MULTIPLE KAKEYA EXPANSIONS

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ABSTRACT. We are interested in expansions of the form $x = \sum c_n t_n$ with digits c_n of zeros and ones, where (t_n) is a given sequence of positive real numbers. Kakeya gave a classical theorem ensuring that under a natural condition on the sequence every $x \in [0, \sum t_n]$ has at least one expansion. We give two stronger conditions ensuring that every $x \in (0, \sum t_n)$ has $2^{\aleph_0}$ expansions. One of them leads to significantly shorter proofs of the existence of $2^{\aleph_0}$ expansions where $(1/t_n)$ is a certain Fibonacci or Lucas type sequence, recently proved by a quasi-ergodic approach. We prove analogous results for non-integer base expansions on arbitrary ternary alphabets.

1. Introduction

In integer bases every real number has at most two expansions. In non-integer bases the situation is completely different. For example (see [10]), if $1 < q < \varphi$, where $\varphi := (1+\sqrt{5})/2$ denotes the Golden ratio, then every real number

$$0 < x < \sum_{i=1}^{\infty} \frac{1}{q^i}$$

has $2^{\aleph_0}$ different expansions of the form

$$x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}, \quad (c_i) \in \{0,1\}^{\mathbb{N}}.$$

This is no more true for $q = \varphi$, because then $x = 1$ has "only" $\aleph_0$ expansions (see [8]), but the property holds for the closely related expansions of the form

$$x = \sum_{i=1}^{\infty} \frac{c_i}{F_i}, \quad (c_i) \in \{0,1\}^{\mathbb{N}},$$

where $(F_i) = 1, 1, 2, 3, 5, \dots$ is the sequence of Fibonacci numbers (see [1]); observe that $F_{i+1}/F_i \to \varphi$.

The first two results were extended by Baker [2] to more general alphabets $\{0, 1, \dots, M\}$ where M is an arbitrary positive integer. The third one was extended in [17] to all odd values of M by adapting a bifurcation principle of Sidorov [23, 24] and Baker [2].

In this paper we apply a different approach for the study of this type of questions. It leads to significantly shorter proofs of the above results, and it also allows us to prove some new theorems for non-regular alphabets.

We adopt the following definitions:

Definitions. By a *Kakeya sequence* we mean a sequence (t_n) of positive numbers, satisfying the relations

$$t_n \to 0 \quad \text{as} \quad n \to \infty, \tag{1.1}$$

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and

$$t_n \le \sum_{i=n+1}^{\infty} t_i \quad \text{for every} \quad n \tag{1.2}$$

If (t_n) is a Kakeya sequence, then by a *Kakeya expansion* of a number x we mean a representation of the form

$$x = \sum_{i=1}^{\infty} c_i t_i \quad \text{with} \quad (c_i) \in \{0,1\}^{\mathbb{N}}. \tag{1.3}$$

Examples 1.1.

(i) $(q^{-n})_{n=1}^{\infty}$ is a Kakeya sequence for every real number $q \in (1,2]$.

(ii) $(1/F_n)_{n=1}^{\infty}$ is a Kakeya sequence by [1, Lemma 3.1].

(iii) The sequence $(1/n^p)_{n=1}^{\infty}$ is a Kakeya sequence if and only if $0 < p \le p_0$, where $p_0 \approx 1.728647$ is the real solution of the equation $\zeta(p) = 2$.

First we observe that (1.1) is satisfied if and only if $p > 0$. For $0 < p \le 1$ (1.2) is also satisfied because the series $\sum 1/n^p)$ is divergent. For $p > 1$ the condition (1.2) for $n = 1$ is equivalent to $\zeta(p) \ge 2$, and this is satisfied if and only if $p \le p_0$. Since $p_0 \in (1,2)$, we complete the proof by showing that (1.2) is satisfied for every $n \ge 2$ and $p \in (1,2]$. Indeed, if $p \in (1,2]$, then

$$\begin{aligned}\left(\sum_{i=n+1}^{\infty} \frac{1}{i^p}\right) - \frac{1}{n^p} > \int_{n+1}^{\infty} \frac{1}{x^p}\,dx - \frac{1}{n^p} &= \frac{(n+1)^{1-p}}{p-1} - \frac{1}{n^p} \\ &= \frac{1}{(n+1)^p}\left(\frac{n+1}{p-1} - \left(1+\frac{1}{n}\right)^p\right)\end{aligned}$$

for every positive integer n, and, since $n \ge 2$ and $p \le 2$,

$$\frac{n+1}{p-1} - \left(1+\frac{1}{n}\right)^p \ge \frac{3}{p-1} - \left(1+\frac{1}{2}\right)^p \ge 3 - \left(1+\frac{1}{2}\right)^2 = \frac{3}{4} > 0.$$

The usefulness of "Kakeya sequences" comes from the following classical theorem:

Theorem 1.2 (Kakeya [13, 14])**.** *If (t_n) is a Kakeya sequence, then every $x \in [0, \sum_{n=1}^{\infty} t_n]$ has at least one expansion of the form* (1.3).

Kakeya stated his theorem under the stronger assumption $\sum t_n < \infty$ instead of $t_n \to 0$; the present, more general result was proved in [16, Proposition 1.1]. For the convenience of the reader we present a short proof of Theorem 1.2 in Section 2.

Let us briefly consider the divergent case. If $\sum t_n = \infty$, then the condition (1.2) is obviously satisfied, so that the Kakeya conditions reduce to the following three assumptions:

$$t_n > 0 \quad \text{for every} \quad n, \quad t_n \to 0, \quad \text{and} \quad \sum_{n=1}^{\infty} t_n = \infty. \tag{1.4}$$

In this case Theorem 1.2 may be strengthened:

Proposition 1.3. *Assume* (1.4)*. Then every $x \in (0,\infty]$ has $2^{\aleph_0}$ Kakeya expansions of the form* (1.3)*.*

Example 1.4. The sequence $(1/n^p)_{n=1}^{\infty}$ satisfies the assumptions of Proposition 1.3 if $0 < p \le 1$.

It is obvious that $x=0$ has only the trivial Kakeya expansion 0^∞ for every Kakeya sequence. Under the assumptions (1.4), by Proposition 1.3 they are the only unique Kakeya expansions. We recall a recent characterization of unique expansions Kakeya's original framework:

Theorem 1.5 (Lai, Loreti [20]). *Let (t_n) be a Kakeya sequence satisfying $\sum t_n < \infty$. An expansion of the form (1.3) is the unique Kakeya expansion of x if and only if the following two conditions are satisfied:*

$$\sum_{i=n+1}^{\infty} c_i t_i < t_n \quad \text{whenever} \quad c_n = 0, \tag{1.5}$$

$$\sum_{i=n+1}^{\infty} (1-c_i) t_i < t_n \quad \text{whenever} \quad c_n = 1. \tag{1.6}$$

See also [5] for a related theorem and [11] for a topological description of the set of numbers having unique Kakeya expansions.

Examples 1.6.

(i) It is well-known that all but countably many $x \in [0,1]$ have unique binary expansions. This may also be deduced from Theorem 1.5 as follows. If $(t_n) = (2^{-n})$, then the conditions (1.5) and (1.6) are satisfied for every sequence (c_i) containing both infinitely many zeros digits and infinitely many one digits. This holds for the expansions of $x \in [0,1]$ that are not binary rationals, hence these numbers have unique expansions.

(ii) The number

$$x := \sum_{i=1}^{\infty} \frac{1}{q^{2i-1}}$$

has a unique expansion in a base $q>1$ if and only if $q>\varphi$. Indeed, applying Theorem 1.5 with $t_n := q^{-n}$, all conditions (1.5) and (1.6) are equivalent to

$$\sum_{i=1}^{\infty} \frac{1}{q^{2i-1}} < 1,$$

i.e., to $q>\varphi$.

Concerning the possible number of Kakeya expansions we extend two earlier theorems of Sidorov [23, 24] and Baker [3, Theorem 4.1] on non-integer base expansions:

Theorem 1.7. *Let (t_n) be a Kakeya sequence.*

(i) *If none of the numbers $x \in (0, \sum t_i)$ has a unique expansion, then* every $x \in (0, \sum t_i)$ *has infinitely many Kakeya expansions of the form* (1.3).

(ii) *The number of Kakeya expansions of a given real number x is a nonnegative integer, $\aleph_0$ or $2^{\aleph_0}$.*

Example 1.8. Consider the Kakeya sequences (q^{-n}) for $q \in (1,2]$. We have already mentioned that $x=1$ has $2^{\aleph_0}$ expansions if $q \in (1,\varphi)$, and $\aleph_0$ expansions if $q=\varphi$. A special case of a theorem in [9] states that if N be an arbitrary positive integer, and $q>1$ is defined by the equation

$$\sum_{i=1}^{\infty} \frac{a_i}{q^i} = 1 \quad \text{with} \quad (a_i) := 1^9(0^8 1)^N (0^4 1)^\infty,$$

then $x=1$ has exactly N expansions in base q.

In the sequel we are mostly investigating the possible existence of $2^{\aleph_0}$ expansions. Our first general result is the following:

Theorem 1.9. *Let (t_n) be a sequence of positive numbers satisfying $\sum t_n < \infty$.*

(i) *If*

$$t_n \le \sum_{i=n+2}^{\infty} t_i \quad \textit{for all} \quad n \ge 1, \tag{1.7}$$

then 0^∞ and 1^∞ are the only unique expansions of the form

$$x = \sum_{i=1}^{\infty} c_i t_i, \quad (c_i) \in \{0,1\}^{\mathbb{N}}. \tag{1.8}$$

(ii) *If*

$$t_n < \sum_{i=n+2}^{\infty} t_i \quad \textit{for all} \quad n \ge 1, \tag{1.9}$$

then every $x \in (0, \sum_{i=1}^{\infty} t_i)$ has $2^{\aleph_0}$ expansions of the form (1.8).

Examples 1.10.

(i) For $t_n := 1/q^n$ the conditions (1.7) and (1.9) are equivalent to $1 < q \le \varphi$ and $1 < q < \varphi$, respectively; in this way we recover two well-known results on q-expansions.

(ii) For $t_n := 1/n^p$ the conditions (1.7) and (1.9) are equivalent to $1 < p \le p_1$ and $1 < p < p_1$, respectively, where $p_1 \approx 1.58$ is defined by $\zeta(p_1) = 2 + 2^{-p_1}$.

Indeed, $\sum 1/n^p$ is convergent for every $p > 1$, and the conditions (1.7) and (1.9) are equivalent to

$$\frac{1}{n^p} \le \sum_{i=n+2}^{\infty} \frac{1}{i^p} \quad \text{for all} \quad n \ge 1, \quad \text{and} \quad \frac{1}{n^p} < \sum_{i=n+2}^{\infty} \frac{1}{i^p} \quad \text{for all} \quad n \ge 1,$$

respectively. For $n = 1$ they are equivalent to $\zeta(p) \ge 2 + \frac{1}{2^p}$ and to $\zeta(p) > 2 + \frac{1}{2^p}$, while they are satisfied for every $n \ge 2$. Indeed,

$$\left(\sum_{i=n+2}^{\infty} \frac{1}{i^p}\right) - \frac{1}{n^p} > \int_{n+2}^{\infty} \frac{1}{x^p}\, dx - \frac{1}{n^p} = \frac{(n+2)^{1-p}}{p-1} - \frac{1}{n^p}$$
$$= \frac{1}{(n+2)^p}\left(\frac{n+2}{p-1} - \left(1 + \frac{2}{n}\right)^p\right)$$

for every positive integer $n \ge 1$. Furthermore, the last expression is positive if $n \ge 2$ and $p \le 2$, because

$$\frac{n+2}{p-1} - \left(1 + \frac{2}{n}\right)^p \ge \frac{4}{p-1} - \left(1 + \frac{2}{2}\right)^p \ge 4 - 2^2 = 0.$$

The next theorem allows us to consider q-expansions with more than two digits:

Theorem 1.11. *Let M be a positive integer, and (p_n) a sequence of positive numbers, converging to zero. Assume that*

$$p_n \le (M+1)p_{n+1} \quad \textit{for every} \quad n, \tag{1.10}$$

and that

$$p_n < \left[\frac{M-1}{2}\right] p_{n+1} + \sum_{i=n+2}^{\infty} M p_i \tag{1.11}$$

for infinitely many indices n, where $[a]$ denotes the (lower) integer part of a. Then every

$$x \in \left(0, M\sum_{i=1}^{\infty} p_i\right)$$

has $2^{\aleph_0}$ expansions of the form

$$x = \sum_{i=1}^{\infty} c_i p_i, \quad (c_i) \in \{0,1,\ldots,M\}^{\mathbb{N}}.$$

For $M = 1$ Theorem 1.11 essentially reduces to [1, Theorem 1.4].

Example 1.12. We rephrase in our setting a short proof of a theorem of Baker, given in [7, Example 4.4 (ii)]. We define

$$\varphi_M := \begin{cases} (\mu + \sqrt{\mu^2+4\mu})/2 & \text{if } M = 2\mu - 1 \text{ is odd,} \\ \mu + 1 & \text{if } M = 2\mu \text{ is even,} \end{cases} \qquad \mu = 1, 2, \ldots,$$

and we observe that

$$1 = \frac{\mu - 1}{\varphi_M} + \sum_{i=2}^{\infty} \frac{M}{\varphi_M^i}. \tag{1.12}$$

Indeed, the right hand side is equal to

$$\frac{\mu-1}{\varphi_M} + \frac{M}{\varphi_M(\varphi_M - 1)} = \frac{(\mu-1)(\varphi_M - 1) + M}{\varphi_M(\varphi_M - 1)},$$

and it remains to show that the numerator is equal to the denominator in the last fraction, or equivalently that

$$\varphi_M^2 = \mu\varphi_M + (M + 1 - \mu).$$

The last equality is obvious if $M = 2\mu$ and $\varphi_M = \mu + 1$. If $M = 2\mu - 1$, then the relation takes the form $\varphi_M^2 - \mu\varphi_M - \mu = 0$, and φ_M is clearly a solution of this second-order equation.

Now we show that the sequence $p_n := q^{-n}$ satisfies the conditions of Theorem 1.11 for every fixed $1 < q < \varphi_M$. Indeed, $p_n \to 0$ because $q > 1$, and $p_n \le (M+1)p_{n+1}$ for every n because

$$\frac{p_n}{p_{n+1}} = q < \varphi_M,$$

and $\varphi_M \le \mu + 1 \le M + 1$ from the definition of φ_M. Finally, the crucial condition (1.11) follows from (1.12) for every n because $\left[\frac{M-1}{2}\right] = \mu - 1$ and $1 < q < \varphi_M$.

Applying Theorem 1.11 we conclude that every $0 < x < \frac{M}{q-1}$ has $2^{\aleph_0}$ expansions of the form

$$x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}, \quad (c_i) \in \{0,1,\ldots,M\}^{\mathbb{N}}$$

for every base $1 < q < \varphi_M$.

Example 1.13. Let μ be a positive integer, $M = 2\mu - 1$, and introduce a sequence of integers F_i by the formulas

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = \mu(F_{n-1} + F_{n-2}) \quad \text{for} \quad n = 2, 3, \ldots .$$

Then the sequence $p_n := 1/F_n$ satisfies the conditions of Theorem 1.11. Indeed, $p_n \to 0$ because $F_n \ge n - 1$ for every n by an easy induction, and $p_n \le (M+1)p_{n+1}$ because $F_{n+1} \le (\mu + 1)F_n$ for every $n \ge 1$ by an easy induction, and hence

$$\frac{p_n}{p_{n+1}} = \frac{F_{n+1}}{F_n} \le \mu + 1 \le M + 1$$

for all $n \ge 1$. Finally, the crucial relation (1.11) holds because

$$\frac{1}{F_n} < \frac{\mu - 1}{F_{n+1}} + \sum_{i=n+2}^{\infty} \frac{M}{F_i}$$

for every odd index n by [17, Lemma 2.1]. This yields a new, shorter proof of [17, Theorem 1.1]: every $0 < x < \sum_{i=1}^{\infty} \frac{M}{F_i}$ has $2^{\aleph_0}$ expansions of the form

$$x = \sum_{i=1}^{\infty} \frac{c_i}{F_i}, \quad (c_i) \in \{0, 1, \ldots, M\}^{\mathbb{N}}.$$

We recall from [17] that $F_{n+1}/F_n \to \varphi_M$, so that the last expansions are close to the expansions in base φ_M, excluded in Example 1.12.

Remark 1.14. Let M be a positive integer, $q \in (1, \varphi_M)$ and $\lambda > 0$ two real numbers, and let F_n denote the closest integer to λq^n. Then there exists a positive integer N such that the sequence $p_n := 1/F_{n+N}$ satisfies the assumptions of Theorem 1.11. Indeed, if $k \to \infty$, then

$$F_k > \lambda q^k \to \infty,$$
$$\frac{F_{k+1}}{F_k} < \frac{\lambda q^{k+1} + 1}{\lambda q^k - 1} \to q < \varphi_M < M + 1,$$

and

$$\left[\frac{M-1}{2}\right] \frac{F_k}{F_{k+1}} + \sum_{i=n+2}^{\infty} \frac{MF_k}{F_i} \to \left[\frac{M-1}{2}\right] \frac{1}{q} + \sum_{j=2}^{\infty} \frac{M}{q^j}$$
$$> \left[\frac{M-1}{2}\right] \frac{1}{\varphi_M} + \sum_{j=2}^{\infty} \frac{M}{\varphi_M^j} = 1,$$

so that $p_n \to 0$, and (1.10)–(1.11) are satisfied if N is sufficiently large.

We do not know whether this remains true for $q = \varphi_M$; this would generalize Example 1.13.

Now we study analogous problems for arbitrary three-letter alphabets [15, 4, 18, 25]. By an elementary normalization argument it is sufficient to consider alphabets $\{0, 1, m\}$ either with real numbers $m \ge 2$, or with real numbers $m \in (1, 2]$.

The following variants of Theorem 1.9 and Theorem 1.11 hold:

Theorem 1.15. *Let (p_n) be a sequence of positive numbers, converging to zero, and $m \in (1, 2]$ a real number. If*

$$p_n < \sum_{i=n+2}^{\infty} p_i \quad \textit{for every} \quad n \ge 1,$$

then every $x \in (0, \sum_{i=1}^{\infty} mp_i)$ *has* $2^{\aleph_0}$ *expansions* $x = \sum_{i=1}^{\infty} c_i p_i$ *with* $c_i \in \{0,1,m\}$.

Theorem 1.16. *Let* (p_n) *be a sequence of positive numbers, converging to zero, and* $m \in (1,2]$ *a real number. Assume that*

$$p_n \le (m+1)p_{n+1} \quad \text{for every} \quad n \ge 1,$$

and

$$p_n < m \sum_{i=n+2}^{\infty} p_i \quad \text{for infinitely many} \quad n.$$

Then every $x \in (0, \sum_{i=1}^{\infty} mp_i)$ *has* $2^{\aleph_0}$ *expansions* $x = \sum_{i=1}^{\infty} c_i p_i$ *with* $c_i \in \{0,1,m\}$.

Example 1.17. Given $m \in (1,2]$ and $q > 1$, we consider expansions of the form

$$x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}, \quad (c_i) \in \{0,1,m\}^{\mathbb{N}}. \tag{1.13}$$

It follows from [21, Proposition 2.1] that if $1 < q \le 1+m$, then every $x \in \left[0, \frac{m}{q-1}\right]$ has at least one expansion. Theorem 1.15 and Theorem 1.16 imply that every $x \in \left(0, \frac{m}{q-1}\right)$ has $2^{\aleph_0}$ expansions if $1 < q < 2$ and $1 < q < \frac{1+\sqrt{1+4m}}{2}$, respectively.

The last theorem of this paper improves the results of Example 1.17. We recall that for each $m > 1$ there exists a critical base G_m such that for $1 < q < G_m$ only the endpoints of $\left[0, \frac{m}{q-1}\right]$ have unique expansions of the form (1.13), while for $q > G_m$ there are other unique expansions as well. These bases G_m have been determined in [15], and then in [4] by a different proof.

Theorem 1.18. *If* $m > 1$ *and* $1 < q < G_m$, *then every* $x \in \left(0, \frac{m}{q-1}\right)$ *has* $2^{\aleph_0}$ *expansions of the form* (1.13).

A slightly weaker result was proved in the first version of this paper; see Remark 4.1 below.

The rest of the paper is organized as follows. In Section 2 we recall a short proof of Theorem 1.2 for the reader's convenience, and we prove Proposition 1.3 and Theorem 1.7. Theorem 1.9 and Theorem 1.11 are proved in Section 3. Finally, Theorem 1.15, Theorem 1.16 and Theorem 1.18 are proved in Section 4.

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2. Proof of Theorem 1.2, Proposition 1.3 and Theorem 1.7

Proof of Theorem 1.2. Using the greedy algorithm, we may construct the lexicographically largest sequence $(c_i) \in \{0,1\}^{\mathbb{N}}$ satisfying

$$\sum_{i=1}^{n} c_i t_i \le x \quad \text{for every} \quad n \ge 1.$$

We observe that (c_i) cannot end with 01^∞ because if $t_n = 0$, and $t_i = 1$ for all $i > n$, then $t_n > t_{n+1} + t_{n+2} + \cdots$ by the lexicographic maximality, contradicting the condition (1.2).

If $t_n = 0$ for infinitely many n, then we conclude by letting $n \to \infty$ in the corresponding inequalities

$$\left(\sum_{i=1}^{n-1} c_i t_i\right) \le x < \left(\sum_{i=1}^{n-1} c_i t_i\right) + t_n.$$

In the only remaining case we have $(c_i) = 1^\infty$, and our choice $x \le \sum t_i$ implies that

$$\sum_{i=1}^{\infty} c_i t_i \le x \le \sum_{i=1}^{\infty} t_i = \sum_{i=1}^{\infty} c_i t_i. \qquad \square$$

Proof of Proposition 1.3. We adapt the proof of [10, Theorem 3]. Given any $x \in (0, \infty]$, using the assumption $t_n \to 0$ we may remove from (t_n) a subsequence (t_{n_j}) such that

$$\sum_{j=1}^{\infty} t_{n_j} < x,$$

and the remaining sequence (p_n) is still infinite. Then (p_n) still satisfies the conditions (1.4). Therefore, for each sequence $(c_{n_j}) \in \{0,1\}^{\mathbb{N}}$, thanks to the inequality $x - \sum_{j=1}^{\infty} c_{n_j} t_{n_j} > 0$ there exists a Kakeya expansion

$$x - \sum_{j=1}^{\infty} c_{n_j} t_{n_j} = \sum_{n=1}^{\infty} c'_n p_n.$$

This may be rewritten in the form (1.3), and each of the $2^{\aleph_0}$ sequences (c_{n_j}) leads to a different Kakeya expansion of x. There cannot be more because there are "only" $2^{\aleph_0}$ sequences $(c_i) \in \{0,1\}^{\mathbb{N}}$. $\square$

Proof of Theorem 1.7. By Proposition 1.3 we may assume that $\sum t_n < \infty$.

(i) Similarly to the proof of Proposition 1.3, it suffices to show that x has at least $2^{\aleph_0}$ expansions. We apply Sidorov's bifurcation method [23, 24], illustrated by Figure 1. We use the notation

$$\pi_n(c_1 \cdots c_k) := \sum_{i=1}^{k} c_i t_{n-1+i}.$$

It follows from the non-uniqueness assumption that for each $n \ge 1$, if $y \in (0, \sum_{i=n}^{\infty} t_i)$, then there exist two expansions of y of the form

$$\sum_{i=n}^{\infty} c_i t_i \quad \text{with} \quad (c_i) \in \{0,1\}^{\mathbb{N}}, \tag{2.1}$$

starting with different words $w_0(n,y)$ and $w_1(n,y)$ of the same length. By choosing these words long enough, we may also assume that $\pi_n(w_0(n,y))$ and $\pi_n(w_1(n,y))$ are larger than $\frac{y}{2}$ if $y < \infty$, and they are larger than one if $y = \infty$. (The latter case occurs only if $\sum_{i=1}^{\infty} t_i = +\infty$.)

Now given an arbitrary point $x \in (0, \sum_{i=1}^{\infty} t_i)$, for each sequence $(a_j) \in \{0,1\}^\infty$ we construct an expansion $x = \sum_{i=1}^{\infty} c_i t_i$ of the form

$$c_1 c_2 \ldots = w_{a_1}(n_1, x_1) w_{a_2}(n_2, x_2) w_{a_3}(n_3, x_3) \ldots.$$

where the parameters $n_1 < n_2 < \cdots$ and $x_1 > x_2 > \cdots$ are defined recursively as follows.

First we define $n_1 := 1$ and $x_1 := x$. If $w_{a_1}(1, x) \dots w_{a_j}(n_j, x_j)$ has already been defined for some $j \geq 1$, then let n_{j+1} denote the length of the word $w_{a_1}(n_1, x_1) \dots w_{a_j}(n_j, x_j)$, and set

$$x_{j+1} = x - \pi(w_{a_1}(n_1, x_1) \dots w_{a_j}(n_j, x_j)).$$

Then, since $w_{a_j}(n_j, x_j)$ is the beginning of an expansion of the form (2.1) with $n = n_j$, $x_{j+1} \in \left(0, \sum_{i=n_{j+1}}^{\infty} c_i t_i\right)$, and therefore $w_{a_{j+1}}(n_{j+1}, x_{j+1})$ is well defined. It follows from the choice of the length of the words $w_0(n, y)$ and $w_1(n, y)$ that $x = \sum_{i=n}^{\infty} c_i t_i$.

We conclude by observing that different sequences (a_j) lead to different expansions of x, so that x has at least $2^{\aleph_0}$ expansions.

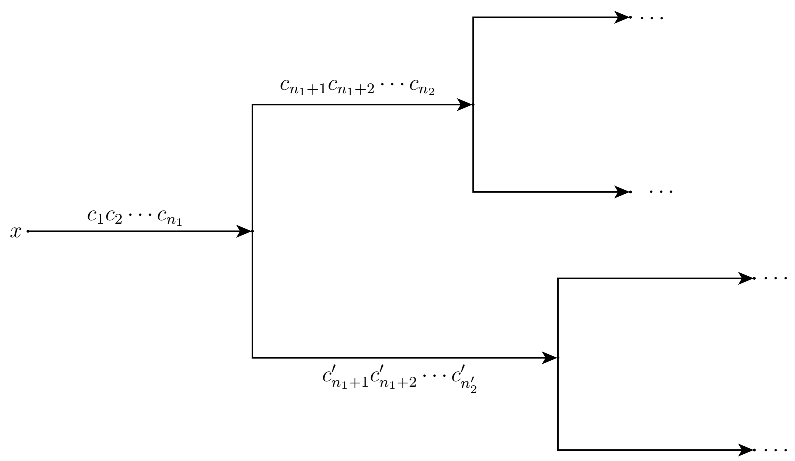

Figure 1. Sidorov's bifurcation method

(ii) We adapt the proof of [6, Theorem 2.3.1] given for expansions in non-integer bases. If $x \notin [0, \sum t_i]$, then x has no expansion. If $x \in [0, \sum t_i]$, then x has at least one expansion. It remains to show that if x has uncountably many expansions, then it has necessarily $2^{\aleph_0}$ expansions.

Consider the discrete topology on $\{0, 1\}$, and endow the set $\{0, 1\}^{\mathbb{N}}$ with the Tychonoff product topology. One easily verifies that the set $E(x)$ of Kakeya expansions of x is a closed subset of the Polish space $\{0, 1\}^{\mathbb{N}}$, and hence $E(x)$ is a Polish space, too. The assertion follows from the well-known fact that uncountable Polish spaces have the cardinality of the continuum. $\square$

3. Proof of Theorem 1.9 and Theorem 1.11

Proof of Theorem 1.9. (i) Let (c_i) be a unique expansion. If $c_n = 0$ and $c_{n+1} = 1$ for some n, then we infer from Theorem 1.5 that

$$\sum_{i=n+1}^{\infty} c_i t_i < t_n \quad \text{and} \quad \sum_{i=n+2}^{\infty} (1 - c_i) t_i < t_{n+1}.$$

Adding them and using the equality $c_{n+1} = 1$ we obtain that

$$\sum_{i=n+2}^{\infty} t_i < t_n,$$

contradicting (1.7). Similarly, if $c_n = 1$ and $c_{n+1} = 0$ for some n, then we infer from Theorem 1.5 that

$$\sum_{i=n+1}^{\infty} (1-c_i)t_i < t_n \quad \text{and} \quad \sum_{i=n+2}^{\infty} c_i t_i < t_{n+1},$$

and they lead to the same contradiction.

(ii) As in the proof of Theorem 1.7 (i), it suffices to show that for each $x \in (0, \sum_{i=1}^{\infty} t_i)$ there exist an integer $n \ge 1$ and $c_1, \dots, c_{n-1} \in \{0,1\}$ such that

$$x - \sum_{i=1}^{n-1} c_i t_i \in \left(t_n, \sum_{i=n+1}^{\infty} t_i \right).$$

It follows from (1.9) that

$$\bigcup_{n=1}^{\infty} \left(t_n, \sum_{i=n+1}^{\infty} t_i \right) = \left(0, \sum_{i=2}^{\infty} t_i \right),$$
$$\bigcup_{n=1}^{\infty} \left(\sum_{i=1}^{n} t_i, \sum_{i=1}^{n-1} t_i + \sum_{i=n+1}^{\infty} t_i \right) = \left(t_1, \sum_{i=1}^{\infty} t_i \right)$$

and

$$\left(0, \sum_{i=2}^{\infty} t_i \right) \bigcup \left(t_1, \sum_{i=1}^{\infty} t_i \right) = \left(0, \sum_{i=1}^{\infty} t_i \right).$$

Therefore for each $x \in (0, \sum_{i=1}^{\infty} t_i)$ there exists an $n \ge 1$ such that

$$x \in \left(t_n, \sum_{i=n+1}^{\infty} t_i \right) \quad \text{or} \quad x \in \left(\sum_{i=1}^{n} t_i, \sum_{i=1}^{n-1} t_i + \sum_{i=n+1}^{\infty} t_i \right);$$

our claim follows with $c_1 = \cdots = c_{n-1} = 0$ in the first case, and with $c_1 = \cdots = c_{n-1} = 1$ in the second case. ☐

Proof of Theorem 1.11. Let us write $m := \left[\frac{M+1}{2}\right]$ for brevity; then $2m - 1 \le M$, and the condition (1.11) takes the form

$$p_n < (m-1)p_{n+1} + \sum_{i=n+2}^{\infty} M p_i \tag{3.1}$$

for infinitely many n.

First step. Given an arbitrarily small $\varepsilon > 0$, using the relation $p_n \to 0$ and the condition (3.1), we may construct by induction an increasing sequence (n_j) of indices such that $\sum p_{n_j} < \varepsilon$, and

$$p_{n_j} < (m-1)p_{n_j+1} + M \sum_{i=n_j+2}^{n_{j+1}} p_i \tag{3.2}$$

for every $j \ge 1$.

Repeating $m-1$ times each p_i with $i = n_j + 1$, and M times each remaining element p_i, we obtain a sequence (t_i) of the form

$$p_1 \cdots p_1 \; p_2 \cdots p_2 \; p_3 \cdots p_3 \; p_4 \cdots$$

with consecutive blocks of $m-1$ or M identical elements.

We claim that (t_i) satisfies the hypotheses of Theorem 1.2. The property $t_i \to 0$ being obvious, it remains to show that

$$t_k \le \sum_{i=k+1}^{\infty} t_i$$

for every $k \ge 1$. This is also obvious if $t_k \le t_{k+1}$, hence it suffices to consider the indices k for which $t_k > t_{k+1}$; then t_k is the last element of a block of consecutive equal elements. If $t_k = p_{n_j}$ for some j, then the condition follows from (3.2):

$$t_k = p_{n_j} < (m-1)p_{n_j+1} + M \sum_{i=n_j+2}^{n_{j+1}} p_i = \sum_{i=k+1}^{k+(m-1)+M(n_{j+1}-n_j-1)} t_i < \sum_{i=k+1}^{\infty} t_i.$$

Since $n_j \to +\infty$, using a backward induction it remains to show that if the inequality holds for the last element t_n of a block of length M, then it also holds for t_{n-M}. This follows from our assumption (1.10):

$$t_{n-M} \le (M+1)t_{n-M+1} = \left(\sum_{i=n-M+1}^{n} t_i \right) + t_n \le \sum_{i=n-M+1}^{\infty} t_i.$$

Second step. Fix $0 < x < M \sum_{i=1}^{\infty} p_i$ arbitrarily, and take the Kakeya sequence (t_i) of the first step with

$$\varepsilon := \min \left\{ \frac{x}{M}, \frac{M(\sum p_i) - x}{M+1-m} \right\};$$

then we have

$$m \sum_{j=1}^{\infty} p_{n_j} < x < M \sum_{i=1}^{\infty} p_i - (M+1-m) \sum_{j=1}^{\infty} p_{n_j} = \sum_{i=1}^{\infty} t_i.$$

Now choose an arbitrary sequence $(\delta_{n_j})_{j=1}^{\infty} \in \{0, m\}^{\mathbb{N}}$. Then we infer from the last inequalities the relations

$$0 < x - \sum_{j=N}^{\infty} \delta_{n_j} p_{n_j} < \sum_{i=1}^{\infty} t_i,$$

and hence, applying Theorem 1.2, the existence of a sequence (ε_i) satisfying the equality

$$x - \sum_{j=N}^{\infty} \delta_{n_j} p_{n_j} = \sum_{i=1}^{\infty} \varepsilon_i p_i,$$

and the extra conditions

$$\varepsilon_{n_j} \in \{0, \ldots, m-1\} \quad \text{for all} \quad j, \quad \text{and} \quad \varepsilon_i \in \{0, \ldots, M\} \quad \text{otherwise.}$$

Setting

$$c_i := \begin{cases} \delta_{n_j} + \varepsilon_{n_j} & \text{if } i = n_j \text{ for some } j, \\ \varepsilon_i & \text{otherwise} \end{cases}$$

we obtain a sequence $(c_i) \in \{0, 1, \ldots, 2m-1\}^{\mathbb{N}} \subset \{0, 1, \ldots, M\}^{\mathbb{N}}$ such that

$$\sum_{i=1}^{\infty} c_i p_i = x.$$

Since there are $2^{\aleph_0}$ sequences $(\delta_{n_j}) \in \{0, m\}^{\mathbb{N}}$, we complete the proof by observing that the map $(\delta_{n_j}) \mapsto (c_i)$ is one-to-one. Indeed,

$$\delta_{n_j} = 0 \Longrightarrow c_{n_j} \in \{0, \dots, m-1\} \quad \text{and} \quad \delta_{n_j} = m \Longrightarrow c_{n_j} \in \{m, \dots, 2m-1\} ;$$

hence $\delta_{n_j} = m \left[\dfrac{c_{n_j}}{m} \right]$ for every j, so that the sequence (δ_{n_j}) is uniquely determined by (c_i). $\square$

4. Proof of Theorem 1.15, Theorem 1.16 and Theorem 1.18

Since the proofs of Theorem 1.15 and Theorem 1.16 are simple adaptations of the proofs of Theorem 1.9 and Theorem 1.11, we only sketch them.

Sketch of proof of Theorem 1.15. As in the proof of Theorem 1.7 (i), it suffices to show that for each $x \in (0, \sum_{i=1}^{\infty} mp_i)$ there exist $c_1, \dots, c_n$ and c_n' in $\{0, 1, m\}$ such that $c_n < c_n'$, and

$$x - \sum_{i=1}^{n} c_i p_i \in \left(0, \sum_{i=n+1}^{\infty} mp_i\right), \quad x - \sum_{i=1}^{n-1} c_i p_i - c_n' p_n \in \left(0, \sum_{i=n+1}^{\infty} mp_i\right).$$

They are equivalent to the existence of $n \geq 1$ and $c_1, \dots, c_{n-1} \in \{0, 1, m\}$ such that

$$x - \sum_{i=1}^{n-1} c_i p_i \in \left(p_n, p_n + \sum_{i=n+1}^{\infty} mp_i\right).$$

We may assume without loss of generality that $p_n \geq p_{n+1}$ for all $n \geq 1$. (Note that reordering the sequence does not change the cardinality of the expansions, and exchanging p_n and p_{n+1} when $p_n < p_{n+1}$ preserves the condition that $p_k < \sum_{i=k+1}^{\infty} p_i$ for all k.) Then for each $x \in (0, \sum_{i=1}^{\infty} mp_i)$ there exists an $n \geq 1$ such that

$$x \in \left(p_n, p_n + \sum_{i=n+1}^{\infty} mp_i\right) \quad \text{or} \quad x \in \left(\sum_{i=1}^{n-1} mp_i + p_n, \sum_{i=1}^{n-1} mp_i + p_n + \sum_{i=n+1}^{\infty} mp_i\right);$$

our claim follows with $c_1 = \cdots = c_{n-1} = 0$ in the first case, and with $c_1 = \cdots = c_{n-1} = 1$ in the second case. $\square$

Sketch of proof of Theorem 1.16. Using the assumptions we may construct a sequence $n_1 < n_2 < \cdots$ of positive integers such that

$$p_{n_j} < m \sum_{i=n_j+2}^{n_{j+1}} p_i$$

for every j, and

$$m \sum_{j=1}^{\infty} p_{n_j} < x < m \sum_{i=1}^{\infty} p_i - m \sum_{j=1}^{\infty} p_{n_j}.$$

Let (t_i) denote the sequence obtained from (p_i) by removing the elements p_{n_j+1}; then

$$t_n \leq m \sum_{i=n+1}^{\infty} t_i \quad \text{for every} \quad n \geq 1.$$

Given any sequence $(\delta_{n_j}) \in \{0, 1, m\}^{\mathbb{N}}$, there exists a lexicographically largest sequence (c_i) satisfying

$$\sum_{i=1}^{\infty} c_i t_i \leq x - \sum_{j=1}^{\infty} \delta_{n_j} p_{n_j}.$$

Similarly to the proof of Theorem 1.2, it may be shown that we have equality here, yielding an expansion of x of the form $\sum_{j=1}^{\infty} c_j p_j$, and different sequences (δ_{n_j}) lead to different expansions of x. $\square$

Proof of Theorem 1.18. We may assume that $m \in (1, 2]$; then $m \le 2 \le G_m < 1 + \sqrt{m}$ and $1 < q < G_m$ by [15] and [4]. Furthermore, $G_2 = 2$, and $m < G_m$ otherwise.

We observe that

$$\frac{1}{q} < \frac{m}{q(q-1)} \quad \text{and} \quad \frac{m}{q} < \frac{1}{q} + \frac{m}{q(q-1)}. \tag{4.1}$$

Indeed, the inequalities are equivalent to $q < 1 + m$ and $q < 1 + \frac{m}{m-1}$, respectively, and both follow from the inequality $q < 1 + \sqrt{m}$.

By Sidorov's bifurcation method it suffices to show that for every $x \in (0, \frac{m}{q-1})$ *there exist* $c_1, \dots, c_{n-1} \in \{0, 1, m\}$ *such that*

$$q^n \left(x - \sum_{i=1}^{n-1} \frac{c_i}{q^i} \right) - c_n \in \left(0, \frac{m}{q-1} \right) \tag{4.2}$$

for at least two choices of $c_n \in \{0, 1, m\}$. This is possible if and only if

$$q^n \left(x - \sum_{i=1}^{n-1} \frac{c_i}{q^i} \right) \in \left(\frac{1}{q}, \frac{m}{q(q-1)} \right) \cup \left(\frac{m}{q}, \frac{1}{q} + \frac{m}{q(q-1)} \right).$$

Suppose that this is not true for some $x \in (0, \frac{m}{q-1})$. Then every expansion $x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}$ satisfies the condition

$$\sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} \in \left[0, \frac{1}{q} \right] \cup \left[\frac{m}{q(q-1)}, \frac{m}{q} \right] \cup \left[\frac{1}{q} + \frac{m}{q(q-1)}, \frac{m}{q-1} \right] \quad \text{for all } n \ge 1. \tag{4.3}$$

(The three intervals are disjoint by (4.1), and the middle interval is empty if $q < 2$.)

We may assume that $x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}$ with

$$c_n = \begin{cases} 0 & \text{if } \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} \in \left[0, \frac{1}{q} \right], \\ 1 & \text{if } \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} \in \left[\frac{m}{q(q-1)}, \frac{m}{q} \right], \\ m & \text{if } \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} \in \left[\frac{1}{q} + \frac{m}{q(q-1)}, \frac{m}{q-1} \right]. \end{cases} \tag{4.4}$$

If $q \ge 2$, then using (4.1) and proceeding by induction on n, we may change $c_n = 1$ to

$$c_n = \begin{cases} 0 & \text{if } \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} = \frac{1}{q}, \\ m & \text{if } \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} = \frac{1}{q} + \frac{m}{q(q-1)}. \end{cases}$$

Then we obtain a new expansion of x satisfying (4.4), and still satisfying (4.3) by our indirect assumption.

Observe that (c_i) does not contain the words $0m$ and $1m$. Indeed, if $c_n = 0$ and $c_{n+1} = m$ for some n, then

$$\sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} \le \frac{1}{q} \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} \ge \frac{1}{q} + \frac{m}{q(q-1)},$$

whence

$$\frac{1}{q} + \frac{m}{q(q-1)} \le \sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} = q \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} \le 1.$$

This is, however, impossible because the relations $1 < q < 1 + \sqrt{m}$ imply $(q-1)^2 < m$, and the latter is equivalent to

$$1 < \frac{1}{q} + \frac{m}{q(q-1)}. \tag{4.5}$$

Next, if $c_n = 1$ and $c_{n+1} = m$ for some n, then

$$\sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} \le \frac{m}{q} \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} \ge \frac{1}{q} + \frac{m}{q(q-1)},$$

whence

$$\frac{1}{q} + \frac{m}{q(q-1)} \le \sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} = q \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} - 1 \le m - 1.$$

Since $m - 1 \le 1$, this contradicts (4.5) again.

There exists a positive integer k such that (c_i) does not contain the blocks $m0^k$ and 10^k. Indeed the existence of such a block would imply

$$\frac{1}{q} + \frac{m}{q(q-1)} \le \frac{m}{q} + \frac{m}{q^{k+1}(q-1)} \quad \text{or} \quad \frac{m}{q(q-1)} \le \frac{1}{q} + \frac{m}{q^{k+1}(q-1)}$$

by (4.4), and none of these inequalities holds large values of k by (4.1). Since $x > 0$, it follows from the above considerations that (c_i) does not end with 0^∞.

We claim that

$$\inf_{n\ :\ c_n=1} \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{G_m^i} \le \frac{m}{G_m(G_m-1)} \quad \text{or} \quad \sup_{n\ :\ c_n=1} \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{G_m^i} \ge \frac{m}{G_m}. \tag{4.6}$$

This obviously holds if $m = 2$, because then $G_m = 2$, so that the right sides of the two inequalities in (4.6) coincide.

Assume on the contrary that (4.6) fails. Then $1 < m < 2$, whence $m < G_m$, and

$$\frac{m}{q(q-1)} < \inf_{n\ :\ c_n=1} \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} \le \sup_{n\ :\ c_n=1} \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} < \frac{m}{q} \tag{4.7}$$

if $m < q < G_m$ is sufficiently close to G_m. This follows from the estimates

$$\sup_{n\ :\ c_n=1} \left| \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{G_m^i} - \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} \right| \le \sum_{i=1}^{\infty} m \left| \frac{1}{q^i} - \frac{1}{G_m^i} \right| = \frac{m}{q-1} - \frac{m}{G_m - 1} \to 0$$

as $q \nearrow G_m$. We infer from (4.7) that

$$\sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} < \frac{1}{q} \quad \text{whenever} \quad c_n = 0. \tag{4.8}$$

Indeed, there exists an $\ell \ge 1$ such that $c_n = \cdots c_{n+\ell-1} = 0$ and $c_{n+\ell} = 1$, and then

$$\sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} = \frac{1}{q^\ell} \sum_{i=1}^{\infty} \frac{c_{n+k-1+i}}{q^i} < \frac{1}{q^\ell} \cdot \frac{m}{q} \le \frac{m}{q^2} < \frac{1}{q}.$$

We deduce from (4.7) and (4.8) that (c_i) is a unique expansion. This is, however, a contradiction because in the bases $< G_m$ the only unique expansions are 0^∞ and m^∞.

Let us rewrite (4.6) in the form

$$\sup_{n\ :\ c_n=1} \sum_{i=1}^{\infty} \frac{m - c_{n+i}}{G_m^i} \ge 1 \quad \text{or} \quad \sup_{n\ :\ c_n=1} \sum_{i=1}^{\infty} \frac{c_{n+i}}{G_m^i} \ge m - 1.$$

Since $q < G_m$ and therefore

$$\sum_{i=1}^{\infty} \frac{m - c_{n+i}}{q^i} - \sum_{i=1}^{\infty} \frac{m - c_{n+i}}{G_m^i} = \sum_{i=1}^{\infty} (m - c_{n+i}) \left(\frac{1}{q^i} - \frac{1}{G_m^i} \right) \geq (m-1) \left(\frac{1}{q} - \frac{1}{G_m} \right) > 0$$

and (using the relation $c_n = 1$)

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} - \sum_{i=1}^{\infty} \frac{c_{n+i}}{G_m^i} = \sum_{i=1}^{\infty} c_{n+i} \left(\frac{1}{q^i} - \frac{1}{G_m^i} \right) \geq \min_{1 \leq \ell \leq k} \left(\frac{1}{q^\ell} - \frac{1}{G_m^\ell} \right) > 0,$$

it follows that

$$\sup_{n \,:\, c_n = 1} \sum_{i=1}^{\infty} \frac{m - c_{n+i}}{q^i} > 1 \quad \text{or} \quad \sup_{n \,:\, c_n = 1} \sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} > m - 1.$$

They are equivalent to

$$\inf_{n \,:\, c_n = 1} \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} < \frac{m}{q(q-1)} \quad \text{or} \quad \sup_{n \,:\, c_n = 1} \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{q^i} > \frac{m}{q},$$

and both inequalities contradict (4.4). □

Remark 4.1. In the first version of this paper the following weaker version of Theorem 1.18 was proved: *If $m \geq 2$ and $1 < q < G_m$, then there exists a countable set $C_{m,q}$ such that every $x \in \left(0, \frac{m}{q-1}\right) \setminus C_{m,q}$ has $2^{\aleph_0}$ expansions. If, moreover, q is transcendental and m is algebraic, then every $x \in \left(0, \frac{m}{q-1}\right)$ has $2^{\aleph_0}$ expansions.*

Let us recall the original short proofs. We recall that if $1 < q < G_m$, then every internal point has an expansion, and none of them is unique. Using this, applying Sidorov's bifurcation argument [23, 24] we obtain that if x has no expansion ending with 0^∞ or m^∞, then x has $2^{\aleph_0}$ expansions, because the orbit of x stays in the open interval $\left(0, \frac{m}{q-1}\right)$.

To prove the last statement of the theorem we argue like Sidorov in the proof of [24, Theorem 2.1]. If x does not have $2^{\aleph_0}$ expansions, then it has two expansions of the form

$$x = \sum_{i=1}^{k} \frac{a_i}{q^i} + \frac{r_1}{q^k(q-1)} = \sum_{i=1}^{k} \frac{b_i}{q^i} + \frac{r_2}{q^k(q-1)}$$

where k is a positive integer,

$$a_i, b_i \in \{0, 1, m\}, \quad a_1 \cdots a_k \neq b_1 \cdots b_k \quad \text{and} \quad r_1, r_2 \in \{0, m\}.$$

Then

$$(q-1) \left(\sum_{i=1}^{k} (a_i - b_i) q^{k-i} \right) + (r_1 - r_2) = 0.$$

But this is impossible because the left side is a nonzero polynomial with algebraic coefficients (because m is algebraic) and q is transcendental.

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