

UNBOUNDED VISIBILITY DOMAINS: METRIC ESTIMATES AND AN APPLICATION

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ABSTRACT. We give an explicit lower bound, in terms of the distance from the boundary, for the Kobayashi metric of a certain class of bounded pseudoconvex domains in \mathbb{C}^n with \mathcal{C}^2 -smooth boundary using the regularity theory for the complex Monge–Ampère equation. Using such an estimate, among other tools, we construct a family of unbounded Kobayashi hyperbolic domains in \mathbb{C}^n having a certain negative-curvature-type property with respect to the Kobayashi distance. As an application, we prove a Picard-type extension theorem for the latter domains.

1. INTRODUCTION AND STATEMENT OF RESULTS

A substantial part of the effort and many of the tools discussed in this paper are directed at the following problems that are seemingly unrelated:

- (a) Using the regularity theory for the complex Monge–Ampère equation on bounded domains $\Omega \Subset \mathbb{C}^n$, $n \geq 2$, to estimate the Kobayashi pseudometric $k_\Omega(z; \cdot)$ in terms of $\text{dist}(z, \partial\Omega)$.
- (b) A Picard-type extension theorem for holomorphic mappings into domains $\Omega \not\subset \mathbb{C}^n$, $n \geq 2$, where Ω is unbounded, but is not the complement of a divisor.

The theme that links these problems is a weak notion of negative curvature for the metric space (Ω, K_Ω) , where K_Ω denotes the Kobayashi pseudodistance (assumed to be a distance on domains considered in this paper). This negative-curvature-type property, called *visibility*, is that, loosely speaking, geodesic lines for K_Ω joining two distinct points in $\partial\Omega$ must bend into Ω with some mild geometric control (reminiscent of the Poincaré disc model of the hyperbolic plane).

If the metric space (Ω, K_Ω) is Cauchy-complete, then any two points in Ω are joined by a geodesic (i.e., a path $\sigma : I \rightarrow \Omega$, where I is an interval, that satisfies $K_\Omega(\sigma(t), \sigma(s)) = |t - s|$ for all $s, t \in I$). But when $n \geq 2$, it is a very hard problem to tell whether, given a domain $\Omega \not\subset \mathbb{C}^n$, (Ω, K_Ω) is Cauchy-complete (even when Ω is pseudoconvex). Therefore, for the domains considered in this paper, (Ω, K_Ω) will **not** be assumed to be Cauchy-complete. Thus, a formal definition of visibility (which will be provided in Section 1.1) needs to be more refined than the picture described above. This raises the question: when does a domain have the visibility property? We begin with this discussion.

1.1. Visibility and a Picard-type extension theorem. One of the objectives of this work is to present a new application of visibility. This will require formalising the rough idea of visibility mentioned above. We shall say that a domain Ω is *Kobayashi hyperbolic* if K_Ω is a distance.

Definition 1.1. Let $\Omega \subset \mathbb{C}^n$ be a (not necessarily bounded) Kobayashi hyperbolic domain.

- (1) Let p and q be two distinct points in $\partial\Omega$. We say that the pair (p, q) has the *visibility property with respect to K_Ω* if there exist neighbourhoods U_p of p and U_q of q in \mathbb{C}^n such that $\overline{U_p} \cap \overline{U_q} = \emptyset$ and such that for each $\lambda \geq 1$ and each $\kappa \geq 0$, there exists a compact set $K \subset \Omega$ such that the image of each (λ, κ) -almost-geodesic $\sigma : [0, T] \rightarrow \Omega$ with $\sigma(0) \in U_p$ and $\sigma(T) \in U_q$ intersects K .

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- (2) We say that $\partial\Omega$ is *visible* if every pair of distinct points $p, q \in \partial\Omega$ has the visibility property with respect to K_Ω .

The property of $\partial\Omega$ being visible is closely related to the notion of Ω being a *visibility domain*, which was introduced by Bharali–Zimmer [3, 4]. The two notions are equivalent. This, in brief, is due to the fact that $\bar{\Omega}$ (resp., the Freudenthal end-compactification of $\bar{\Omega}$) is sequentially compact under the assumptions made in [3] (resp., in [4]). For an alternative argument, see [25, Section 1.3]. We will not define visibility domains here (as they are not germane to the discussion). Instead, we will work with the properties introduced in Definition 1.1, which are adequate for the application presented here. The notion of visibility itself is not new: the property introduced in Definition 1.1 is reminiscent of a property introduced by Eberlein–O’Neill in [14] in the context of Riemannian manifolds having non-positive sectional curvature — formulated in terms of an abstract boundary in place of $\partial\Omega$ and geodesics in place of (λ, κ) -almost-geodesics. This property is also seen in proper geodesic metric spaces that are Gromov hyperbolic; in this setting, the Gromov boundary takes the place of $\partial\Omega$. The latter form of visibility, for domains $\Omega \subsetneq \mathbb{C}^n$ such that (Ω, K_Ω) is a proper (hence geodesic, by the properties of K_Ω) metric space, underlies the proofs of several results that are precursors to the results on holomorphic mappings alluded to in the next paragraph: see, for instance, [1, 6] and [19, Part II]. Results of the latter description are also given by [30], which are more directly linked to the ideas in [14]. Also see [20] for a result on iterative dynamics whose proof relies on a property that could be deduced from the visibility of $\partial\Omega$ (but instead relies on [19]). In the results just cited, (Ω, K_Ω) is assumed to be a geodesic space. But recall the discussion on the difficulty in knowing when (Ω, K_Ω) admits geodesics. This explains the role of (λ, κ) -almost-geodesics (see Section 2 for a definition) in Definition 1.1. They serve as substitutes for geodesics: this is because if Ω is Kobayashi hyperbolic, then (regardless of whether (Ω, K_Ω) is Cauchy-complete) for any $z, w \in \Omega$, $z \neq w$, and any $\kappa > 0$, there exists a (λ, κ) -almost-geodesic joining z and w [4, Proposition 5.3].

Visibility of $\partial\Omega$ has been used to deduce properties of holomorphic mappings into Ω — ranging from their continuous extendability, to the iterative dynamics of such self-maps — which are too numerous to mention here. Instead, we refer readers to [3, 7, 11, 4]. Given this, it is desirable to identify families of unbounded domains $\Omega \subsetneq \mathbb{C}^n$ such that $\partial\Omega$ is visible. A rich collection of **planar** domains with the latter property that also satisfy other metrical conditions, and domains in \mathbb{C}^n , $n \geq 2$, with the latter property and having rather wild boundaries, have been constructed in [4]. But, *given an unbounded domain $\Omega \subsetneq \mathbb{C}^n$, $n \geq 2$, such that $\partial\Omega$ is \mathcal{C}^2 -smooth, Levi pseudoconvex, but **not** strongly Levi pseudoconvex, are there conditions under which $\partial\Omega$ is visible?* One of our theorems addresses this natural question. Why the interest in **unbounded** domains, one may ask. The answer will be evident when we discuss Picard-type theorems.

We present a condition for the visibility of $\partial\Omega$ that answers the question in italics stated above. Our condition, roughly, is that the set of points at which $\partial\Omega$ is weakly Levi pseudoconvex, if non-empty, is small but **not** necessarily totally disconnected. Some notation: if A and B are non-negative quantities, $A \gtrsim B$ will mean that there exists a constant $c > 0$, independent of all variables determining A and B , such that $A \geq cB$. The vector bundle $H(\partial\Omega) := T(\partial\Omega) \cap iT(\partial\Omega)$; so, $H_\xi(\partial\Omega)$ is the maximal complex subspace of $T_\xi(\partial\Omega)$: the tangent space of $\partial\Omega$ at ξ . We now define the *Levi form* of $\partial\Omega$, denoted by \mathcal{L}_Ω . While, abstractly, the Levi form is a vector-valued quadratic form $\mathcal{L}_\Omega(\xi, \cdot) : H_\xi(\partial\Omega) \rightarrow T_\xi(\partial\Omega) \otimes \mathbb{C}/H_\xi(\partial\Omega) \otimes \mathbb{C}$ for $\xi \in \partial\Omega$ — see, for instance, [5, Chapter 10] — since $\partial\Omega$ is a CR hypersurface embedded in \mathbb{C}^n , we can define $\mathcal{L}_\Omega(\xi, \cdot)$ to be \mathbb{R} -valued. This definition makes use of the standard (flat) Hermitian metric on $T(\mathbb{C}^n) \otimes \mathbb{C}$, restricted to $T(\partial\Omega) \otimes \mathbb{C}$, to identify $T_\xi(\partial\Omega) \otimes \mathbb{C}/H_\xi(\partial\Omega) \otimes \mathbb{C}$ with $(T_\xi(\partial\Omega) \otimes \mathbb{C}) \ominus (H_\xi(\partial\Omega) \otimes \mathbb{C})$, the orthogonal complement being given by the above-mentioned metric. Let η_ξ be the outward unit normal vector to $\partial\Omega$ at ξ and let $\mathbb{J}_z (= \mathbb{J}$ for each $z \in \mathbb{C}^n$) denote the standard almost complex structure on $T_z(\mathbb{C}^n) \otimes \mathbb{C}$ for each $z \in \mathbb{C}^n$. Since $\mathbb{J}(\eta_\xi)$ spans

$(T_\xi(\partial\Omega) \otimes \mathbb{C}) \ominus (H_\xi(\partial\Omega) \otimes \mathbb{C})$, the last observation enables us to define the Levi form as

$$\mathcal{L}_\Omega(\xi; v) := (1/2i) \langle [\bar{\mathbf{v}}, \mathbf{v}]_\xi, \mathbb{J}(\eta_\xi) \rangle_\xi \quad \forall v \in H_\xi(\partial\Omega) \text{ and } \forall \xi \in \partial\Omega,$$

where $\langle \cdot, \cdot \rangle_\xi$ is the above-mentioned flat metric on $T_\xi(\partial\Omega) \otimes \mathbb{C}$ and, if $v = (v_1, \dots, v_n)$, \mathbf{v} is any \mathcal{C}^1 -smooth section of $H^{1,0}(\partial\Omega)$ defined around ξ such that $\mathbf{v}(\xi) = \sum_{1 \leq j \leq n} v_j (\partial/\partial z_j|_\xi)$. It is easy to see that the right-hand side above does not depend on the choice of \mathbf{v} . We must mention that the choice of the frame field $\xi \mapsto \eta_\xi$ of $(T(\partial\Omega) \otimes \mathbb{C}) \ominus (H(\partial\Omega) \otimes \mathbb{C})$ is such that if Ω is Levi pseudoconvex, then $\mathcal{L}_\Omega \geq 0$. Then, $w(\partial\Omega)$ is the set of points in $\partial\Omega$ at which $\partial\Omega$ is weakly Levi pseudoconvex: i.e., the set of all $\xi \in \partial\Omega$ at which $\mathcal{L}_\Omega(\xi, \cdot)$ is **not** strictly positive definite.

Theorem 1.2. *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be an unbounded Kobayashi hyperbolic domain that is pseudoconvex and has \mathcal{C}^2 -smooth boundary. Suppose there exists a \mathcal{C}^2 -smooth closed 1-submanifold S of $\partial\Omega$ such that $w(\partial\Omega) \subset S$. Assume that for each $p \in w(\partial\Omega)$, there exists a neighbourhood U_p of p and $m_p > 2$ such that*

$$\mathcal{L}_\Omega(\xi; v) \gtrsim \text{dist}(\xi, S)^{m_p-2} \|v\|^2 \quad \forall v \in H_\xi(\partial\Omega) \text{ and } \forall \xi \in (\partial\Omega \cap U_p) \setminus S. \quad (1.1)$$

Then, $\partial\Omega$ is visible.

Before we can present our next result, we need a general definition.

Definition 1.3. Let Z be a complex manifold and Y a complex submanifold of Z . We say that Y is *hyperbolically imbedded* in Z if for every point $p \in \bar{Y}$ and for each neighbourhood U_p of p in Z , there exists a neighbourhood V_p of p in Z with $V_p \Subset U_p$ such that $K_Y(\bar{V}_p \cap Y, Y \setminus U_p) > 0$. Here, \bar{V}_p is the closure of V_p in Z .

The property of being hyperbolically imbedded is relevant to a class of extension theorems that we wish to examine further. The archetypal results of this class are:

Result 1.4 (Kiernan, [22]: paraphrased for Y, Z manifolds). *Let Z be a complex manifold and let $Y \subset Z$ be a hyperbolically imbedded relatively compact submanifold.*

- (1) *Then, every holomorphic map $f : \mathbb{D}^* \rightarrow Y$ extends as a holomorphic map $\tilde{f} : \mathbb{D} \rightarrow Z$.*
- (2) *Let X be a complex manifold, let $k = \dim_{\mathbb{C}}(X)$, and let $\mathcal{A} \subsetneq X$ be an analytic subvariety of X of dimension $(k-1)$ having at most normal-crossing singularities. Then, any holomorphic map $f : X \setminus \mathcal{A} \rightarrow Y$ extends as a holomorphic map $\tilde{f} : X \rightarrow Z$.*

Kwack had established a result of a similar nature under an analytical hypothesis [24, Theorem 3], whose proof is repurposed in [22] to prove (1) above. Theorems such as Result 1.4 are called *Picard-type extension theorems*. The reason for this terminology is as follows: if $Z = \mathbb{C}\mathbb{P}^1$ and $Y = \mathbb{C} \setminus \{0, 1\}$, then (1) is implied by the Big Picard Theorem.

It is well known that the complement of $(2n+1)$ hyperplanes in general position in $\mathbb{C}\mathbb{P}^n$ is hyperbolically imbedded in $\mathbb{C}\mathbb{P}^n$. To the best of our knowledge, no general techniques are known that tell us when Y is hyperbolically imbedded in Z with $\dim_{\mathbb{C}}(Z)$ being arbitrary—for Y, Z as in Definition 1.3—beyond cases where $Z = \mathbb{C}\mathbb{P}^n$ and Y are complements of certain divisors in $\mathbb{C}\mathbb{P}^n$ (in which case one relies on [23] by Kiernan). It is thus natural to ask: what are some other *explicit* geometric conditions on the pair (Y, Z) that would yield the same conclusions as Result 1.4. A good place to start would be to take $Z = \mathbb{C}^n$ and Y a domain in \mathbb{C}^n . But observe that if $Y \subsetneq \mathbb{C}^n$ is bounded, the extension problem becomes trivial due to Riemann's removable singularities theorem. This is why we consider unbounded domains in our next theorem.

Theorem 1.5. *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be an unbounded Kobayashi hyperbolic domain with the properties stated in Theorem 1.2. Let X be a complex manifold, let $k = \dim_{\mathbb{C}}(X)$, and let $\mathcal{A} \subsetneq X$ be an analytic subvariety of dimension $(k-1)$ having at most normal-crossing singularities. Then, any holomorphic map $f : X \setminus \mathcal{A} \rightarrow \Omega$ extends as a continuous map $\tilde{f} : X \rightarrow \bar{\Omega}^\infty$.*

Here, $\overline{\Omega}^\infty$ denotes the closure of Ω relative to the one-point compactification of \mathbb{C}^n . Both Theorems 1.2 and 1.5 feature the same domain Ω . This is because, as in Result 1.4, Ω being hyperbolically imbedded continues to be crucial to $f : X \setminus \mathcal{A} \rightarrow \Omega$ admitting an extension with any degree of regularity. The latter condition follows if $\partial\Omega$ is visible; see Proposition 2.5. This — in view of the discussion preceding Theorem 1.5 — is the reason for our interest in identifying unbounded visibility domains. Since Ω in Theorem 1.5 is not relatively compact, it does not follow from Result 1.4. Instead, we rely on the work of Joseph–Kwack [18]; see Result 6.1. Their work does not, however, provide any geometric conditions for a domain Ω , whether in \mathbb{C}^n or in some complex manifold, to be hyperbolically imbedded. The focus of [18] is a set of function-theoretic characterisations of hyperbolic imbedding. It is, therefore, natural to seek **geometric** conditions for a domain $\Omega \subsetneq \mathbb{C}^n$ to be hyperbolically imbedded. The hypothesis of Theorems 1.2 and 1.5 provides just such a condition. Domains satisfying this hypothesis are abundant: see, for instance, the examples given by Gaussier in [15, Section 3.2]. We conclude this section with one last question: could one extend Theorem 1.5 to domains $X \setminus \mathcal{A}$ such that \mathcal{A} has worse singularities? We can show that — in the notation of Theorem 1.5 — a holomorphic map $f : X \setminus \mathcal{A} \rightarrow \Omega$ with Ω hyperbolically imbedded does not, in general, extend continuously to X if the singularities of \mathcal{A} are slightly worse than normal-crossing singularities; see Example 6.3.

1.2. Lower bounds for the Kobayashi pseudometric. Estimates for the Kobayashi pseudometric are an essential tool for the project discussed above. This motivates our next theorem and Theorem 4.2. These results are inspired by a well-known result of Diederich–Fornæss [13, Theorem 4], which provides a lower bound for the Kobayashi pseudometric k_Ω of a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with real-analytic boundary. The lower bound that they deduce for $k_\Omega(z; v)$, $(z, v) \in \Omega \times \mathbb{C}^n$, is in terms of some positive power of $(1/\text{dist}(z, \partial\Omega))$, and has many applications. These applications are, in part, the reason for our interest in such an estimate on domains with just \mathcal{C}^2 -smooth boundary. The notation in the theorem below is as described prior to Theorem 1.2. For any $z \in \Omega$, we shall abbreviate $\text{dist}(z, \partial\Omega)$ to $\delta_\Omega(z)$.

Theorem 1.6. *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded pseudoconvex domain having \mathcal{C}^2 -smooth boundary. Assume that there exists a \mathcal{C}^2 -smooth closed submanifold of $\partial\Omega$ such that S is totally-real and such that $w(\partial\Omega) \subset S$. Suppose there exists a number $m > 2$ such that*

$$\mathcal{L}_\Omega(\xi; v) \gtrsim \text{dist}(\xi, S)^{m-2} \|v\|^2 \quad \forall v \in H_\xi(\partial\Omega) \text{ and } \forall \xi \in \partial\Omega \setminus S.$$

Then, there exists a constant $c > 0$ such that

$$k_\Omega(z; v) \geq c \frac{\|v\|}{\delta_\Omega(z)^{1/m}} \quad \forall z \in \Omega \text{ and } \forall v \in \mathbb{C}^n. \quad (1.2)$$

On a first reading, the estimate (1.2) might seem unsurprising. However, to the best of our knowledge, estimates of the form (1.2) seen in the literature that are well-argued do not, for Ω weakly pseudoconvex, provide an **explicit** exponent of δ_Ω . Moreover, there are other significant reasons for placing the estimate (1.2) on record. Namely:

- For bounded, weakly pseudoconvex, finite-type domains Ω with $\partial\Omega$ **not** real analytic, lower bounds for k_Ω resembling (1.2) have been (re)asserted on many occasions — the earliest instance being [12]. Each such claim has, eventually, relied on the difficult half of the paper [8] by Catlin. There seems to be a certain deficit in understanding the latter work — nor is there any alternative exposition on the efficacy of a construction, called a *boundary system*, on which the proofs of the above-mentioned assertions rely.
- We introduce a method relying on the regularity theory for the complex Monge–Ampère equation to derive lower bounds of the form (1.2). One way of deriving such a bound is to construct plurisubharmonic peak functions satisfying certain precise estimates: see, for instance, [13, Theorem 2], [10, Proposition 4.2]. Similar peak functions, but with

less restrictive requirements, suffice to prove the existence and regularity of solutions of the complex Monge–Ampère equation, as one would expect from the proof of [2, Theorem 6.2] (see Remark 4.5). This underlies—in view of a result of Sibony [28]—an effective and simpler idea for deriving lower bounds for the Kobayashi metric.

The latter point is substantiated by a general result whose proof (similar to that of Theorem 1.6) is the method alluded to. For its exact statement, we refer the reader to Theorem 4.2.

2. PRELIMINARIES ON HYPERBOLIC IMBEDDING AND VISIBILITY

This section is devoted to assorted observations of a technical nature that will be needed in our discussion surrounding Theorems 1.2 and 1.5. But we first explain some notation used below and in later sections (some of which has also been used without comment in Section 1).

2.1. Common notations.

- (1) For $v \in \mathbb{R}^d$, $\|v\|$ denotes the Euclidean norm. For any $x \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$, we write $\text{dist}(x, A) := \inf\{\|x - a\| : a \in A\}$.
- (2) Given a point $x \in \mathbb{R}^d$ and $r > 0$, $\mathbb{B}^d(x, r)$ denotes the open Euclidean ball in \mathbb{R}^d with radius r and center x .
- (3) Given a point $z \in \mathbb{C}^n$ and $r > 0$, $B^n(z, r)$ denotes the open Euclidean ball in \mathbb{C}^n with radius r and center z . For simplicity, we write $\mathbb{D} := B^1(0, 1)$. Also, we write $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$.
- (4) Given a \mathcal{C}^2 -smooth function $\phi : \Omega \rightarrow \mathbb{C}$ defined in some domain $\Omega \subset \mathbb{C}^n$, $(\mathfrak{H}_{\mathbb{C}}\phi)(z)$ denotes the complex Hessian of ϕ at $z \in \Omega$.
- (5) $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian inner product on \mathbb{C}^n .

2.2. Definitions and results. We begin with an elementary fact:

Lemma 2.1. *Let Z be a complex manifold. Let X and Y be domains in Z such that $X \subsetneq Y \subsetneq Z$. If X is hyperbolically imbedded in Z , then X is hyperbolically imbedded in Y as well.*

Let p , U_p and V_p be as in Definition 1.3; by the fact that the closure of $V_p \cap Y$ in Y equals $(\overline{V_p \cap Y}) \cap Y$ (where $\overline{V_p \cap Y}$ denotes the closure in Z), the above result follows immediately.

The remainder of this section focuses on definitions and facts related to the property of visibility of $\partial\Omega$, $\Omega \subset \mathbb{C}^n$ being a domain.

Definition 2.2. Let $\Omega \subset \mathbb{C}^n$ be a domain and let $I \subset \mathbb{R}$ be an interval. For $\lambda \geq 1$ and $\kappa \geq 0$, a curve $\sigma : I \rightarrow \Omega$ is called a (λ, κ) -almost-geodesic if

- $\lambda^{-1}|t - s| - \kappa \leq K_{\Omega}(\sigma(s), \sigma(t)) \leq \lambda|t - s| + \kappa$ for every $s, t \in I$, and
- σ is absolutely continuous (whereby $\sigma'(t)$ exists for almost every $t \in I$) and $k_{\Omega}(\sigma(t); \sigma'(t)) \leq \lambda$ for almost every $t \in I$.

Next, we present a definition that formalises one of the sufficient conditions on a domain $\Omega \subset \mathbb{C}^n$ under which $\partial\Omega$ is visible. It is an adaptation, introduced by Bharali–Zimmer [4], to unbounded domains of a well-known property.

Definition 2.3. Let $\Omega \subset \mathbb{C}^n$ be a domain. We say that Ω satisfies a *local interior-cone condition* if for each $R > 0$ there exist constants $r_0 > 0$, $\theta \in (0, \pi)$, and a compact subset $K \subset \Omega$, which depend on R , such that for each $z \in B^n(0, R) \cap (\Omega \setminus K)$, there is a point $\xi_z \in \partial\Omega$ and a unit vector v_z such that

- $z = \xi_z + tv_z$ for some $t \in (0, r_0)$, and
- $(\xi_z + \Gamma(v_z, \theta)) \cap B^n(\xi_z, r_0) \subset \Omega$.

Here, $\Gamma(v_z, \theta)$ denotes the open cone

$$\Gamma(v_z, \theta) := \{w \in \mathbb{C}^n : \operatorname{Re} \langle w, v_z \rangle > \cos(\theta/2) \|w\|\}.$$

The following result is classical in the case when Ω is bounded. But the choice of $K = K(R)$ in Definition 2.3 requires care when Ω is unbounded, for which reason we provide a proof.

Lemma 2.4. *Let $\Omega \subset \mathbb{C}^n$ be an unbounded domain with \mathcal{C}^2 -smooth boundary. Then, Ω satisfies a local interior-cone condition.*

Proof. Since $\partial\Omega$ is \mathcal{C}^2 -smooth, for each $p \in \partial\Omega$, we can find balls $W_p := B^n(p, R_p)$, $V_p := B^n(p, r_p)$ with $0 < r_p < R_p$ such that the following holds:

- (a) For each $z \in V_p \cap \Omega$, there exists unique $\xi_z \in \partial\Omega$ such that $\|\xi_z - z\| = \delta_\Omega(z)$ and $\xi_z \in W_p$,
- (b) If $\xi_1 \neq \xi_2 \in \partial\Omega \cap B^n(p, R_p)$, then $\{\xi_1 + t\eta_{\xi_1} : t \geq 0\} \cap \{\xi_2 + t\eta_{\xi_2} : t \geq 0\} \cap B^n(p, r_p) = \emptyset$ (where η_ξ denotes the inward unit normal at $\xi \in \partial\Omega$).

Fix $R > 0$. If $\overline{B^n(0, R)} \cap \Omega = \emptyset$ or if $\overline{B^n(0, R)} \subset \Omega$, then the two conditions in Definition 2.3 hold true vacuously (taking $K = \overline{B^n(0, R)}$ in the latter case). Hence, fix $R > 0$ such that $\partial\Omega \cap \overline{B^n(0, R)} \neq \emptyset$. Write $S := \partial\Omega \cap \overline{B^n(0, R)}$. Let $W := \bigcup_{p \in S} W_p$ and $V := \bigcup_{p \in S} V_p$. As S is compact, we can find a finite subcover $\{V_1, \dots, V_k\}$ of $\{V_p : p \in S\}$ that covers S . Write $V_j := B^n(p_j, r_j)$ and $W_j := B^n(p_j, R_j)$. We can choose $r > 0$ sufficiently small such that

$$\bigcup_{p \in S} \overline{B^n(p, r)} \subset \bigcup_{j=1}^k V_j. \quad (2.1)$$

Let K be the compact set, $K \subset \Omega$, defined as follows:

$$K := \overline{B^n(0, R)} \cap (\overline{\Omega} \setminus \bigcup_{p \in S} B^n(p, r/2)).$$

Let $r_0 := r/2$. (Note that K and r_0 depend on R .)

Fix $z \in B^n(0, R) \cap (\Omega \setminus K)$. Then, $z \in \bigcup_{p \in S} B^n(p, r/2)$. Hence, by (2.1), $z \in \bigcup_{j=1}^k V_j$. Thus, there exists a unique point in $\partial\Omega$, call it ξ_z , such that $\delta_\Omega(z) = \|z - \xi_z\|$. Let η_{ξ_z} denote the inward unit normal vector to $\partial\Omega$ at ξ_z . Let $z' := \xi_z + (r/2)\eta_{\xi_z}$. If p is a point in S such that $z \in B^n(p, r/2)$, then it is immediate that:

- $\|z - z'\| = |r/2 - \delta_\Omega(z)|$, whereby $z' \in B^n(p, r)$.
- If, for some $j = 1, \dots, k$, V_j contains z' (owing to (2.1)), then (with $\xi_{z'}$ having a meaning analogous to ξ_z) $\xi_z, \xi_{z'} \in B^n(p_j, R_j) =: W_j$.

Thus, by the property of the pair (V_j, W_j) given by (b) above, $\xi_z = \xi_{z'}$; thus $\delta_\Omega(z') = \|z' - \xi_z\| = r/2$. Therefore, $B^n(z', r/2) \subset \Omega$.

Now, clearly, $z = \xi_z + t\eta_{\xi_z}$, where $t = \|z - \xi_z\| < r/2 = r_0$. Also, it is easy to see that there exists a uniform $\theta \in (0, \pi)$ such that

$$(\xi_z + \Gamma(\eta_{\xi_z}, \theta)) \cap B^n(\xi_z, r_0) \subset B^n(z', r_0) \cap B^n(\xi_z, r_0) \subset \Omega.$$

Here, θ is given by the following:

$$\begin{aligned} \cos(\theta/2) &= \operatorname{Re} \langle \eta_{\xi_z}, v \rangle / \|v\| \\ &= \operatorname{Re} \langle \eta_{\xi_z}, v \rangle / r_0, \quad \text{where } v \in \partial B^n(\xi_z, r_0) \cap \partial B^n(z', r_0). \end{aligned}$$

In the above expression, θ is independent of the choice of v as the inner product depends only on r_0 . For this reason, θ is also independent of $z \in B^n(0, R) \cap (\Omega \setminus K)$. This establishes the conditions given in Definition 2.3. \square

The next result is a version of a result due to Sarkar [27, Proposition 3.2-(3)]. However, unlike in [27, Proposition 3.2-(3)], we are given that $\partial\Omega$ is visible as a part of the hypothesis of the result below. This results in a simpler proof than in [27]. Since it is so vital to proving Theorem 1.5, we shall provide a proof of the following

Proposition 2.5. *Let $\Omega \subset \mathbb{C}^n$ be an unbounded Kobayashi hyperbolic domain and suppose $\partial\Omega$ is visible. Then, Ω is hyperbolically imbedded in \mathbb{C}^n .*

Proof. Let $p \in \partial\Omega$. Fix a pair of bounded \mathbb{C}^n -neighbourhoods U_p, V_p of p such that $V_p \Subset U_p$. It suffices to show that $K_\Omega(\overline{V_p} \cap \Omega, \Omega \setminus U_p) > 0$.

We prove the above by contradiction. Assume that $K_\Omega(\overline{V_p} \cap \Omega, \Omega \setminus U_p) = 0$. Then, there exist a pair of sequences $\{z_\nu\} \subset \overline{V_p} \cap \Omega$ and $\{w_\nu\} \subset \Omega \setminus U_p$ such that $K_\Omega(z_\nu, w_\nu) \rightarrow 0$ as $\nu \rightarrow \infty$. As Ω is Kobayashi hyperbolic, by [4, Proposition 5.3], for each ν there exists a $(1, 1/\nu)$ -almost-geodesic $\sigma_\nu : [a_\nu, b_\nu] \rightarrow \Omega$ joining z_ν and w_ν .

Claim. There exist a subsequence $\{(z_{\nu_k}, w_{\nu_k})\}$ of $\{(z_\nu, w_\nu)\}$ and a compact $K \subset \Omega$ such that $\sigma_{\nu_k}([a_{\nu_k}, b_{\nu_k}]) \cap K \neq \emptyset$ for all k .

Proof of claim: Suppose $\{z_\nu : \nu \in \mathbb{Z}_+\} \Subset \Omega$. Let $K := \overline{\{z_\nu : \nu \in \mathbb{Z}_+\}}$, which is contained in Ω and is compact, since V_p is bounded. Clearly, $\sigma_\nu([a_\nu, b_\nu]) \cap K \neq \emptyset$ for all ν .

Now, suppose $\{z_\nu : \nu \in \mathbb{Z}_+\} \not\Subset \Omega$. Then, passing to a subsequence and relabelling, if needed, we may assume that $z_\nu \rightarrow \xi$, for some $\xi \in \partial\Omega \cap \overline{V_p}$. For each ν , define

$$t_\nu := \inf\{t \in [a_\nu, b_\nu] : \sigma_\nu(t) \in \Omega \setminus U_p\}.$$

Clearly, $t_\nu \in (a_\nu, b_\nu)$ and $\sigma_\nu(t_\nu) \in \partial U_p \cap \Omega$. Write $\zeta_\nu := \sigma_\nu(t_\nu)$. As before, if $\{\zeta_\nu : \nu \in \mathbb{Z}_+\} \Subset \Omega$, then $K := \overline{\{\zeta_\nu : \nu \in \mathbb{Z}_+\}}$ is our desired compact set that intersects the image of σ_ν for each ν . If not, then we get a subsequence $\{\zeta_{\nu_k}\}$ of $\{\zeta_\nu\}$ and a point $\eta \in \partial\Omega \cap \partial U_p$ such that $\zeta_{\nu_k} \rightarrow \eta$. Clearly, $\eta \neq \xi = \lim_{k \rightarrow \infty} z_{\nu_k}$. Observe that $\tilde{\sigma}_k := \sigma_{\nu_k}|_{[a_{\nu_k}, t_{\nu_k}]}$ is a $(1, 1)$ -almost-geodesic joining z_{ν_k} and ζ_{ν_k} . Thus, as $\partial\Omega$ is visible and as ξ and η are distinct boundary points, there is a compact $K \subset \Omega$ such that $\text{image}(\tilde{\sigma}_k) \cap K \neq \emptyset$ for every k sufficiently large, from which the claim follows. \blacktriangleleft

Now, let $o_k := \tilde{\sigma}_k(s_k) \in \text{image}(\tilde{\sigma}_k) \cap K$. Without loss of generality, we can assume that there is a point $o \in K$ such that $o_k \rightarrow o$. Using the fact that $\sigma_{\nu_k} : [a_{\nu_k}, b_{\nu_k}] \rightarrow \Omega$ is a $(1, 1/\nu_k)$ -almost-geodesic, we get

$$\begin{aligned} K_\Omega(z_{\nu_k}, o_k) + K_\Omega(o_k, w_{\nu_k}) &\leq (s_k - a_{\nu_k}) + (b_{\nu_k} - s_k) + 2/\nu_k \\ &= (b_{\nu_k} - a_{\nu_k}) + 2/\nu_k \leq K_\Omega(z_{\nu_k}, w_{\nu_k}) + 3/\nu_k \quad \forall k. \end{aligned}$$

By assumption, the right-hand side of the above inequality goes to 0 as $k \rightarrow \infty$. Then, as Ω is Kobayashi hyperbolic, we must have

$$\lim_{k \rightarrow \infty} z_{\nu_k} = \lim_{k \rightarrow \infty} w_{\nu_k} = \lim_{k \rightarrow \infty} o_k = o,$$

which is impossible since $(\overline{V_p} \cap \overline{\Omega}) \cap (\overline{\Omega} \setminus U_p) = \emptyset$. We have arrived at a contradiction. This proves the result. \square

3. ANALYTICAL PRELIMINARIES

This section is devoted to definitions and results that will be essential to the proofs of Theorems 1.6 and 1.2. Recall that the exterior derivative $d = (\partial + \bar{\partial})$ and $d^c := i(\partial - \bar{\partial})$. Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded domain. Given two functions $\phi \in \mathcal{C}(\partial\Omega; \mathbb{R})$ and $h \in \mathcal{C}(\Omega; \mathbb{R})$, $h \geq 0$, the *Dirichlet problem for the complex Monge–Ampère equation* is the non-linear boundary-value problem that seeks a function $u \in \mathcal{C}(\overline{\Omega}; \mathbb{R})$ such that $u|_\Omega$ is plurisubharmonic (which we shall denote as $u \in \text{psh}(\Omega) \cap \mathcal{C}(\overline{\Omega})$) such that

$$\begin{aligned} (dd^c u)^n &:= \underbrace{dd^c u \wedge \cdots \wedge dd^c u}_{n \text{ factors}} = h \mathcal{V}_n, \\ u|_{\partial\Omega} &= \phi, \end{aligned} \tag{3.1}$$

where \mathcal{V}_n is defined as

$$\mathcal{V}_n := (i/2)^n (dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (dz_n \wedge d\bar{z}_n).$$

When $u|_\Omega \notin \mathcal{C}^2(\Omega; \mathbb{R})$, the left-hand side of (3.1) must be interpreted as a current of bidegree (n, n) . That this makes sense when $u \in \text{psh}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ was established by Bedford–Taylor [2].

Our objective in considering the above Dirichlet problem is as follows. With Ω , h , and ϕ as above, any solution of this problem is a function $u \in \text{psh}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ that satisfies $u|_{\partial\Omega} = \phi$; we would like to establish that there exist functions with the latter properties that belong to some Hölder class on $\bar{\Omega}$ —assuming that $\partial\Omega$ is sufficiently “nice” and ϕ is sufficiently regular. The regularity theory for the complex Monge–Ampère equation provides us the means to the latter end. A regularity theorem of the type hinted at for Ω strongly pseudoconvex was established by Bedford–Taylor [2, Theorem 9.1]. Such theorems are much harder to deduce when Ω is *weakly* pseudoconvex. One such theorem is a special case of a result by Ha–Khanh [16]. Recall that $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian inner product on \mathbb{C}^n .

Result 3.1 (special case of [16, Theorem 1.5]). *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded pseudoconvex domain having \mathcal{C}^2 -smooth boundary, let ρ be a defining function of Ω , and let $m \geq 2$. Suppose*

- (*) *there exists a neighbourhood U of $\partial\Omega$, constants $c, C > 0$ and, for each $\delta > 0$ sufficiently small, there exists a plurisubharmonic function φ_δ on U of class \mathcal{C}^2 such that $|\varphi_\delta| \leq 1$ and such that*

$$\langle v, (\mathfrak{H}_{\mathbb{C}}\varphi_\delta)(z)v \rangle \geq c(1/\delta)^{2/m} \|v\|^2 \quad \forall v \in \mathbb{C}^n, \quad (3.2)$$

$$\|D\varphi_\delta(z)\| \leq C/\delta, \quad (3.3)$$

for each $z \in \rho^{-1}((-\delta, 0))$.

Let $\phi \in \mathcal{C}^{s, \alpha}(\partial\Omega)$, $s = 0, 1$, $\alpha \in (0, 1]$. Then, the Dirichlet problem

$$\begin{aligned} (dd^c u)^n &= 0, \\ u|_{\partial\Omega} &= \phi, \end{aligned}$$

has a unique plurisubharmonic solution $u \in \mathcal{C}^{0, (s+\alpha)/m}(\bar{\Omega})$.

The notation $\mathcal{C}^{j, \beta}$, $j \in \mathbb{N}$, $\beta \in (0, 1]$, denotes the class of all **real**-valued functions that are continuously differentiable to order j (the latter being suitably interpreted for the underlying space when $j \geq 1$) and whose j -th partial derivatives satisfy a uniform Hölder condition with exponent β . In what follows, if $j = 0$ and $\beta \in (0, 1]$, we shall denote this class simply as \mathcal{C}^β .

The following result provides the connection between Result 3.1 and the condition on the Levi form in Theorems 1.6 and 1.2.

Result 3.2 (special case of [21, Theorem 2.1]). *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded pseudoconvex domain having \mathcal{C}^2 -smooth boundary. Assume there exists a \mathcal{C}^2 -smooth closed submanifold S of $\partial\Omega$ such that S is totally-real and such that $w(\partial\Omega) \subset S$. Suppose there exists a number $m > 2$ such that*

$$\mathcal{L}_\Omega(\xi; v) \gtrsim \text{dist}(\xi, S)^{m-2} \|v\|^2 \quad \forall v \in H_\xi(\partial\Omega) \text{ and } \forall \xi \in \partial\Omega \setminus S. \quad (3.4)$$

Then, Ω satisfies the condition (*) in Result 3.1.

Remark 3.3. Some comments about Result 3.2 are in order. Firstly, [21, Theorem 2.1] is stated for q -pseudoconvex domains satisfying a somewhat more general condition than (3.4). Result 3.2 is obtained by taking:

- $q = 1$, and
- $F(t) = ct^m$, $t > 0$, for some $c > 0$,

in [21, Theorem 2.1]. (It must be noted that there is a small typo in the description of F in [21]; the asymptotic behaviour required of F is $F(\delta)/\delta^2 \searrow 0$ as $\delta \searrow 0$ and not what is stated on [21, p. 2769].) Secondly, the proof in [21] establishes just the estimate (3.2) (which is condition (1.5) in [21]). However, it is evident from the expression for φ_δ given that, since $m > 2$, the estimate (3.3) is satisfied.

We require one last result for proving Theorems 1.6 and 1.2.

Result 3.4 (paraphrasing [28, Proposition 6]). *Let $\Omega \subset \mathbb{C}^n$ be a domain and $z \in \Omega$. If there exists a negative plurisubharmonic function u on Ω that is of class \mathcal{C}^2 in a neighbourhood of z and satisfies*

$$\langle v, (\mathfrak{H}_{\mathbb{C}} u)(z)v \rangle \geq c\|v\|^2 \quad \forall v \in \mathbb{C}^n,$$

for some $c > 0$, then

$$k_\Omega(z; v) \geq \left(\frac{c}{\alpha}\right)^{1/2} \frac{\|v\|}{|u(z)|^{1/2}} \quad \forall v \in \mathbb{C}^n,$$

where $\alpha > 0$ is a universal constant.

4. LOWER BOUNDS FOR THE KOBAYASHI METRIC

We begin by stating and proving the general result relying on the complex Monge–Ampère equation to estimate the Kobayashi metric that was hinted at in Section 1.2. Before we state it, we need a definition.

Definition 4.1. A function $\omega : [0, \infty) \rightarrow [0, \infty)$ is called a *modulus of continuity* if it is concave, monotone increasing, and such that $\lim_{x \rightarrow 0^+} \omega(x) = \omega(0) = 0$.

Theorem 4.2. *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded domain. Suppose there exists a modulus of continuity $\omega : ([0, \infty), 0) \rightarrow ([0, \infty), 0)$ and that, for each Lipschitz function $\phi : \partial\Omega \rightarrow \mathbb{R}$, there exists a function $u_\phi : \bar{\Omega} \rightarrow \mathbb{R}$ such that $u_\phi|_\Omega$ solves the complex Monge–Ampère equation*

$$\begin{aligned} (dd^c u)^n &= 0, \\ u|_{\partial\Omega} &= \phi, \end{aligned}$$

and satisfies

$$|u_\phi(z_1) - u_\phi(z_2)| \leq C_\phi \omega(\|z_1 - z_2\|) \quad \forall z_1, z_2 \in \bar{\Omega}, \quad (4.1)$$

for some constant $C_\phi > 0$. Then there exists a constant $c > 0$ such that

$$k_\Omega(z; v) \geq c \frac{\|v\|}{\omega(\delta_\Omega(z))^{1/2}} \quad \forall z \in \Omega \text{ and } v \in \mathbb{C}^n. \quad (4.2)$$

Remark 4.3. The hypothesis of Theorem 4.2 is essentially a statement about the geometry of Ω . It is well understood that the boundary-regularity of the solutions of the complex Monge–Ampère equation is influenced by $\partial\Omega$. Moreover, the existence of u_ϕ too is constraint on Ω : for instance, it rules out those Ω that contain analytic varieties of positive dimension in $\partial\Omega$.

Before proving Theorem 4.2 we state the following elementary lemma.

Lemma 4.4. *Let $\omega : [0, \infty) \rightarrow [0, \infty)$ be a concave, monotone increasing function such that $\omega(0) = 0$. Then, for all $\lambda, x \geq 0$, $\omega(\lambda x) \leq (\lambda + 1)\omega(x)$.*

The proof of Theorem 4.2. Define $\phi : \partial\Omega \rightarrow (-\infty, 0]$ by $\phi(z) := -2\|z\|^2$. We note that all the assertions below hold true trivially when Ω is an Euclidean ball with centre $0 \in \mathbb{C}^n$. As this function is Lipschitz, there exists a function $u_\phi : \bar{\Omega} \rightarrow \mathbb{R}$ with the properties stated in the hypothesis of Theorem 4.2. Let us define

$$\Phi(z) := u_\phi(z) + \|z\|^2 \quad \forall z \in \bar{\Omega}.$$

For $\nu \in \mathbb{N}$, write $\Omega_\nu := \{z \in \Omega : \delta_\Omega(z) > 1/2^\nu\}$. Let $\nu_0 \in \mathbb{Z}_+$ and be so large that Ω_ν is connected for every $\nu \geq \nu_0$. It follows from [26, Satz 4.2] by Richberg that there exists a plurisubharmonic function Ψ on Ω of class \mathcal{C}^∞ such that for all $\nu \geq \nu_0$

$$0 \leq \Psi(z) - \Phi(z) \leq \omega(2^{-\nu}) \quad \forall z \in \Omega \setminus \Omega_\nu. \quad (4.3)$$

Clearly, Ψ extends continuously to $\bar{\Omega}$ (we shall refer to this extension as Ψ as well) and

$$\Psi(z) = -\|z\|^2 \quad \forall z \in \partial\Omega. \quad (4.4)$$

Now, let us write $U(z) := \Psi(z) + \|z\|^2$ for each $z \in \bar{\Omega}$. Since Ψ is plurisubharmonic,

$$\langle v, (\mathfrak{H}_{\mathbb{C}}U)(z)v \rangle \geq \|v\|^2 \quad \forall z \in \Omega \text{ and } v \in \mathbb{C}^n. \quad (4.5)$$

Fix a z such that $\delta_\Omega(z) \leq 1/2^{\nu_0}$. As $\partial\Omega$ is compact, there exists a point $\xi_z \in \partial\Omega$ such that $\delta_\Omega(z) = \|z - \xi_z\|$. There exists an integer $\nu_z \geq \nu_0$ such that

$$1/2^{(\nu_z+1)} < \delta_\Omega(z) \leq 1/2^{\nu_z}.$$

It follows from (4.3) that

$$\begin{aligned} |U(z)| &\leq |\Psi(z) - \Phi(z)| + |\Phi(z) + \|z\|^2| \\ &\leq \omega(2^{-\nu_z}) + |(\Phi(z) + \|z\|^2) - (\Phi(\xi_z) + \|\xi_z\|^2)|. \end{aligned} \quad (4.6)$$

Now, owing to our hypothesis on u_ϕ , there exists a constant $C_1 > 0$ such that

$$|(\Phi(z) + \|z\|^2) - (\Phi(\xi_z) + \|\xi_z\|^2)| \leq C_\phi \omega(\delta_\Omega(z)) + C_1 \delta_\Omega(z).$$

Here, we have used the condition (4.1) and the fact that $\|z - \xi_z\| = \delta_\Omega(z)$. Combining the last estimate with (4.6), we get, in view of Lemma 4.4:

$$\begin{aligned} |U(z)| &\leq \left(\frac{2^{-\nu_z}}{\delta_\Omega(z)} + 1 \right) \omega(\delta_\Omega(z)) + C_\phi \omega(\delta_\Omega(z)) + C_1 \delta_\Omega(z) \\ &\leq (3 + C_\phi) \omega(\delta_\Omega(z)) + C_1 \delta_\Omega(z). \end{aligned}$$

From the latter estimate, the fact that ω is concave, and that z —apart from the constraint $\delta_\Omega(z) \leq 1/2^{-\nu_0} \leq 1/2$ —was chosen arbitrarily, we have

$$|U(z)| \leq C \omega(\delta_\Omega(z)) \quad \forall z \in \Omega \text{ such that } \delta_\Omega(z) \leq 1/2^{\nu_0}$$

for some constant $C > 0$. Since the set $\{z \in \Omega : \delta_\Omega(z) \geq 1/2^{\nu_0}\}$ is compact, raising the value of $C > 0$ if needed, we get

$$|U(z)| \leq C \omega(\delta_\Omega(z)) \quad \forall z \in \Omega. \quad (4.7)$$

By (4.4), we get $U|_{\partial\Omega} = 0$. Thus, by the Maximum Principle, U is a smooth negative plurisubharmonic function. Thus, from (4.5), (4.7), and Result 3.4, we conclude that

$$k_\Omega(z; v) \geq \left(\frac{1}{C\alpha} \right)^{1/2} \frac{\|v\|}{\omega(\delta_\Omega(z))^{1/2}} \quad \forall z \in \Omega \text{ and } v \in \mathbb{C}^n,$$

which is the desired lower bound. \square

A substantial part of the proof of Theorem 1.6 is the same as that of the previous theorem. However, Theorem 1.6 is *not* a special case of Theorem 4.2; the assumption on $\partial\Omega$ gives us better boundary behaviour of the solutions of the same Dirichlet problem considered in the proof above. With these words, we give:

The proof of Theorem 1.6. Given our assumptions on $\partial\Omega$, Result 3.2 implies that Ω satisfies the condition (*) in Result 3.1. As in the proof of Theorem 4.2, define $\phi : \partial\Omega \rightarrow (-\infty, 0]$ by $\phi(z) := -2\|z\|^2$. As $\phi \in \mathcal{C}^{1,1}(\partial\Omega)$, taking the values $s = 1$ and $\alpha = 1$ in the conclusion of Result 3.1, we see that the Dirichlet problem stated in Result 3.1, with ϕ as above, has a unique

solution of class $\mathcal{C}^{2/m}(\overline{\Omega})$. Let us denote this solution by u_ϕ . At this stage, exactly the same argument as in the proof of Theorem 4.2 with

$$\omega(r) := r^{2/m}, \quad r \in [0, \infty),$$

gives us a function U defined on $\overline{\Omega}$ such that $U|_\Omega$ is a smooth negative plurisubharmonic function that satisfies the conditions

$$|U(z)| \leq C\delta_\Omega(z)^{2/m}, \quad (4.8)$$

$$\langle v, (\mathfrak{H}_\mathbb{C}U)(z)v \rangle \geq \|v\|^2 \quad (4.9)$$

(for some constant $C > 0$) for every $z \in \Omega$ and $v \in \mathbb{C}^n$. From these inequalities and Result 3.4, we conclude that

$$k_\Omega(z; v) \geq \left(\frac{1}{C\alpha}\right)^{1/2} \frac{\|v\|}{\delta_\Omega(z)^{1/m}} \quad \forall z \in \Omega \text{ and } v \in \mathbb{C}^n,$$

which is the desired lower bound. \square

Remark 4.5. The careful reader may notice that Theorem 1.6 could be proved without reference to Result 3.1 or to the complex Monge–Ampère equation. Given Theorem 3.2, one could instead appeal to [16, Theorem 2.1] which provides a function that, suitably modified, could substitute U in the proof above. However, the **proof** of [16, Theorem 2.1] involves a difficult construction of a family of plurisubharmonic peak functions that must satisfy very restrictive conditions. These are the “plurisubharmonic peak functions satisfying certain precise estimates” alluded to in Section 1.2. Indeed, a proof of Result 3.1 can be given without the use of such precise estimates. Furthermore, such (families of) plurisubharmonic peak functions may not be available to large classes of domains, whereas there are **many** approaches to the existence and boundary-regularity of solutions to the homogeneous complex Monge–Ampère equation. Thus, the complex Monge–Ampère equation may be a useful tool for establishing estimates similar to (1.2). The approach taken in the last proof, and Theorem 4.2, highlight the latter point.

5. THE PROOF OF THEOREM 1.2

Before we can give the proof of Theorem 1.2, we give a definition that will be useful in the latter proof.

Definition 5.1 (Bharali–Zimmer, [4]). Let $\Omega \subset \mathbb{C}^n$ be a Kobayashi hyperbolic domain. Given a subset $A \subset \overline{\Omega}$, we define the function $r \mapsto M_{\Omega, A}(r)$, $r > 0$, as

$$M_{\Omega, A}(r) := \sup \left\{ \frac{1}{k_\Omega(z; v)} : z \in A \cap \Omega, \delta_\Omega(z) \leq r, \|v\| = 1 \right\}.$$

The function $M_{\Omega, A}$ is involved in one of the two conditions that a point $p \in \partial\Omega$, for Ω as in the above definition, must satisfy to be what is called a “local Goldilocks point” by Bharali–Zimmer in [4]; see [4, Definition 1.3]. The connection between local Goldilocks points and the visibility property is given by the following

Result 5.2 (paraphrasing [4, Theorem 1.4]). *Let $\Omega \subset \mathbb{C}^n$ be a Kobayashi hyperbolic domain. If the set of points in $\partial\Omega$ that are not local Goldilocks points is a totally disconnected set, then $\partial\Omega$ is visible.*

Another useful definition:

Definition 5.3. Let Ω be a domain in \mathbb{C}^n and let $p \in \partial\Omega$. A function $\psi : \Delta \rightarrow (-\infty, 0]$, where Δ is a $\overline{\Omega}$ -open neighbourhood of p , is called a *local plurisubharmonic peak function of Ω at p* if $\psi \in \text{psh}(\Delta \cap \Omega) \cap \mathcal{C}(\Delta)$ and satisfies

$$\psi(p) = 0 \quad \text{and} \quad \psi(z) < 0 \quad \forall z \in \Delta \setminus \{p\}.$$

We are now in a position to give

The proof of Theorem 1.2. The proof of Theorem 1.2 will be carried out in two steps.

Step 1. For $p \in \partial\Omega$, constructing a bounded subdomain D_p such that $\partial\Omega \cap \partial D_p \ni p$ and is large

Fix $p \in \partial\Omega$. Consider a unitary change of coordinate $Z = (Z_1, \dots, Z_n)$ centered at p (i.e., $Z(p) = 0$) with respect to which $T_p(\partial\Omega) = \{(Z_1, \dots, Z_n) \in \mathbb{C}^n : \text{Im}(Z_n) = 0\}$, the outward unit normal to $\partial\Omega$ at $p (= 0)$ is $(0, \dots, 0, -i)$, and such that there exist a neighbourhood $U_p^2 := \mathbb{B}^{2n-1}(0, r_2) \times (-r_2, r_2)$ and a function $\varphi_p : (\mathbb{B}^{2n-1}(0, r_2), 0) \rightarrow (\mathbb{R}, 0)$ such that $Z(\partial\Omega) \cap U_p^2$ is connected and

$$Z(\Omega) \cap U_p^2 \subset \{(Z', Z_n) \in \mathbb{B}^{2n-1}(0, r_2) \times \mathbb{R} : \text{Im}(Z_n) > \varphi_p(Z', \text{Re}(Z_n))\}$$

(here, r_2 depends on p but, for simplicity of notation, we will omit suffixes and understand that this dependence is implied). Shrinking r_2 if necessary, we will assume, additionally, that:

- $\varphi_p(Z', x_n) \in (-r_2/3, r_2/3)$ for every $(Z', x_n) \in \mathbb{B}^{2n-1}(0, r_2)$ and $\{Z \in \mathbb{B}^{2n-1}(0, r_2) \times \mathbb{R} : \varphi_p(Z', \text{Re}(Z_n)) < \text{Im}(Z_n) < \varphi_p(Z', \text{Re}(Z_n)) + r_2/3\} \subset Z(\Omega)$,
- $(Z(\partial\Omega) \cap U_p^2) \cap Z(w(\partial\Omega)) = \emptyset$ if $p \notin w(\partial\Omega)$, and $U_p^2 \Subset U_p$ if $p \in w(\partial\Omega)$

(where U_p is as in the statement of Theorem 1.2). Fix $r_1 \in (0, r_2/2)$. Let $\psi_p : \mathbb{R}^{2n-1} \rightarrow [0, \infty)$ be a smooth, non-negative, radial convex function such that

$$\begin{aligned} \psi_p|_{\mathbb{B}^{2n-1}(0, r_1)} &\equiv 0, \text{ and} \\ \psi_p|_{\mathbb{R}^{2n-1} \setminus \mathbb{B}^{2n-1}(0, r_1)} &\text{ is strongly convex.} \end{aligned} \quad (5.1)$$

Clearly,

$$\text{Graph}\left((\varphi_p + \psi_p)|_{\mathbb{B}^{2n-1}(0, r_2) \setminus \overline{\mathbb{B}^{2n-1}(0, r_1)}}\right) \text{ is a strongly Levi pseudoconvex hypersurface.} \quad (5.2)$$

Also, we can find a ψ_p that satisfies all the above conditions and such that $0 \leq \psi_p < r_2/3$ on $\mathbb{B}^{2n-1}(0, r_2)$, due to which

$$\mathcal{S}_p := \text{Graph}\left((\varphi_p + \psi_p)|_{\mathbb{B}^{2n-1}(0, r_2)}\right) \Subset Z(\Omega) \cup \text{Graph}\left(\varphi_p|_{\overline{\mathbb{B}^{2n-1}(0, r_1)}}\right).$$

Owing to this and to (5.2), we can construct a bounded domain \tilde{D}_p such that

- (a) $\tilde{D}_p \subsetneq Z(\Omega)$,
- (b) $\mathcal{S}_p \subset \partial\tilde{D}_p$,
- (c) $\mathcal{S}_p := \text{Graph}\left(\varphi_p|_{\overline{\mathbb{B}^{2n-1}(0, r_1)}}\right) = Z(\partial\Omega) \cap \partial\tilde{D}_p$,
- (d) $\partial\tilde{D}_p$ is strongly Levi pseudoconvex at each $\xi \in \partial\tilde{D}_p \setminus \mathcal{S}_p$ whenever $p \in w(\partial\Omega)$, and \tilde{D}_p is a strongly Levi pseudoconvex domain when $p \notin w(\partial\Omega)$.

Write $D_p := Z^{-1}(\tilde{D}_p)$. Finally, we can extend $S \cap Z^{-1}(\mathcal{S}_p)$, whenever the latter is non-empty, to a \mathcal{C}^2 -smooth closed 1-submanifold of ∂D_p .

Step 2. Showing that each $p \in \partial\Omega$ is a local Goldilocks point.

It is well-known that p is a local Goldilocks point if $p \notin w(\partial\Omega)$. For a $p \in w(\partial\Omega)$, it follows from the discussion in the second paragraph of Step 1 and from the properties (a)–(d) that D_p satisfies all the conditions of Theorem 1.6 with $m = m_p$. Let $\mathcal{U}_p : \overline{D}_p \rightarrow (-\infty, 0]$ denote the function constructed in the proof of Theorem 1.6 that is plurisubharmonic on D_p and satisfies the conditions (4.8), with $m = m_p$, and (4.9). Let W_p be a neighbourhood of p having the following properties (recall that $\partial\Omega$ is \mathcal{C}^2 -smooth):

- $(\overline{W}_p \cap \Omega) \subsetneq D_p$ and $\overline{W}_p \cap \overline{D}_p \cap \partial D_p \subsetneq \partial\Omega \cap \partial D_p$.
- $\delta_\Omega(z) = \delta_{D_p}(z)$ for all $z \in W_p \cap D_p$.
- For each $z \in W_p$, there is a unique point $\pi(z) \in \partial\Omega \cap W_p$ such that $\delta_{D_p}(z) = \|z - \pi(z)\|$.

Fix $\xi \in \partial\Omega \cap W_p$ and $\varepsilon(\xi) \in (0, 1/2)$. Then, owing to (4.9), the function

$$\psi_\xi(z) := \mathcal{U}_p(z) - \varepsilon(\xi)\|z - \xi\|^2, \quad z \in D_p \cup (\partial\Omega \cap \overline{D}_p),$$

is a local plurisubharmonic peak function of Ω at ξ . Applying a construction by Gaussier [15, Section 2], we can find a neighbourhood W_p^* of p , $W_p^* \Subset W_p$, such that for each $\xi \in \partial\Omega \cap W_p^*$, there exist a neighbourhood W_ξ of ξ , $W_\xi \Subset W_p$, such that $\pi(z) \in \partial\Omega \cap W_\xi$ for each $z \in W_\xi \cap D_p$, and an upper-semicontinuous plurisubharmonic function $\tilde{\mathcal{U}}_\xi : \Omega \rightarrow (-\infty, 0)$ such that

$$\tilde{\mathcal{U}}_\xi(z) = \psi_\xi(z) \quad \forall z \in W_\xi \cap D_p. \quad (5.3)$$

Now, define $\widetilde{W}_p := \bigcup_{\xi \in (\partial\Omega \cap W_p^*)} (W_\xi \cap W_p^*)$.

For $z_0 \in \widetilde{W}_p \cap D_p$, by (5.3), Result 3.4, and the inequality (4.9) applied to $\psi_{\pi(z_0)}$, we have

$$k_\Omega(z_0; v) \geq \left(\frac{1/2}{\alpha}\right)^{1/2} \frac{\|v\|}{(|\mathcal{U}_p(z_0)| + \varepsilon(\pi(z_0))\delta_{D_p}(z_0)^2)^{1/2}} \quad \forall v \in \mathbb{C}^n.$$

The above inequality and (4.8) (taking $m = m_p$) imply that there exists a constant $C_p > 0$, independent of $z_0 \in \widetilde{W}_p \cap D_p$, such that

$$k_\Omega(z_0; v) \geq \left(\frac{1/2}{C_p\alpha}\right)^{1/2} \frac{\|v\|}{\delta_{D_p}(z_0)^{1/m_p}} = \left(\frac{1/2}{C_p\alpha}\right)^{1/2} \frac{\|v\|}{\delta_\Omega(z_0)^{1/m_p}} \quad \forall v \in \mathbb{C}^n.$$

The equality above is because $\delta_\Omega(z_0) = \delta_{D_p}(z_0)$. Write $A^p := \widetilde{W}_p \cap \overline{\Omega}$. Since $z_0 \in \widetilde{W}_p \cap D_p$ was arbitrarily chosen, it follows from the above estimate that the quantity M_{Ω, A^p} satisfies

$$M_{\Omega, A^p}(r) \leq c_p r^{1/m_p} \quad \text{for all } r > 0 \text{ sufficiently small,} \quad (5.4)$$

for some $c_p > 0$.

By Lemma 2.4, Ω satisfies a local interior cone condition. Thus, by [4, Lemma 2.2] and (5.4) it follows that p satisfies the conditions for being a local Goldilocks point. Since $p \in w(\partial\Omega)$ was chosen arbitrarily, it follows that every boundary point is a local Goldilocks point. Thus, by Result 5.2, $\partial\Omega$ is visible. \square

6. THE PROOF OF THEOREM 1.5

In this section, we give the (short) proof of Theorem 1.5. This is a Picard-type extension theorem, as discussed in Section 1. Such a theorem relies upon $\Omega \subset \mathbb{C}^n$ being hyperbolically imbedded, just as in Result 1.4, even though Ω is not relatively compact. We state the result by Joseph–Kwack alluded to in Section 1, which clarifies the latter statement. Here, Z^∞ denotes the one-point compactification of Z .

Result 6.1 (Joseph–Kwack, [18, Corollary 7]: paraphrased for Y, Z manifolds). *Let Z be a complex manifold and Y be a complex submanifold of Z such that Y is hyperbolically imbedded in Z . Let X be a complex manifold, let $k = \dim_{\mathbb{C}}(X)$, and let $\mathcal{A} \subsetneq X$ be an analytic subvariety of dimension $(k-1)$ having at most normal-crossing singularities. Then, any holomorphic map $f : X \setminus \mathcal{A} \rightarrow Y$ extends as a continuous map $\tilde{f} : X \rightarrow Z^\infty$.*

Remark 6.2. A comment on some terminology and definitions used in Joseph–Kwack [18] are in order. With Y, Z as above, they defined a notion of when a point in $\overline{Y} \subset Z$ is a *hyperbolic point for Y* (see [18, p. 363] for the definition). Many of the foundational results in [18] give certain necessary and sufficient conditions for a point in \overline{Y} to be a hyperbolic point for Y . A careful reading of the proofs in [18] indicates that the proof of [18, Corollary 7] relies on the latter results, and its conclusion holds true under the assumption that each point in \overline{Y} is a hyperbolic

point for Y . It turns out that Y is hyperbolically imbedded in Z (in the sense of Definition 1.3) if and only if each point in \bar{Y} is a hyperbolic point for Y ; see [17], [29, Corollary 2].

Proof of Theorem 1.5. Since Ω satisfies the hypothesis of Theorem 1.2, $\partial\Omega$ is visible. So, owing to Proposition 2.5, Ω is hyperbolically imbedded in \mathbb{C}^n . Thus, by Result 6.1, the proof follows immediately. \square

6.1. An example. We conclude this discussion with a basic example which shows that, for X , \mathcal{A} as in Section 1, a holomorphic map $f : X \setminus \mathcal{A} \rightarrow \Omega$, where Ω is a hyperbolically imbedded domain, does not, in general, extend continuously to X if the singularities of \mathcal{A} are even slightly worse than normal-crossing singularities.

Example 6.3. *An example of an unbounded planar domain Ω that is hyperbolically imbedded in \mathbb{C} and a holomorphic function $f : (\mathbb{D}^2 \setminus \mathcal{A}) \rightarrow \Omega$, where \mathcal{A} is a closed analytic set in \mathbb{D}^2 of codimension 1 containing singular points, but not normal-crossing singularities, such that f does **not** extend:*

- either to a holomorphic function on \mathbb{D}^2 ,
- or to a continuous map from \mathbb{D}^2 to \mathbb{C}^∞ .

Let $\Omega := \mathbb{C} \setminus \{0, 1\}$. It is a long-established fact that $\mathbb{CP}^1 \setminus \{[0 : 1], [1 : 0], [1 : 1]\}$ is hyperbolically imbedded in \mathbb{CP}^1 . So, it follows by Lemma 2.1 that Ω is hyperbolically imbedded in \mathbb{C} . Define

$$\mathcal{A} := \{(z, w) \in \mathbb{D}^2 : z(w - z)w = 0\}.$$

If we define $f : (\mathbb{D}^2 \setminus \mathcal{A}) \rightarrow \mathbb{C}$ by

$$f(z, w) := z/w \quad \forall (z, w) \in (\mathbb{D}^2 \setminus \mathcal{A}),$$

then it is elementary to see that, by construction, f is holomorphic and that $\text{range}(f) \subseteq \Omega$. For each fixed $\lambda \in \mathbb{C} \setminus \{0, 1\}$, $(\lambda\zeta, \zeta) \in \mathbb{D}^2 \setminus \mathcal{A}$ for every $\zeta \in \mathbb{C}^*$ with sufficiently small $|\zeta|$. We have

$$\lim_{\zeta \rightarrow 0} f(\lambda\zeta, \zeta) = \lambda.$$

Since $\lambda \in \mathbb{C} \setminus \{0, 1\}$ was arbitrary, the above shows that $(0, 0)$ is a point of indeterminacy of f . Hence, the extension of f to \mathbb{D}^2 in either of the two above-mentioned ways is impossible. \blacktriangleleft

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