Dynamics inside Parabolic Basins

Mi Hu

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Abstract

In our previous paper, we investigated the orbits inside attracting basins for polynomials in $\mathbb{C}$. Suppose $f(z)$ is a polynomial of degree at least 2 on $\mathbb{C}$, p is an attracting fixed point of $f(z)$, Ω_1 is the immediate basin of attraction of p, $\{f^{-1}(p)\} \cap \Omega_1 \neq \{p\}$. Let $\mathcal{A}(p)$ be the basin of attraction of p, and let $\Omega_i (i = 1, 2, \cdots)$ denote the connected components of $\mathcal{A}(p)$. Then there exists a constant $\tilde{C}$ such that for every point z_0 inside any Ω_i, there exists a point $q \in \bigcup_k \{f^{-k}(p)\}_{k\in\mathbb{N}}$ inside Ω_i such that $d_{\Omega_i}(z_0, q) \leq \tilde{C}$, where d_{Ω_i} denotes the hyperbolic distance on Ω_i. If $\{f^{-1}(p)\} \cap \Omega_1 = \{p\}$, then we proved a suitably modified version, we choose a point p' very close to p. In this case, there exists a $q \in \cup_k \{f^{-k}(p')\}_{k\in\mathbb{N}}$ inside Ω_i such that $d_{\Omega_i}(z_0, q) \leq \tilde{C}$.

In this paper, we obtained opposite results about the behavior of orbits inside parabolic basins of polynomials in $\mathbb{C}$. Let $f(z) = z + az^{m+1} + \text{(higher terms)}, m \geq 1, a \neq 0$. A complex number $\mathbf{v_j}$ is called an attraction vector if $ma\mathbf{v_j}^m = -1$. Suppose $\mathcal{P}_j$ is an *attracting petal* for $f(z)$ for the vector $\mathbf{v_j}$ at 0, $\mathcal{A}_j = \mathcal{A}(0, \mathbf{v_j})$ is the parabolic basin of attraction associated to $\mathbf{v_j}$, and Ω_j is the immediate basin of $\mathcal{A}_j$. We choose an arbitrary constant $C > 0$ and an arbitrary point $q = a\mathbf{v_j} \in \mathcal{P}_j$, a is a small positive real number. Then there exists a point $z_0 \in \Omega_j$ so that for any $\tilde{q} \in Q := \cup_{l=0}^{\infty} \{f^{-l}(f^k(q))\} \cap \Omega_j$ (l, k are non-negative integers), the hyperbolic distance $d_{\Omega_j}(z_0, \tilde{q}) \geq C$, where d_{Ω_j} is the hyperbolic distance on Ω_j.

In conclusion, for attracting basins, the preimages of the fixed point p or a point p' inside the immediate basin of the attracting fixed point p will intersect all hyperbolic disks in the basin with some fixed radius C. However, for parabolic basins, the inverse images of $f^k(q)$, where q is any point on an attraction vector, will avoid arbitrary large hyperbolic disks in the basin. Note that $f^k(q)$ can be arbitrarily close to the parabolic fixed point p.

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1 Introduction

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}, f(z) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a nonconstant holomorphic map with degree at least 2, and $f^n(z) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be its n-fold iterate. For $z \in \hat{\mathbb{C}}$, we call the set $\{z_n\}_{n\in\mathbb{N}} = \{z_1 = f(z_0), z_2 = f^2(z_0), \cdots\}$ the orbit of the point $z = z_0$. If $z_N = z_0$ for some integer N, we say that z_0 is a periodic point of $f(z)$. If $N = 1$, then z_0 is a fixed point of $f(z)$.

In complex dynamics [3, 7, 9, 10, 19], there are two crucial disjoint invariant sets, the *Julia set*, and the *Fatou set* of $f(z)$, which partition the sphere $\hat{\mathbb{C}}$. The Fatou set of $f(z)$

is defined as the largest open set $\mathcal{F}$ where the family of iterates is locally normal. In other words, for any point $z \in \mathcal{F}$, there exists some neighborhood U of z so that the sequence of iterates of the map, restricted to U, forms a normal family, so the orbits of iteration are well-behaved. The complement of the Fatou set is called the Julia set.

There have been many studies on probability measures that can describe the dynamics on the Julia set. We define $\{f^{-1}(z)\} = \{w|f(w) = z\}$, that is the set of preimages of the point z, and $\{f^{-n}(z)\}_{n\in\mathbb{N}} = \{w|f^n(w) = z\}$. For example, if z is any non-exceptional point, the inverse orbits $\{f^{-n}(z)\}_{n\in\mathbb{N}}$ equidistribute toward the Green measure μ which lives on the Julia set. This was proved already by Brolin [6] in 1965, and many improvements and generalizations have been made [8, 11]. However, this equidistribution toward μ is in the weak sense, and hence it is with respect to the Euclidean metric. Therefore, it is a reasonable question to ask how dense $\{f^{-n}(z)\}_{n\in\mathbb{N}}$ is in $\mathcal{F}$ near the boundary of $\mathcal{F}$ if we use finer metrics, for instance, the hyperbolic metric. The hyperbolic metric is an important tool in complex dynamics, see examples in [1, 2, 4, 5].

There are some classical results about the behavior of a rational function on the Fatou set $\mathcal{F}$ as well. The connected components of the Fatou set of $f(z)$ are called Fatou components. A Fatou component $\Omega \subset \hat{\mathbb{C}}$ of $f(z)$ is invariant if $f(\Omega) = \Omega$. At the beginning of the 20th century, Fatou [12, 13, 14] classified all possible invariant Fatou components of rational functions on the Riemann sphere. He proved that only three cases can occur: attracting, parabolic, and rotation. And in the '80s, Sullivan [20] completed the classification of Fatou components. He proved that every Fatou component of a rational map is eventually periodic, i.e., there are $n, m \in \mathbb{N}$ such that $f^{n+m}(\Omega) = f^m(\Omega)$.

The orbits in the rotation domains are easy to describe since the functions are conjugate to irrational rotations.

The orbits in attracting basins in $\mathbb{C}$ are much more complicated. Near the attracting fixed point, there has a uniform estimate about how fast the orbit converges to the attracting fixed point, for more details, readers can read Theorem 8.2, Chapter 8 in [19]. If the map $f(z)$ is hyperbolic on the Julia set, then there is a uniform estimate about how fast the orbit escapes near the boundary, we refer to Lemma 2.1, Chapter V in [7]. In the paper [16], we investigated the behavior of orbits inside attracting basins, no matter whether $f(z)$ is hyperbolic or not, and obtained the following theorem:

Theorem 1.1. *Suppose $f(z)$ is a polynomial of degree at least 2 on $\mathbb{C}$, p is an attracting fixed point of $f(z)$, Ω_1 is the immediate basin of attraction of p, $\{f^{-1}(p)\} \cap \Omega_1 \neq \{p\}$, and $\mathcal{A}(p)$ is the basin of attraction of p, $\Omega_i (i = 1, 2, \cdots)$ are the connected components of $\mathcal{A}(p)$. Then there is a constant $\tilde{C}$, which only depends on $f(z)$ and p, such that for every point z_0 inside any Ω_i, there exists a point $q \in \cup_k\{f^{-k}(p)\}_{k\in\mathbb{N}}$ inside Ω_i such that $d_{\Omega_i}(z_0, q) \leq \tilde{C}$, where d_{Ω_i} is the hyperbolic distance on Ω_i.*

This Theorem 1.1 essentially shows that any arbitrary orbit can be tracked by an orbit of one preimage of the fixed point p. Note that $\{f^{-1}(p)\} \cap \Omega_1 \neq \{p\}$ means the set of preimages of p inside Ω_1 should not be only the point p itself. Otherwise, $q = p$, then $d_{\Omega_1}(z_0, q) \to \infty$ as z_0 approaches the boundary of Ω_1. In the case where $\{f^{-1}(p)\} \cap \Omega_1 = \{p\}$, the set of preimages $\{f^{-k}(p)\}_{k\in\mathbb{N}}$ that intersect Ω_1 consists of only a single point. In this situation, it does not make sense to consider the inverse orbit of p. Thus, instead of considering the inverse orbit of p, we choose a point $\hat{p}$ which is inside the immediate basin of the attracting fixed point p. Then we can consider the orbit $\{f^{-k}(\hat{p})\}_{k\in\mathbb{N}}$. In this way, we proved a suitably modified version of Theorem 1.1:

Theorem 1.2. *Suppose $g(z) = e^{i\theta}z^m, m \geq 2$. We pick a point $\hat{p} \in \triangle \setminus \{0\}$. Then there exists a constant $C_0 > 0$ such that for every point $z_0 \in \triangle$, there exists $q \in \cup_k\{g^{-k}(\hat{p})\}, k \geq 0$ satisfying $d_\triangle(z_0, q) \leq C_0$, where $d_\triangle$ denotes the hyperbolic distance on the unit disk $\triangle$.*

We also studied the orbits in attracting basins in $\mathbb{C}^2$ in the paper [17]. There are various interesting results in $\mathbb{C}^2$: Theorem 1.1 holds for some holomorphic mappings, but it fails for some other holomorphic mappings.

It is a natural question to ask if Theorem 1.1 in paper [16] can be generalized to parabolic basins. In the present paper, we show in the following Theorems A, B and C that the answer is negative.

Theorem A. Let $f(z) = z + z^2$. We choose an arbitrary constant $C > 0$ and the point $q = -\frac{1}{2} \in \mathcal{A}$. Then there exists a point $z_0 \in \mathcal{A}$ such that for any $\tilde{q} \in Q := \cup_{l,k=0}^{\infty}\{f^{-l}(f^k(-\frac{1}{2}))\}$ (l, k are non negative integers), the hyperbolic distance satisfies $d_\mathcal{A}(z_0, \tilde{q}) \geq C$, where $d_\mathcal{A}$ is the hyperbolic distance on $\mathcal{A}$.

Here, $\mathcal{A} = \mathcal{A}(0, -1)$ in Theorem A denotes the parabolic basin of $f(z) = z + z^2$ with the attraction vector $\mathbf{v} = -1$, see Definitions 2.1 and 2.4 below in section 2. Note that for $f(z) = z + z^2$, there is only one attracting vector, that is $\mathbf{v_j} = \mathbf{v} = -1$.

In this Theorem A, we cannot directly choose to iterate the inverse of the parabolic fixed point 0, since all preimages $\{f^{-k}(0)\}_{k\in\mathbb{N}}$ are inside the Julia set of $f(z)$. Then Theorem A is trivial. However, we still aim to apply the same approach as in Theorem 1.2 for attracting basins, where we choose to iterate the inverse of a point inside the immediate basin of the attracting fixed point p. Hence, we choose $q \in \mathcal{A}$ and iterate it k times. Then $f^k(q)$ is getting arbitrarily close to the parabolic fixed point as $k \to \infty$. Then we consider the preimages of $f^k(q)$. To simplify, here we choose $q = -\frac{1}{2}$.

By Theorem A and Proposition 2.7, we obtain the following Corollary which states that the set of all such points z_0 clusters at every point in the boundary of the parabolic basin of $f(z) = z + z^2$:

Corollary 3.4. Let $X \subset \mathcal{A}$ be the set of all $z_0 \in \mathcal{A}$ such that $d_\mathcal{A}(z_0, \tilde{q}) \geq C$ for any $\tilde{q} \in \mathcal{A}$. If $z \in X$, then any point $w \in f^{-1}(z)$ is in X. Therefore, X is dense in the boundary of $\mathcal{A}$.

We also generalize Theorem A to the case of several petals inside the parabolic basin in Theorem B:

Theorem B. Let $f(z) = z + az^{m+1}, m \geq 1, a \neq 0$, and Ω_j be the immediate basin of $\mathcal{A}_j$. We choose an arbitrary constant $C > 0$ and an arbitrary point $q = a\mathbf{v_j} \in \mathcal{P}_j$, a is a small positive real number. Then there exists a point $z_0 \in \Omega_j$ such that for any $\tilde{q} \in Q := \cup_{l=0}^{\infty}\{f^{-l}(f^k(q))\} \cap \Omega_j$ (l, k are non-negative integers), the hyperbolic distance satisfies $d_{\Omega_j}(z_0, \tilde{q}) \geq C$, where d_{Ω_j} denotes the hyperbolic distance on Ω_j.

Here $\mathbf{v_j}$ is an attraction vector in the tangent space of $\mathbb{C}$ at 0, $\mathcal{P}_j$ is an *attracting petal* for $f(z)$ corresponding to the vector $\mathbf{v_j}$ at 0, and $\mathcal{A}_j = \mathcal{A}(0, \mathbf{v_j})$ is the parabolic basin of attraction associated with $\mathbf{v_j}$, see Definitions 2.1, 2.4 and 2.5.

In the end, we consider the behavior of orbits inside parabolic basins of general polynomials.

Theorem C. Let $f(z) = z + az^{m+1} + (\text{higher terms}), m \geq 1, a \neq 0$, and Ω_j be the immediate basin of $\mathcal{A}_j$. We choose an arbitrary constant $C > 0$ and an arbitrary point $q = a\mathbf{v_j} \in \mathcal{P}_j$, a is a small positive real number. Then there exists a point $z_0 \in \Omega_j$ such that for any $\tilde{q} \in Q := \cup_{l=0}^{\infty}\{f^{-l}(f^k(q))\} \cap \Omega_j$ $(l, k$ are non-negative integers), the hyperbolic distance satisfies $d_{\Omega_j}(z_0, \tilde{q}) \geq C$, where d_{Ω_j} denotes the hyperbolic distance on Ω_j.

This paper is organized as follows. In section 2, we recall some definitions and results [19] about holomorphic dynamics of polynomials in a neighborhood of the parabolic fixed point and the hyperbolic metric. In section 3, we prove our main results, Theorem A, Theorem B, and Theorem C.

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2 Preliminary

2.1 Holomorphic dynamics of polynomials in a neighborhood of a parabolic fixed point.

Let us first recall some definitions and results [19] about holomorphic dynamics of a polynomial $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ in a neighborhood of the parabolic fixed point 0.

Definition 2.1. *Let $f(z) = z + az^{m+1} + (\textit{higher terms}), m \geq 1, a \neq 0$. A complex number $\mathbf{v}$ will be called an* attraction vector *at the origin if $ma\mathbf{v}^m = -1$, and a* repulsion vector *at the origin if $ma\mathbf{v}^m = 1$. Note here that $\mathbf{v}$ should be thought of as a tangent vector at the origin. We say that some orbit $z_0 \mapsto z_1 \mapsto z_2 \mapsto \cdots$ for the map $f(z)$ converges to zero* nontrivially *if $z_n \to 0$ as $n \to \infty$, but no z_n is actually equal to zero. There are m equally spaced attraction vectors at the origin.*

Lemma 2.2. *If an orbit of $f(z) : z_0 \mapsto z_1 \mapsto \cdots$ converges to zero nontrivially, then z_k is asymptotic to $\mathbf{v_j}/\sqrt[m]{k}$ as $k \to +\infty$ for one of the m attraction vectors $\mathbf{v_j}$.*

Proof. See the proof in chapter 10 of Milnor's book [19]. □

Definition 2.3. *If an orbit $z_0 \mapsto z_1 \mapsto \cdots$ under $f(z)$ converges to zero with $z_k \sim \mathbf{v_j}/\sqrt[m]{k}$, we will say that this orbit $\{z_k\}_{k\in\mathbb{N}}$ tends to zero along the direction $\mathbf{v_j}$.*

Definition 2.4. *Given an attraction vector* $\mathbf{v_j}$ *in the tangent space of* $\mathbb{C}$ *at* 0*, the associated* parabolic basin of attraction $\mathcal{A}_j = \mathcal{A}(0, \mathbf{v_j})$ *is defined to be the set consisting of all* $z_0 \in \mathbb{C}$ *for which the orbit* $z_0 \mapsto z_1 \mapsto \cdots$ *converges to* 0 *along the direction* $\mathbf{v_j}$*.*

Definition 2.5. *Suppose* $f(z) = z + az^{m+1} + (\text{ higher terms}), m \geq 1, a \neq 0$ *is defined and univalent on some neighborhood* N *at* 0 *on* $\mathbb{C}$*. An open set* $\mathcal{P}_j \subset N$ *is called an* attracting petal *for* $f(z)$ *for the* $\mathbf{v_j}$ *at* 0 *if*

(1) $f(z)$ *maps* $\mathcal{P}_j$ *into itself, and*

(2) an orbit $z_0 \mapsto z_1 \mapsto \cdots$ *under* $f(z)$ *is eventually absorbed by* $\mathcal{P}_j$ *if and only if it converges to* 0 *along the direction* $\mathbf{v_j}$*.*

2.2 The hyperbolic metric

Definition 2.6. *The metric*

$$F_{\mathbb{D}} = \frac{2|dz|}{1-|z|^2} \quad \textit{for} \quad z \in \mathbb{D}$$

is called hyperbolic (or Poincaré) metric on the unit disk $\mathbb{D}$*.*

Let $\gamma : [0,1] \to \hat{\Omega}$ *be a piecewise smooth curve. The* hyperbolic length *of* γ *is defined to be*

$$L_{\hat{\Omega}}(\gamma) = \int_{\gamma} F_{\hat{\Omega}}(z, \xi)|dz| = \int_0^1 F_{\hat{\Omega}}\big(\gamma(t), \gamma'(t)\big)|\gamma'(t)|dt.$$

For any two points z_1 *and* z_2 *in* $\hat{\Omega}$*, the* hyperbolic distance *between* z_1 *and* z_2 *is defined to be*

$$d_{\hat{\Omega}}(z_1, z_2) = \inf\{L_{\hat{\Omega}}(\gamma) : \gamma \textit{ is a piecewise smooth curve connecting } z_1 \textit{ and } z_2\}.$$

Note that $d_{\hat{\Omega}}(z_1, z_2)$ *is defined when* z_1 *and* z_2 *are in the same connected component of* $\hat{\Omega}$*.*

Let $g : \triangle \to S$ be a covering map, we define a hyperbolic metric F_S on any hyperbolic Riemann surface S by declaring that g induces an isometry at every point. In other words, let

$$F_{\mathbb{D}}(z) = \frac{2|dw|}{1-|w|^2} \quad \text{for} \quad z = g(w) \in S.$$

If φ is a local determination for g^{-1}, then

$$F_{\mathbb{D}}(z) = \frac{2|\varphi'(z)|}{1-|\varphi|^2}|dz|, \quad z \in S.$$

For example, when S is the upper half-plane, then we can use the conformal map $\varphi(z) = \frac{z-i}{z+i}$ of the upper half-plane $\mathbb{H}$ onto the unit disk $\triangle$. Hence,

$$F_{\mathbb{H}} = \frac{2|\varphi'(z)|}{1-|\varphi|^2}|dz| = \frac{2\left|\left(\frac{z-i}{z+i}\right)'\right|}{1-\left|\frac{z-i}{z+i}\right|^2}|dz| = \frac{|dz|}{y}, \quad z = x+iy \in \mathbb{H}$$

is the hyperbolic metric on the upper half-plane $\mathbb{H}$. We refer to page 12, chapter I.4 in [7] for more calculation details about $F_{\mathbb{H}}$.

Proposition 2.7 (The distance decreasing property of the hyperbolic metric)**.** *Suppose* Ω_1 *and* Ω_2 *are domains in* $\mathbb{C}$, $z, \omega \in \Omega_1, \xi \in \mathbb{C}$, *and* $f(z) : \Omega_1 \to \Omega_2$ *is holomorphic. Then*

$$F_{\Omega_2}(f(z), f'(z)\xi) \leq F_{\Omega_1}(z, \xi), \quad d_{\Omega_2}(f(z), f(\omega)) \leq d_{\Omega_1}(z, \omega).$$

Proof. We refer to Theorem 4.1 on page 12, Chapter I.4 in [7]. □

Corollary 2.8. *Suppose* $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}$. *Then for any* $z, \omega \in \Omega_1$ *and* $\xi \in \mathbb{C}$, *we have*

$$F_{\Omega_2}(z, \xi) \leq F_{\Omega_1}(z, \xi), \quad d_{\Omega_2}(z, \omega) \leq d_{\Omega_1}(z, \omega).$$

Proof. We refer to Theorem 4.2 on page 13, Chapter I.4 in [7]. □

3 Proof of the main theorems

In this section, we will prove our main theorems: Theorem A, Theorem B, and Theorem C.

3.1 Dynamics inside the parabolic basin of $f(z) = z + z^2$

Let us recall the statement of our main Theorem A:

Theorem A. Let $f(z) = z + z^2$. We choose an arbitrary constant $C > 0$ and the point $q = -\frac{1}{2} \in \mathcal{A}$. Then there exists a point $z_0 \in \mathcal{A}$ such that for any $\tilde{q} \in Q := \cup_{l,k=0}^{\infty}\{f^{-l}(f^k(-\frac{1}{2}))\}$ (l, k are non negative integers), the hyperbolic distance satisfies $d_{\mathcal{A}}(z_0, \tilde{q}) \geq C$, where $d_{\mathcal{A}}$ is the hyperbolic distance on $\mathcal{A}$.

Proof. Let $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^- = (-\infty, 0]$ be the positive and negative real axis, respectively. Then the parabolic basin $\mathcal{A} \subsetneqq (\mathbb{C} \setminus \mathbb{R}^+)$. Let $\mathbb{H}$ be the upper half-plane, $\varphi(z) : \mathbb{C} \setminus \mathbb{R}^+ \to \mathbb{H}$ with $\varphi(z) = \sqrt{z}$ (see Figure 3.8). By Definition 2.6., we know the hyperbolic metric on the upper half plane is

$$F_{\mathbb{H}} = \frac{|dw|}{v}$$

for $w = u + iv \in \mathbb{H}$. Hence the hyperbolic metric on $\mathbb{C} \setminus \mathbb{R}^+$ is

$$F_{\mathbb{C}\setminus\mathbb{R}^+} = \frac{|\varphi'|}{\operatorname{Im}\varphi}|dz| = \frac{1}{2|\sqrt{z}|\operatorname{Im}\sqrt{z}}|dz| = \frac{1}{2r\sin\frac{\theta}{2}}|dz|$$

for $z = re^{i\theta} \in \mathbb{C} \setminus \mathbb{R}^+, i.e., \theta \in (0, 2\pi)$.

We know that $f^k(q)$ will always be on the negative real axis for any integer $k \geq 0$. We choose $0 < \theta_0 < \frac{\pi}{2}$ and two rays $l_1 := \{z = re^{i\theta_0}, r > 0\}, l_2 := \{z = re^{-i\theta_0}, r > 0\}$. Then we denote by $T := \{z = re^{i\theta}, r > 0, \theta_0 \leq \theta \leq 2\pi - \theta_0\}$, a sector inside $\mathbb{C} \setminus \mathbb{R}^+$.

Before continuing with the proof of Theorem A, we have the following well-known Lemma 3.2. For the reader's convenience and to introduce notation, we include the proof and define a Left/Right Pac-Man for an easy explanation of the proof.

Definition 3.1. *We call a domain* $D_R := \{z = re^{i\theta}, 0 < r \leq R, \theta_0 < \theta < 2\pi - \theta_0\}$ *a* Left Pac-Man *and* $\tilde{D}_R := \{z = re^{i\theta}, 0 < r \leq R, -\pi + \theta_0 < \theta < \pi - \theta_0\}$ *a* Right Pac-Man.

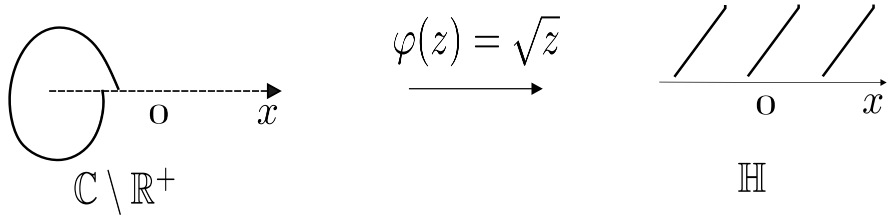

Figure 1: The map φ

Lemma 3.2. *For any $\theta_0 \in (0, \pi/2)$, there exists R_0 such that the Left Pac-Man $D_{R_0} \subsetneq T \cap \mathcal{A}$.*

Proof. We want to know how orbits go precisely near the parabolic fixed point at 0. Let $\omega = \varphi(z) = -1/z$ send 0 to ∞, then the conjugated map has the expansion

$$F(\omega) = \varphi \circ f \circ \varphi^{-1}(\omega) = \omega + 1 + o(1) \quad \text{as} \quad |\omega| \to \infty.$$

We have $l_1(l_2)$ is mapped to two new rays $l_1^\omega := \{z = re^{i(\pi-\theta_0)}, r > 0\}(l_2^\omega := \{z = re^{i(-\pi+\theta_0)}\})$; T is mapped to $T' = \{re^{i\theta}, r > 0, -\pi + \theta_0 \leq \theta \leq \pi - \theta_0\}$; the Left Pac-Man D_R is mapped to $T' \setminus \tilde{D}_{\frac{1}{R}}$ for any radius R (see Figure 2).

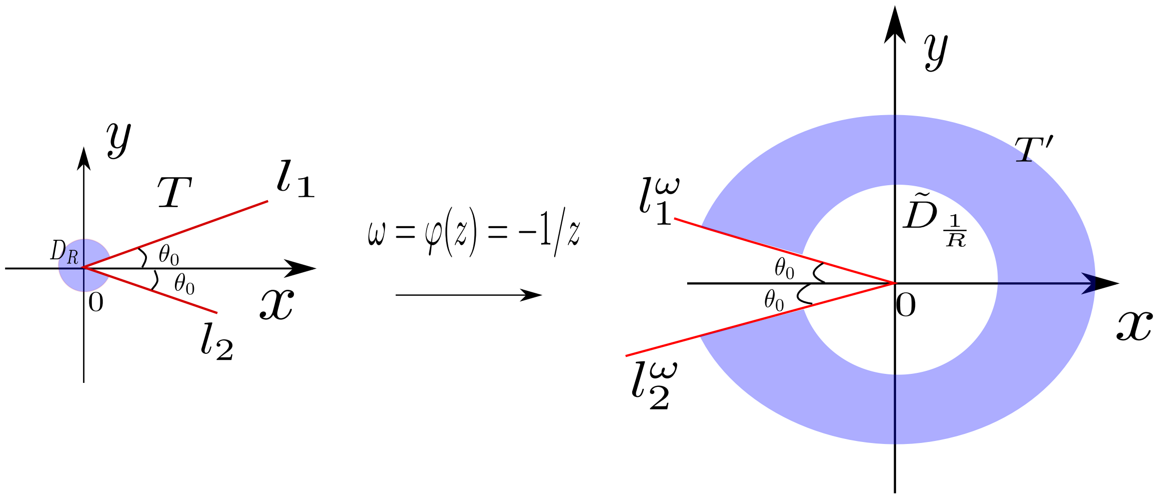

Figure 2: The image of a Left Pac-Man

We choose a Right Pac-Man $\tilde{D}_{\frac{1}{r_0}}$ such that for any $\omega \in T' \setminus \tilde{D}_{\frac{1}{r_0}}$, we have $|o(1)| < \frac{\theta_0}{3}$. Note here, actually, $r_0 = r_0(\theta_0)$ should be sufficiently small. Then we draw the upper tangent line L_0 of $\tilde{D}_{\frac{1}{r_0}}$ such that the angle between L_0 and the real axis is $\frac{\theta_0}{2}$. Then L_0 will intersect l_1^ω and the real axis, we denote these two intersect points A_0 and B_0, respectively. Let $\frac{1}{r} = \max\{|OA_0|, |OB_0|\} > \frac{1}{r_0}$, here O is the origin zero, then we choose the Right Pac-Man $\tilde{D}_{\frac{1}{r}}$ (see Figure 3).

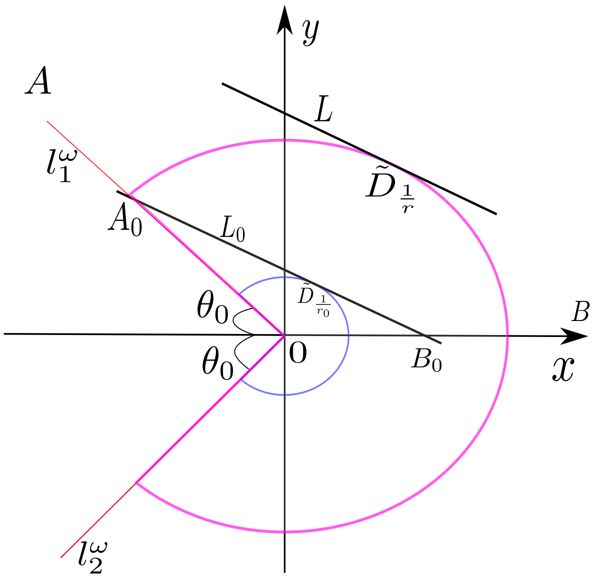

Figure 3: The choice of the Right Pac-Man

If we take any $\omega_0 \in T' \setminus \tilde{D}_{\frac{1}{r}}$, $F^n(\omega_0) = \omega_0 + n + o(1)$ for all positive integers n such that $|F^n(\omega_0)| \geq |\omega_0 + n - n\frac{\theta_0}{3}|$, then we know that $F^n(\omega_0)$ will never go inside $\tilde{D}_{\frac{1}{r_0}}$.

Therefore, let $R_0 = r, \tilde{R}_0 = r_0$, then we have for any $z_0 \in D_{R_0}, f^n(z_0) \to 0$, hence $D_{R_0} \subsetneqq T \cap \mathcal{A}$, and we also know that $f^n(D_{R_0}) \subseteq D_{\tilde{R}_0}$.

□

Lemma 3.3. *We can choose a Left Pac-Man $D_{R_0'} \subseteq D_{R_0}$ such that $f^n(D_{R_0'}) \subseteq D_{R_0}$ for all $n = 1, 2, \ldots$*

Proof. On the procedure for proving Lemma 3.2, we can draw another upper tangent line L of $\tilde{D}_{R_0}$ such that the angle between L and the real axis is $\frac{\theta_0}{2}$. Then L will also intersect l_1^ω and the real axis, we denote these two intersect points A and B respectively. Let $\frac{1}{r'} = \max\{|OA|, |OB|\}$, then we choose the Right Pac-Man $\tilde{D}_{\frac{1}{r'}}$. If we take any $\omega \in T' \setminus \tilde{D}_{\frac{1}{r'}}$, we know that $F^n(\omega)$ will never go inside of $\tilde{D}_{\frac{1}{r}}$. Hence, let $R_0' = \frac{1}{r'}$, we have $f^n(D_{R_0'}) \subseteq D_{R_0}$.

□

We continue with the proof of Theorem A. The idea of the proof is to find a point $z_0 \in D_\varepsilon \setminus \mathbb{R}^-$ such that for any $\tilde{q} \in Q$, we have $d_{\mathcal{A}}(z_0, \tilde{q}) \geq C$.

Now, we will consider the following cases of $\tilde{q}$ inside three subsets of $\mathcal{A}$ (see Figure 4):

Case 1: $\tilde{q} \in \big((T \cap \mathcal{A}) \setminus D_{R_0'}\big) \setminus \mathbb{R}^-$. Let $d^1_{\mathcal{A}}(z_0, \tilde{z})$ be the hyperbolic distance between z_0 and any point $\tilde{z} \in \partial D_{R_0'} \setminus R^-$ (see the blue curve in Figure 4). Then we prove that $d^1_{\mathcal{A}}(z_0, \tilde{z}) \geq C$.

Case 2: $\tilde{q} \in \mathbb{R}^-$. Let $d^2_{\mathcal{A}}(z_0, z')$ be the hyperbolic distance between z_0 and any point $z' \in \mathbb{R}^- \cap \mathcal{A}$ (see the pink curve in Figure 4). Then we prove that $d^2_{\mathcal{A}}(z_0, z') \geq C$.

Case 3: $\tilde{q} \in \mathcal{A} \cap \{S' := \{z = re^{i\theta}, r > 0, 0 < \theta \leq \theta_0\}\}$. Let $d^3_{\mathcal{A}}(z_0, \hat{z})$ be the hyperbolic distance between z_0 and any point $\hat{z} \in l_1$ (see the green curve in Figure 4). Then we prove that $d^3_{\mathcal{A}}(z_0, \hat{z}) \geq C$.

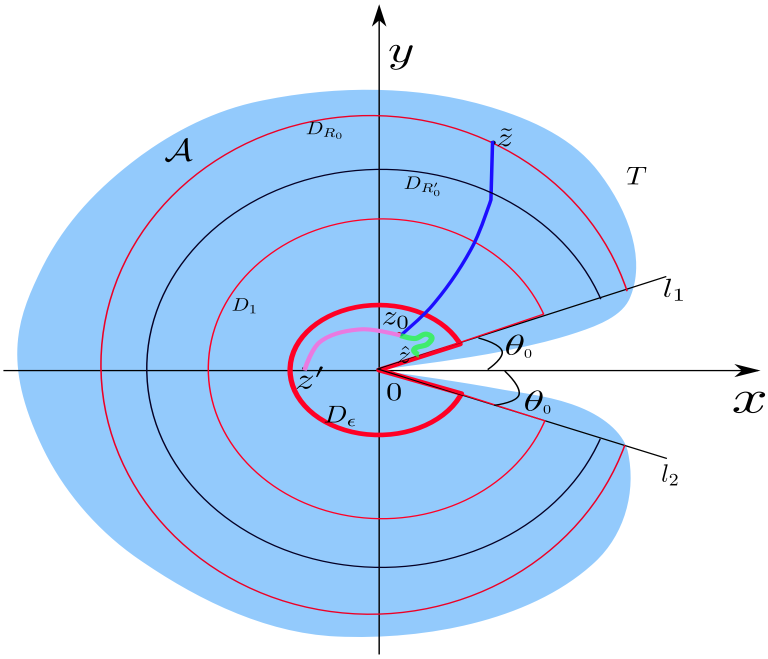

Figure 4: The three hyperbolic distances in $\mathcal{A}$

Remark:

1) In the investigation of these three cases, it will become clear how small θ_0 needs to be.

2) $\partial D_{R_0'}$ means the boundary of the Left Pac-Man $D_{R_0'}$. It is the circular curve of $D_{R_0'}$, not including the mouth of $D_{R_0'}$ which belongs to l_1 and l_2;

3) $\tilde{q}$ can never be inside $D_{R_0'} \setminus \mathbb{R}^-$: If $\tilde{q} \in D_{R_0'} \setminus \mathbb{R}^-$, more precisely, suppose $f^{-l}(f^k(q)) \in D_{R_0'} \setminus \mathbb{R}^-$ for some integers $l, k \geq 0$, we iterate l times of $f^{-l}(f^k(q))$, we have $f^k(q) \in D_{R_0} \setminus \mathbb{R}^-$ since we know $f^n(D_{R_0'}) \subseteq D_{R_0}$ for any positive integer n by Lemma 3.2 and $f^k(q) \notin \mathbb{R}^-$ since $R_0' < R_0 = r < r_0$, and $f(z)$ is biholomorphic near 0. Hence $\text{Im}(f^k(q)) \neq 0$. However, this contradicts $\text{Im}(f^k(q)) = \text{Im}(f^k(-\frac{1}{2})) = 0$. Thus $\tilde{q} \notin D_{R_0'} \setminus \mathbb{R}^-$;

4) if $\tilde{q} \in \mathcal{A} \setminus D_{R_0'}$, $\tilde{q}$ is far away from $\partial D_{R_0'}$, then we know that $d_{\mathcal{A}}(z_0, \tilde{q}) \geq d^1_{\mathcal{A}}(z_0, \tilde{z})$. This is because $d^1_{\mathcal{A}}(z_0, \tilde{z})$ is the minimum distance between z_0 and any $\tilde{q} \in \big((T \cap \mathcal{A}) \setminus D_{R_0'}\big) \setminus \mathbb{R}^-$.

Hence, we need to prove that we can choose z_0 so that all these three hyperbolic distances $d^1_{\mathcal{A}}(z_0, \tilde{z}), d^2_{\mathcal{A}}(z_0, z'), d^3_{\mathcal{A}}(z_0, \hat{z}) \geq C$ for the given constant C. Next, we will estimate these three hyperbolic distances.

First, we estimate $d^1_{\mathcal{A}}(z_0, \tilde{z})$. Suppose $z_0 \in D_\varepsilon$,

$$\begin{aligned} d^1_{\mathcal{A}}(z_0, \tilde{z}) \geq & d_{\mathbb{C}\backslash\mathbb{R}^+}(z_0, \tilde{z}) \\ = & \inf \int_{\gamma(t)} F_{\mathbb{C}\backslash\mathbb{R}^+}(\gamma(t))|\gamma'(t)|dt \\ = & \inf \int_{\gamma(t)} \frac{1}{2|\gamma(t)|\sin\frac{\arg(\gamma(t))}{2}}|\gamma'(t)|dt \\ \geq & \inf \int_{\gamma(t)} \frac{1}{2|\gamma(t)|}|\gamma'(t)|dt \\ = & \frac{1}{2}\inf \int_{\varepsilon}^{R_0'} \frac{|dr|}{r} \\ \geq & \frac{1}{2}(\ln R_0' - \ln \varepsilon), \end{aligned}$$

where $\gamma(t)$ is a smooth path joining z_0 to $\tilde{z}$. The last inequality holds since there might have some derivatives of the path $\gamma(t)$ are negative in some pieces. In addition, we can see that $d^1_{\mathcal{A}}(z_0, \tilde{z}) \to \infty$ as $\varepsilon \to 0$.

Second, we estimate $d^2_{\mathcal{A}}(z_0, z')$. Let $\varepsilon = \varepsilon_0$, i.e., fix ε, and let $D_1 \subset T$ be a scaling of D_{ε_0} by $S(z) = \frac{z}{|z_0|}$, sending z_0, z' to $\tilde{z}_0 := \frac{z_0}{|z_0|}, \frac{z'}{|z_0|}$, respectively. By homogeneity, we know the hyperbolic distance $d_{\mathbb{C}\backslash\mathbb{R}^+}(z_0, z') = d_{\mathbb{C}\backslash\mathbb{R}^+}(\tilde{z}_0, z'/|z_0|)$. Since we hope to prove $d^2_{\mathcal{A}}(z_0, z') \geq C$, we need z_0 to be far from $\mathbb{R}^-$, and so does $\tilde{z}_0$. Let $S_T := \{z = e^{i\theta}, \theta_0 < \theta < \frac{\pi}{2} - \theta_0\}$, assume $\tilde{z}_0 \in S_T$ and $\operatorname{Re} \tilde{z}_0 > \frac{1}{2}$, then any curve from $\tilde{z}_0$ to $\frac{z'}{|z_0|}$ must pass through a point $\tilde{z}'$ on the positive imaginary axis, i.e., $\operatorname{Re} \tilde{z}' = 0$. For simplicity, we assume this curve and $\tilde{z}'$ lie in the upper half plane. Hence $d^2_{\mathcal{A}}(z_0, z') \geq d_{\mathbb{C}\backslash\mathbb{R}^+}(z_0, z') = d_{\mathbb{C}\backslash\mathbb{R}^+}(\tilde{z}_0, z'/|z_0|) \geq d_{\mathbb{C}\backslash\mathbb{R}^+}(\tilde{z}_0, \tilde{z}')$. We have

$$\begin{aligned} d^2_{\mathcal{A}}(z_0, z') &\geq d_{\mathbb{C}\backslash\mathbb{R}^+}(\tilde{z}_0, \tilde{z}') \\ &= \inf \int_{\tilde{z}_0}^{\tilde{z}'} F_{\mathbb{C}\backslash\mathbb{R}^+}(z) \\ &= \inf \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{|z|2\sin(\theta/2)} \\ &\geq \sqrt{2}\inf \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{|z|\sin\theta} \\ &\geq \sqrt{2}\inf \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{\operatorname{Im} z} \\ &> \sqrt{2}|\ln(\operatorname{Im}\tilde{z}') - \ln(\operatorname{Im}\tilde{z}_0)|. \end{aligned}$$

Then there are three situations for the hyperbolic distance between $\tilde{z}_0$ and $\tilde{z}'$:

1) If $\operatorname{Im}\tilde{z}' \geq e^C|\operatorname{Im}\tilde{z}_0|$ or $\operatorname{Im}\tilde{z}' \leq \frac{|\operatorname{Im}\tilde{z}_0|}{e^C}$ for the constant $C > 0$ (see the blue curves on Figure 5), then $|\ln(\operatorname{Im}\tilde{z}') - \ln(\operatorname{Im}\tilde{z}_0)| \geq C$, hence $d^2_{\mathcal{A}}(z_0, z') \geq C$ is true.

2) If $\tilde{z}' \in \mathcal{L} := \{z = x + iy, \frac{|\operatorname{Im}\tilde{z}_0|}{e^C} < y < e^C|\operatorname{Im}\tilde{z}_0|\}$ (see the green curve on Figure 5). We prove that $d_{\mathcal{A}}(\tilde{z}_0, \tilde{z}') \geq \sqrt{2}\inf\int_{\tilde{z}_0}^{\tilde{z}'}\frac{|dz|}{\operatorname{Im} z} \geq C$ for $z \in \mathcal{L}$.

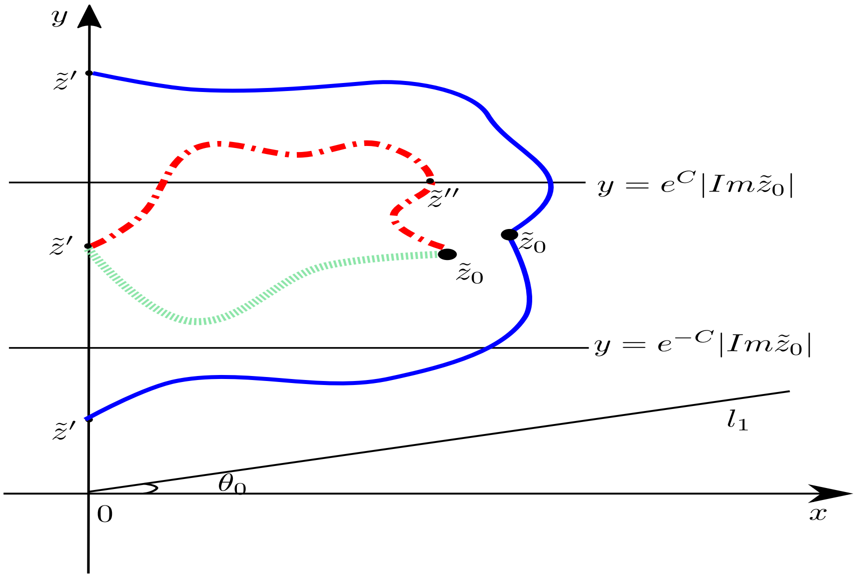

Figure 5: Three situations for the hyperbolic distance between $\tilde z_0$ and $\tilde z'$

Let $z = x + iy \in \mathcal{L}$, then $\operatorname{Im} z \le e^C|\operatorname{Im} \tilde z_0|$. Hence we have

$$\int_{\tilde z_0}^{\tilde z'} \frac{|dz|}{\operatorname{Im} z} \ge \int_{\tilde z_0}^{\tilde z'} \frac{|dx|}{\operatorname{Im} z} \ge \int_{\tilde z_0}^{\tilde z'} \frac{|dx|}{e^C|\operatorname{Im} \tilde z_0|} = \frac{\operatorname{Re} \tilde z_0}{e^C} \frac{1}{|\operatorname{Im} \tilde z_0|} > \frac{1}{2e^C|\operatorname{Im} \tilde z_0|}.$$

Note that the last inequality holds because we choose $\operatorname{Re} \tilde z_0 > 1/2$.

As long as $\theta_0 < \frac{1}{2Ce^C}$, we can choose $\tilde z_0$ so that $\operatorname{Im} \tilde z_0 < \frac{1}{2Ce^C}$, hence $\int_{\tilde z_0}^{\tilde z'} \frac{|dz|}{\operatorname{Im} z} > C$.

3) If $\tilde z' \in \mathcal{L}$, but the curve γ between $\tilde z_0$ and $\tilde z'$ gets outside of $\mathcal{L}$ starting at some point $\tilde z'' \in \gamma \cap \mathcal{L}$ for a while (see the red curve in Figure 5), then enter back to $\mathcal{L}$ again, then we still have $d^2_{\mathcal{A}}(z_0, z') \ge C$ is true because $d_{\mathbb{C}\setminus\mathbb{R}^+}(z_0, z') \ge d_{\mathbb{C}\setminus\mathbb{R}^+}(\tilde z_0, \tilde z'') \ge C$. The last inequality holds since 1) is valid.

After these calculations, we fix z_0 so that $d^1_{\mathcal{A}}(z_0, \tilde z)$ and $d^2_{\mathcal{A}}(z_0, z')$ are both bigger or equal to C. To obtain $d^3_{\mathcal{A}}(z_0, \hat z) \ge C$, we will need an even smaller θ_0', see the following calculation.

We know $d^3_{\mathcal{A}}(z_0, \hat z) \ge d_{\mathbb{C}\setminus\mathbb{R}^+}(z_0, \hat z) = d_{\mathbb{C}\setminus\mathbb{R}^+}(\tilde z_0, \hat z/|z_0|)$. Hence to have $d^3_{\mathcal{A}}(z_0, \hat z) \ge C$, we need to estimate $d_{\mathbb{C}\setminus\mathbb{R}^+}(\tilde z_0, \hat z/|z_0|)$.

When we estimate $d^2_{\mathcal{A}}(z_0, z')$, we send z_0 to $\tilde z_0$ and choose $\operatorname{Re} \tilde z_0 > 1/2$ and $\tilde z$ close to l_1. Hence $d_{\mathbb{C}\setminus\mathbb{R}^+}(\tilde z_0, \hat z/|z_0|)$ might be very small. To handle this situation, we first choose a disk $\Delta^K_{\mathbb{C}\setminus\mathbb{R}^+}(\tilde z_0, C)$ centered at $\tilde z_0$ with radius C in the hyperbolic distance. Since this disk $\Delta^K_{\mathbb{C}\setminus\mathbb{R}^+}(\tilde z_0, C)$ is a compact subset of $\mathbb{C} \setminus \mathbb{R}^+$, there exists a sector $S'' := \{z = re^{i\theta}, r > 0, 0 < \theta < \theta_0'\}$ such that $S'' \cap \Delta^K_{\mathbb{C}\setminus\mathbb{R}^+}(\tilde z_0, C) = \emptyset$, here we can assume that $\theta_0' < \theta_0$. Therefore, $d_{\mathbb{C}\setminus\mathbb{R}^+}(\tilde z_0, \hat z/|z_0|) \ge C$ for any $\hat z/|z_0| \in \mathcal{A} \cap S''$. Furthermore, for this new θ_0', it does not change the conclusion of the estimation of $d^1_{\mathcal{A}}(z_0, \tilde z)$ and $d^2_{\mathcal{A}}(z_0, z')$.

Therefore, there is a point z_0 such that all these three distances $d^1_{\mathcal{A}}(z_0, \tilde z), d^2_{\mathcal{A}}(z_0, z'), d^3_{\mathcal{A}}(z_0, \hat z) \ge C$ for the given constant C.

$\square$

Corollary 3.4. *Let $X \subset \mathcal{A}$ be the set of all $z_0 \in \mathcal{A}$ such that $d_{\mathcal{A}}(z_0, \tilde{q}) \geq C$ for any $\tilde{q} \in \mathcal{A}$. If $z \in X$, then any point $w \in f^{-1}(z)$ is in X. Therefore, X is dense in the boundary of $\mathcal{A}$.*

Proof. By Theorem A, we know that $X \neq \emptyset$. Suppose $w \notin X$, then there exists $\tilde{q} \in Q$ so that $d_{\mathcal{A}}(w, \tilde{q}) < C$. Since $f(z)$ is distance decreasing, see Proposition 2.7, we have $d_{\mathcal{A}}(f(w), f(\tilde{q})) = d_{\mathcal{A}}(z, \tilde{q}) < C$. This contradicts $z \in X$.

Furthermore, we know that $\{f^{-n}(z)\}_{n\in\mathbb{N}}$ clusters at every point in Julia set. In particular, this is true if $z \in X$. More precisely, $\{f^{-n}(z)\}_{n\in\mathbb{N}}$ equidistributes toward the Green measure. Therefore, the closure of X contains the boundary of $\mathcal{A}$.

□

3.2 Dynamics inside the parabolic basin of $f(z) = z + az^{m+1}, m \geq 1, a \neq 0$

In this subsection, we generalize Theorem A to the case of several petals inside the parabolic basin. Let us recall the statement of our main Theorem B:

Theorem B. Let $f(z) = z + az^{m+1}, m \geq 1, a \neq 0$, and Ω_j be the immediate basin of $\mathcal{A}_j$. We choose an arbitrary constant $C > 0$ and an arbitrary point $q = a\mathbf{v_j} \in \mathcal{P}_j$, a is a small positive real number. Then there exists a point $z_0 \in \Omega_j$ such that for any $\tilde{q} \in Q := \cup_{l=0}^{\infty}\{f^{-l}(f^k(q))\} \cap \Omega_j$ (l, k are non-negative integers), the hyperbolic distance satisfies $d_{\Omega_j}(z_0, \tilde{q}) \geq C$, where d_{Ω_j} denotes the hyperbolic distance on Ω_j.

Proof. We conjugate $f(z)$ using rotation:

$$\begin{array}{ccc} z & \xrightarrow{f(z)=z+az^{m+1}} & f(z) \\ {\scriptstyle e^{i\theta}z}\downarrow & & \downarrow{\scriptstyle e^{i\theta}z} \\ w & \xrightarrow{\quad g \quad} & g(w) \end{array}$$

$w \mapsto e^{i\theta}(e^{-i\theta}w + a(e^{-i\theta}w)^{m+1}) = w + ae^{-im\theta}w^{m+1}$. Suppose $a = re^{i\psi}$, then we can choose $\theta = \frac{\psi}{m}$ such that $ae^{-im\theta}$ is a real positive number. Then we can assume $S := \{z = re^{i\theta}, r > 0, 0 < \theta < \frac{2\pi}{m}\}$ be the sector with angle $2\pi/m$, including the attracting petal $\mathcal{P}_j$, then the angle between $\mathbf{v_j}$ and $\mathbb{R}^+$ is $\frac{\pi}{m}$. We denote the boundary rays of S by $l^1_{\mathbb{R}^+} := \{z = r > 0\}$ and $l^2_{\mathbb{R}^+} := \{z = re^{i\frac{2\pi}{m}}, r > 0\}$. We choose $\theta_0 > 0$ and two rays $l_1 := \{z = re^{i\theta_0}, r > 0\}, l_2 := \{z = re^{i(\frac{2\pi}{m} - \theta_0)}, r > 0\}$. Then we denote by $T := \{z = re^{i\theta}, r > 0, \theta_0 < \theta < \frac{2\pi}{m} - \theta_0\}$ a sector inside S.

Let $\varphi(z) : S \to \mathbb{H}$ with $\varphi(z) = z^{m/2}$. Then by Definition 2.6., the hyperbolic metric F_S is

$$F_S = \frac{|\varphi'|}{\operatorname{Im}\varphi}|dz| = \frac{m}{2r\sin(\frac{\theta m}{2})}|dz|.$$

As in the proof of Theorem A, we can choose two analogous "Pac-Man" $D^m_{R'_0} := \{z = re^{i\theta}, 0 < r < R'_0, \theta_0 < \theta < \frac{2\pi}{m} - \theta_0\}, D^m_{R_0} := \{z = re^{i\theta}, 0 < r < R_0, \theta_0 < \theta < \frac{2\pi}{m} - \theta_0\}$ central at 0 with radius $R'_0, R_0 > 0 (R'_0 < R_0)$, respectively, such that $D^m_{R_0} \subsetneq T \cap \Omega_j$ and $f^n(D^m_{R'_0}) \subset D^m_{R_0}$.

Then similarly, we need to estimate the three hyperbolic distances from z_0 to any point $\tilde{z} \in \partial D^m_{R_0}$ (see the blue curve in Figure 6), $z' \in \mathbf{v_j}$ (see the pink curve in Figure 6), and $\hat{z} \in \Omega_j \cap \left\{S' := \{z = re^{i\theta}, r > 0, 0 < \theta < \theta_0\}\right\}$ (see the green curve in Figure 6), and show that all of them are not less than C.

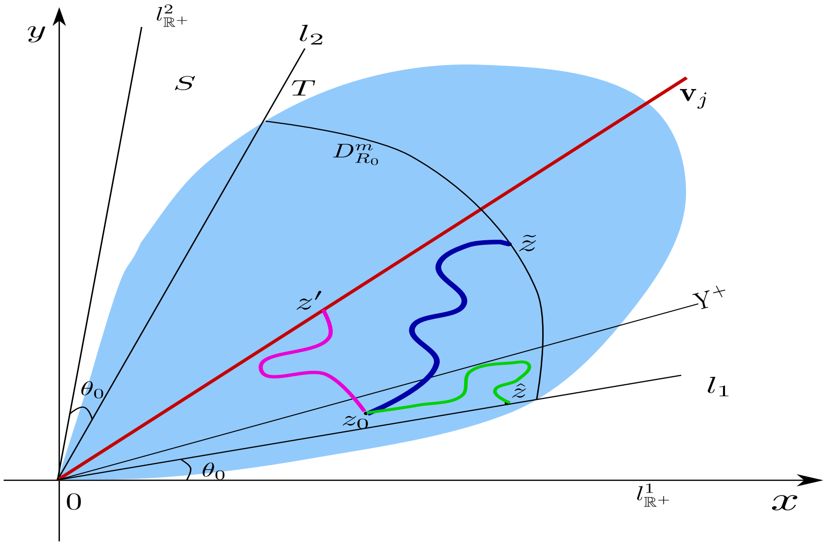

Figure 6: The three hyperbolic distances in Ω_j

First, suppose $z_0 \in D^m_\varepsilon := \{z = re^{i\theta}, 0 < r < \varepsilon, \theta_0 < \theta < \frac{2\pi}{m} - \theta_0\}, \varepsilon << R'_0$. Let us estimate the hyperbolic distance from z_0 to any point $\tilde{z}$ on the boundary of $D^m_{R_0}$, and we denote this distance by $d^1_{\Omega_j}(z_0, \tilde{z})$.

$$\begin{aligned} d^1_{\Omega_j}(z_0, \tilde{z}) &\geq d_S(z_0, \tilde{z}) = \inf \int_{\gamma(t)} F_S(\gamma(t))|\gamma'(t)|dt \\ &= \inf \int_{\gamma(t)} \frac{m}{2|\gamma(t)| \sin \frac{m \arg(\gamma(t))}{2}} |\gamma'(t)|dt \geq \inf \int_{\gamma(t)} \frac{m}{2|\gamma(t)|}|\gamma'(t)|dt \\ &= \frac{m}{2} \inf \int_\varepsilon^{R_0} \frac{|dr|}{r} \geq \frac{m}{2}(\ln R_0 - \ln \varepsilon), \end{aligned}$$

where $\gamma(t)$ is a smooth path joining z_0 to $\tilde{z}$. In addition, we can see that $d^1_{\Omega_j}(z_0, \tilde{z}) \to \infty$ as $\varepsilon \to 0$.

Second, we calculate the hyperbolic distance from z_0 to any point $z' \in \mathbf{v_j}$ denoted by $d^2_{\Omega_j}(z_0, z')$. Let $\varepsilon = \varepsilon_0$, i.e., fix ε, and $D^m_1 \subset T$ be a scaling of $D^m_{\varepsilon_0}$ by $S(z) = \frac{z}{|z_0|}$, sending z_0, z' to $\tilde{z}_0 := \frac{z_0}{|z_0|}, \frac{z'}{|z_0|}$, respectively. By homogeneity, we know the hyperbolic distance $d_S(z_0, z') = d_S(\tilde{z}_0, z'/|z_0|)$. Since we hope to prove $d^2_{\Omega_j}(z_0, z') > C$, we need z_0 to be far from $\mathbf{v_j}$, and so does $\tilde{z}_0$. Let $S_T := \{z = e^{i\theta}, \theta_0 < \theta < \frac{\pi}{2m} - \theta_0\}$, assume $\tilde{z}_0 \in S_T$ and $\operatorname{Re} \tilde{z}_0$ is sufficiently big, then any curve from $\tilde{z}_0$ to $\frac{z'}{|z_0|}$ must pass through a point $\tilde{z}'$ on the ray $Y^+ := \{re^{i\frac{\pi}{2m}}, r > 0\}$.

Hence $d^2_{\Omega_j}(z_0, z') \geq d_S(z_0, z') = d_S(\tilde{z}_0, z'/|z_0|) \geq d_S(\tilde{z}_0, \tilde{z}')$. Then we have

$$\begin{aligned} d^2_{\Omega_j}(z_0, z') \geq d_S(\tilde{z}_0, \tilde{z}') &= \inf \int_{\tilde{z}_0}^{\tilde{z}'} F_S(z) \\ &= \inf \int_{\tilde{z}_0}^{\tilde{z}'} \frac{m|dz|}{2r\sin(m\theta/2)} \\ &= \inf \int_{\tilde{z}_0}^{\tilde{z}'} \frac{\frac{m\theta}{2}}{\sin(\frac{m\theta}{2})} \cdot \frac{m|dz|}{2r(m\theta/2)} \\ &= c_1 \inf \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{r\theta} \\ &= c_1 \inf \int_{\tilde{z}_0}^{\tilde{z}'} \frac{\sin\theta}{\theta} \cdot \frac{|dz|}{r\sin\theta} \\ &= c_1 c_2 \inf \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{r\sin\theta} \\ &= c_1 c_2 \inf \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{\operatorname{Im} z} \\ &\geq c_1 c_2 |\ln(\operatorname{Im} \tilde{z}') - \ln(\operatorname{Im} \tilde{z}_0)|. \end{aligned}$$

where $c_1 := \inf \frac{\frac{m\theta}{2}}{\sin\frac{m\theta}{2}}, c_2 := \inf \frac{\sin\theta}{\theta}$.

Then there are three situations for the hyperbolic distance between $\tilde{z}_0$ and $\tilde{z}'$:

1) If $\operatorname{Im} \tilde{z}' \geq e^C |\operatorname{Im} \tilde{z}_0|$ or $\operatorname{Im} \tilde{z}' \leq \frac{|\operatorname{Im} \tilde{z}_0|}{e^C}$ for some constant $C > 1$, then $|\ln(\operatorname{Im} \tilde{z}') - \ln(\operatorname{Im} \tilde{z}_0)| \geq C$, hence $d^2_{\Omega_j}(z_0, z') \geq C$ is true.

2) If $\tilde{z}' \in \mathcal{L} := \{z = x + iy, \frac{|\operatorname{Im} \tilde{z}_0|}{e^C} < y < e^C |\operatorname{Im} \tilde{z}_0|\}$. We need to prove that $d_{\Omega_j}(\tilde{z}_0, \tilde{z}') \geq c_1 c_2 \inf \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{\operatorname{Im} z} \geq C$ for $z \in \mathcal{L}$.

Let $z = x + iy \in \mathcal{L}$, then $\operatorname{Im} z \leq e^C |\operatorname{Im} \tilde{z}_0|$. Hence we have

$$\int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{\operatorname{Im} z} \geq \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dx|}{\operatorname{Im} z} \geq \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dx|}{e^C |\operatorname{Im} \tilde{z}_0|} = \frac{|\operatorname{Re} \tilde{z}' - \operatorname{Re} \tilde{z}_0|}{e^C} \frac{1}{|\operatorname{Im} \tilde{z}_0|} > \frac{1}{2e^C |\operatorname{Im} \tilde{z}_0|},$$

Note that the last inequality holds since we choose $\operatorname{Re} \tilde{z}_0$ sufficiently big so that $\tilde{z}_0$ is close to l_1, and $|\operatorname{Re} \tilde{z}' - \operatorname{Re} \tilde{z}_0| > 1/2$ because $\tilde{z}'$ will have to be close to 0 since it lies on $Y^+ \cap \mathcal{L}$ and its imaginary part is close to 0, which makes $d^2_{\Omega_j}(z_0, z')$ as big as we want.

In other words, as long as $\theta_0 < \frac{1}{2Ce^C}$, we have $\operatorname{Im} \tilde{z}_0 < \frac{1}{2Ce^C}$. In addition, $|\operatorname{Re} \tilde{z}' - \operatorname{Re} \tilde{z}_0| > 1/2$, we obtain $\int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{\operatorname{Im} z} > C$.

3) If $\tilde{z}' \in \mathcal{L}$, but the curve γ between $\tilde{z}_0$ and $\tilde{z}'$ get outside of $\mathcal{L}$ starting at some point $\tilde{z}'' \in \gamma \cap \mathcal{L}$ for a while, then enter back to $\mathcal{L}$ again, then we still have $d^2_{\Omega_j}(z_0, z') \geq C$ is true because $d_S(z_0, z') \geq d_S(\tilde{z}_0, \tilde{z}'') \geq C$. The last inequality holds since 1) is valid. We have a conclusion as same as $d^2_{\mathcal{A}}(z_0, z') \geq C$ in the proof of Theorem A.

At last, we estimate the hyperbolic distance from z_0 to any point $\hat{z} \in \Omega_j \cap \{S' :=$

$\{z = re^{i\theta}, r > 0, 0 < \theta < \theta_0\}\}$ denoted by $d^3_{\Omega_j}(z_0, \hat{z})$. We know $d^3_{\Omega_j}(z_0, \hat{z}) \geq d_S(z_0, \hat{z}) = d_S(\tilde{z}_0, \hat{z}/|z_0|)$.

We use the method for computing $d^3_{\Omega_j}(z_0, \hat{z})$ as same as $d^3_{\mathcal{A}}(z_0, \hat{z}) \geq C$ in the proof of Theorem A. We first choose a disk $\Delta^K_S(\tilde{z}_0, C)$ centered at $\tilde{z}_0$ with radius C in the hyperbolic distance. Since this disk $\Delta^K_S(\tilde{z}_0, C)$ is a compact subset of S, there exists a sector $S'' := \{z = re^{i\theta}, r > 0, \pi - \frac{\pi}{m} < \theta < \pi - \frac{\pi}{m} + \theta'_0\}$ such that $S'' \cap \Delta^K_S(\tilde{z}_0, C) = \emptyset$, here we can assume that $\theta'_0 < \theta_0$. Therefore, $d_S(\tilde{z}_0, \hat{z}/|z_0|) \geq C$ for any $\hat{z}/|z_0| \in \Omega_j \cap S''$. Furthermore, for this new θ'_0, it does not change the conclusion of the estimation of $d^1_{\Omega_j}(z_0, \tilde{z})$ and $d^2_{\Omega_j}(z_0, z')$.

□

3.3 Dynamics inside the parabolic basin of $f(z) = z + az^{m+1} +$ (higher order terms), $m \geq 1, a \neq 0$

Finally, in this subsection, we consider the behavior of orbits inside parabolic basins of general polynomials. Let us recall the statement of our main Theorem C:

Theorem C. Let $f(z) = z + az^{m+1} + \text{(higher terms)}, m \geq 1, a \neq 0$, and Ω_j be the immediate basin of $\mathcal{A}_j$. We choose an arbitrary constant $C > 0$ and an arbitrary point $q = a\mathbf{v_j} \in \mathcal{P}_j$, a is a small positive real number. Then there exists a point $z_0 \in \Omega_j$ such that for any $\tilde{q} \in Q := \cup_{l=0}^{\infty}\{f^{-l}(f^k(q))\} \cap \Omega_j$ (l, k are non-negative integers), the hyperbolic distance satisfies $d_{\Omega_j}(z_0, \tilde{q}) \geq C$, where d_{Ω_j} denotes the hyperbolic distance on Ω_j.

Proof. The essential idea to prove Theorem C is the same as the proof of Theorem A and Theorem B. However, we cannot draw the parabolic basin of $f(z) = z + az^{m+1} +$ (higher order terms) directly as Figure 4 and Figure 6 since there are higher order terms of $f(z)$, the parabolic basin can be more complicated.

To simplify the discussion, we first consider the case $m = 1$:

$$f(z) = z + z^2 + \text{(higher order terms)}.$$

When there are no higher order terms, the crucial estimate of the hyperbolic metric comes from the fact that the parabolic basin is contained in $\mathbb{C} \setminus \mathbb{R}^+$. Hence we could compare it with the hyperbolic metric on $\mathbb{C} \setminus \mathbb{R}^+$.

In the case of higher order terms, the parabolic basin can be more complicated. However, we can, instead of $\mathbb{C} \setminus \mathbb{R}^+$, use the double sheeted domain

$$V_R := \{z = re^{i\theta}, 0 < r < R, -\theta_0 < \theta < 2\pi + \theta_0\}.$$

Next, we investigate the properties of V_R to explain why we choose the double sheeted domain V_R as above.

Proposition 3.5. *Let $\bar{D}_R := \{z = re^{i\theta}, 0 < r < R, -\theta_0 < \theta < \theta_0\}$, $\mathcal{A}$ be the whole basin of $f(z)$, S_1 be the connected component of $\mathcal{A} \cap \bar{D}_R$ which contains $\{z = re^{i\theta_0}, 0 < r < R\}$, and S_2 be the connected component of $\mathcal{A} \cap \bar{D}_R$ which contains $\{z = re^{-i\theta_0}, 0 < r < R\}$. Then any two pieces S_1, S_2 (see the left of Figure 7) are disjoint in $\bar{D}_R$.*

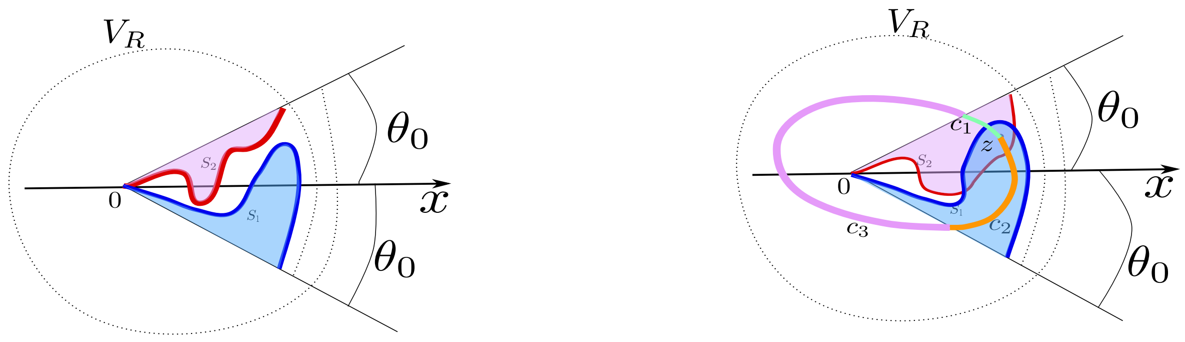

Figure 7: Two pieces S_1 and S_2 in $\bar{D}_R$

Proof. We know that, inside V_R and near the origin, $\mathcal{A}$ contains the Left Pac-Man $D_R := \{z = re^{i\theta}, 0 < r < R, \theta_0 < \theta < 2\pi - \theta_0\}$.

If S_1 intersects S_2, then there is a point $z \in S_1 \cap S_2$. We can draw three curves, c_1 from z to $l_1 := \{z = re^{i\theta_0}\}$, c_2 from z to $l_2 := \{z = re^{-i\theta_0}\}$, and $c_3 \in D_R$ which connect c_1 and c_2. Hence $\mathcal{A}$ contains a closed curve $\gamma_c := c_1 + c_2 + c_3$ with the winding number 1 around the origin (see the right of Figure 7).

We know that $f^n(z) \to 0$ when $z \in \gamma_c$, since $\gamma_c \in \mathcal{A}$. In addition, by the maximum principle, we have $f^n(z) \to 0$ when z is inside the domain bounded by γ_c. Hence $\mathcal{A}$ contains a neighborhood of 0, then 0 is an attracting fixed point. However, this contradicts that 0 is a parabolic fixed point of $f(z)$.

□

By Proposition 3.5, we can use the hyperbolic metric on V_R instead of $\mathbb{C} \setminus \mathbb{R}^+$.

First, we know that V_R can be mapped to a sector $S := \{z = re^{i\theta}, r > 0, \theta_1 < \theta < \theta_2, \theta_1 << \theta_2 < \pi/2\}$ by $\varphi_1(z) = z^{c/2}$ when c is sufficiently small. Second, we can change c such that $\theta_2 = \pi - \theta_1$ by some map $\varphi_2(z)$. At last, by some rotation map φ_3, we can map S to the upper half plane $\mathbb{H}$ (see the Figure 8).

Therefore, the map $\varphi(z) := \varphi_3 \circ \varphi_2 \circ \varphi_1$ from V_R to the upper half plane $\mathbb{H}$ becomes $\varphi(z) = e^{i\psi} z^{\frac{c}{2}}$, instead of $\varphi(z) = z^{1/2}$, where c is very close to 1. Then, with the above setting, the rest of the estimation goes through as in Theorem B.

If $m > 1$, it is difficult to draw the specific parabolic basins of $f(z)$ or the attracting petals. Let

$$V_R =: \{z = re^{i\theta}, r > 0, -\theta_0 < \theta < \frac{2\pi}{m} + \theta_0\}.$$

We use Figure 9 to illustrate how we can choose V_R (see the domain with pink curves as its argument). We want to map V_R to the upper half-plane. First, Let $z \to e^{i\theta_0} z$, then V_R is mapped to

$$V_R' := \{z = re^{i\theta}, r > 0, 0 < \theta < \frac{2\pi}{m} + 2\theta_0\}.$$

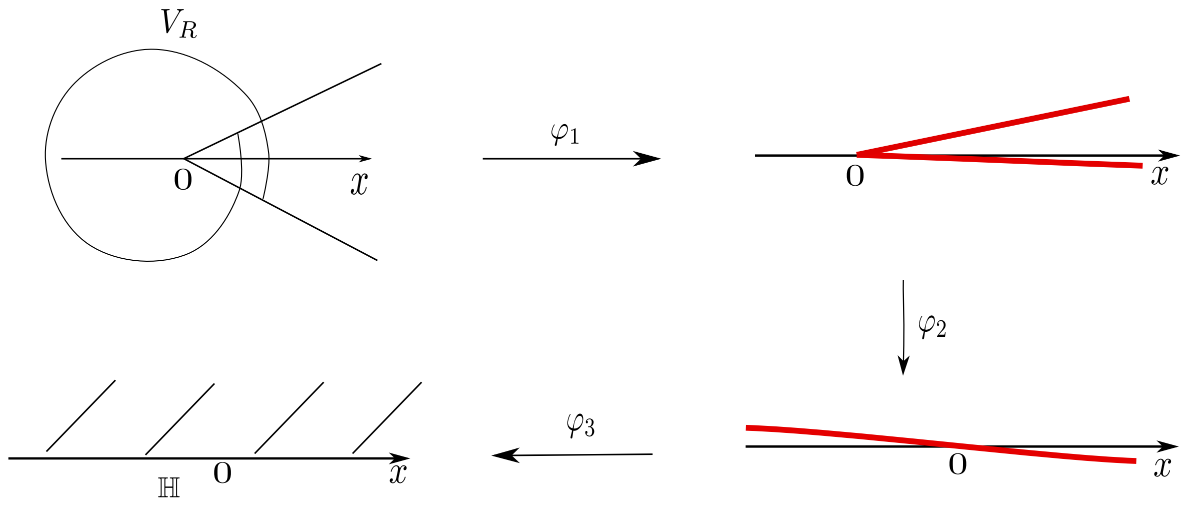

Figure 8: The maps φ_1, φ_2 and φ_3

We define $\varphi(z) = (e^{i\theta_0} z)^{\frac{\pi}{\frac{2\pi}{m}+2\theta_0}}$. Then the hyperbolic metric on V_R is

$$F_{V_R} = \frac{|\varphi'(z)|}{\operatorname{Im}\varphi}|d_z|.$$

And similarly, we have the same properties of S_1, S_2 as in Proposition 3.5 (see the red curve and blue curve on Figure 9). Then the rest of the estimation goes through as in

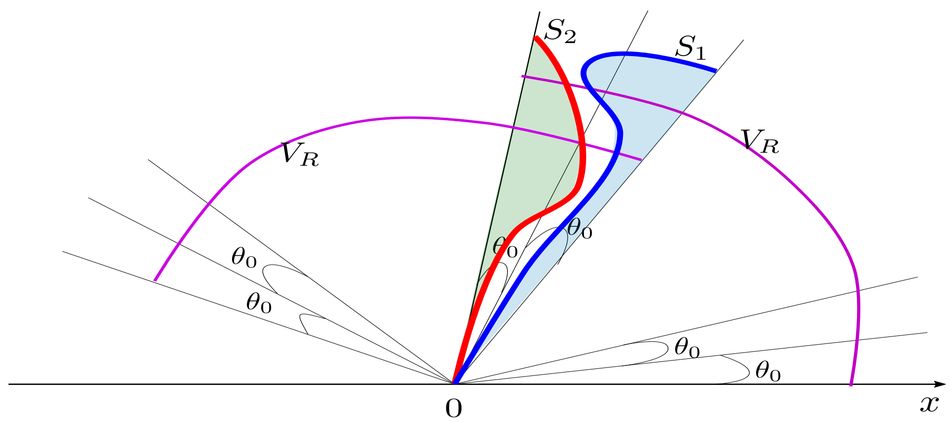

Figure 9: Two pieces S_1 and S_2

Theorem B. Thus, we are done.

□

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University of Parma, Department of Mathematical, Physical and Computer Sciences, Parco Area delle Scienze, 53/A, 43124 Parma PR, Italy

Email address: humihqu@gmail.com