HARMONIC ANALYSIS FOR A MULTIDIMENSIONAL DISCRETE LAPLACIAN

ÓSCAR CIAURRI

ABSTRACT. In this paper we analyze some classical operators in harmonic analysis associated to the multidimensional discrete Laplacian

$$\Delta_N f(\mathbf{n}) = \sum_{i=1}^{N} (f(\mathbf{n}+\mathbf{e}_i) - 2f(\mathbf{n}) + f(\mathbf{n}-\mathbf{e}_i)), \qquad \mathbf{n} \in \mathbb{Z}^N.$$

We deal with the heat and Poisson semigroups, the fractional integrals, the Riesz transforms, the fractional powers of the Laplacian, and the g_k-square functions.

Contents

2010 *Mathematics Subject Classification.* Primary: 42C10.

Key words and phrases. Discrete harmonic analysis, heat semigroup, Poisson semigroup, fractional integral, Riesz transform, fractional Laplacian, g_k-function.

The author was supported by grant PID2021-124332NB-C22 AEI, from Spanish Government.

1. Introduction

Our target is to analyze mapping properties of some operators related to the multidimensional discrete Laplacian

$$\Delta_N f(\mathbf{n}) = \sum_{i=1}^{N} \Delta_{N,i} f(\mathbf{n}),$$

where

$$\Delta_{N,i} f(\mathbf{n}) = f(\mathbf{n}+\mathbf{e}_i) - 2f(\mathbf{n}) + f(\mathbf{n}-\mathbf{e}_i),$$

with $\{\mathbf{e}_i\}_{i=1,\dots,N}$ being N-dimensional vectors each of whose components are all zero, except the i-th that equals one. This operator is the natural extension to higher dimensions of the one-dimensional discrete Laplacian studied in [7] and [8]. We have to observe that

$$\Delta_{N,i} f(\mathbf{n}) = \delta_i^- \delta_i^+ f(\mathbf{n}) \tag{1}$$

with

$$\delta_i^+ f(\mathbf{n}) = f(\mathbf{n}+\mathbf{e}_i) - f(\mathbf{n}) \qquad \text{and} \qquad \delta_i^- f(\mathbf{n}) = f(\mathbf{n}) - f(\mathbf{n}-\mathbf{e}_i).$$

Note that $\delta_i^- \delta_i^+ = \delta_i^+ \delta_i^-$. When we consider sequences of one variable we write $\delta^+ f(n) = f(n+1) - f(n)$ and similarly for $\delta^- f$ and Δf.

We will focus in this paper on the study of the heat and Poisson semigroups, the fractional integrals, the Riesz transforms, the fractional powers of the Laplacian, and the g_k-square functions related to Δ_N.

The heat semigroup $W_t = e^{t\Delta_N}$ is the solution of the N-dimensional heat equation

$$\begin{cases} \Delta_N u(\mathbf{n},t) = \dfrac{d}{dt} u(\mathbf{n},t), \\ u(\mathbf{n},t) = f(\mathbf{n}), \end{cases}$$

and it is given by (see [12] for the one-dimensional case)

$$W_t f(\mathbf{n}) = \sum_{\mathbf{k}\in\mathbb{Z}^N} G_{t,N}(\mathbf{n}-\mathbf{k}) f(\mathbf{k}), \tag{2}$$

where, for $\mathbf{n} = (n_1, \dots, n_N)$,

$$G_{t,N}(\mathbf{n}) = \prod_{k=1}^{N} (e^{-2t} I_{n_k}(2t)),$$

with I_a being the modified Bessel function of first kind and order a. This semigroup will be the main tool to analyze our discrete multidimensional operators.

M. Riesz initiated the study of discrete operators in harmonic analysis in [21], where he proved the boundedness of the Hilbert transform in $L^p(\mathbb{R})$ and its discrete analogue in $\ell^p(\mathbb{Z})$. Later, in [5] A. P. Calderón and A. Zygmund obtained the boundedness in $\ell^p(\mathbb{Z}^N)$ of the discrete analogues of the singular integrals from his result in $L^p(\mathbb{R}^N)$. Weighted inequalities for the discrete Hilbert and discrete maximal operator in the one-dimensional case were proved by R. Hunt, B. Muckenhoupt, and R. Wheeden in [13]. In the last few decades, several discrete analogues of continuous operators have been analyzed; some significative contributions are due to I. Arkhipov and K. I. Oskolkov [3], J. Bourgain [4], and E. M. Stein and S. Wainger [26, 27]. In [13], we can find a brief history and a nice exposition on discrete analogues of classical operators in harmonic analysis.

In general, along the paper will work on weighted Lebesgue spaces of sequences on $\mathbb{Z}^N$. More precisely, we define

$$\ell^p(\mathbb{Z}^n, w) = \left\{ f = \{f(\mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^N} : \|f\|_{\ell^p(\mathbb{Z}^N,w)} < \infty \right\}, \qquad 1 \le p \le \infty,$$

where the weight $w = \{w(\mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^N}$ is a sequence of positive numbers and

$$\|f\|_{\ell^p(\mathbb{Z}^N,w)} = \left(\sum_{\mathbf{n}\in\mathbb{Z}^N} |f(\mathbf{n})|^p w(\mathbf{n}) \right)^{1/p}, \qquad 1 \le p < \infty,$$

and

$$\|f\|_{\ell^\infty(\mathbb{Z}^N,w)} = \sup\{|f(\mathbf{n})| : \mathbf{n} \in \mathbb{Z}^N\}.$$

When $w(\mathbf{n}) = 1$ for all $\mathbf{n} \in \mathbb{Z}^N$ we will write $\ell^p(\mathbb{Z}^N)$ only.

The paper is organized as follows. In the next section we present the main tools to deal with the kernel $G_{t,N}$. Section 3 will be focused on the analysis of the heat and Poisson semigroups. Section 4 contains our results about the fractional integrals and the Riesz transforms. In Section 5 we present some results on the fractional powers of Δ_N. Finally, in Section 6 we analyze the mapping properties of the g_k-square functions.

2. Some facts about the modified Bessel functions of first kind

The modified Bessel function of first kind and order a, with $a \in \mathbb{R}$, is given by

$$I_a(x) = \sum_{k=0}^{\infty} \frac{c_{k,a}}{k!} \left(\frac{x}{2}\right)^{2k+a}, \qquad x > 0,$$

where

$$c_{k,a} = \frac{1}{\Gamma(a+k+1)},$$

understanding that for $a \le -1$

$$c_{k,a} = \frac{(a+k+1)\cdots(a+[-a])}{\Gamma(a+[-a]+1)}, \qquad 0 \le k < \lfloor -a \rfloor.$$

In this section we compile some relevant inequalities for the modified Bessel functions of first kind. They will be the main tools to obtain our results.

2.1. An AM-GM type inequality for the modified Bessel functions of first kind.

In [28], we find the identity

$$I_a(x)I_b(x) = \sum_{k=0}^{\infty} \binom{a+b+2k}{k} c_{k,a}c_{k,b} \left(\frac{x}{2}\right)^{2k+a+b}, \tag{3}$$

which can be proved by applying the Cauchy product of two absolutely convergent series. From this fact, we can obtain the next inequalities for the modified Bessel functions.

Proposition 1. *For $a_1, \dots, a_n > -1$ and $x \ge 0$ the inequalities*

$$\frac{\left(\Gamma\left(\frac{a_1+\cdots+a_n}{n}+1\right)\right)^n}{\Gamma(a_1+1)\cdots\Gamma(a_n+1)} \left(I_{\frac{a_1+\cdots+a_n}{n}}(x)\right)^n \le I_{a_1}(x)\cdots I_{a_n}(x) \le \left(I_{\frac{a_1+\cdots+a_n}{n}}(x)\right)^n \tag{4}$$

hold.

Proof. We will provide an *à la Cauchy* proof of this AM-GM type inequality, obtaining the cases $n = 2^m$ first and extending the result for general n. To prove the case $n = 2$ we are going to check that

$$\frac{c_{0,a_1} c_{0,a_2}}{\left(c_{0,\frac{a_1+a_2}{2}}\right)^2} \leq \frac{c_{k,a_1} c_{k,a_2}}{\left(c_{k,\frac{a_1+a_2}{2}}\right)^2} \leq 1. \tag{5}$$

From the Weierstrass' product formula for the Gamma function

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(\left(1 + \frac{z}{n}\right)^{-1} e^{z/n} \right), \qquad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\},$$

we have

$$\frac{c_{k,a_1} c_{k,a_2}}{\left(c_{k,\frac{a_1+a_2}{2}}\right)^2} = \prod_{n=0}^{\infty} \frac{(n + a_1 + k + 1)(n + a_2 + k + 1)}{\left(n + \frac{a_1+a_2}{2} + k + 1\right)^2}.$$

Then, by using that

$$\frac{(n + a_1 + 1)(n + a_2 + 1)}{\left(n + \frac{a_1+a_2}{2} + 1\right)^2} \leq \frac{(n + a_1 + k + 1)(n + a_2 + k + 1)}{\left(n + \frac{a_1+a_2}{2} + k + 1\right)^2} \leq 1$$

the inequalities in (5) follow.

Now, from (3), (5) and the identity

$$\frac{c_{0,a_1} c_{0,a_2}}{\left(c_{0,\frac{a_1+a_2}{2}}\right)^2} = \frac{\left(\Gamma\left(\frac{a_1+a_2}{2} + 1\right)\right)^2}{\Gamma(a_1 + 1)\,\Gamma(a_2 + 1)},$$

we have

$$\begin{aligned} &\frac{\left(\Gamma\left(\frac{a_1+a_2}{2} + 1\right)\right)^2}{\Gamma(a_1 + 1)\,\Gamma(a_2 + 1)} \binom{(a_1 + a_2)/2 + (a_1 + a_2)/2 + 2k}{k} \left(c_{k,\frac{a_1+a_2}{2}}\right)^2 \\ &\quad \leq \binom{a_1 + a_2 + 2k}{k} c_{k,a_1} c_{k,a_2} \leq \binom{(a_1 + a_2)/2 + (a_1 + a_2)/2 + 2k}{k} \left(c_{k,\frac{a_1+a_2}{2}}\right)^2 \end{aligned}$$

and

$$\frac{\left(\Gamma\left(\frac{a_1+a_2}{2} + 1\right)\right)^2}{\Gamma(a_1 + 1)\,\Gamma(a_2 + 1)} \left(I_{\frac{a_1+a_2}{2}}(x)\right)^2 \leq I_{a_1}(x) I_{a_2}(x) \leq \left(I_{\frac{a_1+a_2}{2}}(x)\right)^2.$$

With an elementary induction process, it can be deduced (4) for $n = 2^m$. When n is not a power of two, we can find m such that $n < 2^m$. Taking the values $b_1 = a_1$, $\dots$, $b_n = a_n$, and

$$b_k = \frac{a_1 + \cdots + a_n}{n}, \qquad n + 1 \leq k \leq 2^m,$$

from (4) with $n = 2^m$, we have

$$\begin{aligned} &\frac{\left(\Gamma\left(\frac{b_1+\cdots+b_{2^m}}{2^m} + 1\right)\right)^{2^m}}{\Gamma(b_1 + 1) \cdots \Gamma(b_{2^m} + 1)} \left(I_{\frac{b_1+\cdots+b_{2^m}}{2^m}}(x)\right)^{2^m} \\ &\qquad \leq I_{b_1}(x) I_{b_2}(x) \cdots I_{b_n}(x) I_{b_{n+1}}(x) \cdots I_{b_{2^m}}(x) \leq \left(I_{\frac{b_1+\cdots+b_{2^m}}{2^m}}(x)\right)^{2^m}. \end{aligned}$$

Finally, applying that

$$\frac{b_1 + \cdots + b_{2^m}}{2^m} = \frac{a_1 + \cdots + a_n}{n}$$

and

$$I_{b_{n+1}}(x)\cdots I_{b_{2^m}}(x) = \left(I_{\frac{a_1+\cdots+a_n}{n}}(x)\right)^{2^m-n}$$

the general case of (4) follows. □

The result in the previous proposition is a particular case of [29, Theorem 2.7]. We have included this new proof because it is elementary and to do this paper self-contained.

Taking the normalized modified Bessel functions of the first kind

$$\mathcal{I}_a(x) = \frac{2^a\Gamma(a+1)}{x^a} I_a(x),$$

that verify $\mathcal{I}_a(0) = 1$, the left inequality in (4) reads

$$\left(\mathcal{I}_{\frac{a_1+\cdots+a_n}{n}}(x)\right)^n \leq \mathcal{I}_{a_1}(x)\cdots\mathcal{I}_{a_n}(x).$$

2.2. **Some inequalities for the difference of modified Bessel functions of first kind.** It is easy to check that the modified Bessel functions decrease with the order; i.e., $I_a(x) < I_b(x)$ when $-1 < b < a$ and $x > 0$ (note that $c_{k,a} < c_{k,b}$ in this case). We will use this fact sometimes along the paper without explicit mention to it.

In our next result we analyze the difference of two consecutive modified Bessel functions of first kind.

Lemma 2. *For $a > -1$ and $x > 0$ the inequalities*

$$0 < I_a(x) - I_{a+1}(x) < \frac{2(a+1)}{a+1+x} I_a(x) \tag{6}$$

hold.

Proof. The lower bound is a consequence of the monotonicity respect to the order of the functions $I_a(x)$. To obtain the upper bound, we use the inequalities [24, Theorem 2] (see also [2] and [18])

$$f_{a+1}(x) < \frac{I_{a+1}(x)}{I_a(x)} < f_{a+1/2}(x), \tag{7}$$

where

$$f_\alpha(x) = \frac{x}{\alpha + \sqrt{\alpha^2 + x^2}}$$

and the upper bound holds for $a \geq -1/2$ and the lower one for $a \geq -1$. Then, it is clear that

$$g_{a+1/2}(x) < 1 - \frac{I_{a+1}(x)}{I_a(x)} < g_{a+1}(x), \tag{8}$$

with

$$g_\alpha(x) = 1 - f_\alpha(x) = \frac{2\alpha}{\alpha + x + \sqrt{\alpha^2 + x^2}}$$

and now the upper bound holds for $a \geq -1$ and the lower one for $a \geq -1/2$. Then, by using the upper bound in (8), for $a > -1$, we have

$$I_a(x) - I_{a+1}(x) = \left(1 - \frac{I_{a+1}(x)}{I_a(x)}\right) I_a(x) < g_{a+1}(x) I_a(x) \leq \frac{2(a+1)}{a+1+x} I_a(x). \quad \square$$

Lemma 3. *For $a \geq -1/2$ and $x > 0$ the inequality*

$$|I_a(x) - 2I_{a+1}(x) + I_{a+2}(x)| < \left(\frac{3}{1+x} + \frac{4(a+1)(a+2)}{(a+1+x)^2}\right) I_a(x) \tag{9}$$

holds.

Proof. We estimate

$$\left|1 - 2\frac{I_{a+1}(x)}{I_a(x)} + \frac{I_{a+2}(x)}{I_a(x)}\right|.$$

To do this we use the obvious identity

$$1 - 2\frac{I_{a+1}(x)}{I_a(x)} + \frac{I_{a+2}(x)}{I_a(x)} = S_1 + S_2$$

with

$$S_1 = \frac{I_{a+2}(x)}{I_{a+1}(x)} - \frac{I_{a+1}(x)}{I_a(x)}$$

and

$$S_2 = \left(1 - \frac{I_{a+1}(x)}{I_a(x)}\right)\left(1 - \frac{I_{a+2}(x)}{I_{a+1}(x)}\right).$$

By using (7) and that $f_\alpha(x) < f_\beta(x)$ when $\beta < \alpha$, for $a \geq -1/2$ we have

$$|S_1| < \max\{f_{a+1}(x) - f_{a+3/2}(x), f_{a+1/2}(x) - f_{a+2}(x)\} = f_{a+1/2}(x) - f_{a+2}(x).$$

Now, with the bound

$$f_a(x) - f_{a+p}(x) = \frac{2px}{(a+\sqrt{a^2+x^2})(p+\sqrt{a^2+x^2}+\sqrt{(a+p)^2+x^2})}$$

$$\leq \frac{2p}{p+x}, \qquad p > 0, \tag{10}$$

we have $|S_1| < 3/(1+x)$.

For S_2 we use (8) to obtain that

$$S_2 < g_{a+1}(x)g_{a+2}(x) \leq \frac{4(a+1)(a+2)}{(a+1+x)(a+2+x)} < \frac{4(a+1)(a+2)}{(a+1+x)^2}$$

and the proof is finished. □

Finally, we estimate an alternating sum of four consecutive modified Bessel functions of first kind.

Lemma 4. *For $a \geq -1/2$ and $x > 0$ the inequality*

$$|I_a(x) - 3I_{a+1}(x) + 3I_{a+2}(x) - I_{a+3}(x)| < C\left(\frac{a+2}{(a+1+x)(1+x)} + \frac{(a+1)(a+2)(a+3)}{(a+1+x)^3}\right) I_a(x) \tag{11}$$

holds.

Proof. As in the previous proof, we focus on an appropriate bound for

$$\left|1 - 3\frac{I_{a+1}(x)}{I_a(x)} + 3\frac{I_{a+2}(x)}{I_a(x)} - \frac{I_{a+3}(x)}{I_a(x)}\right|.$$

In this time, we have

$$1 - 3\frac{I_{a+1}(x)}{I_a(x)} + 3\frac{I_{a+2}(x)}{I_a(x)} - \frac{I_{a+3}(x)}{I_a(x)} = T_1 + T_2 + T_3,$$

with

$$T_1 = -\frac{I_{a+1}(x)}{I_a(x)} + 2\frac{I_{a+2}(x)}{I_{a+1}(x)} - \frac{I_{a+3}(x)}{I_{a+2}(x)},$$

$$T_2 = \left(1 - \frac{I_{a+1}(x)}{I_a(x)}\right)\left(\frac{I_{a+3}(x)}{I_{a+2}(x)} - \frac{I_{a+2}(x)}{I_{a+1}(x)}\right) + \left(1 - \frac{I_{a+2}(x)}{I_{a+1}(x)}\right)\left(\frac{I_{a+3}(x)}{I_{a+2}(x)} - \frac{I_{a+1}(x)}{I_a(x)}\right)$$

and

$$T_3 = \left(1 - \frac{I_{a+1}(x)}{I_a(x)}\right)\left(1 - \frac{I_{a+2}(x)}{I_{a+1}(x)}\right)\left(1 - \frac{I_{a+3}(x)}{I_{a+2}(x)}\right).$$

We are going to estimate $|T_1|$ by using continued fractions. Let

$$\xi := \langle q_0, q_1, q_2, \dots \rangle = q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \cdots}}}$$

be a continued fraction. If

$$\langle q_0, q_1, \dots, q_n \rangle = \frac{P_n}{Q_n},$$

it is known [15, Ch. 2, Theorem 9] that

$$\left|\xi - \frac{P_n}{Q_n}\right| \le \frac{1}{Q_n Q_{n+1}}, \qquad n \ge 0. \tag{12}$$

In [19, 10.33.1], we find the expansion

$$\frac{I_{a+1}(x)}{I_a(x)} = \langle q_0(a,x), q_1(a,x), q_2(a,x), \dots \rangle, \qquad x > 0, \quad a > -1,$$

with $q_0(a,x) = 0$ and $q_k(a,x) = 2(a+k)/x$ for $k \ge 1$. Then, taking $n = 2$ in (12),

$$|T_1| \le \frac{1}{Q_2(a,x)Q_3(a,x)} + \frac{2}{Q_2(a+1,x)Q_3(a+1,x)} + \frac{1}{Q_2(a+2,x)Q_3(a+2,x)} + \left|\frac{P_2(a,x)}{Q_2(a,x)} - 2\frac{P_2(a+1,x)}{Q_2(a+1,x)} + \frac{P_2(a+2,x)}{Q_2(a+2,x)}\right|.$$

By using that

$$Q_2(a,x) = 4(a+1)(a+2) + x^2 \qquad \text{and} \qquad Q_3(a,x) = 4(a+2)(2(a+1)(a+3) + x^2)$$

and applying the inequalities

$$(a+1)^2 + x^2 \le Q_2(a,x) \le Q_2(a+1,x)$$

and

$$4(a+2)((a+1)^2 + x^2) \le Q_3(a,x) \le Q_3(a+1,x),$$

for $a > -1$ and $x > 0$, we have

$$\frac{1}{Q_2(a,x)Q_3(a,x)} + \frac{2}{Q_2(a+1,x)Q_3(a+1,x)} + \frac{1}{Q_2(a+2,x)Q_3(a+2,x)} \le \frac{1}{(a+2)((a+1)^2 + x^2)^2}. \tag{13}$$

Moreover, due to $P_2(a,x) = 2(a+2)x$, for $a \ge -1/2$

$$\frac{P_2(a,x)}{Q_2(a,x)} - 2\frac{P_2(a+1,x)}{Q_2(a+1,x)} + \frac{P_2(a+2,x)}{Q_2(a+2,x)} = \frac{16x(4(a+2)(a+3)(a+4) - x^2(8+3a))}{Q_2(a,x)Q_2(a+1,x)Q_2(a+2,x)}$$

and

$$\left|\frac{P_2(a,x)}{Q_2(a,x)} - 2\frac{P_2(a+1,x)}{Q_2(a+1,x)} + \frac{P_2(a+2,x)}{Q_2(a+2,x)}\right| \le C\frac{x(a+3)((a+2)(a+4)+x^2)}{((a+1)^2+x^2)^3} \le \frac{C}{(a+1)^2+x^2}. \tag{14}$$

In this way, from (13) and (14), we obtain that

$$|T_1| \le \frac{C}{(a+1)^2+x^2}, \qquad a \ge -1/2.$$

Now, from (7) and (8), it is clear that

$$|T_2| \le g_{a+1}(x)(f_{a+3/2}(x) - f_{a+3}(x)) + g_{a+2}(x)(f_{a+1/2}(x) - f_{a+3}(x))$$

and applying (10) we have

$$|T_2| \le \frac{C}{1+x}(g_{a+1}(x) + g_{a+2}(x)) \le C\frac{a+2}{(a+1+x)(1+x)}.$$

Finally, from (8), the bound

$$T_3 < g_{a+1}(x)g_{a+2}(x)g_{a+3}(x) \le \frac{8(a+1)(a+2)(a+3)}{(a+1+x)^3}$$

can be deduced and the proof of (11) is completed. □

2.3. **A uniform bound for the modified Bessel functions of first kind.** Now, we present a uniform bound respect to the order for the function $I_a(x)$. It will be fundamental all along the paper.

Lemma 5. *For $a > -1/2$, let α be a real number such that $-1/2 \le \alpha < a$. Then the inequality*

$$x^{-\alpha}e^{-x}I_a(x) \le \frac{C}{(a+1)^{2\alpha+1}} \tag{15}$$

holds with a constant C independent of a.

Proof. From the identity [19, 10.32.2]

$$I_a(x) = \frac{x^a}{\sqrt{\pi}2^a\Gamma(a+1/2)}\int_{-1}^{1} e^{-xs}(1-s^2)^{a-1/2}\,ds, \qquad a > -1/2,$$

by using the change of variable $x(1+s) = 2w$, we can deduce that

$$\begin{aligned} x^{-\alpha}e^{-x}I_a(x) &= \frac{2^a x^{-\alpha-1/2}}{\sqrt{\pi}\Gamma(a+1/2)}\int_0^x e^{-2w}w^{a-1/2}\left(1-\frac{w}{x}\right)^{a-1/2}dw \\ &= \frac{2^a}{\sqrt{\pi}\Gamma(a+1/2)}\int_0^x e^{-2w}w^{a-1-\alpha}\left(\frac{w}{x}\right)^{\alpha+1/2}\left(1-\frac{w}{x}\right)^{a-1/2}dw. \end{aligned}$$

We start with the case $a \geq 1/2$. Analyzing the maximum of the function $z^p(1-z)^q$, we deduce the inequality

$$z^p(1-z)^q \leq \frac{p^p q^q}{(p+q)^{p+q}}, \qquad p, q \geq 0, \quad 0 < z < 1,$$

where in the cases $p = 0$, $q = 0$, and $p = q = 0$ the bound has to be understood as one, and we have

$$\left(\frac{w}{x}\right)^{\alpha+1/2} \left(1 - \frac{w}{x}\right)^{a-1/2} \leq \frac{C}{a^{\alpha+1/2}},$$

with C depending on α but not on a. In this way,

$$\begin{aligned} x^{-\alpha} e^{-x} I_a(x) &\leq C \frac{2^a}{\Gamma(a+1/2) a^{\alpha+1/2}} \int_0^t e^{-2w} w^{a-1-\alpha}\, dw \\ &\leq \frac{C}{\Gamma(a+1/2) a^{\alpha+1/2}} \int_0^\infty e^{-r} r^{a-1-\alpha}\, dr \\ &= C \frac{\Gamma(a-\alpha)}{\Gamma(a+1/2) a^{\alpha+1/2}} \leq \frac{C}{a^{2\alpha+1}}, \end{aligned}$$

where in the last step we have applied that $\Gamma(r+p)/\Gamma(r+q) \sim r^{p-q}$, and this is enough for our purpose.

For $-1/2 < a < 1/2$ we have to prove that $x^{-\alpha} e^{-x} I_a(x) \leq C$. We consider the decomposition

$$\begin{aligned} x^{-\alpha} e^{-x} I_a(x) = {}& \frac{2^a}{\sqrt{\pi}\Gamma(a+1/2)} \\ & \times \left(\int_0^{x/2} + \int_{x/2}^x e^{-2w} w^{a-1-\alpha} \left(\frac{w}{x}\right)^{\alpha+1/2} \left(1 - \frac{w}{x}\right)^{a-1/2} dw \right). \end{aligned}$$

The first integral can be controlled easily. In fact,

$$\begin{aligned} & \int_0^{x/2} e^{-2w} w^{a-1-\alpha} \left(\frac{w}{x}\right)^{\alpha+1/2} \left(1 - \frac{w}{x}\right)^{a-1/2} dw \\ & \qquad \leq \frac{1}{2^{\alpha+a}} \int_0^{x/2} e^{-2w} w^{a-1-\alpha}\, dw \leq \frac{\Gamma(a-\alpha)}{2^{2a}}. \end{aligned}$$

For the second integral, with the change of variable $w = x(1-r)$, we have

$$\begin{aligned} & \int_{x/2}^x e^{-2w} w^{a-1-\alpha} \left(\frac{w}{x}\right)^{\alpha+1/2} \left(1 - \frac{w}{x}\right)^{a-1/2} dw \\ & \qquad \leq \left(\frac{x}{2}\right)^{a-1-\alpha} \int_{x/2}^x e^{-2w} \left(1 - \frac{w}{x}\right)^{a-1/2} dw \\ & \qquad = \frac{x^{a-\alpha}}{2^{a-1-\alpha}} \int_0^{1/2} e^{-2x(1-r)} r^{a-1/2}\, dr \\ & \qquad \leq \frac{x^{a-\alpha}}{2^{a-1-\alpha}} e^{-x} \int_0^{1/2} r^{a-1/2}\, dr \leq \frac{(a-\alpha)^{a-\alpha} e^{-a+\alpha}}{2^{a-1-\alpha}(a+1/2)}, \end{aligned}$$

where in the last step we have applied that $x^b e^{-x} \leq b^b e^{-b}$ for $x, b > 0$. Then, using that $0 < a - \alpha < 1$

$$x^{-\alpha} e^{-x} I_a(x) \leq \frac{2^a}{\sqrt{\pi}\Gamma(a+1/2)} \left(\frac{\Gamma(a-\alpha)}{2^{2a}} + \frac{(a-\alpha)^{a-\alpha} e^{-a+\alpha}}{2^{a-1-\alpha}(a+1/2)} \right) \leq C$$

and the proof is finished. □

2.4. **Further properties of the modified Bessel functions of first kind.** In this subsection we collect some known properties of the modified Bessel functions which will be used in this paper.

For each $k \in \mathbb{Z}$, it is verified that (see [19, 10.27.1])

$$I_{-k}(t) = I_k(t). \tag{16}$$

Moreover, from its definition it is clear that $I_k(t) > 0$ for $t > 0$,

$$I_k(0) = 0, \qquad k \neq 0, \tag{17}$$

and $I_0(0) = 1$.

We have the generating function (see [19, 10.35.1])

$$e^{t(u+u^{-1})/2} = \sum_{k\in\mathbb{Z}} u^k I_k(t) \tag{18}$$

and it implies, taking $u = 1$,

$$\sum_{k\in\mathbb{Z}} e^{-t} I_k(t) = 1. \tag{19}$$

Another easy consequence of the given generating function is the Neumann's identity (see [11, Chapter II, 7.10] or [19, 10.44.3])

$$I_n(t_1 + t_2) = \sum_{k\in\mathbb{Z}} I_k(t_1) I_{n-k}(t_2). \tag{20}$$

It can be obtained from (18) by using the Cauchy product of two series.

The asymptotic expansion for large order and t fix (see [19, 10.41.1])

$$I_\nu(t) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{et}{2\nu}\right)^\nu \tag{21}$$

will be used in some points.

3. The heat semigroup and beyond

Basic properties of the operator (2) can be deduced easily. First, by using (19), we have

$$\sum_{\mathbf{k}\in\mathbb{Z}^N} G_{t,N}(\mathbf{k}) = 1 \tag{22}$$

and, due to the positivity of $G_{t,N}$,

$$|W_t f(\mathbf{n})| \leq \|f\|_{\ell^\infty(\mathbb{Z}^N)} \sum_{\mathbf{k}\in\mathbb{Z}^N} G_{t,N}(\mathbf{n}-\mathbf{k}) \leq \|f\|_{\ell^\infty(\mathbb{Z}^N)}.$$

So W_t is well defined for any sequence in $\ell^\infty(\mathbb{Z}^N)$. Moreover, from (17) and (20), we deduce the properties

$$W_0 f(\mathbf{n}) = f(\mathbf{n}) \qquad \text{and} \qquad W_{t_1} W_{t_2} f(\mathbf{n}) = W_{t_1+t_2} f(\mathbf{n})$$

and the family of operators W_t is a semigroup. By using again the positive of the heat kernel, we obtain that W_t is a positive operator (it is positive for positive sequences) and from (19) it is possible to prove that it is Markovian; i. e.,

$$W_t \mathbf{1}(\mathbf{n}) = \mathbf{1}(\mathbf{n}),$$

where $\mathbf{1}(\mathbf{n})$ is the sequence having one in all its entries.

Defining the convolution on $\mathbb{Z}^N$ by

$$f * g(\mathbf{n}) = \sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{n}-\mathbf{k})g(\mathbf{k}),$$

we can observe that

$$W_t f(\mathbf{n}) = G_{t,N} * f(\mathbf{n}).$$

Then, by Young's inequality

$$\|f * g\|_{\ell^p(\mathbb{Z}^N)} \le \|f\|_{\ell^q(\mathbb{Z}^N)}\|g\|_{\ell^r(\mathbb{Z}^N)}, \tag{23}$$

with $1/q + 1/r = 1 + 1/p$ and $1 \le p, q, r \le \infty$, we obtain that

$$\|W_t f\|_{\ell^p(\mathbb{Z}^N)} \le \|G_{t,N}\|_{\ell^1(\mathbb{Z}^N)}\|f\|_{\ell^p(\mathbb{Z}^N)} \le \|f\|_{\ell^p(\mathbb{Z}^N)}, \qquad 1 \le p \le \infty \tag{24}$$

where in the last step we have applied (22).

For a sequence $f \in \ell^1(\mathbb{Z}^N)$, its Fourier transform is given by

$$\mathcal{F}f(x) = \sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{k})e^{2\pi i\langle \mathbf{k},x\rangle}, \qquad x \in [-1/2, 1/2]^N.$$

It is well known that $\mathcal{F}$ can be extended as an isometry from the space $\ell^2(\mathbb{Z}^N)$ into $L^2([-1/2, 1/2]^N)$ with inverse

$$\mathcal{F}^{-1}f(\mathbf{n}) = \int_{[-1/2,1/2]^N} f(x)e^{-2\pi i\langle \mathbf{n},x\rangle}\,dx, \qquad f \in L^2([-1/2, 1/2]^N).$$

Moreover, the Parseval's identity

$$\|\mathcal{F}f\|_{L^2([-1/2,1/2]^N)} = \|f\|_{\ell^2(\mathbb{Z}^N)}$$

holds.

Now, from the integral representation [19, 10.32.3]

$$I_n(t) = \frac{1}{\pi}\int_0^{\pi} e^{t\cos\theta}\cos(n\theta)\,d\theta, \qquad n \in \mathbb{Z}, \tag{25}$$

we get

$$e^{-t}I_n(t) = \int_{-1/2}^{1/2} e^{-2t\sin^2(\pi\theta)}e^{-2\pi in\theta}\,d\theta.$$

and

$$G_{t,N}(\mathbf{n}) = \int_{[-1/2,1/2]^N} e^{-4t\sum_{k=1}^N \sin^2(\pi x_k)}e^{-2\pi i\langle \mathbf{n},x\rangle}\,dx. \tag{26}$$

Then

$$\mathcal{F}(G_{t,N})(x) = e^{-4t\sum_{k=1}^N \sin^2(\pi x_k)}.$$

Hence, using

$$\mathcal{F}(f * g) = \mathcal{F}f \cdot \mathcal{F}g,$$

we deduce that

$$\lim_{t\to 0}\|W_t f - f\|_{\ell^2(\mathbb{Z}^N)} = \lim_{t\to 0}\left\|(e^{-4t\sum_{k=1}^N \sin^2(\pi x_k)} - 1)\mathcal{F}f\right\|_{L^2([-1/2,1/2]^N)} = 0$$

and we have the convergence in $\ell^2(\mathbb{Z}^N)$ of the heat semigroup. Moreover, due to the structure of the spaces $\ell^p(\mathbb{Z}^N)$, we have the pointwise convergence

$$W_t f(\mathbf{n}) \longrightarrow f(\mathbf{n}), \qquad \mathbf{n} \in \mathbb{Z}^N,$$

because

$$|W_t f(\mathbf{n}) - f(\mathbf{n})| \le \|W_t f - f\|_{\ell^2(\mathbb{Z}^N)}.$$

Note that, by using the inequality $1 - e^{-s} \le s$ with $s \ge 0$, it is verified that

$$|W_t f(\mathbf{n}) - f(\mathbf{n})| \le Ct\|f\|_{\ell^2(\mathbb{Z}^N)}. \tag{27}$$

3.1. **Decay of the heat semigroup.** Now we focus on the study of the decay of the heat semigroup in time. This is a natural question when we are dealing with this operator. We start with a $\ell^q - \ell^p$ estimate. The analogous for $\mathbb{R}^N$ can be seen, for example, in [6, Ch. 3, p. 44]. Our discrete result can be found in [14] but we include here it to do this paper self-contained and to provide a new and simple proof.

Theorem 6. *For $1 \le q \le p \le \infty$, the inequality*

$$\|W_t f\|_{\ell^p(\mathbb{Z}^N)} \le Ct^{-N/2(1/q-1/p)}\|f\|_{\ell^q(\mathbb{Z}^N)} \tag{28}$$

holds.

Proof. From the Young's inequality for the convolution, it is enough to prove that

$$\|G_{t,N}\|_{\ell^r(\mathbb{Z}^N)} \le Ct^{-N/2(1-1/r)} \tag{29}$$

and to do that we use Littlewood's inequality

$$\|f\|_{\ell^r(\mathbb{Z}^N)} \le \|f\|^{1/r}_{\ell^1(\mathbb{Z}^N)}\|f\|^{1-1/r}_{\ell^\infty(\mathbb{Z}^N)}, \qquad 1 \le r \le \infty, \tag{30}$$

the identity (22) and the bound $G_{t,N}(\mathbf{n}) \le Ct^{-N/2}$. This last estimate can be deduced from (4) and (15) with $\alpha = -1/2$. Indeed, using the notations $\mathbf{n}_+ = (|n_1|, \dots, |n_N|)$ and $|\mathbf{n}| = |n_1| + \dots + |n_N|$ and taking into account that $G_{t,N}(\mathbf{n}) = G_{t,N}(\mathbf{n}_+)$, it is verified that

$$G_{t,N}(\mathbf{n}) \le \left(e^{-2t} I_{|\mathbf{n}|/N}(2t)\right)^N = t^{-N/2}\left(t^{1/2}e^{-2t}I_{|\mathbf{n}|/N}(2t)\right)^N \le Ct^{-N/2}. \tag{31}$$ □

Remark 1. (a) By using (4) and (15) with $\alpha = 0$, it is easy to check that

$$G_{t,N}(\mathbf{n}) \le C(1 + |\mathbf{n}|)^{-N} \le C. \tag{32}$$

Then, for $0 < t < T$, with T finite, we deduce the bound

$$\|G_{t,N}\|_{\ell^r(\mathbb{Z}^N)} \le C \tag{33}$$

and the estimate

$$\|W_t f\|_{\ell^p(\mathbb{Z}^N)} \le C\|f\|_{\ell^q(\mathbb{Z}^N)}.$$

Moreover, combining (28) and (33), it is possible to obtain the inequality

$$\|W_t f\|_{\ell^p(\mathbb{Z}^N)} \le C\min\{1, t^{-N/2(1/q-1/p)}\}\|f\|_{\ell^q(\mathbb{Z}^N)}, \qquad 1 \le q \le p \le \infty.$$

(b) From the previous theorem it is possible to prove that for $t > 1$

$$\|G_{t,N}\|_{\ell^r(\mathbb{Z}^N)} \sim t^{-N/2(1-1/r)}, \qquad 1 \le r \le 2. \tag{34}$$

The upper bound is given in (29). To prove the lower one, we consider the sequence $f(\mathbf{k}) = G_{t,N}(\mathbf{k})$ to obtain

$$W_t f(\mathbf{n}) = \sum_{\mathbf{k}\in\mathbb{Z}^N} G_{t,N}(\mathbf{k})G_{t,N}(\mathbf{n}-\mathbf{k}) = G_{2t,N}(\mathbf{n}),$$

where in the las step we have applied (20). Then, from (28), we deduce that

$$\|G_{2t,N}\|_{\ell^2(\mathbb{Z}^N)} \le Ct^{-N/2(1/r-1/2)}\|G_{t,N}\|_{\ell^r(\mathbb{Z}^N)}, \qquad 1 \le r \le 2.$$

In this way, proving that

$$\|G_{2t,N}\|_{\ell^2(\mathbb{Z}^N)} \geq Ct^{-N/4} \tag{35}$$

the proof of the lower bound in (34) will be completed. From (26) and Parseval's identity

$$\begin{aligned}\|G_{2t,N}\|^2_{\ell^2(\mathbb{Z}^N)} &= \int_{[-1/2,1/2]^N} e^{-8t\sum_{k=1}^N \sin^2(\pi x_k)}\,dx = \left(\int_{-1/2}^{1/2} e^{-8t\sin^2(\pi s)}\,ds\right)^N \\ &\geq \left(\int_{-1/(2\sqrt{t})}^{1/(2\sqrt{t})} e^{-8t\sin^2(\pi s)}\,ds\right)^N \geq Ct^{-N/2},\end{aligned}$$

where we have used that $e^{-8t\sin^2(\pi s)} \geq K$ for $s \in [-1/(2\sqrt{t}), 1/(2\sqrt{t})]$, and the proof of (35) is finished.

The next result contains the main novelty of this subsection. For the classical heat equation in $\mathbb{R}^N$, with an initial data having polynomial decay at infinity, in [9] the authors study how the mass of the solution is distributed for large value of t. In fact, it is proved the inequality

$$\left\|H_t f - \left(\int_{\mathbb{R}^N} f(x)\,dx\right) G_t\right\|_{L^p(\mathbb{R}^N)} \leq Ct^{-1/2-N/2(1/q-1/p)}\||\cdot|f\|_{L^q(\mathbb{R}^N)},$$

where H_t is the solution of the heat equation, G_t its kernel (the Gaussian function), $1 \leq q \leq p \leq \infty$, $1 \leq q < N/(N-1)$, $f \in L^1(\mathbb{R}^N)$, and $|\cdot|f \in L^q(\mathbb{R}^N)$. Remember that the mass of the solution H_t is a conservative quantity; i.e.,

$$\int_{\mathbb{R}^N} H_t f(x)\,dx = \int_{\mathbb{R}^N} f(x)\,dx$$

Note that (22) implies that

$$\sum_{\mathbf{k}\in\mathbb{Z}^N} W_t f(\mathbf{k}) = \sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{k}),$$

so in the discrete setting the mass of the solution of the heat equation is conservative also. The discrete analogous of this result is given in [14] only in the case $q = 1$, with $1 \leq p \leq \infty$, and the case $1 < q \leq p$ is left as an open problem. Our next result closes this problem completely.

Theorem 7. *Let $N \geq 1$, $1 \leq q \leq p \leq \infty$, $1 \leq q < N/(N-1)$, and $|\cdot|f \in \ell^q(\mathbb{Z}^N)$. Then the inequality*

$$\left\|W_t f - \left(\sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{k})\right) G_{t,N}\right\|_{\ell^p(\mathbb{Z}^N)} \leq Ct^{-1/2-N/2(1/q-1/p)}\||\cdot|f\|_{\ell^q(\mathbb{Z}^N)} \tag{36}$$

holds.

To prove our result we need some preliminary lemmas.

Lemma 8. *Let $N \geq 1$, $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^N$ such that $|\mathbf{n}| > 2|\mathbf{n}-\mathbf{m}|$, and*

$$H_{t,N}(z) = \frac{z}{1+t}\left(G_{t,1}(z)\right)^N, \qquad z > 0. \tag{37}$$

Then the inequality

$$|G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m})| \le C|\mathbf{n}-\mathbf{m}| H_{t,N}\left(\frac{|\mathbf{n}|+|\mathbf{m}|}{K}\right) \tag{38}$$

holds with constants C and $K>1$ independent of $\mathbf{n}$ and $\mathbf{m}$.

Proof. Without lost of generality we suppose that $n_i \neq m_i$ for $i=1,\dots,N$. In the opposite case the situation can be obviously reduced to this one easily. Indeed, let us suppose that $n_i = m_i$ for j values of i with $0<j<N$, then we consider the decompositions

$$\mathbf{n} = \mathbf{n}_1 \cup \mathbf{n}_2 \qquad \text{and} \qquad \mathbf{m} = \mathbf{n}_1 \cup \mathbf{m}_2,$$

where $\mathbf{n}_1$ are the common values of $\mathbf{n}$ and $\mathbf{m}$ and $\mathbf{n}_2$ and $\mathbf{m}_2$ are the other components. Then, if we suppose that the result has been proved for different values, we have

$$\begin{aligned}|G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m})| &= G_{t,j}(\mathbf{n}_1)|G_{t,N-j}(\mathbf{n}_2) - G_{t,N-j}(\mathbf{m}_2)| \\ &\le CG_{t,j}(\mathbf{n}_1)|\mathbf{n}_2-\mathbf{m}_2| H_{t,N-j}\left(\frac{|\mathbf{n}_2|+|\mathbf{m}_2|}{K}\right) \\ &\le C|\mathbf{n}-\mathbf{m}| H_{t,N}\left(\frac{|\mathbf{n}|+|\mathbf{m}|}{2KN}\right).\end{aligned}$$

To check the last inequality in the previous chain we use (4) to have

$$\begin{aligned}&G_{t,j}(\mathbf{n}_1)\left(G_{t,1}\left(\frac{|\mathbf{n}_2|+|\mathbf{m}_2|}{K}\right)\right)^{N-j} \\ &\qquad\qquad \le \left(G_{t,1}\left(\frac{(N-j)(|\mathbf{n}_2|+|\mathbf{m}_2|)/K + |\mathbf{n}_1|}{N}\right)\right)^{N},\end{aligned}$$

then the inequality follows applying the monotonicity of the functions I_a and the estimate

$$\begin{aligned}\frac{(N-j)(|\mathbf{n}_2|+|\mathbf{m}_2|)}{K} + |\mathbf{n}_1| &\ge \frac{(N-j)(|\mathbf{n}_2|+|\mathbf{m}_2|) + |\mathbf{n}_1|}{K} \\ &\ge \frac{|\mathbf{n}_2|+|\mathbf{m}_2|+2|\mathbf{n}_1|}{2K} = \frac{|\mathbf{n}|+|\mathbf{m}|}{2K}.\end{aligned}$$

From (16), it is clear that $G_{t,N}(\mathbf{n}) = G_{t,N}(\mathbf{n}_+)$. Moreover, we have to observe that $G_{t,N}(\mathbf{n})$ is invariant under permutations of $\mathbf{n}$. With these previous reductions, we can consider $\mathbf{n} = (n_1,\dots,n_N)$ and $\mathbf{m} = (m_1,\cdots,m_N)$ with

$$0 \le n_1 \le n_2 < \cdots \le n_N \qquad \text{and} \qquad 0 \le m_N \le m_{N-1} \le \cdots \le m_1. \tag{39}$$

An important tool in the proof will be the identity

$$G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m}) = \sum_{i=1}^{N} (G_{t,1}(n_i) - G_{t,1}(m_i)) \prod_{j=1}^{i-1} G_{t,1}(m_j) \prod_{k=i+1}^{N} G_{t,1}(n_k), \tag{40}$$

where the empty products has to be understood as one. The proof of this relation we can done by induction on N. To do that we consider $\tilde{\mathbf{n}} = (\mathbf{n}, n_{N+1})$ and $\tilde{\mathbf{m}} = (\mathbf{m}, m_{N+1})$ and we apply the identity

$$\begin{aligned}G_{t,N+1}(\tilde{\mathbf{n}}) - G_{t,N+1}(\tilde{\mathbf{m}}) &= (G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m}))G_{t,1}(n_{N+1}) \\ &\quad + G_{t,N}(\mathbf{m})(G_{t,1}(n_{N+1}) - G_{t,1}(m_{N+1})).\end{aligned}$$

For $n_i < m_i$, applying the monotonicity of the modified Bessel with respect to order and the bound

$$\delta^+ G_{t,1}(j) \le C \frac{j+1}{1+t} G_{t,1}(j),$$

which is deduced by (6), we have

$$\begin{aligned} |G_{t,1}(n_i) - G_{t,1}(m_i)| &\le \sum_{j=n_i}^{m_i-1} |\delta^+ G_{t,1}(j)| \\ &\le \frac{C}{1+t} \sum_{j=n_i}^{m_i-1} (j+1) G_{t,1}(j) \\ &\le C \frac{(m_i-n_i)(m_i+n_i+1)}{1+t} G_{t,1}(n_i). \end{aligned}$$

Then, in general, we obtain the bound

$$|G_{t,1}(n_i) - G_{t,1}(m_i)| \le C \frac{(m_i-n_i)(m_i+n_i+1)}{1+t} \max\{G_{t,1}(n_i), G_{t,1}(m_i)\} \tag{41}$$

and, from (40),

$$\begin{aligned} |G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m})| &\le C \sum_{i=1}^{N} \frac{|m_i-n_i|(m_i+n_i+1)}{1+t} \\ &\quad \times \max\{G_{t,1}(n_i), G_{t,1}(m_i)\} \prod_{j=1}^{i-1} G_{t,1}(m_i) \prod_{k=i+1}^{N} G_{t,1}(n_k). \end{aligned} \tag{42}$$

Now, applying (4), we deduce the estimate

$$\begin{aligned} &\max\{G_{t,1}(n_i), G_{t,1}(m_i)\} \prod_{j=1}^{i-1} G_{t,1}(m_j) \prod_{k=i+1}^{N} G_{t,1}(n_k) \\ &\qquad \le \left(G_{t,1}\left(\frac{m_1 + \dots + m_{i-1} + \min\{m_i, n_i\} + n_{i+1} + \dots + n_N}{N} \right) \right)^N. \end{aligned}$$

From the condition $|\mathbf{n}| > 2|\mathbf{n}-\mathbf{m}|$, we can deduce that $|\mathbf{n}|/2 < |\mathbf{m}| < 3|\mathbf{n}|/2$. Then if $n_1 < m_1$

$$\min\{m_1, n_1\} + n_2 + \dots + n_N = |\mathbf{n}| > \frac{2}{5}(|\mathbf{n}| + |\mathbf{m}|),$$

and if $m_1 < n_1$, by our supposition (39), we can prove easily that

$$\min\{m_1, n_1\} + n_2 + \dots + n_N \ge \frac{|\mathbf{n}| + |\mathbf{m}|}{N+1}.$$

Similarly, for $i = 2, \dots, N-1$, by applying again (39), we have

$$m_1 + \dots + m_{i-1} + \min\{m_i, n_i\} + n_{i+1} + \dots + n_N \ge \frac{|\mathbf{n}| + |\mathbf{m}|}{N+1}.$$

Moreover, for $n_N < m_N$ it is verified that

$$m_1 + m_2 + \dots + \min\{m_N, n_N\} \ge \frac{|\mathbf{n}| + |\mathbf{m}|}{N+1},$$

and for $m_N < n_N$

$$m_1 + m_2 + \dots + \min\{m_N, n_N\} = |\mathbf{m}| > \frac{|\mathbf{n}| + |\mathbf{m}|}{3}.$$

As conclusion,

$$\frac{m_1+\cdots+m_{i-1}+\min\{m_i,n_i\}+n_{i+1}+\cdots+n_N}{N}\geq\frac{|\mathbf{n}|+|\mathbf{m}|}{K}$$

with $K>1$. In this way

$$\max\{G_{t,1}(n_i),G_{t,1}(m_i)\}\prod_{j=1}^{i-1}G_{t,1}(m_j)\prod_{k=i+1}^{N}G_{t,1}(n_k)\leq\left(G_{t,1}\left(\frac{|\mathbf{n}|+|\mathbf{m}|}{K}\right)\right)^N$$

and by (42) the proof is completed. □

Remark 2. (a) From our proof of (38) it is clear that it holds for $|\mathbf{n}|\sim|\mathbf{m}|$ also.

(b) From (41) and (42), using (15) with $\alpha=-1/2$, it is easy to deduce that

$$|G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{m})|\leq C|\mathbf{m}-\mathbf{n}|(|\mathbf{n}|+|\mathbf{m}|)t^{-N/2-1},\qquad \mathbf{n},\mathbf{m}\in\mathbb{Z}^N. \tag{43}$$

(c) Moreover, we can check that for $|\mathbf{n}|\geq B|\mathbf{m}|$, for some positive constant B, the inequality

$$|G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{m})|\leq C\frac{|\mathbf{n}-\mathbf{m}|(|\mathbf{n}|+|\mathbf{m}|)}{1+t}\left(G_{t,1}\left(\frac{|\mathbf{m}|}{K}\right)\right)^N, \tag{44}$$

holds for some constant K.

The following estimate will be useful in the proof of Theorem 7.

Lemma 9. *The inequality*

$$\frac{|G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{k})|}{|\mathbf{n}-\mathbf{k}|}\leq Ct^{-1/2-N/2} \tag{45}$$

holds

Proof. From (26) we have

$$\begin{aligned}\frac{|G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{k})|}{|\mathbf{n}-\mathbf{k}|}&\leq\int_{[-1/2,1/2]^N}e^{-4t\sum_{i=1}^N\sin^2(\pi x_i)}\frac{|e^{-2\pi i\langle\mathbf{n}-\mathbf{k},x\rangle}-1|}{|\mathbf{n}-\mathbf{k}|}\,dx\\&\leq C\int_{[-1/2,1/2]^N}e^{-4t\sum_{i=1}^N\sin^2(\pi x_i)}|x|\,dx,\end{aligned}$$

where we have applied that $|(e^{-iz}-1)/z|\leq C$ for $z\in\mathbb{R}$. Now, with the estimates

$$\int_{-1/2}^{1/2}e^{-4t\sin^2(\pi x)}\,dx\leq Ct^{-1/2}\qquad\text{and}\qquad\int_{-1/2}^{1/2}e^{-4t\sin^2(\pi x)}|x|\,dx\leq Ct^{-1}$$

the result is obtained. □

The function $H_{t,N}$, see (37), in Lemma 8 will appear in convolution with a specific sequence in the proof of Theorem 7, so we need the size of its norm.

Lemma 10. *For $N\geq1$ and $1\leq r\leq\infty$, it is verified that*

$$\left\|H_{t,N}\left(\frac{|\cdot|}{K}\right)\right\|_{\ell^r(\mathbb{Z}^N)}\leq Ct^{-1/2-N/2(1-1/r)}, \tag{46}$$

with the constant C being independent of t.

Proof. From (15) with $\alpha = -1/2 + 1/(2N)$, we have

$$\left| H_{t,N}\left(\frac{|\mathbf{n}|}{K}\right)\right| \le Ct^{-N/2-1/2}|\mathbf{n}|\left(t^{1/2-1/(2N)}G_{t,1}\left(\frac{|\mathbf{n}|}{K}\right)\right)^N \le Ct^{-N/2-1/2} \tag{47}$$

and

$$\left\| H_{t,N}\left(\frac{|\cdot|}{K}\right)\right\|_{\ell^\infty(\mathbb{Z}^N)} \le Ct^{-N/2-1/2},$$

Now, we are going to prove that

$$\left| H_{t,N}\left(\frac{|\mathbf{n}|}{K}\right)\right| \le \frac{C}{(|\mathbf{n}|+1)^{N+1}} \tag{48}$$

and to do that we distinguish two cases. First, by applying again (15) with $\alpha = 1/N$, for $|\mathbf{n}|/K > 1/N$ we deduce that

$$\left| H_{t,N}\left(\frac{|\mathbf{n}|}{K}\right)\right| \le C|\mathbf{n}|\left(t^{-1/N}G_{t,1}\left(\frac{|\mathbf{n}|}{K}\right)\right)^N \le \frac{C}{(|\mathbf{n}|+1)^{N+1}}.$$

In the case $|\mathbf{n}|/K \le 1/N$, we use the estimate

$$G_{t,1}\left(\frac{|\mathbf{n}|}{K}\right) \le G_{t,1}(0) \le C$$

to obtain that

$$\left| H_{t,N}\left(\frac{|\mathbf{n}|}{K}\right)\right| \le C\frac{|\mathbf{n}|}{1+t} \le \frac{C}{(|\mathbf{n}|+1)^{N+1}}.$$

Then, from (48), for $0 < t \le T$ with $T < 100$, for example, it is obtained that

$$\left\| H_{t,N}\left(\frac{|\cdot|}{K}\right)\right\|_{\ell^1(\mathbb{Z}^N)} \le C\sum_{\mathbf{n}\in\mathbb{Z}^N}\frac{1}{(|\mathbf{n}|+1)^{N+1}} \le C\sum_{m=1}^{\infty}\frac{1}{m^2} \le C.$$

For $t > T$ we consider the decomposition

$$\left\| H_{t,N}\left(\frac{|\cdot|}{K}\right)\right\|_{\ell^1(\mathbb{Z}^N)} = S_1 + S_2,$$

where

$$S_1 = \sum_{|\mathbf{n}|\le n_t}\left| H_{t,N}\left(\frac{|\mathbf{n}|}{K}\right)\right| \qquad \text{and} \qquad S_2 = \sum_{|\mathbf{n}|> n_t}\left| H_{t,N}\left(\frac{|\mathbf{n}|}{K}\right)\right|,$$

where $n_t = \lfloor\sqrt{t}\rfloor$ (this notation will be used in some points along the paper without explicit mention to it). For S_1, applying (47), we have

$$S_1 \le Ct^{-N/2-1/2}\sum_{|\mathbf{n}|\le n_t} 1 \le Ct^{-N/2-1/2}\sum_{m=0}^{n_t}(m+1)^{N-1} \le Ct^{-1/2}.$$

Now, by using (48), for S_2 we obtain the estimate

$$S_2 \le C\sum_{|\mathbf{n}|>n_t}\frac{1}{|\mathbf{n}|^{N+1}} \le C\sum_{m=n_t}^{\infty}\frac{1}{m^2} \le Ct^{-1/2}.$$

In this way

$$\left\| H_{t,N}\left(\frac{|\cdot|}{K}\right)\right\|_{\ell^1(\mathbb{Z}^N)} \le Ct^{-1/2}.$$

and the result follows from (30). ∎

In some points we will need the weighted estimate of the heat kernel in the following lemma.

Lemma 11. *For $N \geq 1$, $1 \leq r \leq \infty$, and $N(1-r) < \gamma < N$, it is verified that*

$$\left\| \left(G_{t,1}\left(\frac{|\cdot|}{N}\right)\right)^N \right\|_{\ell^r(\mathbb{Z}^N,(|\mathbf{n}|+1)^{-\gamma})} \leq Ct^{-1/2((\gamma-N)/r+N)}, \tag{49}$$

with the constant C being independent of t.

Proof. From the bound (31) the result for $r=\infty$ is clear, so we analyze the estimate for $1 \leq r < \infty$. From (15) with $\alpha = -1/2$ and $\alpha = 0$, we get

$$\begin{aligned}
&\left\| \left(G_{t,1}\left(\frac{|\cdot|}{N}\right)\right)^N \right\|^r_{\ell^r(\mathbb{Z}^N,(|\mathbf{n}|+1)^{-\gamma})} \\
&\qquad \leq C\left(t^{-Nr/2} \sum_{|\mathbf{n}|\leq n_t} (|\mathbf{n}|+1)^{-\gamma} + \sum_{|\mathbf{n}|\geq n_t} (|\mathbf{n}|+1)^{-\gamma-Nr}\right) \\
&\qquad \leq C\left(t^{-Nr/2} \sum_{m=0}^{n_t} (m+1)^{-\gamma+N-1} + \sum_{m=n_t}^{\infty} (m+1)^{-\gamma-Nr+N-1}\right) \\
&\qquad \leq Ct^{-1/2(\gamma-N+Nr)}
\end{aligned}$$

finishing the proof. □

The proof of our main result will be divided in the cases $N=1$ and $N>1$ and to deal with the first one we will apply the next estimate.

Lemma 12. *For $1 \leq r \leq \infty$, the inequality*

$$\|\delta^+ G_{t,1}\|_{\ell^r(\mathbb{Z})} \leq Ct^{-1/2-(1-1/r)/2} \tag{50}$$

holds.

Proof. As usual, we prove the inequality for $r=1$ and $r=\infty$ and the result follows from (30). From (6) and (15) with $\alpha=0$, we have

$$|\delta^+ G_{t,1}(n)| \leq Ct^{-1}(|n|+1)G_{t,1}(n) \leq Ct^{-1}$$

and this is enough for $r=\infty$. For $r=1$ we use (6) and decompose the sum

$$\|\delta^+ G_{t,1}\|_{\ell^1(\mathbb{Z})} \leq Ct^{-1}(B_1+B_2),$$

where

$$B_1 = \sum_{|n|\leq n_t} (|n|+1)G_{t,1}(n) \qquad \text{and} \qquad B_2 = \sum_{|n|>n_t} (|n|+1)G_{t,1}(n).$$

To treat B_1, by using (15) with $\alpha=-1/2$, we deduce that

$$t^{-1}B_1 \leq t^{-3/2} \sum_{|n|\leq n_t} (|n|+1) \leq Ct^{-1/2}$$

and this is enough for our purpose. The factor B_2 can be analyzed with the estimate (15) taking $\alpha=1$ (note that we can assume $|n|>1$). Indeed,

$$t^{-1}B_2 \leq C \sum_{|n|>n_t} \frac{1}{(|n|+1)^2} \leq Ct^{-1/2}$$

and the proof is complete. □

Proof of the Theorem 7. From the restriction $1 \le q < N/(N-1)$, applying Hölder inequality, we have

$$\|f\|_{\ell^1(\mathbb{Z}^N)} \le C\||\cdot|f\|_{\ell^q(\mathbb{Z}^N)},$$

so, in particular $f \in \ell^1(\mathbb{Z}^N)$ and the left hand side of (36) is finite.

We start showing the inequality (36) for $N = 1$. In [1, Theorem 6.1] we find (with our notation) the identity

$$W_t f(n) - \left(\sum_{n\in\mathbb{Z}} f(n)\right) G_{t,1}(n) = \left(\delta^+ G_{t,1}\right) * F(n),$$

where

$$F(n) = \sum_{j=-\infty}^{n-1} f(j), \quad \text{for } n \le 0, \qquad \text{and} \qquad F(n) = -\sum_{j=n}^{\infty} f(j), \quad \text{for } n \ge 1.$$

Then, by using Young's inequality (23) and (50), we have

$$\left\| W_t f - \left(\sum_{k\in\mathbb{Z}} f(k)\right) G_{t,1} \right\|_{\ell^p(\mathbb{Z})} \le C t^{-1/2-(1/q-1/p)/2} \|F\|_{\ell^q(\mathbb{Z})}.$$

Finally, applying the well known inequality

$$\sum_{n=1}^{\infty} \left| \sum_{j=n}^{\infty} f(j) \right|^q \le C \sum_{n=1}^{\infty} |nf(n)|^q, \qquad 1 \le q < \infty,$$

which is a consequence of the classical Hardy's inequality for $q > 1$ and for $q = 1$ is obvious, the result follows in this case because

$$\begin{aligned}\|F\|^q_{\ell^q(\mathbb{Z})} &\le \sum_{n=1}^{\infty} \left(\left(\sum_{j=n}^{\infty} |f(-j)| \right)^q + \left(\sum_{j=n}^{\infty} |f(j)| \right)^q \right) \\ &\le C \sum_{n=1}^{\infty} n^q (|f(-n)|^q + |f(n)|^q) = C\||\cdot|f\|^q_{\ell^q(\mathbb{Z})}.\end{aligned}$$

Now, we consider $N > 1$. Firstly, we are going to prove the inequality

$$\left\| W_t f - \left(\sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{k})\right) G_{t,N} \right\|_{\ell^p(\mathbb{Z}^N)} \le C t^{-1/2} \||\cdot|f\|_{\ell^p(\mathbb{Z}^N)} \tag{51}$$

for $1 \le p < N/(N-1)$. To do that, for a fix $\mathbf{n} \in \mathbb{Z}^N$, we consider the sets $A_1 = \{\mathbf{k} \in \mathbb{Z}^N : |\mathbf{k}| \le |\mathbf{n}|/2\}$, $A_2 = \{\mathbf{k} \in \mathbb{Z}^N : |\mathbf{n}|/2 < |\mathbf{k}| < 2|\mathbf{n}|\}$, and $A_3 = \{\mathbf{k} \in \mathbb{Z}^N : |\mathbf{k}| \ge 2|\mathbf{n}|\}$. With them we do the decomposition

$$\left| W_t f(\mathbf{n}) - \left(\sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{k})\right) G_{t,N}(\mathbf{n}) \right| \le P_1 f(\mathbf{n}) + P_2 f(\mathbf{n}) + P_3 f(\mathbf{n}),$$

with

$$P_i f(\mathbf{n}) = \left| \sum_{\mathbf{k}\in\mathbb{Z}^N} \chi_{A_i}(\mathbf{n},\mathbf{k}) f(\mathbf{n}-\mathbf{k}) (G_{t,N}(\mathbf{k}) - G_{t,N}(\mathbf{n})) \right|, \qquad i = 1, 2, 3,$$

where χ_{A_i} denotes the characteristic function of the set A_i.

From (4), in A_1 it is verified that $|G_{t,N}(\mathbf{k})-G_{t,N}(\mathbf{n})|\le C(G_{t,1}(|\mathbf{k}|/N))^N$ and

$$P_1f(\mathbf{n})\le C\sum_{|\mathbf{k}|\le|\mathbf{n}|/2}|f(\mathbf{n}-\mathbf{k})|(G_{t,1}(|\mathbf{k}|/N))^N \tag{52}$$
$$\le C\sum_{|\mathbf{k}|\le|\mathbf{n}|/2}|\mathbf{n}-\mathbf{k}||f(\mathbf{n}-\mathbf{k})|\frac{(G_{t,1}(|\mathbf{k}|/N))^N}{|\mathbf{k}|+1}.$$

Then, by Minkowski integral inequality and Lemma 11 (with $r=\gamma=1$), we obtain that

$$\|P_1f\|_{\ell^p(\mathbb{Z}^N)}\le C\||\cdot|f\|_{\ell^p(\mathbb{Z}^N)}\sum_{k\in\mathbb{Z}^N}\frac{(G_{t,1}(|\mathbf{k}|/N))^N}{|\mathbf{k}|+1}\le Ct^{-1/2}\||\cdot|f\|_{\ell^p(\mathbb{Z}^N)}.$$

For P_2f, by using that $|\mathbf{k}|\sim|\mathbf{n}|$, we can apply Remark 2 (a) to get

$$P_2(\mathbf{n})\le C\sum_{\mathbf{k}\in\mathbb{Z}^N}|\mathbf{n}-\mathbf{k}||f(\mathbf{n}-\mathbf{k})|H_{t,N}(\mathbf{k})=CF*H_{t,N}(\mathbf{n}),$$

where $F(\mathbf{n})=|\mathbf{n}||f(\mathbf{n})|$. Then, by applying (46) and the Young inequality (23) for the convolution, we have

$$\|P_2f\|_{\ell^p(\mathbb{Z}^N)}\le Ct^{-1/2}\||\cdot|f\|_{\ell^p(\mathbb{Z}^N)}.$$

Finally, we treat P_3f. First, we have to observe that for $p<N/(N-1)$, by Hölder's inequality, we obtain that

$$\sum_{|\mathbf{k}|\ge2|\mathbf{n}|}|f(\mathbf{n}-\mathbf{k})|\le\||\cdot|f\|_{\ell^p(\mathbb{Z}^N)}\left(\sum_{|\mathbf{k}|\ge2|\mathbf{n}|}\frac{1}{|\mathbf{n}-\mathbf{k}|^{p'}}\right)^{1/p'}$$
$$\le C\||\cdot|f\|_{\ell^p(\mathbb{Z}^N)}(|\mathbf{n}|+1)^{N/p'-1}.$$

Then, by using that in A_3 it is satisfied the estimate $|G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{k})|\le C(G_{t,N}(|\mathbf{n}|/N))^N$,

$$P_3f(\mathbf{n})\le C(G_{t,1}(|\mathbf{n}|/N))^N(|\mathbf{n}|+1)^{N/p'-1}\||\cdot|f\|_{\ell^p(\mathbb{Z}^N)}. \tag{53}$$

In this way, applying Lemma 11 (with $r=p$ and $\gamma=p(1-N/p')$) the inequality

$$\|P_3f\|_{\ell^p(\mathbb{Z}^N)}\le Ct^{-1/2}\||\cdot|f\|_{\ell^p(\mathbb{Z}^N)}$$

is attained and the proof of (51) is completed.

Now, we are going to check that

$$\left\|W_tf-\left(\sum_{\mathbf{k}\in\mathbb{Z}^N}f(\mathbf{k})\right)G_{t,N}\right\|_{\ell^\infty(\mathbb{Z}^N)}\le Ct^{-1/2-N/(2q)}\||\cdot|f\|_{\ell^q(\mathbb{Z}^N)} \tag{54}$$

for $1\le q<N/(N-1)$. We consider the same decomposition that in the previous case. To analyze P_1, we consider the decomposition

$$\|P_1f\|_{\ell^\infty(\mathbb{Z}^N)}=P_{1,1}+P_{1,2}$$

where

$$P_{1,1}=\sup_{|\mathbf{n}|/2\le n_t}|P_1f(\mathbf{n})|\qquad\text{and}\qquad P_{1,2}=\sup_{|\mathbf{n}|/2>n_t}|P_1f(\mathbf{n})|.$$

So, from (43) and Hölder's inequality, it is verified that

$$\begin{aligned} P_{1,1} &\le Ct^{-N/2-1} \sup_{|\mathbf{n}|/2\le n_t} |\mathbf{n}| \sum_{|\mathbf{k}|\le |\mathbf{n}|/2} |\mathbf{n}-\mathbf{k}||f(\mathbf{n}-\mathbf{k})| \\ &\le Ct^{-N/2-1} \||\cdot|f\|_{\ell^q(\mathbb{Z}^N)} \sup_{|\mathbf{n}|/2\le n_t} |\mathbf{n}| \left(\sum_{|\mathbf{k}|\le|\mathbf{n}|/2} 1\right)^{1/q'} \\ &\le Ct^{-N/2-1} \||\cdot|f\|_{\ell^q(\mathbb{Z}^N)} \sup_{|\mathbf{n}|/2\le n_t} |\mathbf{n}|^{1+N/q'} \le Ct^{-1/2-N/(2q)} \||\cdot|f\|_{\ell^q(\mathbb{Z}^N)} \end{aligned}$$

and this is enough. Now, to deal with $P_{1,2}$ we split the sum defining $P_1 f$ in two pieces: one for $|\mathbf{k}|\le n_t$, denoted $P'_{1,2}$, and another one for $n_t<|\mathbf{k}|\le|\mathbf{n}|/2$, with the notation $P''_{1,2}$ for this one. Then, from (45) and using Hölder's inequality, we get

$$\begin{aligned} P'_{1,2} &\le Ct^{-1/2-N/2} \sup_{|\mathbf{n}|/2> n_t} \sum_{|\mathbf{k}|\le n_t} |\mathbf{n}-\mathbf{k}||f(\mathbf{n}-\mathbf{k})| \\ &\le Ct^{-1/2-N/2} \||\cdot|f\|_{\ell^q(\mathbb{Z}^N)} \left(\sum_{|\mathbf{k}|\le n_t} 1\right)^{1/q'} \le Ct^{-1/2-N/(2q)} \||\cdot|f\|_{\ell^q(\mathbb{Z}^N)}. \end{aligned}$$

Now, proceeding as in the proof of (52), we deduce that

$$\begin{aligned} P''_{1,2} &\le C \sup_{|\mathbf{n}|/2> n_t} \sum_{n_t<|\mathbf{k}|\le|\mathbf{n}|/2} |\mathbf{n}-\mathbf{k}||f(\mathbf{n}-\mathbf{k})| \frac{(G_{t,1}(|\mathbf{k}|/N))^N}{|\mathbf{k}|+1} \\ &\le C\||\cdot|f\|_{\ell^q(\mathbb{Z}^N)} \left(\sum_{n_t<|\mathbf{k}|} \frac{(G_{t,1}(|\mathbf{k}|/N))^{q'N}}{(|\mathbf{k}|+1)^{q'}}\right)^{1/q'}. \end{aligned}$$

Finally, the required estimate is obtained by using that $G_{t,1}(|\mathbf{k}|/N)\le C/(|\mathbf{k}|+1)$, which is obtained by (15) with $\alpha=0$. Indeed,

$$\begin{aligned} P''_{1,2} &\le C\||\cdot|f\|_{\ell^q(\mathbb{Z}^N)} \left(\sum_{|\mathbf{k}|>n_t} \frac{1}{(|\mathbf{k}|+1)^{q'(N+1)}}\right)^{1/q'} \\ &\le Ct^{-1/2-N/(2q)} \||\cdot|f\|_{\ell^q(\mathbb{Z}^N)}. \end{aligned}$$

To obtain the estimate

$$\|P_2 f\|_{\ell^\infty(\mathbb{Z}^N)} \le Ct^{-1/2-N/(2q)} \||\cdot|f\|_{\ell^q(\mathbb{Z}^N)}$$

we proceed as in the proof of (51), so we omit the details.

Finally, we have to bound $P_3 f$ and to do this we take the decomposition

$$P_3 f(\mathbf{n}) \le P_{3,1} + P_{3,2},$$

with

$$P_{3,1} = \sup_{2|\mathbf{n}|\le n_t} |P_3 f(\mathbf{n})| \qquad \text{and} \qquad P_{3,2} = \sup_{2|\mathbf{n}|> n_t} |P_3 f(\mathbf{n})|.$$

For $P_{3,2}$, proceeding as in (53) and using the bound (31), we deduce that

$$P_{3,2} \le C\||\cdot|f\|_{\ell^q(\mathbb{Z}^N)} \sup_{2|\mathbf{n}|>n_t} (G_{t,1}(|\mathbf{n}|/N))^N (|\mathbf{n}|+1)^{N/q'-1}$$

$$\leq Ct^{-1/2-N/(2q)}\||\cdot|f\|_{\ell^q(\mathbb{Z}^N)}.$$

Now, to treat $P_{3,1}$ we decompose the sum defining $P_3 f$ in two pieces, $P'_{3,1}$ and $P''_{3,1}$, where $2|\mathbf{n}| \leq |\mathbf{k}| \leq n_t$ and $n_t < |\mathbf{k}|$, respectively. For $P'_{3,1}$ the required bound can be obtained as for $P'_{1,2}$ and we omit the details. To conclude the result for $P''_{3,1}$ we observe that, due to the restriction $q < N/(N-1)$,

$$\sum_{|\mathbf{k}|>n_t} |f(\mathbf{n}-\mathbf{k})| \leq \||\cdot|f\|_{\ell^q(\mathbb{Z}^N)} \left(\sum_{|\mathbf{k}|>n_t} \frac{1}{|\mathbf{n}-\mathbf{k}|^{q'}}\right)^{1/q'}$$
$$\leq Ct^{-1/2+N/(2q')}\||\cdot|f\|_{\ell^q(\mathbb{Z}^N)}.$$

Then, applying (31),

$$P''_{3,1} \leq Ct^{-1/2+N/(2q')}\||\cdot|f\|_{\ell^q(\mathbb{Z}^N)} \sup_{2|\mathbf{n}|\leq n_t} (G_{t,1}(|\mathbf{n}|/N))^N$$
$$\leq Ct^{-1/2-N/(2q)}\||\cdot|f\|_{\ell^q(\mathbb{Z}^N)}$$

and the proof of (54) is completed.

The proof of the theorem will be finished interpolating between (51) and (54). □

Remark 3. By using (33) and Minkowski integral inequality, for $0 < t < T$ (with $T > 100$ for example) and $1 \leq q < N/(N-1)$ we obtain

$$\left\| W_t f - \left(\sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{k})\right) G_{t,N} \right\|_{\ell^p(\mathbb{Z}^N)}$$
$$\leq \|f\|_{\ell^1(\mathbb{Z}^N)} \left(\sup_{\mathbf{k}\in\mathbb{Z}^N} \|G_{t,N}(\cdot-\mathbf{k})\|_{\ell^p(\mathbb{Z}^N)} + \|G_{t,N}\|_{\ell^p(\mathbb{Z}^N)}\right) \leq C\||\cdot|f\|_{\ell^q(\mathbb{Z}^N)}.$$

Then, combining this fact with (36) and with the hypotheses of Theorem 7, we get

$$\left\| W_t f - \left(\sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{k})\right) G_{t,N} \right\|_{\ell^p(\mathbb{Z}^N)} \leq C \min\{1, t^{-1/2-N/2(1/q-1/p)}\}\||\cdot|f\|_{\ell^q(\mathbb{Z}^N)}. \tag{55}$$

3.2. **Maximal operators for the heat and the Poisson semigroups.** From the well known identity

$$e^{-|\beta|t} = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-t^2\beta^2/(4u)}\,du = \frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-t^2/(4v)}}{\sqrt{v}} e^{-v\beta^2}\,\frac{dv}{v},$$

we can define by subordination the Poisson semigroup

$$P_t f(\mathbf{n}) = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} W_{t^2/(4u)} f(\mathbf{n})\,du = \frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-t^2/(4v)}}{\sqrt{v}} W_v f(\mathbf{n})\frac{dv}{v}. \tag{56}$$

It is an easy exercise to show that P_t satisfies the Laplace type equation

$$\frac{d^2}{dt^2}P_t f(\mathbf{n}) + \Delta_N P_t f(\mathbf{n}) = 0.$$

In this part of the paper we are interested in the maximal operators

$$W^\star f(\mathbf{n}) = \sup_{t>0} |W_t f(\mathbf{n})|$$

and

$$P^\star f(\mathbf{n}) = \sup_{t>0} |P_t f(\mathbf{n})|.$$

In fact, we want analyze weighted inequalities for these operators with weights in the Muckenhoupt class. We say that a positive sequence w is a weight in $A_p(\mathbb{Z}^N)$ for $1 < p < \infty$ when

$$\left(\frac{1}{\mu(Q)} \sum_{\mathbf{n}\in Q} w(\mathbf{n})\right) \left(\frac{1}{\mu(Q)} \sum_{\mathbf{n}\in Q} (w(\mathbf{n}))^{-1/(p-1)}\right)^{p-1} \le C,$$

where Q is any cube in $\mathbb{Z}^N$ and $\mu(A) = \sum_{\mathbf{n}\in A} 1$ for any $A \subset \mathbb{Z}^N$. The weight w belongs to $A_1(\mathbb{Z}^N)$ when

$$\left(\frac{1}{\mu(Q)} \sum_{\mathbf{n}\in Q} w(\mathbf{n})\right) \sup\left\{(w(\mathbf{n})^{-1} : \mathbf{n} \in \mathbb{Z}^N\right\} \le C.$$

To obtain our result, we will use the Calderón-Zygmund theory adapts to our setting. We say that K, defined on $\mathbb{Z}^N \setminus \{\mathbf{0}\}$ and taking values in a Banach space A, is a standard kernel when

$$\|K(\mathbf{n})\|_A \le \frac{C}{|\mathbf{n}|^N} \tag{57}$$

and

$$\|K(\mathbf{n}) - K(\mathbf{n}+\mathbf{m})\|_A \le C\frac{|\mathbf{m}|}{|\mathbf{n}|^{N+1}}, \qquad |\mathbf{n}| > 2|\mathbf{m}|. \tag{58}$$

Let T be an operator bounded from $\ell^q(\mathbb{Z}^N)$ to $\ell^q_A(\mathbb{Z}^N)$ (where we understand that $f \in \ell^q_A(\mathbb{Z}^N)$ when $\|f\|_A \in \ell^q(\mathbb{Z}^N)$) for some $1 < q < \infty$ and for $f \in \ell^q(\mathbb{Z}^N)$ with compact support

$$Tf(\mathbf{n}) = K * f(\mathbf{n}), \qquad \mathbf{n} \notin \operatorname{supp} f,$$

where K is a standard kernel taking values in A, we say that T is Calderón-Zygmund operator.

A Calderón-Zygmung operator Tf is bounded from $\ell^p(\mathbb{Z}^N, w)$ into $\ell^p_A(\mathbb{Z}^N, w)$ for $1 < p < \infty$ and $w \in A_p(\mathbb{Z}^N)$, and from $\ell^1(\mathbb{Z}^N, w)$ into $\ell^{1,\infty}_A(\mathbb{Z}^N, w)$ for $w \in A_1(\mathbb{Z}^N)$ (see [22, 23]). As usual the space $\ell^{1,\infty}(\mathbb{Z}^N, w)$ is defined as the space of sequences such that $\|f\|_{\ell^{1,\infty}(\mathbb{Z}^N,w)} < \infty$, where

$$\|f\|_{\ell^{1,\infty}(\mathbb{Z}^N,w)} = \sup_{\lambda>0} \left(\lambda \sum_{\mathbf{n}\in A_\lambda} w(\mathbf{n})\right)$$

and

$$A_\lambda = \{\mathbf{n} \in \mathbb{Z}^N : |f(\mathbf{n})| > \lambda\}.$$

Theorem 13. *For T equal $W^\star$ or $P^\star$ and $w \in A_p(\mathbb{Z}^N)$, the inequalities*

$$\|Tf\|_{\ell^p(\mathbb{Z}^N,w)} \le \|f\|_{\ell^p(\mathbb{Z}^N,w)}, \qquad 1 < p < \infty, \qquad \ell^2(\mathbb{Z}^N) \cap \ell^p(\mathbb{Z}^N, w),$$

and

$$\|Tf\|_{\ell^{1,\infty}(\mathbb{Z}^N,w)} \le \|f\|_{\ell^1(\mathbb{Z}^N,w)}, \qquad \ell^2(\mathbb{Z}^N) \cap \ell^1(\mathbb{Z}^N, w),$$

hold.

Proof. We focus on $W^\star$ because the result for $P^\star$ follows from its definition by subordination (56) (note that it is clear that $P^\star f \le W^\star f$). By using that

$$W^* f(\mathbf{n}) = \|W_t f(\mathbf{n})\|_{L^\infty((0,\infty))},$$

we prove that $W^\star$ is a Calderón-Zygmund operator taking values in $L^\infty((0,\infty))$. From (24), applying Stein's Maximal Theorem of diffusion semigroups (see [25, Chapter III, Section 3, p. 73]), we have

$$\|W^* f\|_{\ell^p(\mathbb{Z}^N)} \le C\|f\|_{\ell^p(\mathbb{Z}^N)}, \qquad 1 < p < \infty.$$

In particular we have the case $p = 2$. To conclude it is enough to check the estimates

$$\|G_{t,N}(\mathbf{n})\|_{L^\infty((0,\infty))} \le \frac{C}{(|\mathbf{n}|+1)^N} \tag{59}$$

and

$$\|G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{n}+\mathbf{m})\|_{L^\infty((0,\infty))} \le C\frac{|\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{n}+\mathbf{m}|)^{N+1}}, \qquad |\mathbf{n}| > 2|\mathbf{m}|. \tag{60}$$

The bound (59) has been already proved in (32).

Now, from (38), we deduce that for $|\mathbf{n}| > 2|\mathbf{m}|$ it is verified that

$$|G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{n}+\mathbf{m})| \le C\frac{|\mathbf{m}|(|\mathbf{n}|+|\mathbf{n}+\mathbf{m}|)}{1+t}\left(G_{t,1}\left(\frac{|\mathbf{n}|+|\mathbf{n}+\mathbf{m}|}{K}\right)\right)^N.$$

In the case $(|\mathbf{n}|+|\mathbf{n}+\mathbf{m}|)/K > 1/N$, applying (15) with $\alpha = 1/N$, we have

$$\begin{aligned}|G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{n}+\mathbf{m})| &\le C|\mathbf{m}|(|\mathbf{n}|+|\mathbf{n}+\mathbf{m}|)\left(t^{-1/N}G_{t,1}\left(\frac{|\mathbf{n}|+|\mathbf{n}+\mathbf{m}|}{K}\right)\right)^N\\ &\le C\frac{|\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{n}+\mathbf{m}|)^{N+1}}.\end{aligned}$$

When $(|\mathbf{n}|+|\mathbf{n}+\mathbf{m}|)/K \le 1/N$, it is clear that

$$G_{t,1}\left(\frac{|\mathbf{n}|+|\mathbf{n}+\mathbf{m}|}{K}\right) \le G_{t,1}(0) \le C$$

and

$$|G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{n}+\mathbf{m})| \le C\frac{|\mathbf{m}|(|\mathbf{n}|+|\mathbf{n}+\mathbf{m}|)}{1+t} \le \frac{|\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{n}+\mathbf{m}|)^{N+1}}.$$

Then, the proof of (60) is completed. □

4. The fractional integrals and the Riesz transforms

From the identity

$$\frac{1}{\lambda^\sigma} = \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-\lambda t}t^{\sigma-1}\,dt,$$

we define the fractional integrals (or negative powers of the Laplacian Δ_N) by the relation

$$(-\Delta_N)^{-\sigma}f(\mathbf{n}) = \frac{1}{\Gamma(\sigma)}\int_0^\infty W_t f(\mathbf{n})t^{\sigma-1}\,dt.$$

By using that (see [16] or take $\alpha = -1/2$ in (15))

$$e^{-t}I_k(t) \sim t^{-1/2}, \qquad t \to \infty,$$

it is clear that the powers $(-\Delta_N)^{-\sigma}$ are well defined for $0 < 2\sigma < N$.

Having these operators, for $N \geq 2$ we can define the Riesz transforms as

$$R_i f(\mathbf{n}) = \delta_i^+(-\Delta_N)^{-1/2} f(\mathbf{n}), \qquad i = 1, \dots, N.$$

In [7] the Riesz transform for $N = 1$ was analyzed defining it as a limit. In fact, it was defined as

$$Rf(n) = \lim_{\sigma \to (1/2)^-} \delta^+(-\Delta_1)^{-\sigma} f(n). \qquad n \in \mathbb{Z}.$$

From this, the identity

$$Rf(n) = \sum_{k \in \mathbb{Z}} \frac{f(n)}{n - k + 1/2}$$

was obtained; i.e., it matches with the discrete Hilbert transform.

In this section, firstly, we show some basic properties of the fractional integrals and we provide a Hardy-Littlewood-Sobolev inequality in our setting and an extension of it. We continue with the study of multidimensional discrete Schauder inequalities for $(-\Delta_N)^{-\sigma}$. To conclude we analyze the mapping properties of the Riesz transforms on the Hölder classes and on $\ell^p(\mathbb{Z}^N, w)$

4.1. **Basic properties of the fractional integrals.** In this part, we provide some properties of the fractional integrals, we prove a Hardy-Littlewood-Sobolev inequality for them and an extension of it.

In some points a long the paper we will consider the spaces

$$\ell_\sigma(\mathbb{Z}^N) = \ell^1(\mathbb{Z}^N, w_\sigma) \tag{61}$$

with $w_\sigma(\mathbf{n}) = (1 + |\mathbf{n}|)^{2\sigma - N}$. Moreover, we will denote $\|f\|_{\ell_\sigma(\mathbb{Z}^N)} = \|f\|_{\ell^1(\mathbb{Z}^N, w_\sigma)}$.

Theorem 14. *Let $0 < \sigma < N/2$ and $f \in \ell_\sigma(\mathbb{Z}^N)$. We have the pointwise formula*

$$(-\Delta_N)^{-\sigma} f(\mathbf{n}) = \sum_{\mathbf{k} \in \mathbb{Z}^N} f(\mathbf{n} - \mathbf{k}) K_\sigma(\mathbf{k}) \tag{62}$$

where

$$K_\sigma(\mathbf{n}) = \frac{1}{\Gamma(\sigma)} \int_0^\infty G_{t,N}(\mathbf{n}) t^{\sigma - 1}\, dt$$

and it verifies that

$$0 < K_\sigma(\mathbf{n}) \leq \frac{C}{(|\mathbf{n}| + 1)^{N - 2\sigma}}. \tag{63}$$

Proof. First we are going to check that the heat operator is well defined for $f \in \ell_\sigma(\mathbb{Z}^N)$. Indeed, for each $L > 0$ with t fixed, by using (4) and the asymptotic (21),

$$\begin{aligned}
\sum_{|\mathbf{k}| > L} G_{t,N}(\mathbf{k}) |f(\mathbf{n} - \mathbf{k})| &\leq C e^{-2Nt} N^{N/2} \sum_{|\mathbf{k}| > L} \left(\frac{(Net)^{|\mathbf{k}|/N}}{|\mathbf{k}|^{|\mathbf{k}|/N + 1/2}} \right)^N |f(\mathbf{n} - \mathbf{k})| \\
&\leq C \|f\|_{\ell_\sigma(\mathbb{Z}^N)} e^{-2Nt} N^{N/2} \sup_{|\mathbf{k}| > L} \frac{(Net)^{|\mathbf{k}|} (1 + |\mathbf{n} - \mathbf{k}|)^{N - 2\sigma}}{|\mathbf{k}|^{|\mathbf{k}| + N/2}} \\
&\leq C \|f\|_{\ell_\sigma(\mathbb{Z}^N)},
\end{aligned}$$

where the last constant depends on t, N, L, $\mathbf{n}$, and σ but not on f.

Let us prove (62). To do it we can suppose that f is a non negative sequence because in other case we can consider its decomposition in positive and negative parts. Then, applying Tonelli's theorem, we have

$$\begin{aligned}(-\Delta_N)^{-\sigma} f(\mathbf{n}) &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{n}-\mathbf{k}) G_{t,N}(\mathbf{k}) t^{\sigma-1}\, dt \\ &= \frac{1}{\Gamma(\sigma)} \sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{n}-\mathbf{k}) \int_0^\infty G_{t,N}(\mathbf{k}) t^{\sigma-1}\, dt = \sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{n}-\mathbf{k}) K_\sigma(\mathbf{k}).\end{aligned}$$

Now, let us go with the proof of (63). The positivity of K_σ is a simple consequence of the positivity of $G_{t,N}$. Now, to obtain the upper bound for the kernel, we consider the decomposition

$$K_\sigma(\mathbf{n}) = I_1 + I_2$$

with

$$I_1 = \frac{1}{\Gamma(\sigma)} \int_0^{|\mathbf{n}|^2} G_{t,N}(\mathbf{n}) t^{\sigma-1}\, dt \qquad \text{and} \qquad I_2 = \frac{1}{\Gamma(\sigma)} \int_{|\mathbf{n}|^2}^\infty G_{t,N}(\mathbf{n}) t^{\sigma-1}\, dt.$$

We consider $\mathbf{n} \neq \mathbf{0}$, because the bound $K_\sigma(\mathbf{0}) \le C$ is clear. For I_1 we use (32) to have

$$I_1 \le \frac{C}{|\mathbf{n}|^N} \int_0^{|\mathbf{n}|^2} t^{\sigma-1}\, dt = \frac{C}{|\mathbf{n}|^{N-2\sigma}},$$

and for I_2 we apply the bound in (31) to obtain that

$$I_2 \le C \int_{|\mathbf{n}|^2}^\infty \frac{dt}{t^{N/2+1-\sigma}} = \frac{C}{|\mathbf{n}|^{N-2\sigma}}. \qquad \square$$

To prove our next theorem we need the following result that we can see in [10, Theorem 1.6].

Theorem 15. *Let be*

$$\mathcal{I}_{2\sigma} f(\mathbf{n}) = \sum_{\substack{\mathbf{k}\in\mathbb{Z}^N \\ \mathbf{k}\neq\mathbf{n}}} \frac{f(\mathbf{k})}{|\mathbf{n}-\mathbf{k}|^{n-2\sigma}}, \qquad 0 < 2\sigma < N,$$

and $\langle \mathbf{n} \rangle = (1+|\mathbf{n}|^2)^{1/2}$. *Taking* $1 \le p, q < \infty$, $s, t \in \mathbb{R}$ *and the sets of conditions*

$$(\mathcal{C}_1) \qquad \begin{cases} s \le t, \\ \dfrac{2\sigma}{N} + \max\left\{\dfrac{1}{p} + \dfrac{s}{N}, 0\right\} \le \max\left\{\dfrac{1}{q} + \dfrac{t}{N}, 0\right\}, \\ \dfrac{2\sigma}{N} + \dfrac{1}{p} + \dfrac{s}{N} \le 1, \end{cases}$$

$$(\mathcal{C}_2) \qquad \begin{cases} s \le t = 2\sigma - N, q = 1, \\ \dfrac{1}{p} + \dfrac{s}{N} < 0, \end{cases}$$

and

$$(\mathcal{C}_3) \qquad \begin{cases} s \le t, \\ \dfrac{2\sigma}{N} + \dfrac{1}{p} + \dfrac{s}{N} = \dfrac{1}{q} + \dfrac{t}{N}, \\ \dfrac{1}{p} \le \dfrac{1}{q}, \quad 0 < \dfrac{1}{p} + \dfrac{s}{N}, \quad \dfrac{1}{q} + \dfrac{t}{N} < 1, \end{cases}$$

it is verified that

$$\|\langle\cdot\rangle^s \mathcal{I}_{2\sigma} f\|_{\ell^p(\mathbb{Z}^N)} \le \|\langle\cdot\rangle^t f\|_{\ell^q(\mathbb{Z}^N)}$$

if and only if (p,q,s,t) *satisfies one of the conditions* $\mathcal{C}_1$, $\mathcal{C}_2$, *and* $\mathcal{C}_3$.

Remark 4. The previous theorem in its full version includes the case $p,q=\infty$ but the definition of the Lebesgue spaces for such value in [10] is different from ours one, so we omit this particular value from the theorem.

Now, we can present the weighted Hardy-Littlewood-Sobolev for the fractional powers $(-\Delta_N)^{-\sigma}$.

Theorem 16. *Let* $0<2\sigma<N$, $1\le p,q<\infty$ *and* $s,t\in\mathbb{R}$. *If* (q,p,s,t) *verify one of the conditions* $\mathcal{C}_1$, $\mathcal{C}_2$, *and* $\mathcal{C}_3$, *then*

$$\|\langle\cdot\rangle^s(-\Delta_N)^{-\sigma} f\|_{\ell^p(\mathbb{Z}^N)} \le \|\langle\cdot\rangle^t f\|_{\ell^q(\mathbb{Z}^N)}.$$

Proof. The result is immediate from the previous theorem because, by (63), it is verified that

$$|(-\Delta_N)^{-\sigma} f(\mathbf{n})| \le C\left(|\mathcal{I}_{2\sigma} f(\mathbf{n})| + |f(\mathbf{n})|\right). \qquad \square$$

Taking $t=s=0$ in the previous theorem, we recover the classical Hardy-Littlewood-Sobolev inequality in our setting. However, we can go further.

Corollary 17. *Let* $0<2\sigma<N$, $1<q<p<\infty$ *with* $1/p\le 1/q-2\sigma/N$, *then the inequality*

$$\|(-\Delta_N)^{-\sigma} f\|_{\ell^p(\mathbb{Z}^N)} \le C\|f\|_{\ell^q(\mathbb{Z}^N)}$$

holds.

Moreover, for $1<q<p\le 2$ *the Hardy-Littlewood-Sobolev implies* $1/p\le 1/q-2\sigma/N$.

Proof. We have to proof the second part of the result. To do this, we consider the sequence $f(\mathbf{n})=G_{t,N}(\mathbf{n})$ with $t>1$. In this way,

$$\begin{aligned}(-\Delta_N)^{-\sigma} f(\mathbf{n}) &= \int_0^\infty \sum_{\mathbf{k}\in\mathbb{Z}^N} G_{t,N}(\mathbf{k})G_{s,N}(\mathbf{n}-\mathbf{k})s^{\sigma-1}\,ds\\ &= \int_0^\infty G_{t+s,N}(\mathbf{n})s^{\sigma-1}\,ds\end{aligned}$$

From the integral representation [19, 5.10.22]

$$I_\nu(t)=\frac{t^\nu}{\sqrt{\pi}2^\nu\Gamma(\nu+1/2)}\int_{-1}^1 e^{-tz}(1-z^2)^{\nu-1/2}\,dz, \qquad t>0, \quad \nu>-1/2,$$

we can deduce that

$$G_{t+s,N}(\mathbf{n})\ge G_{2t,N}(\mathbf{n}), \qquad t/2\le s\le t.$$

In this way,

$$(-\Delta_N)^{-\sigma} f(\mathbf{n}) \ge G_{2t,N}(\mathbf{n})\int_{t/2}^t s^{\sigma-1}\,ds \ge CG_{2t,N}(\mathbf{n})t^\sigma.$$

Then, by the Hardy-Littlewood-Sobolev, we have

$$t^\sigma\|G_{2t,N}\|_{\ell^p(\mathbb{Z}^N)} \le C\|G_{t,N}\|_{\ell^q(\mathbb{Z}^N)}$$

and, applying (34), it becomes $t^{\sigma-N/2(1/q-1/p)}\le C$ and this implies $1/p\le 1/q-2\sigma/N$. $\square$

To finish this subsection we show a new inequality extending the classical Hardy-Littlewood-Sobolev one.

Theorem 18. *Let $0 < 2\sigma < N$, $1 \le q \le p \le \infty$ with $1 \le q < N/(N-1)$ and $1/p < 1/q - (2\sigma-1)/N$, and $|\cdot|f \in \ell^q(\mathbb{Z}^N)$, then the inequality*

$$\left\| (-\Delta_N)^{-\sigma} f - \left(\sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{k}) \right) K_\sigma \right\|_{\ell^p(\mathbb{Z}^N)} \le C \| |\cdot| f\|_{\ell^q(\mathbb{Z}^N)}$$

holds.

Proof. The result follows from Minkowski integral inequality and (55). Indeed,

$$\begin{aligned} &\left\| (-\Delta_N)^{-\sigma} f - \left(\sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{k}) \right) K_\sigma \right\|_{\ell^p(\mathbb{Z}^N)} \\ &\qquad \le \frac{1}{\Gamma(\sigma)} \int_0^\infty \left\| W_t f - \sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{k}) G_{t,N} \right\|_{\ell^p(\mathbb{Z}^N)} t^{\sigma-1}\, dt \\ &\qquad \le C \| |\cdot| f\|_{\ell^q(\mathbb{Z}^N)} \int_0^\infty \min\{1, t^{-1/2-N/2(1/q-1/p)}\} t^{\sigma-1}\, dt \\ &\qquad \le C \| |\cdot| f\|_{\ell^q(\mathbb{Z}^N)}. \end{aligned}$$

□

4.2. **Discrete Schauder estimates for the fractional integrals.** Now, we prove multidimensional discrete Schauder estimates for the fractional integrals and to do that we need to define the Hölder spaces. Given a sequence $f = \{f(\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^N}$ and $0 < \alpha \le 1$, we say that f belongs to the Hölder space $C^{0,\alpha}(\mathbb{Z}^N)$ when

$$[f]_{C^{0,\alpha}(\mathbb{Z}^N)} := \sup_{\mathbf{n},\mathbf{m}\in\mathbb{Z}} \frac{|f(\mathbf{n}) - f(\mathbf{m})|}{|\mathbf{n}-\mathbf{m}|^\alpha} < \infty.$$

For a natural number k such that $k \ge 1$ we say that $f \in C^{k,\alpha}(\mathbb{Z}^N)$ when it is verified that

$$[f]_{C^{k,\alpha}(\mathbb{Z}^N)} := \sum_{m_1+\cdots+m_N = k} [(\delta_1^+)^{m_1} (\delta_2^+)^{m_2} \cdots (\delta_N^+)^{m_N} f]_{C^{0,\alpha}(\mathbb{Z}^N)} < \infty.$$

It is easy to check that $\ell^\infty(\mathbb{Z}^N) \subset C^{0,\alpha}(\mathbb{Z}^N) \subset C^{1,\alpha}(\mathbb{Z}^N) \subset C^{2,\alpha}(\mathbb{Z}^N) \subset \cdots$.

Sometimes it is common write the definition of Hölder classes combining δ_i^+ and δ_i^- but such definitions are equivalents to the our one due to the relation $\delta_i^- f(\mathbf{n}) = \delta_i^+ f(\mathbf{n} - \mathbf{e}_i)$.

The next result is the multidimensional version of [8, Theorem 1.6].

Theorem 19. *Let $k \ge 0$, $0 < \alpha \le 1$, $0 < \sigma < 1/2$ and $f \in \ell_\sigma(\mathbb{Z}^N)$.*

a) *If $f \in C^{k,\alpha}(\mathbb{Z}^N)$ and $2\sigma + \alpha < 1$ then $(-\Delta)^{-\sigma} f \in C^{k,2\sigma+\alpha}(\mathbb{Z}^N)$ and*
$$[(-\Delta)^{-\sigma} f]_{C^{k,2\sigma+\alpha}(\mathbb{Z}^N)} \le C [f]_{C^{k,\alpha}(\mathbb{Z}^N)}.$$

b) *If $f \in C^{k,\alpha}(\mathbb{Z}^N)$ and $2\sigma + \alpha > 1$ then $(-\Delta)^{-\sigma} f \in C^{k+1,2\sigma+\alpha-1}(\mathbb{Z}^N)$ and*
$$[(-\Delta)^{-\sigma} f]_{C^{k+1,2\sigma+\alpha-1}(\mathbb{Z}^N)} \le C [f]_{C^{k,\alpha}(\mathbb{Z}^N)}.$$

c) *If $f \in \ell^\infty(\mathbb{Z}^N)$ then $(-\Delta)^{-\sigma} f \in C^{0,2\sigma}(\mathbb{Z}^N)$ and*
$$[(-\Delta)^{-\sigma} f]_{C^{0,2\sigma}(\mathbb{Z}^N)} \le C \|f\|_{\ell^\infty(\mathbb{Z}^N)}.$$

Before starting with the proof of the Schauder estimates, we need the four following lemmas.

Lemma 20. *For $N \geq 1$, $\mathbf{n} \in \mathbb{Z}^N$, and $0 < \sigma < 1/2$, we have*

$$\sum_{\mathbf{k}\in\mathbb{Z}^N} (K_\sigma(\mathbf{n}-\mathbf{k}) - K_\sigma(\mathbf{k})) = 0.$$

Proof. From the obvious identity

$$\sum_{\mathbf{k}\in\mathbb{Z}^N} (G_{t,N}(\mathbf{n}-\mathbf{k}) - G_{t,N}(\mathbf{k})) = 0,$$

we deduce that

$$\int_0^\infty \sum_{\mathbf{k}\in\mathbb{Z}^N} (G_{t,N}(\mathbf{n}-\mathbf{k}) - G_{t,N}(\mathbf{k}))t^{\sigma-1}\,dt = 0,$$

Let us see now that, for $0 < \sigma < 1/2$, we can apply Fubini theorem. From (22), it is clear that

$$\int_0^{|\mathbf{n}|^2} \sum_{\mathbf{k}\in\mathbb{Z}^N} |G_{t,N}(\mathbf{n}-\mathbf{k}) - G_{t,N}(\mathbf{k})|t^{\sigma-1}\,dt \leq 2\int_0^{|\mathbf{n}|^2} t^{\sigma-1}\,dt \leq C|\mathbf{n}|^{2\sigma} < \infty,$$

so we have to focus on the integral in the interval $(|\mathbf{n}|^2, \infty)$. To do that we consider

$$I_1 = \int_{|\mathbf{n}|^2}^\infty \sum_{|\mathbf{k}|\leq 2|\mathbf{n}|} |G_{t,N}(\mathbf{n}-\mathbf{k}) - G_{t,N}(\mathbf{k})|t^{\sigma-1}\,dt$$

and

$$I_2 = \int_{|\mathbf{n}|^2}^\infty \sum_{|\mathbf{k}|> 2|\mathbf{n}|} |G_{t,N}(\mathbf{n}-\mathbf{k}) - G_{t,N}(\mathbf{k})|t^{\sigma-1}\,dt.$$

For I_1 we use the bound (43) and the restriction $|\mathbf{k}| \leq 2|\mathbf{n}|$ to obtain that

$$\begin{aligned} I_1 &\leq C|\mathbf{n}|^2 \int_{|\mathbf{n}|^2}^\infty \left(\sum_{|\mathbf{k}|\leq 2|\mathbf{n}|} 1\right) t^{\sigma-N/2-2}\,dt \\ &\leq C|\mathbf{n}|^{2+N} \int_{|\mathbf{n}|^2}^\infty t^{\sigma-N/2-2}\,dt \leq C|\mathbf{n}|^{2\sigma} < \infty. \end{aligned}$$

To analyze I_2 we apply the identity $G_{t,N}(\mathbf{n}-\mathbf{k}) = G_{t,N}(\mathbf{k}-\mathbf{n})$, (38) and (46) with $r = 1$ to deduce the estimate

$$\begin{aligned} I_2 &\leq C|\mathbf{n}| \int_{|\mathbf{n}|^2}^\infty \left(\sum_{|\mathbf{k}|>2|\mathbf{n}|} H_{t,N}\left(\frac{|\mathbf{k}|}{K}\right)\right) t^{\sigma-1}\,dt \\ &\leq C|\mathbf{n}| \int_{|\mathbf{n}|^2}^\infty t^{\sigma-3/2}\,dt \leq C|n|^{2\sigma} < \infty. \end{aligned}$$

In this way,

$$\sum_{k\in\mathbb{Z}^N} (K_\sigma(\mathbf{n}-\mathbf{k}) - K_\sigma(\mathbf{k})) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \sum_{\mathbf{k}\in\mathbb{Z}^N} (G_{t,N}(\mathbf{n}-\mathbf{k}) - G_{t,N}(\mathbf{k}))t^{\sigma-1}\,dt = 0$$

and the proof is complete. □

Lemma 21. *Let $N \geq 1$, $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^N$ such that $|\mathbf{n}| > 2|\mathbf{n}-\mathbf{m}|$, and $0 < 2\sigma < N$. Then the inequality*

$$|K_\sigma(\mathbf{n}) - K_\sigma(\mathbf{m})| \leq \frac{C|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+1-2\sigma}} \tag{64}$$

holds with a constant C independent of $\mathbf{n}$ and $\mathbf{m}$.

Proof. It is clear that

$$\Gamma(\sigma)|K_\sigma(\mathbf{n}) - K_\sigma(\mathbf{m})| \leq \int_0^\infty |G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m})| t^{\sigma-1}\, dt = J_1 + J_2,$$

where

$$J_1 = \int_0^{(|\mathbf{n}|+|\mathbf{m}|)^2} |G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m})| t^{\sigma-1}\, dt$$

and

$$J_2 = \int_{(|\mathbf{n}|+|\mathbf{m}|)^2}^\infty |G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m})| t^{\sigma-1}\, dt.$$

To estimate J_1 we apply Lemma 8 to have

$$|G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m})| \leq C\frac{|\mathbf{n}-\mathbf{m}|(|\mathbf{n}|+|\mathbf{m}|)}{1+t}\left(G_{t,1}\left(\frac{|\mathbf{n}|+|\mathbf{m}|}{K}\right)\right)^N.$$

As we did in the proof of Theorem 13, we analyze separately the cases $(|\mathbf{n}|+|\mathbf{m}|)/K > 1/N$, where we can use (15) with $\alpha = 1/N$, and $(|\mathbf{n}|+|\mathbf{m}|)/K \leq 1/N$ to deduce that

$$|G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m})| \leq C\frac{|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+1}}.$$

In this way

$$J_1 \leq \frac{C|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+1}} \int_0^{(|\mathbf{n}|+|\mathbf{m}|)^2} t^{\sigma-1}\, dt \leq \frac{C|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+1-2\sigma}}.$$

The bound for J_2 is obtained from (43) in the following way

$$J_2 \leq |\mathbf{n}-\mathbf{m}|(|\mathbf{n}|+|\mathbf{m}|)\int_{(|\mathbf{n}|+|\mathbf{m}|)^2}^\infty t^{\sigma-N/2-2}\, dt \leq \frac{C|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+1-2\sigma}}. \qquad \square$$

Lemma 22. *Let $N \geq 1$, $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^N$ such that $|\mathbf{n}| > 2|\mathbf{n}-\mathbf{m}|$, and*

$$\mathcal{H}_{t,N}(z) = \left(\frac{1}{1+t} + \frac{z^2}{(1+t)^2}\right)(G_{t,1}(z))^N, \qquad z > 0.$$

Then the inequality

$$|\delta_i^+(G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m}))| \leq C|\mathbf{n}-\mathbf{m}|\mathcal{H}_{t,N}\left(\frac{|\mathbf{n}|+|\mathbf{m}|}{K}\right), \qquad i = 1, \dots, N, \tag{65}$$

holds with constants C and $K > 1$ independent of $\mathbf{n}$ and $\mathbf{m}$.

Proof. We consider $i = 1$ because the other cases can be done in the same way. We can prove the result for $n_i \neq m_i$ for $i = 1, \dots, N$ because this situation implies the general case. In fact, to deduce the general case from this one, as in Lemma 8, let us suppose that $n_i = m_i$ for j values of i, with $1 < j < N$ and consider the decompositions

$$\mathbf{n} = \mathbf{n}_1 \cup \mathbf{n}_2 \qquad \text{and} \qquad \mathbf{m} = \mathbf{n}_1 \cup \mathbf{m}_2,$$

where $\mathbf{n}_1$ are the common values of $\mathbf{n}$ and $\mathbf{m}$ and $\mathbf{n}_2$ and $\mathbf{m}_2$ are the other components. Now, we have to distinguish two possibilities $n_1 \in \mathbf{n}_1$ and $n_1 \notin \mathbf{n}_1$. In the first situation, by using (6) and Lemma 8, we have

$$\begin{aligned}|\delta_1^+(G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m}))| &\le C\frac{|n_1|+1}{1+t}G_{t,j}(\mathbf{n}_1)|G_{t,N-j}(\mathbf{n}_2) - G_{t,N-j}(\mathbf{m}_2)| \\ &\le C|\mathbf{n}_2 - \mathbf{m}_2|\frac{(|\mathbf{n}_1|+1)(|\mathbf{n}_2|+|\mathbf{m}_2|)}{(1+t)^2}G_{t,j}(\mathbf{n}_1)\left(G_{t,1}\left(\frac{|\mathbf{n}_2|+|\mathbf{m}_2|}{K}\right)\right)^{N-j} \\ &\le C|\mathbf{n}-\mathbf{m}|\left(\frac{1}{1+t}+\frac{(|\mathbf{n}|+|\mathbf{m}|)^2}{(1+t)^2}\right)G_{t,j}(\mathbf{n}_1)\left(G_{t,1}\left(\frac{|\mathbf{n}_2|+|\mathbf{m}_2|}{K}\right)\right)^{N-j}\end{aligned}$$

and we conclude as in Lemma 8. When $n_1 \notin \mathbf{n}_1$, if we consider proved the case with different components, it is verified that

$$\begin{aligned}|\delta_1^+(G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m}))| &= G_{t,j}(\mathbf{n}_1)|\delta_1^+(G_{t,N-j}(\mathbf{n}_2) - G_{t,N-j}(\mathbf{m}_2))| \\ &\le CG_{t,j}(\mathbf{n}_1)|\mathbf{n}_2 - \mathbf{m}_2|\mathcal{H}_{t,N-j}\left(\frac{|\mathbf{n}_2|+|\mathbf{m}_2|}{K}\right)\end{aligned}$$

and the proof of this case is finished again as in Lemma 8.

Now, we assume the restrictions (39) on $\mathbf{n}$ and $\mathbf{m}$ as in the proof of Lemma 8. Then, using (40), we get

$$\begin{aligned}\delta_1^+(G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m})) &= \delta^+(G_{t,1}(n_1) - G_{t,1}(m_1))\prod_{k=2}^N G_{t,1}(n_k) \\ &+ \delta^+G_{t,1}(m_1)\sum_{i=2}^N (G_{t,1}(n_i) - G_{t,1}(m_i))\prod_{j=2}^{i-1}G_{t,1}(m_j)\prod_{k=i+1}^N G_{t,1}(n_k).\end{aligned} \tag{66}$$

Proceeding as in Lemma 8 and using (6), we have

$$\begin{aligned}&|\delta^+G_{t,1}(m_1)(G_{t,1}(n_i) - G_{t,1}(m_i))| \\ &\quad\le C\frac{(m_1+1)|n_i - m_i|(m_i+n_i+1)}{(1+t)^2}G_{t,1}(m_1)\max\{G_{t,1}(n_i), G_{t,1}(m_i)\}.\end{aligned} \tag{67}$$

Now, taking $n_1 < m_1$, it is obtained that

$$\begin{aligned}\delta^+(G_{t,1}(n_1) - G_{t,1}(m_1)) &= -\sum_{k=n_1}^{m_1-1}(\delta^+G_{t,1}(k+1) - \delta^+G_{t,1}(k)) \\ &= -\sum_{k=n_1}^{m_1-1}(G_{t,1}(k+2) - 2G_{t,1}(k+1) + G_{t,1}(k))\end{aligned}$$

and, applying (9),

$$\begin{aligned}|\delta^+(G_{t,1}(n_1) - G_{t,1}(m_1))| &\le C\sum_{k=n_1}^{m_1-1}\left(\frac{1}{1+t}+\frac{(k+1)(k+2)}{(1+t)^2}\right)G_{t,1}(k) \\ &\le C(m_1-n_1)\left(\frac{1}{1+t}+\frac{(m_1+n_1)^2}{(1+t)^2}\right)G_{t,1}(n_1).\end{aligned}$$

In general,

$$|\delta^+(G_{t,1}(n_1) - G_{t,1}(m_1))| \tag{68}$$

$$\leq C|m_1-n_1|\left(\frac{1}{1+t}+\frac{(m_1+n_1)^2}{(1+t)^2}\right)\max\{G_{t,1}(n_1),G_{t,1}(m_1)\}.$$

With (66), (67), and (68) the result can be concluded as in Lemma 8. □

Remark 5. Again, following the proof of the previous lemma, it is easy to check that

(69)
$$|\delta_i^+(G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{m}))|\leq C|\mathbf{n}-\mathbf{m}|\left(\frac{1}{1+t}+\frac{(|\mathbf{n}|+|\mathbf{m}|)^2}{(1+t)^2}\right)t^{-N/2},\quad \mathbf{n},\mathbf{m}\in\mathbb{Z}^N.$$

Lemma 23. *Let $N\geq 1$, $\mathbf{n},\mathbf{m}\in\mathbb{Z}^N$ such that $|\mathbf{n}|>2|\mathbf{n}-\mathbf{m}|$, and $0<2\sigma<N$. Then the inequality*

(70)
$$|\delta_i^+(K_\sigma(\mathbf{n})-K_\sigma(\mathbf{m}))|\leq\frac{C|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+2-2\sigma}},\qquad i=1,\dots,N,$$

holds with a constant C independent of $\mathbf{n}$ and $\mathbf{m}$.

Proof. It is clear that

$$\Gamma(\sigma)|\delta_i^+(K_\sigma(\mathbf{n})-K_\sigma(\mathbf{m}))|\leq\int_0^\infty|\delta_i^+(G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{m}))|t^{\sigma-1}\,dt=Q_1+Q_2,$$

with

$$Q_1=\int_0^{(|\mathbf{n}|+|\mathbf{m}|)^2}|\delta_i^+(G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{m}))|t^{\sigma-1}\,dt$$

and

$$Q_2=\int_{(|\mathbf{n}|+|\mathbf{m}|)^2}^\infty|\delta_i^+(G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{m}))|t^{\sigma-1}\,dt.$$

To obtain the required bound for Q_1 we start proving that

(71)
$$|\delta_i^+(G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{m}))|\leq C\frac{|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+2}}.$$

First, we consider Lemma 22 to have

$$|\delta_i^+(G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{m}))|\leq C|\mathbf{n}-\mathbf{m}|\left(\frac{1}{1+t}+\frac{(|\mathbf{n}|+|\mathbf{m}|)^2}{(1+t)^2}\right)\left(G_{t,1}\left(\frac{|\mathbf{n}|+|\mathbf{m}|}{K}\right)\right)^N.$$

In the case $(|\mathbf{n}|+|\mathbf{m}|)/K>2/N$, applying (15) with $\alpha=1/N$ and $\alpha=2/N$, (71) follows immediately. Indeed,

$$\begin{aligned}|\delta_i^+(G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{m}))|&\leq C|\mathbf{n}-\mathbf{m}|\left(\left(t^{-1/N}G_{t,1}\left(\frac{|\mathbf{n}|+|\mathbf{m}|}{K}\right)\right)^N\right.\\&\qquad\left.+(|\mathbf{n}|+|\mathbf{m}|)^2\left(t^{-2/N}G_{t,1}\left(\frac{|\mathbf{n}|+|\mathbf{m}|}{K}\right)\right)^N\right)\\&\leq C\frac{|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+2}}.\end{aligned}$$

In the case $(|\mathbf{n}|+|\mathbf{m}|)/K\leq 2/N$, taking into account that

$$G_{t,1}\left(\frac{|\mathbf{n}|+|\mathbf{m}|}{K}\right)\leq G_{t,1}(0)\leq C,$$

it is obtained that

$$|\delta_i^+(G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{m}))| \le C|\mathbf{n}-\mathbf{m}|\left(\frac{1}{1+t}+\frac{(|\mathbf{n}|+|\mathbf{m}|)^2}{(1+t)^2}\right) \le C\frac{|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+2}}$$

and the proof of (71) is concluded. Then, applying (71), we have

$$Q_1 \le C\frac{|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+2}}\int_0^{(|\mathbf{n}|+|\mathbf{m}|)^2} t^{\sigma-1}\,dt \le \frac{C|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+2-2\sigma}}.$$

The estimate for Q_2 can be deduced from (69) in the following way

$$\begin{aligned} Q_2 &\le |\mathbf{n}-\mathbf{m}|\left(\int_{(|\mathbf{n}|+|\mathbf{m}|)^2}^{\infty} t^{\sigma-N/2-2}\,dt+(|\mathbf{n}|+|\mathbf{m}|)^2\int_{(|\mathbf{n}|+|\mathbf{m}|)^2}^{\infty} t^{\sigma-N/2-3}\,dt\right)\\ &\le \frac{C|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+2-2\sigma}}. \end{aligned}$$

□

Proof of Theorem 19. a) By using that

$$(\delta_i^+)^k(-\Delta_N)^{-\sigma}f(\mathbf{n}) = (-\Delta_N)(\delta_i^+)^k f(\mathbf{n}),$$

it is enough to prove the result for $k=0$. Now, from Lemma 20 we have

$$\begin{aligned} \sum_{\mathbf{k}\in\mathbb{Z}^N}(K_\sigma(\mathbf{n}-\mathbf{k})-K_\sigma(\mathbf{m}-\mathbf{k})) &= \sum_{\mathbf{k}\in\mathbb{Z}^N}(K_\sigma(\mathbf{n}-\mathbf{k})-K_\sigma(\mathbf{k}-\mathbf{m}))\\ &= \sum_{\mathbf{j}\in\mathbb{Z}^N}(K_\sigma(\mathbf{n}-\mathbf{m}-\mathbf{j})-K_\sigma(\mathbf{j}))=0, \end{aligned}$$

and it is verified that

$$(-\Delta_N)^{-\sigma}f(\mathbf{n})-(-\Delta_N)^{-\sigma}f(\mathbf{m}) = \sum_{\mathbf{k}\in\mathbb{Z}^N}(K_\sigma(\mathbf{n}-\mathbf{k})-K_\sigma(\mathbf{m}-\mathbf{k}))(f(\mathbf{k})-f(\mathbf{n})).$$

Now, we split the sum on $0<|\mathbf{n}-\mathbf{k}|\le 2|\mathbf{n}-\mathbf{m}|$ and its complementary. By using (63) we get

$$\begin{aligned} &\sum_{0<|\mathbf{n}-\mathbf{k}|\le 2|\mathbf{n}-\mathbf{m}|} K_\sigma(\mathbf{n}-\mathbf{k})|f(\mathbf{k})-f(\mathbf{n})|\\ &\qquad\le C[f]_{C^{0,\alpha}(\mathbb{Z}^N)}\sum_{0<|\mathbf{n}-\mathbf{k}|\le 2|\mathbf{n}-\mathbf{m}|}\frac{|\mathbf{n}-\mathbf{k}|^\alpha}{|\mathbf{n}-\mathbf{k}|^{N-2\sigma}}\\ &\qquad\le C[f]_{C^{0,\alpha}(\mathbb{Z}^N)}\sum_{0<|\mathbf{j}|\le 2|\mathbf{n}-\mathbf{m}|}|\mathbf{j}|^{\alpha+2\sigma-N}\\ &\qquad\le C[f]_{C^{0,\alpha}(\mathbb{Z}^N)}|\mathbf{n}-\mathbf{m}|^{\alpha+2\sigma}. \end{aligned}$$

For the other term in the first part of the decomposition, by using that $|\mathbf{n}-\mathbf{k}|\le 2|\mathbf{n}-\mathbf{m}|$ implies $|\mathbf{m}-\mathbf{k}|\le 3|\mathbf{n}-\mathbf{m}|$ and (63), we have

$$\sum_{0<|\mathbf{n}-\mathbf{k}|\le 2|\mathbf{n}-\mathbf{m}|} K_\sigma(\mathbf{m}-\mathbf{k})|f(\mathbf{k})-f(\mathbf{n})|$$

$$\leq C[f]_{C^{0,\alpha}(\mathbb{Z}^N)} \sum_{|\mathbf{m}-\mathbf{k}|\leq 3|\mathbf{n}-\mathbf{m}|} \frac{|\mathbf{n}-\mathbf{k}|^\alpha}{(|\mathbf{m}-\mathbf{k}|+1)^{N-2\sigma}}$$
$$\leq C[f]_{C^{0,\alpha}(\mathbb{Z}^N)} \left(\sum_{|\mathbf{m}-\mathbf{k}|\leq 3|\mathbf{n}-\mathbf{m}|} \frac{|\mathbf{n}-\mathbf{m}|^\alpha}{(|\mathbf{m}-\mathbf{k}|+1)^{N-2\sigma}} + \sum_{|\mathbf{m}-\mathbf{k}|\leq 3|\mathbf{n}-\mathbf{m}|} (|\mathbf{m}-\mathbf{k}|+1)^{\alpha+2\sigma-N} \right)$$
$$\leq C[f]_{C^{0,\alpha}(\mathbb{Z}^N)} |\mathbf{n}-\mathbf{m}|^{\alpha+2\sigma}.$$

Finally, applying (64), we have

$$\sum_{|\mathbf{n}-\mathbf{k}|\geq 2|\mathbf{n}-\mathbf{m}|} |K_\sigma(\mathbf{n}-\mathbf{k}) - K_\sigma(\mathbf{m}-\mathbf{k})||f(\mathbf{k}) - f(\mathbf{n})|$$
$$\leq C[f]_{C^{0,\alpha}(\mathbb{Z}^N)} |\mathbf{n}-\mathbf{m}| \sum_{|\mathbf{n}-\mathbf{k}|\geq 2|\mathbf{n}-\mathbf{m}|} \frac{|\mathbf{n}-\mathbf{k}|^\alpha}{|\mathbf{n}-\mathbf{k}|^{N+1-2\sigma}}$$
$$\leq C[f]_{C^{0,\alpha}(\mathbb{Z}^N)} |\mathbf{n}-\mathbf{m}|^{\alpha+2\sigma}$$

and we have concluded this part.

b) Again, we can reduce the proof to the case $k=0$. Now, from Lemma 20, for any $i = 1, \dots, n$ we have

$$\delta_i^+((-\Delta_N)^{-\sigma} f(\mathbf{n}) - (-\Delta_N)^{-\sigma} f(\mathbf{m})) = \sum_{\mathbf{k}\in\mathbb{Z}^N} \delta_i^+(K_\sigma(\mathbf{n}-\mathbf{k}) - K_\sigma(\mathbf{m}-\mathbf{k}))(f(\mathbf{k}) - f(\mathbf{n}))).$$

Splitting the sum as in a), the proof follows the same steps but using in the region $|\mathbf{n}-\mathbf{k}| \leq 2|\mathbf{n}-\mathbf{m}|$ the bounds (deduced from (64))

$$|K_\sigma(\mathbf{n}+\mathbf{e_i}-\mathbf{k}) - K_\sigma(\mathbf{n}-\mathbf{k})| \leq C|\mathbf{n}-\mathbf{k}|^{2\sigma-N-1}, \qquad \mathbf{n},\mathbf{k} \in \mathbb{Z},$$

and

$$|K_\sigma(\mathbf{m}+\mathbf{e_i}-\mathbf{k}) - K_\sigma(\mathbf{m}-\mathbf{k})| \leq C|\mathbf{m}-\mathbf{k}|^{2\sigma-N-1}, \qquad \mathbf{m},\mathbf{k} \in \mathbb{Z},$$

and in the region $|\mathbf{n}-\mathbf{k}| > 2|\mathbf{n}-\mathbf{m}|$ the estimate (70).

c) The proof of this part can be done as in a) but changing the bound $|f(\mathbf{n}) - f(\mathbf{k})| \leq C[f]_{C^{0,\alpha}(\mathbb{Z}^N)}|\mathbf{n}-\mathbf{k}|^\alpha$ by $|f(\mathbf{n}) - f(\mathbf{k})| \leq 2\|f\|_{\ell^\infty(\mathbb{Z}^N)}$. □

4.3. **The Riesz transforms.** As we have said at the beginning of this section the Riesz transform are well defined, because the kernel

$$\int_0^\infty \delta_i^+ G_{t,N}(\mathbf{n}) t^{-1/2}\, dt$$

is absolutely convergent.

Of course the Riesz transforms can be written for $N \geq 2$ as

$$R_i f(\mathbf{n}) = \mathcal{R}_i * f(\mathbf{n}),$$

with

$$\mathcal{R}_i(\mathbf{n}) = \frac{1}{\sqrt{\pi}} \int_0^\infty \delta_i^+ G_{t,N}(\mathbf{n}) t^{-1/2}\, dt.$$

To present our results about the Riesz transforms we need two preliminary lemmas.

Lemma 24. *Let $N \geq 2$, then the inequalities*

$$|\mathcal{R}_i(\mathbf{n})| \leq \frac{C}{(|\mathbf{n}|+1)^N} \tag{72}$$

and

$$|\mathcal{R}_i(\mathbf{n}) - \mathcal{R}_i(\mathbf{m})| \leq C\frac{|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+1}}, \qquad |\mathbf{n}| > 2|\mathbf{n}-\mathbf{m}|, \tag{73}$$

hold.

Proof. To prove (72) we consider $\mathbf{n} \neq \mathbf{0}$ because the bound $|\mathcal{R}_i(\mathbf{0})| \leq C$ is obvious. Now, from (6) and (4), it is deduced that

$$|\delta_i^+ G_{t,N}(\mathbf{n})| \leq C\frac{|n_i|+1}{t} G_{t,N}(\mathbf{n}) \leq C\frac{|\mathbf{n}|}{t}\left(G_{t,1}\left(\frac{|\mathbf{n}|}{N}\right)\right)^N.$$

Then, using (15) with $\alpha = 1/N$ and $\alpha = -1/2$, we deduce that

$$|\mathcal{R}_i(\mathbf{n})| \leq C|\mathbf{n}|\left(\frac{1}{|\mathbf{n}|^{N+2}}\int_0^{|\mathbf{n}|^2} t^{-1/2}\,dt + \int_{|\mathbf{n}|^2}^\infty t^{-N/2-3/2}\,dt\right) \leq \frac{C}{|\mathbf{n}|^N}.$$

Now, we decompose the difference in (73) in the following way

$$\begin{aligned}|\mathcal{R}_i(\mathbf{n}) - \mathcal{R}_i(\mathbf{m})| \leq &\int_0^{(|\mathbf{n}|+|\mathbf{m}|)^2} |\delta_i^+(G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m}))| t^{-1/2}\,dt \\ &+ \int_{(|\mathbf{n}|+|\mathbf{m}|)^2}^\infty |\delta_i^+(G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m}))| t^{-1/2}\,dt := L_1 + L_2.\end{aligned}$$

To analyze L_1 we use (71) to obtain that

$$L_1 \leq C\frac{|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+2}}\int_0^{(|\mathbf{n}|+|\mathbf{m}|)^2} t^{-1/2}\,dt \leq C\frac{|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+1}}.$$

From (69), we have

$$\begin{aligned}L_2 &\leq C|\mathbf{n}-\mathbf{m}|\left(\int_{(|\mathbf{n}|+|\mathbf{m}|)^2}^\infty t^{-N/2-3/2}\,dt + (|\mathbf{n}|+|\mathbf{m}|)^2\int_{(|\mathbf{n}|+|\mathbf{m}|)^2}^\infty t^{-N/2-5/2}\,dt\right) \\ &\leq C\frac{|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+1}}\end{aligned}$$

and the proof is finished. □

Lemma 25. *For $N \geq 2$ it is verified that*

$$\sum_{\mathbf{k}\in\mathbb{Z}^N} \mathcal{R}_i(\mathbf{k}) = 0, \qquad i = 1, \dots, N, \tag{74}$$

and, for a fix value $\ell \in \mathbb{N}$,

$$\left|\sum_{|\mathbf{k}|\geq \ell} \mathcal{R}_i(\mathbf{k})\right| \leq C, \qquad i = 1, \dots, N. \tag{75}$$

Proof. To prove the result we focus on $i = 1$ because the other cases are similar. We consider $M \in \mathbb{N}$ big enough and the partial sums

$$S_M = \sum_{|\mathbf{k}|\le M} \mathcal{R}_1(\mathbf{k}).$$

Now, we observe that

$$\mathcal{R}_1(-\mathbf{k}) = -\mathcal{R}_1(\mathbf{k}-\mathbf{e}_1). \tag{76}$$

Indeed, it is clear that

$$\begin{aligned}\delta_1^+ G_{t,N}(-\mathbf{k}) &= (G_{t,1}(-k_1+1) - G_{t,1}(-k_1)) \prod_{j=2}^N G_{t,1}(-k_j)\\ &= -(G_{t,1}(k_1) - G_{t,1}(k_1-1)) \prod_{j=2}^N G_{t,1}(k_j) = -\delta_1^+ G_{t,N}(\mathbf{k}-\mathbf{e}_1)\end{aligned}$$

and (76) follows immediately. With (76) we have

$$\begin{aligned}2S_M &= \sum_{|\mathbf{k}|\le M} \mathcal{R}_1(\mathbf{k}) + \sum_{|\mathbf{k}|\le M} \mathcal{R}_1(-\mathbf{k})\\ &= \sum_{|\mathbf{k}|\le M} \mathcal{R}_1(\mathbf{k}) - \sum_{|\mathbf{k}|\le M} \mathcal{R}_1(\mathbf{k}-\mathbf{e}_1) = \sum_{|\mathbf{k}|\le M} \mathcal{R}_1(\mathbf{k}) - \sum_{|\mathbf{k}+\mathbf{e}_1|\le M} \mathcal{R}_1(\mathbf{k}).\end{aligned}$$

Then, applying (72) we get

$$|S_M| \le C \sum_{M-1\le|\mathbf{k}|\le M+1} |\mathcal{R}_1(\mathbf{k})| \le \frac{C}{M}$$

and

$$\sum_{\mathbf{k}\in\mathbb{Z}^N} \mathcal{R}_1(\mathbf{k}) = \lim_{M\to\infty} S_M = 0$$

proving (74).

To analyze (75), we have to note that

$$\left|\sum_{|\mathbf{k}|\ge\ell} \mathcal{R}_i(\mathbf{k})\right| = \left|\sum_{\mathbf{k}\in\mathbb{Z}^N} \mathcal{R}_i(\mathbf{k}) - S_{\ell-1}\right| = |S_{\ell-1}| \le \frac{C}{\ell-1}$$

and the result follows for $\ell > 1$. The case $\ell = 1$ is obvious. □

First, we present the behaviour of the Riesz transforms on the Hölder classes.

Theorem 26. *Let $N \ge 2$, $0 < \alpha < 1$, and $f \in C^{0,\alpha}(\mathbb{Z}^N)$. Then for $1 \le i \le N$ the inequality*

$$[R_i f]_{C^{0,\alpha}(\mathbb{Z}^N)} \le C[f]_{C^{0,\alpha}(\mathbb{Z}^N)}$$

holds.

Proof. By using Lemma 25, for $\mathbf{n},\mathbf{m} \in \mathbb{Z}^N$ it is easy to check that

$$R_i f(\mathbf{n}) - R_i f(\mathbf{m}) = \sum_{\mathbf{k}\in\mathbb{Z}^N} (f(\mathbf{k}) - f(\mathbf{n}))\mathcal{R}_i(\mathbf{n}-\mathbf{k}) - \sum_{\mathbf{k}\in\mathbb{Z}^N} (f(\mathbf{k}) - f(\mathbf{m}))\mathcal{R}_i(\mathbf{m}-\mathbf{k})$$

$$\begin{aligned}
&= \sum_{0<|\mathbf{n}-\mathbf{k}|\le 2|\mathbf{n}-\mathbf{m}|} ((f(\mathbf{k})-f(\mathbf{n}))\mathcal{R}_i(\mathbf{n}-\mathbf{k}) - (f(\mathbf{k})-f(\mathbf{m}))\mathcal{R}_i(\mathbf{m}-\mathbf{k}))\\
&\quad + \sum_{|\mathbf{n}-\mathbf{k}|>2|\mathbf{n}-\mathbf{m}|} ((f(\mathbf{k})-f(\mathbf{n}))\mathcal{R}_i(\mathbf{n}-\mathbf{k}) - (f(\mathbf{k})-f(\mathbf{m}))\mathcal{R}_i(\mathbf{m}-\mathbf{k}))\\
&:= S_1 + S_2,
\end{aligned}$$

For S_1, by using that $|\mathbf{n}-\mathbf{k}| \le 2|\mathbf{n}-\mathbf{m}|$ implies $|\mathbf{m}-\mathbf{k}| \le 3|\mathbf{n}-\mathbf{m}|$ and the estimate (72), it is verified that

$$\begin{aligned}
|S_1| &\le \sum_{0<|\mathbf{n}-\mathbf{k}|\le 2|\mathbf{n}-\mathbf{m}|} (|f(\mathbf{k})-f(\mathbf{n})||\mathcal{R}_i(\mathbf{n}-\mathbf{k})| + |f(\mathbf{k})-f(\mathbf{m})||\mathcal{R}_i(\mathbf{m}-\mathbf{k})|)\\
&\le C[f]_{C^{0,\alpha}(\mathbb{Z}^N)} \left(\sum_{0<|\mathbf{n}-\mathbf{k}|\le 2|\mathbf{n}-\mathbf{m}|} |\mathbf{k}-\mathbf{n}|^{\alpha-N} + \sum_{0<|\mathbf{m}-\mathbf{k}|\le 3|\mathbf{n}-\mathbf{m}|} |\mathbf{k}-\mathbf{m}|^{\alpha-N} \right)\\
&\le C[f]_{C^{0,\alpha}(\mathbb{Z}^N)} |\mathbf{n}-\mathbf{m}|^{\alpha}.
\end{aligned}$$

To treat S_2 we add the term $\pm f(\mathbf{m})\mathcal{R}_i(\mathbf{n}-\mathbf{k})$ and apply (73) and (75) to have

$$\begin{aligned}
|S_2| &\le \sum_{|\mathbf{n}-\mathbf{k}|>2|\mathbf{n}-\mathbf{m}|} |f(\mathbf{k})-f(\mathbf{m})||\mathcal{R}_i(\mathbf{n}-\mathbf{k}) - \mathcal{R}_i(\mathbf{m}-\mathbf{k})|\\
&\quad + |f(\mathbf{n})-f(\mathbf{m})| \left| \sum_{|\mathbf{n}-\mathbf{k}|>2|\mathbf{n}-\mathbf{m}|} \mathcal{R}_i(\mathbf{n}-\mathbf{k}) \right|\\
&\le C[f]_{C^{0,\alpha}(\mathbb{Z}^N)} \left(|\mathbf{n}-\mathbf{m}| \sum_{|\mathbf{n}-\mathbf{k}|>2|\mathbf{n}-\mathbf{m}|} |\mathbf{n}-\mathbf{k}|^{\alpha-N-1} + |\mathbf{n}-\mathbf{m}|^{\alpha} \right)\\
&\le C[f]_{C^{0,\alpha}(\mathbb{Z}^N)} |\mathbf{n}-\mathbf{m}|^{\alpha}
\end{aligned}$$

and this concludes the proof. □

It would be possible to define another Riesz transform by mean

$$\overline{R}_i f(\mathbf{n}) = \delta_i^-(-\Delta_N)^{-1/2} f(\mathbf{n}).$$

The identity $\delta_i^- f(\mathbf{n}) = \delta_i^+ f(\mathbf{n}-\mathbf{e}_i)$ proves the following result.

Corollary 27. *Let $N \ge 2$, $0 < \alpha < 1$, and $f \in C^{0,\alpha}(\mathbb{Z}^N)$. Then for $1 \le i \le N$ the inequality*

$$[\overline{R}_i f]_{C^{0,\alpha}(\mathbb{Z}^N)} \le C[f]_{C^{0,\alpha}(\mathbb{Z}^N)}$$

holds.

To finish this section we give a result about the boundedness with weights of the Riesz transforms.

Theorem 28. *Let $N \ge 2$, $w \in A_p(\mathbb{Z}^N)$, and $1 \le i \le N$. Then the inequalities*

$$\|R_i f\|_{\ell^p(\mathbb{Z}^n,w)} \le C\|f\|_{\ell^p(\mathbb{Z}^n,w)}, \qquad 1 < p < \infty, \qquad f \in \ell^2(\mathbb{Z}^N) \cap \ell^p(\mathbb{Z}^N, w),$$

and

$$\|R_i f\|_{\ell^{1,\infty}(\mathbb{Z}^n,w)} \le C\|f\|_{\ell^1(\mathbb{Z}^n,w)}, \qquad \ell^2(\mathbb{Z}^N) \cap \ell^1(\mathbb{Z}^N, w),$$

hold.

Proof. We will prove that each R_i is a Calderón-Zygmund operator.

First, we check that R_i is bounded from $\ell^2(\mathbb{Z}^N)$ into itself. For $f \in \ell^2(\mathbb{Z}^N)$, we have

$$\mathcal{F}(\delta_i f)(x) = \sum_{\mathbf{n}\in\mathbb{Z}^N} f(\mathbf{n}+\mathbf{e}_i)e^{2\pi i\langle \mathbf{n},x\rangle} - \sum_{\mathbf{n}\in\mathbb{Z}^N} f(\mathbf{n})e^{2\pi i\langle \mathbf{n},x\rangle} = (e^{-2\pi i x_i}-1)\mathcal{F}f(x)$$

and

$$\begin{aligned}\mathcal{F}((-\Delta_N)^{-1/2}f)(x) &= \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-4t(\sum_{k=1}^N \sin^2(\pi x_k))}\mathcal{F}f(x)t^{-1/2}\,dt\\ &= \left(4\sum_{k=1}^N \sin^2(\pi x_k)\right)^{-1/2}\mathcal{F}f(x).\end{aligned}$$

With this we deduce that δ_i^+ and $(-\Delta_N)^{-1/2}$ can be seen as multipliers of the functions $e^{-2\pi i x_i}-1$ and $\left(4\sum_{k=1}^N \sin^2(\pi x_k)\right)^{-1/2}$, respectively. Then, the operator R_i is associated with the multiplier

$$\frac{e^{-2\pi i x_i}-1}{\left(4\sum_{k=1}^N \sin^2(\pi x_k)\right)^{1/2}};$$

i.e.,

$$\begin{aligned}\mathcal{F}(R_i f)(x) &= \frac{e^{-2\pi i x_i}-1}{\left(4\sum_{k=1}^N \sin^2(\pi x_k)\right)^{1/2}}\mathcal{F}f\\ &= \frac{-ie^{\pi i x_i}\sin(\pi x_i)}{\left(\sum_{k=1}^N \sin^2(\pi x_k)\right)^{1/2}}\mathcal{F}f, \qquad i=1,\dots,N.\end{aligned}$$

In this way, using the bound

$$\frac{|\sin(\pi x_i)|}{\left(\sum_{k=1}^N \sin^2(\pi x_k)\right)^{1/2}} \le 1,$$

by Plancherel identity we have

$$\|R_i f\|_{\ell^2(\mathbb{Z}^N)} = \|\mathcal{F}(R_i f)\|_{L^2([-1/2,1/2]^N)} \le C\|\mathcal{F}f\|_{L^2([-1/2,1/2]^N)} = \|f\|_{\ell^2(\mathbb{Z}^N)}.$$

Finally, the estimates (57) and (58) can be deduced from (72) and (73) and we have concluded. □

For the Riesz transforms $\overline{R}_i$ we have the next result.

Corollary 29. *Let $N \ge 2$, $w \in A_p(\mathbb{Z}^N)$, and $1 \le i \le N$. Then the inequalities*

$$\|\overline{R}_i f\|_{\ell^p(\mathbb{Z}^n,w)} \le C\|f\|_{\ell^p(\mathbb{Z}^n,w)}, \qquad 1<p<\infty, \qquad f \in \ell^2(\mathbb{Z}^N)\cap \ell^p(\mathbb{Z}^N,w),$$

and

$$\|\overline{R}_i f\|_{\ell^{1,\infty}(\mathbb{Z}^n,w)} \le C\|f\|_{\ell^1(\mathbb{Z}^n,w)}, \qquad f \in \ell^2(\mathbb{Z}^N)\cap \ell^1(\mathbb{Z}^N,w),$$

hold.

5. Fractional powers of the Laplacian

From the identity

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-\lambda t} - 1) \frac{dt}{t^{s+1}}, \qquad 0 < s < 1,$$

we define the fractional powers (in fact, positive powers) of $-\Delta_N$ by

$$(-\Delta_N)^s f(\mathbf{n}) = \frac{1}{\Gamma(-s)} \int_0^\infty (W_t f(\mathbf{n}) - f(\mathbf{n})) \frac{dt}{t^{s+1}}, \qquad f \in \ell^2(\mathbb{Z}^N). \tag{77}$$

By using (22), it is clear that

$$W_t f(\mathbf{n}) - f(\mathbf{n}) = \sum_{\mathbf{k} \in \mathbb{Z}^N} G_{t,N}(\mathbf{n} - \mathbf{k})(f(\mathbf{k}) - f(\mathbf{n})). \tag{78}$$

Moreover, we can prove that the operators $(-\Delta_N)^s$ are well defined in $\ell^2(\mathbb{Z}^N)$ by (77). Indeed, we consider the decomposition

$$\int_0^\infty |W_t f(\mathbf{n}) - f(\mathbf{n})| \frac{dt}{t^{s+1}} \le F_1 + F_2$$

where

$$F_1 = \int_0^1 |W_t f(\mathbf{n}) - f(\mathbf{n})| \frac{dt}{t^{s+1}}$$

and F_2 is the integral on $[1, \infty)$ of the same function. For F_1, applying (27), we have

$$F_1 \le C\|f\|_{\ell^2(\mathbb{Z}^N)} \int_0^1 t^{-s}\, dt \le C\|f\|_{\ell^2(\mathbb{Z}^N)}.$$

Now, from Theorem 6 with $p = \infty$ and $q = 2$, we deduce that

$$|W_t f(\mathbf{n}) - f(\mathbf{n})| \le C(t^{-N/4} + 1)\|f\|_{\ell^2(\mathbb{Z}^N)}$$

and

$$F_2 \le C\|f\|_{\ell^2(\mathbb{Z}^N)} \int_1^\infty (t^{-N/4} + 1) t^{-s-1}\, dt \le C\|f\|_{\ell^2(\mathbb{Z}^N)}.$$

In this section we will focus on the study of some aspects of the operators $(-\Delta_N)^s$. Firstly, we will show some basic properties of the fractional powers of the Laplacian, including a maximum principle. In the last part of the section we will deal with the mapping properties of $(-\Delta_N)^s$ on the Hölder classes.

5.1. Basic properties of the fractional powers of the Laplacian.

In this section we present some elementary properties of the fractional powers $(-\Delta_N)^s$. They are collected in the next theorem.

Theorem 30. *Let $N \ge 1$ and $0 < s < 1$.*

i) *For $f \in \ell_{-s}(\mathbb{Z}^N)$, it is verified that*

$$(-\Delta_N)^s f(\mathbf{n}) = \sum_{\substack{\mathbf{k} \in \mathbb{Z}^N \\ \mathbf{k} \neq \mathbf{n}}} \mathcal{K}_s(\mathbf{n} - \mathbf{k})(f(\mathbf{n}) - f(\mathbf{k})),$$

where

$$\mathcal{K}_s(\mathbf{n}) = \frac{1}{|\Gamma(-s)|} \int_0^\infty G_{t,N}(\mathbf{n}) \frac{dt}{t^{s+1}}, \qquad \mathbf{n} \neq \mathbf{0}, \tag{79}$$

and $\mathcal{K}_s(\mathbf{0}) = 0$.

ii) *For* $\mathbf{n}\neq\mathbf{0}$, *the inequalities*

$$0<\mathcal{K}_s(\mathbf{n})\leq\frac{C}{|\mathbf{n}|^{N+2s}}\frac{1}{|\Gamma(-s)|}\left(\frac{1}{1-s}+\frac{2}{N+2s}\right) \tag{80}$$

hold with a constant C *independent of* s *and* $\mathbf{n}$.

iii) *For* $f\in\ell_0(\mathbb{Z}^N)$, *it is verified that*

$$\lim_{s\to 0^+}(-\Delta_N)^s f(\mathbf{n})=f(\mathbf{n}).$$

iv) *For* $f\in\ell^\infty(\mathbb{Z}^N)$, *it is verified that*

$$\lim_{s\to 1^-}(-\Delta_N)^s f(\mathbf{n})=-\Delta_N f(\mathbf{n}).$$

Proof. First, we observe that $W_t f$ is well defined on the spaces $\ell_{-s}(\mathbb{Z}^N)$ for $0\leq s\leq 1$. Effectively, for $M>0$ big enough, with t fix, we have

$$\sum_{|\mathbf{k}|>M}G_{t,N}(\mathbf{k})f(\mathbf{n}-\mathbf{k})\leq C\|f\|_{\ell_{-s}(\mathbb{Z}^N)}\sup_{|\mathbf{k}|>M}G_{t,N}(\mathbf{k})(1+|\mathbf{n}-\mathbf{k}|)^{N+2s}.$$

From (4) and the asymptotic expansion (21), we deduce that

$$\begin{aligned}\sup_{|\mathbf{k}|>M}&G_{t,N}(\mathbf{k})(1+|\mathbf{n}-\mathbf{k}|)^{N+2s}\\&\leq Ce^{-2Nt}N^{N/2}\sup_{|\mathbf{k}|>M}\frac{(Net)^{|\mathbf{k}|}(1+|\mathbf{n}|+|\mathbf{k}|)^{N+2s}}{|\mathbf{k}|^{|\mathbf{k}|+N/2}}\leq C_{t,M,|\mathbf{n}|,s,N}\end{aligned}$$

and we conclude.

To prove i) we use (78) to get

$$\begin{aligned}(-\Delta_N)^s f(\mathbf{n})&=\frac{1}{\Gamma(-s)}\int_0^\infty\sum_{\mathbf{k}\neq\mathbf{n}}G_{t,N}(\mathbf{n}-\mathbf{k})(f(\mathbf{k})-f(\mathbf{n}))\frac{dt}{t^{s+1}}\\&=\sum_{\mathbf{k}\neq\mathbf{n}}\mathcal{K}_s(\mathbf{n}-\mathbf{k})(f(\mathbf{n})-f(\mathbf{k}))\end{aligned}$$

where $\mathcal{K}_s$ is given by (79). The proof will be completed justifying the application of Fubini theorem in the second identity. To do this we need the upper bound in ii), so we proceed with its proof.

From (4) and (15) with $\alpha=1/N$, we obtain that

$$G_{t,N}(\mathbf{n})t^{-1}\leq\frac{C}{|\mathbf{n}|^{N+2}}.$$

Then, with (31), we deduce that

$$\begin{aligned}\mathcal{K}_s(\mathbf{n})&\leq\frac{C}{|\Gamma(-s)|}\left(\frac{1}{|\mathbf{n}|^{N+2}}\int_0^{|\mathbf{n}|^2}t^{-s}\,dt+\int_{|\mathbf{n}|^2}^\infty t^{-N/2-s-1}\,dt\right)\\&=\frac{C}{|\mathbf{n}|^{N+2s}}\frac{1}{|\Gamma(-s)|}\left(\frac{1}{1-s}+\frac{2}{N+2s}\right).\end{aligned}$$

The positivity of the kernel $\mathcal{K}_s$ is clear.

Let us justify the interchange of the sum and the integral. First, we apply (80) and Tonelli's theorem in the term

$$\int_0^\infty\sum_{\mathbf{k}\neq\mathbf{n}}G_{t,N}(\mathbf{n}-\mathbf{k})|f(\mathbf{k})|\frac{dt}{t^{s+1}}$$

to have that it is bounded by

$$C\sum_{\mathbf{k}\neq\mathbf{n}}\frac{|f(\mathbf{k})|}{|\mathbf{n}-\mathbf{k}|^{N+2s}}$$

and this term is finite for each $\mathbf{n}$ because $f\in\ell_{-s}(\mathbb{Z}^N)$. Similarly for the term

$$|f(\mathbf{n})|\int_0^\infty\sum_{\mathbf{k}\neq\mathbf{n}}G_{t,N}(\mathbf{n}-\mathbf{k})\frac{dt}{t^{s+1}},$$

we obtain the bound

$$C|f(\mathbf{n})|\sum_{\mathbf{k}\neq\mathbf{n}}\frac{1}{|\mathbf{n}-\mathbf{k}|^{N+2s}}$$

and this last sum is finite.

To prove iii) we check that

$$\lim_{s\to0^+}N_1=1 \qquad\text{and}\qquad \lim_{s\to0^+}N_2=0$$

where

$$N_1=\sum_{\mathbf{k}\neq\mathbf{n}}\mathcal{K}_s(\mathbf{n}-\mathbf{k}) \qquad\text{and}\qquad N_2=\sum_{\mathbf{k}\neq\mathbf{n}}\mathcal{K}_s(\mathbf{n}-\mathbf{k})|f(\mathbf{k})|.$$

We start with N_2. From (80), we have

$$N_2\le\frac{C}{|\Gamma(-s)|}\left(\frac{1}{1-s}+\frac{2}{N+2s}\right)\sum_{\mathbf{k}\neq\mathbf{n}}\frac{|f(\mathbf{k})|}{|\mathbf{n}-\mathbf{k}|^{N+2s}}.$$

Now, using that $f\in\ell_0(\mathbb{Z}^N)$, applying dominated convergence, we deduce that for each $\mathbf{n}$ it is verified that

$$\sum_{\mathbf{k}\neq\mathbf{n}}\frac{|f(\mathbf{k})|}{|\mathbf{n}-\mathbf{k}|^{N+2s}}\le C\|f\|_{\ell_0(\mathbb{Z}^N)}$$

and then $\lim_{s\to0^+}N_2=0$. To treat N_1, we consider the decomposition $N_1=N_{1,1}+N_{1,2}$ where

$$N_{1,1}=\frac{1}{|\Gamma(-s)|}\sum_{\mathbf{k}\neq\mathbf{n}}\int_0^1 G_{t,N}(\mathbf{n}-\mathbf{k})\frac{dt}{t^{s+1}}$$

and $N_{1,2}$ is the same sum but with the integral on the interval $[1,\infty)$. To analyze $N_{1,1}$ we apply (4) and (15) with $\alpha=1/(2N)$ and we consider $0<s<1/2$. In this way,

$$N_{1,1}\le\frac{C}{|\Gamma(-s)|}\sum_{\mathbf{k}\neq\mathbf{n}}\frac{1}{|\mathbf{n}-\mathbf{k}|^{N+1}}\int_0^1 t^{-1/2-s}\,dt\le\frac{C}{|\Gamma(-s)|(1-2s)}$$

and $\lim_{s\to0^+}N_{1,1}=0$. Now, using that

$$\sum_{\mathbf{k}\neq\mathbf{n}}G_{t,N}(\mathbf{n}-\mathbf{k})=1-G_{t,N}(\mathbf{0}),$$

we obtain that

$$N_{1,2}=\frac{1}{|\Gamma(-s)|}\left(\frac{1}{s}-\int_1^\infty G_{t,N}(\mathbf{0})\frac{dt}{t^{s+1}}\right).$$

With the bound (31) we get

$$\int_1^\infty G_{t,N}(\mathbf{0})\frac{dt}{t^{s+1}}\le C\int_1^\infty t^{-N/2-s-1}\,dt=\frac{C}{N+2s}$$

and, with the identity $|\Gamma(-s)|s = \Gamma(1-s)$ we conclude that $\lim_{s\to 0^+} N_{1,2} = 1$.

To prove iv) we start observing

$$(-\Delta_N)^s f(\mathbf{n}) = \sum_{k=1}^{N} (\mathcal{K}_s(\mathbf{e}_k))(f(\mathbf{n}) - f(\mathbf{n}-\mathbf{e}_k)) + \mathcal{K}_s(-\mathbf{e}_k))(f(\mathbf{n}) - f(\mathbf{n}+\mathbf{e}_k))) \\ + \sum_{\substack{\mathbf{k}\in\mathbb{Z}^N \\ \mathbf{k}\notin\{\mathbf{0},\pm\mathbf{e}_1,\dots,\pm\mathbf{e}_N\}}} \mathcal{K}_s(\mathbf{k})(f(\mathbf{n}) - f(\mathbf{n}-\mathbf{k})) := M_1 + M_2.$$

Taking into account that $\mathcal{K}_s(-\mathbf{n}) = \mathcal{K}_s(\mathbf{n})$, we have

$$M_1 = -\sum_{k=1}^{N} \mathcal{K}_s(\mathbf{e}_k)\Delta_{N,k} f(\mathbf{n}),$$

where $\Delta_{N,k}$ is defined in (1). Now, using that for any continuous and bounded function on $[0,\infty)$,

$$\lim_{s\to 1^-} \frac{1}{\Gamma(1-s)} \int_0^\infty e^{-t} f(t) \frac{dt}{t^s} = f(0)$$

(the identity is clear for polynomials and with the hypotheses on f the general result is obtained), taking the function

$$f(z) = \frac{2I_1(z)}{z} (I_0(z))^{N-1},$$

we deduce that

$$\begin{aligned}\lim_{s\to 1^-} \mathcal{K}_s(\mathbf{e}_i) &= \lim_{s\to 1^-} \frac{1}{\Gamma(1-s)} \int_0^\infty e^{-2Nt} \frac{I_1(2t)}{t} (I_0(2t))^{N-1} \frac{dt}{t^s} \\ &= \lim_{s\to 1^-} \frac{(2N)^{s-1}}{\Gamma(1-s)} \int_0^\infty e^{-w} f\left(\frac{w}{N}\right) \frac{dw}{w^s} = f(0) = 1.\end{aligned}$$

Then,

$$\lim_{s\to 1^-} M_1 = -\Delta_N f(\mathbf{n}).$$

Finally, let us check that $\lim_{s\to 1^-} M_2 = 0$. By using that $G_{t,N}(\mathbf{n}) t^{-3/2} \le C|\mathbf{n}|^{-N-3}$ for $|\mathbf{n}| > 1$ (which can be deduced from (4) and (15) with $\alpha = 3/(2N)$) and proceeding as in the proof of (80), we can obtain the bound

$$\mathcal{K}_s(\mathbf{n}) \le \frac{C}{|\mathbf{n}|^{N+2s}} \frac{1}{|\Gamma(-s)|} \left(\frac{1}{3-2s} + \frac{1}{N+2s}\right).$$

Then

$$M_2 \le \frac{C}{|\Gamma(-s)|} \|f\|_{\ell^\infty(\mathbb{Z}^N)} \sum_{|\mathbf{k}|>1} \frac{1}{|\mathbf{k}|^{N+2s}} \le \frac{C}{|\Gamma(-s)|} \|f\|_{\ell^\infty(\mathbb{Z}^N)}$$

and $\lim_{s\to 1^-} M_2 = 0$. □

To conclude this section, we present maximum and comparison principles for the fractional powers which are obvious consequences of the positivity of the kernel $\mathcal{K}_s$.

Theorem 31. *Let $N \ge 1$ and $0 < s < 1$.*

i) *(Maximum principle) Let* $f \in \ell^2(\mathbb{Z}^N)$ *be such that* $f \geq 0$ *and* $f(\mathbf{n}_0) = 0$ *for some* $\mathbf{n}_0 \in \mathbb{Z}^N$*. Then*

$$(-\Delta_N)^s f(\mathbf{n}_0) \leq 0.$$

Moreover, $(-\Delta_N)^s f(\mathbf{n}_0) = 0$ *only if* $f(\mathbf{n}) = 0$ *for all* $\mathbf{n} \in \mathbb{Z}^N$*.*

ii) *(Comparison principle) Let* $f, g \in \ell^2(\mathbb{Z}^N)$ *be such that* $f \geq g$ *and* $f(\mathbf{n}_0) = g(\mathbf{n}_0)$ *for some* $\mathbf{n}_0 \in \mathbb{Z}^N$*. Then*

$$(-\Delta_N)^s f(\mathbf{n}_0) \leq (-\Delta_N)^s g(\mathbf{n}_0).$$

Moreover, $(-\Delta_N)^s f(\mathbf{n}_0) = (-\Delta_N)^s g(\mathbf{n}_0)$ *only if* $f(\mathbf{n}) = g(\mathbf{n})$ *for all* $\mathbf{n} \in \mathbb{Z}^N$*.*

5.2. **The fractional powers of the Laplacian on the Hölder classes.** The target of this subsection is the analysis of the behaviour of $(-\Delta_N)^s$ in the Hölder classes and it is contained in the next result.

Theorem 32. *Let* $N \geq 1$*,* $k \geq 0$*,* $0 < s < 1$*, and* $f \in \ell_{-s}(\mathbb{Z}^N)$*.*

i) *If* $f \in C^{k,\alpha}(\mathbb{Z}^N)$*,* $0 < \alpha \leq 1$*, and* $2s < \alpha$ *then* $(-\Delta_N)^s f \in C^{k,\alpha-2s}(\mathbb{Z}^N)$ *and*

$$[(-\Delta_N)^s f]_{C^{k,\alpha-2s}(\mathbb{Z}^N)} \leq C[f]_{C^{k,\alpha}(\mathbb{Z}^N)}.$$

ii) *If* $f \in C^{k+1,\alpha}(\mathbb{Z}^N)$*,* $0 < \alpha < 1$*, and* $\alpha < 2s < \alpha + 1$ *then* $(-\Delta_N)^s f \in C^{k,\alpha-2s+1}(\mathbb{Z}^N)$ *and*

$$[(-\Delta_N)^s f]_{C^{k,\alpha-2s+1}(\mathbb{Z}^N)} \leq C[f]_{C^{k+1,\alpha}(\mathbb{Z}^N)}.$$

Proof. We will prove both results for $k = 0$ because the other cases can be deduced by using that $\delta_i^+(-\Delta_N)^s = (-\Delta_N)^s \delta_i^+$.

To prove i), we consider the identity

$$|(-\Delta_N)^s f(\mathbf{n}) - (-\Delta_N)^s f(\mathbf{m})| = |B_1 + B_2|,$$

where

$$B_1 = \sum_{\substack{\mathbf{k} \in \mathbb{Z}^N \\ 1 \leq |\mathbf{k}| \leq |\mathbf{n}-\mathbf{m}|}} (f(\mathbf{n}) - f(\mathbf{n}+\mathbf{k}) - f(\mathbf{m}) + f(\mathbf{m}+\mathbf{k}))\mathcal{K}_s(\mathbf{k})$$

and B_2 is the sum on $|\mathbf{k}| > |\mathbf{n} - \mathbf{m}|$. First, from the estimate

$$\begin{aligned}|f(\mathbf{n}) - f(\mathbf{n}+\mathbf{k}) - f(\mathbf{m}) + f(\mathbf{m}+\mathbf{k})| &\leq |f(\mathbf{n}) - f(\mathbf{n}+\mathbf{k})| + |f(\mathbf{m}) - f(\mathbf{m}+\mathbf{k})| \\ &\leq C\|f\|_{C^{0,\alpha}(\mathbb{Z}^N)}|\mathbf{k}|^\alpha\end{aligned}$$

and (80), we get

$$B_1 \leq C[f]_{C^{0,\alpha}(\mathbb{Z}^N)} \sum_{\substack{\mathbf{k} \in \mathbb{Z}^N \\ 1 \leq |\mathbf{k}| \leq |\mathbf{n}-\mathbf{m}|}} |\mathbf{k}|^{\alpha-N-2s} \leq C[f]_{C^{0,\alpha}(\mathbb{Z}^N)}|\mathbf{n}-\mathbf{m}|^{\alpha-2s}$$

and this is appropriate for our purpose. To treat B_2, we use that

$$\begin{aligned}|f(\mathbf{n}) - f(\mathbf{n}+\mathbf{k}) - f(\mathbf{m}) + f(\mathbf{m}+\mathbf{k})| &\leq |f(\mathbf{n}) - f(\mathbf{m})| + |f(\mathbf{n}+\mathbf{k}) - f(\mathbf{m}+\mathbf{k})| \\ &\leq C\|f\|_{C^{0,\alpha}(\mathbb{Z}^N)}|\mathbf{n}-\mathbf{m}|^\alpha\end{aligned}$$

and the bound (80), to get

$$B_2 \leq C[f]_{C^{0,\alpha}(\mathbb{Z}^N)}|\mathbf{n}-\mathbf{m}|^\alpha \sum_{\substack{\mathbf{k} \in \mathbb{Z}^N \\ |\mathbf{k}| > |\mathbf{n}-\mathbf{m}|}} |\mathbf{k}|^{-N-2s} \leq C[f]_{C^{0,\alpha}(\mathbb{Z}^N)}|\mathbf{n}-\mathbf{m}|^{\alpha-2s}$$

and the proof of i) is completed.

Now, we prove ii) for $N \geq 2$ because the case $N = 1$ can be found in [8, Theorem 1.5]. We consider the identity

$$(-\Delta_N)^s = (-\Delta_N)^{s-1/2}(-\Delta_N)^{-1/2}(-\Delta_N) = (-\Delta_N)^{s-1/2}(-\Delta_N)^{-1/2}\sum_{i=1}^{N}\delta_i^-\delta_i^+$$
$$= (-\Delta_N)^{s-1/2}\sum_{i=1}^{N}\overline{R}_i\delta_i^+$$

and we distinguish two cases. For $s-1/2 > 0$ it is verified that $0 < \alpha-2(s-1/2) < 1$ and, applying i) and Corollary 27, we deduce that

$$[(-\Delta_N)^s f]_{C^{0,\alpha-2s+1}(\mathbb{Z}^N)} \leq C\sum_{i=1}^{N}[\overline{R}_i\delta_i^+ f]_{C^{0,\alpha}(\mathbb{Z}^N)}$$
$$\leq C\sum_{i=1}^{N}[\delta_i^+ f]_{C^{0,\alpha}(\mathbb{Z}^N)} \leq C[f]_{C^{1,\alpha}(\mathbb{Z}^N)}.$$

In the case $s-1/2 < 0$ we have $0 < \alpha+2(-s+1/2)$ and we can apply a) in Theorem 19 and Corollary 27 to conclude

$$[(-\Delta_N)^s f]_{C^{0,\alpha-2s+1}(\mathbb{Z}^N)} = [(-\Delta_N)^{-(-s+1/2)}\sum_{i=1}^{N}\overline{R}_i\delta_i^+ f]_{C^{0,\alpha-2s+1}(\mathbb{Z}^N)}$$
$$\leq C\sum_{i=1}^{N}[\overline{R}_i\delta_i^+ f]_{C^{0,\alpha}(\mathbb{Z}^N)}$$
$$\leq C\sum_{i=1}^{N}[\delta_i^+ f]_{C^{0,\alpha}(\mathbb{Z}^N)} \leq C[f]_{C^{1,\alpha}(\mathbb{Z}^N)}.$$

For $s = 1/2$ the result is a direct consequence of Corollary 27. □

6. The discrete square functions

In this section we will prove weighted inequalities for the discrete Littlewood-Paley-Stein g_k-square functions associated to the heat semigroup and for the $\mathfrak{g}_k$-square functions associated to the Poisson semigroup. Moreover, we conclude with a result about Laplace type multipliers.

6.1. **The g_k-square functions associated to the heat semigroup.**

The discrete g_k-square functions associated to the heat semigroup are defined by

$$g_k(f)(\mathbf{n}) = \left(\int_0^\infty t^{2k-1}\left|\frac{\partial^k}{\partial t^k}W_t f(\mathbf{n})\right|^2 dt\right)^{1/2}, \qquad k = 1, 2, \dots.$$

Note that taking the Banach space $\mathbb{B}_k = L^2((0,\infty), t^{2k-1}\,dt)$, we have

$$g_k f(\mathbf{n}) = \|P_{t,k}f(\mathbf{n})\|_{\mathbb{B}_k}$$

with

$$P_{t,k}f(\mathbf{n}) = \sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{k})\mathcal{P}_{t,k}(\mathbf{n}-\mathbf{k})$$

and

$$\mathcal{P}_{t,k}(\mathbf{n}) = \frac{d^k}{dt^k} G_{t,N}(\mathbf{n}) = \frac{d^k}{dt^k} \int_{[-1/2,1/2]^N} e^{-4t\sum_{i=1}^N \sin^2(\pi x_i)} e^{-2\pi i \langle x, \mathbf{n}\rangle}\, dt$$
$$= \int_{[-1/2,1/2]^N} F_{t,k}(x) e^{-2\pi i \langle x, \mathbf{n}\rangle}\, dt,$$

being

$$F_{t,k}(x) = (-4)^k \left(\sum_{i=1}^N \sin^2(\pi x_i)\right)^k e^{-4t\sum_{i=1}^N \sin^2(\pi x_i)}.$$

Our main result about the g_k-square functions is the following one.

Theorem 33. *Let $N \geq 1$, $1 < p < \infty$, $w \in A_p(\mathbb{Z}^N)$, and $k \in \mathbb{N}$. Then the inequalities*

(81)
$$C_1 \|f\|_{\ell^p(\mathbb{Z}^N,w)} \leq \|g_k(f)\|_{\ell^p(\mathbb{Z}^N,w)} \leq C_2 \|f\|_{\ell^p(\mathbb{Z}^N,w)}, \qquad f \in \ell^2(\mathbb{Z}^N) \cap \ell^p(\mathbb{Z}^N, w),$$

hold.

To obtain the proof of the equivalence in the previous theorem, we start proving the boundedness of the g_k-square functions from $\ell^2(\mathbb{Z}^N)$ into itself.

Lemma 34. *For $N \geq 1$ and $k \in \mathbb{N}$, it is verified that*

(82)
$$\|g_k(f)\|^2_{\ell^2(\mathbb{Z}^N)} = \frac{\Gamma(2k)}{2^{2k}} \|f\|^2_{\ell^2(\mathbb{Z}^N)}.$$

Proof. For each sequence in $\ell^2(\mathbb{Z}^N)$, we have

$$P_{t,k} f = \mathcal{F}^{-1}(F_{t,k} \mathcal{F} f).$$

Then

$$\|g_k(f)\|^2_{\ell^2(\mathbb{Z}^N)} = \sum_{\mathbf{k}\in\mathbb{Z}^N} \|P_{t,k} f(\mathbf{k})\|^2_{\mathbb{B}_k} = \left\| \sum_{\mathbf{k}\in\mathbb{Z}^N} (P_{t,k} f(\mathbf{k}))^2 \right\|^2_{\mathbb{B}_k}.$$

By applying Parseval's identity,

$$\sum_{\mathbf{k}\in\mathbb{Z}^N} (P_{t,k} f(\mathbf{k}))^2 = \int_{[-1/2,1/2]^N} (F_{t,k}(x))^2 (\mathcal{F} f(x))^2\, dx$$

and

$$\|g_k(f)\|^2_{\ell^2(\mathbb{Z}^N)} = \int_0^\infty t^{2k-1} \int_{[-1/2,1/2]^N} (F_{t,k}(x))^2 (\mathcal{F} f(x))^2\, dx\, dt$$
$$= \int_{[-1/2,1/2]^N} (\mathcal{F} f(x))^2 \int_0^\infty t^{2k-1} (F_{t,k}(x))^2\, dt\, dx$$
$$= \frac{\Gamma(2k)}{2^{2k}} \int_{[-1/2,1/2]^N} (\mathcal{F} f(x))^2\, dx = \frac{\Gamma(2k)}{2^{2k}} \|f\|^2_{\ell^2(\mathbb{Z}^N)}. \qquad \square$$

Now, we proceed with two reductions to simplify the proof of Theorem 33.

Lemma 35. *Let $N \geq 1$, $1 < p < \infty$, $w \in A_p(\mathbb{Z}^N)$, and $k \in \mathbb{N}$. Then the inequality*

(83)
$$\|g_k(f)\|_{\ell^p(\mathbb{Z}^N,w)} \leq C \|f\|_{\ell^p(\mathbb{Z}^N,w)}, \qquad f \in \ell^2(\mathbb{Z}^N) \cap \ell^p(\mathbb{Z}^N, w)$$

implies

(84)
$$\|f\|_{\ell^p(\mathbb{Z}^N,w)} \leq C \|g_k(f)\|_{\ell^p(\mathbb{Z}^N,w)}, \qquad f \in \ell^2(\mathbb{Z}^N) \cap \ell^p(\mathbb{Z}^N, w).$$

Proof. Polarising the identity (82), we have

$$\sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{k})h(\mathbf{k}) = \frac{2^{2k}}{\Gamma(2k)} \sum_{\mathbf{k}\in\mathbb{Z}^N} \int_0^\infty t^{2k-1} \left(\frac{d^k}{dt^k} W_t f(\mathbf{k})\right) \left(\frac{d^k}{dt^k} W_t h(\mathbf{k})\right) dt$$

and

$$\left|\sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{k})h(\mathbf{k})\right| \le C \sum_{\mathbf{k}\in\mathbb{Z}^N} g_k(f)(\mathbf{k}) g_k(h)(\mathbf{k}).$$

Taking $h(\mathbf{k}) = w^{1/p}(\mathbf{k})h_1(\mathbf{k})$ with $h_1 \in c_{00}$, the space of sequences having a finite number of non-null terms, we deduce that

$$\begin{aligned}\left|\sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{k})w^{1/p}(\mathbf{k})h_1(\mathbf{k})\right| &\le C \sum_{\mathbf{k}\in\mathbb{Z}^N} g_k(f)(\mathbf{k}) g_k(w^{1/p}h_1)(\mathbf{k}) \\ &= C \sum_{\mathbf{k}\in\mathbb{Z}^N} g_k(f)(\mathbf{k}) w^{1/p}(\mathbf{k}) w^{-1/p}(\mathbf{k}) g_k(w^{1/p}h_1)(\mathbf{k}) \\ &\le C \|g_k(f)\|_{\ell^p(\mathbb{Z}^N,w)} \|g_k(w^{1/p}h_1)\|_{\ell^{p'}(\mathbb{Z}^N,w')},\end{aligned}$$

where $w' = w^{-1/(p-1)}$ and p' is the conjugate exponent of p. It is clear that (83) implies

$$\|g_k(f)\|_{\ell^{p'}(\mathbb{Z}^N,w')} \le C\|f\|_{\ell^{p'}(\mathbb{Z}^N,w')}, \qquad f \in \ell^2(\mathbb{Z}^N) \cap \ell^{p'}(\mathbb{Z}^N, w').$$

Then,

$$\|g_k(w^{1/p}h_1)\|_{\ell^{p'}(\mathbb{Z}^N,w')} \le C\|w^{1/p}h_1\|_{\ell^{p'}(\mathbb{Z}^N,w')} = \|h_1\|_{\ell^{p'}(\mathbb{Z}^N)}$$

and

$$\left|\sum_{\mathbf{k}\in\mathbb{Z}^N} f(\mathbf{k})w^{1/p}(\mathbf{k})f_1(\mathbf{k})\right| \le C\|g_k(f)\|_{\ell^p(\mathbb{Z}^N,w)} \|f_1\|_{\ell^{p'}(\mathbb{Z}^n)} < \infty$$

and taking the supremum over all $h_1 \in c_{00}$ such that $\|h_1\|_{\ell^{p'}(\mathbb{Z}^N)} \le 1$ the inequality (84) is proved. □

Lemma 36. *Let $N \ge 1$, $1 < p < \infty$, and $w \in A_p(\mathbb{Z}^N)$. Then the inequality*

$$\|g_1(f)\|_{\ell^p(\mathbb{Z}^N,w)} \le C\|f\|_{\ell^p(\mathbb{Z}^N,w)}, \qquad f \in \ell^2(\mathbb{Z}^N) \cap \ell^p(\mathbb{Z}^N, w),$$

implies

$$\|g_k(f)\|_{\ell^p(\mathbb{Z}^N,w)} \le C\|f\|_{\ell^p(\mathbb{Z}^N,w)}, \qquad k > 1, \qquad f \in \ell^2(\mathbb{Z}^N) \cap \ell^p(\mathbb{Z}^N, w).$$

Proof. We use an induction argument to prove the result. Let us suppose that the operator $P_{t,k}$ is bounded from $\ell^p(\mathbb{Z}^N, w)$ into $\ell^p_{\mathbb{B}_k}(\mathbb{Z}^N, w)$. Taking $k = 1$ and applying Krivine's theorem (see [17, Theorem 1.f.14]), we deduce that the operator $\overline{P}_{t,1} : \ell^p_{\mathbb{B}_k}(\mathbb{Z}^N, w) \longrightarrow \ell^p_{\mathbb{B}_k\times\mathbb{B}_1}(\mathbb{Z}^N, w)$, given by

$$\{f_s(n)\}_{s\ge 0} \longmapsto \{P_{t,1}f_s\}_{t,s\ge 0},$$

is bounded. Moreover, $\overline{P}_{t,1} \circ P_{s,k}$ is a bounded operator from $\ell^p(\mathbb{Z}^N, w)$ into $\ell^p_{\mathbb{B}_k\times\mathbb{B}_1}(\mathbb{Z}^N, w)$. With the Chapman-Kolmogorov type identity

$$\sum_{\mathbf{k}\in\mathbb{Z}^N} G_{t,N}(\mathbf{k})G_{s,N}(\mathbf{n}-\mathbf{k}) = G_{t+s,N}(\mathbf{n}),$$

which can be deduced from Neumann's identity (20) and corresponds to the semigroup property, we obtain that

$$\frac{\partial}{\partial t}\left(W_t\left(\frac{\partial^k}{\partial s^k}W_s f\right)\right) = \frac{\partial^{k+1}}{\partial u^{k+1}}W_u f\bigg|_{u=s+t},$$

and using this we have

$$\begin{aligned}\left\|\overline{P}_{t,1}\circ P_{s,k}f\right\|^2_{\mathbb{B}_k\times\mathbb{B}_1} &= \int_0^\infty\int_0^\infty ts^{2k-1}\left|\frac{\partial^{k+1}}{\partial u^{k+1}}W_u f\bigg|_{u=s+t}\right|^2 ds\,dt\\ &= \int_0^\infty\int_t^\infty t(r-t)^{2k-1}\left|\frac{\partial^{k+1}}{\partial u^{k+1}}W_u f\bigg|_{u=r}\right|^2 dr\,dt\\ &= \int_0^\infty\left|\frac{\partial^{k+1}}{\partial r^{k+1}}W_r f\right|^2\int_0^r t(r-t)^{2k-1}\,dt\,dr\\ &= \frac{1}{(2k+1)(2k)}\int_0^\infty r^{2k+1}\left|\frac{\partial^{k+1}}{\partial r^{k+1}}W_r f\right|^2 dr\\ &= \frac{(g_{k+1}(f))^2}{(2k+1)(2k)}\end{aligned}$$

and the proof is finished. □

After the reductions in the previous lemmas, Theorem 33 will be a consequence of the following result.

Theorem 37. *Let $N \geq 1$ and $w \in A_p$. Then the inequalities*

$$\|g_1(f)\|_{\ell^p(\mathbb{Z}^n,w)} \leq C\|f\|_{\ell^p(\mathbb{Z}^n,w)}, \qquad 1<p<\infty, \qquad \ell^2(\mathbb{Z}^N)\cap\ell^p(\mathbb{Z}^N,w)$$

and

$$\|g_1(f)\|_{\ell^{1,\infty}(\mathbb{Z}^n,w)} \leq C\|f\|_{\ell^1(\mathbb{Z}^n,w)}, \qquad \ell^2(\mathbb{Z}^N)\cap\ell^1(\mathbb{Z}^N,w),$$

hold.

To prove this theorem we check that g_1 is a Calderón-Zygmund operator. With Lemma 34 and the estimates in the following lemma this fact will be clear.

Lemma 38. *Let $N \geq 1$, then inequalities*

$$\|\mathcal{P}_{t,1}(\mathbf{n})\|_{\mathbb{B}_1} \leq \frac{C}{(|\mathbf{n}|+1)^N} \tag{85}$$

and

$$\|\mathcal{P}_{t,1}(\mathbf{n})-\mathcal{P}_{t,1}(\mathbf{m})\|_{\mathbb{B}_1} \leq C\frac{|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+1}}, \qquad |\mathbf{n}|>2|\mathbf{n}-\mathbf{m}| \tag{86}$$

hold.

Before proving the previous lemma, we need another one.

Lemma 39. *Let $N \geq 1$, $\mathbf{n},\mathbf{m}\in\mathbb{Z}^N$ such that $|\mathbf{n}|>2|\mathbf{n}-\mathbf{m}|$, and*

$$\mathfrak{H}_{t,N}(z) = \left(\frac{z}{(1+t)^2}+\frac{z^3}{(1+t)^3}\right)(G_{t,1}(z))^N, \qquad z>0.$$

Then the inequality

$$|\Delta_{N,i}(G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{m}))| \leq C|\mathbf{n}-\mathbf{m}|\mathfrak{H}_{t,N}\left(\frac{|\mathbf{n}|+|\mathbf{m}|}{K}\right), \qquad i=1,\dots,N, \tag{87}$$

holds with constants C and $K>1$ independent of $\mathbf{n}$ *and* $\mathbf{m}$.

Proof. Again, it is enough to analyze the case $i=1$ because the other ones work similarly.

By using the ideas in Lemma 8 and Lemma 22, we can reduce the proof to the case where $n_i \neq m_i$ for $i=1,\dots,N$. Moreover, the assumptions (39) on $\mathbf{n}$ and $\mathbf{m}$ in the proof of Lemma 8 can be considered also.

We can assume that $m_1 \geq 1$ because, due to (39) and $|\mathbf{n}|>2|\mathbf{n}-\mathbf{m}|$, in the case $m_1=0$ it is verified that $\mathbf{n}=\mathbf{m}=\mathbf{0}$ and this is not possible.

From (40) we deduce that

$$\Delta_{N,1}(G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{m})) = \Delta(G_{t,1}(n_1)-G_{t,1}(m_1))\prod_{k=2}^{N} G_{t,1}(n_k) + \sum_{i=2}^{N} \Delta G_{t,1}(m_1)(G_{t,1}(n_i)-G_{t,1}(m_i))\prod_{j=2}^{i-1} G_{t,1}(m_j)\prod_{k=i+1}^{N} G_{t,1}(n_k) := Y_1+Y_2.$$

We can control $G_{t,1}(n_i)-G_{t,1}(m_i)$ as in (41) and, applying (9), we have

$$|\Delta G_{t,1}(m_1)(G_{t,1}(n_i)-G_{t,1}(m_i))| \leq C\frac{|n_i-m_i|(m_i+n_i+1)}{1+t}\left(\frac{1}{1+t}+\frac{m_1^2}{(1+t)^2}\right) \times G_{t,1}(m_1-1)\max\{G_{t,1}(n_i),G_{t,1}(m_i)\}. \tag{88}$$

Then.

$$|Y_2| \leq C|\mathbf{n}-\mathbf{m}|\left(\frac{|\mathbf{n}|+|\mathbf{m}|}{(1+t)^2}+\frac{(|\mathbf{n}|+|\mathbf{m}|)^3}{(1+t)^3}\right) \sum_{i=2}^{N}\left(G_{t,1}\left(\frac{m_1-1+\dots+\min\{n_i,m_j\}+\dots+n_N}{N}\right)\right)^N$$

and to obtain the estimate

$$G_{t,1}\left(\frac{m_1-1+\dots+\min\{n_i,m_j\}+\dots+n_N}{N}\right) \leq G_{t,1}\left(\frac{|\mathbf{n}|+|\mathbf{m}|}{K}\right),$$

for $j=2,\dots,N$, we proceed as in the proof of Lemma 8 analyzing separately the cases $m_1=1$, where the study is elementary, and $m_1\geq 2$, where we can change m_1-1 by $m_1/2$ to obtain the result because

$$G_{t,1}\left(\frac{m_1-1+\dots+\min\{n_i,m_j\}+\dots+n_N}{N}\right) \geq G_{t,1}\left(\frac{m_1/2+\dots+\min\{n_i,m_j\}+\dots+n_N}{N}\right)$$

For $n_1<m_1$, we can obtain that

$$\Delta(G_{t,1}(n_1)-G_{t,1}(m_1)) = -\sum_{k=n_1}^{m_1-1}\Delta(G_{t,1}(k+1)-G_{t,1}(k))) = -\sum_{k=n_1-1}^{m_1}(G_{t,1}(k+3)-3G_{t,1}(k+2)+3G_{t,1}(k+1)-G_{t,1}(k))$$

and, applying (11),

$$|\Delta(G_{t,1}(n_1)-G_{t,1}(m_1))| \le C\sum_{k=n_1-1}^{m_1}\left(\frac{k+2}{(1+t)^2}+\frac{(k+1)(k+2)(k+3)}{(1+t)^3}\right)G_{t,1}(k)$$
$$\le C(m_1-n_1)\left(\frac{m_1+n_1}{(1+t)^2}+\frac{m_1^3+n_1^3}{(1+t)^3}\right)G_{t,1}(n_1).$$

More generally,

$$|\Delta(G_{t,1}(n_1)-G_{t,1}(m_1))|$$
$$\le C|m_1-n_1|\left(\frac{m_1+n_1}{(1+t)^2}+\frac{m_1^3+n_1^3}{(1+t)^3}\right)\max\{G_{t,1}(n_1),G_{t,1}(m_1)\}$$

and

$$|Y_1| \le C|\mathbf{n}-\mathbf{m}|\left(\frac{|\mathbf{n}|+\mathbf{m}}{(1+t)^2}+\frac{(|\mathbf{n}|+|\mathbf{m}|)^3}{(1+t)^3}\right)$$
$$\times\left(G_{t,1}\left(\frac{\min\{m_1,n_1\}+n_2+\cdots+n_N}{N}\right)\right)^N.$$

Now, the estimate for this term can be concluded as in Lemma 8 again. □

Remark 6. From the proof of the previous result, it is clear that

$$|\Delta_{N,i}(G_{t,N}(\mathbf{n})-G_{t,N}(\mathbf{m}))| \le C|\mathbf{n}-\mathbf{m}|\left(\frac{|\mathbf{n}|+|\mathbf{m}|}{(1+t)^2}+\frac{(|\mathbf{n}|+|\mathbf{m}|)^3}{(1+t)^3}\right)t^{-N/2}, \tag{89}$$

for $\mathbf{n},\mathbf{m}\in\mathbb{Z}^N$.

Proof of Lemma 38. First, it is easy to check that

$$\|\mathcal{P}_{t,1}(\mathbf{n})\|_{\mathbb{B}_1}^2 = \left\|\frac{\partial}{\partial t}G_{t,N}(\mathbf{n})\right\|_{\mathbb{B}_1}^2 = \|\Delta_N G_{t,N}(\mathbf{n})\|_{\mathbb{B}_1}^2 \le \sum_{i=1}^N \|\Delta_{N,i}G_{t,N}(\mathbf{n})\|_{\mathbb{B}_1}^2.$$

Now, for $|\mathbf{n}|\ge 1$, by using (9), we have

$$|\Delta_{N,i}G_{t,N}(\mathbf{n})| \le C\left(\frac{1}{1+t}+\frac{|\mathbf{n}|^2}{(1+t)^2}\right)\left(G_{t,1}\left(\frac{|\mathbf{n}|-1}{N}\right)\right)^N.$$

We consider the decomposition

$$\|\Delta_{N,i}G_{t,N}(\mathbf{n})\|_{\mathbb{B}_1}^2 \le C(D_1+D_2),$$

where

$$D_1=\int_0^{|\mathbf{n}|^2} t\left(\frac{1}{1+t}+\frac{|\mathbf{n}|^2}{(1+t)^2}\right)^2\left(G_{t,1}\left(\frac{|\mathbf{n}|-1}{N}\right)\right)^{2N}dt$$

and D_2 the integral of the same function on $[|\mathbf{n}|^2,\infty)$. Applying (15) with $\alpha = 3/(2N)$ we deduce that

$$t\left(\frac{1}{1+t}+\frac{|\mathbf{n}|^2}{(1+t)^2}\right)^2\left(G_{t,1}\left(\frac{|\mathbf{n}|-1}{N}\right)\right)^{2N}$$
$$\le C(t+|\mathbf{n}|^2)^2\left(t^{-3/(2N)}G_{t,1}\left(\frac{|\mathbf{n}|-1}{N}\right)\right)^{2N} \le C\frac{(t+|\mathbf{n}|^2)^2}{|\mathbf{n}|^{2N+6}}$$

for $(|\mathbf{n}|-1)/N > 3/(2N)$. By using that

$$G_{t,1}\left(\frac{|\mathbf{n}|-1}{N}\right) \le G_{t,1}(0) \le C,$$

we obtain the same estimate for $(|\mathbf{n}|-1)/N \le 3/(2N)$. In this way,

$$D_1 \le \frac{C}{|\mathbf{n}|^{2N+6}} \int_0^{|\mathbf{n}|^2} (t+|\mathbf{n}|^2)^2\,dt = \frac{C}{|\mathbf{n}|^{2N}}.$$

For D_2, using the bound $(G_{t,1}((|\mathbf{n}|-1)/N))^N \le Ct^{-N/2}$, we obtain the estimate

$$D_2 \le C\left(\int_{|\mathbf{n}|^2}^{\infty} t^{-N-1}\,dt + |\mathbf{n}|^4 \int_{|\mathbf{n}|^2}^{\infty} t^{-N-3}\,dt\right) = \frac{C}{|\mathbf{n}|^{2N}}.$$

The case $|\mathbf{n}| = 0$ is clear and the proof of (85) is completed.

To obtain (86), it is enough to prove that

$$\|\Delta_{N,i}(G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m}))\|_{\mathbb{B}_1} \le C\frac{|\mathbf{n}-\mathbf{m}|}{(|\mathbf{n}|+|\mathbf{m}|)^{N+1}}, \qquad |\mathbf{n}| > 2|\mathbf{n}-\mathbf{m}|. \tag{90}$$

To do this we consider the decomposition

$$\|\Delta_{N,i}(G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m}))\|_{\mathbb{B}_1}^2 = E_1 + E_2,$$

with

$$E_1 = \int_0^{(|\mathbf{n}|+|\mathbf{m}|)^2} t(\Delta_{N,i}(G_{t,N}(\mathbf{n}) - G_{t,N}(\mathbf{m})))^2\,dt$$

and E_2 equals to the integral of the same function on $[(|\mathbf{n}|+|\mathbf{m}|)^2, \infty)$. By using (87), we have

$$\begin{aligned} E_1 &\le C|\mathbf{n}-\mathbf{m}|^2 \\ &\quad \times \int_0^{(|\mathbf{n}|+|\mathbf{m}|)^2} t\left(\frac{|\mathbf{n}|+|\mathbf{m}|}{(1+t)^2} + \frac{(|\mathbf{n}|+|\mathbf{m}|)^3}{(1+t)^3}\right)^2 \left(\mathfrak{H}_{t,N}\left(\frac{|\mathbf{n}|+|\mathbf{m}|}{K}\right)\right)^2 dt, \end{aligned}$$

and applying (15) with $\alpha = 5/(2N)$ and $(|\mathbf{n}|+|\mathbf{m}|)/K > 5/(2N)$ it is deduced that

$$\begin{aligned} E_1 &\le C\frac{|\mathbf{n}-\mathbf{m}|^2}{(|\mathbf{n}|+|\mathbf{m}|)^{2N+10}} \int_0^{(|\mathbf{n}|+|\mathbf{m}|)^2} (t(|\mathbf{n}|+|\mathbf{m}|) + (|\mathbf{n}|+|\mathbf{m}|)^3)^2\,dt \\ &\le C\frac{|\mathbf{n}-\mathbf{m}|^2}{(|\mathbf{n}|+|\mathbf{m}|)^{2N+2}}. \end{aligned}$$

To analyze the case $(|\mathbf{n}|+|\mathbf{m}|)/K \le 5/(2N)$ we proceed as we did for D_1. Finally, to estimate E_2 we use (89) to obtain the required bound. Indeed,

$$\begin{aligned} E_2 &\le C|\mathbf{n}-\mathbf{m}|^2 \\ &\quad \times \left((|\mathbf{n}|+|\mathbf{m}|)^2 \int_{(|\mathbf{n}|+|\mathbf{m}|)^2}^{\infty} t^{-N-3}\,dt + (|\mathbf{n}|+|\mathbf{m}|)^6 \int_{(|\mathbf{n}|+|\mathbf{m}|)^2}^{\infty} t^{-N-5}\,dt\right) \\ &\le C\frac{|\mathbf{n}-\mathbf{m}|^2}{(|\mathbf{n}|+|\mathbf{m}|)^{2N+2}} \end{aligned}$$

and the proof of (90) is finished. □

6.2. **The $\mathfrak{g}_k$-square functions associated to the Poisson semigroup.** It is very common to define some square functions in terms of the Poisson semigroup instead of the heat semigroup. In our case such $\mathfrak{g}_k$-square functions are given by

$$\mathfrak{g}_k(f)(\mathbf{n}) = \left(\int_0^\infty t^{2k-1} \left| \frac{\partial^k}{\partial t^k} P_t f(\mathbf{n}) \right|^2 dt \right)^{1/2}, \qquad k = 1, 2, \dots.$$

By using that

$$P_t f(\mathbf{n}) = \sum_{\mathbf{k} \in \mathbb{Z}^N} f(\mathbf{k}) Q_t(\mathbf{n} - \mathbf{k})$$

with

$$Q_t(\mathbf{n}) = \int_{[-1/2,1/2]^N} e^{-2t\sqrt{\sum_{i=1}^N \sin^2(\pi x_i)}} e^{-2\pi i \langle x, \mathbf{n} \rangle} \, dx,$$

it is clear that

$$\mathfrak{g}_k(f)(\mathbf{n}) = \| Z_{t,k} f(\mathbf{n}) \|_{\mathbb{B}_k} \tag{91}$$

where

$$Z_{t,k} f(\mathbf{n}) = \sum_{\mathbf{k} \in \mathbb{Z}^N} f(\mathbf{k}) Q_{t,k}(\mathbf{n} - \mathbf{k})$$

being

$$Q_{t,k}(\mathbf{n}) = \frac{d^k}{dt^k} Q_t(\mathbf{n}) = \int_{[-1/2,1/2]^N} S_{t,k}(x) e^{-2\pi i \langle x, \mathbf{n} \rangle} \, dx$$

and

$$S_{t,k}(x) = (-2)^k \left(\sum_{i=1}^N \sin^2(\pi x_i) \right)^{k/2} e^{-2t\sqrt{\sum_{i=1}^N \sin^2(\pi x_i)}}.$$

For this family of square functions we have the following result.

Theorem 40. *Let $N \geq 1$, $1 < p < \infty$, $w \in A_p(\mathbb{Z}^N)$, and $k \in \mathbb{N}$. Then the inequalities*

$$C_1 \|f\|_{\ell^p(\mathbb{Z}^N,w)} \leq \|\mathfrak{g}_k(f)\|_{\ell^p(\mathbb{Z}^N,w)} \leq C_2 \|f\|_{\ell^p(\mathbb{Z}^N,w)}, \qquad f \in \ell^2(\mathbb{Z}^N) \cap \ell^p(\mathbb{Z}^N, w),$$

hold.

To prove the previous result we need the two next lemmas.

Lemma 41. *For $N \geq 1$ and $k \in \mathbb{N}$, it is verified that*

$$\|\mathfrak{g}_k(f)\|^2_{\ell^2(\mathbb{Z}^N)} = \frac{\Gamma(2k)}{2^{2k}} \|f\|^2_{\ell^2(\mathbb{Z}^N)}. \tag{92}$$

The proof of this lemma can be done exactly as the proof of Lemma 34 but using (91) so we omit the details.

Lemma 42. *Let $N \geq 1$, $1 < p < \infty$, $w \in A_p(\mathbb{Z}^N)$, and $k \in \mathbb{N}$. Then the inequality*

$$\|\mathfrak{g}_k(f)\|_{\ell^p(\mathbb{Z}^N,w)} \leq C \|f\|_{\ell^p(\mathbb{Z}^N,w)}, \qquad f \in \ell^2(\mathbb{Z}^N) \cap \ell^p(\mathbb{Z}^N, w),$$

implies

$$\|f\|_{\ell^p(\mathbb{Z}^N,w)} \leq C \|\mathfrak{g}_k(f)\|_{\ell^p(\mathbb{Z}^N,w)}, \qquad f \in \ell^2(\mathbb{Z}^N) \cap \ell^p(\mathbb{Z}^N, w).$$

In this case, the proof of this lemma can be obtained polarising the identity (92) and using duality. Again we omit the details.

Finally, to conclude the proof of Theorem 40 we will use the estimate

$$\|g_k(f)\|_{\ell^p(\mathbb{Z}^N,w)} \leq C\|f\|_{\ell^p(\mathbb{Z}^N,w)}, \qquad f \in \ell^2(\mathbb{Z}^N) \cap \ell^p(\mathbb{Z}^N,w)$$

and the next result.

Lemma 43. *Let $k \in \mathbb{N}$, then*

$$\mathfrak{g}_k(f)(\mathbf{n}) \leq \sum_{j=0}^{\lfloor k/2 \rfloor} A_{k,j} g_{k-j}(f)(\mathbf{n}),$$

where $A_{k,j}$ are some positive constants and $\lfloor \cdot \rfloor$ denotes the floor function.

Proof. First, we observe that for a function h with k derivatives it is verified that

$$\frac{\partial^k}{\partial t^k} h\left(\frac{t^2}{4u}\right) = \sum_{j=0}^{[k/2]} B_{k,j} \left.\frac{\partial^{k-j}}{\partial s^{k-j}} h(s)\right|_{s=\frac{t^2}{4u}} \frac{t^{k-2j}}{(4u)^{k-j}},$$

for some constants $B_{k,j}$. Then, from (56), we have

$$\frac{\partial^k}{\partial t^k} P_t f(\mathbf{n}) = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{[k/2]} B_{k,j} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \left(\left.\frac{\partial^{k-j}}{\partial s^{k-j}} W_s f(\mathbf{n})\right|_{s=\frac{t^2}{4u}} \right) \frac{t^{k-2j}}{(4u)^{k-j}}\, du$$

and, by Minkowski's integral inequality,

$$\mathfrak{g}_k(f)(\mathbf{n}) \leq \sum_{j=0}^{[k/2]} B_{k,j} S_j(\mathbf{n})$$

where

$$S_j(\mathbf{n}) \\ = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}(4u)^{k-j}} \left(\int_0^\infty t^{4k-4j-1} \left(\left.\frac{\partial^{k-j}}{\partial s^{k-j}} W_s f(\mathbf{n})\right|_{s=\frac{t^2}{4u}} \right)^2 dt \right)^{1/2} du.$$

Now, by using an appropriate change of variables, we have

$$\begin{aligned} S_j(\mathbf{n}) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \left(\int_0^\infty s^{2k-2j-1} \left(\frac{\partial^{k-j}}{\partial s^{k-j}} W_s f(\mathbf{n}) \right)^2 ds \right)^{1/2} du \\ &= \frac{1}{\sqrt{2}} g_{k-j}(f)(\mathbf{n}) \end{aligned}$$

and the result follows. □

6.3. **The Laplace type multipliers.** Given a bounded function M defined on $[0, 4N]$, the multiplier associated with M is the operator, initially defined on $\ell^2(\mathbb{N})$, given by the identity

$$T_M f(\mathbf{n}) = \int_{[-1/2,1/2]^N} M\left(4\sum_{i=1}^N \sin^2(\pi x_i)\right) \mathcal{F}f(x) e^{-2\pi i \langle x, \mathbf{n}\rangle}\, dx.$$

We say that T_M is a Laplace type multiplier when

$$M(x) = x\int_0^\infty e^{-xt}a(t)\,dt,$$

with a being a bounded function.

The Laplace type multipliers were introduced by Stein in [25, Ch. 2]. There, it is observed that they verify $|x^k M^{(k)}(x)| \le C_k$ for $k = 0, 1, \dots$, and then form a subclass of Marcinkiewicz multipliers. For the operators T_M we have the following result.

Theorem 44. *Let $1 < p < \infty$ and $w \in A_p(\mathbb{Z}^N)$. Then,*

$$\|T_M f\|_{\ell^p(\mathbb{Z}^N,w)} \le C\|f\|_{\ell^p(\mathbb{Z}^N,w)}, \qquad f \in \ell^2(\mathbb{Z}^N)\cap \ell^p(\mathbb{Z}^N,w),$$

where C is a constant independent of f.

From the identity

$$x^{i\gamma} = \frac{x}{\Gamma(1-i\gamma)}\int_0^\infty e^{-xt}t^{-i\gamma}\,dt, \qquad \gamma \in \mathbb{R},$$

we deduce the following corollary.

Corollary 45. *Let $1 < p < \infty$, $\gamma \in \mathbb{R}$, and $w \in A_p(\mathbb{Z}^N)$. Then,*

$$\|(-\Delta_N)^{i\gamma} f\|_{\ell^p(\mathbb{Z}^N,w)} \le C\|f\|_{\ell^p(\mathbb{Z}^N,w)}, \qquad f \in \ell^2(\mathbb{Z}^N)\cap \ell^p(\mathbb{Z}^N,w),$$

where C is a constant independent of f.

Proof of Theorem 44. We only need prove that

$$g_1(T_M f)(\mathbf{n}) \le C g_2(f)(\mathbf{n}), \tag{93}$$

since by Theorem 33 we get that

$$\|T_M f\|_{\ell^p(\mathbb{Z}^N,w)} \le C\|g_1(T_M f)\|_{\ell^p(\mathbb{Z}^N,w)} \le C\|g_2(f)\|_{\ell^p(\mathbb{Z}^N,w)} \le C\|f\|_{\ell^p(\mathbb{Z}^N,w)}.$$

Moreover, it is enough to prove (93) for sequences in c_{00}, the space of sequences having a finite number of non-null terms. First, we have

$$T_M f(\mathbf{n}) = -\int_0^\infty a(s)\frac{\partial}{\partial s}W_s f(\mathbf{n})\,ds,$$

which is an elementary consequence of the relation

$$\begin{aligned}
&\int_{[-1/2,1/2]^N} M\left(4\sum_{i=1}^N \sin^2(\pi x_i)\right)e^{-2\pi i\langle x,\mathbf{n}\rangle}\,dx\\
&\qquad = \int_0^\infty a(s)\int_{[-1/2,1/2]^N} 4\sum_{i=1}^N \sin^2(\pi x_i) e^{-4s\sum_{i=1}^N \sin^2(\pi x_i)}e^{-2\pi i\langle x,\mathbf{n}\rangle}\,dx\,ds\\
&\qquad = -\int_0^\infty a(s)\frac{\partial}{\partial s}\int_{[-1/2,1/2]^N} e^{-4s\sum_{i=1}^N \sin^2(\pi x_i)}e^{-2\pi i\langle x,\mathbf{n}\rangle}\,dx\,ds\\
&\qquad = -\int_0^\infty a(s)\frac{\partial}{\partial s}G_{s,N}(\mathbf{n})\,ds.
\end{aligned}$$

Then, applying the semigroup property of W_t we obtain

$$W_t(T_M f)(\mathbf{n}) = -\int_0^\infty a(s)\frac{\partial}{\partial s}W_{s+t}f(\mathbf{n})\,ds$$

and hence,

$$\frac{\partial}{\partial t} W_t(T_M f)(\mathbf{n}) = -\int_0^\infty a(s) \frac{\partial}{\partial t}\frac{\partial}{\partial s} W_{s+t} f(\mathbf{n})\, ds = -\int_0^\infty a(s) \frac{\partial^2}{\partial s^2} W_{s+t} f(\mathbf{n})\, ds.$$

In this way,

$$\begin{aligned}\left|\frac{\partial}{\partial t} W_t(T_M f)(\mathbf{n})\right| &\le C \int_t^\infty s \left|\frac{\partial^2}{\partial s^2} W_s f(\mathbf{n})\right| \frac{ds}{s} \\ &\le C t^{-1/2} \left(\int_t^\infty s^2 \left|\frac{\partial^2}{\partial s^2} W_s f(\mathbf{n})\right|^2 ds \right)^{1/2}.\end{aligned}$$

Finally,

$$\begin{aligned}(g_1(T_M f)(\mathbf{n}))^2 &= \int_0^\infty t \left|\frac{\partial}{\partial t} W_t(T_M f)(\mathbf{n})\right|^2 dt \le C \int_0^\infty \int_t^\infty s^2 \left|\frac{\partial^2}{\partial s^2} W_s f(\mathbf{n})\right|^2 ds\, dt \\ &= C \int_0^\infty s^3 \left|\frac{\partial^2}{\partial s^2} W_s f(\mathbf{n})\right|^2 ds = C (g_2(f)(\mathbf{n}))^2\end{aligned}$$

and the proof of (93) is completed. □

References

[1] L. Abadias, J. González-Camus, P. J. Miana, and J. C. Pozo, *Large time behaviour for the heat equation on ℤ, moments and decay rates*, J. Math. Anal. Appl. **500** (2021), Paper No. 125137, 25 pp.

[2] D. E. Amos, *Computation of modified Bessel functions and their ratios*, Math. Comp. **28** (1974), 239–251.

[3] G. I. Arkhipov and K. I. Oskolkov, *A special trigonometric series and its applications*, Mat. Sb. (N.S.) **134(176)** (1987), 147–157, 287; translation in Math. USSR-Sb. **62** (1989), 145–155.

[4] J. Bourgain, *Pointwise ergodic theorems for arithmetic sets*, Inst. Hautes Etudes Sci. Publ. Math. **69** (1989), 5–45.

[5] A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.

[6] T. Cazenave and A. Haraux, "An introduction to semilinear evolution equations", Oxford Lecture Series in Mathematics and its Applications 13, The Clarendon Press, Oxford University Press, New York, 1998.

[7] Ó. Ciaurri, T. A. Gillespie, L. Roncal, J. L. Torrea, and J. L. Varona, *Harmonic analysis associated with a discrete Laplacian*, J. Anal. Math. **132** (2017), 109–131.

[8] Ó. Ciaurri, L. Roncal, P. R. Stinga, J. L. Torrea, and J. L. Varona, *Nonlocal discrete diffusion equations and the fractional discrete Laplacian, regularity and applications*, Adv. Math. **330** (2018), 688–738.

[9] J. Duoandikoetxea and E. Zuazua, *Moments, masses de Dirac et décomposition de fonctions (Moments, Dirac deltas and expansion of functions)*, C. R. Acad. Sci. Paris Sér. I Math. **315** (1992), 693–698.

[10] W. Guo, D. Fan, Dashan, H. Wu, and G. Zhao, *Sharp weighted convolution inequalities and some applications*, Studia Math. **241** (2018), 201–239.

[11] W. Feller, "An Introduction to Probability Theory and its Applications, Vol. 2", second ed., Wiley, New York, 1971.

[12] F. A. Grünbaum and P. Iliev, *Heat kernel expansions on the integers*, Math. Phys. Anal. Geom. **5** (2002), 183–200.

[13] R. Hunt, B. Muckenhoupt, and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. **176** (1973), 227–251.

[14] L. I. Ignat, *Qualitative properties of a numerical scheme for the heat equation*, In: "Numerical mathematics and advanced applications", Alfredo Bermúdez de Castro, Dolores Gómez, Peregrina Quintela, and Pilar Salgado (eds.), Springer, Berlin, 2006, 593–600.

[15] A. Ya. Khinchin, "Continued fractions", University of Chicago Press, 1964.
[16] N. N. Lebedev, "Special functions and its applications", Dover, New York, 1972.
[17] J. Lindenstrauss and L. Tzafriri, "Classical Banach spaces II", Springer-Verlag, Berlin, 1979.
[18] I. Nåsell, *Rational bounds for ratios of modified Bessel functions*, SIAM J. Math. Anal. **9** (1978), 1–11.
[19] F. W. J. Olver (editor-in-chief), "NIST Handbook of Mathematical Functions", Cambridge University Press, New York, 2010. Available online in `https://dlmf.nist.gov/`.
[20] L. B. Pierce, "Discrete Analogues in Harmonic Analysis", Ph.D. thesis, Princeton University, Princeton, 2009.
[21] M. Riesz, *Sur les fonctions conjuguées*, Math. Z. **27** (1928), 218–244.
[22] J. L. Rubio de Francia, F. J. Ruiz, and J. L. Torrea, *Calderón-Zygmund theory for operator-valued kernels*, Adv. in Math. **62** (1986), 7–48.
[23] F. J. Ruiz and J. L. Torrea, *Vector-valued Calderón-Zygmund theory and Carleson measures on spaces of homogeneous nature*, Studia Math. **88** (1988), 221–243.
[24] D. Ruiz-Antolín and J. Segura, *A new type of sharp bounds for ratios of modified Bessel functions*, J. Math. Anal. Appl. **443** (2016) 1232–1246.
[25] E. M. Stein, "Topics in Harmonic Analysis Related to the Littlewood-Paley Theory", Princeton Univ. Press, Princeton, NJ, 1970.
[26] E. M. Stein and S. Wainger, *Discrete analogues in harmonic analysis, I: ℓ^2 estimates for singular Radon transforms*, Amer. J. Math. **121** (1999), 1291–1336.
[27] E. M. Stein and S. Wainger, *Discrete analogues in harmonic analysis II: fractional integration*, J. Anal. Math. **80** (2000), 335–355.
[28] V. R. Thiruvenkatachar and T. S. Nanjundiah, *Inequalities concerning Bessel functions and orthogonal polynomials*, Proc. Indian Acad. Sci. A **33** (1951), 373–384.
[29] Z. H. Yang and S. Z. Zheng, *Monotonicity and convexity of the ratios of the first kind modified Bessel functions and applications*, Math. Inequal. Appl. **21** (2018), 107–125.

Departamento de Matemáticas y Computación, Universidad de La Rioja, Complejo Científico-Tecnológico, Calle Madre de Dios 53, 26006 Logroño, Spain
Email address: `oscar.ciaurri@unirioja.es`