Critical exponent in a fully parabolic Keller-Segel system with Dirichlet boundary condition for signal

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Abstract: The fully parabolic Keller-Segel system

$$\begin{cases}\partial_t u = \Delta u - \nabla \cdot \big(u(1+u)^{-\alpha}\nabla v\big), & (x,t) \in \Omega \times (0,\infty),\\ \partial_t v = \Delta v - v + u, & (x,t) \in \Omega \times (0,\infty)\end{cases}$$

is examined within a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$ ($N \geq 3$). Unlike much of the existing literature that focus on the no-flux/no-flux boundary value problem for chemotaxis systems, we impose a homogeneous/inhomogeneous Dirichlet signal boundary condition:

$$v(x,t) = v_*, \qquad (x,t) \in \partial\Omega \times (0,\infty).$$

Here, v_* is a nonnegative constant, aligning with suggestions from modeling literature. We establish that for suitably regular initial data, the associated no-flux/Dirichlet initial-boundary value problem possesses a globally bounded classical solution if $\alpha > 1 - \frac{2}{N}$. Conversely, we construct a finite-time blow-up solution in the radially symmetric setting when $0 < \alpha < 1 - \frac{2}{N}$. Our findings underscore $\alpha_c := 1 - \frac{2}{N}$ as a critical exponent in our model, distinguishing between global solvability and finite-time blow-up singularity.

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1 Introduction

Chemotaxis is a biological phenomenon where cells or species move towards a more favorable chemical environment. A prominent mathematical model describing the spatial dynamics of this effect is the Keller-Segel (KS) system:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot \big(u \nabla v\big), \\ \partial_t v = \Delta v - v + u. \end{cases} \tag{1.1}$$

Here, $u = u(x,t)$ represents the density of a cell population, and $v = v(x,t)$ denotes the concentration of a chemical signal attracting cells. Since Keller and Segel's seminal work in the 1970s [20, 21], numerous studies have contributed significantly to understanding this system (see [1, 12, 14] for surveys). Research in this field has focused on excluding (e.g., [26, 28, 42]) or detecting [25, 43] finite-time blow-up phenomena and exploring additional qualitative properties (e.g., [29, 47]) within system (1.1) and related variants, including simplified parabolic-elliptic versions [26, 11, 18, 27, 32].

Blow-up detecting and excluding for KS system with no-flux/no-flux boundary conditions. The Keller-Segel (KS) system (1.1), when posed in bounded domains, is typically accompanied by no-flux boundary conditions:

$$\nabla u \cdot \mathbf{n} = \nabla v \cdot \mathbf{n} = 0, \qquad (x,t) \in \partial\Omega \times (0,\infty), \tag{1.2}$$

where $\mathbf{n}$ represents the unit outward normal vector field on $\partial\Omega$. It is well-established that system (1.1)-(1.2) in bounded planar domains admits globally bounded solutions provided the initial data $(u(\cdot,0), v(\cdot,0))$ are suitably regular and satisfy $\int_\Omega u(\cdot,0) =: m < 4\pi$ ([26]). However, for each $m \in (4\pi,\infty) \setminus \{4k\pi | k \in \mathbb{N}\}$, there exist initial data such that $\int_\Omega u(\cdot,0) = m$, leading to unbounded solutions either in finite or infinite time ([15]). In the three-dimensional case, no mass threshold phenomenon occurs. Specifically, when the spatial domain Ω is a ball in $\mathbb{R}^N$ with $N \geq 3$, it is proven that for any prescribed $m > 0$, there exist radially symmetric positive initial data $(u(\cdot,0), v(\cdot,0))$ with $\int_\Omega u(\cdot,0) = m$ such that the corresponding solution blows up in finite time [43]. Furthermore, an essentially explicit blow-up criterion demonstrates that within the space of all radial functions, the set of such initial data enforcing blow-up is indeed large in an appropriate sense.

Given biological contexts where unbounded population densities are deemed unrealistic, significant efforts have been directed towards devising modified variants that preemptively prevent explosive behaviors. One frequently explored refinement involves altering cell motility to depend differently on population density, particularly at high densities, leading to saturation effects in the cross-diffusion term, as exemplified in the KS variant:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot \big(u S(u) \nabla v\big), \\ \partial_t v = \Delta v - v + u. \end{cases} \tag{1.3}$$

Here, $S(u)$ is a nonnegative function, possibly diminishing at large u, such as $S(u) \lesssim (1+u)^{-\alpha}$ with $\alpha > 0$. For the associated no-flux/no-flux boundary-value problem in smoothly bounded domains $\Omega \subset \mathbb{R}^N (N \geq 2)$, it is established that if

$$S(u) \leq \overline{C}(1+u)^{-\alpha} \quad \text{for all } u \geq 0, \tag{1.4}$$

with some $\overline{C} > 0$ and $\alpha > 1 - \frac{2}{N}$, then global bounded classical solutions exist for all suitably smooth initial data [16, 19]. Conversely, if there exist $\underline{C} > 0$ and $\alpha < 1 - \frac{2}{N}$ satisfying

$$S(u) \geq \underline{C}(1+u)^{-\alpha} \quad \text{for all } u \geq 0, \tag{1.5}$$

then some solutions may become unbounded for $N \geq 3$ [5, 7] and $N \geq 2$ [6, 16, 41].

The saturation effect at high cell densities also enhances the global solvability of the KS system when coupled with an additional Stokes or Navier-Stokes fluid model, which accounts for chemotactic movement in a fluid environment. Specifically, it has been demonstrated that if the chemotactic sensitivity $S(u)$ satisfies (1.4), the no-flux/no-flux/Dirichlet boundary-value problem of the two-dimensional Keller-Segel-(Navier-)Stokes system yields a unique global bounded classical solution for arbitrarily large initial data whenever $\alpha > \frac{1}{2}$ [35, 37]. In the three-dimensional case, global weak solutions can be constructed for the Keller-Segel-Navier-Stokes system [24, 34], while global classical solutions are obtained for the Keller-Segel-Stokes version [45] provided $\alpha > \frac{1}{3}$.

Dirichlet boundary conditions for the signal in chemotaxis system. While homogeneous Neumann boundary conditions are often used for their mathematical tractability in chemotaxis systems, recent studies have underscored the relevance of alternative boundary conditions. Notably, there is a suggestion to adopt more realistic boundary conditions for the chemical signal, as oxygen, for instance, diffuses significantly faster in air than in water. In studies like [3, 4, 8, 9, 33], it is proposed to maintain a fixed oxygen concentration on relevant boundary parts. Assuming, for simplicity, that the entire fluid is surrounded by air, recent works have explored chemotaxis-consumption models where prescribed signal concentrations on the boundary necessitate coupling the chemotaxis systems with the boundaries:

$$(\nabla u - u\nabla v) \cdot \mathbf{n} = 0, \quad v = v_*, \qquad (x,t) \in \partial\Omega \times (0,\infty), \tag{1.6}$$

where v_* is a given nonnegative constant or a function dependent on x and t (see [23, 2, 38, 39, 40]). This modification in boundary conditions leads to the loss of the global energy structure crucial for the development of existence theories and asymptotic behavior in chemotaxis(-fluid) systems. In the context of the 3D chemotaxis-Stokes system with prescribed signal on the boundary, only global generalized solutions are constructed via local energy estimates [38]. Global classical solvability is derived under smallness assumptions on initial data in two-dimensional bounded domains [39]. For the fluid-free chemotaxis-consumption system, in radially symmetric settings, global existence of bounded classical solutions is established for $N = 2$, while global weak solutions are constructed for $N \in \{3, 4, 5\}$ [23].

In the context of the chemotaxis model with signal production akin to the KS system (1.1), the inclusion of Dirichlet signal boundary coupling can also capture realistic phenomena in biology or physics [30, 9]. However, it may correspond to significantly different mathematical features. For example, by replacing the second equation in (1.1) with $0 = \Delta v + u$ and considering boundary conditions (1.6) with $v_* = 0$ in two-dimensional bounded domains, Suzuki [30] demonstrated that the solution exhibits a collapse in infinite time when the initial total mass is 8π and the domain is close to a disc. Notably, the solution to such a problem remains bounded near the boundary, indicating that the blowup set consists of a finite number of interior points. In [9], the parabolic-elliptic simplification version of (1.1):

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u\nabla v), \\ 0 = \Delta v - v + u \end{cases} \tag{1.7}$$

augmented with boundary conditions (1.6) with $v_* = 0$ is investigated. The authors unveiled a further dynamical facet linked to the presence of a secondary mass threshold: in the case of $N \geq 3$, there exists $M^* := M^*(\Omega) > 0$ such that all nontrivial solutions blow up whenever the initial mass $\int_\Omega u(\cdot, 0) \geq M^*$. This scenario differs from the corresponding no-flux/no-flux boundary-value problem of the parabolic-elliptic KS system with the second equation in (1.7) replaced by $0 = \Delta v - \frac{1}{|\Omega|}\int_\Omega u + u$. While a similar secondary critical mass also exists in this alternative setting, blow-up can only be triggered by initial data that are radially symmetric and, in a specifically defined sense, more concentrated than the associated spatially homogeneous equilibrium [46].

Main results. Motivated by previous works, this study delves into the fully parabolic KS system with power-type nonlinear chemotaxis sensitivity, incorporating no-flux boundary conditions for

u and Dirichlet boundary conditions for v. Specifically, we consider the initial-boundary value problem:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot \big(uS(u)\nabla v\big), & (x,t) \in \Omega \times (0,\infty), \\ \partial_t v = \Delta v - v + u, & (x,t) \in \Omega \times (0,\infty), \\ \big(\nabla u - uS(u)\nabla v\big) \cdot \mathbf{n} = 0, \quad v = v_*, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.8}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, v_* is a nonnegative constant, and the chemotaxis sensitivity function $S \in C^2\big([0,\infty)\big)$ satisfies

$$\big|S(s)\big| \le K_S(1+s)^{-\alpha} \qquad \text{for all} \qquad s \ge 0, \tag{1.9}$$

where K_S and α are positive constants. Our objective is to explore how the value of α influences the global well-posedness of the KS system in this setting involving a Dirichlet boundary condition for signal. Particularly, we aim to identify a critical value for α that distinguishes global existence from finite-time blowup for system (1.8) with $S(s) = (1+s)^{-\alpha}$.

To ensure coherent interpretation of our findings, we specify that the initial data adhere to

$$\begin{cases} (u_0, v_0) \in C^0\big(\overline{\Omega}\big) \times W^{1,\infty}(\Omega), \\ u_0 > 0 \text{ in } \overline{\Omega}, \quad v_0 \ge v_* \text{ in } \Omega \text{ with } v_0 = v_* \text{ on } \partial\Omega. \end{cases} \tag{1.10}$$

Then, for suitably large α, system (1.8) sustains global bounded classical solutions. Precisely, we have

Theorem 1.1 *Let $N \ge 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Suppose that* (1.9) *holds with $\alpha > 1 - \frac{2}{N}$. Then for any (u_0, v_0) fulfilling* (1.10)*, there exists a unique nonnegative classical solution (u,v) to system* (1.8)*, which is global and uniformly bounded in the sense that*

$$\big\|u(\cdot,t)\big\|_{L^\infty(\Omega)} + \big\|v(\cdot,t)\big\|_{W^{1,\infty}(\Omega)} \le C, \qquad t \in (0,\infty)$$

for some positive constant C.

Next, we aim to identify solutions that blow up at finite time for system (1.8) with a chemotaxis sensitivity $S \in C^2([0,\infty))$ given by

$$S(s) := K_S(1+s)^{-\alpha}, \qquad s \ge 0, \tag{1.11}$$

where $\alpha > 0$ and $K_S > 0$. To achieve this, we concentrate on the radial symmetry setting, considering $\Omega := B_R \subset \mathbb{R}^N$ ($N \ge 3$) with $R > 0$. Inspired by [13, 14], we introduce the functionals

$$\mathcal{F}(u,v) := \frac{1}{2}\int_{B_R} |\nabla v|^2 + \frac{1}{2}\int_{B_R} v^2 - \int_{B_R} uv + \int_{B_R} G(u) \tag{1.12}$$

and

$$\mathcal{D}(u,v) := \int_{B_R} \big|\Delta v - v + u\big|^2 + \int_{B_R} \Big|\frac{\nabla u}{\sqrt{uS(u)}} - \sqrt{uS(u)}\nabla v\Big|^2 \tag{1.13}$$

for suitable u and v, where

$$G(s) := \int_1^s \int_1^\sigma \frac{1}{\tau S(\tau)} d\tau d\sigma, \qquad s \ge 0. \tag{1.14}$$

To state our result, we further introduce the set

$$\mathcal{B}(m, \overline{M}, v_*) := \Big\{(u_0, v_0) \in C^0(\overline{B_R}) \times W^{1,\infty}(B_R) \,\Big|\, u_0 \text{ and } v_0 \text{ are radially symmetric and}$$

$$\text{positive in } B_R,\ v_0 \ge v_* \text{ in } B_R,\quad v_0 = v_* \text{ on } \partial B_R, \text{ and satisfy}$$
$$\|u_0\|_{L^1(B_R)} = m,\ \|v_0\|_{W^{1,2}(B_R)} \le \overline{M},\ \text{and}$$
$$\mathcal{F}(u_0, v_0) \le -K\big(\overline{M}^{\frac{2N+4}{N}} + 1\big) \text{ for some } K = K(m, v_*)\Big\}.$$

Then we have the following blow-up result.

Theorem 1.2 *Suppose that* (1.11) *holds with* $0 < \alpha < 1 - \frac{2}{N}$ *and* $N \ge 3$*. For any given positive constants* m *and* $\overline{M}$*, and* $v_* \ge 0$*, there exist two positive constants* $K(m, v_*)$ *and* $T(m, \overline{M}, v_*)$ *such that for any* $(u_0, v_0) \in \mathcal{B}(m, \overline{M}, v_*)$*, the corresponding solution* (u, v) *to system* (1.8) *blows up at some finite time* $T_{\max} \in (0, \infty)$*, i.e.*

$$\limsup_{t \to T_{\max}} \|u(\cdot, t)\|_{L^\infty(B_R)} = \infty,$$

and the maximal time of existence satisfies $T_{\max} \le T(m, \overline{M}, v_*)$*.*

Remark 1.1 *For any* $m > 0$ *and* $v_* \ge 0$*, we can follow the proof of Theorem 1.2 in [5] to find* $\overline{M} > 0$ *such that the set* $\mathcal{B}(m, \overline{M}, v_*)$ *is nonempty (see also Lemma 6.1 in [43]).*

Remark 1.2 *Theorem 1.1 and Theorem 1.2 imply that in the radial setting* $\alpha_c := 1 - \frac{2}{N}$ *will be the critical blow-up exponent for system* (1.8) *with* $S(s) = K_S(1+s)^{-\alpha}$ *for some* $\alpha > 0$ *and* $K_S > 0$*.*

Remark 1.3 *In the two-dimensional setting, we can establish that* $\alpha > 0$ *is sufficient to ensure the global existence and boundedness of solutions to system* (1.8) *by following a similar proof strategy as in Section 3. However, the proof presented in Section 4 is not valid for the case* $\alpha < 0$ *(see Lemma 4.5). Consequently, it remains an open question whether* $\alpha_c = 1 - \frac{2}{N} = 0$ *for* $N = 2$ *serves as a critical blow-up exponent for system* (1.8)*, even within the radial framework.*

Main idea and some plans of the paper. In the scenario where $\alpha > 1 - \frac{2}{N}$, we first devise an approximating problem for system (1.8) (see (3.2) below) and then establish the existence of a globally bounded approximate solution $(u_\varepsilon, v_\varepsilon)$ via an iteration argument. Our pivotal focus lies in a coupled estimate for the time evolution of u_ε and $\nabla \widehat{v}_\varepsilon$ (Lemma 3.3 - Lemma 3.5). The global existence of bounded solutions to the original system (1.8) will be secured under additional *a priori* estimates along with a limit process.

On the other hand, for $0 < \alpha < 1 - \frac{2}{N}$ in a radial setting, we will adapt methods from [43, 5] to construct finite-time blowup solutions. We first establish that the functional $\mathcal{F}(u, v)$ defined as in (1.12) serves as a Lyapunov functional for (1.8) with a dissipation rate $\mathcal{D}$ (Lemma 4.1), meaning

$$\frac{d}{dt}\mathcal{F}(u(\cdot, t), v(\cdot, t)) = -\mathcal{D}(u(\cdot, t), v(\cdot, t)) \qquad \text{for all } t \in (0, T_{\max}).$$

The subsequent technical approach relies on estimating the dissipation rate $\mathcal{D}$ in terms of $\mathcal{F}$ (Lemma 4.8). After establishing an ODI for $-\mathcal{F}$ (Lemma 4.9), we directly obtain the desired blow-up result.

2 Preliminaries

In this section, we state some basic preliminaries and establish the local solvability of system (1.8). We begin with recalling two trace theorems.

Lemma 2.1 (Lemma 2.3 in [17]) *Let* $N \ge 2$ *and* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with smooth boundary and let* $r \in (0, \infty)$*. Then*

$$W^{r,2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$$

is a compact embedding.

Lemma 2.2 (Lemma 2.4 in [17]) *Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. If $r > 0$, then there exists a linear and bounded map from $W^{r+\frac{1}{2},2}(\Omega)$ onto $W^{r,2}(\partial\Omega)$.*

Then we present the following pointwise inequality for normal derivatives, which will be used to estimate the boundary integrals appearing in Lemma 3.3.

Lemma 2.3 (Lemma 3.4 in [2]) *Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary of class C^2, and let $\mathcal{K} \in \mathbb{R}$ denote the maximum of the curvatures on $\partial\Omega$. Then whenever $\omega \in C^2(\overline{\Omega})$ and $\omega_* \in \mathbb{R}$ are such that $\omega = \omega_*$ on $\partial\Omega$,*

$$\nabla|\nabla\omega|^2 \cdot \mathbf{n} \leq 2\Delta\omega\nabla\omega \cdot \mathbf{n} + 2\mathcal{K}|\nabla\omega \cdot \mathbf{n}|^2 \qquad \text{on} \quad \partial\Omega.$$

The following lemma gives the local well-posedness of system (1.8).

Lemma 2.4 *Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Then for each (u_0, v_0) fulfilling (1.10), there exist a maximal time of existence $T_{\max} \in (0, \infty]$ and a uniquely determined (u, v) of functions such that*

$$(u, v) \in C\big(\overline{\Omega} \times [0, T_{\max})\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, T_{\max})\big) \times C\big(\overline{\Omega} \times [0, T_{\max})\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, T_{\max})\big),$$

that (u, v) solves system (1.8) in the classical sense in $\Omega \times (0, T_{\max})$, and that $u > 0$ in $\overline{\Omega} \times (0, T_{\max})$, $v > 0$ in $\Omega \times (0, T_{\max})$ and $v \geq v_*$ in $\Omega \times (0, T_{\max})$. If $T_{\max} < \infty$, then

$$\lim_{t \nearrow T_{\max}} \big(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)}\big) = \infty.$$

Furthermore, we have

$$\int_\Omega u(\cdot, t) = \int_\Omega u_0 \qquad \text{for all} \quad t \in (0, T_{\max}). \tag{2.1}$$

Proof. The proof is based on a straightforward modification of the arguments in Proposition 2.1 of Fuhrmann-Lankeit-Winkler [9] after homogenizing the boundary conditions. For this purpose, we first set

$$\widehat{v}(x, t) := v(x, t) - v_*$$

and rewrite system (1.8) as its equivalent form

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot \big(uS(u)\nabla\widehat{v}\big), & (x, t) \in \Omega \times (0, \infty), \\ \partial_t \widehat{v} = \Delta\widehat{v} - \widehat{v} - v_* + u, & (x, t) \in \Omega \times (0, \infty), \\ (\nabla u - uS(u)\nabla\widehat{v}) \cdot \mathbf{n} = \widehat{v} = 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x),\ \widehat{v}(x, 0) = v_0(x) - v_*, & x \in \Omega. \end{cases} \tag{2.2}$$

Then the statements concerning local existence, uniqueness and regularity of solution $(u, \widehat{v})$ to system (2.2) as well as extensibility criterion can be proved by a standard contraction argument established in [9]. Therefore, transferring back to the variable $v = \widehat{v} + v_*$, we obtain the corresponding properties for system (1.8).

The positivity of u and v as well as its comparison with v_* follows from an application of the strong maximum principle to system (1.8) due to $v_* \geq 0$.

Finally, the mass conservation (2.1) can be deduced from upon integrating the first equation in (1.8) over Ω. $\square$

We end this section by stating a quite basic but important property of the second component v.

Lemma 2.5 *Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. If the initial data (u_0, v_0) fulfills (1.10) and $v_* \geq 0$ is a constant, then the classical solution (u, v) satisfies that*

$$\int_\Omega v(\cdot, t) \leq M := \max\Big\{\int_\Omega v_0, \int_\Omega u_0\Big\} \qquad \text{for all} \quad t \in \big(0, T_{\max,\varepsilon}\big). \tag{2.3}$$

Proof. The fact $v(x,t) = v_*$ on $\partial\Omega \times (0, T_{\max})$ and $v(x,t) \geq v_*$ in $\Omega \times (0, T_{\max})$ imply that $\nabla v = -|\nabla v|\mathbf{n}$ on $\partial\Omega \times (0, T_{\max})$ and thus $\nabla v \cdot \mathbf{n} = -|\nabla v|$. Then by integrating the second equation in (1.8) over Ω, we obtain

$$\frac{d}{dt}\int_\Omega v = \int_{\partial\Omega} \nabla v \cdot \mathbf{n} - \int_\Omega v + \int_\Omega u = -\int_{\partial\Omega} |\nabla v| - \int_\Omega v + \int_\Omega u \leq -\int_\Omega v + \int_\Omega u_0$$

for all $t \in (0, T_{\max})$. Therefore, we can employ a direct comparison argument to conclude that indeed (2.3) holds. $\square$

3 Global existence and uniform boundedness

This section investigates the global existence and uniform boundedness of solutions to system (1.8). The establishment of the main result will be based on a series of *a priori* estimates of approximate solutions to a suitable regularized system.

3.1 Construction of regularized solutions

Let $\{\eta_\varepsilon\}_{\varepsilon\in(0,1)} \subset C_0^\infty(\Omega)$ be a family of standard cut-off functions fulfilling $0 \leq \eta_\varepsilon \leq 1$ in Ω for all $\varepsilon \in (0,1)$ and $\eta_\varepsilon \nearrow 1$ in Ω pointwisely as $\varepsilon \searrow 0$, and define

$$S_\varepsilon(u) = \eta_\varepsilon(x) S(u), \qquad (x,u) \in \overline{\Omega} \times [0,\infty)$$

for each $\varepsilon \in (0,1)$. Then we see that S_ε vanishes on $\partial\Omega$ and satisfies

$$\big|S_\varepsilon(u)\big| \leq K_S (1+u)^{-\alpha} \tag{3.1}$$

due to (1.9). In order to construct global solutions to system (1.8), we simplify its boundary conditions via the above cur-off functions and consider the regularized system

$$\begin{cases} \partial_t u_\varepsilon = \Delta u_\varepsilon - \nabla \cdot \big(u_\varepsilon S_\varepsilon(u_\varepsilon)\nabla \widehat{v}_\varepsilon\big), & (x,t) \in \Omega \times (0,\infty), \\ \partial_t \widehat{v}_\varepsilon = \Delta \widehat{v}_\varepsilon - \widehat{v}_\varepsilon - v_* + u_\varepsilon, & (x,t) \in \Omega \times (0,\infty), \\ \nabla u_\varepsilon \cdot \mathbf{n} = \widehat{v}_\varepsilon = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u_\varepsilon(x,0) = u_0(x), \quad \widehat{v}_\varepsilon(x,0) = v_0(x) - v_*, & x \in \Omega. \end{cases} \tag{3.2}$$

Then for each $\varepsilon \in (0,1)$, the following local solvability will be a direct consequence of Lemma 2.4.

Lemma 3.1 *Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Suppose that (u_0, v_0) satisfies (1.10), and that (3.1) holds with some $\alpha > 0$. Then for each fixed $\varepsilon \in (0,1)$, there exist $T_{\max,\varepsilon} \in (0,\infty]$ and functions*

$$\begin{cases} u_\varepsilon \in C\big(\overline{\Omega} \times [0, T_{\max,\varepsilon})\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, T_{\max,\varepsilon})\big), \\ \widehat{v}_\varepsilon \in C\big(\overline{\Omega} \times [0, T_{\max,\varepsilon})\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, T_{\max,\varepsilon})\big) \end{cases}$$

such that $u_\varepsilon > 0$ and $\widehat{v}_\varepsilon + v_* \geq 0$ in $\overline{\Omega} \times (0, T_{\max,\varepsilon})$, that $(u_\varepsilon, \widehat{v}_\varepsilon)$ solves system (3.2) in $\Omega \times (0, T_{\max,\varepsilon})$ in the classical sense, and that if $T_{\max,\varepsilon} < \infty$, then

$$\lim_{t \nearrow T_{\max,\varepsilon}} \Big(\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|\widehat{v}_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \Big) = +\infty.$$

Furthermore, the following mass conservation for u_ε holds:

$$\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0 \qquad \text{for all} \quad t \in \big(0, T_{\max,\varepsilon}\big). \tag{3.3}$$

3.2 *A priori* estimates to the regularized problems

In this subsection, we will devote ourselves to deriving some *a priori* estimates for system (3.2), which will eventually result in the global existence and uniform boundedness of $(u_\varepsilon, \widehat{v}_\varepsilon)$. We begin with showing an estimate for $\widehat{v}_\varepsilon$ from a supposedly present appropriate boundedness property of u_ε.

Lemma 3.2 *Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Assume that $(u_\varepsilon, \widehat{v}_\varepsilon)$ is a solution to system (3.2) in $\Omega \times (0, T_{\max,\varepsilon})$. If*

$$\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq L \qquad \text{for all} \quad t \in (0, T_{\max,\varepsilon})$$

for some $p \geq 1$ and positive constant L, then for all $q \geq 1$ satisfying

$$\begin{cases} q \in \Big[1, \dfrac{Np}{N-p}\Big) & \text{if} \quad p \leq N, \\ q \in [1, \infty] & \text{if} \quad p > N, \end{cases}$$

there exists a positive constant $C = C(p, q, L)$ such that

$$\|\widehat{v}_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} \leq C \qquad \text{for all} \quad t \in (0, T_{\max,\varepsilon}). \tag{3.4}$$

Proof. Due to

$$\frac{1}{2} + \frac{N}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big) < 1,$$

we can apply the gradient estimate of the Dirichlet heat semigroup (see, e.g., Lemma 2.4 in [10]) to the second equation in (3.2) to find some positive constant C_1 fulfilling

$$\|\nabla\widehat{v}_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \leq C_1\Big(1 + \sup_{s\in(0,t)} \|v_* - u_\varepsilon(\cdot,s)\|_{L^p(\Omega)}\Big) \leq C_1\,(1 + v_* + L) \quad \text{for all } t \in (0, T_{\max,\varepsilon}).$$

Since $\widehat{v}_\varepsilon = 0$ on $\partial\Omega$, we can also infer from the Poincaré inequality that

$$\|\widehat{v}_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \leq C_2\|\nabla\widehat{v}_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \leq C_1C_2\,(1 + v_* + L)$$

with some $C_2 > 0$. Combining the above two inequalities, we obtain (3.4) immediately. □

We next derive a coupled estimate involving the time evolution of $\|u_\varepsilon\|_{L^p}$ and $\|\nabla\widehat{v}_\varepsilon\|_{L^{2q}}$. The coupled estimate of this type has been widely used in establishing the global existence of the pure no-flux initial-boundary value problem of KS system (see e.g. [31, 35, 36]). In our current setting, we need some new tricks to deal with additional integrals on the boundary arising from the Dirichlet signal boundary.

Lemma 3.3 *Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Assume that $(u_\varepsilon, \widehat{v}_\varepsilon)$ is a solution to system (3.2) in $\Omega \times (0, T_{\max,\varepsilon})$. Then for any $p \geq 1$ and $q \geq 1$, there exists some positive constant $\widetilde{C}$ such that*

$$\begin{aligned} &\frac{d}{dt}\Big(\int_\Omega u_\varepsilon^p(\cdot,t) + \frac{1}{q}\int_\Omega |\nabla\widehat{v}_\varepsilon(\cdot,t)|^{2q}\Big) + \frac{3(p-1)}{p}\int_\Omega \big|\nabla u_\varepsilon^{\frac{p}{2}}(\cdot,t)\big|^2 + \frac{q-1}{q^2}\int_\Omega |\nabla|\nabla\widehat{v}_\varepsilon(\cdot,t)|^q|^2 \\ &\leq \widetilde{C}\int_\Omega u_\varepsilon^{p-2\alpha}(\cdot,t)|\nabla\widehat{v}_\varepsilon(\cdot,t)|^2 + \widetilde{C}\int_\Omega u_\varepsilon^2(\cdot,t)|\nabla\widehat{v}_\varepsilon(\cdot,t)|^{2q-2} + \widetilde{C}\int_\Omega |\nabla\widehat{v}_\varepsilon(\cdot,t)|^{2q} + \widetilde{C} \end{aligned} \tag{3.5}$$

for all $t \in (0, T_{\max,\varepsilon})$.

Proof. Noting that $S_\varepsilon(u_\varepsilon) = 0$ on $\partial\Omega \times (0,\infty)$ due to $\eta_\varepsilon \in C_0^\infty(\Omega)$, we can test the first equation in (3.2) by u_ε^{p-1} and then use the Young inequality to obtain that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u_\varepsilon^p + (p-1)\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 = (p-1)\int_\Omega u_\varepsilon^{p-1}S_\varepsilon(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\widehat{v}_\varepsilon$$

$$\leq \frac{p-1}{4}\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + (p-1)\int_\Omega u_\varepsilon^p |S_\varepsilon(u_\varepsilon)|^2|\nabla \widehat{v}_\varepsilon|^2$$

for all $t \in (0, T_{\max,\varepsilon})$, which implies that

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{3(p-1)}{p}\int_\Omega \big|\nabla u_\varepsilon^{\frac{p}{2}}\big|^2 \leq p(p-1)K_S^2\int_\Omega u_\varepsilon^{p-2\alpha}|\nabla \widehat{v}_\varepsilon|^2 \tag{3.6}$$

for all $t \in (0, T_{\max,\varepsilon})$ thanks to (3.1).

On the other hand, by applying ∇ to the second equation of (3.2) and multiplying the resulting equation by $2\nabla\widehat{v}_\varepsilon$ on both sides, we have

$$\partial_t\big(|\nabla\widehat{v}_\varepsilon|^2\big) = 2\nabla\widehat{v}_\varepsilon\cdot\nabla\Delta\widehat{v}_\varepsilon - 2|\nabla\widehat{v}_\varepsilon|^2 + 2\nabla\widehat{v}_\varepsilon\cdot\nabla u_\varepsilon,$$

which together with the identity $2\nabla\widehat{v}_\varepsilon\cdot\nabla\Delta\widehat{v}_\varepsilon = \Delta|\nabla\widehat{v}_\varepsilon|^2 - 2|D^2\widehat{v}_\varepsilon|^2$ yields that

$$\partial_t\big(|\nabla\widehat{v}_\varepsilon|^2\big) + 2|D^2\widehat{v}_\varepsilon|^2 + 2|\nabla\widehat{v}_\varepsilon|^2 = \Delta|\nabla\widehat{v}_\varepsilon|^2 + 2\nabla\widehat{v}_\varepsilon\cdot\nabla u_\varepsilon \tag{3.7}$$

for all $t \in (0, T_{\max,\varepsilon})$. We then test (3.7) by $|\nabla\widehat{v}_\varepsilon|^{2q-2}$ and integrate by parts over Ω to deduce that

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\int_\Omega|\nabla\widehat{v}_\varepsilon|^{2q} + (q-1)\int_\Omega|\nabla\widehat{v}_\varepsilon|^{2q-4}\big|\nabla|\nabla\widehat{v}_\varepsilon|^2\big|^2 + 2\int_\Omega|\nabla\widehat{v}_\varepsilon|^{2q-2}|D^2\widehat{v}_\varepsilon|^2 + 2\int_\Omega|\nabla\widehat{v}_\varepsilon|^{2q}\\
&= \int_{\partial\Omega}|\nabla\widehat{v}_\varepsilon|^{2q-2}\nabla|\nabla\widehat{v}_\varepsilon|^2\cdot\mathbf{n} + 2\int_\Omega|\nabla\widehat{v}_\varepsilon|^{2q-2}\nabla\widehat{v}_\varepsilon\cdot\nabla u_\varepsilon\\
&= \int_{\partial\Omega}|\nabla\widehat{v}_\varepsilon|^{2q-2}\nabla|\nabla\widehat{v}_\varepsilon|^2\cdot\mathbf{n} + 2\int_{\partial\Omega}u_\varepsilon|\nabla\widehat{v}_\varepsilon|^{2q-2}\nabla\widehat{v}_\varepsilon\cdot\mathbf{n}\\
&\quad - 2\int_\Omega u_\varepsilon|\nabla\widehat{v}_\varepsilon|^{2q-2}\Delta\widehat{v}_\varepsilon - 2(q-1)\int_\Omega u_\varepsilon|\nabla\widehat{v}_\varepsilon|^{2q-4}\nabla\widehat{v}_\varepsilon\cdot\nabla|\nabla\widehat{v}_\varepsilon|^2
\end{aligned}\tag{3.8}$$

for all $t \in (0, T_{\max,\varepsilon})$. For the last two terms on the right hand side of (3.8), it follows from the Young inequality and the pointwise inequality $|\Delta\widehat{v}_\varepsilon|^2 \leq N|D^2\widehat{v}_\varepsilon|^2$ that

$$\begin{aligned}
&-2\int_\Omega u_\varepsilon|\nabla\widehat{v}_\varepsilon|^{2q-2}\Delta\widehat{v}_\varepsilon - 2(q-1)\int_\Omega u_\varepsilon|\nabla\widehat{v}_\varepsilon|^{2q-4}\nabla\widehat{v}_\varepsilon\cdot\nabla|\nabla\widehat{v}_\varepsilon|^2\\
&\leq \frac{1}{N}\int_\Omega|\nabla\widehat{v}_\varepsilon|^{2q-2}|\Delta\widehat{v}_\varepsilon|^2 + N\int_\Omega u_\varepsilon^2|\nabla\widehat{v}_\varepsilon|^{2q-2} + 4(q-1)\int_\Omega u_\varepsilon^2|\nabla\widehat{v}_\varepsilon|^{2q-2}\\
&\quad + \frac{q-1}{4}\int_\Omega|\nabla\widehat{v}_\varepsilon|^{2q-4}\big|\nabla|\nabla\widehat{v}_\varepsilon|^2\big|^2\\
&\leq \int_\Omega|\nabla\widehat{v}_\varepsilon|^{2q-2}|D^2\widehat{v}_\varepsilon|^2 + \big(4(q-1)+N\big)\int_\Omega u_\varepsilon^2|\nabla\widehat{v}_\varepsilon|^{2q-2} + \frac{q-1}{4}\int_\Omega|\nabla\widehat{v}_\varepsilon|^{2q-4}\big|\nabla|\nabla\widehat{v}_\varepsilon|^2\big|^2.
\end{aligned}\tag{3.9}$$

We now deal with the boundary integrals on the right hand side of (3.8). Due to the homogeneous Dirichlet boundary condition for $\widehat{v}_\varepsilon$, we have

$$\Delta\widehat{v}_\varepsilon = \partial_t\widehat{v}_\varepsilon + \widehat{v}_\varepsilon + v_* - u_\varepsilon = v_* - u_\varepsilon \qquad \text{on} \quad \partial\Omega\times(0,T_{\max,\varepsilon}),$$

which together with Lemma 2.3 entails that

$$\nabla|\nabla\widehat{v}_\varepsilon|^2\cdot\mathbf{n} \leq 2\Delta\widehat{v}_\varepsilon\nabla\widehat{v}_\varepsilon\cdot\mathbf{n} + 2\mathcal{K}|\nabla\widehat{v}_\varepsilon\cdot\mathbf{n}|^2 = 2(v_*-u_\varepsilon)\nabla\widehat{v}_\varepsilon\cdot\mathbf{n} + 2\mathcal{K}|\nabla\widehat{v}_\varepsilon\cdot\mathbf{n}|^2$$

and thus that

$$\begin{aligned}
&\int_{\partial\Omega}|\nabla\widehat{v}_\varepsilon|^{2q-2}\nabla|\nabla\widehat{v}_\varepsilon|^2\cdot\mathbf{n} + 2\int_{\partial\Omega}u_\varepsilon|\nabla\widehat{v}_\varepsilon|^{2q-2}\nabla\widehat{v}_\varepsilon\cdot\mathbf{n}\\
&\leq 2v_*\int_{\partial\Omega}|\nabla\widehat{v}_\varepsilon|^{2q-1} + 2\mathcal{K}\int_{\partial\Omega}|\nabla\widehat{v}_\varepsilon|^{2q}\\
&\leq C_1\int_{\partial\Omega}|\nabla\widehat{v}_\varepsilon|^{2q} + C_1 \qquad \text{on} \quad \partial\Omega\times(0,T_{\max,\varepsilon})
\end{aligned}\tag{3.10}$$

with some positive constant C_1. Noticing that Lemma 2.1 and Lemma 2.2 yields

$$C_1 \int_{\partial\Omega} |\nabla \widehat{v}_\varepsilon|^{2q} \le C_2 \big\| |\nabla \widehat{v}_\varepsilon|^q \big\|^2_{W^{\frac{3}{4},2}(\Omega)} \le \frac{q-1}{q^2} \big\| \nabla |\nabla \widehat{v}_\varepsilon|^q \big\|^2_{L^2(\Omega)} + C_3 \big\| |\nabla \widehat{v}_\varepsilon|^q \big\|^2_{L^2(\Omega)} \tag{3.11}$$

with some positive constants C_2 and C_3, we can substitute (3.9)-(3.11) into (3.8) to find some positive constant C_4 such that

$$\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla \widehat{v}_\varepsilon|^{2q} + \frac{q-1}{q^2}\int_\Omega |\nabla|\nabla \widehat{v}_\varepsilon|^q|^2 + 2\int_\Omega |\nabla \widehat{v}_\varepsilon|^{2q} \le C_4 \int_\Omega u_\varepsilon^2 |\nabla \widehat{v}_\varepsilon|^{2q-2} + C_4 \int_\Omega |\nabla \widehat{v}_\varepsilon|^{2q} + C_4$$

for all $t \in (0, T_{\max,\varepsilon})$. This inequality together with (3.6) yields (3.5). □

To close the estimate in Lemma 3.3, we need to choose some parameters appropriately.

Lemma 3.4 *Let $N \ge 3$, $\alpha > 1 - \frac{2}{N}$, $\bar{p} \ge 1$ and $\bar{q} \ge 2$. Then there exist parameters $p \ge \max\{\bar{p}, N\}$, $q \ge \bar{q}$, $1 \le s < \frac{N}{N-1}$, $r \in \left(1, \frac{N}{N-2}\right)$ and $b > \frac{N}{2}$ such that*

$$p > 2\alpha + 1, \tag{3.12}$$

$$\frac{Nq}{Nq - N + 2} < r < \frac{Np}{(N-2)(p-2\alpha)}, \tag{3.13}$$

$$\frac{Nq}{2q + N - 2} < b < \frac{Np}{2(N-2)} \tag{3.14}$$

and

$$\frac{p - 2\alpha - \frac{1}{r}}{1 - \frac{N}{2} + \frac{pN}{2}} + \frac{\frac{2}{s} - 1 + \frac{1}{r}}{1 - \frac{N}{2} + \frac{Nq}{s}} < \frac{2}{N}, \tag{3.15}$$

as well as

$$\frac{2 - \frac{1}{b}}{1 - \frac{N}{2} + \frac{pN}{2}} + \frac{\frac{2(q-1)}{s} - 1 + \frac{1}{b}}{1 - \frac{N}{2} + \frac{Nq}{s}} < \frac{2}{N}. \tag{3.16}$$

Proof. Let

$$q_0(p) := \frac{Np}{2(N-1)} \qquad \text{for} \quad p \ge 1.$$

We first fix $r \in \left(1, \frac{N}{N-2}\right)$ and $b > \frac{N}{2}$. Then we can find some large $p \ge \max\{\bar{p}, N\}$ satisfying

$$q_0(p) > \bar{q}, \tag{3.17}$$

and

$$p > 2\alpha + 1$$

such that

$$\frac{Nq_0(p)}{Nq_0(p) - N + 2} < r < \frac{Np}{(N-2)(p-2\alpha)} \tag{3.18}$$

and

$$\frac{Nq_0(p)}{2q_0(p) + N - 2} < b < \frac{Np}{2(N-2)}. \tag{3.19}$$

Indeed, thanks to $r < \frac{N}{N-2}$ and b being fixed, we have

$$r < \frac{Np}{(N-2)(p-2\alpha)} \qquad \text{and} \qquad b < \frac{Np}{2(N-2)} \tag{3.20}$$

for all sufficiently large p. According to the definition of $q_0(p)$, it is clear that $q_0(p) \to \infty$ as $p \to \infty$, which combined with $r > 1$ and $b > \frac{N}{2}$ yields that

$$r > \frac{Nq_0(p)}{Nq_0(p) - N + 2} \qquad \text{and} \qquad b > \frac{Nq_0(p)}{2q_0(p) + N - 2} \tag{3.21}$$

for all sufficiently large p. It then follows from (3.20) and (3.21) that (3.18) and (3.19) are valid for sufficiently large p.

On the other hand, we define

$$h_1(q,s) := \frac{p - 2\alpha - \frac{1}{r}}{1 - \frac{N}{2} + \frac{pN}{2}} + \frac{\frac{2}{s} - 1 + \frac{1}{r}}{1 - \frac{N}{2} + \frac{Nq}{s}} \qquad \text{for} \quad (q,s) \in [2, +\infty) \times \Big[1, \frac{N}{N-1}\Big]$$

and

$$h_2(q,s) := \frac{2 - \frac{1}{b}}{1 - \frac{N}{2} + \frac{pN}{2}} + \frac{\frac{2(q-1)}{s} - 1 + \frac{1}{b}}{1 - \frac{N}{2} + \frac{Nq}{s}} \qquad \text{for} \quad (q,s) \in [2, +\infty) \times \Big[1, \frac{N}{N-1}\Big]$$

It follows from $\alpha > 1 - \frac{2}{N}$ that

$$h_1\Big(q_0(p), \frac{N}{N-1}\Big) = \frac{p - 2\alpha - \frac{1}{r}}{1 - \frac{N}{2} + \frac{pN}{2}} + \frac{1 - \frac{2}{N} + \frac{1}{r}}{1 - \frac{N}{2} + \frac{pN}{2}} = \frac{p - 2\alpha + 1 - \frac{2}{N}}{1 - \frac{N}{2} + \frac{pN}{2}} < \frac{p - 1 + \frac{2}{N}}{1 - \frac{N}{2} + \frac{pN}{2}} = \frac{2}{N}.$$

By a continuity argument and (3.17), we can choose q close enough to $q_0(p)$ such that

$$\bar{q} < q < q_0(p) \tag{3.22}$$

and

$$h_1\Big(q, \frac{N}{N-1}\Big) < \frac{2}{N}, \tag{3.23}$$

and that both (3.13) and (3.14) hold thanks to (3.18) and (3.19).

By using a continuity argument again, we can also see from (3.23) that for $s \in \big[1, \frac{N}{N-1}\big)$ but closing enough to $\frac{N}{N-1}$, it holds that

$$h_1(q,s) < \frac{2}{N}, \tag{3.24}$$

which implies that (3.15) is valid.

Finally, the facts

$$h_2\Big(q_0(p), \frac{N}{N-1}\Big) = \frac{2 - \frac{1}{b}}{1 - \frac{N}{2} + \frac{pN}{2}} + \frac{p - 3 + \frac{2}{N} + \frac{1}{b}}{1 - \frac{N}{2} + \frac{pN}{2}} = \frac{p + \frac{2}{N} - 1}{1 - \frac{N}{2} + \frac{pN}{2}} = \frac{2}{N}$$

and

$$\begin{aligned} \frac{\partial h_2}{\partial q}\Big(q, \frac{N}{N-1}\Big) &= \frac{\frac{2(N-1)}{N}\left(1 - \frac{N}{2} + (N-1)q\right) - \left(\frac{2(N-1)(q-1)}{N} - 1 + \frac{1}{b}\right)(N-1)}{(1 - \frac{N}{2} + (N-1)q)^2} \\ &= \frac{(N-1)(2 - \frac{1}{b})}{(1 - \frac{N}{2} + (N-1)q)^2} > 0 \qquad \text{for} \quad q \in [2, +\infty) \end{aligned}$$

imply that

$$h_2\Big(q, \frac{N}{N-1}\Big) < \frac{2}{N} \qquad \text{for} \quad q \in \big(2, q_0(p)\big).$$

In particular, for q determined by (3.22) and (3.23), it also holds that

$$h_2\Big(q, \frac{N}{N-1}\Big) < \frac{2}{N}.$$

Thus for s determined by (3.24), it holds that

$$h_2(q,s) < \frac{2}{N},$$

which implies that (3.16) holds. This completes the proof of Lemma 3.4. $\square$

With Lemma 3.4 at hand, we can now choose appropriate parameters to close the coupled estimates in Lemma 3.3.

Lemma 3.5 *Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Assume that $(u_\varepsilon, \widehat{v}_\varepsilon)$ is a solution to system (3.2) in $\Omega \times (0, T_{\max,\varepsilon})$. Let (3.1) hold with some $\alpha > 1 - \frac{2}{N}$. Then for all $(p,q) \in [1,+\infty) \times [1,+\infty)$, there exists some positive constant C such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq C \qquad \text{for all} \quad t \in (0, T_{\max,\varepsilon}) \tag{3.25}$$

and

$$\|\nabla \widehat{v}_\varepsilon(\cdot,t)\|_{L^{2q}(\Omega)} \leq C \qquad \text{for all} \quad t \in (0, T_{\max,\varepsilon}). \tag{3.26}$$

Proof. For any $p_0 > 1$ and $q_0 > 2$, we claim that there exist $p > p_0$ and $q > q_0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq C_1 \qquad \text{for all} \quad t \in (0, T_{\max,\varepsilon}) \tag{3.27}$$

and

$$\|\nabla \widehat{v}_\varepsilon(\cdot,t)\|_{L^{2q}(\Omega)} \leq C_1 \qquad \text{for all} \quad t \in (0, T_{\max,\varepsilon}) \tag{3.28}$$

for some positive constant C_1. Indeed, to achieve this, we let

$$\bar{p} := p_0, \qquad \bar{q} := q_0,$$

and then fix $p > \bar{p} \geq 1$, $q > \bar{q} \geq 2$, $1 \leq s < \frac{N}{N-1}$, $r > 1$ and $b > 1$ as provided by Lemma 3.4.

For the first and second integral on the right-hand side of (3.5), we apply the Hölder inequality to obtain that

$$\int_\Omega u_\varepsilon^{p-2\alpha} |\nabla \widehat{v}_\varepsilon|^2 \leq \Big(\int_\Omega u_\varepsilon^{r(p-2\alpha)}\Big)^{\frac{1}{r}} \Big(\int_\Omega |\nabla \widehat{v}_\varepsilon|^{\frac{2r}{r-1}}\Big)^{\frac{r-1}{r}} \tag{3.29}$$

and

$$\int_\Omega u_\varepsilon^2 |\nabla \widehat{v}_\varepsilon|^{2q-2} \leq \Big(\int_\Omega u_\varepsilon^{2b}\Big)^{\frac{1}{b}} \Big(\int_\Omega |\nabla \widehat{v}_\varepsilon|^{\frac{2(q-1)b}{b-1}}\Big)^{\frac{b-1}{b}}. \tag{3.30}$$

By using the Gagliardo-Nirenberg inequality and the mass conservation (3.3), we see

$$\begin{aligned}
\Big(\int_\Omega u_\varepsilon^{r(p-2\alpha)}\Big)^{\frac{1}{r}} &= \big\|u_\varepsilon^{\frac{p}{2}}\big\|_{L^{\frac{2r(p-2\alpha)}{p}}(\Omega)}^{\frac{2(p-2\alpha)}{p}} \\
&\leq C_1\Big(\big\|\nabla u_\varepsilon^{\frac{p}{2}}\big\|_{L^2(\Omega)}^{\beta_1} \big\|u_\varepsilon^{\frac{p}{2}}\big\|_{L^{\frac{2}{p}}(\Omega)}^{1-\beta_1} + \big\|u_\varepsilon^{\frac{p}{2}}\big\|_{L^{\frac{2}{p}}(\Omega)}\Big)^{\frac{2(p-2\alpha)}{p}} \\
&= C_1\Big(\big\|\nabla u_\varepsilon^{\frac{p}{2}}\big\|_{L^2(\Omega)}^{\beta_1} \|u_0\|_{L^1(\Omega)}^{\frac{p}{2}(1-\beta_1)} + \|u_0\|_{L^1(\Omega)}^{\frac{p}{2}}\Big)^{\frac{2(p-2\alpha)}{p}} \\
&\leq C_2\Big(\big\|\nabla u_\varepsilon^{\frac{p}{2}}\big\|_{L^2(\Omega)}^{\beta_1} + 1\Big)^{\frac{2(p-2\alpha)}{p}}
\end{aligned}$$

$$\leq C_2\Big(\big\|\nabla u_\varepsilon^{\frac{p}{2}}\big\|_{L^2(\Omega)}^{\frac{2\beta_1(p-2\alpha)}{p}}+1\Big) \tag{3.31}$$

with some positive constants C_1 and C_2, where

$$\beta_1 := \frac{pN - \frac{pN}{r(p-2\alpha)}}{pN - N + 2} \in (0,1)$$

due to $(p-2\alpha)r > 1$ from $r > 1$ and (3.12), and $r < \frac{pN}{(N-2)(p-2\alpha)}$ from (3.13). Similarly, we have

$$\begin{aligned}
\Big(\int_\Omega u_\varepsilon^{2b}\Big)^{\frac{1}{b}} &= \big\|u_\varepsilon^{\frac{p}{2}}\big\|_{L^{\frac{4b}{p}}(\Omega)}^{\frac{4}{p}} \\
&\leq C_3\Big(\big\|\nabla u_\varepsilon^{\frac{p}{2}}\big\|_{L^2(\Omega)}^{\beta_2}\big\|u_\varepsilon^{\frac{p}{2}}\big\|_{L^{\frac{2}{p}}(\Omega)}^{1-\beta_2}+\big\|u_\varepsilon^{\frac{p}{2}}\big\|_{L^{\frac{2}{p}}(\Omega)}\Big)^{\frac{4}{p}} \\
&= C_3\Big(\big\|\nabla u_\varepsilon^{\frac{p}{2}}\big\|_{L^2(\Omega)}^{\beta_2}\|u_0\|_{L^1(\Omega)}^{\frac{p}{2}(1-\beta_2)}+\|u_0\|_{L^1(\Omega)}^{\frac{p}{2}}\Big)^{\frac{4}{p}} \\
&\leq C_4\Big(\big\|\nabla u_\varepsilon^{\frac{p}{2}}\big\|_{L^2(\Omega)}^{\frac{4\beta_2}{p}}+1\Big)
\end{aligned} \tag{3.32}$$

for some positive constants C_3 and C_4, where

$$\beta_2 := \frac{pN - \frac{pN}{2b}}{pN - N + 2} \in (0,1)$$

due to $b > 1$ and the fact $b < \frac{Np}{2(N-2)}$ from (3.14).

Next, we estimate the terms involving $\nabla\widehat{v}_\varepsilon$ on the right-hand side of (3.29) and (3.30). From Lemma 3.2 and the mass conservation (3.3), we know that there exists a positive constant C_5 such that

$$\|\widehat{v}_\varepsilon\|_{W^{1,s}(\Omega)} \leq C_5 \tag{3.33}$$

for any $1 \leq s < \frac{N}{N-1} < 2$ and thus we can employ the Gagliardo-Nirenberg inequality to find some positive constants C_6 and C_7 such that

$$\begin{aligned}
\Big(\int_\Omega |\nabla\widehat{v}_\varepsilon|^{\frac{2r}{r-1}}\Big)^{\frac{r-1}{r}} &= \big\||\nabla\widehat{v}_\varepsilon|^q\big\|_{L^{\frac{2r}{q(r-1)}}(\Omega)}^{\frac{2}{q}} \\
&\leq C_6\Big(\big\|\nabla|\nabla\widehat{v}_\varepsilon|^q\big\|_{L^2(\Omega)}^{\beta_3}\big\||\nabla\widehat{v}_\varepsilon|^q\big\|_{L^{\frac{s}{q}}(\Omega)}^{1-\beta_3}+\big\||\nabla\widehat{v}_\varepsilon|^q\big\|_{L^{\frac{s}{q}}(\Omega)}\Big)^{\frac{2}{q}} \\
&\leq C_6\Big(C_5\big\|\nabla|\nabla\widehat{v}_\varepsilon|^q\big\|_{L^2(\Omega)}^{\beta_3}+C_5\Big)^{\frac{2}{q}} \\
&\leq C_7\Big(\big\|\nabla|\nabla\widehat{v}_\varepsilon|^q\big\|_{L^2(\Omega)}^{\frac{2\beta_3}{q}}+1\Big),
\end{aligned} \tag{3.34}$$

where

$$\beta_3 := \frac{Nq\big(\frac{1}{s}-\frac{r-1}{2r}\big)}{\frac{Nq}{s}+1-\frac{N}{2}} \in (0,1)$$

due to $r > 1$, $1 \leq s < 2$ and the fact $r > \frac{Nq}{Nq-N+2}$ from (3.13). Similarly, we have

$$\begin{aligned}
\Big(\int_\Omega |\nabla\widehat{v}_\varepsilon|^{\frac{2(q-1)b}{b-1}}\Big)^{\frac{b-1}{b}} &= \big\||\nabla\widehat{v}_\varepsilon|^q\big\|_{L^{\frac{2(q-1)b}{q(b-1)}}(\Omega)}^{\frac{2(q-1)}{q}} \\
&\leq C_8\Big(\big\|\nabla|\nabla\widehat{v}_\varepsilon|^q\big\|_{L^2(\Omega)}^{\beta_4}\big\||\nabla\widehat{v}_\varepsilon|^q\big\|_{L^{\frac{s}{q}}(\Omega)}^{1-\beta_4}+\big\||\nabla\widehat{v}_\varepsilon|^q\big\|_{L^{\frac{s}{q}}(\Omega)}\Big)^{\frac{2(q-1)}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\le C_8\Big(C_5\big\|\nabla|\nabla\widehat{v}_\varepsilon|^q\big\|_{L^2(\Omega)}^{\beta_4}+C_5\Big)^{\frac{2(q-1)}{q}}\\
&\le C_9\Big(\big\|\nabla|\nabla\widehat{v}_\varepsilon|^q\big\|_{L^2(\Omega)}^{\frac{2(q-1)\beta_4}{q}}+1\Big)
\end{aligned}\tag{3.35}
$$

for some positive constants C_8 and C_9, where

$$
\beta_4:=\frac{Nq\big(\frac{1}{s}-\frac{b-1}{2(q-1)b}\big)}{\frac{Nq}{s}+1-\frac{N}{2}}\in(0,1)
$$

due to $b>1$, $1\le s<2$, $q>\bar{q}\ge 2$ and the fact $b>\frac{Nq}{2q+N-2}$ from (3.14).

Collecting the estimates (3.29)-(3.35), and then using the Young inequality, we can find positive constants C_{10} and C_{11} such that

$$
\begin{aligned}
&\int_\Omega u_\varepsilon^{p-2\alpha}|\nabla\widehat{v}_\varepsilon|^2+\int_\Omega u_\varepsilon^2|\nabla\widehat{v}_\varepsilon|^{2q-2}\\
&\le C_2C_7\Big(\|\nabla u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2\beta_1(p-2\alpha)}{p}}+1\Big)\Big(\big\|\nabla|\nabla\widehat{v}_\varepsilon|^q\big\|_{L^2(\Omega)}^{\frac{2\beta_3}{q}}+1\Big)\\
&\quad+C_4C_9\Big(\|\nabla u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{4\beta_2}{p}}+1\Big)\Big(\big\|\nabla|\nabla\widehat{v}_\varepsilon|^q\big\|_{L^2(\Omega)}^{\frac{2(q-1)\beta_4}{q}}+1\Big)\\
&\le C_{10}\Big(\int_\Omega|\nabla u_\varepsilon^{\frac{p}{2}}|^2\Big)^{\frac{\beta_1(p-2\alpha)}{p}}\Big(\int_\Omega\big|\nabla|\nabla\widehat{v}_\varepsilon|^q\big|^2\Big)^{\frac{\beta_3}{q}}+C_{10}\Big(\int_\Omega|\nabla u_\varepsilon^{\frac{p}{2}}|^2\Big)^{\frac{2\beta_2}{p}}\Big(\int_\Omega\big|\nabla|\nabla\widehat{v}_\varepsilon|^q\big|^2\Big)^{\frac{(q-1)\beta_4}{q}}\\
&\quad+C_{10}\Big(\int_\Omega|\nabla u_\varepsilon^{\frac{p}{2}}|^2\Big)^{\frac{\beta_1(p-2\alpha)}{p}}+C_{10}\Big(\int_\Omega\big|\nabla|\nabla\widehat{v}_\varepsilon|^q\big|^2\Big)^{\frac{\beta_3}{q}}+C_{10}\Big(\int_\Omega|\nabla u_\varepsilon^{\frac{p}{2}}|^2\Big)^{\frac{2\beta_2}{p}}\\
&\quad+C_{10}\Big(\int_\Omega\big|\nabla|\nabla\widehat{v}_\varepsilon|^q\big|^2\Big)^{\frac{(q-1)\beta_4}{q}}+C_{10}\\
&\le\frac{p-1}{p\widetilde{C}}\int_\Omega|\nabla u_\varepsilon^{\frac{p}{2}}|^2+\frac{q-1}{2q^2\widetilde{C}}\int_\Omega\big|\nabla|\nabla\widehat{v}_\varepsilon|^q\big|^2+C_{11}
\end{aligned}\tag{3.36}
$$

with $\widetilde{C}$ as fixed in (3.5). Here we used the facts that

$$
\frac{\beta_1(p-2\alpha)}{p}+\frac{\beta_3}{q}=\frac{N}{2}\Big(\frac{p-2\alpha-\frac{1}{r}}{1-\frac{N}{2}+\frac{pN}{2}}+\frac{\frac{2}{s}-1+\frac{1}{r}}{1-\frac{N}{2}+\frac{Nq}{s}}\Big)<1
$$

and

$$
\frac{2\beta_2}{p}+\frac{(q-1)\beta_4}{q}=\frac{N}{2}\Big(\frac{2-\frac{1}{b}}{1-\frac{N}{2}+\frac{pN}{2}}+\frac{\frac{2(q-1)}{s}-1+\frac{1}{b}}{1-\frac{N}{2}+\frac{Nq}{s}}\Big)<1
$$

due to (3.15) and (3.16).

Combining (3.5) and (3.36), we can find positive constant C_{12} such that

$$
\frac{d}{dt}\Big(\int_\Omega u_\varepsilon^p+\frac{1}{q}\int_\Omega|\nabla\widehat{v}_\varepsilon|^{2q}\Big)+\frac{2(p-1)}{p}\int_\Omega\big|\nabla u_\varepsilon^{\frac{p}{2}}\big|^2+\frac{q-1}{2q^2}\int_\Omega\big|\nabla|\nabla\widehat{v}_\varepsilon|^q\big|^2\le C_{12}\int_\Omega|\nabla\widehat{v}_\varepsilon|^{2q}+C_{12}\tag{3.37}
$$

for all $t\in(0,T_{\max,\varepsilon})$. By the Gagliardo-Nirenberg inequality, (3.3) and the Young inequality, we have

$$
\begin{aligned}
\int_\Omega u_\varepsilon^p=\|u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2&\le C_{13}\Big(\|\nabla u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{2\beta_5}\|u_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{2(1-\beta_5)}+\|u_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^2\Big)\\
&=C_{13}\Big(\|\nabla u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{2\beta_5}\|u_0\|_{L^1(\Omega)}^{p(1-\beta_5)}+\|u_0\|_{L^1(\Omega)}^p\Big)\\
&\le C_{14}\Big(\|\nabla u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{2\beta_5}+1\Big)
\end{aligned}
$$

$$\leq \frac{p-1}{p}\int_\Omega \left|\nabla u_\varepsilon^{\frac{p}{2}}\right|^2 + C_{15} \tag{3.38}$$

for some positive constants C_{13}, C_{14} and C_{15}, where

$$\beta_5 := \frac{(p-1)N}{2+(p-1)N} \in (0,1)$$

due to $p > \bar{p} > 1$. Similarly, we can use (3.33) to find positive constants C_{16}, C_{17} and C_{18} such that

$$\begin{aligned} C_{12}\int_\Omega |\nabla\widehat{v}_\varepsilon|^{2q} = C_{12}\big\||\nabla\widehat{v}_\varepsilon|^q\big\|^2_{L^2(\Omega)} &\leq C_{16}\Big(\big\|\nabla|\nabla\widehat{v}_\varepsilon|^q\big\|^{2\beta_6}_{L^2(\Omega)}\big\||\nabla\widehat{v}_\varepsilon|^q\big\|^{2(1-\beta_6)}_{L^{\frac{s}{q}}(\Omega)} + \big\||\nabla\widehat{v}_\varepsilon|^q\big\|^2_{L^{\frac{s}{q}}(\Omega)}\Big) \\ &\leq C_{17}\Big(\big\|\nabla|\nabla\widehat{v}_\varepsilon|^q\big\|^{2\beta_6}_{L^2(\Omega)} + 1\Big) \\ &\leq \frac{q-1}{8q^2}\int_\Omega |\nabla|\nabla\widehat{v}_\varepsilon|^q|^2 + C_{18}, \end{aligned} \tag{3.39}$$

where

$$\beta_6 := \frac{\frac{2Nq}{s} - N}{2 - N + \frac{2Nq}{s}} \in (0,1)$$

due to $q > \bar{q} > 2$ and $s < \frac{N}{N-1} < 2$.

Substituting (3.38) and (3.39) into (3.37) and letting

$$y(t) := \int_\Omega u_\varepsilon^p(\cdot,t) + \frac{1}{q}\int_\Omega |\nabla\widehat{v}_\varepsilon(\cdot,t)|^{2q} \qquad \text{for all} \quad t\in(0,T_{\max,\varepsilon}),$$

we can find positive constants C_{19} and C_{20} such that

$$y'(t) + C_{19}y(t) \leq C_{20} \qquad \text{for all} \quad t\in(0,T_{\max,\varepsilon}).$$

Then an ODE comparison argument implies that

$$y(t) \leq \max\Big\{\int_\Omega u_0^p + \int_\Omega |\nabla v_0|^{2q},\, \frac{C_{20}}{C_{19}}\Big\} \qquad \text{for all} \quad t\in(0,T_{\max,\varepsilon}),$$

which entails (3.27) and (3.28) for some (p,q) satisfying $p > p_0$ and $q > q_0$.

For general $(p,q) \in [1,+\infty)\times[1,+\infty)$, (3.27) and (3.28) can be deduced from the arbitrariness and the Hölder inequality. This completes the proof of Lemma 3.5. □

Based on the coupled estimates at hand, we can establish the key L^∞-estimate of the component u_ε.

Lemma 3.6 *Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Assume that $(u_\varepsilon, \widehat{v}_\varepsilon)$ is a solution to system (3.2) in $\Omega\times(0,T_{\max,\varepsilon})$. Let (3.1) hold with some $\alpha > 1 - \frac{2}{N}$. Then there exists a positive constant C such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|\widehat{v}_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C \qquad \text{for all} \quad t\in(0,T_{\max,\varepsilon}).$$

Proof. In view of Lemma 3.5, we can first fix a $p > N$ and find some positive constant C_1 such that

$$\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq C_1 \qquad \text{for all} \quad t\in(0,T_{\max,\varepsilon}), \tag{3.40}$$

which combined with Lemma 3.2 implies that

$$\|\widehat{v}_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C_2 \qquad \text{for all} \quad t\in(0,T_{\max,\varepsilon}) \tag{3.41}$$

for some positive constant C_2.

Then applying the variation-of-constants formula to u_ε and using the smoothing estimate for the Neumann heat semigroup in Ω (see e.g. Lemma 1.3 in [42]), we can find some positive constant C_3 such that

$$\begin{aligned}
&\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\\
&\le \left\|e^{t\Delta}u_0\right\|_{L^\infty(\Omega)} + \int_0^t \left\|e^{(t-\tau)\Delta}\nabla\cdot\left(u_\varepsilon(\cdot,\tau)S_\varepsilon(u_\varepsilon)(\cdot,\tau)\nabla\widehat{v}_\varepsilon(\cdot,\tau)\right)\right\|_{L^\infty(\Omega)} d\tau\\
&\le \|u_0\|_{L^\infty(\Omega)} + C_3\int_0^t \left(1+(t-\tau)^{-\frac{1}{2}-\frac{N}{2p}}\right)e^{-\lambda_1(t-\tau)}\left\|u_\varepsilon(\cdot,\tau)S_\varepsilon(u_\varepsilon)(\cdot,\tau)\nabla\widehat{v}_\varepsilon(\cdot,\tau)\right\|_{L^p(\Omega)} d\tau \qquad (3.42)
\end{aligned}$$

for all $t\in(0,T_{\max,\varepsilon})$, where λ_1 is the first non-zero eigenvalue of $-\Delta$ in Ω under the homogeneous Neumann boundary conditions. Noticing (3.1), (3.40) and (3.41) imply that

$$\left\|u_\varepsilon S_\varepsilon(u_\varepsilon)\nabla\widehat{v}_\varepsilon\right\|_{L^p(\Omega)} \le K_S\|u_\varepsilon\nabla\widehat{v}_\varepsilon\|_{L^p(\Omega)} \le K_S\|u_\varepsilon\|_{L^p(\Omega)}\|\nabla\widehat{v}_\varepsilon\|_{L^\infty(\Omega)} \le K_S C_4$$

for some positive constant C_4, we see from the fact $p>N$ that

$$\begin{aligned}
&\int_0^t \left(1+(t-\tau)^{-\frac{1}{2}-\frac{N}{2p}}\right)e^{-\lambda_1(t-\tau)}\left\|u_\varepsilon(\cdot,\tau)S_\varepsilon(u_\varepsilon)(\cdot,\tau)\nabla\widehat{v}_\varepsilon(\cdot,\tau)\right\|_{L^p(\Omega)} d\tau\\
&\le K_S C_4\int_0^t \left(1+(t-\tau)^{-\frac{1}{2}-\frac{N}{2p}}\right)e^{-\lambda_1(t-\tau)} d\tau\\
&\le C_5
\end{aligned}$$

for some positive constant C_5. Therefore, we can conclude from (3.42) that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_6 \qquad \text{for all} \quad t\in(0,T_{\max,\varepsilon})$$

for some positive constant C_6. This together with (3.41) completes the proof of this lemma. □

We now end this subsection by stating the global existence and uniform boundedness of classical solutions to system (3.2), which is a direct consequence of the boundedness in Lemma 3.6 together with the blow-up criterion in Lemma 3.1.

Proposition 3.1 *Let $N\ge 3$ and $\Omega\subset\mathbb{R}^N$ be a bounded domain with smooth boundary and* (3.1) *hold for some $\alpha>1-\frac{2}{N}$. Then for any (u_0,v_0) fulfilling* (1.10)*, system* (3.2) *admits a global classical solution $(u_\varepsilon,\widehat{v}_\varepsilon)$, which is unique and uniformly bounded in the sense that*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|\widehat{v}_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \qquad \text{for all} \quad t\in(0,\infty) \qquad (3.43)$$

for some $C>0$ and any $\varepsilon\in(0,1)$.

3.3 Passing to the limit: Proof of Theorem 1.1.

In this section, we use an approximate procedure to construct the global bounded solution to system (1.8). For this purpose, we first combine the uniform bounds in (3.43) and the standard parabolic regularity theory to establish the estimates in Lemma 3.7-Lemma 3.9 below.

Lemma 3.7 *Suppose that the assumptions of Theorem 1.1 hold. Then for all $T>0$, there exists a positive constant $C(T)$ such that for all $\varepsilon\in(0,1)$, we have*

$$\int_0^T\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \le C(T). \qquad (3.44)$$

Proof. Multiplying the first equation in (3.2) by u_ε, integrating by parts over Ω, and then using (3.1), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_\varepsilon^2+\int_\Omega|\nabla u_\varepsilon|^2\le K_S\int_\Omega|u_\varepsilon||\nabla\widehat{v}_\varepsilon||\nabla u_\varepsilon|\le\frac{1}{2}\int_\Omega|\nabla u_\varepsilon|^2+\frac{K_S^2}{2}\int_\Omega u_\varepsilon^2|\nabla\widehat{v}_\varepsilon|^2$$

for all $t\in(0,\infty)$, which together with (3.43) entails that

$$\frac{d}{dt}\int_\Omega u_\varepsilon^2+\int_\Omega|\nabla u_\varepsilon|^2\le C\qquad\text{for all}\quad t\in(0,\infty)$$

for all $\varepsilon\in(0,1)$. By integrating the above inequality from 0 to T, we obtain (3.44). $\square$

Lemma 3.8 *Suppose that the assumptions of Theorem 1.1 hold. Then there exists a positive constant C such that for all $\varepsilon\in(0,1)$, we have*

$$\|\partial_t u_\varepsilon(\cdot,t)\|_{\left(W_0^{2,2}(\Omega)\right)^*}\le C\qquad\text{for all}\quad t\in(0,\infty). \tag{3.45}$$

Proof. For any given $\phi\in W_0^{2,2}(\Omega)$, we deduce from the first equation in (3.2) that

$$\int_\Omega\partial_t u_\varepsilon\phi=-\int_\Omega\nabla u_\varepsilon\cdot\nabla\phi+\int_\Omega(u_\varepsilon S_\varepsilon(u_\varepsilon)\nabla\widehat{v}_\varepsilon)\cdot\nabla\phi=\int_\Omega u_\varepsilon\Delta\phi+\int_\Omega(u_\varepsilon S_\varepsilon(u_\varepsilon)\nabla\widehat{v}_\varepsilon)\cdot\nabla\phi$$

for all $t\in(0,\infty)$ and $\varepsilon\in(0,1)$. By using the Hölder inequality and (3.43), we obtain

$$\begin{aligned}\left|\int_\Omega\partial_t u_\varepsilon\phi\right|&\le\|u_\varepsilon\|_{L^\infty(\Omega)}\|\Delta\phi\|_{L^1(\Omega)}+K_S\|u_\varepsilon\|_{L^\infty(\Omega)}\|\nabla\widehat{v}_\varepsilon\|_{L^\infty(\Omega)}\|\nabla\phi\|_{L^1(\Omega)}\\&\le C_1\|\Delta\phi\|_{L^2(\Omega)}+C_1\|\nabla\phi\|_{L^2(\Omega)}\le C_2\|\phi\|_{W_0^{2,2}(\Omega)}\end{aligned}$$

with some positive constants C_1 and C_2 for all $t\in(0,\infty)$, which implies (3.45). $\square$

Lemma 3.9 *Suppose that the assumptions of Theorem 1.1 hold. Then there exist some constants $C>0$ and $\delta\in(0,1)$ such that for all $\varepsilon\in(0,1)$, we have*

$$\|\widehat{v}_\varepsilon(\cdot,t)\|_{C^{\delta,\frac{\delta}{2}}(\overline{\Omega}\times[t,t+1])}\le C\qquad\text{for all}\quad t>0. \tag{3.46}$$

Moreover, for each $t_0>0$, we can find $\widetilde{C}(t_0)>0$ such that

$$\|\nabla\widehat{v}_\varepsilon(\cdot,t)\|_{C^{\delta,\frac{\delta}{2}}(\overline{\Omega}\times[t,t+1])}\le\widetilde{C}(t_0)\qquad\text{for all}\quad t>t_0. \tag{3.47}$$

Proof. Re-interpreting the second equation in (3.2) as

$$\partial_t\widehat{v}_\varepsilon=\Delta\widehat{v}_\varepsilon-\widehat{v}_\varepsilon-v_*+u_\varepsilon=:\Delta\widehat{v}_\varepsilon+h_\varepsilon(x,t),\qquad x\in\Omega,\ t>0.$$

Since h_ε is bounded in $L^\infty\big(\Omega\times(0,\infty)\big)$ by (3.43), we can draw on the standard parabolic regularity theory to obtain (3.46) and (3.47). $\square$

Based on the estimates proved so far, we can now pass to the limit by a standard subsequence extraction procedure.

Lemma 3.10 *Suppose that the assumptions of Theorem 1.1 hold. Then there exists a pair of function $(u,\widehat{v})$ and a subsequence $\{\varepsilon_j\}_{j\in\mathbb{N}}\subset(0,1)$ with $\varepsilon_j\searrow0$ as $j\to\infty$ such that*

$$u_\varepsilon\to u\qquad\text{in}\quad L^2_{\rm loc}\big(\overline{\Omega}\times[0,\infty)\big), \tag{3.48}$$

$$\nabla u_\varepsilon\rightharpoonup\nabla u\qquad\text{in}\quad L^2_{\rm loc}\big(\overline{\Omega}\times[0,\infty)\big), \tag{3.49}$$

$$u_\varepsilon\to u\qquad\text{a.e. in}\quad\Omega\times(0,\infty), \tag{3.50}$$

$$\widehat{v}_\varepsilon\to\widehat{v}\qquad\text{in}\quad C_{\rm loc}\big(\overline{\Omega}\times[0,\infty)\big), \tag{3.51}$$

$$\nabla\widehat{v}_\varepsilon\to\nabla\widehat{v}\qquad\text{in}\quad C_{\rm loc}\big(\overline{\Omega}\times[0,\infty)\big), \tag{3.52}$$

$$\nabla\widehat{v}_\varepsilon\to\nabla\widehat{v}\qquad\text{weak}^*\ \text{in}\quad L^\infty\big(\Omega\times(0,\infty)\big) \tag{3.53}$$

as $\varepsilon=\varepsilon_j\searrow0$.

Proof. In view of Lemma 3.9, the Arzelà-Ascoli theorem along with a standard extraction procedure yields a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0,1)$ with $\varepsilon_j \searrow 0$ as $j \to \infty$ such that (3.51) and (3.52) hold with some limit function $\widehat{v}$ belonging to the indicated spaces. The estimate (3.43) shows that (3.53) holds along a further subsequence.

The estimate (3.44) in Lemma 3.7 implies that (3.49) holds for some limit function u belonging to the indicated spaces after a further extraction of an adequate subsequence. Since $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ is bounded in $L^2\big((0,T); W^{1,2}(\Omega)\big)$, we can use Lemma 3.8 and the Aubin-Lions lemma to obtain the strong precompactness (3.48) of $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ in $L^2_{\mathrm{loc}}\big(\overline{\Omega}\times[0,\infty)\big)$, which combined with Egorov's theorem entails (3.50). This completes the proof of Lemma 3.10. $\square$

Proof of Theorem 1.1. We aim to show that solutions to the regularized problems (3.2) will approach a classical solution of system (2.2) as $\varepsilon \to 0$. To this end, we only need to show that the limit function pair $(u, \widehat{v})$ obtained in Lemma 3.10 is a weak solution of system (2.2) in a natural weak sense by relying on Lemma 3.10. Indeed, testing the first equation in (3.2) by $\varphi \in C_0^\infty\big(\overline{\Omega}\times[0,\infty)\big)$, we obtain

$$-\int_0^\infty\int_\Omega u_\varepsilon\partial_t\varphi = \int_\Omega u_0\varphi(\cdot,0) - \int_0^\infty\int_\Omega \nabla u_\varepsilon\cdot\nabla\varphi + \int_0^\infty\int_\Omega u_\varepsilon S_\varepsilon(u_\varepsilon)\nabla\widehat{v}_\varepsilon\cdot\nabla\varphi$$

for all $\varepsilon \in (0,1)$. Thanks to (3.48) and (3.49), we have

$$\int_0^\infty\int_\Omega u_\varepsilon\partial_t\varphi \to \int_0^\infty\int_\Omega u\partial_t\varphi,$$
$$\int_0^\infty\int_\Omega \nabla u_\varepsilon\cdot\nabla\varphi \to \int_0^\infty\int_\Omega \nabla u\cdot\nabla\varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$, where ε_j is as provided by Lemma 3.10. Since $S_\varepsilon \in C^2([0,\infty))$, we have $S_\varepsilon(u_\varepsilon) \to S(u)$ a.e. in $\Omega\times(0,T)$ due to (3.50). In view of $S_\varepsilon \in C^2\big([0,\infty)\big)$ satisfying condition (1.9) and $u_\varepsilon > 0$, we obtain the boundedness of $S_\varepsilon(u_\varepsilon)$ in $L^\infty\big(\Omega\times(0,\infty)\big)$. Then by continuity of S_ε, the strong precompactness of u_ε in $L^2_{\mathrm{loc}}\big(\overline{\Omega}\times[0,\infty)\big)$ and a well-known argument (Lemma A.4 in [44]), we conclude that

$$u_\varepsilon S_\varepsilon(u_\varepsilon) \to uS(u) \qquad \text{in } L^2\big(\Omega\times(0,T)\big).$$

as $\varepsilon = \varepsilon_j \searrow 0$, which combined with (3.52) ensures that

$$\int_0^\infty\int_\Omega u_\varepsilon S_\varepsilon(u_\varepsilon)\nabla\widehat{v}_\varepsilon\cdot\nabla\varphi \to \int_0^\infty\int_\Omega uS(u)\nabla\widehat{v}\cdot\nabla\varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$. These results of convergence imply that

$$-\int_0^\infty\int_\Omega u\partial_t\varphi = \int_\Omega u_0\varphi(\cdot,0) - \int_0^\infty\int_\Omega \nabla u\cdot\nabla\varphi + \int_0^\infty\int_\Omega uS(u)\nabla\widehat{v}\cdot\nabla\varphi.$$

Taking a similar procedure to the second equation in (3.2), we can see

$$-\int_0^\infty\int_\Omega \widehat{v}\partial_t\varphi = \int_\Omega \widehat{v}_0\varphi - \int_0^\infty\int_\Omega \nabla\widehat{v}\cdot\nabla\varphi - \int_0^\infty\int_\Omega (\widehat{v}+v_*-u)\varphi.$$

Next according to the standard parabolic regularity theory (see chapter IV in [22]), we can prove the higher regularity of solutions $(u, \widehat{v})$. By combining the uniform bound estimate (3.43) with the convergence results of the solutions of the approximating system (3.2), we can conclude that $(u, \widehat{v})$ is a global bounded classical solution of system (2.2).

Finally, transforming back via $v := \widehat{v} + v_*$, we can establish the boundedness of the classical solution of system (1.8). This completes the proof of Theorem 1.1. $\square$

4 Finite-time blowup

The purpose of this section is to address the blow-up problem for system (1.8) in the radially symmetric framework. Throughout the sequel, we assume $\Omega = B_R \subset \mathbb{R}^N$ $(N \geq 3)$, and the chemotaxis sensitivity function $S \in C^2\big([0,\infty)\big)$ satisfies (1.11), i.e.,

$$S(s) = K_S(1+s)^{-\alpha}, \qquad s \geq 0$$

for some positive constants K_S and α. Let the initial data $u_0 \in C^0(\overline{B_R})$ and $v_0 \in W^{1,\infty}(B_R)$ be positive and radially symmetric. According to Lemma 2.4, we know that for such an initial data, there exists a positive classical solution (u,v) in $B_R \times (0, T_{\max})$. Indeed, such a solution is radially symmetric.

4.1 The Lyapunov functional

To establish the finite-time blow-up result, we shall first show that the functional $\mathcal{F}(u,v)$ defined by (1.12) is actually a Lyapunov functional for (1.8) with dissipation rate $\mathcal{D}$ defined by (1.13).

Lemma 4.1 *Assume that (u,v) is a classical solution to system* (1.8) *in $B_R \times (0,T_{\max})$. Then we have*

$$\frac{d}{dt}\mathcal{F}\big(u(\cdot,t), v(\cdot,t)\big) = -\mathcal{D}\big(u(\cdot,t), v(\cdot,t)\big) \qquad \text{for all} \quad t \in (0, T_{\max}), \tag{4.1}$$

where $\mathcal{F}$ and $\mathcal{D}$ are as defined in (1.12) *and* (1.13)*, respectively.*

Proof. It follows from (1.8) and the definition of G in (1.14) that

$$\begin{aligned}
\int_{B_R} \partial_t(G(u)) = \int_{B_R} \Big(\int_1^u \frac{1}{\tau S(\tau)}\Big)\partial_t u &= \int_{B_R} \Big(\int_1^u \frac{1}{\tau S(\tau)}\Big)\nabla\cdot\big(\nabla u - uS(u)\nabla v\big) \\
&= -\int_{B_R} \frac{1}{uS(u)}\nabla u \cdot \big(\nabla u - uS(u)\nabla v\big) \\
&= -\int_{B_R} \frac{|\nabla u|^2}{uS(u)} + \int_{B_R} \nabla u \cdot \nabla v
\end{aligned}$$

for all $t \in (0, T_{\max})$. Then differentiating (1.12) with regard to t directly and using (1.8) again, we integrate by parts to obtain that

$$\begin{aligned}
\frac{d}{dt}\mathcal{F} &= \int_{B_R} \nabla v \cdot \nabla \partial_t v + \int_{B_R} v\partial_t v - \int_{B_R} v\partial_t u - \int_{B_R} u\partial_t v + \int_{B_R} \partial_t(G(u)) \\
&= \int_{B_R} \nabla v \cdot \nabla \partial_t v + \int_{B_R} (v-u)\partial_t v - \int_{B_R} (\Delta u - \nabla\cdot(uS(u)\nabla v))v + \int_{B_R} \partial_t(G(u)) \\
&= \int_{B_R} \nabla v \cdot \nabla \partial_t v + \int_{B_R} (\Delta v - \partial_t v)\partial_t v + \int_{B_R} (\nabla u - uS(u)\nabla v)\cdot\nabla v + \int_{B_R} \nabla u\cdot\nabla v - \int_{B_R} \frac{|\nabla u|^2}{uS(u)} \\
&= \int_{B_R} \nabla v \cdot \nabla \partial_t v - \int_{B_R} \nabla v \cdot \nabla \partial_t v - \int_{B_R} |\partial_t v|^2 + 2\int_{B_R} \nabla u \cdot \nabla v - \int_{B_R} uS(u)|\nabla v|^2 - \int_{B_R} \frac{|\nabla u|^2}{uS(u)} \\
&= -\int_{B_R} |\partial_t v|^2 - \int_{B_R} uS(u)\Big|\frac{\nabla u}{uS(u)} - \nabla v\Big|^2 \\
&= -\mathcal{D}(u,v) \qquad \text{for all} \quad t \in (0, T_{\max}),
\end{aligned}$$

where we used the fact that v_* is a constant and thus that $\partial_t v = \partial_t v_* = 0$ on $\partial B_R \times (0, T_{\max})$. This completes the proof of Lemma 4.1. $\square$

4.2 Estimates for the Lyapunov functional

The main idea to derive finite-time blow-up result is to prove that $-\mathcal{F}(u,v)$ satisfies an ODI with nonlinear growth from (4.1). To this end, we devote ourselves to some estimates for the Lyapunov functional. The main step towards this will be provided by some upper estimate for $\int_\Omega uv$. To establish the latter estimate, we first show a pointwise upper bound for the second solution component v of system (1.8).

Lemma 4.2 *Let $\kappa > N-2$. Then there exists a positive constant $C = C(\kappa)$ such that for all radially symmetric and positive functions $u_0 \in C(\overline{B_R})$ and $v_0 \in W^{1,\infty}(B_R)$ satisfying $v_0 \geq v_*$ in B_R and $v_0 = v_*$ on ∂B_R, the corresponding solution (u,v) to system (1.8) satisfies*

$$v(r,t) \leq C(\kappa)\Big(\|u_0\|_{L^1(B_R)} + v_*|B_R| + \|v_0\|_{L^1(B_R)} + \|\nabla v_0\|_{L^2(B_R)}\Big) r^{-\kappa} \tag{4.2}$$

for some positive constant $C(\kappa)$ and all $(r,t) \in (0,R)\times(0,T_{\max})$.

Proof. We first represent $\widehat{v}$ by using the variation-of-constants formula as follows

$$\widehat{v}(\cdot,t) = e^{t(\Delta-1)}\widehat{v}_0 - \int_0^t e^{(t-\tau)(\Delta-1)}\left(v_* - u(\cdot,\tau)\right) d\tau \qquad \text{for all} \quad t\in(0,T_{\max}).$$

According to the decay estimate of the Dirichlet heat semigroup $(e^{t\Delta_D})_{t\geq 0}$ in B_R (see e.g. Lemma 2.1 in [48]), the Poincaré inequality and (2.1), we can find positive constants C_1, C_2 and C_3 depending only on p such that

$$\begin{aligned}
&\|\nabla\widehat{v}(\cdot,t)\|_{L^p(B_R)}\\
&\leq \big\|\nabla e^{t(\Delta-1)}\widehat{v}_0\big\|_{L^p(B_R)} + \int_0^t \big\|\nabla e^{(t-\tau)(\Delta-1)}(v_* - u(\cdot,\tau))\big\|_{L^p(B_R)} d\tau\\
&\leq C_1\|\nabla\widehat{v}_0\|_{L^p(B_R)} + C_1\int_0^t \Big(1+(t-\tau)^{-\frac12-\frac N2(1-\frac1p)}\Big)e^{-\lambda_2(t-\tau)}\|v_*-u(\cdot,\tau)\|_{L^1(B_R)}d\tau\\
&\leq C_2\|\nabla\widehat{v}_0\|_{L^2(B_R)} + C_2\big(\|v_*\|_{L^1(B_R)} + \|u_0\|_{L^1(B_R)}\big)\int_0^t \Big(1+(t-\tau)^{-\frac12-\frac N2(1-\frac1p)}\Big)e^{-\lambda_2(t-\tau)}d\tau\\
&\leq C_3\big(\|\nabla\widehat{v}_0\|_{L^2(B_R)} + \|v_*\|_{L^1(B_R)} + \|u_0\|_{L^1(B_R)}\big) \qquad \text{for all} \quad t\in(0,T_{\max}),
\end{aligned}$$

where λ_2 is the first eigenvalue of $-\Delta$ in B_R under the homogeneous Dirichlet boundary conditions, while $p \in \big(1, \frac{N}{N-1}\big)$ entailing $\frac12 + \frac N2\big(1-\frac1p\big) < 1$ is to be determined. Thus,

$$\|\nabla v\|_{L^p(B_R)} = \|\nabla\widehat{v}\|_{L^p(B_R)} \leq C_3\big(\|\nabla\widehat{v}_0\|_{L^2(B_R)} + \|v_*\|_{L^1(B_R)} + \|u_0\|_{L^1(B_R)}\big).$$

Then thanks to Lemma 2.5, we can directly repeat the proof of Lemma 3.2 in [43] to confirm that

$$v(r,t) \leq C_4(p)\Big(\|u_0\|_{L^1(B_R)} + \|v_*\|_{L^1(B_R)} + \|v_0\|_{L^1(B_R)} + \|\nabla v_0\|_{L^2(B_R)}\Big) r^{-\frac{N-p}{p}} \tag{4.3}$$

for some positive constant $C_4(p)$ and all $(r,t)\in(0,R)\times(0,T_{\max})$. Due to $\kappa > N-2$, we can fix $p \in \big(1,\frac{N}{N-1}\big)$ such that $p \geq \frac{N}{\kappa+1}$, which implies that $\frac{N-p}{p} \leq \kappa$ and thus that

$$r^{-\frac{(N-p)}{p}} \leq R^{\kappa - \frac{(N-p)}{p}} r^{-\kappa} \leq C(\kappa) r^{-\kappa} \qquad \text{for all} \quad r\in(0,R],$$

which together with (4.3) entails (4.2). $\square$

Lemma 4.2 tells that for $\kappa > N-2$, there exists $B>0$ depending only on u_0,v_0, v_*, R and κ such that

$$v(x,t) \leq B|x|^{-\kappa} \qquad \text{for all} \quad (x,t)\in B_R\times(0,T_{\max}). \tag{4.4}$$

This together with (2.1) and (2.3) inspires us to introduce the space

$$\begin{aligned}\mathcal{S}(m,M,B,\kappa,v_*) := \Big\{(u,v)\in C^2\big(\overline{B_R}\big)\times C^2\big(\overline{B_R}\big)\,\Big|\, & u \text{ and } v \text{ are positive and radially symmetric in } B_R,\\ & \text{and satisfy that } \ \|u\|_{L^1(B_R)} = m \text{ and } \|v\|_{L^1(B_R)} \le M,\\ & \text{and that } v(x)\le B|x|^{-\kappa} \text{ in } B_R\backslash\{0\} \text{ and } v = v_* \text{ on } \partial B_R\Big\}. \end{aligned} \tag{4.5}$$

We now devote ourselves to establishing the inequality

$$\frac{\mathcal{F}(u,v)}{\mathcal{D}^\theta(u,v)+1} \ge -C(m,M,B,\kappa,v_*), \qquad\qquad (u,v)\in\mathcal{S}(m,M,B,\kappa,v_*)$$

for some $\theta\in(0,1)$ and positive constant $C(m,M,B,\kappa,v_*)$, where $\mathcal{F}$ and $\mathcal{D}$ are defined by (1.12) and (1.13), respectively. Our main ideas are inspired by [43] and [7], in which the blow-up solutions are constructed for the homogeneous Neumann boundary value problem. Of course, we need some new trick due to the different boundary conditions. Note that in the subsequent Lemmata, say, Lemma 4.3-Lemma 4.8, the symbol (u,v) does not necessarily denote the solution to system (1.8). We just do some estimates of $\mathcal{F}(u,v)$ for $(u,v)\in\mathcal{S}(m,M,B,\kappa,v_*)$.

Lemma 4.3 *Assume that $(u,v)\in\mathcal{S}(m,M,B,\kappa,v_*)$ with $\kappa > N-2$. Then for any $\eta\in(0,1)$, there exists a positive constant $C(\eta,m,v_*)$ such that*

$$\int_{B_R} uv \le (1+\eta)\|\nabla v\|^2_{L^2(B_R)} + C(\eta,m,v_*)\big(M^2+1\big)\Big(\|\Delta v - v + u\|^{\frac{2N+4}{N+4}}_{L^2(B_R)}+1\Big). \tag{4.6}$$

Proof. For convenience, we set

$$f := -\Delta v + v - u \tag{4.7}$$

for $(u,v)\in\mathcal{S}(m,M,B,\kappa,v_*)$. Multiplying (4.7) by v and integrating by parts over B_R, we have

$$\int_{B_R} uv = -\int_{\partial B_R} v_*\nabla v\cdot\mathbf{n} + \int_{B_R}|\nabla v|^2 + \int_{B_R}|v|^2 - \int_{B_R} fv. \tag{4.8}$$

Noticing that the radial symmetry of u and v entails

$$\big(r^{N-1}v_r\big)_r = -r^{N-1}u - r^{N-1}f + r^{N-1}v, \tag{4.9}$$

we deduce from (4.9) and (4.5) that

$$\begin{aligned} -\int_{\partial B_R} v_*\nabla v\cdot\mathbf{n} &= -v_* v_r(R)R^{N-1}\omega_{N-1} = -v_*\omega_{N-1}\int_0^R \big(r^{N-1}v_r\big)_r dr\\ &= v_*\omega_{N-1}\Big(\int_0^R r^{N-1}u dr + \int_0^R r^{N-1} f dr - \int_0^R r^{N-1} v dr\Big)\\ &\le v_*\big(\|u\|_{L^1(B_R)} + \|f\|_{L^1(B_R)} + \|v\|_{L^1(B_R)}\big)\\ &\le v_*\|f\|_{L^1(B_R)} + v_*(m+M), \end{aligned} \tag{4.10}$$

where ω_{N-1} is the surface area of $\partial B(0,1)$. On the other hand, by the Gagliardo-Nirenberg inequality, the Hölder inequality and (4.5), we can find a positive constant C_1 such that

$$\begin{aligned} &\int_{B_R}|v|^2 - \int_{B_R} fv\\ &\le C_1\Big(\|\nabla v\|^{\frac{2N}{N+2}}_{L^2(B_R)}\|v\|^{\frac{4}{N+2}}_{L^1(B_R)} + \|v\|^2_{L^1(B_R)}\Big) + C_1\Big(\|\nabla v\|^{\frac{N}{N+2}}_{L^2(B_R)}\|v\|^{\frac{2}{N+2}}_{L^1(B_R)} + \|v\|_{L^1(B_R)}\Big)\|f\|_{L^2(B_R)} \end{aligned}$$

$$\le C_1\Big(\|\nabla v\|_{L^2(B_R)}^{\frac{2N}{N+2}} M^{\frac{4}{N+2}} + M^2\Big) + C_1\Big(\|\nabla v\|_{L^2(B_R)}^{\frac{N}{N+2}} M^{\frac{2}{N+2}} + M\Big)\|f\|_{L^2(B_R)}. \tag{4.11}$$

Combining (4.8), (4.10) and (4.11), we can apply the Young inequality to show that

$$\begin{aligned}
\int_{B_R} uv &\le v_*\|f\|_{L^1(B_R)} + v_*(m+M) + \|\nabla v\|_{L^2(B_R)}^2 \\
&\quad + C_1\Big(\|\nabla v\|_{L^2(B_R)}^{\frac{2N}{N+2}} M^{\frac{4}{N+2}} + M^2\Big) + C_1\Big(\|\nabla v\|_{L^2(B_R)}^{\frac{N}{N+2}} M^{\frac{2}{N+2}} + M\Big)\|f\|_{L^2(B_R)} \\
&\le v_*|B_R|^{\frac12}\|f\|_{L^2(B_R)} + v_*(m+M) + (1+\eta)\|\nabla v\|_{L^2(B_R)}^2 + C_2(\eta)M^2 \\
&\quad + C_2(\eta)M^{\frac{4}{N+4}}\|f\|_{L^2(B_R)}^{\frac{2N+4}{N+4}} + C_1 M\|f\|_{L^2(B_R)} \\
&\le (1+\eta)\|\nabla v\|_{L^2(B_R)}^2 + C_3(\eta, m, v_*)\big(M^2+1\big)\Big(\|f\|_{L^2(B_R)}^{\frac{2N+4}{N+4}} + 1\Big)
\end{aligned}$$

for any $\eta \in (0,1)$. This completes the proof of (4.6). □

To appropriately estimate $\|\nabla v\|_{L^2(B_R)}^2$ appeared in the right hand side of (4.6), we split it into an integral over a suitable small inner ball B_{r_0} and a corresponding outer annulus $B_R \backslash B_{r_0}$. We begin with the outer case.

Lemma 4.4 *Assume that $(u,v) \in \mathcal{S}(m, M, B, \kappa, v_*)$ with $\kappa > N-2$. Then for any $r_0 \in (0,R)$ and $\eta \in (0,1)$, there exists a positive constant $C(\eta, m, \kappa, v_*)$ such that*

$$\begin{aligned}
\int_{B_R\backslash B_{r_0}} |\nabla v|^2 &\le \eta\|uv\|_{L^1(B_R)} + \eta\|\nabla v\|_{L^2(B_R)}^2 \\
&\quad + C(\eta, m, \kappa, v_*)\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)\Big(r_0^{-\frac{2N+4}{N}\kappa} + \big\|\Delta v - v + u\big\|_{L^2(B_R)}^{\frac{2N+4}{N+4}} + 1\Big).
\end{aligned}$$

Proof. For $(u,v) \in \mathcal{S}(m, M, B, \kappa, v_*)$, we multiply

$$f := -\Delta v + v - u,$$

by $v^{\frac12}$ and integrate by parts over Ω to obtain

$$\int_{B_R} f v^{\frac12} = -\int_{\partial B_R} v^{\frac12}\nabla v\cdot \mathbf{n} + \frac12\int_{B_R} v^{-\frac12}|\nabla v|^2 + \int_{B_R} v^{\frac32} - \int_{B_R} uv^{\frac12},$$

which together with the boundary condition and a similar estimate as (4.10) implies that

$$\begin{aligned}
\frac12\int_{B_R} v^{-\frac12}|\nabla v|^2 + \int_{B_R} v^{\frac32} &= v_*^{\frac12}\int_{\partial B_R}\nabla v\cdot\mathbf{n} + \int_{B_R} uv^{\frac12}(\cdot,t) + \int_{B_R} fv^{\frac12} \\
&\le v_*^{\frac12}\big(\|f\|_{L^1(B_R)} + m + M\big) + \int_{B_R} uv^{\frac12} + \int_{B_R} fv^{\frac12}.
\end{aligned} \tag{4.12}$$

The first term on the left hand side will provide a gradient estimate for v in the annulus. Indeed, in view of the fact (4.4), we have

$$\int_{B_R} v^{-\frac12}|\nabla v|^2 \ge \int_{B_R\backslash B_{r_0}} v^{-\frac12}|\nabla v|^2 \ge B^{-\frac12} r_0^{\frac12\kappa}\int_{B_R\backslash B_{r_0}} |\nabla v|^2,$$

which combined with (4.12) yields

$$\int_{B_R\backslash B_{r_0}} |\nabla v|^2 \le 2v_*^{\frac12} B^{\frac12} r_0^{-\frac12\kappa}\big(\|f\|_{L^1(B_R)} + m + M\big) + 2B^{\frac12} r_0^{-\frac12\kappa}\int_{B_R} uv^{\frac12} + 2B^{\frac12} r_0^{-\frac12\kappa}\int_{B_R} fv^{\frac12}. \tag{4.13}$$

We only need to further estimate the right hand side of (4.13). For any $\eta > 0$, we can use the Hölder inequality and the Young inequality to find positive constants $C_1, C_2, \cdots, C_5$ depending on η, m, M, B, κ and v_* such that

$$\begin{aligned}
&2v_*^{\frac{1}{2}} B^{\frac{1}{2}} r_0^{-\frac{1}{2}\kappa} \big(\|f\|_{L^1(B_R)} + m + M\big) \\
&\le 2v_*^{\frac{1}{2}} B^{\frac{1}{2}} r_0^{-\frac{1}{2}\kappa} \big(\|f\|_{L^2(B_R)} |B_R|^{\frac{1}{2}} + m + M\big) \\
&\le C_1 B^{\frac{1}{2}} \Big(r_0^{-\frac{N+2}{N}\kappa} + \|f\|_{L^2(B_R)}^{\frac{2N+4}{N+4}}\Big) + C_2(M+1) B^{\frac{1}{2}} \Big(r_0^{-\frac{2N+4}{N}\kappa} + 1\Big) \\
&\le C_3(M+1) B^{\frac{1}{2}} \Big(r_0^{-\frac{2N+4}{N}\kappa} + \|f\|_{L^2(B_R)}^{\frac{2N+4}{N+4}} + 1\Big),
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
2B^{\frac{1}{2}} r_0^{-\frac{1}{2}\kappa} \int_{B_R} u v^{\frac{1}{2}} = 2B^{\frac{1}{2}} \int_{B_R} (uv)^{\frac{1}{2}} (r_0^{-\kappa} u)^{\frac{1}{2}} &\le \eta \int_{B_R} uv + \frac{B}{\eta} r_0^{-\kappa} \int_{B_R} u \\
&\le \eta \int_{B_R} uv + \frac{mB}{\eta} r_0^{-\frac{2N+4}{N}\kappa} R^{\frac{N+4}{N}\kappa},
\end{aligned} \tag{4.15}$$

and that

$$\begin{aligned}
2B^{\frac{1}{2}} r_0^{-\frac{1}{2}\kappa} \int_{B_R} f v^{\frac{1}{2}} &\le 2B^{\frac{1}{2}} r_0^{-\frac{1}{2}\kappa} \int_{B_R} (|f|v)^{\frac{1}{2}} |f|^{\frac{1}{2}} \\
&\le \int_{B_R} |f| v + B r_0^{-\kappa} \int_{B_R} |f| \\
&\le \|f\|_{L^2(B_R)} \|v\|_{L^2(B_R)} + B r_0^{-\kappa} |B_R|^{\frac{1}{2}} \|f\|_{L^2(B_R)} \\
&\le C_4 \|f\|_{L^2(B_R)} \Big(\|\nabla v\|_{L^2(B_R)}^{\frac{N}{N+2}} \|v\|_{L^1(B_R)}^{\frac{2}{N+2}} + \|v\|_{L^1(B_R)}\Big) + B r_0^{-\kappa} |B_R|^{\frac{1}{2}} \|f\|_{L^2(B_R)} \\
&\le C_5 \|f\|_{L^2(B_R)} \Big(\|\nabla v\|_{L^2(B_R)}^{\frac{N}{N+2}} M^{\frac{2}{N+2}} + M\Big) + B r_0^{-\kappa} |B_R|^{\frac{1}{2}} \|f\|_{L^2(B_R)} \\
&\le \eta \|\nabla v\|_{L^2(B_R)}^2 + C_6\big(M + B + 1\big) \Big(r_0^{-\frac{2N+4}{N}\kappa} + \|f\|_{L^2(B_R)}^{\frac{2N+4}{N+4}} + 1\Big).
\end{aligned} \tag{4.16}$$

Substituting (4.14)-(4.16) into (4.13), we complete the proof of Lemma 4.4. □

To perform the L^2 estimate of ∇v in the inner ball B_{r_0}, the assumptions $N \ge 3$ and $0 < \alpha < 1 - \frac{2}{N}$ will be crucially needed.

Lemma 4.5 *Let (1.11) hold with $0 < \alpha < 1 - \frac{2}{N}$. Assume that $(u, v) \in \mathcal{S}(m, M, B, \kappa, v_*)$ with $\kappa > N - 2$. Then there exist $\mu \in (1, 2)$ and positive constant $C(m)$ such that*

$$\begin{aligned}
\int_{B_{r_0}} |\nabla v|^2 \le C(m) \Big(r_0 \big\|\Delta v - v + u\big\|_{L^2(B_R)}^2 + r_0 \Big\|\frac{\nabla u}{\sqrt{uS(u)}} - \sqrt{uS(u)} \nabla v\Big\|_{L^2(B_R)}^2 + \|v\|_{L^2(B_R)}^2 + 1\Big) \\
+ \mu \big\|G(u)\big\|_{L^1(B_R)}
\end{aligned}$$

for all $r_0 \in (0, R)$, where G is defined by (1.14).

Proof. Noticing that $\big(\frac{4(N-1)}{N-2} - 2\big)\alpha \in (0, 2)$ due to $\alpha \in \big(0, 1 - \frac{2}{N}\big)$, we can pick $\eta_1 \in \big(0, \frac{2N-2}{R}\big)$ small enough such that

$$\mu_1 := \Big(\frac{4(N-1)}{N-2} e^{\eta_1 R} - 2\Big)\alpha \in (0, 2).$$

Then by letting $\eta_2 := \frac{2-\mu_1}{2R} \in \big(0, \frac{1}{R}\big)$, we have

$$\mu := \eta_2 R + \mu_1 = 1 + \frac{1}{2}\mu_1 \in (1, 2). \tag{4.17}$$

As before, we set

$$f := -\Delta v + v - u \qquad \text{and} \qquad g := \Big(\frac{\nabla u}{\sqrt{uS(u)}} - \sqrt{uS(u)}\nabla v \Big) \cdot \frac{x}{|x|}, \quad x \neq 0. \tag{4.18}$$

Since both u and v are radially symmetric, we rewrite the above two equalities to

$$\big(r^{N-1}v_r\big)_r = -r^{N-1}u - r^{N-1}f + r^{N-1}v \qquad \text{and} \qquad v_r = \frac{u_r}{uS(u)} - \frac{g}{\sqrt{uS(u)}},$$

respectively. Then the Young inequality entails that

$$\begin{aligned}\frac{1}{2}\big((r^{N-1}v_r)^2\big)_r &= -r^{2N-2}uv_r - r^{2N-2}fv_r + r^{2N-2}vv_r \\ &\le -r^{2N-2}\frac{u_r}{S(u)} + r^{2N-2}\frac{|g|u}{\sqrt{uS(u)}} + \frac{\eta_1}{2}\big(r^{N-1}v_r\big)^2 + \frac{1}{2\eta_1}r^{2N-2}f^2 + \frac{1}{2}r^{2N-2}(v^2)_r\end{aligned}$$

for all $r \in (0, R)$. Then by setting

$$y(r) := r^{2N-2}v_r^2(r), \qquad r \in [0, R],$$

we have

$$y_r - \eta_1 y \le -2r^{2N-2}\frac{u_r}{S(u)} + 2r^{2N-2}\frac{|g|u}{\sqrt{uS(u)}} + \frac{1}{\eta_1}r^{2N-2}f^2 + r^{2N-2}(v^2)_r$$

for all $r \in (0, R)$. By integrating the above inequality from 0 to r and using $y(0) = 0$, we obtain

$$\begin{aligned}r^{2N-2}v_r^2(r) = y(r) &\le -2\int_0^r e^{\eta_1(r-\rho)}\rho^{2N-2}\frac{u_\rho(\rho)}{S\big(u(\rho)\big)}d\rho + 2\int_0^r e^{\eta_1(r-\rho)}\rho^{2N-2}\frac{\big|g(\rho)\big|u(\rho)}{\sqrt{u(\rho)S\big(u(\rho)\big)}}d\rho \\ &\quad + \frac{1}{\eta_1}\int_0^r e^{\eta_1(r-\rho)}\rho^{2N-2}f^2(\rho)d\rho + \int_0^r e^{\eta_1(r-\rho)}\rho^{2N-2}\big(v^2(\rho)\big)_\rho d\rho \\ &=: I_1 + I_2 + I_3 + I_4 \qquad \text{for all} \quad r \in (0, R).\end{aligned} \tag{4.19}$$

Before estimating the integrals on the right-hand side of (4.19), we first claim that

$$G(s) \ge \frac{1}{K_S\alpha(\alpha+1)}(1+s)^{1+\alpha} - \frac{2^\alpha}{K_S\alpha}s - \frac{2^{\alpha+1}}{K_S\alpha(\alpha+1)} \qquad \text{for all} \quad s \ge 0. \tag{4.20}$$

Indeed, in the case of $s \ge 1$, we see from the definition of $G(s)$ and (1.11) that

$$\begin{aligned}G(s) = \frac{1}{K_S}\int_1^s\int_1^\delta \frac{1}{\tau(1+\tau)^{-\alpha}}d\tau d\delta &\ge \frac{1}{K_S}\int_1^s\int_1^\delta \frac{(1+\tau)^\alpha}{1+\tau}d\tau d\delta \\ &\ge \frac{1}{K_S\alpha(\alpha+1)}(1+s)^{1+\alpha} - \frac{2^\alpha}{K_S\alpha}s - \frac{2^{\alpha+1}}{K_S\alpha(\alpha+1)},\end{aligned}$$

while in the case of $0 < s < 1$, we also have

$$\begin{aligned}G(s) = \frac{1}{K_S}\int_s^1\int_\delta^1 \frac{1}{\tau(1+\tau)^{-\alpha}}d\tau d\delta &\ge \frac{1}{K_S}\int_s^1\int_\delta^1 \frac{(1+\tau)^\alpha}{1+\tau}d\tau d\delta \\ &\ge \frac{1}{K_S\alpha(\alpha+1)}(1+s)^{1+\alpha} - \frac{2^\alpha}{K_S\alpha}s - \frac{2^{\alpha+1}}{K_S\alpha(\alpha+1)}.\end{aligned}$$

This confirms our claim. On the other hand, by letting

$$H(s) := \int_0^s \frac{d\delta}{S(\delta)}, \qquad s \ge 0,$$

we will have

$$H(s)=\frac{1}{K_S}\int_0^s(1+\delta)^\alpha d\delta=\frac{1}{K_S(\alpha+1)}(1+s)^{1+\alpha}-\frac{1}{K_S(\alpha+1)}\geq 0\qquad\text{for all}\quad s\geq 0. \tag{4.21}$$

We now estimate the integrals on the right-hand side of (4.19) one by one. For I_1, it follows from the definition of H and the integration by parts that

$$\begin{aligned}I_1&=-2\int_0^r e^{\eta_1(r-\rho)}\rho^{2N-2}\big(H(u(\rho))\big)_\rho d\rho\\&=-2r^{2N-2}H\big(u(r)\big)+4(N-1)\int_0^r e^{\eta_1(r-\rho)}\rho^{2N-3}H\big(u(\rho)\big)d\rho-2\eta_1\int_0^r e^{\eta_1(r-\rho)}\rho^{2N-2}H\big(u(\rho)\big)d\rho\\&\leq -2r^{2N-2}H\big(u(r)\big)+4(N-1)e^{\eta_1 R}\int_0^r\rho^{2N-3}H\big(u(\rho)\big)d\rho\qquad\text{for all}\quad r\in(0,R).\end{aligned}$$

For I_2, we can first utilize the Hölder inequality and (1.11) to deduce that

$$\begin{aligned}I_2&\leq 2\Big(\int_0^r\rho^{N-1}\frac{u(\rho)}{S\big(u(\rho)\big)}d\rho\Big)^{\frac12}\Big(\int_0^r e^{2\eta_1(r-\rho)}\rho^{3N-3}g^2(\rho)d\rho\Big)^{\frac12}\\&\leq 2\Big(\frac{1}{K_S}\int_0^r\rho^{N-1}\big(u(\rho)+1\big)^{1+\alpha}d\rho\Big)^{\frac12}\Big(e^{2\eta_1 R}r^{2N-2}\int_0^r\rho^{N-1}g^2(\rho)d\rho\Big)^{\frac12}\\&\leq\frac{2e^{\eta_1 R}}{\omega_{N-1}\sqrt{K_S}}r^{N-1}\big\|(u+1)^{1+\alpha}\big\|^{\frac12}_{L^1(B_R)}\|g\|_{L^2(B_R)}\\&\leq\frac{\eta_2}{\omega_{N-1}K_S\alpha(\alpha+1)}r^{N-1}\big\|(u+1)^{1+\alpha}\big\|_{L^1(B_R)}+\frac{\alpha(\alpha+1)e^{2\eta_1 R}}{\eta_2\omega_{N-1}}r^{N-1}\|g\|^2_{L^2(B_R)}\end{aligned}$$

for all $r\in(0,R)$. Then by

$$(s+1)^{1+\alpha}\leq K_S\alpha(\alpha+1)G(s)+2^\alpha(\alpha+1)s+2^{\alpha+1}\qquad\text{for all}\quad s\geq 0,$$

which follows from (4.20), we obtain

$$\begin{aligned}I_2&\leq\frac{\eta_2}{\omega_{N-1}}r^{N-1}\Big(\|G(u)\|_{L^1(B_R)}+\frac{2^\alpha}{K_S\alpha}\|u\|_{L^1(B_R)}+\frac{2^{\alpha+1}|B_R|}{K_S\alpha(\alpha+1)}\Big)+\frac{\alpha(\alpha+1)e^{2\eta_1 R}}{\eta_2\omega_{N-1}}r^{N-1}\|g\|^2_{L^2(B_R)}\\&\leq\frac{\eta_2}{\omega_{N-1}}r^{N-1}\Big(\|G(u)\|_{L^1(B_R)}+C_1(m)\Big)+\frac{\alpha(\alpha+1)e^{2\eta_1 R}}{\eta_2\omega_{N-1}}r^{N-1}\|g\|^2_{L^2(B_R)}\qquad\text{for all }r\in(0,R)\end{aligned}$$

with some constant $C_1(m)>0$. For I_3 and I_4, a direct calculation yields that

$$I_3\leq\frac{e^{\eta_1 R}}{\eta_1}r^{N-1}\int_0^r\rho^{N-1}f^2(\rho)d\rho=\frac{e^{\eta_1 R}}{\eta_1\omega_{N-1}}r^{N-1}\|f\|^2_{L^2(B_R)}\qquad\text{for all }r\in(0,R)$$

and

$$I_4=r^{2N-2}v^2(r)-\int_0^r\big((2N-2)\rho^{2N-3}-\eta_1\rho^{2N-2}\big)e^{\eta_1(r-\rho)}v^2(\rho)d\rho\leq r^{2N-2}v^2(r)\quad\text{for all }r\in(0,R),$$

where we used $(2N-2)\rho^{2N-3}\geq\eta_1\rho^{2N-2}$ in the last inequality due to $\eta_1\leq\frac{2N-2}{R}$.

Substituting the above estimates into (4.19), we can find a positive constant $C_2=C_2(m)$ such that

$$\begin{aligned}r^{N-1}v_r^2(r)\leq&-2r^{N-1}H\big(u(r)\big)+4(N-1)e^{\eta_1 R}\frac{1}{r^{N-1}}\int_0^r\rho^{2N-3}H\big(u(\rho)\big)d\rho+\frac{\eta_2}{\omega_{N-1}}\|G(u)\|_{L^1(B_R)}\\&+\frac{C_2}{\omega_{N-1}}+\frac{C_2}{\omega_{N-1}}\|g\|^2_{L^2(B_R)}+\frac{C_2}{\omega_{N-1}}\|f\|^2_{L^2(B_R)}+r^{N-1}v^2(r)\end{aligned}$$

for all $r \in (0, R)$. Then a direct integration entails that

$$\begin{aligned}
\int_{B_{r_0}} |\nabla v|^2 &= \omega_{N-1} \int_0^{r_0} r^{N-1} v_r^2(r) dr \\
&\le -2\omega_{N-1} \int_0^{r_0} r^{N-1} H\big(u(r)\big) dr + 4(N-1)e^{\eta_1 R} \omega_{N-1} \int_0^{r_0} r^{1-N} \int_0^r \rho^{2N-3} H\big(u(\rho)\big) d\rho dr \\
&\quad + \eta_2 r_0 \|G(u)\|_{L^1(B_R)} + C_2 r_0 + C_2 r_0 \|g\|^2_{L^2(B_R)} + C_2 r_0 \|f\|^2_{L^2(B_R)} \\
&\quad + \omega_{N-1} \int_0^{r_0} r^{N-1} v^2(r) dr \\
&\le -2 \int_{B_{r_0}} H(u) + 4(N-1) e^{\eta_1 R} \omega_{N-1} \int_0^{r_0} \rho^{2N-3} H\big(u(\rho)\big) \Big(\int_\rho^{r_0} r^{1-N} dr \Big) d\rho \\
&\quad + \eta_2 R \|G(u)\|_{L^1(B_R)} + C_2 r_0 \big(1 + \|g\|^2_{L^2(B_R)} + \|f\|^2_{L^2(B_R)}\big) + \|v\|^2_{L^2(B_R)} \qquad (4.22)
\end{aligned}$$

for all $r_0 \in (0, R)$, where we used the Fubini's theorem in the last inequality. Noticing that

$$\begin{aligned}
& -2 \int_{B_{r_0}} H(u) + 4(N-1) e^{\eta_1 R} \omega_{N-1} \int_0^{r_0} \rho^{2N-3} H\big(u(\rho)\big) \Big(\int_\rho^{r_0} r^{1-N} dr \Big) d\rho \\
&= -2 \int_{B_{r_0}} H(u) + \frac{4(N-1)}{N-2} e^{\eta_1 R} \omega_{N-1} \int_0^{r_0} \big(\rho^{2-N} - r_0^{2-N}\big) \rho^{2N-3} H\big(u(\rho)\big) d\rho \\
&\le -2 \int_{B_{r_0}} H(u) + \frac{4(N-1)}{N-2} e^{\eta_1 R} \omega_{N-1} \int_0^{r_0} \rho^{N-1} H\big(u(\rho)\big) d\rho \\
&\le \Big(\frac{4(N-1)}{N-2} e^{\eta_1 R} - 2 \Big) \int_\Omega H(u),
\end{aligned}$$

we can deduce from (4.21) and (4.20) that

$$\begin{aligned}
& -2 \int_{B_{r_0}} H(u) + 4(N-1) e^{\eta_1 R} \omega_{N-1} \int_0^{r_0} \rho^{2N-3} H\big(u(\rho)\big) \Big(\int_\rho^{r_0} r^{1-N} dr \Big) d\rho \\
&\le \Big(\frac{4(N-1)}{N-2} e^{\eta_1 R} - 2 \Big) \Big(\alpha \|G(u)\|_{L^1(B_R)} + \frac{2^\alpha}{K_S} \|u\|_{L^1(B_R)} + \frac{2^{\alpha+1}}{K_S(\alpha+1)} |B_R| \Big) \\
&\le \Big(\frac{4(N-1)}{N-2} e^{\eta_1 R} - 2 \Big) \alpha \|G(u)\|_{L^1(B_R)} + C_3 = \mu_1 \|G(u)\|_{L^1(B_R)} + C_3 \qquad (4.23)
\end{aligned}$$

for some positive constant $C_3 = C_3(m)$. By substituting (4.23) into (4.22), we obtain

$$\|\nabla v\|^2_{L^2(B_{r_0})} \le \mu \|G(u)\|_{L^1(B_R)} + C_2 r_0 \big(1 + \|g\|^2_{L^2(B_R)} + \|f\|^2_{L^2(B_R)}\big) + \|v\|^2_{L^2(B_R)} + C_3$$

due to (4.17). This completes the proof of Lemma 4.5. $\square$

We further fix a suitable small r_0 such that $r_0\big(\|f\|^2_{L^2(B_R)} + \|g\|^2_{L^2(B_R)}\big)$ essentially becomes a suitable subquadratic power of $\|f\|^2_{L^2(B_R)} + \|g\|^2_{L^2(B_R)}$ for f and g defined by (4.18).

Lemma 4.6 *Let* (1.11) *hold with* $0 < \alpha < 1 - \frac{2}{N}$. *Assume that* $(u, v) \in \mathcal{S}(m, M, B, \kappa, v_*)$ *with* $\kappa > N - 2$. *Then for any* $\eta \in \big(0, \frac{1}{2}\big)$, *there exist positive constants* $C(\eta, m, \kappa, v_*)$ *and*

$$\theta := \frac{\kappa(2N+4)}{(2N+4)\kappa + N} \in \Big(\frac{N+2}{N+4}, 1 \Big) \subset \Big(\frac{1}{2}, 1 \Big). \qquad (4.24)$$

such that

$$\begin{aligned}
\int_{B_R} |\nabla v|^2 &\le C(\eta, m, \kappa, v_*) \Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1 \Big) \Big(\big\| \Delta v - v + u \big\|^{2\theta}_{L^2(B_R)} \\
&\quad + \Big\| \frac{\nabla u}{\sqrt{uS(u)}} - \sqrt{uS(u)} \nabla v \Big\|^{2\theta}_{L^2(B_R)} + 1 \Big) + \frac{\mu}{1-2\eta} \|G(u)\|_{L^1(B_R)} + \frac{\eta}{1-2\eta} \|uv\|_{L^1(B_R)},
\end{aligned}$$

where $\mu \in (1, 2)$ *is determined by Lemma 4.5.*

Proof. As before, we set

$$f := -\Delta v + v - u \qquad \text{and} \qquad g := \Big(\frac{\nabla u}{\sqrt{uS(u)}} - \sqrt{uS(u)}\nabla v \Big) \cdot \frac{x}{|x|}, \quad x \neq 0.$$

Define

$$r_0 := \min\Big\{ \frac{R}{2}, \, \big(\|f\|_{L^2(B_R)} + \|g\|_{L^2(B_R)}\big)^{-\frac{2N}{(2N+4)\kappa+N}} \Big\} \in (0, R).$$

Then for any fixed $\eta \in (0, \frac{1}{2})$, in view of Lemma 4.4 and Lemma 4.5, we can find positive constants $C_1 = C_1(\eta, m, \kappa, v_*)$ and $C_2 = C_2(m)$ such that

$$\begin{aligned}(1-\eta)\int_{B_R} |\nabla v|^2 \le\ & C_1\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)\|f\|_{L^2(B_R)}^{\frac{2N+4}{N+4}} + C_2 r_0\big(\|f\|_{L^2(B_R)}^2 + \|g\|_{L^2(B_R)}^2\big) \\ & + C_1\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big) r_0^{-\frac{2N+4}{N}\kappa} + C_2\|v\|_{L^2(B_R)}^2 \\ & + \mu\big\|G(u)\big\|_{L^1(B_R)} + \eta\|uv\|_{L^1(B_R)} + C_1\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)\end{aligned}$$

Since the Gagliardo-Nirenberg inequality, (4.5) and the Young inequality entail that

$$\begin{aligned}C_2\|v\|_{L^2(B_R)}^2 &\le C_3\Big(\|\nabla v\|_{L^2(B_R)}^{\frac{N}{N+2}}\|v\|_{L^1(B_R)}^{\frac{2}{N+2}} + \|v\|_{L^1(B_R)}\Big)^2 \\ &\le 2C_3\Big(\|\nabla v\|_{L^2(B_R)}^{\frac{2N}{N+2}} M^{\frac{4}{N+2}} + M^2\Big) \\ &\le \eta\int_{B_R}|\nabla v|^2 + C_4 M^2\end{aligned}$$

for some positive constants $C_3 = C_3(m)$ and $C_4 = C_4(\eta, m)$, we have

$$\begin{aligned}(1-2\eta)\int_{B_R} |\nabla v|^2 \le\ & C_1\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)\|f\|_{L^2(B_R)}^{\frac{2N+4}{N+4}} + C_2 r_0\big(\|f\|_{L^2(B_R)}^2 + \|g\|_{L^2(B_R)}^2\big) \\ & + C_1\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big) r_0^{-\frac{2N+4}{N}\kappa} \\ & + \mu\big\|G(u)\big\|_{L^1(B_R)} + \eta\|uv\|_{L^1(B_R)} + C_5\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big) \qquad (4.25)\end{aligned}$$

for some positive constant $C_5 = C_5(\eta, m, \kappa, v_*)$.

If $\|f\|_{L^2(B_R)} + \|g\|_{L^2(B_R)} \le \big(\frac{2}{R}\big)^{\frac{(2N+4)\kappa+N}{2N}}$, then $r_0 = \frac{R}{2}$. In this case, it is clear that

$$\begin{aligned}& C_1\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)\|f\|_{L^2(B_R)}^{\frac{2N+4}{N+4}} + C_2 r_0\big(\|f\|_{L^2(B_R)}^2 + \|g\|_{L^2(B_R)}^2\big) \\ & \quad + C_1\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big) r_0^{-\frac{2N+4}{N}\kappa} \\ & \le C_1\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)\Big(\frac{2}{R}\Big)^{\frac{(2N+4)\kappa+N}{2N}\cdot\frac{2N+4}{N+4}} + C_2 R\Big(\frac{2}{R}\Big)^{\frac{(2N+4)\kappa+N}{N}} \\ & \quad + C_1\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)\Big(\frac{R}{2}\Big)^{-\frac{2N+4}{N}\kappa},\end{aligned}$$

which together with (4.25) implies that there exist a positive constant $C_6 = C_6(\eta, m, \kappa, v_*)$ such that

$$\begin{aligned}\int_{B_R}|\nabla v|^2 &\le C_6\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big) + \frac{\mu}{1-2\eta}\|G(u)\|_{L^1(B_R)} + \frac{\eta}{1-2\eta}\|uv\|_{L^1(B_R)} \\ &\le C_6\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)\Big(\|f\|_{L^2(B_R)}^{2\theta} + \|g\|_{L^2(B_R)}^{2\theta} + 1\Big)\end{aligned}$$

$$+\frac{\mu}{1-2\eta}\|G(u)\|_{L^1(B_R)}+\frac{\eta}{1-2\eta}\|uv\|_{L^1(B_R)} \tag{4.26}$$

for any $\theta>0$. On the other hand, if $\|f\|_{L^2(B_R)}+\|g\|_{L^2(B_R)}>\left(\frac{2}{R}\right)^{\frac{(2N+4)\kappa+N}{2N}}$, then $r_0=\left(\|f\|_{L^2(B_R)}+\|g\|_{L^2(B_R)}\right)^{-\frac{2N}{(2N+4)\kappa+N}}$. In this case, we can first find a positive constant $C_7=C_7(\eta,m,\kappa,v_*)$ such that

$$\begin{aligned}
&C_1\Big(M^{\frac{2N+4}{N}}+B^{\frac{N+2}{N}}+1\Big)\|f\|_{L^2(B_R)}^{\frac{2N+4}{N+4}}+C_2r_0\big(\|f\|^2_{L^2(B_R)}+\|g\|^2_{L^2(B_R)}\big)\\
&\quad+C_1\Big(M^{\frac{2N+4}{N}}+B^{\frac{N+2}{N}}+1\Big)r_0^{-\frac{2N+4}{N}\kappa}\\
&\le C_1\Big(M^{\frac{2N+4}{N}}+B^{\frac{N+2}{N}}+1\Big)\|f\|_{L^2(B_R)}^{\frac{2N+4}{N+4}}+C_2\big(\|f\|_{L^2(B_R)}+\|g\|_{L^2(B_R)}\big)^{2-\frac{2N}{(2N+4)\kappa+N}}_{L^2(B_R)}\\
&\quad+C_1\Big(M^{\frac{2N+4}{N}}+B^{\frac{N+2}{N}}+1\Big)\big(\|f\|_{L^2(B_R)}+\|g\|_{L^2(B_R)}\big)^{\frac{2\kappa(2N+4)}{(2N+4)\kappa+N}}\\
&\le C_1\Big(M^{\frac{2N+4}{N}}+B^{\frac{N+2}{N}}+1\Big)\|f\|_{L^2(B_R)}^{\frac{2N+4}{N+4}}+C_7\Big(M^{\frac{2N+4}{N}}+B^{\frac{N+2}{N}}+1\Big)\Big(\|f\|^{\frac{2\kappa(2N+4)}{(2N+4)\kappa+N}}_{L^2(B_R)}+\|g\|^{\frac{2\kappa(2N+4)}{(2N+4)\kappa+N}}_{L^2(B_R)}\Big).
\end{aligned}$$

Then thanks to $\kappa>N-2$ and $N\ge 3$, we can take θ as (4.24) and further use the Young inequality to obtain

$$\begin{aligned}
&C_1\Big(M^{\frac{2N+4}{N}}+B^{\frac{N+2}{N}}+1\Big)\|f\|_{L^2(B_R)}^{\frac{2N+4}{N+4}}+C_2r_0\big(\|f\|^2_{L^2(B_R)}+\|g\|^2_{L^2(B_R)}\big)\\
&\quad+C_1\Big(M^{\frac{2N+4}{N}}+B^{\frac{N+2}{N}}+1\Big)r_0^{-\frac{2N+4}{N}\kappa}\\
&\le C_1\Big(M^{\frac{2N+4}{N}}+B^{\frac{N+2}{N}}+1\Big)\|f\|_{L^2(B_R)}^{\frac{2N+4}{N+4}}+C_7\Big(M^{\frac{2N+4}{N}}+B^{\frac{N+2}{N}}+1\Big)\Big(\|f\|^{2\theta}_{L^2(B_R)}+\|g\|^{2\theta}_{L^2(B_R)}\Big)\\
&\le (C_1+C_7)\Big(M^{\frac{2N+4}{N}}+B^{\frac{N+2}{N}}+1\Big)\Big(\|f\|^{2\theta}_{L^2(B_R)}+\|g\|^{2\theta}_{L^2(B_R)}+1\Big),
\end{aligned}$$

which implies that

$$\begin{aligned}
\int_{B_R}|\nabla v|^2\le{}& C_8\Big(M^{\frac{2N+4}{N}}+B^{\frac{N+2}{N}}+1\Big)\Big(\|f\|^{2\theta}_{L^2(B_R)}+\|g\|^{2\theta}_{L^2(B_R)}+1\Big)\\
&+\frac{\mu}{1-2\eta}\|G(u)\|_{L^1(B_R)}+\frac{\eta}{1-2\eta}\|uv\|_{L^1(B_R)}+C_8
\end{aligned}\tag{4.27}$$

for some positive constant $C_8=C_8(\eta,m,\kappa,v_*)$.

Combining (4.26) and (4.27), and fixing θ by (4.24), we end the proof of Lemma 4.6. □

The final step is to control $\|uv\|_{L^1(B_R)}$ appeared in Lemma 4.6.

Lemma 4.7 *Let* (1.11) *hold with* $0<\alpha<1-\frac{2}{N}$ *and* $\theta:=\frac{\kappa(2N+4)}{(2N+4)\kappa+N}$. *Assume that* $(u,v)\in\mathcal{S}(m,M,B,\kappa,v_*)$ *with* $\kappa>N-2$. *Then there exists a positive constant* $C(m,\kappa,v_*)$ *such that*

$$\begin{aligned}
\int_{B_R}uv\le{}& C(m,\kappa,v_*)\Big(M^{\frac{2N+4}{N}}+B^{\frac{N+2}{N}}+1\Big)\Big(\big\|\Delta v-v+u\big\|^{2\theta}_{L^2(B_R)}\\
&+\Big\|\frac{\nabla u}{\sqrt{uS(u)}}-\sqrt{uS(u)}\nabla v\Big\|^{2\theta}_{L^2(B_R)}+1\Big)+\|G(u)\|_{L^1(B_R)}+\frac{1}{2}\|\nabla v\|^2_{L^2(B_R)}.
\end{aligned}\tag{4.28}$$

Proof. As before, we still set

$$f:=-\Delta v+v-u\qquad\text{and}\qquad g:=\Big(\frac{\nabla u}{\sqrt{uS(u)}}-\sqrt{uS(u)}\nabla v\Big)\cdot\frac{x}{|x|},\quad x\ne 0$$

for simplicity and let $\mu\in(1,2)$ be the constant determined by Lemma 4.5. Then we can first pick a constant $\eta_1\in\left(0,\frac{1}{2}\right)$ fulfilling $(1-\eta_1)\mu<1$ and then use a continuity argument to find $\eta_2\in\left(0,\frac{1}{4}\right)$ small enough such that

$$\frac{\eta_1(1-2\eta_2)}{1-3\eta_2+\eta_1\eta_2-\eta_2^2}<\frac{1}{2}\qquad\text{and}\qquad\frac{(1-\eta_1+\eta_2)\mu}{1-3\eta_2+\eta_1\eta_2-\eta_2^2}<1. \tag{4.29}$$

Thus, we can use Lemma 4.3 to find a positive constant $C_1 = C_1(m, v_*)$ such that

$$\|uv\|_{L^1(B_R)} \le \eta_1 \|\nabla v\|^2_{L^2(B_R)} + (1-\eta_1+\eta_2)\|\nabla v\|^2_{L^2(B_R)} + C_1\big(M^2+1\big)\Big(\|f\|^{\frac{2N+4}{N+4}}_{L^2(B_R)} + 1\Big), \quad (4.30)$$

and similarly, use Lemma 4.6 to find positive constants $C_2 = C_2(m, \kappa, v_*)$ and $\theta = \frac{\kappa(2N+4)}{(2N+4)\kappa+N}$ such that

$$\begin{aligned}\|\nabla v\|^2_{L^2(B_R)} \le\ & C_2\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)\Big(\|f\|^{2\theta}_{L^2(B_R)} + \|g\|^{2\theta}_{L^2(B_R)} + 1\Big) \\ & + \frac{\mu}{1-2\eta_2}\big\|G(u)\big\|_{L^1(B_R)} + \frac{\eta_2}{1-2\eta_2}\big\|uv\big\|_{L^1(B_R)}.\end{aligned}$$

Substituting the above inequality into the second term on the right hand side of (4.30), we can obtain

$$\begin{aligned}&\Big(1 - \frac{\eta_2(1-\eta_1+\eta_2)}{1-2\eta_2}\Big)\|uv\|_{L^1(B_R)} \\ &\le \eta_1\|\nabla v\|^2_{L^2(B_R)} + (1-\eta_1+\eta_2)C_2\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)\Big(\|f\|^{2\theta}_{L^2(B_R)} + \|g\|^{2\theta}_{L^2(B_R)}\Big) \\ &\quad + \frac{(1-\eta_1+\eta_2)\mu}{1-2\eta_2}\big\|G(u)\big\|_{L^1(B_R)} + C_1\big(M^2+1\big)\|f\|^{\frac{2N+4}{N+4}}_{L^2(B_R)} \\ &\quad + \Big((1-\eta_1+\eta_2)C_2\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big) + C_1\big(M^2+1\big)\Big),\end{aligned}$$

which together with (4.29) implies that there exists positive constants $C_3 = C_3(m, \kappa, v_*)$ and $C_4 = C_4(m, \kappa, v_*)$ such that

$$\begin{aligned}\|uv\|_{L^1(B_R)} &\le \frac{\eta_1(1-2\eta_2)}{1-3\eta_2+\eta_1\eta_2-\eta_2^2}\|\nabla v\|^2_{L^2(B_R)} + C_3\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)\Big(\|f\|^{2\theta}_{L^2(B_R)} + \|g\|^{2\theta}_{L^2(B_R)}\Big) \\ &\quad + \frac{(1-\eta_1+\eta_2)\mu}{1-3\eta_2+\eta_1\eta_2-\eta_2^2}\big\|G(u)\big\|_{L^1(B_R)} + C_3\big(M^2+1\big)\|f\|^{\frac{2N+4}{N+4}}_{L^2(B_R)} \\ &\quad + C_3\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big) \\ &\le \frac{1}{2}\|\nabla v\|^2_{L^2(B_R)} + C_3\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)\Big(\|f\|^{2\theta}_{L^2(B_R)} + \|g\|^{2\theta}_{L^2(B_R)}\Big) \\ &\quad + \big\|G(u)\big\|_{L^1(B_R)} + C_3\big(M^2+1\big)\|f\|^{\frac{2N+4}{N+4}}_{L^2(B_R)} + C_3\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big) \\ &\le \frac{1}{2}\|\nabla v\|^2_{L^2(B_R)} + C_4\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)\Big(\|f\|^{2\theta}_{L^2(B_R)} + \|g\|^{2\theta}_{L^2(B_R)} + 1\Big) + \big\|G(u)\big\|_{L^1(B_R)}.\end{aligned}$$

Here in the last inequality, we used the fact $\theta \in \big(\frac{N+2}{N+4}, 1\big)$ and the Young inequality. This completes the proof of Lemma 4.7. $\square$

We are now in the position to estimate $\mathcal{F}(u, v)$ in terms of some sublinear power of $\mathcal{D}(u, v)$.

Lemma 4.8 *Let* (1.11) *hold with* $0 < \alpha < 1 - \frac{2}{N}$ *and* $\theta := \frac{\kappa(2N+4)}{(2N+4)\kappa+N}$. *Assume that* $(u, v) \in \mathcal{S}(m, M, B, \kappa, v_*)$ *with* $\kappa > N-2$. *Then there exist a positive constant* $C(m, \kappa, v_*)$ *such that*

$$\mathcal{F}(u, v) \ge -C(m, \kappa, v_*)\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)\Big(\mathcal{D}^\theta(u, v) + 1\Big),$$

where $\mathcal{F}$ *and* $\mathcal{D}$ *are given by* (1.12) *and* (1.13), *respectively.*

Proof. By (4.28), we can find two positive constants $C_1 = C_1(m, \kappa, v_*)$ and $\theta = \frac{\kappa(2N+4)}{(2N+4)\kappa+N}$ such that

$$\|uv\|_{L^1(B_R)} \le C_1\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)\Big(\big\|\Delta v - v + u\big\|^{2\theta}_{L^2(B_R)}$$

$$+\left\|\frac{\nabla u}{\sqrt{uS(u)}}-\sqrt{uS(u)}\nabla v\right\|_{L^2(B_R)}^{2\theta}+1\Big)+\|G(u)\|_{L^1(B_R)}+\frac{1}{2}\|\nabla v\|_{L^2(B_R)}^2.$$

Then by the definitions of $\mathcal{F}$ and $\mathcal{D}$, we deduce that

$$\begin{aligned}\mathcal{F}(u,v)&\geq \frac{1}{2}\|\nabla v\|_{L^2(B_R)}^2-\|uv\|_{L^1(B_R)}+\big\|G(u)\big\|_{L^1(B_R)}\\&\geq -C_1\Big(M^{\frac{2N+4}{N}}+B^{\frac{N+2}{N}}+1\Big)\Big(\|\Delta v-v+u\|_{L^2(B_R)}^{2\theta}+\left\|\frac{\nabla u}{\sqrt{uS(u)}}-\sqrt{uS(u)}\nabla v\right\|_{L^2(B_R)}^{2\theta}+1\Big)\\&\geq -C_2\Big(M^{\frac{2N+4}{N}}+B^{\frac{N+2}{N}}+1\Big)\Big(\mathcal{D}^{\theta}(u,v)+1\Big)\end{aligned}$$

for some positive constant $C_2=C_2(m,\kappa,v_*)$. This completes the proof of Lemma 4.8. □

4.3 Blowup in finite time: Proof of Theorem 1.2

In view of Lemma 4.8 and Lemma 4.1, we can prove the existence of solution (u,v) blowing up in finite time. We denote by

$$\widetilde{\mathcal{B}}(m,\overline{M},K,v_*):=\Big\{(u_0,v_0)\in C^0(\overline{B_R})\times W^{1,\infty}(B_R)\,\Big|\,u_0 \text{ and } v_0 \text{ are radially symmetric and positive in } B_R,\ v_0\geq v_* \text{ in } B_R,\ \ v_0=v_* \text{ on } \partial B_R, \text{ and satisfy } \|u_0\|_{L^1(B_R)}=m,\ \|v_0\|_{W^{1,2}(B_R)}\leq \overline{M},\ \text{and } \mathcal{F}(u_0,v_0)\leq -K\Big\}.$$

Lemma 4.9 *Let* (1.11) *hold with* $0<\alpha<1-\frac{2}{N}$ *and* $\kappa>N-2$. *For any given positive constants* m *and* $\overline{M}$, *and* $v_*\geq 0$, *there exist a positive constants* $K=K(m,\kappa,v_*)\big(\overline{M}^{\frac{2N+4}{N}}+1\big)$ *such that for any* $(u_0,v_0)\in\widetilde{\mathcal{B}}(m,\overline{M},K,v_*)$, *the corresponding solution* (u,v) *of system* (1.8) *fulfills*

$$\mathcal{F}(u,v)\leq \frac{\mathcal{F}(u_0,v_0)}{(1-Ct)^{\frac{\theta}{1-\theta}}}\qquad \text{for all}\quad t\in(0,T_{\max}), \tag{4.31}$$

where $\theta=\frac{(2N+4)\kappa}{(2N+4)\kappa+N}$ *and* $C=C(\kappa,K)\big(-\mathcal{F}(u_0,v_0)\big)^{\frac{1-\theta}{\theta}}$ *with some constant* $C(\kappa,K)$ *depending on* κ *and* K.

Proof. Recalling Lemma 4.2, we see that there exists a positive constant $C_1=C_1(\kappa)$ such that the second solution component v of system (1.8) satisfies

$$\begin{aligned}v(x,t)&\leq C_1\Big(\|u_0\|_{L^1(B_R)}+v_*|B_R|+\|v_0\|_{L^1(B_R)}+\|\nabla v_0\|_{L^2(B_R)}\Big)|x|^{-\kappa}\\&\leq C_1\Big(\|u_0\|_{L^1(B_R)}+v_*|B_R|+\|v_0\|_{L^2(B_R)}|B_R|^{\frac{1}{2}}+\|\nabla v_0\|_{L^2(B_R)}\Big)|x|^{-\kappa}\end{aligned}\tag{4.32}$$

for all $(x,t)\in B_R\backslash\{0\}\times(0,T_{\max})$. Thus if we set

$$m:=\|u_0\|_{L^1(B_R)},\quad M:=\max\big\{m,\overline{M}|B_R|^{\frac{1}{2}}\big\},\quad B:=C_1\big(m+v_*|B_R|+\overline{M}|B_R|^{\frac{1}{2}}+\overline{M}\big),$$

then we see from (2.1), (2.3) and (4.32) that

$$\big(u(\cdot,t),v(\cdot,t)\big)\in\mathcal{S}(m,M,B,\kappa,v_*)\qquad \text{for all}\quad t\in(0,T_{\max})$$

due to $\kappa>N-2$, and that

$$M\leq C_2\big(\overline{M}+1\big)\qquad\text{and}\qquad B\leq C_2\big(\overline{M}+1\big)$$

for some positive constant $C_2 = C_2(m, \kappa, v_*)$. Thanks to Lemma 4.8, the assumption (1.11) with $0 < \alpha < 1 - \frac{2}{N}$ entails that

$$\mathcal{D}^\theta\big(u(\cdot,t), v(\cdot,t)\big) \geq \frac{-\mathcal{F}\big(u(\cdot,t), v(\cdot,t)\big)}{C_3\Big(M^{\frac{2N+4}{N}} + B^{\frac{N+2}{N}} + 1\Big)} - 1 \geq \frac{-\mathcal{F}\big(u(\cdot,t), v(\cdot,t)\big)}{C_4\Big(\overline{M}^{\frac{2N+4}{N}} + 1\Big)} - 1$$

for some positive constants $C_3 = C_3(m, \kappa, v_*)$ and $C_4 = C_4(m, \kappa, v_*)$ and all $t \in (0, T_{\max})$, where $\theta = \frac{(2N+4)\kappa}{(2N+4)\kappa + N}$. Setting

$$K = 2C_4\Big(\overline{M}^{\frac{2N+4}{N}} + 1\Big) =: K(m, \kappa, v_*)\Big(\overline{M}^{\frac{2N+4}{N}} + 1\Big)$$

and noticing that $-\mathcal{F}\big(u(\cdot,t), v(\cdot,t)\big)$ is nondecreasing with respect to time t due to (4.1) and thus that

$$-\mathcal{F}\big(u(\cdot,t), v(\cdot,t)\big) \geq -\mathcal{F}\big(u_0, v_0\big),$$

we see

$$\begin{aligned}
\mathcal{D}^\theta\big(u(\cdot,t), v(\cdot,t)\big) &\geq \frac{-\mathcal{F}\big(u(\cdot,t), v(\cdot,t)\big)}{K} + \Big(\frac{-\mathcal{F}\big(u(\cdot,t), v(\cdot,t)\big)}{K} - 1\Big) \\
&\geq \frac{-\mathcal{F}\big(u(\cdot,t), v(\cdot,t)\big)}{K} + \Big(\frac{-\mathcal{F}\big(u_0, v_0\big)}{K} - 1\Big) \\
&\geq \frac{-\mathcal{F}\big(u(\cdot,t), v(\cdot,t)\big)}{K} + \Big(\frac{K}{K} - 1\Big) \\
&= \frac{-\mathcal{F}\big(u(\cdot,t), v(\cdot,t)\big)}{K} \qquad \text{for all} \quad t \in (0, T_{\max}).
\end{aligned}$$

It then follows from (4.1) that

$$\frac{d}{dt}\Big(-\mathcal{F}\big(u(\cdot,t), v(\cdot,t)\big)\Big) = \mathcal{D}(u, v) \geq \Big(\frac{-\mathcal{F}\big(u(\cdot,t), v(\cdot,t)\big)}{K}\Big)^{\frac{1}{\theta}} \qquad \text{for all} \quad t \in (0, T_{\max}).$$

A direct calculation shows that

$$\mathcal{F}\big(u(\cdot,t), v(\cdot,t)\big) \leq \mathcal{F}\big(u_0, v_0\big)\Big(1 - \frac{1-\theta}{\theta K^{\frac{1}{\theta}}}\Big(-\mathcal{F}\big(u_0, v_0\big)\Big)^{\frac{1-\theta}{\theta}} t\Big)^{-\frac{\theta}{1-\theta}}$$

for all $t \in (0, T_{\max})$. This confirms (4.31). $\square$

Proof of Theorem 1.2. We first fix an arbitrary constant $\kappa > N - 2$ and then let $K(m, v_*) := K(m, \kappa, v_*)$ and $T(m, \overline{M}, v_*) := \Big(C(\kappa, K)\big(- \mathcal{F}(u_0, v_0)\big)^{\frac{1-\theta}{\theta}}\Big)^{-1}$ with $K(m, \kappa, v_*)$ and $C(\kappa, K)$ determined by Lemma 4.9. Therefore, Lemma 4.9 implies that the solution (u, v) will blow up before the finite time $T(m, \overline{M}, v_*)$. This completes the proof of Theorem 1.2. $\square$

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