{P}^2=\mathbf{P}\begin{pmatrix}y_1&0&0\\y_2&0&0\\y_3&0&0\end{pmatrix}$ and $\ell\subset\mathbf{P}^2$ is defined by the equation $y_3=0$. Hence, the preimage of this $\mathbf{P}^2$ via $\Phi|_{\mathbf{P}^8\smallsetminus\mathscr{M}_1}$ is the complement in

$$\mathbf{P}^5 = \mathbf{P}\begin{pmatrix}0&x_{12}&x_{13}\\0&x_{22}&x_{23}\\0&x_{32}&x_{33}\end{pmatrix}\subset\mathscr{M}_2$$

of the subvariety $\mathscr{M}_1\cap\mathbf{P}^5$, and the preimage of ℓ is defined by the quadratic equation $x_{12}x_{23}-x_{22}x_{13}=0$. Note that $\mathbf{P}(K)\cap\mathbf{P}^5$ is a linear space that contains the three distinct points x, y, z but does not meet the codimension 2 subvariety $\mathscr{M}_1\cap\mathbf{P}^5$. Therefore, $\mathbf{P}(K)\cap\mathbf{P}^5$ is a line and, since it meets the quadric $\Phi^{-1}(\ell)$ in three distinct points, it is contained in it. In particular, ℓ is the image of this line contained in $\mathbf{P}(K)$, as wanted.

We now consider the case $\ell \not\subset \mathscr{M}_1$. In this case, Φ^{-1} defines a rational morphism on ℓ whose image C is an irreducible curve. Recall that, outside $\mathscr{M}_2$, the composition $\Phi^{-1} \circ \Phi$ is the identity. Moreover, since Φ is an automorphism of $\mathbf{P}^8 \smallsetminus \mathscr{M}_2$, we have inclusions

$$\Phi(\mathbf{P}(K)) \cap \mathscr{M}_2 = \Phi(\mathbf{P}(K) \cap \mathscr{M}_2) \subset \Phi(\mathscr{M}_2 \smallsetminus \mathscr{M}_1) = \mathscr{M}_1.$$

In particular, $\Phi(\mathbf{P}(K)) \cap \mathscr{M}_2 = \Phi(\mathbf{P}(K)) \cap \mathscr{M}_1$ and $\Phi^{-1}\Phi(w) = w$ for some $w \in \mathbf{P}(K)$ if and only if $\Phi(w) \notin \mathscr{M}_1$. We consider the points $\Phi(x)$, $\Phi(y)$ and $\Phi(z)$ of ℓ and distinguish four cases.

- The three points are all in $\mathscr{M}_1$. This case cannot actually happen: since $\mathscr{M}_1$ is an intersection of quadrics, each line is either contained in $\mathscr{M}_1$ or meets it in at most 2 points.
- None of the three points are in $\mathscr{M}_1$, hence the curve C contains x, y and z. Since the morphism Φ^{-1} is defined by quadrics, C is a line or a conic that meets the linear space $\mathbf{P}(K)$ in x, y, z. Since all conics are planar, it follows that C is contained in $\mathbf{P}(K)$ and, of course, $\ell = \Phi(C)$.
- Exactly one of these points, say $\Phi(x)$, is contained in $\mathscr{M}_1$. Hence, all quadric polynomials that define Φ^{-1} vanish on $\Phi(x)$, therefore the curve C is a line that meets the linear space $\mathbf{P}(K)$ in the two distinct point y and z, thus it is contained in it.
- Exactly two of these points are contained in $\mathscr{M}_1$. In this case, all quadric polynomials that define Φ^{-1} vanish at two points of ℓ, hence Φ is constant on ℓ, which is absurd.

Therefore, $\ell = \Phi(C)$ for some curve C as wanted, and we obtain a contradiction, as explained above. □

2.3. **Local description of Z^k_A and proof of Theorem ??(d).** We are now ready to give a local description of the analytic germ of Z^k_A at a point $[U_0]$ of Z^ℓ_A (with $\ell \geq k$) and to prove Theorem **??**(d). Actually, we will prove, following [**?**, Claim 3.8], a more general result that will be needed later on.

Keeping the notation (**??**), we define the families

$$\mathscr{Z}^{\geq k} := \{([A],[U]) \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \times \mathrm{Gr}(3, V_6) \mid \dim(A \cap T_U) \geq k\}$$

of degeneracy loci $\mathsf{Z}^{\geq k}_A$ and we define $\mathscr{Z}^k$ analogously.

As in [**?**, Claim 3.8], we study the local structure of $\mathscr{Z}^{\geq k}$ at a point $([A_0],[U_0])$ of $\mathscr{Z}^\ell$ for $\ell \geq k$.

Proposition 2.6. *Let $A_0 \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and let $([A_0],[U_0])$ be a point of $\mathscr{Z}^\ell$, with $\ell \in \{0,\dots,4\}$. Set $K := A_0 \cap T_{U_0}$ and let $\mathscr{S}_{\leq i} \subset \mathrm{Sym}^2 K^\vee$ be the subscheme of quadratic forms of rank $\leq i$.*

For any $k \in \{0,\dots,\ell\}$, the analytic germ of the scheme $\mathscr{Z}^{\geq k}$ at the point $([A_0],[U_0])$ is isomorphic to the product of a smooth germ $(M,0)$ and the germ $(\mathscr{S}_{\leq \ell-k}, 0)$.

*Under this isomorphism, the germ of $\mathsf{Z}^{\geq k}_{A_0} \subset \mathscr{Z}^{\geq k}$ at $[U_0]$ corresponds to the germ of $M' \times (\mathrm{Im}(r_K) \cap \mathscr{S}_{\leq \ell-k})$ at $(0,0)$, where the morphism r_K is defined in (**??**) and $(M',0) \subset (M,0)$ is smooth of dimension* $\max(0, 9 - \ell(\ell+1)/2)$.

In particular, for $[A_0] \in \Gamma \smallsetminus \Sigma$, the germ of $\mathsf{Z}^{\geq 2}_{A_0}$ at a point of $\mathsf{Z}^4_{A_0}$ is the germ of $\mathrm{Im}(r_K) \cap \mathscr{S}_{\leq 2}$ *at* 0.

Proof. Choose a 3-dimensional subspace U_∞ of V_6 complementary to U_0 and such that T_{U_∞} is complementary to A_0 (and also to T_{U_0}). As in the proof of Proposition **??**, we consider affine open neighborhoods $\mathscr{U} \subset \mathrm{Gr}(3, V_6)$ of $[U_0]$ and $\mathscr{V} \subset \mathrm{LG}(\bigwedge^3 V_6)$ of $[A_0]$, with identifications $\mathrm{Hom}(U_0, U_\infty) \simeq \mathscr{U}$ and $\mathrm{Sym}^2 T_{U_0}^\vee \simeq \mathscr{V}$.

If $([A], [U]) \in \mathscr{V} \times \mathscr{U}$, we denote by q_U and q_A the quadratic forms on T_{U_0} associated with the Lagrangian subspaces T_U and A (which are elements of $\mathscr{V}$) respectively.

Consider the smooth morphism

$$\Psi \colon \mathscr{V} \times \mathscr{U} \longrightarrow \mathrm{Sym}^2 T_{U_0}^\vee$$

sending $([A], [U])$ to the quadratic form $q_U - q_A$. It is then easy to check (see for example the beginning of the proof of [?, Lemma 2.9]) that the intersection $\mathscr{Z}^{\geq k} \cap (\mathscr{V} \times \mathscr{U})$ is equal to $\Psi^{-1}(\Sigma_k)$, where $\Sigma_k \subset \mathrm{Sym}^2 T_{U_0}^\vee$ is the corank k degeneracy locus.

Fix a subspace $J \subset T_{U_0}$ such that $K \oplus J = T_{U_0}$. The quadratic form $\Psi([A_0], [U_0])$ is nondegenerate on J. Consider the vector bundle $\widetilde{K} \to \mathscr{V} \times \mathscr{U}$ whose fiber

$$\widetilde{K}_{([A],[U])} \coloneqq \{\omega \in T_{U_0} \mid \forall \alpha \in J \quad \Psi([A],[U])(\omega, \alpha) = 0\} \subset T_{U_0}$$

over $([A], [U])$ is the orthogonal complement of J with respect to the quadratic form $\Psi([A], [U])$.

Upon shrinking $\mathscr{V}$ and $\mathscr{U}$, we can suppose that $\widetilde{K}$ is trivial over $\mathscr{V} \times \mathscr{U}$ (with fiber K) and that $\Psi([A], [U])$ is nondegenerate on J for all $[A] \in \mathscr{V}$, $[U] \in \mathscr{U}$. Hence, the corank of $\Psi([A], [U])$ is equal to the corank of the restriction of $\Psi([A], [U])$ to $\widetilde{K}_{([A],[U])} \simeq K$. Therefore, the degeneracy loci of Ψ are equal to the degeneracy loci of the smooth morphism

$$\Psi_K \colon \mathscr{V} \times \mathscr{U} \xrightarrow{\Psi} \mathrm{Sym}^2 T_{U_0}^\vee \xrightarrow{\pi_K} \mathrm{Sym}^2 K^\vee,$$

where π_K is the restriction morphism.

In particular, $\mathscr{Z}^{\geq k} \cap (\mathscr{V} \times \mathscr{U}) = \Psi_K^{-1}(\mathscr{S}_{\leq \ell - k})$. This implies the first claim. Analogously, the scheme $\mathsf{Z}_{A_0}^{\geq k}$ is the preimage of $\mathscr{S}_{\leq \ell - k}$ via the morphism

$$\Psi_K([A_0], \cdot) \colon \mathscr{U} \longrightarrow \mathrm{Sym}^2 K^\vee.$$

Note that the morphism $\Psi([A_0], \cdot) \colon \mathscr{U} \to \mathrm{Sym}^2 T_{U_0}^\vee$ is induced by the morphism T from Section **??**, whose tangent map induces the isomorphism Θ from Proposition **??**. Therefore, the tangent map to $\Psi_K([A_0], \cdot)$ at $[U_0]$ is the morphism r_K of (**??**). Hence, analytically locally around $[U_0] \in \mathsf{Z}_{A_0}^{\ell}$, the image of the subscheme $\mathsf{Z}_{A_0}^{\geq k} \subset \mathscr{Z}^{\geq k}$ can be identified with $\mathrm{Im}(r_K) \cap \mathscr{S}_{\leq \ell - k} \subset \mathscr{S}_{\leq \ell - k}$.

This proves that the germ of $\mathsf{Z}_{A_0}^{\geq k}$ at $[U_0]$ is the product of the germ $\mathrm{Im}(r_K) \cap \mathscr{S}_{\leq \ell - k}$ at 0 with a smooth germ whose dimension can be computed using the fact, proved in Proposition **??**, that, since A contains no decomposable vectors, r_K has maximal rank. □

The following corollary proves in particular Item (d) of Theorem **??**.

Corollary 2.7. *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and let $[U_0] \in \mathsf{Z}_A^k$. The scheme Z_A^k is smooth at $[U_0]$, of dimension $9 - \frac{k(k+1)}{2}$ when $k \in \{0, 1, 2, 3\}$, and of dimension 0 when $k = 4$.*

Proof. This is just the case $k = \ell$ of the proposition. □

3. Divisors in the Lagrangian Grassmannian $\mathrm{LG}(\bigwedge^3 V_6)$

3.1. **The divisors Γ, Σ, and Δ in $\mathrm{LG}(\bigwedge^3 V_6)$.** O'Grady introduced in [?, Section 1] the divisors

$$\begin{aligned} \Sigma &:= \{[A] \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \mid A \text{ contains a decomposable vector}\}, \\ \Delta &:= \{[A] \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \mid \mathsf{Y}_A^{\geq 3} \neq \varnothing\} \end{aligned}$$

in $\mathrm{LG}(\bigwedge^3 V_6)$ and proved in [?, Proposition 3.1] and [?, Proposition 2.2] that they are irreducible and distinct. Here $\mathsf{Y}_A^{\geq 3}$ is the third stratum of the EPW sextic stratification constructed in [?].

In [?, Lemmas 3.6 and 3.7], it is proven that

$$\Gamma := \{[A] \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \mid \mathsf{Z}_A^{\geq 4} \neq \varnothing\} \tag{5}$$

is a divisor in $\mathrm{LG}(\bigwedge^3 V_6)$ which is distinct from O'Grady's irreducible divisor Δ. We prove here a stronger version this result.

Proposition 3.1. *The loci Γ, Σ, and Δ are mutually distinct irreducible divisors in $\mathrm{LG}(\bigwedge^3 V_6)$.*

Proof. Consider the closed subvariety

$$\widetilde{\Gamma} := \{([A],[U],[B]) \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \times \mathrm{Gr}(3, V_6) \times \mathrm{Gr}(4, \bigwedge^3 V_6) \mid B \subset A \cap T_U\},$$

where T_U was defined in (??), and the chain of morphisms

$$\begin{array}{ccccc} \widetilde{\Gamma} & \xrightarrow{q} & \mathrm{Gr}(4, T_{\mathscr{U}}) & \longrightarrow & \mathrm{Gr}(3, V_6) \\ ([A],[U],[B]) & \longmapsto & ([U],[B]) & \longmapsto & [U]. \end{array}$$

where $\mathrm{Gr}(4, T_{\mathscr{U}})$ is the $\mathrm{Gr}(4,10)$-bundle over $\mathrm{Gr}(3, V_6)$ whose fiber at $[U]$ is equal to $\mathrm{Gr}(4, T_U)$.

The fiber of q over $B \subset T_U$ is isomorphic to the set of Lagrangian subspaces A such that $B \subset A$. Since $B \subset T_U$ is totally isotropic, this fiber is isomorphic to the Lagrangian Grassmannian $\mathrm{LG}(B^\perp/B)$, smooth irreducible of dimension 21.

Since $\mathrm{Gr}(4, T_{\mathscr{U}})$ is smooth irreducible of dimension 33, it follows that $\widetilde{\Gamma}$ is also smooth irreducible, of dimension 54. Projecting $\widetilde{\Gamma}$ to $\mathrm{LG}(\bigwedge^3 V_6)$, we obtain that its image Γ is also irreducible.

For the fact that Γ, Σ, and Δ are mutually distinct, we refer to Proposition **??** (proved independently of the rest of the paper, of course). ☐

3.2. **The divisor Γ: parameter count.** Following [?, Section 2.1], we show that for $[A] \in \Gamma$ general, $\mathsf{Z}_A^{\geq 4}$ consists of a single point.

Lemma 3.2. *The subvariety*

$$\Gamma_+ = \{[A] \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \mid |\mathsf{Z}_A^{\geq 4}| > 1\}$$

of $\mathrm{LG}(\bigwedge^3 V_6)$ has codimension at least 2. Therefore, for $[A] \in \Gamma$ general, the scheme $\mathsf{Z}_A^{\geq 4}$ consists of a single smooth point, which is the only singular point of the scheme $\widetilde{\mathsf{Z}}_A^{\geq 2}$, defined in Equation (??).

Proof. Consider the locally closed subvariety

$$\widetilde{\Gamma}_+ = \{([A],[U_1],[U_2]) \mid [U_1] \neq [U_2],\ \dim(A \cap T_{U_1}) = \dim(A \cap T_{U_2}) = 4\}$$

of $\mathrm{LG}(\bigwedge^3 V_6) \times \mathrm{Gr}(3, V_6)^2$. It follows from Theorem **??**(e) that Γ_+ is contained in the union of $\Gamma \cap \Sigma$ and the image of the first projection $\pi_+ \colon \widetilde{\Gamma}_+ \to \mathrm{LG}(\bigwedge^3 V_6)$. Since Γ and Σ are distinct irreducible divisors by Proposition **??**, it is therefore enough to show that $\widetilde{\Gamma}_+$ has dimension at most 53, .

We consider the morphism

$$\begin{aligned} \eta\colon \widetilde{\Gamma}_+ &\longrightarrow \mathrm{Gr}(4, \textstyle\bigwedge^3 V_6)^2 \times \mathrm{Gr}(3, V_6)^2 \\ ([A],[U_1],[U_2]) &\longmapsto ([A\cap T_{U_1}],[A\cap T_{U_2}],[U_1],[U_2]). \end{aligned}$$

Since A is a Lagrangian subspace, the image of η is contained in

$$N = \left\{ ([K_1],[K_2],[U_1],[U_2]) \;\middle|\; \begin{matrix} [U_1]\neq[U_2], & [K_1]\in \mathrm{Gr}(4,T_{U_1}), & [K_2]\in\mathrm{Gr}(4,T_{U_2}), \\ K_1\perp K_2, & & K_1\cap T_{U_2} = K_1\cap K_2 = K_2\cap T_{U_1} \end{matrix} \right\}.$$

Observe that the fiber of an element $([K_1],[K_2],[U_1],[U_2])$ of N is isomorphic to

$$\{[A]\in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \mid K_1 = A\cap T_{U_1},\ K_2 = A\cap T_{U_2}\},$$

which, since K_1+K_2 is totally isotropic, is contained in $\mathrm{LG}((K_1+K_2)^\perp/(K_1+K_2))$. Therefore, the fiber $\eta^{-1}([K_1],[K_2],[U_1],[U_2])$ has dimension at most $\frac{(2+k)(3+k)}{2}$, where $k=\dim(K_1\cap K_2)$.

We stratify N according to $k=\dim(K_1\cap K_2)$ and $\ell=\dim(U_1\cap U_2)$, and denote by

$$N_{k,\ell} = \{([K_1],[K_2],[U_1],[U_2])\in N \mid \ell=\dim(U_1\cap U_2),\ k=\dim(K_1\cap K_2)\}$$

the stratum corresponding to k and ℓ. Clearly we have the inclusion $K_1\cap K_2\subset T_{U_1}\cap T_{U_2}$, where the dimension $d(\ell)$ of $T_{U_1}\cap T_{U_2}$ is 0 for $\ell=0$, and $2(\ell+1)$ for $\ell\in\{1,2\}$. Moreover, the morphism η has equidimensional fibers over $N_{k,\ell}$. Thus, the dimension of $\widetilde{\Gamma}_+$ is the maximum of $\dim(\eta^{-1}(N_{k,\ell}))$ for $0\le k\le d(\ell)$ and $0\le\ell\le 2$, namely

$$\dim(\widetilde{\Gamma}_+) = \max_{0\le k\le d(\ell)} \frac{(2+k)(3+k)}{2} + \dim(N_{k,\ell}). \tag{6}$$

We count parameters to determine the dimension of $N_{k,\ell}$. The space of pairs $([U_1],[U_2])\in \mathrm{Gr}(3,V_6)^2$ whose intersection has dimension ℓ is a Schubert cell of dimension $18-\ell^2$. Having fixed such a pair $([U_1],[U_2])$, the quadruple $([K_1],[K_2],[U_1],[U_2])$ is in $N_{k,\ell}$ if and only if $K_1\cap T_{U_2}$ has dimension k and

$$K_1\cap T_{U_2}\subset K_2\subset K_1^\perp\cap T_{U_2}\ , \quad K_2\cap T_{U_1} = K_1\cap T_{U_2}.$$

Hence, given $[K]=[K_1\cap T_{U_2}]$ in $\mathrm{Gr}(k, T_{U_1}\cap T_{U_2})$, and $[K_1]$ in $\mathrm{Gr}(4-k, T_{U_1}/K)$, the point $[K_2]$ lies in $\mathrm{Gr}(4-k,(K_1^\perp\cap T_{U_2})/K)$. Note that, since $K_1\cap T_{U_2}$ has dimension k, the space $K_1^\perp\cap T_{U_2}$ has dimension $6+k$. Putting all together, we obtain

$$\begin{aligned} \dim(N_{k,\ell}) &\le 18-\ell^2 + k(d(\ell)-k) + 6(4-k) + (4-k)(2+k) \\ &= 50 - 2(k+2)k + d(\ell)k - \ell^2. \end{aligned}$$

Hence, the dimension of $\eta^{-1}(N_{k,\ell})$ is at most $53-\ell^2-\frac{3}{2}k(k-\frac{2d(\ell)-3}{3})$, whose maximum for $0\le k\le \max\{d(\ell),4\}$ and $0\le\ell\le 2$ is 53. Therefore, $\widetilde{\Gamma}_+$ has dimension at most 53, as wanted. ☐

4. The double cover $\widetilde{\mathsf{Z}}_A^{\ge 2}\to\mathsf{Z}_A^{\ge 2}$

In [?, Theorem 5.7(2)], the authors construct, for each $[A]\in\mathrm{LG}(\bigwedge^3 V_6)$ with no decomposable vectors, a canonical double cover

$$g\colon \widetilde{\mathsf{Z}}_A^{\ge 2}\longrightarrow \mathsf{Z}_A^{\ge 2}$$

branched over $\mathsf{Z}_A^{\geq 3}$. The variety $\widetilde{\mathsf{Z}}_A^{\geq 2}$ is integral, normal, and smooth away from $g^{-1}(\mathsf{Z}_A^4)$.

When moreover $[A] \notin \Gamma$, this double cover coincides with the EPW cube of [?] (see [?, Lemma 5.8]); it is therefore a smooth hyper-Kähler variety of K3$^{[3]}$-type with a polarization H_A of square 4 and divisibility 2. In this section, we study the singular sixfold $\widetilde{\mathsf{Z}}_A^{\geq 2}$ when $[A] \in \Gamma \smallsetminus \Sigma$ and describe its blowup along its singular locus and some smooth hyper-Kähler resolutions.

We will study these double covers and the desingularization of $\widetilde{\mathsf{Z}}_A^{\geq 2}$ in families. So we start with the families

$$\mathscr{Z}^{\geq k} \longrightarrow \mathrm{LG}(\textstyle\bigwedge^3 V_6)$$

defined in Section **??**.

Let $[A_0] \in \Gamma \smallsetminus \Sigma$ and let $\mathscr{V} \subset \mathrm{LG}(\bigwedge^3 V_6) \smallsetminus \Sigma$ be a (sufficiently small classical) open neighborhood of $[A_0]$. We set $\mathscr{Z}_{\mathscr{V}}^{\geq k} := \mathscr{Z}^{\geq k} \cap (\mathscr{V} \times \mathrm{Gr}(3, V_6))$. As in the case of EPW sextics (see the proof of [?, Theorem 5.2]), there exists a unique double cover

$$f_{\mathscr{V}} \colon \widetilde{\mathscr{Z}_{\mathscr{V}}} \longrightarrow \mathscr{Z}_{\mathscr{V}}^{\geq 2}$$

branched along $\mathscr{Z}_{\mathscr{V}}^{\geq 3}$, whose restriction over $\mathsf{Z}_A^{\geq 2}$ is the morphism $g \colon \widetilde{\mathsf{Z}}_A^{\geq 2} \to \mathsf{Z}_A^{\geq 2}$.

Given $[U_0] \in \mathsf{Z}_A^{\geq 4}$, we constructed in the proof of Proposition **??** a smooth morphism

$$\Psi_K \colon \mathscr{Z}^{\geq 2} \cap (\mathscr{V} \times \mathscr{U}) \longrightarrow \mathscr{S}_{\leq 2},$$

where $\mathscr{U}$ is a neighborhood of $[U_0]$ in $\mathrm{Gr}(3, V_6)$, and $\mathscr{S}_{\leq 2}$ is the scheme of (4×4)-matrices of rank ≤ 2. We prove that the double cover $f_{\mathscr{V}}$ is the pullback by Ψ_K of a certain "universal double cover" of $\widetilde{\mathscr{S}_{\leq 2}} \to \mathscr{S}_{\leq 2}$ studied in Section **??**. In particular, we obtain a description of the blowup of $\widetilde{\mathscr{Z}^{\geq 2}}$ along its singular locus $f_{\mathscr{V}}^{-1}(\mathscr{Z}_{\mathscr{V}}^{\geq 4})$, from which we deduce a description of the blowup of $\widetilde{\mathsf{Z}}_A^{\geq 2}$.

4.1. **Local model of double cover.** In this section, we consider a "canonical example" of double cover. Let V be a vector space of dimension n and consider the universal quadratic form

$$q \colon V \otimes \mathscr{O}_{\mathrm{Sym}^2 V^\vee} \longrightarrow V^\vee \otimes \mathscr{O}_{\mathrm{Sym}^2 V^\vee}.$$

Denote by $\mathscr{S}_{\leq i} \subset \mathrm{Sym}^2 V^\vee$ the rank i degeneracy locus of q, so that

- $\mathscr{S}_{\leq 2}$ is the affine cone over the secant variety to the second Veronese embedding $\mathbf{P}(V^\vee) \subset \mathbf{P}(\mathrm{Sym}^2 V^\vee)$,
- $\mathscr{S}_{\leq 1}$ is the affine cone over the same embedding $\mathbf{P}(V^\vee) \subset \mathbf{P}(\mathrm{Sym}^2 V^\vee)$,
- $\mathscr{S}_{\leq 0} = \{0\}$.

In this situation, [?, Theorem 3.1] provides a canonical double cover $\widetilde{f} \colon \widetilde{\mathscr{S}_{\leq 2}} \to \mathscr{S}_{\leq 2}$.

Lemma 4.1. *The affine variety* $(V^\vee \otimes V^\vee)_1 = \{\mu \in V^\vee \otimes V^\vee \mid \mathrm{rk}(\mu) \leq 1\}$ *is the cone over the Segre variety* $\mathbf{P}(V^\vee) \times \mathbf{P}(V^\vee)$. *The map*

$$\begin{aligned} g_2 \colon (V^\vee \otimes V^\vee)_1 &\longrightarrow \mathscr{S}_{\leq 2} \\ \mu \quad &\longmapsto \mu + \iota(\mu), \end{aligned}$$

where ι *is the involution that exchanges the two factors, is the canonical double cover* $\widetilde{f} \colon \widetilde{\mathscr{S}_{\leq 2}} \to \mathscr{S}_{\leq 2}$. *In particular, the unique singular point of the variety* $\widetilde{\mathscr{S}_{\leq 2}}$ *is* 0.

Proof. We follow the proof of [**?**, Lemma 3.4]. Note that f is a double cover because it is the quotient for the action of the involution ι on $(V^\vee \otimes V^\vee)_1$; it is moreover étale over $\mathscr{S}^0_{\leq 2}$. Thus, since $(V^\vee \otimes V^\vee)_1$ is normal, in order to show that g_2 and $\widetilde{f}$ are isomorphic, it is enough to show that their associated reflexive sheaves are the same.

Let R be the ring of functions of the affine variety $(V^\vee \otimes V^\vee)_1$. If we denote by $S^\bullet(V \otimes V)$ the symmetric algebra of $V \otimes V$, we can identify R with the quotient $S^\bullet(V \otimes V)/I$, where I is the ideal generated by $\{(v \otimes w)(v' \otimes w') - (v \otimes w')(v' \otimes w) \mid v, w, v', w' \in V\}$. Note that there is an isomorphism

$$R \simeq \bigoplus_{i=0}^{\infty} S^i(V) \otimes S^i(V) \tag{7}$$

that sends the class of a monomial $(v_1 \otimes w_1) \cdots (v_j \otimes w_j) \in S^j(V \otimes V)$ to $(v_1 \cdots v_j) \otimes (w_1 \cdots w_j)$.

Given a subset P of $S^\bullet(V \otimes V)$, we denote by $[P]$ its image in $S^\bullet(V \otimes V)/I$.

The action of $\mathbf{Z}/2\mathbf{Z}$ on $(V^\vee \otimes V^\vee)_1$ induces an action on R given by the involution ι that sends $[v \otimes w]$ to $[w \otimes v]$. We have a decomposition

$$R = R_+ \oplus R_-, \tag{8}$$

where R_+ is the invariant part and R_- is the antiinvariant part with respect to this action. Under the isomorphism (**??**), their graded part are given by

$$R_+ = \bigoplus_{i=0}^{\infty} S^2(S^i V) \quad \text{and} \quad R_- = \bigoplus_{i=0}^{\infty} \bigwedge^2(S^i V).$$

Note that R_- is the vector space generated by $[\alpha - \iota(\alpha)]$, for all monomials α in $S^\bullet(V \otimes V)$. Any monomial α in $S^j(V \otimes V)$ can be written as $\alpha = (v \otimes w)\beta$, where $\beta \in S^{j-1}(V \otimes V)$. Thus, by induction on j using the equality

$$2(\alpha - \iota(\alpha)) = (v \otimes w - w \otimes v)(\beta + \iota(\beta)) + (v \otimes w + w \otimes v)(\beta - \iota(\beta)),$$

we see that R_- is generated by $[\bigwedge^2 V]$ as an R_+-module.

Then $\mathscr{S}_{\leq 2} = \operatorname{Spec}(R_+)$, the direct sum (**??**) provides the decomposition of $g_{2,*}\mathscr{O}_{(V^\vee \otimes V^\vee)_1}$ into invariant and antiinvariant parts, and the reflexive sheaf associated to g_2 is the sheaf associated with the module R_-.

The restriction of q to $\mathscr{S}_{\leq 2}$ is induced by the morphism

$$\begin{array}{rcl} q\colon V \otimes R_+ & \longrightarrow & V^\vee \otimes R_+ = \operatorname{Hom}(V \otimes R_+, R_+) \\ v \otimes 1 & \longmapsto & (w \otimes 1 \mapsto \frac{1}{2}[v \otimes w + w \otimes v]) \end{array}$$

of free R_+-modules. Set $\mathscr{S}_2 \coloneqq \mathscr{S}_{\leq 2} \smallsetminus \mathscr{S}_{\leq 1}$. We show that the sheaf $\det(\operatorname{Im}(q|_{\mathscr{S}_2}))$ (which is, by the construction of $\widetilde{f}$ in [**?**, Theorem 3.1], the reflexive sheaf associated to $\widetilde{f}$) is the sheaf associated to the R_+-module R_-. Indeed, over $\mathscr{S}_2$, the image of q is locally free of rank 2, thus

$$\det(\operatorname{Im}(q|_{\mathscr{S}_2})) = \bigwedge^2 \operatorname{Im}(q|_{\mathscr{S}_2}) = \operatorname{Im}\big(\bigwedge^2 q|_{\mathscr{S}_2}\big).$$

The morphism

$$\bigwedge^2 q\colon \bigwedge^2 V \otimes R_+ \longrightarrow \operatorname{Hom}(\bigwedge^2 V \otimes R_+, R_+)$$

sends an element $(v_1 \wedge v_2) \otimes 1 \in \bigwedge^2 V \otimes R_+$ to the dual form defined by

$$(w_1 \wedge w_2) \otimes 1 \longmapsto q(v_1, w_1)q(v_2, w_2) - q(v_1, w_2)q(v_2, w_1).$$

By developing the latter expression, we obtain that $\bigwedge^2 q$ factors (up to a constant) as

$$\begin{array}{ccccc} \bigwedge^2 V\otimes R_+ & \longrightarrow & R_- & \longrightarrow & \mathrm{Hom}(\bigwedge^2 V\otimes R_+, R_+) \\ (v_1\wedge v_2)\otimes 1 & \longmapsto & v_1\wedge v_2 & \longmapsto & \big((w_1\wedge w_2)\otimes 1\mapsto -(w_1\wedge w_2)(v_1\wedge v_2)\big), \end{array}$$

where we identify $\bigwedge^2 V$ with $(R_-)_1$.

Note that the first morphism is surjective, since R_- is generated by $\bigwedge^2 V$ as an R_+-module, while the second morphism is injective. Thus $\mathrm{Im}\big(\bigwedge^2 q|_{\mathscr{S}_2}\big)$ is the sheaf associated to R_-, as wanted. □

Let $\rho\colon \widetilde{T}\to \widetilde{\mathscr{S}_{\leq 2}}$ be the blowup of the unique singular point 0 of $\widetilde{\mathscr{S}_{\leq 2}}$ and let $H\subset \mathrm{Sym}^2 V^\vee$ be a general hyperplane.

Proposition 4.2. *The exceptional divisor E of ρ is isomorphic to $\mathbf{P}(V^\vee)\times\mathbf{P}(V^\vee)$, and the normal bundle of E in $\widetilde{T}$ is isomorphic to $\mathscr{O}_{\mathbf{P}(V^\vee)\times\mathbf{P}(V^\vee)}(-1,-1)$.*

Moreover, the intersection of the strict transform of $g_2^{-1}(H)$ with E is a general hyperplane section of E.

Proof. The description in Lemma **??** of $\widetilde{\mathscr{S}_{\leq 2}}$ as the cone over the Segre variety $\mathbf{P}(V^\vee)\times\mathbf{P}(V^\vee)$ implies the description of the exceptional divisor E and its normal bundle.

If we choose coordinates $(x_i)_{1\leq i\leq n}$ on $V^\vee$, which induce coordinates $(s_{ij})_{1\leq i\leq j\leq n}$ on $\mathrm{Sym}^2 V^\vee$, then $g_2^{-1}(H)$ can be identified with the cone over the preimage of the hyperplane $s_{11}+\cdots+s_{nn}=0$ via the morphism

$$\begin{aligned} \mathbf{P}(V^\vee)\times\mathbf{P}(V^\vee) &\longrightarrow \mathbf{P}(\mathscr{S}_{\leq 2})\subset \mathbf{P}(\mathrm{Sym}^2 V^\vee) \\ ([(x_i)],[(y_j)]) &\longmapsto [(x_iy_j+y_jx_i)]_{1\leq i\leq j\leq n}, \end{aligned}$$

which is given by the equation $x_1y_1+\cdots+x_ny_n=0$. Thus, the strict transform of $g_2^{-1}(H)$ intersects the exceptional divisor $\mathbf{P}(V^\vee)\times\mathbf{P}(V^\vee)$ as a general hyperplane section. □

4.2. **Singularities of $\widetilde{\mathsf{Z}}_A^{\geq 2}$ for $[A]$ in $\Gamma\smallsetminus\Sigma$ and simultaneous resolution.** We now go back to the double cover

$$\widetilde{\mathscr{Z}_{\mathscr{V}}}\xrightarrow{f_{\mathscr{V}}}\mathscr{Z}_{\mathscr{V}}^{\geq 2}\longrightarrow\mathscr{V}$$

branched along $\mathscr{Z}_{\mathscr{V}}^{\geq 3}$ mentioned at the beginning of Section **??** (recall that $\mathscr{V}\subset \mathrm{LG}(\bigwedge^3 V_6)\smallsetminus\Sigma$ is a sufficiently small classical open neighborhood of a given point $[A_0]\in\Gamma\smallsetminus\Sigma$). The next theorem explains the geometry of $\widetilde{\mathscr{Z}_{\mathscr{V}}}$; note that the situation is completely analogous (in dimension 6 instead of 4) to the situation for double EPW sextics described in [**?**, Claim 3.8].

Theorem 4.3. *With notation as previously, we have the following.*

(a) *The singular locus of $\widetilde{\mathscr{Z}_{\mathscr{V}}}$ is $f_{\mathscr{V}}^{-1}(\mathscr{Z}_{\mathscr{V}}^{\geq 4})$ and it is finite over $\mathscr{V}$.*
(b) *The blowup*
$$\rho_{\mathscr{V}}\colon\widetilde{\mathscr{X}_{\mathscr{V}}}\longrightarrow\widetilde{\mathscr{Z}_{\mathscr{V}}}$$
of $f_{\mathscr{V}}^{-1}(\mathscr{Z}_{\mathscr{V}}^{\geq 4})$ in $\widetilde{\mathscr{Z}_{\mathscr{V}}}$ is smooth.
(c) *The morphism $f_{\mathscr{V}}\circ\rho_{\mathscr{V}}$ restricted to the exceptional divisor $E_{\mathscr{V}}$ of $\rho_{\mathscr{V}}$ induces a locally trivial fibration*
$$E_{\mathscr{V}}\longrightarrow\mathscr{Z}_{\mathscr{V}}^{\geq 4}$$
with fibers $\mathbf{P}^3\times\mathbf{P}^3$, and the restriction of the normal bundle to $E_{\mathscr{V}}$ in $\widetilde{\mathscr{X}_{\mathscr{V}}}$ to each fiber is $\mathscr{O}(-1,-1)$.

(d) *For any* $[A] \in \mathscr{V} \cap \Gamma$, *the strict transform* $\widetilde{\mathsf{X}}_A$ *of* $\widetilde{\mathsf{Z}}_A^{\geq 2} \subset \widetilde{\mathscr{Z}_{\mathscr{V}}}$ *is smooth and intersects* $E_{\mathscr{V}}$ *in a general hyperplane section of* $\mathbf{P}^3 \times \mathbf{P}^3$.

Proof. For any $([A],[U]) \notin \mathscr{Z}_{\mathscr{V}}^{\geq 4}$, consider an open neighborhood $\mathscr{V}' \times \mathscr{U} \subset \mathscr{V} \times \mathrm{Gr}(3, V_6)$ of $([A],[U])$ such that $\mathscr{Z}^{\geq 4} \cap (\mathscr{V}' \times \mathscr{U}) = \varnothing$. By uniqueness of the double cover constructed in [?], the variety $\widetilde{\mathscr{Z}_{\mathscr{V}}} \cap (\mathscr{V}' \times \mathscr{U})$ is the double cover of $\mathscr{Z}^{\geq 2} \cap (\mathscr{V}' \times \mathscr{U})$, hence is smooth by [?, Corollary 4.8]. Therefore, the singular locus of $\widetilde{\mathscr{Z}_{\mathscr{V}}}$ is contained in $f^{-1}(\mathscr{Z}_{\mathscr{V}}^{\geq 4})$.

Let us work now around a point $([A_0],[U_0])$ of $\mathscr{Z}_{\mathscr{V}}^{\geq 4}$. By Proposition **??** (with $k = 2$ and $\ell = 4$), the analytic germ of $\mathscr{Z}_{\mathscr{V}}^{\geq 2}$ at that point is the product of a smooth germ and the germ of $\mathscr{S}_{\leq 2}$ at 0, where $\mathscr{S}_{\leq 2} \subset \mathrm{Sym}^2(A_0 \cap T_{U_0})^\vee$ is the subscheme of quadratic forms of rank ≤ 2. In particular, locally, we have a diagram

$$\begin{array}{ccc} \widetilde{\mathscr{Z}_{\mathscr{V}}} & \longrightarrow & \widetilde{\mathscr{S}_{\leq 2}} \\ {\scriptstyle f_{\mathscr{V}}}\big\downarrow & & \big\downarrow{\scriptstyle g_2} \\ \mathscr{Z}_{\mathscr{V}}^{\geq 2} & \xrightarrow{\Psi_K} & \mathscr{S}_{\leq 2}, \end{array}$$

where Ψ_K is smooth and g_2 is the canonical double cover of $\mathscr{S}_{\leq 2}$ (see Lemma **??**). Moreover, the variety $\widetilde{\mathsf{Z}}_A^{\geq 2}$ is the product of a smooth germ with $g_2^{-1}(\mathrm{Im}(r_K) \cap \mathscr{S}_{\leq 2})$, where $\mathrm{Im}(r_K) \subset \mathrm{Sym}^2(A_0 \cap T_{U_0})^\vee$ is a general hyperplane (Lemma **??**).

Therefore, it is enough to show (a), (b), (c), and (d) for the scheme $\widetilde{\mathscr{S}_{\leq 2}}$ described in Lemma **??**) and the preimage of a general hyperplane section of $\mathscr{S}_{\leq 2}$. This was done in Section **??**. ∎

Given $[A_0] \in \Gamma \smallsetminus \Sigma$ with $\mathsf{Z}_{A_0}^{\geq 4} = \{z_1, \dots, z_r\}$, we have shown that the blowup of $\widetilde{\mathsf{Z}}_{A_0}^{\geq 2}$ along it singular locus $\mathsf{Z}_{A_0}^{\geq 4}$ is

$$\tau \colon \widetilde{\mathsf{X}}_{A_0} \longrightarrow \widetilde{\mathsf{Z}}_{A_0}^{\geq 2},$$

with exceptional divisor $E = E_1 \sqcup \cdots \sqcup E_r$, the disjoint union of r copies of the incidence variety $I \subset \mathbf{P}^3 \times (\mathbf{P}^3)^\vee$ (this is a general hyperplane section), and $\mathscr{O}(E)|_{E_i} = \mathscr{O}_{E_i}(-1,-1)$.

The situation is therefore exactly as in [?, Section 3.2], which we follow without repeating the proofs. For each $i \in \{1, \dots, r\}$, choose a projection $E_i \to \mathbf{P}^3$. We call that a choice of projections ε for $\mathsf{Z}_{A_0}^{\geq 2}$.

Proposition 4.4. *Let* $[A_0] \in \Gamma \smallsetminus \Sigma$ *and let* ε *be a choice of* $\mathbf{P}^3$*-fibrations for* $\mathsf{Z}_{A_0}^{\geq 2}$*. Then, there exists a classical open neighborhood* $\mathscr{V}$ *of* $[A_0]$ *in* $\mathrm{LG}(\bigwedge^3 V_6) \smallsetminus \Sigma$ *such that*

(a) *There is a factorization*

$$\rho_{\mathscr{V}} \colon \widetilde{\mathscr{X}_{\mathscr{V}}} \xrightarrow{d_\varepsilon} \mathscr{X}_{\mathscr{V}}^{\varepsilon} \xrightarrow{c_\varepsilon} \widetilde{\mathscr{Z}_{\mathscr{V}}^{\geq 2}},$$

where $\mathscr{X}_{\mathscr{V}}^{\varepsilon}$ *is a complex manifold and* d_ε *induces on each* $E_i \subset \widetilde{\mathsf{X}}_{A_0} \subset \widetilde{\mathscr{X}_{\mathscr{V}}}$*, the chosen projection* $E_i \to \mathbf{P}^3$*,*

(b) *The composition*

$$\varphi_{\mathscr{V}} \colon \mathscr{X}_{\mathscr{V}}^{\varepsilon} \xrightarrow{c_\varepsilon} \widetilde{\mathscr{Z}_{\mathscr{V}}^{\geq 2}} \longrightarrow \mathscr{V}$$

is smooth and

– *the fiber of* $[A] \in \mathscr{V} \smallsetminus \Gamma$ *is the EPW cube* $\widetilde{\mathsf{Z}}_A^{\geq 2}$*,*

– *the fiber* X_A^ε *of* $[A] \in \mathscr{V} \cap \Gamma$ *is a small resolution of the singular EPW cube* $\widetilde{\mathsf{Z}}_A^{\geq 2}$*, with exceptional locus a disjoint union of* $\mathbf{P}^3$ *(one for each singular point of* $\widetilde{\mathsf{Z}}_A^{\geq 2}$*).*

Given another choice ε' of $\mathbf{P}^3$-fibrations, there exists a commutative diagram

$$\begin{array}{ccc} & \widetilde{\mathsf{X}}_{A_0} & \\ {\scriptstyle d_\varepsilon}\swarrow & & \searrow{\scriptstyle d_{\varepsilon'}} \\ \mathsf{X}_{A_0}^{\varepsilon} & \dashrightarrow & \mathsf{X}_{A_0}^{\varepsilon'} \\ {\scriptstyle c_\varepsilon}\searrow & & \swarrow{\scriptstyle c_{\varepsilon'}} \\ & \widetilde{\mathsf{Z}}_{A_0}^{\geq 2}, & \end{array}$$

where the horizontal dashed arrow is a birational morphism which is the Mukai flop of the exceptional $\mathbf{P}^3$ for which $\varepsilon_i \neq \varepsilon_i'$. Note that we do not claim that $\mathsf{X}_{A_0}^\varepsilon$ is projective. In Section **??**, we will show that at least one $\mathsf{X}_{A_0}^\varepsilon$ (for a suitable choice of ε) is indeed projective.

5. Projective resolutions of $\widetilde{\mathsf{Z}}_A^{\geq 2}$

Given a Lagrangian $[A] \in \Gamma \smallsetminus \Sigma$, we described in Section **??** the blowup $\tau\colon \widetilde{\mathsf{X}}_A \to \widetilde{\mathsf{Z}}_A^{\geq 2}$ of the singular locus of the EPW cube $\widetilde{\mathsf{Z}}_A^{\geq 2}$ and, *in the analytic category,* some small resolutions $h_\varepsilon\colon \mathsf{X}_A^\varepsilon \to \widetilde{\mathsf{Z}}_A^{\geq 2}$ together with factorizations

$$\tau\colon \widetilde{\mathsf{X}}_A \xrightarrow{d_\varepsilon} \mathsf{X}_A^\varepsilon \xrightarrow{c_\varepsilon} \widetilde{\mathsf{Z}}_A^{\geq 2}.$$

The aim of this section is to show that there exists ε such that the contraction c_ε is projective. We do so by constructing a smooth projective hyper-Kähler variety Y_A with a small contraction $\mathsf{Y}_A \to \widetilde{\mathsf{Z}}_A^{\geq 2}$ and then showing that the blowup map τ factors though it. Our construction will use properties of the period map for EPW cubes which we now explain.

5.1. **Period maps for EPW cubes and double EPW sextics.** Smooth EPW cubes are hyper-Kähler sixfolds of $\mathrm{K3}^{[3]}$-type with a polarization of Beauville–Bogomolov square 4 and divisibility 2. There is a 19-dimensional quasi-projective irreducible coarse moduli space ${}^{[3]}\mathscr{M}_4^{(2)}$ for these polarized sixfolds (see [**?**, Theorem 3.5]), hence a *moduli morphism*

$$\mu_3\colon \operatorname{LG}(\textstyle\bigwedge^3 V_6) \smallsetminus (\Gamma \cup \Sigma) \longrightarrow {}^{[3]}\mathscr{M}_4^{(2)}$$

that sends a Lagrangian $[A] \notin \Gamma \cup \Sigma$ to the point of the moduli space corresponding to the smooth EPW cube $\widetilde{\mathsf{Z}}_A^{\geq 2}$ with its canonical polarization.

There is also a 19-dimensional quasi-projective period domain ${}^{[3]}\mathscr{P}_4^{(2)}$ and a *period map*

$$\wp_3\colon {}^{[3]}\mathscr{M}_4^{(2)} \hookrightarrow {}^{[3]}\mathscr{P}_4^{(2)}$$

(see Section **??** for more details) which is injective by the Torelli theorem ([**?**, **?**, **?**]).

The situation for double EPW sextics is completely analogous and we describe it only briefly. Given a Lagrangian subspace $A \subset \bigwedge^3 V_6$ not in $\Delta \cup \Sigma$ (see Section **??** for the definition of the divisors Δ and Σ), O'Grady constructed in [**?**] a smooth double cover of a certain sextic hypersurface in $\mathbf{P}(V_6)$. This double cover is a hyper-Kähler fourfold with a canonical polarization of Beauville–Bogomolov square 2 and divisibility 1. So, as above, we have a moduli morphism

$$\mu_2\colon \mathrm{LG}(\textstyle\bigwedge^3 V_6) \smallsetminus (\Delta \cup \Sigma) \longrightarrow {}^{[2]}\mathscr{M}_2^{(1)}$$

and an injective period map

$$\wp_2\colon {}^{[2]}\mathscr{M}_2^{(1)} \hookrightarrow {}^{[2]}\mathscr{P}_2^{(1)}.$$

O'Grady proved in [?, Proposition 4.8] that the composition $\wp_2 \circ \mu_2$ extends to a regular morphism

$$\overline{\wp}_2\colon \mathrm{LG}(\textstyle\bigwedge^3 V_6) \smallsetminus \Sigma' \longrightarrow {}^{[2]}\mathscr{P}_2^{(1)}, \tag{9}$$

where $\Sigma' \subsetneq \Sigma$ is a proper closed subset.

There is a degree-2 finite map

$$\rho\colon {}^{[2]}\mathscr{P}_2^{(1)} \longrightarrow {}^{[3]}\mathscr{P}_4^{(2)}$$

defined in Equation (**??**) and the main theorem of [?, Corollary 6.1] says that

$$\rho \circ \wp_2 \circ \mu_2 = \wp_3 \circ \mu_3 \tag{10}$$

on $\mathrm{LG}(\bigwedge^3 V_6) \smallsetminus (\Delta \cup \Gamma \cup \Sigma)$. This implies that the composition $\wp_3 \circ \mu_3$ extends to a regular morphism

$$\overline{\wp}_3 \coloneqq \rho \circ \overline{\wp}_2\colon \mathrm{LG}(\textstyle\bigwedge^3 V_6) \smallsetminus \Sigma' \longrightarrow {}^{[3]}\mathscr{P}_4^{(2)}. \tag{11}$$

5.2. **Projective resolutions of singular EPW cubes.** We show here that one of the analytic contractions c_ε of Proposition **??** is projective.

Proposition 5.1. *Let $[A] \in \Gamma \smallsetminus \Sigma$. There exists a smooth quasi-polarized projective hyper-Kähler variety (Y, H) such that the big and nef line bundle H defines a small contraction $c\colon \mathsf{Y}_A \to \widetilde{\mathsf{Z}}_A^{\geq 2}$.*

Proof. We use the following lemma, whose proof goes as the proof of [?, Theorem 6.1].

Lemma 5.2. *Given a point $x \in {}^{[3]}\mathscr{P}_4^{(2)}$, there exists a smooth projective hyper-Kähler sixfold Y with a big and nef line bundle H of square 4 and divisibility 2 with period x.*

Let (Y_A, H) be the quasi-polarized hyper-Kähler sixfold associated to the period point $x \coloneqq \overline{\wp}_3([A])$ given by the lemma.

Consider the local universal deformation $f\colon (\mathscr{Y}, \mathscr{H}) \to (B, 0)$ of the pair (Y_A, H). For b general in B, the line bundle $\mathscr{H}_b$ is ample on $\mathscr{Y}_b$ and the pair $(\mathscr{Y}_b, \mathscr{H}_b)$ defines an element of the moduli space ${}^{[3]}\mathscr{M}_4^{(2)}$. Up to shrinking B to a smooth curve containing 0, we can suppose that for each $b \in B$, the point $\wp_3([\mathscr{Y}_b])$ is contained in the image of $\overline{\wp}_3$ and is, for all $b \neq 0$, the period of a (smooth) EPW cube.

Since the line bundle $\mathscr{H}$ is f-big and f-nef, the relative Kawamata–Viehweg vanishing theorem ([?, Theorem 1-2-5 and Remark 1-2-6]) implies $\mathbf{R}^{>0} f_* \mathscr{H} = 0$. Furthermore, by the relative Kawamata base-point-free theorem ([?, Theorem 3-1-1 and Remark 3-1-2]), the $\mathscr{O}_B$-algebra $\bigoplus_{m\geq 0} f_*\mathscr{H}^m$ is finitely generated and there is a B-morphism $\varphi\colon \mathscr{Y} \to \mathscr{Z}'$ to its projective spectrum such that the line bundle $\mathscr{H}$ is the pullback of a relatively ample line bundle on $\mathscr{Z}'$. The morphism φ then induces isomorphisms $\mathscr{Y}_b \xrightarrow{\sim} \mathscr{Z}'_b$ for $b \in B \smallsetminus \{0\}$, and a projective contraction $\mathsf{Y}_A \to \mathsf{Z}'$ on central fibres.

Consider the families

$$
(12)\qquad \begin{array}{ccc} \widetilde{\mathscr{X}}_B^{\geq 2} & & \mathscr{X}' \\ & \searrow \quad \swarrow & \\ & B, & \end{array}
$$

where the morphism $\widetilde{\mathscr{X}}_B^{\geq 2} \to B$, with central fiber $\widetilde{\mathsf{Z}}_A^{\geq 2}$, is obtained by lifting the curve B to $\mathrm{LG}(\bigwedge^3 V_6)$ using the period map $\wp_3$ as in [**?**], and comes with a relatively ample line bundle (which is the pullback of the hyperplane section of $\mathsf{Z}_A^{\geq 2} \subset \mathrm{Gr}(3, V_6)$ on each fiber).

Over each point of $B \smallsetminus \{0\}$, the fibers of the two morphisms in the diagram (**??**) are isomorphic polarized hyper-Kähler manifolds with the same period map hence, by the Torelli theorem, there exists an isomorphism between the two families that respects the polarizations. By separatedness of the moduli space of polarized varieties with trivial canonical bundle (see for example [**?**, Theorem 2.1]), the central fibers $\widetilde{\mathsf{Z}}_A^{\geq 2}$ and Z' of these two morphisms are isomorphic.

Therefore, we have a projective birational contraction $c\colon \mathsf{Y}_A \to \widetilde{\mathsf{Z}}_A^{\geq 2}$, defined by the sections of a sufficiently large multiple mH of H. Taking the top self-intersections of both sides of the equality $c^* H_A = mH$, we obtain $m = 1$, so in fact, H defines c.

In particular, $\widetilde{\mathsf{Z}}_A^{\geq 2}$ is isomorphic outside a closed subset of codimension ≥ 2 to an open subset of Y_A, which has trivial canonical bundle. It follows that its smooth locus $(\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}}$ also has trivial canonical bundle.

It remains to show that c is small. For that, we first show that the image of the exceptional locus of c is the singular locus of $\widetilde{\mathsf{Z}}_A^{\geq 2}$. Consider the restriction

$$
c'\colon c^{-1}\big((\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}}\big) \longrightarrow (\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}}
$$

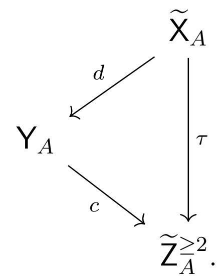
of c to the preimage of the smooth locus of $\widetilde{\mathsf{Z}}_A^{\geq 2}$. It is a birational morphism between smooth varieties, hence

$$
K_{c^{-1}((\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}})} \underset{\mathrm{lin}}{\equiv} K_{(\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}}} + \mathrm{Exc}(c').
$$

Since the canonical divisors $K_{c^{-1}((\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}})}$ and $K_{(\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}}}$ are trivial, we obtain that c' is an isomorphism. Therefore, c is an isomorphism over $(\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}}$. This implies that the image of the exceptional locus of c is the finite set of singular points of $\widetilde{\mathsf{Z}}_A^{\geq 2}$, and therefore the contraction is small by [**?**, Proposition (1.4)]. □

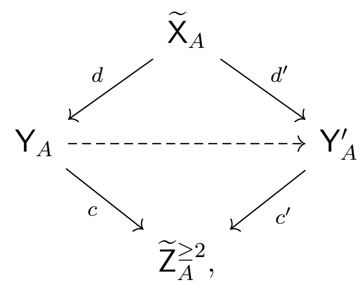
Given $[A] \in \Gamma \smallsetminus \Sigma$, we now prove that the blowup $\tau\colon \widetilde{\mathsf{X}}_A \to \widetilde{\mathsf{Z}}_A^{\geq 2}$ of $\widetilde{\mathsf{Z}}_A^{\geq 2}$ along its finite singular locus Z_A^4 factors through the small contraction c constructed above.

Theorem 5.3. *Let $[A] \in \Gamma \smallsetminus \Sigma$ be a Lagrangian subspace of $\bigwedge^3 V_6$. The resolution $\tau\colon \widetilde{\mathsf{X}}_A \to \widetilde{\mathsf{Z}}_A^{\geq 2}$ obtained by blowing up the finite singular locus of the EPW cube $\widetilde{\mathsf{Z}}_A^{\geq 2}$ factors through the small* *contraction c constructed in Proposition **??**:*



The morphism $d\colon \widetilde{\mathsf{X}}_A \to \mathsf{Y}_A$ contracts each component $E_i \subset \mathbf{P}^3 \times \mathbf{P}^3$ of the exceptional locus of τ to a copy of $\mathbf{P}^3$ and the small contraction $c\colon \mathsf{Y}_A \to \widetilde{\mathsf{Z}}_A^{\geq 2}$ contracts each such $\mathbf{P}^3$ to a singular point of $\widetilde{\mathsf{Z}}_A^{\geq 2}$.

Moreover, there exists another projective variety Y'_A and a diagram



where $\mathsf{Y}_A \dashrightarrow \mathsf{Y}'_A$ is the flop of every $\mathbf{P}^3$ in the exceptional locus of c.

We will need the following lemma.

Lemma 5.4. *Let D be a divisor on $\widetilde{\mathsf{X}}_A$ such that the line bundle $\mathscr{O}_{E_i}(D|_{E_i})$ has nonzero sections for each component E_i of E. For any sufficiently ample divisor H on $\widetilde{\mathsf{Z}}_A^{\geq 2}$, the line bundle $\mathscr{O}_{\widetilde{\mathsf{X}}_A}(D + \tau^*H)$ is generated by its global sections.*

Proof. Consider the divisor τ_*D on $\widetilde{\mathsf{Z}}_A^{\geq 2}$. For H sufficiently ample, τ_*D+H is globally generated on $\widetilde{\mathsf{Z}}_A^{\geq 2}$. Write $\tau^*(\tau_*D) = D + D'$, where the support of D' is contained in E. For H sufficiently ample, we may also suppose that the linear system $|\tau^*H - D'|$ has no base points outside of E.

The first condition on H implies that, for any $x \notin E$, there exists an effective divisor F on $\widetilde{\mathsf{Z}}_A^{\geq 2}$ linearly equivalent to $\tau_*D + H$, whose support does not contain $\tau(x)$. We then have $\tau^*F = D + D' + \tau^*H$, hence $D + 2\tau^*H \underset{\text{lin}}{\equiv} \tau^*F + \tau^*H - D'$. The second condition on H implies that x is not in the base locus of $D + 2\tau^*H$.

It remains to show that $D+2\tau^*H$ has no base-points on E. The divisor $(D+2\tau^*H)|_{E_i} = D|_{E_i}$ is effective by hypothesis hence is globally generated on E_i (like every effective divisor on E_i). Therefore, it is enough to show that we can lift the sections of $(D + 2\tau^*H)|_E$ to $\widetilde{\mathsf{X}}_A$. For that, the vanishing

$$H^1(\widetilde{\mathsf{X}}_A, D + 2\tau^*H - E) = 0$$

is enough. Using the Leray spectral sequence, it is enough to show that

$$H^1\big(\widetilde{\mathsf{Z}}_A^{\geq 2}, (\tau_*\mathscr{O}_{\widetilde{\mathsf{X}}_A}(D - E))(2H)\big) = H^0\big(\widetilde{\mathsf{Z}}_A^{\geq 2}, (R^1\tau_*\mathscr{O}_{\widetilde{\mathsf{X}}_A}(D - E))(2H)\big) = 0.$$

The vanishing of the H^1 holds for H sufficiently ample ([**?**, Proposition 5.3]). For the vanishing of the H^0, we need to show $R^1\tau_*\mathscr{O}_{\widetilde{\mathsf{X}}_A}(D-E)=0$. This follows from the theorem on formal functions ([**?**, Theorem 11.1]), noticing that $H^1(E,D-mE)=0$ for all $m\geq 0$ (because $E|_{E_i}=\mathscr{O}(-1,-1)$ and, again, $D|_{E_i}$ is effective). $\square$

Proof of the theorem. By [**?**, Corollary A.2], small normal partial resolutions of $\widetilde{\mathsf{Z}}_A^{\geq 2}$ correspond to nontrivial classes in $\mathrm{Cl}(\widetilde{\mathsf{Z}}_A^{\geq 2})/\operatorname{Pic}(\widetilde{\mathsf{Z}}_A^{\geq 2})$ modulo multiplication by positive integers. In particular, the existence of c implies that there exists a Weil divisor class $[D]$ on $\widetilde{\mathsf{Z}}_A^{\geq 2}$ which is not Cartier at any singular point (and such that Y_A is the blowup of $[D]$).

As before, we denote by $z_1,\dots,z_r$ the points of Z_A^4 and by E_i the component of the exceptional divisor E of τ over the point z_i.

Construction of a divisor L on $\widetilde{\mathsf{X}}_A$ whose restriction to each E_i is of the form $\mathscr{O}_{E_i}(a_i,b_i)$ with $a_i\neq b_i$. The blowup τ induces an isomorphism $\mathrm{Cl}(\widetilde{\mathsf{Z}}_A^{\geq 2})\simeq \operatorname{Pic}(\widetilde{\mathsf{X}}_A)/\bigoplus_i \mathbf{Z}E_i$. Let L be a divisor on $\widetilde{\mathsf{X}}_A$ whose class corresponds to the nontrivial Weil divisor class $[D]\in\mathrm{Cl}(\widetilde{\mathsf{Z}}_A^{\geq 2})$ and whose support does not contain any E_i.

Choose a neighborhood $\mathsf{Z}_i\subset\widetilde{\mathsf{Z}}_A^{\geq 2}$ of $g^{-1}(z_i)$ that contains no other singular points and set $\mathsf{X}_i\coloneqq\widetilde{\mathsf{X}}_A\times_{\widetilde{\mathsf{Z}}_A^{\geq 2}}\mathsf{Z}_i$ and $\mathsf{Y}_i\coloneqq\mathsf{Y}_A\times_{\widetilde{\mathsf{Z}}_A^{\geq 2}}\mathsf{Z}_i$. Then, $\mathsf{X}_i\to\mathsf{Z}_i$ is the blowup of the unique singular point $g^{-1}(z_i)$ of Z_i and $c_i\colon\mathsf{Y}_i\to\mathsf{Z}_i$ is a small projective resolution.

Again by [**?**, Corollary A.2], the contraction c_i corresponds to the nontrivial class of the Weil divisor $D_i\coloneqq D\cap\mathsf{Z}_i$ in $\mathrm{Cl}(\mathsf{Z}_i)/\operatorname{Pic}(\mathsf{Z}_i)$. As in [**?**, Equation (1.1)], there is an injection

$$j_i\colon\ \mathrm{Cl}(\mathsf{Z}_i)/\operatorname{Pic}(\mathsf{Z}_i)\hookrightarrow\operatorname{Pic}(E_i)/\mathbf{Z}(E_i|_{E_i})\simeq\mathbf{Z},$$

where the last isomorphism follows from the Lefschetz hyperplane theorem.

Therefore, $\mathrm{Cl}(\mathsf{Z}_i)/\operatorname{Pic}(\mathsf{Z}_i)\simeq\mathbf{Z}$ has rank 1 and is generated by the class of D_i. Since j_i is injective, the divisor class $j_i([D_i])=[L|_{E_i}]$ is not a multiple of $[E_i|_{E_i}]\simeq\mathscr{O}(-1,-1)$, hence is of the form $\mathscr{O}_{E_i}(a_i,b_i)$ with $a_i\neq b_i$.

Construction of smooth projective varieties Y_+ and Y_- and divisorial contractions $d_\pm\colon\widetilde{\mathsf{X}}_A\to\mathsf{Y}_\pm$. Let L be the divisor constructed above. There exist integers p_i and q_i such that $L+p_iE_i$ and $-L+q_iE_i$ restrict on E_i to $\mathscr{O}_{E_i}(c_i,0)$ and $\mathscr{O}_{E_i}(0,c_i)$, with $c_i>0$.

We set $L_1\coloneqq L+p_1E_1+\cdots+p_rE_r$ and $L_2\coloneqq -L+q_1E_1+\cdots+q_rE_r$. These divisors restrict to positive multiples of $\mathscr{O}(1,0)$ and $\mathscr{O}(0,1)$ on each E_i. By Lemma **??**, after adding to L_1 and L_2 the pullback of a sufficiently ample divisor on $\widetilde{\mathsf{Z}}_A^{\geq 2}$, we can suppose that they are globally generated. Therefore, by [**?**, Theorem 1 and Corollary 2], they define morphisms $d_\pm\colon\widetilde{\mathsf{X}}_A\to\mathsf{Y}_\pm$, with $\mathsf{Y}_\pm$ smooth projective, that map each $E_i\subset\mathbf{P}^3\times\mathbf{P}^3$ to either one of the two $\mathbf{P}^3$ and fit into factorizations

$$\tau\colon\widetilde{\mathsf{X}}_A\xrightarrow{d_\pm}\mathsf{Y}_\pm\xrightarrow{c_\pm}\widetilde{\mathsf{Z}}_A^{\geq 2},$$

where c_+ and c_- are small projective contractions that are not isomorphic over each Z_i.

End of the proof. Because of the isomorphism $\mathrm{Cl}(\mathsf{Z}_i)/\operatorname{Pic}(\mathsf{Z}_i)\simeq\mathbf{Z}$, the results of [**?**, Appendix A] imply that Z_i has exactly two projective small resolutions, namely $\mathsf{Y}_i^+=\mathrm{Bl}_{[D_i]}(\mathsf{Z}_i)$ and $\mathsf{Y}_i^-=\mathrm{Bl}_{[-D_i]}(\mathsf{Z}_i)$, and Y_i is one of those. Over each Z_i, the contractions c_+ and c_- are two distinct normal small projective resolutions of Z_i, hence they must coincide with Y_i^+ and Y_i^-.

Since Y_i is one of those, we have a morphism $\mathsf{X}_i \to \mathsf{Y}_i$. In this way, we extend the rational map $d := c^{-1} \circ \tau \colon \widetilde{\mathsf{X}}_A \dashrightarrow \mathsf{Y}_A$ to each X_i, hence to $\widetilde{\mathsf{X}}_A$, as a regular morphism.

Finally, Y_A is the blowup of, say, $[D]$, and [?, Corollary A.4.] shows that the blowup Y'_A of $[-D]$ is the flop of Y_A, and the same argument shows that there exists a morphism $d' \colon \widetilde{\mathsf{X}}_A \to \mathsf{Y}'_A$. ☐

6. No associated K3 surface

As mentioned earlier, O'Grady obtained in [?] results similar to our Theorem **??**, but for double EPW sextics: when $[A] \in \Delta \smallsetminus \Sigma$, the double EPW sextic associated to A is singular with small projective resolutions. In addition, he constructs a smooth K3 surface $S_A \subset \mathbf{P}^6$ and proves that the double EPW sextic is birationally isomorphic to the Hilbert square $S_A^{[2]}$.

In our case, we prove below that, on the contrary, when $[A] \in \Gamma \smallsetminus \Sigma$ is very general, the singular EPW cube $\widetilde{\mathsf{Z}}_A^{\geq 2}$ is *not* birationally isomorphic to any smooth moduli space of stable sheaves on a K3 surface. For this, we use the determination of the image by the period map of the divisor Γ obtained in Proposition **??** and results from [?].

Proposition 6.1. *When $[A] \in \Gamma \smallsetminus \Sigma$ is very general, the singular EPW cube $\widetilde{\mathsf{Z}}_A^{\geq 2}$ is not birationally isomorphic to any smooth moduli space of stable sheaves on a K3 surface.*

Proof. By [?, Theorem 3.7], a smooth hyper-Kähler variety X of $\mathrm{K3}^{[3]}$-type is birationally isomorphic to a smooth moduli space $M_v(S,H)$ for some K3 surface S if and only if X and S have Hodge isometric transcendental lattices. We will determine these lattices when X is a smooth model of a very general singular EPW cube $\widetilde{\mathsf{Z}}_A^{\geq 2}$ as in the proposition and when S is a (necessarily very general) polarized K3 surface, and prove that they cannot be isomorphic.

As shown in Proposition **??**, when $[A] \in \Gamma \smallsetminus \Sigma$ is very general, the singular EPW cube $\widetilde{\mathsf{Z}}_A^{\geq 2}$ is birationally isomorphic to a smooth quasi-polarized projective hyper-Kähler variety (Y_A, H) of $\mathrm{K3}^{[3]}$-type.

Let $\Lambda_{\mathrm{K3}^{[3]}} = \Lambda_{\mathrm{K3}} \oplus \mathbf{Z}(-4)$ be the lattice associated with hyper-Kähler manifolds of $\mathrm{K3}^{[3]}$-type and let h be a vector of square 4 and divisibility 2 in $\Lambda_{\mathrm{K3}^{[3]}}$. By Proposition **??**, the period point of Y_A is a very general point of the Heegner divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,12}$, hence the transcendental lattice of Y_A is isometric to $K^\perp \subset \Lambda$, where $K \subset \Lambda$ is a discriminant 12 lattice that contains h.

As explained in the proof of Proposition **??**, the lattice $h^\perp$ is of the form $M \oplus \mathbf{Z}k \oplus \mathbf{Z}\ell$, where $M = N \oplus U$ is unimodular and k and ℓ have square -2, and the lattice $K^\perp$ is equal to $\beta^\perp \subset h^\perp$ for some vector $\beta \in h^\perp$ of square 12, divisibility 2 and class $\beta_* = (1,1) \in A_{h^\perp}$ (up to isometries of $O(h^\perp)$, this vector is unique by [?, Satz 10.4]). Therefore, we can take $\beta = 2m + k + l$ with $m = u - v$, where (u,v) is a basis of $U \subset M$. The transcendental lattice of X is equal to

$$T = N \oplus \begin{pmatrix} 2 & 0 & 1 \\ 0 & -4 & -2 \\ 1 & -2 & -2 \end{pmatrix},$$

where a basis of the nonunimodular part is given by $(u+v, k-\ell, v+k)$.

The transcendental lattice of a very general polarized K3 surface of degree $2e$ is isomorphic to $N \oplus U \oplus \mathbf{Z}(-2e)$. We show that the lattice T defined above is not of this form by proving that there exist no vectors $w \in T$ of divisibility 1 and square 0 orthogonal to N. Such a vector

w would be of the form $x(u+v)+y(k-\ell)+z(v+k)$. We compute its divisibility and square in terms of x,y,z:

$$\operatorname{div}(w)=(2x+z,-4y-2z,x-2y-2z)$$

and

$$w^2=2x^2-4y^2-2z^2+2xz-4yz=2(x^2-z^2+xz-2(y^2+yz)).$$

We now impose $w^2=0$ and $\operatorname{div}(w)=1$. The first condition implies

$$x^2-z^2+xz-2(y^2+yz)=0.$$

In particular, x^2+z^2+xz is even, hence x and z are both even, which is absurd since w has divisibility 1. □

Remark 6.2. More generally, [**?**, Corollary 4.4] shows that for $[A]\in\Gamma\smallsetminus\Sigma$ very general, the singular EPW cube $\widetilde{\mathsf{Z}}_A^{\geq 2}$ is not birationally isomorphic to any moduli space of *twisted* sheaves on a K3 surface.

Appendix A. Divisors in the period domain ${}^{[3]}\mathscr{P}_4^{(2)}$

In this appendix, we define Heegner divisors in period domains and prove some of their properties in cases that concern us. We then analyze the images in the period domains of the divisors Δ, Γ, and Σ defined in Section **??** and prove that they are Heegner divisors.

A.1. Heegner divisors in the period domains ${}^{[2]}\mathscr{P}_2^{(1)}$ and ${}^{[3]}\mathscr{P}_4^{(2)}$.

We start by introducing some notation, following [**?**, Sections 3.9 and 3.10].

Let $\Lambda_{K3^{[m]}}$ be the lattice associated to hyper-Kähler manifolds of K3$^{[m]}$-type, and let τ be a polarization type, namely the $O(\Lambda_{K3^{[m]}})$-orbit of a vector $h\in\Lambda_{K3^{[m]}}$. The period domain of polarized hyper-Kähler manifolds of K3$^{[m]}$-type and polarization type τ is the quotient

$${}^{[m]}\mathscr{P}_\tau:=\mathscr{D}_{h^\perp}/\widehat{O}(\Lambda_{K3^{[m]}},h),$$

where $\mathscr{D}_{h^\perp}\subset\mathbf{P}(h^\perp\otimes\mathbf{C})$ is the open set of $x\in\mathbf{P}(h^\perp\otimes\mathbf{C})$ such that $x^2=0$ and $x\cdot\bar{x}>0$, and $\widehat{O}(\Lambda_{K3^{[m]}},h)$ is the group of isometries of $\Lambda_{K3^{[m]}}$ that fix the vector h and act as $\pm\mathsf{id}$ on the discriminant group of $\Lambda_{K3^{[m]}}$.

When the divisibility $\operatorname{div}(h)$ of h is equal to 1 or 2, the polarization type τ of h is uniquely determined by h^2 and $\operatorname{div}(h)$ (see [**?**, Remark 3.23]). Therefore, we denote by ${}^{[2]}\mathscr{P}_2^{(1)}$ the period space of polarized hyper-Kähler manifolds of K3$^{[2]}$-type with polarization of square 2 and divisibility 1, and by ${}^{[3]}\mathscr{P}_4^{(2)}$ the period space of polarized hyper-Kähler manifolds of K3$^{[3]}$-type with polarization of square 4 and divisibility 2.

There exists a finite degree 2 map

$$\rho\colon {}^{[2]}\mathscr{P}_2^{(1)}\to{}^{[3]}\mathscr{P}_4^{(2)}. \tag{13}$$

Indeed, the period spaces ${}^{[2]}\mathscr{P}_2^{(1)}$ and ${}^{[3]}\mathscr{P}_4^{(2)}$ are obtained as quotients of the same domain $\mathscr{D}$ by two groups of isometries Γ_2 and Γ_3, such that Γ_2 has index 2 in Γ_3 ([**?**, Proposition 6.3]).

Definition A.1 (Heegner divisors). Let $K\subset\Lambda_{K3^{[m]}}$ be a rank-2 lattice of real signature $(1,1)$ that contains h. The *Heegner divisor* ${}^{[m]}\mathscr{D}_{\tau,K}$ of ${}^{[m]}\mathscr{P}_\tau$ is the image of the hyperplane section $\mathbf{P}(K^\perp\otimes\mathbf{C})\cap\mathscr{D}_{h^\perp}$ in the quotient $\mathscr{D}_{h^\perp}/\widehat{O}(\Lambda_{K3^{[m]}},h)$. Moreover, for $d>0$, we set

$${}^{[m]}\mathscr{D}_{\tau,d}:=\bigcup_{K,\ \operatorname{disc}(K^\perp)=-d}{}^{[m]}\mathscr{D}_{\tau,K},$$

where K runs over the rank-2 sublattices of $\Lambda_{K3^{[m]}}$ with real signature $(1,1)$ that contain h. Note that the Heegner divisor ${}^{[m]}\mathscr{D}_{\tau,K}$ only depends on $\beta^\perp$, where β is any nonzero vector in $K \cap h^\perp$.

Following the notation of [?], we denote by ${}^{[2]}\mathscr{D}^{(1)}_{2,d}$ the Heegner divisors in the period space ${}^{[2]}\mathscr{P}^{(1)}_2$, and by ${}^{[3]}\mathscr{D}^{(2)}_{4,d}$ the Heegner divisors in the period space ${}^{[3]}\mathscr{P}^{(2)}_4$. We compare these Heegner divisors using the double cover $\rho\colon {}^{[2]}\mathscr{P}^{(1)}_2 \to {}^{[3]}\mathscr{P}^{(2)}_4$ defined in (??).

Heegner divisors in ${}^{[2]}\mathscr{P}^{(1)}_2$ are described in [?, Proposition 3.29]: the divisor ${}^{[2]}\mathscr{D}^{(1)}_{2,d}$ is nonempty if and only $d = 2e$ and $e \not\equiv 3 \pmod 4$. Moreover, if e is even, the divisor ${}^{[2]}\mathscr{D}^{(1)}_{2,2e}$ is irreducible, while it has 2 components for $e \equiv 1 \pmod 4$. We give a similar description of Heegner divisors in ${}^{[3]}\mathscr{P}^{(2)}_4$.

Proposition A.2. *Let d be a positive integer. The Heegner divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,d} \subset {}^{[3]}\mathscr{P}^{(2)}_4$ is nonempty if and only if d is an even number $2e$ such that $e \not\equiv 3 \pmod 4$. When this is the case, ${}^{[3]}\mathscr{D}^{(2)}_{4,2e}$ is irreducible and is the image by ρ of the Heegner divisor ${}^{[2]}\mathscr{D}^{(1)}_{2,2e} \subset {}^{[2]}\mathscr{P}^{(1)}_2$.*

Proof. Recall that ${}^{[3]}\mathscr{P}^{(2)}_4 = \mathscr{D}_{h^\perp}/\widehat{O}(\Lambda_{\mathrm{K3}^{[3]}}, h)$, where $h \in \Lambda_{\mathrm{K3}^{[3]}}$ has square 4 and divisibility 2. As remarked in [?, Section 6.1], the lattice $h^\perp$ decomposes as

$$h^\perp = M \oplus \mathbf{Z}k \oplus \mathbf{Z}\ell, \tag{14}$$

where k and ℓ are vectors of square -2 and M is a unimodular lattice. The discriminant group $A_{h^\perp}$ is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Moreover the group $\widehat{O}(\Lambda_{\mathrm{K3}^{[3]}}, h)$ is equal to the whole group of isometries $O(h^\perp)$. Finally, the group $O(h^\perp)$ is generated by its subgroup $\widetilde{O}(h^\perp)$ of isometries that acts trivially on the discriminant group $A_{h^\perp}$, and the reflection r with respect to $k+\ell$, which exchanges k and ℓ and is the identity on M.

Let K be a rank-2 sublattice of $\Lambda_{\mathrm{K3}^{[3]}}$ of signature (1,1) that contains h, and denote by β a primitive generator of $K \cap h^\perp$. We determine when the Heegner divisor defined by K is contained in ${}^{[3]}\mathscr{D}^{(2)}_{4,d}$. Since β is in $h^\perp$, we can write, with respect to the decomposition (??),

$$\beta = am + bk + c\ell,$$

where $m \in M$ is a primitive vector and $(a,b,c) = 1$. Let $2t := m^2$. We compute $\beta^2 = 2ta^2 - 2b^2 - 2c^2$ and $\operatorname{div}(\beta) = (a, 2b, 2c) = (a, 2)$. In particular, β has divisibility 2 if and only if $2 \mid a$, and in this case, either b or c must be odd.

Using [?, Lemma 7.2], we compute the discriminant

$$\operatorname{disc}(\beta^\perp) = \frac{-\beta^2 \operatorname{disc}(h^\perp)}{\operatorname{div}(\beta)^2} = \frac{4}{\operatorname{div}(\beta)^2} \cdot 2(-ta^2 + b^2 + c^2), \tag{15}$$

of the lattice $\beta^\perp \subset h^\perp$, where we used $\operatorname{disc}(h^\perp) = 4$. Since $\operatorname{div}(\beta) \mid 2$, it follows that $\operatorname{disc}(\beta^\perp)$ is equal to $-\beta^2$ when $2 \mid a$, and to $-4\beta^2$ when $2 \nmid a$. In particular, since β^2 is even, $\operatorname{disc}(\beta^\perp)$ is an even number $2e$.

Moreover, e cannot be equal to 3 modulo 4. Indeed, if $\operatorname{disc}(\beta^\perp) \equiv 6 \pmod 8$, then a must be even and $\operatorname{disc}(\beta^\perp) = -\beta^2 = -2ta^2 + b^2 + c^2$, which is never equal to 6 modulo 8.

Conversely, we now show that for any $e \not\equiv 3 \pmod 4$, there exists a unique $O(h^\perp)$-orbit in $h^\perp$ of vectors β such that $\beta^\perp$ has discriminant $2e$. Using [?, Lemma 3.3], this implies that ${}^{[3]}\mathscr{D}^{(2)}_{4,2e}$ is irreducible.

We start by studying the orbit of a primitive vector $\beta \in h^\perp$ of square β^2 and class $\beta_* = [\beta/\operatorname{div}(\beta)] = (x,y)$ in $A_{h^\perp} = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. It is equal to

$$O(h^\perp)\beta = \widetilde{O}(h^\perp)\beta \cup \widetilde{O}(h^\perp)r(\beta).$$

By Eichler's Lemma, the orbit $\widetilde{O}(h^\perp)\beta$ is the set of vectors v in $h^\perp$ of square β^2 and class $v_* \in A_{h^\perp}$ equal to β_*. Hence, since r acts on $A_{h^\perp}$ by exchanging the components, the orbit $O(h^\perp)\beta$ is the set of primitive vectors $v \in h^\perp$ with square β^2 and class $v_* \in A_{h^\perp}$ equal to (x,y) or (y,x). Therefore, there are at most three $O(h^\perp)$-orbits of primitive vectors of given square, respectively of class $(0,0)$, $(0,1)$ or $(1,0)$, and $(1,1)$, in the discriminant group.

We compute the discriminant of $\beta^\perp$ in each case, using Equation (**??**):

(a) $\operatorname{div}(\beta) = 1$ hence $\beta_* = (0,0)$; in this case, $\operatorname{disc}(\beta^\perp) = -4\beta^2 = 2e$ for some $e \equiv 0 \pmod 4$.
(b) $\operatorname{div}(\beta) = 2$ and $\beta_* \in \{(0,1),(1,0)\}$; in this case, a is even and exactly one among b and c is odd. Therefore, $\operatorname{disc}(\beta^\perp) = -\beta^2 = 2e$ for $e = -ta^2 + b^2 + c^2 \equiv 1 \pmod 4$.
(c) $\operatorname{div}(\beta) = 2$ and $\beta_* = (1,1)$; in this case, a is even, and b and c are odd, hence $\operatorname{disc}(\beta^\perp) = -\beta^2 = 2e$ for $e = -ta^2 + b^2 + c^2 \equiv 2 \pmod 4$.

In particular, $\operatorname{disc}(\beta^\perp)$ determines β^2 and $\beta_* \in \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ up to coordinates swap, hence the $O(h^\perp)$-orbit of β.

Finally, to compare Heegner divisors of ${}^{[2]}\mathscr{P}_2^{(1)}$ and ${}^{[3]}\mathscr{P}_4^{(2)}$, recall from [**?**, Proposition 6.3] that ${}^{[2]}\mathscr{P}_2^{(1)} = \mathscr{D}_{h^\perp}/\widetilde{O}(h^\perp)$ and the morphism ρ is induced by the action of the reflection r on ${}^{[2]}\mathscr{P}_2^{(1)}$. Hence, ${}^{[3]}\mathscr{D}_{4,2e}^{(2)}$ is the image of ${}^{[2]}\mathscr{D}_{2,2e}^{(1)}$ by ρ. □

Remark A.3. The previous proof also shows that the primitive vectors $\beta \in h^\perp$ associated to a Heegner divisor ${}^{[3]}\mathscr{D}_{4,2e}^{(2)}$ (where $e \not\equiv 3 \pmod 4$) satisfy the following:

- if $e \equiv 0 \pmod 4$, the vectors β have divisibility 1 and square $e/2$;
- if $e \equiv 1, 2 \pmod 4$, the vectors β have divisibility 2 and square $2e$.

A.2. **Image by $\overline{\wp}_3$ of the divisors Δ, Γ, and Σ.** In this section, we prove that the images of the divisors Δ, Γ, and Σ defined in Section **??** by the period map

$$\overline{\wp}_3 \colon \operatorname{LG}(\textstyle\bigwedge^3 V_6) \smallsetminus \Sigma' \longrightarrow {}^{[3]}\mathscr{P}_4^{(2)}.$$

defined in (**??**) are distinct Heegner divisors. For this, we will use Gushel–Mukai varieties and their periods (for definitions, the reader may consult [**?**]).

Proposition A.4. *One has*

$$\overline{\overline{\wp}_3(\Delta \smallsetminus \Sigma')} = {}^{[3]}\mathscr{D}_{4,10}^{(2)} \quad , \quad \overline{\overline{\wp}_3(\Gamma \smallsetminus \Sigma')} = {}^{[3]}\mathscr{D}_{4,12}^{(2)} \quad , \quad \overline{\overline{\wp}_3(\Sigma \smallsetminus \Sigma')} = {}^{[3]}\mathscr{D}_{4,8}^{(2)}.$$

In particular, the divisors Δ, Γ, and Σ are mutually distinct.

Proof. The images of these divisors in ${}^{[2]}\mathscr{P}_2^{(1)}$ by the period map $\overline{\wp}_2$ were determined in [**?**, Remark 5.29] and they are Heegner divisors:

$$\begin{aligned}
\overline{\overline{\wp}_2(\Delta \smallsetminus \Sigma')} &= {}^{[2]}\mathscr{D}_{2,10}^{\prime\prime(1)} = \overline{\{\text{periods of GM 4folds containing a } \sigma\text{-plane}\}} \\
&= \overline{\{\text{periods of GM sixfolds containing a } \mathbf{P}^3\}}, \\
\overline{\overline{\wp}_2(\Gamma \smallsetminus \Sigma')} &= {}^{[2]}\mathscr{D}_{2,12}^{(1)} = \overline{\{\text{periods of GM 4folds containing a } \tau\text{-plane}\}}, \\
\overline{\overline{\wp}_2(\Sigma \smallsetminus \Sigma')} &= {}^{[2]}\mathscr{D}_{2,8}^{(1)} = \overline{\{\text{periods of nodal GM 4folds}\}}
\end{aligned}$$

(where ${}^{[2]}\mathscr{D}''^{(1)}_{2,10}$ is one of the components of the reducible Heegner divisor ${}^{[2]}\mathscr{D}^{(1)}_{2,10}$). To conclude, it is then enough to apply Proposition **??**, the relation (**??**), and the fact that the Heegner divisors ${}^{[3]}\mathscr{D}^{(2)}_{4,10}$, ${}^{[3]}\mathscr{D}^{(2)}_{4,12}$, and ${}^{[3]}\mathscr{D}^{(2)}_{4,8}$ are mutually distinct. □

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