OPTIMAL SINGULARITIES OF INITIAL DATA OF A FRACTIONAL SEMILINEAR HEAT EQUATION IN OPEN SETS

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Abstract. We consider necessary conditions and sufficient conditions on the local-in-time solvability of the Cauchy–Dirichlet problem for a fractional semilinear heat equation in open sets (possibly unbounded and disconnected) with a smooth boundary. Our conditions enable us to identify the optimal strength of the admissible singularity of initial data for the local-in-time solvability and they differ in the interior of the set and on the boundary of the set.

1. Introduction

1.1. **Introduction.** This paper is concerned with the local-in-time solvability of the Cauchy–Dirichlet problem for

$$\begin{cases} \partial_t u + (-\Delta)^{\frac{\theta}{2}}|_\Omega u = u^p, & x \in \Omega,\ t > 0, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where Ω is an open set in $\mathbb{R}^N$ (possibly unbounded and disconnected) with a nonempty $C^{1,1}$ boundary, $N \geq 1$, $p > 1$, and $0 < \theta < 2$. In this paper all solutions are assumed to be nonnegative. For $0 < \theta < 2$, the fractional Laplacian $(-\Delta)^{\theta/2}$ can be written in the form

$$(-\Delta)^{\frac{\theta}{2}} u(x) = c \lim_{\epsilon \to +0} \int_{\{y \in \mathbb{R}^N; |x-y| > \epsilon\}} \frac{u(x) - u(y)}{|x-y|^{N+\theta}}\, dy$$

for some specific constant $c = c(N, \theta) > 0$. Furthermore, $(-\Delta)^{\theta/2}|_\Omega$ denotes the fractional Laplacian with zero exterior condition. For more details, see, for example, [7], which summarizes many properties of the fractional Laplacian $(-\Delta)^{\theta/2}$. In this paper we attempt to give necessary conditions and sufficient conditions on the local-in-time solvability of the Cauchy–Dirichlet problem for (1.1) and identify the optimal

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strength of the admissible singularity of initial data for the local-in-time solvability. In particular, an initial value with such singularity may not be an L^1_{loc}-function. To address this problem, we follow the arguments in [11]. In the following, we describe their ideas and results. Throughout this paper, we denote

$$p_\alpha(d,l) := 1 + \frac{\alpha}{d+l}$$

for $\alpha > 0$, $d \geq 1$, and $l \geq 0$. For any $x \in \overline{\Omega}$ and $r > 0$, set

$$B(x,r) := \{y \in \mathbb{R}^N; |x-y| < r\}, \quad B_\Omega(x,r) := B(x,r) \cap \overline{\Omega}.$$

For a Borel set $A \subset \mathbb{R}^N$, $\chi_A(x)$ denotes the characteristic function of A.

The solvability of the Cauchy–Dirichlet problem for (1.1) (including the case of $\theta \geq 2$ and the case of $\Omega = \mathbb{R}^N$) has been studied in many papers. See, for example, [1, 3, 4, 9–11, 13–18, 20–28] and references therein. Of course, in the case of $\Omega = \mathbb{R}^N$, we ignore the boundary/exterior condition, and in the case where θ is a positive even integer, $\mathbb{R}^N \setminus \Omega$ in the exterior condition is replaced by $\partial\Omega$. Among them, the author of this paper, Ishige, and Takahashi [11] considered the solvability of the Cauchy–Dirichlet problem for

$$\begin{cases} \partial_t u - \Delta u = u^p, & x \in \mathbb{R}^N_+, \ t > 0, \\ u = 0, & x \in \partial\mathbb{R}^N_+, \ t > 0, \end{cases} \tag{1.2}$$

where $N \geq 1$, $p > 1$, and

$$\mathbb{R}^N_+ := \begin{cases} \mathbb{R}^{N-1} \times (0,\infty) & \text{if} \quad N \geq 2, \\ (0,\infty) & \text{if} \quad N = 1. \end{cases}$$

For $d = 1, 2, \cdots$, let g_d be the heat kernel in $\mathbb{R}^d \times (0,\infty)$, that is,

$$g_d(x,t) := \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{4t}\right)$$

for $x \in \mathbb{R}^d$ and $t \in (0,\infty)$. Let $p = p(x,y,t)$ be the Dirichlet heat kernel in $\overline{\mathbb{R}^N_+} \times (0,\infty)$, that is,

$$p(x,y,t) := g_{N-1}(x'-y',t)[g_1(x_N - y_N, t) - g_1(x_N + y_N, t)]$$

for $x = (x', x_N)$, $y = (y', y_N) \in \overline{\mathbb{R}^N_+}$, and $t > 0$. The Cauchy–Dirichlet problem for (1.2) can possess a solution even if $u(\cdot,0)$ is not a Radon measure on $\overline{\mathbb{R}^N_+}$ due to the boundary condition. For example, Tayachi and Weissler [24] proved that if $1 < p < p_2(N,1)$ and $u(\cdot,0)$ satisfies

$$u(\cdot,0) = -\kappa \partial_{x_N} \delta_N \quad \text{on} \quad \overline{\mathbb{R}^N_+} \tag{1.3}$$

for sufficiently small $\kappa > 0$, then problem (1.2) with (1.3) possesses a local-in-time solution, where δ_N is the N-dimensional Dirac measure concentrated at the origin. For this reason, we could not treat initial data of (1.2) in the framework of the Radon measure on $\overline{\mathbb{R}^N_+}$. In order to overcome this difficulty, the authors of [11] introduced the following idea. By the explicit formula of $p(x,y,t)$ we see that for $y = (y',0) \in \partial\mathbb{R}^N_+$,

$$0 < \lim_{y_N \to +0} \frac{p(x,y',y_N,t)}{y_N} = \partial_{y_N} p(x,y',0,t) < \infty$$

for $x \in \mathbb{R}^N_+$ and $t \in (0,\infty)$. Therefore, the function

$$k(x,y,t) := \begin{cases} \dfrac{p(x,y,t)}{y_N} & \text{if} \quad (x,y,t) \in \overline{\Omega} \times \Omega \times (0,\infty), \\ \partial_{y_N} p(x,y,t) & \text{if} \quad (x,y,t) \in \overline{\Omega} \times \partial\Omega \times (0,\infty), \end{cases}$$

is well-defined and continuous on $\overline{\mathbb{R}^N_+} \times \overline{\mathbb{R}^N_+} \times (0,\infty)$. Using this function, the solution of the heat equation on $\mathbb{R}^N_+ \times (0,\infty)$ can be rewritten as

$$\int_{\overline{\mathbb{R}^N_+}} p(x,y,t) u(y,0)\, dy = \int_{\overline{\mathbb{R}^N_+}} k(x,y,t) y_N u(y,0)\, dy.$$

Since $k(x,y,t)$ is positive and finite for $(x,y,t) \in \mathbb{R}^N_+ \times \overline{\mathbb{R}^N_+} \times (0,\infty)$, they gave an initial condition of Radon measure on $\overline{\mathbb{R}^N_+}$ to $x_N u(\cdot,0)$, instead of $u(\cdot,0)$ itself, that is,

$$x_N u(\cdot,0) = \mu \quad \text{on} \quad \overline{\mathbb{R}^N_+}, \tag{1.4}$$

where μ is a Radon measure on $\overline{\mathbb{R}^N_+}$. Thanks to this idea, we can treat the initial condition of (1.2) in the framework of the Radon measures. Indeed, in this representation, (1.3) corresponds to $\mu = \kappa\delta_N$ on $\overline{\mathbb{R}^N_+}$. They obtained necessary conditions for μ on the solvability of problem (1.2) with (1.4).

Theorem A (H.–Ishige–Takahashi [11])**.** *Let $N \geq 1$ and $p > 1$. Assume problem (1.2) with (1.4) possesses a supersolution in $\mathbb{R}^N_+ \times [0,T)$, where $T \in (0,\infty)$. Then there exists $\gamma = \gamma(N,p) > 0$ such that*

$$\mu(B_{\mathbb{R}^N_+}(z,\sigma)) \leq \gamma \sigma^{-\frac{2}{p-1}} \int_{B_{\mathbb{R}^N_+}(z,\sigma)} y_N\, dy$$

for all $z \in \overline{\mathbb{R}^N_+}$ and $\sigma \in (0, T^{1/2})$. In addition,

(i) *if* $p = p_2(N, 0)$, *then there exists* $\gamma' = \gamma'(N) > 0$ *such that*

$$z_N^{-1}\mu(B_{\mathbb{R}^N_+}(z,\sigma)) \le \gamma' \left[\log\left(e + \frac{T^{\frac{1}{2}}}{\sigma}\right)\right]^{-\frac{N}{2}}$$

for all $z = (z', z_N) \in \mathbb{R}^N_+$ *with* $z_N \ge 3\sigma$ *and* $\sigma \in (0, T^{1/2})$.

(ii) *if* $p = p_2(N, 1)$, *then there exists* $\gamma'' = \gamma''(N) > 0$ *such that*

$$\mu(B_\Omega(z,\sigma)) \le \gamma'' \left[\log\left(e + \frac{T^{\frac{1}{2}}}{\sigma}\right)\right]^{-\frac{N+1}{2}}$$

for all $z \in \partial\mathbb{R}^N_+$ *and* $\sigma \in (0, T^{1/2})$.

(iii) *if* $p \ge 2$, *then* $\mu(\partial\Omega) = 0$.

See also [1, 10]. In addition, they also obtained sufficient conditions for μ on the solvability of problem (1.2) with (1.4). These conditions enable us to identify the optimal strength of the admissible singularity of initial data μ for the local-in-time solvability. Namely, they proved the following theorems:

Theorem B (H.–Ishige–Takahashi [11])**.** *Let* $z \in \mathbb{R}^N_+$. *Set*

$$f_z(x) := \begin{cases} |x-z|^{-\frac{2}{p-1}}\chi_{B_\Omega(z,1)}(x) & \text{if} \quad p > p_2(N,0), \\ |x-z|^{-N}|\log|x-z||^{-\frac{N}{2}-1}\chi_{B_\Omega(z,1/2)}(x) & \text{if} \quad p = p_2(N,0), \end{cases}$$

for $x \in \mathbb{R}^N_+$. *Then there exists* $\kappa_z > 0$ *with the following properties:*

(i) *problem* (1.2) *with* (1.4) *possesses a local-in-time solution with* $\mu = \kappa x_N f_z(x)$ *if* $0 < \kappa < \kappa_z$;

(ii) *problem* (1.2) *with* (1.4) *possesses no local-in-time solutions with* $\mu = \kappa x_N f_z(x)$ *if* $\kappa > \kappa_z$.

Here $\sup_{z\in\mathbb{R}^N_+} \kappa_z < \infty$.

Theorem C (H.–Ishige–Takahashi [11])**.** *Set*

$$f_0(x) := \begin{cases} |x|^{-\frac{2}{p-1}}\chi_{B_\Omega(0,1)}(x) & \text{if} \quad p > p_2(N,1), \\ |x|^{-N-1}|\log|x||^{-\frac{N+1}{2}-1}\chi_{B_\Omega(0,1/2)}(x) & \text{if} \quad p = p_2(N,1), \end{cases}$$

for $x \in \mathbb{R}^N_+$. *Then there exists* $\kappa_0 > 0$ *with the following properties:*

(i) *problem* (1.2) *with* (1.4) *possesses a local-in-time solution with* $\mu = \kappa x_N f_0(x)$ *if* $0 < \kappa < \kappa_0$;

(ii) *problem* (1.2) *with* (1.4) *possesses no local-in-time solutions with* $\mu = \kappa x_N f_0(x)$ *if* $\kappa > \kappa_0$.

The authors of [11] termed the singularity of the function f_z at $x = z$ as in the above theorems an *optimal singularity* of initial data for the local-in-time solvability of problem (1.2) with (1.4) at $x = z$. Since optimal singularities can be identified using their necessary conditions and sufficient conditions on the solvability, we can say that these conditions are sharp.

We shall go back to (1.1). In the arguments in [11], the explicit formula of the Dirichlet heat kernel $p(x, y, t)$ was used. However, this is not expected when Ω is general or $0 < \theta < 2$, and only estimates of the Dirichlet heat kernel are known (see Theorem D below). One of the novelties of this paper is that we obtain analogous results to those of [11] in this situation. Furthermore, applying these conditions, we can identify optimal singularities of initial data.

1.2. **Notation and the definition of solutions.** In order to state our main results, we introduce some notation and formulate the definition of solutions. We denote by $\mathcal{M}$ the set of nonnegative Radon measures on $\overline{\Omega}$. For any $L^1_{loc}(\Omega)$-function μ, we often identify $d\mu = \mu(x)\,dx$ in $\mathcal{M}$. For any $T \in (0, \infty)$, we set $Q_T := \Omega \times (0, T)$. For two nonnegative functions f and g, the notation $f \asymp g$ means that there exist positive constants $c_1 < c_2$ such that $c_1 g(x) \le f(x) \le c_2 g(x)$ in the common domain of definition of f and g. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$.

Let $\Gamma_\theta = \Gamma_\theta(x, t)$ be the fundamental solution of

$$\partial_t v + (-\Delta)^{\frac{\theta}{2}} v = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, \infty),$$

where $N \ge 1$ and $0 < \theta < 2$. For $x, y \in \overline{\Omega}$ and $t > 0$, let $G = G(x, y, t)$ be the Dirichlet heat kernel on Ω. Then, G is continuous on $\overline{\Omega} \times \overline{\Omega} \times (0, \infty)$ and satisfies

$$\int_\Omega G(x, y, t)\,dy \le 1, \tag{1.5}$$

$$\int_\Omega G(x, z, t) G(z, y, s)\,dz = G(x, y, t + s), \tag{1.6}$$

for all $x, y \in \Omega$ and $s, t > 0$, and

$$\begin{cases} G(x, y, t) = G(y, x, t) & \text{if} \quad (x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, \infty), \\ G(x, y, t) > 0 & \text{if} \quad (x, y, t) \in \Omega \times \Omega \times (0, \infty), \\ G(x, y, t) = 0 & \text{if} \quad (x, y, t) \in \Omega^c \times \mathbb{R}^N \times (0, \infty). \end{cases}$$

See e.g. [4]. No explicit formulas of G can be expected in this situation but Chen, Kim, and Song [5] obtained the following two-sided estimate of G:

Theorem D. *Let Ω be a open set in $\mathbb{R}^N$ with a $C^{1,1}$ boundary and* $d(x) := \mathrm{dist}(x, \partial\Omega)$.

(i) *There exists $T' > 0$ depending only on Ω such that*

$$G(x,y,t) \asymp \left(1 \wedge \frac{d(x)^{\frac{\theta}{2}}}{\sqrt{t}}\right)\left(1 \wedge \frac{d(y)^{\frac{\theta}{2}}}{\sqrt{t}}\right)\Gamma_\theta(x-y,t) \tag{1.7}$$

for all $x, y \in \overline{\Omega}$ and $t \in (0, T']$.

(ii) *When Ω is bounded and $t > T'$, one has*

$$G(x,y,t) \asymp d(x)^{\frac{\theta}{2}} d(y)^{\frac{\theta}{2}} e^{-\lambda_1 t}$$

for all $x, y \in \overline{\Omega}$ and $t > T'$. Here, $\lambda_1 > 0$ is the smallest eigenvalue of the Dirichlet fractional Laplacian $(-\Delta)^{\theta/2}|_\Omega$.

See also [2,4,6]. For $x \in \overline{\Omega}$, $y \in \partial\Omega$, and $t > 0$, define the $\theta/2$-normal derivative as

$$D_{\frac{\theta}{2}} G(x,y,t) := \lim_{\tilde{y}\in\Omega, \tilde{y}\to y} \frac{G(x,\tilde{y},t)}{d(\tilde{y})^{\frac{\theta}{2}}},$$

and in virtue of the result in [4], this limit exists for all $x \in \overline{\Omega}$, $y \in \partial\Omega$, and $t > 0$. Furthermore, $D_{\theta/2}G$ is positive and finite for all $x \in \Omega$, $y \in \partial\Omega$, and $t > 0$. Define

$$K(x,y,t) := \begin{cases} \dfrac{G(x,y,t)}{d(y)^{\frac{\theta}{2}}} & \text{if} \quad (x,y,t) \in \overline{\Omega} \times \Omega \times (0,\infty), \\ D_{\frac{\theta}{2}} G(x,y,t) & \text{if} \quad (x,y,t) \in \overline{\Omega} \times \partial\Omega \times (0,\infty). \end{cases}$$

Then $K \in C(\overline{\Omega} \times \overline{\Omega} \times (0,\infty))$ and

$$\begin{cases} K(x,y,t) > 0 & \text{if} \quad (x,y,t) \in \Omega \times \overline{\Omega} \times (0,\infty), \\ K(x,y,t) = 0 & \text{if} \quad (x,y,t) \in \partial\Omega \times \overline{\Omega} \times (0,\infty). \end{cases}$$

Furthermore, it follows from (1.7) that K satisfies

$$K(x,y,t) \asymp \left(1 \wedge \frac{d(x)^{\frac{\theta}{2}}}{\sqrt{t}}\right)\left(\frac{1}{d(y)^{\frac{\theta}{2}}} \wedge \frac{1}{\sqrt{t}}\right)\Gamma_\theta(x-y,t) \tag{1.8}$$

for all $x, y \in \overline{\Omega}$ and $t \in (0, T']$. From the analogy of the result [11], we give an initial condition to $d(x)^{\theta/2}u(\cdot,0)$, instead of $u(\cdot,0)$. Namely, this paper is concerned with the local-in-time solvability of the Cauchy–Dirichlet problem

$$\text{(SHE)} \qquad \begin{cases} \partial_t u + (-\Delta)^{\frac{\theta}{2}}|_\Omega u = u^p, & x \in \Omega,\ t \in (0,T), \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega,\ t \in (0,T), \\ d(x)^{\frac{\theta}{2}} u(\cdot,0) = \mu & \text{in} \quad \overline{\Omega}, \end{cases}$$

where $N \geq 1$, $0 < \theta < 2$, $p > 1$, $T > 0$, and μ is a nonnegative Radon measure on $\overline{\Omega}$. Therefore, an optimal singularity in this paper is defined as follows:

Definition 1.1. *Let $z \in \overline{\Omega}$ and f_z be a nonnegative measurable function on Ω. We say that the singularity of f_z at $x = z$ is an optimal singularity if f_z satisfies the following:*

- *there exists $R > 0$ such that f_z is continuous in $B_\Omega(z, R) \setminus \{z\}$ and $f_z = 0$ outside $B_\Omega(z, R)$;*
- *there exists $\kappa_z > 0$ such that problem (SHE) with $\mu = \kappa d(x)^{\theta/2} f_z(x)$, possesses a local-in-time solution if $0 < \kappa < \kappa_z$ and it possesses no local-in-time solutions if $\kappa > \kappa_z$.*

Next, we formulate the definition of solutions of (SHE).

Definition 1.2. *Let u be a nonnegative measurable function in $\Omega \times (0, T)$, where $0 < T < \infty$. We say that u is a solution of (SHE) in Q_T if u satisfies*

$$\begin{aligned} \infty > u(x,t) = & \int_{\overline{\Omega}} K(x,y,t)\, d\mu(y) \\ & + \int_0^t \int_\Omega G(x,y,t-s) u(y,s)^p \, dyds \end{aligned} \tag{1.9}$$

for almost all $(x,t) \in Q_T$. If u satisfies the above equality with $=$ replaced by $\geq$, then u is said to be a supersolution of (SHE) in $(x,t) \in Q_T$.

1.3. **Main results.** For $L > 0$ define

$$\Omega_L := \{x \in \Omega; d(x) \geq L\} \quad \text{and} \quad L_\Omega := \sup\{L > 0; \Omega_L \neq \emptyset\}.$$

In what follows, denote $T_* > 0$ by

$$T_* := \min\left\{1, T', \frac{(\operatorname{diam}\Omega)^\theta}{16}, \frac{L_\Omega^\theta}{16}\right\}$$

Now we are ready to state our main results of this paper. In the first theorem, we obtain necessary conditions on the local-in-time solvability of problem (SHE). Compare with Theorem A.

Theorem 1.1. *Let $N \geq 1$, $0 < \theta < 2$, and $p > 1$. Assume problem (SHE) possesses a supersolution in Q_T, where $T \in (0, T_*]$. Then there exists $\gamma_1 = \gamma_1(\Omega, N, \theta, p) > 0$ such that*

$$\mu(B_\Omega(z,\sigma)) \leq \gamma_1 \sigma^{-\frac{\theta}{p-1}} \int_{B_\Omega(z,\sigma)} d(y)^{\frac{\theta}{2}}\, dy \tag{1.10}$$

for all $z \in \overline{\Omega}$ and $\sigma \in (0, T^{1/\theta})$. In addition,

(i) *if* $p = p_\theta(N,0)$, *then there exists* $\gamma_1' = \gamma_1'(\Omega, N, \theta) > 0$ *such that*

$$d(z)^{-\frac{\theta}{2}}\mu(B_\Omega(z,\sigma)) \le \gamma_1' \left[\log\left(e + \frac{T^{\frac{1}{\theta}}}{\sigma}\right)\right]^{-\frac{N}{\theta}} \tag{1.11}$$

for all $z \in \Omega_{3\sigma}$ *and* $\sigma \in (0, T^{1/\theta})$.

(ii) *if* $p = p_\theta(N, \theta/2)$, *then there exists* $\gamma_1'' = \gamma_1''(\Omega, N, \theta) > 0$ *such that*

$$\mu(B_\Omega(z,\sigma)) \le \gamma_1'' \left[\log\left(e + \frac{T^{\frac{1}{\theta}}}{\sigma}\right)\right]^{-\frac{2N+\theta}{2\theta}} \tag{1.12}$$

for all $z \in \partial\Omega$ *and* $\sigma \in (0, T^{1/\theta})$.

Remark 1.1. *The corresponding result for Theorem* A-(iii) *has not been proved. The proof of Theorem* A-(iii) *strongly depends on the speciality of the domain* $\mathbb{R}^N_+$ *and the Dirichlet heat kernel* p *on* $\mathbb{R}^N_+$, *and we have not yet established a method to overcome this problem. However, the following result is expected to hold:*

- *Let* $p \ge p_\theta(1, \theta/2)$. *If problem* (SHE) *possesses a local-in-time solution, then* $\mu(\partial\Omega) = 0$ *must hold.*

Actually, in the case where $\theta = 2$ *and* $\Omega = \mathbb{R}^N_+$, *this statement coincides with Theorem* A-(iii). *Solving this problem will be our future work. It should be commented that if* μ *is a nonnegative measurable function on* Ω, *this problem does not occur and therefore does not affect the identification of optimal singularities.*

In the second theorem, we identify optimal singularities in the interior of Ω of initial data for the local-in-time solvability of problem (SHE). Compare with Theorem B.

Theorem 1.2. *Let* $z \in \Omega$. *Set*

$$\varphi_z(x) := \begin{cases} |x-z|^{-\frac{\theta}{p-1}}\chi_{B_\Omega(z,1)}(x), & \text{if} \quad p > p_\theta(N,0), \\ |x-z|^{-N}|\log|x-z||^{-\frac{N}{\theta}-1}\chi_{B_\Omega(z,1/2)}(x), & \text{if} \quad p = p_\theta(N,0), \end{cases}$$

for $x \in \Omega$. *Then there exists* $\kappa_z > 0$ *with the following properties:*

(i) *If* $p < p_\theta(N,0)$, *for any* $\nu \in \mathcal{M}$ *problem* (SHE) *possesses a local-in-time solution with* $\mu = d(x)^{\theta/2}\nu$;

(ii) *problem* (SHE) *possesses a local-in-time solution with* $\mu = \kappa d(x)^{\theta/2}\varphi_z(x)$ *if* $\kappa < \kappa_z$;

(iii) *problem* (SHE) *possesses no local-in-time solutions with* $\mu = \kappa d(x)^{\theta/2}\varphi_z(x)$ *if* $\kappa > \kappa_z$.

Namely, φ_z is an optimal singularity in the interior of Ω. Here, $\sup_{z\in\Omega}\kappa_z<\infty$.

In the third theorem, we identify optimal singularities on the boundary of Ω of initial data for the local-in-time solvability of problem (SHE). Compare with Theorem C.

Theorem 1.3. *Let $z\in\partial\Omega$. Set*

$$\psi_z(x):=\begin{cases} |x-z|^{-\frac{\theta}{p-1}}\chi_{B_\Omega(z,1)}(x), & \text{if} \quad p>p_\theta(N,\theta/2),\\ |x-z|^{-N-\frac{\theta}{2}}|\log|x-z||^{-\frac{2N+\theta}{2\theta}-1}\chi_{B_\Omega(z,1/2)}(x), & \text{if} \quad p=p_\theta(N,\theta/2),\end{cases}$$

for $x\in\Omega$. Then there exists $\kappa_z>0$ with the following properties:

(i) *If $p<p_\theta(N,\theta/2)$, problem* (SHE) *possesses a local-in-time solution for all $\mu\in\mathcal{M}$;*
(ii) *problem* (SHE) *possesses a local-in-time solution with $\mu=\kappa d(x)^{\theta/2}\psi_z(x)$ if $\kappa<\kappa_z$;*
(iii) *problem* (SHE) *possesses no local-in-time solutions with $\mu=\kappa d(x)^{\theta/2}\psi_z(x)$ if $\kappa>\kappa_z$.*

Namely, ψ_z is an optimal singularity on the boundary $\partial\Omega$. Here, $\sup_{z\in\partial\Omega}\kappa_z<\infty$.

Remark 1.2.

- *Theorem* 1.2-(i) *and Theorem* 1.3-(i) *with $\theta=2$ have already been shown in* [11]*. These results can therefore be seen as extensions to the case of $0<\theta<2$;*
- *Note that Theorem* B *and Theorem* 1.2-(ii), (iii)*, and Theorem* C *and Theorem* 1.3-(ii), (iii) *correspond. Indeed, if we take $\theta=2$ in Theorems* 1.2-(ii), (iii) *and* 1.3-(ii), (iii)*, these coincide with Theorems* B *and* C*, respectively.*

We first prove Theorem 1.1 and obtain sufficient conditions on the local-in-time solvability of problem (SHE) (see Theorems 4.1, 4.3, and 4.4 below). We then apply them to prove Theorems 1.2 and 1.3.

The rest of this paper is organized as follows. In Section 2 we collect some properties of the kernels G and K and prove some preliminary lemmas. In Section 3 we prove Theorem 1.1. In Section 4 we obtain sufficient conditions on the local-in-time solvability of problem (SHE). In Section 5 by applying these conditions, we prove Theorems 1.2 and 1.3.

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2. Preliminaries.

In what follows we will use C to denote generic positive constants. The letter C may take different values within a calculation. We first prove the following covering lemma.

Lemma 2.1. *Let $N \geq 1$ and $\delta \in (0,1)$. Then there exists $m \in \{1,2,\cdots\}$ with the following properties.*

(i) *For any $z \in \mathbb{R}^N$ and $r > 0$, there exists $\{z_i\}_{i=1}^m \subset \mathbb{R}^N$ such that*
$$B(z,r) \subset \bigcup_{i=1}^m B(z_i, \delta r).$$

(ii) *For any $z \in \mathbb{R}^N$ and $r > 0$, there exists $\{\overline{z}_i\}_{i=1}^m \subset B_\Omega(z,2r)$ such that*
$$B_\Omega(z,r) \subset \bigcup_{i=1}^m B_\Omega(\overline{z}_i, \delta r).$$

Proof. Assertion (i) has been already proved in [11]. We prove assertion (ii). We find $m \in \{1,2,\cdots\}$ and $\{\tilde{z}_i\}_{i=1}^m \subset B_\Omega(0,1)$ such that $B_\Omega(0,1) \subset \cup_{i=1}^m B_\Omega(\tilde{z}_i, \delta/2)$, so that
$$B_\Omega(z,r) \subset \bigcup_{i=1}^m B_\Omega(z + r\tilde{z}_i, \delta r/2). \tag{2.1}$$
Set $\overline{z}_i := z + r\tilde{z}_i$ if $z + r\tilde{z}_i \in \overline{\Omega}$ and $\overline{z}_i \in B_\Omega(z + r\tilde{z}_i, \delta r/2) \cap \partial\Omega$ if $z + r\tilde{z}_i \notin \overline{\Omega}$. Then
$$\overline{z}_i \in B_\Omega(z,2r), \quad \text{and} \quad B_\Omega(z + r\tilde{z}_i, \delta r/2) \subset B_\Omega(\overline{z}_i, \delta r)$$
if $B_\Omega(z + r\tilde{z}_i, \delta r/2) \neq \emptyset$. This together with (2.1) implies that
$$B_\Omega(z,r) \subset \bigcup_{i=1}^m B_\Omega(\overline{z}_i, \delta r).$$
Then assertion (ii) follows, and the proof is complete. □

Next, we collect some properties of the kernels G and K and prepare preliminary lemmas. We see that Γ_θ satisfies
$$\Gamma_\theta(x,t) \asymp t^{-\frac{N}{\theta}} \wedge \frac{t}{|x|^{N+\theta}}, \tag{2.2}$$
$$\int_{\mathbb{R}^N} \Gamma_\theta(x,t)\, dx = 1, \tag{2.3}$$

for all $x \in \mathbb{R}^N$ and $t > 0$ (see e.g., [2, 10]). Denote $D(x,t)$ by

$$D(x,t) := \frac{d(x)^{\frac{\theta}{2}}}{d(x)^{\frac{\theta}{2}} + \sqrt{t}} \tag{2.4}$$

for $(x,t) \in \overline{\Omega} \times (0,\infty)$. Then the following lemmas hold.

Lemma 2.2.

(i) *There exists* $C_1 > 0$ *such that*

$$\int_{\mathbb{R}^N} \Gamma_\theta(x-y,t)\, dm_1(y) \le C_1 t^{-\frac{N}{\theta}} \sup_{z \in \mathbb{R}^N} m_1(B_\Omega(z, t^{\frac{1}{\theta}}))$$

for all nonnegative Radon measure m_1 *on* $\mathbb{R}^N$ *and* $(x,t) \in \mathbb{R}^N \times (0,\infty)$.

(ii) *There exists* $C_2 > 0$ *such that*

$$K(x,y,t) \le C_2 \frac{D(x,t)}{d(y)^{\frac{\theta}{2}} + \sqrt{t}} \Gamma_\theta(x-y,t) \tag{2.5}$$

for all $(x,y,t) \in \overline{\Omega} \times \overline{\Omega} \times (0,T']$. *Furthermore, there exists* $C_3 > 0$ *such that*

$$\int_{\overline{\Omega}} \frac{K(x,y,t)}{D(x,t)}\, dm_2(y) \le C_3 t^{-\frac{N}{\theta}} \sup_{z \in \overline{\Omega}} \int_{B_\Omega(z,t^{\frac{1}{\theta}})} \frac{dm_2(y)}{d(y)^{\frac{\theta}{2}} + \sqrt{t}} \tag{2.6}$$

for all $m_2 \in \mathcal{M}$ *and* $(x,t) \in \overline{\Omega} \times (0,T']$.

Proof. Assertion (i) follows from [10, Lemma 2.1]. It follows that

$$1 \wedge ab \le (1 \wedge a)(1 \wedge b)(1 + |a-b|) \le \frac{4ab(1+|a-b|)}{(1+a)(1+b)}$$

for all $a, b > 0$ (see e.g. [19, Section 1.1]). Let $t \in (0,T']$. Then, by (1.7) we have

$$\begin{aligned} G(x,y,t) &\le C\left(1 \wedge \frac{d(x)^{\frac{\theta}{2}}}{\sqrt{t}}\right)\left(1 \wedge \frac{d(y)^{\frac{\theta}{2}}}{\sqrt{t}}\right)\Gamma_\theta(x-y,t) \\ &\le \frac{Cd(x)^{\frac{\theta}{2}} d(y)^{\frac{\theta}{2}}}{(d(x)^{\frac{\theta}{2}} + \sqrt{t})(d(y)^{\frac{\theta}{2}} + \sqrt{t})}\Gamma_\theta(x-y,t) \\ &= CD(x,t)D(y,t)\Gamma_\theta(x-y,t) \end{aligned}$$

for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$ and $t \in (0,T']$. This implies that (2.5) holds. (2.6) follows from (2.5) and assertion (i) with $m_1 = m_2 \chi_{\overline{\Omega}}(y)/(d(y)^{\theta/2} + \sqrt{t})$. $\square$

Lemma 2.3. *The integral kernels G and K satisfy*

$$\int_\Omega K(x,y,t)\,dx \le C_4 t^{-\frac{1}{2}} \quad \textit{for} \quad (y,t)\in\partial\Omega\times(0,T']; \tag{2.7}$$

$$\int_\Omega G(z,x,s)K(x,y,t)\,dx = K(z,y,t+s) \tag{2.8}$$

$$\textit{for} \quad (z,y,t,s)\in\Omega\times\overline{\Omega}\times(0,\infty)^2,$$

where $C_4>0$ is constant depending only on Ω, N, and θ.

Proof. Let $y\in\partial\Omega$. By (2.3) and (2.5) we have

$$\begin{aligned}\int_\Omega K(x,y,t)\,dx &\le C\int_\Omega \frac{D(x,t)}{d(y)^{\frac{\theta}{2}}+\sqrt{t}}\Gamma_\theta(x-y,t)\,dx\\ &\le Ct^{-\frac{1}{2}}\int_{\mathbb{R}^N}\Gamma_\theta(x-y,t)\,dx = Ct^{-\frac{1}{2}}\end{aligned}$$

for all $y\in\partial\Omega$ and $t\in(0,T']$. Then (2.7) follows.

Let $y\in\Omega$. By (1.6) we have

$$\begin{aligned}\int_\Omega G(z,x,s)K(x,y,t)\,dx &= \frac{1}{d(y)^{\frac{\theta}{2}}}\int_\Omega G(z,x,s)G(x,y,t)\,dx\\ &= \frac{G(z,y,t+s)}{d(y)^{\frac{\theta}{2}}} = K(z,y,t+s).\end{aligned}$$

Let $y\in\partial\Omega$. By (1.6), (1.7), and the dominated convergence theorem we have

$$\begin{aligned}\int_\Omega G(z,x,s)K(x,y,t)\,dx &= \int_\Omega G(z,x,s)\lim_{\tilde{y}\in\Omega,\tilde{y}\to y}\frac{G(x,\tilde{y},t)}{d(\tilde{y})^{\frac{\theta}{2}}}\,dx\\ &= \lim_{\tilde{y}\in\Omega,\tilde{y}\to y}\frac{1}{d(\tilde{y})^{\frac{\theta}{2}}}\int_\Omega G(z,x,s)G(x,\tilde{y},t)\,dx\\ &= \lim_{\tilde{y}\in\Omega,\tilde{y}\to y}\frac{G(z,\tilde{y},t+s)}{d(\tilde{y})^{\frac{\theta}{2}}} = K(z,y,t+s).\end{aligned}$$

Then (2.8) follows and the proof is complete. □

At the end of this section we prepare a lemma on an integral inequality, which has been already proved in [11]. This idea of using this kind of lemma is due to [17]. See also [8, 11, 12].

Lemma 2.4. *Let ζ be a nonnegative measurable function in $(0,T)$, where $T>0$. Assume that*

$$\infty > \zeta(t) \ge c_1 + c_2\int_{t_*}^t s^{-\alpha}\zeta(s)^\beta\,ds \quad \textit{for almost all} \quad t\in(t_*,T),$$

where $c_1, c_2 > 0$, $\alpha \ge 0$, $\beta > 1$, and $t_* \in (0, T/2)$. Then there exists $C = C(\alpha, \beta) > 0$ such that

$$c_1 \le C c_1^{-\frac{1}{\beta-1}} t_*^{\frac{\alpha-1}{\beta-1}}.$$

In addition, if $\alpha = 1$, then

$$c_1 \le (c_2(\beta-1))^{-\frac{1}{\beta-1}} \left[\log \frac{T}{2t_*}\right]^{-\frac{1}{\beta-1}}.$$

3. Necessary conditions for the local-in-time solvability.

In this section we prove Theorem 1.1. In order to do that, we first modify the arguments in [11] and prove Proposition 3.1 below.

Proposition 3.1. *Assume that there exists a supersolution of problem* (SHE) *in Q_T, where $T \in (0, T_*]$. Then there exists $\gamma > 0$ such that*

$$d(z)^{-\frac{\theta}{2}} \mu(B_\Omega(z,\sigma)) \le \gamma \sigma^{N - \frac{\theta}{p-1}} \tag{3.1}$$

for all $z \in \Omega_{T^{1/\theta}}$ and $\sigma \in (0, T^{1/\theta}/16)$. Furthermore, if $p = p_\theta(N, 0)$, there exists $\gamma' > 0$ such that

$$d(z)^{-\frac{\theta}{2}} \mu(B_\Omega(z,\sigma)) \le \gamma' \left[\log\left(e + \frac{T^{\frac{1}{\theta}}}{\sigma}\right)\right]^{-\frac{N}{\theta}} \tag{3.2}$$

for all $z \in \Omega_{T^{1/\theta}}$ and $\sigma \in (0, T^{1/\theta}/16)$.

In order to prove Proposition 3.1, we prepare two lemmas on the integral kernels.

Lemma 3.1. *For any $\epsilon \in (0, 1/2)$, there exists $C > 0$ such that*

$$\int_{B_\Omega(z,\sigma)} K(z, y, \sigma^\theta)\, d\mu(y) \ge C \sigma^{-N} d(z)^{-\frac{\theta}{2}} \mu(B_\Omega(z,\sigma))$$

for all $\mu \in \mathcal{M}$, $z \in \Omega_{T^{1/\theta}}$, $\sigma \in (0, \epsilon T^{1/\theta})$, and $T \in (0, T_*]$.

Proof. Let $\epsilon \in (0, 1/2)$, $\sigma \in (0, \epsilon T^{1/\theta})$, $z \in \Omega_{T^{1/\theta}}$, and $y \in B_\Omega(z,\sigma)$. By (1.8) and (2.2) we have

$$K(z, y, \sigma^\theta) \ge C \left(1 \wedge \frac{d(z)^{\frac{\theta}{2}}}{\sigma^{\frac{\theta}{2}}}\right) \left(\frac{1}{d(y)^{\frac{\theta}{2}}} \wedge \frac{1}{\sigma^{\frac{\theta}{2}}}\right) \left(\sigma^{-N} \wedge \frac{\sigma^\theta}{|z-y|^{N+\theta}}\right).$$

Since

$$d(z)^{\frac{\theta}{2}} \ge T^{\frac{1}{2}} > \epsilon^{-\frac{\theta}{2}} \sigma^{\frac{\theta}{2}} > \sigma^{\frac{\theta}{2}},$$

$$d(y) > d(z) - \sigma \ge T^{\frac{1}{\theta}} - \sigma > (\epsilon^{-1} - 1)\sigma > \sigma,$$

and $|z-y|<\sigma$, we have

$$K(z,y,\sigma^\theta)\geq C\sigma^{-N}d(y)^{-\frac{\theta}{2}}.$$

Furthermore, since

$$d(y)<d(z)+\sigma\leq d(z)+T^{\frac{1}{\theta}}\leq 2d(z),$$

we obtain

$$K(z,y,\sigma^\theta)\geq C\sigma^{-N}d(z)^{-\frac{\theta}{2}}.$$

Thus, Lemma 3.1 follows. □

Lemma 3.2.

(i) *One has*

$$\Gamma_\theta(x,2t-s)\geq\left(\frac{s}{2t}\right)^{\frac{N}{\theta}}\Gamma_\theta(x,s)$$

for all $x\in\mathbb{R}^N$ *and* $s,t>0$ *with* $s<t$.

(ii) *There exists* $C>0$ *such that*

$$G(z,y,2t-s)\geq C\left(\frac{s}{2t}\right)^{\frac{N}{\theta}}G(z,y,s) \tag{3.3}$$

for all $z\in\Omega_{T^{1/\theta}}$, $y\in\Omega$, $s,t\in(0,T/32)$ *with* $s<t$, *and* $T\in(0,T_*]$.

Proof. Assertion (i) has been already proved in [10]. We prove assertion (ii). Let $z\in\Omega_{T^{1/\theta}}$, $y\in\Omega$, $s,t\in(0,T/32)$ with $s<t$, and $T\in(0,T_*]$. By (1.7) we have

$$\begin{aligned}&G(z,y,2t-s)\\&\geq C\left(1\wedge\frac{d(z)^{\frac{\theta}{2}}}{\sqrt{2t-s}}\right)\left(1\wedge\frac{d(y)^{\frac{\theta}{2}}}{\sqrt{2t-s}}\right)\Gamma_\theta(z-y,2t-s).\end{aligned} \tag{3.4}$$

Since $d(z)\geq T^{1/\theta}>(2t-s)^{1/\theta}>s^{1/\theta}$, we see that

$$\left(1\wedge\frac{d(z)^{\frac{\theta}{2}}}{\sqrt{2t-s}}\right)=\left(1\wedge\frac{d(z)^{\frac{\theta}{2}}}{\sqrt{s}}\right)=1.$$

Assume that $d(y)\geq(2t-s)^{1/\theta}(>s^{1/\theta})$. Similarly,

$$\left(1\wedge\frac{d(y)^{\frac{\theta}{2}}}{\sqrt{2t-s}}\right)=\left(1\wedge\frac{d(y)^{\frac{\theta}{2}}}{\sqrt{s}}\right)=1.$$

By (3.4) and assertion (i) we then have

$$\begin{aligned}
&G(z,y,2t-s)\\
&\ge C\left(1\wedge\frac{d(z)^{\frac{\theta}{2}}}{\sqrt{s}}\right)\left(1\wedge\frac{d(y)^{\frac{\theta}{2}}}{\sqrt{s}}\right)\Gamma_\theta(z-y,2t-s)\\
&= C\left(\frac{s}{2t}\right)^{\frac{N}{\theta}}\left(1\wedge\frac{d(z)^{\frac{\theta}{2}}}{\sqrt{s}}\right)\left(1\wedge\frac{d(y)^{\frac{\theta}{2}}}{\sqrt{s}}\right)\Gamma_\theta(z-y,s)\\
&\ge C\left(\frac{s}{2t}\right)^{\frac{N}{\theta}} G(z,y,s).
\end{aligned}$$

We obtain the desired inequality.

On the other hand, assume that $d(y)\le(2t-s)^{1/\theta}$. Note that

$$\begin{aligned}
|z-y|^{N+\theta} &> (d(z)-d(y))^{N+\theta}\\
&> (T^{\frac{1}{\theta}}-(2t-s)^{\frac{1}{\theta}})^{N+\theta}\\
&> (2t-s)^{\frac{N}{\theta}+1}.
\end{aligned}$$

By (2.2) and (3.4) we then have

$$\begin{aligned}
G(z,y,2t-s) &\ge C\left(1\wedge\frac{d(z)^{\frac{\theta}{2}}}{\sqrt{s}}\right)\frac{d(y)^{\frac{\theta}{2}}}{\sqrt{2t-s}}\Gamma_\theta(z-y,2t-s)\\
&= C\left(1\wedge\frac{d(z)^{\frac{\theta}{2}}}{\sqrt{s}}\right)\frac{d(y)^{\frac{\theta}{2}}}{\sqrt{2t-s}}\frac{2t-s}{|z-y|^{N+\theta}}\\
&= C\sqrt{\frac{2t-s}{s}}\left(1\wedge\frac{d(z)^{\frac{\theta}{2}}}{\sqrt{s}}\right)\frac{d(y)^{\frac{\theta}{2}}}{\sqrt{s}}\frac{s}{|z-y|^{N+\theta}}\\
&\ge C\left(1\wedge\frac{d(z)^{\frac{\theta}{2}}}{\sqrt{s}}\right)\left(1\wedge\frac{d(y)^{\frac{\theta}{2}}}{\sqrt{s}}\right)\left(s^{-\frac{N}{\theta}}\wedge\frac{s}{|z-y|^{N+\theta}}\right)\\
&\ge C\left(1\wedge\frac{d(z)^{\frac{\theta}{2}}}{\sqrt{s}}\right)\left(1\wedge\frac{d(y)^{\frac{\theta}{2}}}{\sqrt{s}}\right)\Gamma_\theta(z-y,s)\\
&\ge CG(z,y,s)\ge C\left(\frac{s}{2t}\right)^{\frac{N}{\theta}} G(z,y,s).
\end{aligned}$$

We obtain the desired inequality and the proof is complete. □

Proof of Proposition 3.1. Let u be a supersolution of problem (SHE) in Q_T, where $T\in(0,T_*]$. Let $\sigma\in(0,T^{1/\theta}/16)$ and $z\in\Omega_{T^{1/\theta}}$. It follows from (1.6) and Lemmas 2.3 and 3.2 that

$$\begin{aligned}
&\int_\Omega G(z,x,t)u(x,t)\,ds\\
&\quad\ge \int_{\overline{\Omega}}\int_\Omega G(z,x,t)K(x,y,t)\,dxd\mu(y)\\
&\qquad\qquad + \int_0^t\int_\Omega\int_\Omega G(z,x,t)G(x,y,t-s)u(y,s)^p\,dxdyds\\
&\quad\ge \int_{\overline{\Omega}} K(z,y,2t)\,d\mu(y) + \int_{\sigma^\theta}^t G(z,y,2t-s)u(y,s)^p\,dyds\\
&\quad\ge \int_{\overline{\Omega}} K(z,y,2t)\,d\mu(y) + C\int_{\sigma^\theta}^t \left(\frac{s}{2t}\right)^{\frac{N}{\theta}}\int_\Omega G(z,y,s)u(y,s)^p\,dyds
\end{aligned}$$

for almost all $t \in (\sigma^\theta, T/32)$. Furthermore, Jensen's inequality with (1.5) implies that

$$\int_\Omega G(z,y,s)u(y,s)^p\,dy \ge \left(\int_\Omega G(z,y,s)u(y,s)\,dy\right)^p$$

for almost all $s > 0$. Then we obtain

$$\begin{aligned}
&\int_\Omega G(z,x,t)u(x,t)\,ds\\
&\quad\ge \int_{\overline{\Omega}} K(z,y,2t)\,d\mu(y)\\
&\qquad\qquad + Ct^{-\frac{N}{\theta}}\int_{\sigma^\theta}^t s^{\frac{N}{\theta}}\left(\int_\Omega G(z,y,s)u(y,s)\,dy\right)^p ds
\end{aligned} \tag{3.5}$$

for almost all $t \in (\sigma^\theta, T/32)$. By Lemma 3.1 we have

$$\begin{aligned}
\int_{\overline{\Omega}} K(z,y,2t)\,d\mu(y) &\ge \int_{B_\Omega(z,(2t)^{\frac{1}{\theta}})} K(z,y,2t)\,d\mu(y)\\
&\ge Ct^{-\frac{N}{\theta}}d(z)^{-\frac{\theta}{2}}\mu(B_\Omega(z,(2t)^{\frac{1}{\theta}}))\\
&\ge Ct^{-\frac{N}{\theta}}d(z)^{-\frac{\theta}{2}}\mu(B_\Omega(z,\sigma))
\end{aligned} \tag{3.6}$$

for all $t \in (\sigma^\theta, T/32)$. Therefore, setting

$$U(t) := t^{\frac{N}{\theta}}\int_\Omega G(z,y,t)u(y,t)\,dy,$$

(3.5) and (3.6) yield

$$U(t) \ge Cd(z)^{-\frac{\theta}{2}}\mu(B_\Omega(z,\sigma)) + C\int_{\sigma^\theta}^t s^{-\frac{N}{\theta}(p-1)}U(s)^p\,ds$$

for almost all $t \in (\sigma^\theta, T/32)$. Applying Lemma 2.4, we obtain

$$d(z)^{-\frac{\theta}{2}}\mu(B_\Omega(z,\sigma)) \le C(\sigma^\theta)^{\frac{N}{\theta}-\frac{1}{p-1}} = C\sigma^{N-\frac{\theta}{p-1}}$$

for all $\sigma \in (0, T^{1/\theta}/16)$ and almost all $z \in \Omega_{T^{1/\theta}}$, so that (3.1) holds. Furthermore, in the case of $p = p_\theta(N,0)$, we have

$$d(z)^{-\frac{\theta}{2}}\mu(B_\Omega(z,\sigma)) \le C\left[\log\frac{T}{2\sigma^\theta}\right]^{-\frac{N}{\theta}} \le C\left[\log\left(e+\frac{T^{\frac{1}{\theta}}}{\sigma}\right)\right]^{-\frac{N}{\theta}}$$

for all $\sigma \in (0, T^{1/\theta}/16)$ and almost all $z \in \Omega_{T^{1/\theta}}$, so that (3.2) holds. Thus, Proposition 3.1 holds. □

Next we prove Proposition 3.2 on the behavior of μ near the boundary.

Proposition 3.2. *Assume that there exists a supersolution of problem* (SHE) *in Q_T, where $T \in (0, T_*]$. Then there exist $\gamma > 0$ and $\epsilon \in (0,1)$ such that*

$$\mu(B_\Omega(z,\sigma)) \le \gamma\sigma^{N+\frac{\theta}{2}-\frac{\theta}{p-1}}$$

for all $z \in \partial\Omega$ and $\sigma \in (0, \epsilon T^{\frac{1}{\theta}})$.

In order to prove Proposition 3.2, we prepare Lemma 3.3.

Lemma 3.3. *Let u be a solution of problem* (SHE) *in Q_T, where $T \in (0, T_*]$. Then there exists $C > 0$ such that*

$$u(x, (2\sigma)^\theta) \ge C\sigma^{-N-\frac{\theta}{2}}\mu(B_\Omega(z,\sigma))$$

for all $z \in \partial\Omega$, almost all $x \in B_\Omega(z, 8\sigma)$ with $d(x) \in (2\sigma, 4\sigma)$, and almost all $\sigma \in (0, T^{1/\theta}/16)$.

Proof. Let $z \in \partial\Omega$. For any $x \in B_\Omega(z, 8\sigma)$ with $d(x) \in (2\sigma, 4\sigma)$ and $y \in B_\Omega(z,\sigma)$, (1.8) and (2.2) yield

$$K(x,y,(2\sigma)^\theta) \ge C\left(1 \wedge \frac{d(x)^{\frac{\theta}{2}}}{(2\sigma)^{\frac{\theta}{2}}}\right)\left(\frac{1}{d(y)^{\frac{\theta}{2}}} \wedge \frac{1}{(2\sigma)^{\frac{\theta}{2}}}\right)\left((2\sigma)^{-N} \wedge \frac{(2\sigma)^\theta}{|x-y|^{N+\theta}}\right). \tag{3.7}$$

Since

$$d(x) > 2\sigma, \quad d(y) < \sigma, \quad \text{and} \quad |x-y| \le |x-z| + |z-y| \le 9\sigma,$$

(3.7) implies that

$$K(x,y,(2\sigma)^\theta) \ge C\sigma^{-N-\frac{\theta}{2}}.$$

Then it follows from Definition 1.2 that

$$u(x,(2\sigma)^\theta) \ge \int_{B_\Omega(z,\sigma)} K(x,y,(2\sigma)^\theta)\,d\mu(y) \ge C\sigma^{-N-\frac{\theta}{2}}\mu(B_\Omega(z,\sigma))$$

for all $z \in \partial\Omega$, almost all $x \in B_\Omega(z, 8\sigma)$ with $d(x) \in (2\sigma, 4\sigma)$, and almost all $\sigma \in (0, T^{1/\theta}/16)$. Thus, Lemma 3.3 follows. $\square$

Proof of Proposition 3.2. Assume that there exists a supersolution of problem (SHE) in Q_T, where $T \in (0, T_*]$. Let $\epsilon \in (0, 1/16)$. For $\sigma \in (0, \epsilon T^{1/\theta})$, we have

$$T - (2\sigma)^\theta > (1 - 4\epsilon^\theta)T > \frac{T}{2}.$$

Set $\tilde{u}(x,t) := u(x, t + (2\sigma)^\theta)$. Then the function $\tilde{u}$ is a supersolution of problem (SHE) with $\mu = d(x)^{\theta/2} u(x, (2\sigma)^\theta)$ in $Q_{T/2}$ for almost all $\sigma \in (0, \epsilon T^{1/\theta})$. For $z \in \partial\Omega$, let $\tilde{z} \in \Omega$ be such that $\tilde{z} \in \partial B_\Omega(z, 3\sigma)$ and $d(\tilde{z}) = 3\sigma$. Let $\delta \in (0, 3/16)$. Since $\epsilon T^{1/\theta} < T^{1/\theta}/16$ and $y \in B_\Omega(\tilde{z}, \delta\sigma)$ satisfies $y \in B_\Omega(z, 8\sigma)$ and $d(y) \in (2\sigma, 4\sigma)$, by Lemma 3.3 we have

$$\begin{aligned} &\int_{B_\Omega(\tilde{z},\delta\sigma)} d(y)^{\frac{\theta}{2}} u(y, (2\sigma)^\theta)\, dy \\ &\ge C\sigma^{-N-\frac{\theta}{2}} \mu(B_\Omega(z,\sigma)) \int_{B_\Omega(\tilde{z},\delta\sigma)} d(y)^{\frac{\theta}{2}}\, dy \ge C\mu(B_\Omega(z,\sigma)). \end{aligned} \tag{3.8}$$

In addition, by Proposition 3.1 with $T = (3\sigma)^\theta$ we have

$$\begin{aligned} d(\tilde{z})^{-\frac{\theta}{2}} \int_{B_\Omega(\tilde{z},\delta\sigma)} d(y)^{\frac{\theta}{2}} u(y, (2\sigma)^\theta)\, dy &= d(\tilde{z})^{-\frac{\theta}{2}} \int_{B_\Omega(\tilde{z},\delta\sigma)} d(y)^{\frac{\theta}{2}} \tilde{u}(y, 0)\, dy \\ &\le C\sigma^{N-\frac{\theta}{p-1}}. \end{aligned}$$

This together with (3.8) implies that

$$\mu(B_\Omega(z,\sigma)) \le C\sigma^{N+\frac{\theta}{2}-\frac{\theta}{p-1}}$$

for all $z \in \partial\Omega$ and almost all $\sigma \in (0, \epsilon T^{1/\theta})$. Then we obtain the desired inequality for all $z \in \partial\Omega$ and all $\sigma \in (0, \epsilon T^{1/\theta})$. Thus, Proposition 3.2 follows. $\square$

Now we are ready to complete the proof of Theorem 1.1.

Proofs of (1.10) *and* (1.11). By Propositions 3.1 and 3.2 we find $\delta \in (0, 1/3)$ such that

$$\begin{aligned} &\sup_{z\in\Omega, d(z)\ge\sigma} d(z)^{-\frac{\theta}{2}} \mu(B_\Omega(z, \delta\sigma)) \le C\sigma^{N-\frac{\theta}{p-1}}, \\ &\sup_{z\in\partial\Omega} \mu(B_\Omega(z,\delta\sigma)) \le C\sigma^{N+\frac{\theta}{2}-\frac{\theta}{p-1}}, \end{aligned} \tag{3.9}$$

for all $\sigma \in (0, T^{1/\theta})$. Furthermore, if $p = p_\theta(N, 0)$, then

$$\sup_{z\in\Omega, d(z)\geq\sigma} d(z)^{-\frac{\theta}{2}} \mu(B_\Omega(z, \delta\sigma)) \leq C \left[\log\left(e + \frac{T^{\frac{1}{\theta}}}{\sigma}\right)\right]^{-\frac{N}{\theta}} \tag{3.10}$$

for all $\sigma \in (0, T^{1/\theta})$.

Let $\sigma \in (0, T^{1/\theta})$ and $z \in \overline{\Omega}$. Consider the case of $0 \leq d(z) \leq \delta\sigma/2$. Since $0 < \delta < 1/3$, we have

$$B_\Omega(z, \delta\sigma/2) \subset B_\Omega(\zeta, \delta\sigma) \subset B_\Omega(z, \sigma),$$

where $\zeta \in \overline{B_\Omega(z, \delta\sigma/2)} \cap \partial\Omega \neq \emptyset$. Then by (3.9) we obtain

$$\begin{aligned} \mu(B_\Omega(z, \delta\sigma/2)) \leq \mu(B_\Omega(\zeta, \delta\sigma)) &\leq C\sigma^{N+\frac{\theta}{2}-\frac{\theta}{p-1}} \\ &\leq C\sigma^{-\frac{\theta}{p-1}} \int_{B_\Omega(\zeta,\delta\sigma)\cap\{y\in\Omega; d(y)\geq\delta\sigma/3\}} d(y)^{\frac{\theta}{2}}\, dy \\ &\leq C\sigma^{-\frac{\theta}{p-1}} \int_{B_\Omega(\zeta,\delta\sigma)} d(y)^{\frac{\theta}{2}}\, dy \\ &\leq C\sigma^{-\frac{\theta}{p-1}} \int_{B_\Omega(z,\sigma)} d(y)^{\frac{\theta}{2}}\, dy. \end{aligned} \tag{3.11}$$

Consider the case of $d(z) > \delta\sigma/2$. Then, by (3.9) we have

$$\begin{aligned} \mu(B_\Omega(z, \delta^2\sigma)) \leq Cd(z)^{\frac{\theta}{2}}\sigma^{N-\frac{\theta}{p-1}} &\leq Cd(z)^{\frac{\theta}{2}}\sigma^{-\frac{\theta}{p-1}} \int_{B_\Omega(z,\delta^2\sigma/4)} dy \\ &\leq C\sigma^{-\frac{\theta}{p-1}} \int_{B_\Omega(\delta^2\sigma/4)} d(y)^{\frac{\theta}{2}}\, dy \\ &\leq C\sigma^{-\frac{\theta}{p-1}} \int_{B_\Omega(z,\sigma)} d(y)^{\frac{\theta}{2}}\, dy. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\mu(B_\Omega(z, \delta^2\sigma/2)) \leq C\sigma^{-\frac{\theta}{p-1}} \int_{B_\Omega(z,\sigma)} d(y)^{\frac{\theta}{2}}\, dy \tag{3.13}$$

for $z \in \overline{\Omega}$ and $\sigma \in (0, T^{1/\theta})$. Therefore, by Lemma 2.1 (ii) and (3.13), for any $z \in \overline{\Omega}$, we find $\{\overline{z}_i\}_{i=1}^m \subset B_\Omega(z, 2\sigma)$ such that

$$\begin{aligned}
\mu(B_\Omega(z,\sigma)) &\le \sum_{i=1}^m \mu(B_\Omega(\overline{z}_i, \delta^2\sigma/2)) \\
&\le C\sigma^{-\frac{\theta}{p-1}} \sum_{i=1}^m \int_{B_\Omega(\overline{z}_i,\sigma)} d(y)^{\frac{\theta}{2}}\, dy \\
&\le C\sigma^{-\frac{\theta}{p-1}} \int_{B_\Omega(z,3\sigma)} d(y)^{\frac{\theta}{2}}\, dy \\
&\le C\sigma^{-\frac{\theta}{p-1}} \int_{B_\Omega(z,\sigma)} d(y)^{\frac{\theta}{2}}\, dy.
\end{aligned}$$

This implies assertion (i).

Let $p = p_\theta(N, 0)$. Similarly, by Lemma 2.1, for any $z \in \Omega_{3\sigma}$, we find $\{\tilde{z}_i\}_{i=1}^{m'} \subset B_\Omega(z, 2\sigma)$ such that

$$\mu(B_\Omega(z,\sigma)) \le \sum_{i=1}^{m'} \mu(B_\Omega(\tilde{z}_i, \delta\sigma)).$$

Since $\tilde{z}_i$ satisfies $d(\tilde{z}_i) \ge \sigma$ and $0 < \delta < 1/3$, we deduce from (3.10) that

$$\begin{aligned}
d(z)^{-\frac{\theta}{2}}\mu(B_\Omega(z,\sigma)) &\le C\sum_{i=1}^{m'} \left(\frac{d(z)+2\sigma}{d(z)}\right)^{\frac{\theta}{2}} \left[\log\left(e + \frac{T^{\frac{1}{\theta}}}{\sigma}\right)\right]^{-\frac{N}{\theta}} \\
&\le C\left[\log\left(e + \frac{T^{\frac{1}{\theta}}}{\sigma}\right)\right]^{-\frac{N}{\theta}}
\end{aligned}$$

for all $z \in \Omega_{3\sigma}$ and $\sigma \in (0, T^{1/\theta})$. This implies assertion (ii), and the proof is complete. □

In the case of $p = p_\theta(N, \theta/2)$, we obtain more delicate estimates of μ near the boundary than those of (1.10).

Proof of (1.12). Let $p = p_\theta(N, \theta/2)$. Assume that there exists a supersolution of problem (SHE) in Q_T, where $T \in (0, T_*]$. By Lemma 2.4, for almost all $\sigma \in (0, T^{1/\theta}/3)$, the function $v(x,t) := u(x, t + (2\sigma)^\theta)$ is a solution of problem (SHE) in $Q_{T-(2\sigma)^\theta}$. Let $z \in \partial\Omega$. It follows from (1.10) that

$$\int_{B_\Omega(z,r)} d(y)^{\frac{\theta}{2}} v(y,t)\, dy \le Cr^{-\frac{\theta}{p-1}} \int_{B_\Omega(z,r)} d(y)^{\frac{\theta}{2}}\, dy \tag{3.14}$$

for all $r \in (0, (T-(2\sigma^\theta)-t))^{1/\theta}$ and almost all $t \in (0, T-(2\sigma)^\theta)$. Then

$$V(t) := t^{\frac{N}{\theta}+1} \int_\Omega K(x,z,t)v(x,t)\,dx < \infty$$

holds for almost all $t \in (\sigma^\theta, (T-(2\sigma)^\theta)/2)$. Indeed, Lemma 2.2 and (3.14) yield

$$\begin{aligned} \int_\Omega K(x,z,t)v(x,t)\,dx &\le Ct^{-1}\int_\Omega \Gamma_\theta(x-z,t)d(x)^{\frac{\theta}{2}}v(x,t)\,dx \\ &\le Ct^{-\frac{N}{\theta}-1}\sup_{z\in\overline{\Omega}}\int_{B_\Omega(z,(2t)^{\frac{1}{\theta}})} d(y)^{\frac{\theta}{2}}v(y,t)\,dy < \infty \end{aligned}$$

for almost all $t \in (\sigma^\theta, (T-(2\sigma)^\theta)/2)$.

We derive an integral inequality for V. Fubini's theorem and (2.8) yield

$$\begin{aligned} &\int_\Omega K(x,z,t)v(x,t)\,dx \\ &\ge \int_\Omega\int_\Omega K(x,z,t)G(x,y,t)v(y,0)\,dxdy \\ &\qquad + \int_0^t\int_\Omega\int_\Omega K(x,z,t)G(x,y,t-s)v(y,s)^p\,dxdyds \\ &\ge \int_\Omega K(y,z,2t)v(y,0)\,dy + \int_0^t\int_\Omega K(y,z,2t-s)v(y,s)^p\,dyds. \end{aligned} \tag{3.15}$$

Set

$$I := \{y \in B_\Omega(z,8\sigma); d(y) \in (2\sigma,4\sigma)\}.$$

For $y \in I$, by (1.8) and (2.2) we have

$$K(y,z,2t) \ge C\left(1 \wedge \frac{d(y)^{\frac{\theta}{2}}}{\sqrt{2t}}\right)\frac{1}{\sqrt{2t}}\left((2t)^{-\frac{N}{\theta}} \wedge \frac{2t}{|y-z|^{N+\theta}}\right). \tag{3.16}$$

Since

$$d(y)^{\frac{\theta}{2}} < 4^{\frac{\theta}{2}}\sigma^{\frac{\theta}{2}} < 4^{\frac{\theta}{2}}\sqrt{t} \quad \text{and} \quad |y-z| < 8\sigma < 8t^{\frac{1}{\theta}}$$

for $y \in I$ and $t \in (\sigma^\theta, (T-(2\sigma)^\theta)/2)$, (3.16) implies that

$$K(y,z,2t) \ge Cd(y)^{\frac{\theta}{2}}t^{-\frac{N}{\theta}-1}$$

for $y \in I$ and $t \in (\sigma^\theta, (T-(2\sigma)^\theta)/2)$. Then we have

$$\int_\Omega K(y,z,2t)v(y,0)\,dy \ge Ct^{-\frac{N}{\theta}-1}\int_I d(y)^{\frac{\theta}{2}}v(y,0)\,dy \tag{3.17}$$

for all $t \in (\sigma^\theta, (T-(2\sigma)^\theta)/2)$. On the other hand, by the same argument as in the proof of (3.3), we have

$$K(y,z,2t-s) \geq C\left(\frac{s}{2t}\right)^{-\frac{N}{\theta}+1} K(y,z,s)$$

for all $y \in \Omega$, $z \in \partial\Omega$, and $s,t \in (0,T)$ with $s < t$. Then Jensen's inequality with (2.7) implies that

$$\begin{aligned}
&\int_0^t \int_\Omega K(y,z,2t-s)v(y,s)^p \, dyds \\
&\geq \int_0^t \left(\frac{s}{2t}\right)^{\frac{N}{\theta}+1} C_4 s^{-\frac{1}{2}} \int_\Omega C_4^{-1} s^{\frac{1}{2}} K(y,z,s)v(y,s)^p \, dyds \\
&\geq C\int_0^t \left(\frac{s}{2t}\right)^{\frac{N}{\theta}+1} s^{-\frac{1}{2}} \left(\int_\Omega s^{\frac{1}{2}} K(y,z,s)v(y,s)\, dy\right)^p ds \\
&\geq Ct^{-\frac{N}{\theta}-1} \int_{\sigma^\theta}^t s^{-\left(\frac{N}{\theta}+\frac{1}{2}\right)(p-1)} \left(\int_\Omega s^{\frac{N}{\theta}+1} K(y,z,s)v(y,s)\, dy\right)^p ds.
\end{aligned} \tag{3.18}$$

Since $p = p_\theta(N,\theta/2)$, by (3.15), (3.17), and (3.18) we see that

$$V(t) \geq C\int_I d(y)^{\frac{\theta}{2}} u(y,(2\sigma)^\theta)\, dy + C\int_{\sigma^\theta}^t s^{-1} V(s)^p \, ds \tag{3.19}$$

for almost all $t \in (\sigma^\theta, (T-(2\sigma)^\theta)/3)$ and almost all $\sigma \in (0, T^{1/\theta}/3)$.

Let $\epsilon \in (0,1/2)$. We apply Lemma 2.4 to inequality (3.19) to obtain

$$\begin{aligned}
\int_I d(y)^{\frac{\theta}{2}} u(y,(2\sigma)^\theta)\, dy &\leq C\left[\log\frac{T}{\sigma^\theta}\right]^{-\frac{2N+\theta}{2\theta}} \\
&\leq C\left[\log\left(e+\frac{T^{\frac{1}{\theta}}}{\sigma}\right)\right]^{-\frac{2N+\theta}{2\theta}}
\end{aligned} \tag{3.20}$$

for almost all $\sigma \in (0, \epsilon T^{1/\theta})$ Therefore, by Lemma 3.3, taking small enough $\epsilon > 0$ if necessary, we have

$$\begin{aligned}
&\int_I d(y)^{\frac{\theta}{2}} u(y,(2\sigma)^\theta)\, dy \\
&\geq C\sigma^{-N-\frac{\theta}{2}} \mu(B_\Omega(z,\sigma)) \int_I d(y)^{\frac{\theta}{2}}\, dy \geq C\mu(B_\Omega(z,\sigma))
\end{aligned} \tag{3.21}$$

for almost all $\sigma \in (0, \epsilon T^{1/\theta})$.

Combining (3.20) and (3.21), we find $\delta \in (0,1)$ such that

$$\sup_{z\in\partial\Omega} \mu(B_\Omega(z,\delta\sigma)) \le C\left[\log\left(e+\frac{T^{\frac{1}{\theta}}}{\sigma}\right)\right]^{-\frac{2N+\theta}{2\theta}}$$

for almost all $\sigma \in (0, T^{1/\theta})$. This together with Lemma 2.1 implies that

$$\sup_{z\in\partial\Omega} \mu(B_\Omega(z,\sigma)) \le \sum_{i=1}^{m'} \mu(B_\Omega(z_i',\delta\sigma)) \le C\left[\log\left(e+\frac{T^{\frac{1}{\theta}}}{\sigma}\right)\right]^{-\frac{2N+\theta}{2\theta}}$$

for all $\sigma \in (0, T^{1/\theta})$. Thus, (1.12) follows. □

4. Sufficient conditions for the local-in-time solvability.

In this section we study sufficient conditions on the solvability of problem (SHE). Denote $\mathcal{L}$ by the set of nonnegative measurable functions on Ω. For $\mu \in \mathcal{M}$, define

$$[\mathsf{G}(t)\mu](x) := \int_\Omega G(x,y,t)\,d\mu(y),$$

for $x \in \overline{\Omega}$.

We first show that the existence of solutions and supersolutions of problem (SHE) are equivalent. The arguments in the proofs of sufficient conditions are based on Lemma 4.1.

Lemma 4.1. *Assume that there exists a supersolution v of problem* (SHE) *in Q_T. Then problem* (SHE) *possesses a solution u in Q_T such that $u \le v$ in Q_T.*

Proof. This lemma can be proved by the same argument as in [10, Lemma 2.2]. Define

$$u_1(x,t) := \int_{\overline{\Omega}} K(x,y,t)\,d\mu(y),$$
$$u_{j+1} := u_1(x,t) + \int_0^t \int_\Omega G(x,y,t-s)u_j(y,s)^p\,dyds, \quad j=1,2,\cdots,$$

for almost all $(x,t) \in Q_T$. Thanks to (1.9) and the nonnegativity of K and G, by induction we obtain

$$0 \le u_1(x,t) \le u_2(x,t) \le \cdots \le u_j(x,t) \le \cdots \le v(x,t) < \infty$$

for almost all $(x,t) \in Q_T$. Then the limit function

$$u(x,t) := \lim_{j\to\infty} u_j(x,t)$$

is well-defined for almost all $(x,t)\in Q_T$ and it is a solution of problem (SHE) in Q_T such that $u(x,t)\le v(x,t)$ for almost all $(x,t)\in Q_T$. Then the proof is complete. □

4.1. **The case of $\mu\in\mathcal{M}$.** We begin with the case of $\mu\in\mathcal{M}$.

Theorem 4.1. *Let $N\ge 1$, $p>1$, and $0<\theta<2$. Then there exists $\gamma=\gamma(\Omega,N,p,\theta)>0$ such that, if $\mu\in\mathcal{M}$ satisfies*

$$\int_0^T s^{-\frac{N}{\theta}(p-1)}\left(\sup_{z\in\overline{\Omega}}\int_{B_\Omega(z,s^{\frac{1}{\theta}})}\frac{d\mu(y)}{d(y)^{\frac{\theta}{2}}+\sqrt{s}}\right)^{p-1}ds\le\gamma \tag{4.1}$$

for some $T\in(0,T_*]$, then problem (SHE) *possesses a solution in Q_T.*

Proof. Assume (4.1). Let $T\in(0,T_*]$ and

$$w(x,t):=2\int_{\overline{\Omega}}K(x,y,t)\,d\mu(y).$$

It follows from (2.6) that

$$\|w(t)\|_{L^\infty(\Omega)}\le Ct^{-\frac{N}{\theta}}\sup_{z\in\overline{\Omega}}\int_{B_\Omega(z,t^{\frac{1}{\theta}})}\frac{d\mu(y)}{d(y)^{\frac{\theta}{2}}+\sqrt{t}}$$

for all $t\in(0,T_*]$. Then, by (2.8) we have

$$\begin{aligned}
&\int_{\overline{\Omega}}K(x,y,t)\,d\mu(y)+\int_0^t\int_\Omega G(x,y,t-s)w(y,s)^p\,dyds\\
&\le\frac{1}{2}w(x,t)+\int_0^t\|w(s)\|_{L^\infty(\Omega)}^{p-1}\int_\Omega G(x,y,t-s)w(y,s)\,dyds\\
&\le\frac{1}{2}w(x,t)+C\int_0^t\|w(s)\|_{L^\infty(\Omega)}^{p-1}\int_{\overline{\Omega}}\int_\Omega G(x,y,t-s)K(y,z,s)\,dyd\mu(z)ds\\
&=\frac{1}{2}w(x,t)+C\int_0^t\|w(s)\|_{L^\infty(\Omega)}^{p-1}\,ds\int_{\overline{\Omega}}K(x,z,t)\,d\mu(z)\\
&\le\frac{1}{2}w(x,t)\\
&+C\int_0^T s^{-\frac{N}{\theta}(p-1)}\left(\sup_{z\in\overline{\Omega}}\int_{B_\Omega(z,s^{\frac{1}{\theta}})}\frac{d\mu(y)}{d(y)^{\frac{\theta}{2}}+\sqrt{s}}\right)^{p-1}ds\int_{\overline{\Omega}}K(x,z,t)\,d\mu(z)\\
&\le\frac{1}{2}w(x,t)+C\gamma w(x,t)
\end{aligned}$$

for almost all $(x,t)\in Q_T$. Taking sufficiently small $\gamma>0$ if necessary, we see that w is a supersolution of (SHE). Thus, Theorem 4.1 follows from Lemma 4.1. □

As a corollary of Theorems 1.1 and 4.1, we can obtain the local-in-time solvability when μ is the N-dimensional Dirac measure concentrated at some point of $\partial\Omega$.

Corollary 4.1. *Let $N \geq 1$ and δ_N be the N-dimensional Dirac measure concentrated at the origin. Let $\kappa > 0$ and $z \in \partial\Omega$. If $\mu = \kappa\delta_N(\cdot - z)$ on $\overline{\Omega}$, then the following holds:*

(i) *If $p \geq p_\theta(N, \theta/2)$, then problem* (SHE) *possesses no local-in-time solutions;*
(ii) *If $1 < p < p_\theta(N, \theta/2)$, then problem* (SHE) *possesses a local-in-time solution.*

Proof. We shall prove assertion (i). Assume problem (SHE) possesses a solution in Q_T, where $T > 0$ is sufficiently small. It follows from Theorem (1.10) and (1.12) that the inequality

$$\kappa = \sup_{\zeta\in\partial\Omega} \mu(B_\Omega(\zeta,\sigma)) \leq \begin{cases} \gamma\sigma^{N+\frac{\theta}{2}-\frac{\theta}{p-1}} & \text{if} \quad p > p_\theta(N,\theta/2), \\ \gamma\left[\log\left(e + \dfrac{T^{\frac{1}{\theta}}}{\sigma}\right)\right]^{-\frac{2N+\theta}{2\theta}} & \text{if} \quad p = p_\theta(N,\theta/2), \end{cases}$$

must hold for all $\sigma \in (0, T^{1/\theta})$. However, the right hand side of the above inequality goes to 0 as $\sigma \to 0$. This implies that problem (SHE) possesses no local-in-time solutions since the above inequality does not hold for sufficiently small $\sigma > 0$.

We shall prove assertion (ii). Since $1 < p < p_\theta(N, \theta/2)$, we have

$$\begin{aligned} &\int_0^T s^{-\frac{N}{\theta}(p-1)} \left(\sup_{\zeta\in\overline{\Omega}} \int_{B_\Omega(\zeta,s^{\frac{1}{\theta}})} \frac{d\mu(y)}{d(y)^{\frac{\theta}{2}} + \sqrt{s}}\right)^{p-1} ds \\ &= \kappa^{p-1} \int_0^T s^{-(\frac{N}{\theta}+\frac{1}{2})(p-1)}\, ds \\ &\leq C\kappa^{p-1} T^{1-(\frac{N}{\theta}+\frac{1}{2})(p-1)} \to 0 \quad \text{as} \quad T \to 0. \end{aligned}$$

This implies that (4.1) holds if $T > 0$ is sufficiently small, which implies that problem (SHE) possesses a local-in-time solution. Then the proof is complete. □

4.2. **The case of $\mu \in \mathcal{L}$.** In this subsection we modify the arguments in [8, 10, 11, 21] to obtain Theorem 4.2 on sufficient conditions on the local-in-time solvability of problem (SHE).

Theorem 4.2. *Let $f \in \mathcal{L}$. Consider problem* (SHE) *with*

$$\mu = d(x)^{\frac{\theta}{2}} f(x) \in \mathcal{L}. \tag{4.2}$$

Let Ψ be a strictly increasing, nonnegative, and convex function on $[0,\infty)$. Set

$$v(x,t) := 2\Psi^{-1}([\mathsf{G}(t)\Psi(f)](x))$$

for $(x,t) \in Q_\infty$. Define

$$A(\tau) := \frac{\Psi^{-1}(\tau)^p}{\tau}, \quad B(\tau) := \frac{\tau}{\Psi^{-1}(\tau)}, \quad \textit{for} \quad \tau > 0.$$

If

$$\sup_{t\in(0,T)} \left(\|B(\mathsf{G}(t)\Psi(f))\|_{L^\infty(\Omega)} \int_0^t \|A(\mathsf{G}(s)\Psi(f))\|_{L^\infty(\Omega)}\, ds \right) \le \epsilon \tag{4.3}$$

for some $T \in (0, T_*]$ and a sufficiently small $\epsilon > 0$, then problem (SHE) *possesses a solution u in Q_T such that*

$$0 \le u(x,t) \le v(x,t) \quad \textit{for almost all} \quad (x,t) \in Q_T.$$

Proof. Let μ be as in (4.2). We show that v is a supersolution of problem (SHE) in Q_T. By Jensen's inequality with the convexity of Ψ and (1.5) we have

$$\begin{aligned}\int_{\overline{\Omega}} K(x,y,t)\, d\mu(y) &\le \int_\Omega G(x,y,t) f(y)\, dy \\ &= [\mathsf{G}(t)f](x) \\ &\le \Psi^{-1}([\mathsf{G}(t)\Psi(f)](x)) = \frac{1}{2} v(x,t)\end{aligned}$$

for all $(x,t) \in Q_\infty$. By the semigroup property of G and (4.3) we see that

$$\begin{aligned}&\int_0^t \mathsf{G}(t-s) v(s)^p\, ds \\ &\le 2^p \int_0^t \mathsf{G}(t-s) \left\| \frac{[\Psi^{-1}(\mathsf{G}(s)\Psi(f))]^p}{\mathsf{G}(s)\Psi(f)} \right\|_{L^\infty(\Omega)} \mathsf{G}(s)\Psi(f)\, ds \\ &= 2^p \mathsf{G}(t)\Psi(f) \int_0^t \left\| \frac{[\Psi^{-1}(\mathsf{G}(s)\Psi(f))]^p}{\mathsf{G}(s)\Psi(f)} \right\|_{L^\infty(\Omega)} ds \\ &\le 2^{p-1} v(t) \left\| \frac{\mathsf{G}(t)\Psi(f)}{\Psi^{-1}(\mathsf{G}(t)\Psi(f))} \right\|_{L^\infty(\Omega)} \int_0^t \left\| \frac{[\Psi^{-1}(\mathsf{G}(s)\Psi(f))]^p}{\mathsf{G}(s)\Psi(f)} \right\|_{L^\infty(\Omega)} ds \\ &\le C\epsilon v(t).\end{aligned}$$

Taking a sufficiently small $\epsilon > 0$ if necessary,

$$\int_{\overline{\Omega}} K(\cdot,y,t)\, d\mu(y) + \int_0^t \mathsf{G}(t-s) v(s)^p\, ds \le v(t)$$

holds for all $t \in (0,T)$. This means that v is a supersolution of problem (SHE) in Q_T. Then Lemma 4.1 implies that problem (SHE) possesses a solution in Q_T. Thus, Theorem 4.2 follows. $\square$

Next, as an application of Theorem 4.2, we obtain sufficient conditions on the solvability of problem (SHE).

Theorem 4.3. *Let $f \in \mathcal{L}$. For any $q > 1$, there exists $\gamma = \gamma(\Omega, N, \theta, p, q) > 0$ with the following property: if there exists $T \in (0, T_*]$ such that*

$$\sup_{z\in\overline{\Omega}} \int_{B_\Omega(z,\sigma)} D(y,\sigma^\theta) f(y)^q \, dy \le \gamma \sigma^{N-\frac{\theta q}{p-1}}, \tag{4.4}$$

for all $\sigma \in (0, T^{1/\theta})$, then problem (SHE) *with* (4.2) *possesses a solution in Q_T, with u satisfying*

$$0 \le u(x,t) \le 2[\mathsf{G}(t) f^q](x)^{\frac{1}{q}} \tag{4.5}$$

for almost all $(x,t) \in Q_T$, where D is as in (2.4).

Proof. Assume (4.4). We can assume without loss of generality, that $q \in (1,p)$. Indeed, if $q \ge p$, then, for any $1 < q' < p$, we apply Hölder's inequality to obtain

$$\begin{aligned}
&\sup_{z\in\overline{\Omega}} \int_{B_\Omega(z,\sigma)} D(y,\sigma) f(y)^{q'} \, dy \\
&\le \sup_{z\in\overline{\Omega}} \left[\int_{B_\Omega(z,\sigma)} D(y,\sigma)\, dy\right]^{1-\frac{q'}{q}} \left[\int_{B_\Omega(z,\sigma)} D(y,\sigma) f(y)^q \, dy\right]^{\frac{q'}{q}} \\
&\le C \gamma^{\frac{q'}{q}} \sigma^{N-\frac{\theta q'}{p-1}}
\end{aligned}$$

for all $\sigma \in (0, T^{1/\theta})$. Then (4.4) holds with q replaced by q'. Furthermore, if (4.5) holds for some $q' \in (1,p)$, then, since

$$[\mathsf{G}(t) f^{q'}](x)^{\frac{1}{q'}} \le [\mathsf{G}(t) f^q](x)^{\frac{1}{q}}$$

for $x \in \Omega$ and $t > 0$, the desired inequality (4.5) holds.

We apply Theorem 4.2 to prove Theorem 4.3. Let A and B be as in Theorem 4.2 with $\Psi(\tau) = \tau^q$. Then $A(\tau) = \tau^{(p/q)-1}$ and $B(\tau) = \tau^{1-(1/q)}$. Set

$$v(x,t) := 2[\mathsf{G}(t) f^q](x)^{\frac{1}{q}}.$$

for all $(x,t) \in Q_T$. It follows from (2.6) that

$$\begin{aligned}
[\mathsf{G}(t) f^q](x) &= \int_\Omega K(x,y,t) d(y)^{\frac{\theta}{2}} f(y)^q \, dy \\
&\le C t^{-\frac{N}{\theta}} \sup_{z\in\overline{\Omega}} \int_{B_\Omega(z,t^{\frac{1}{\theta}})} D(y,t) f(y)^q \, dy \le C\gamma t^{-\frac{q}{p-1}}
\end{aligned}$$

for all $t \in (0, T^{1/\theta})$. Then thanks to $q \in (1,p)$, we have

$$\begin{aligned}
&\|B(\mathsf{G}(t)\Psi(f))\|_{L^\infty(\Omega)} \int_0^t \|A(\mathsf{G}(s)\Psi(f))\|_{L^\infty(\Omega)}\, ds \\
&= \|\mathsf{G}(t)f^q\|_{L^\infty(\Omega)}^{1-\frac{1}{q}} \int_0^t \|\mathsf{G}(s)f^q\|_{L^\infty(\Omega)}^{\frac{p}{q}-1}\, ds \\
&\le C\gamma^{\frac{p-1}{q}} t^{-\frac{q-1}{p-1}} \int_0^t s^{-\frac{p-q}{p-1}}\, ds \le C\gamma^{\frac{p-1}{q}}
\end{aligned}$$

for all $t \in (0, T^{1/\theta})$. Then we apply Theorem 4.2 to obtain the desired conclusion. Thus, the proof is complete. $\square$

Theorem 4.4. *Let $p = p_\theta(N,l)$ with $l \in \{0, \theta/2\}$. Let $r > 0$ and set $\Phi(\tau) := \tau[\log(e+\tau)]^r$ for $\tau \ge 0$. For any $T > 0$, there exists $\gamma = \gamma(\Omega, N, \theta, r, T, l) > 0$ such that, if $f \in \mathcal{L}$ satisfies*

$$\sup_{z\in\overline{\Omega}} \int_{B_\Omega(z,\sigma)} d(y)^l \Phi(T^{\frac{1}{p-1}} f(y))\, dy \le \gamma T^{\frac{N+l}{\theta}} \left[\log\left(e + \frac{T^{\frac{1}{\theta}}}{\sigma}\right)\right]^{r-\frac{N+l}{\theta}}$$

for all $\sigma \in (0, T^{1/\theta})$, then problem (SHE) with $\mu = d(x)^{\theta/2} f(x)$ possesses a solution u in Q_T, with u satisfying

$$0 \le u(x,t) \le C\Phi^{-1}\left([\mathsf{G}(t)\Phi(T^{\frac{1}{p-1}})f](x)\right)$$

for almost all $(x,t) \in Q_T$ for some $C > 0$.

Proof. Let $0 < \epsilon < p-1$. We find $L \in [e, \infty)$ with the following properties:

(a) $\Psi(s) := s[\log(L+s)]^r$ is positive and convex in $(0,\infty)$;
(b) $s^p/\Psi(s)$ is increasing in $(0,\infty)$;
(c) $s^\epsilon[\log(L+s)]^{-pr}$ is increasing in $(0,\infty)$.

Since $C^{-1}\Phi(s) \le \Psi(s) \le C\Phi(s)$ for $s \in (0,\infty)$, we see that

$$\sup_{z\in\overline{\Omega}} \int_{B_\Omega(z,\sigma)} d(y)^l \Psi(T^{\frac{1}{p-1}} f(y))\, dy \le \gamma T^{\frac{N+l}{\theta}} \left[\log\left(e + \frac{T^{\frac{1}{\theta}}}{\sigma}\right)\right]^{r-\frac{N+l}{\theta}}$$

for all $\sigma \in (0, T^{1/\theta})$. Here we can assume, without loss of generality, that $\gamma \in (0,1)$. Set

$$z(x,t) := \left[\mathsf{G}(t)\Psi(T^{\frac{1}{p-1}} f)\right](x) = \int_\Omega K(x,y,t) d(y)^{\frac{\theta}{2}} \Psi(T^{\frac{1}{p-1}} f(y))\, dy.$$

By (2.6) we have

$$
\begin{aligned}
\|z(t)\|_{L^\infty(\Omega)} &\le Ct^{-\frac{N}{\theta}} \sup_{z\in\overline{\Omega}} \int_{B_\Omega(z,t^{\frac{1}{\theta}})} D(y,t)\Psi(T^{\frac{1}{p-1}}f(y))\,dy \\
&\le Ct^{-\frac{N+l}{\theta}} \sup_{z\in\overline{\Omega}} \int_{B_\Omega(z,t^{\frac{1}{\theta}})} d(y)^l\Psi(T^{\frac{1}{p-1}}f(y))\,dy \\
&\le C\gamma t_T{}^{-\frac{N+l}{\theta}}|\log t_T|^{r-\frac{N+l}{\theta}} \le Ct_T{}^{-\frac{N+l}{\theta}}|\log t_T|^{r-\frac{N+l}{\theta}}
\end{aligned}
$$

for all $t \in (0,T)$, where $t_T := t/(2T) \in (0,1/2)$. Since

$$
C^{-1}\tau[\log(L+\tau)]^{-r} \le \Psi^{-1}(\tau) \le C\tau[\log(L+\tau)]^{-r}
$$

for $\tau > 0$,

$$
\begin{aligned}
A(z(x,t)) &= \frac{\Psi^{-1}(z(x,t))^p}{z(x,t)} \le Cz(x,t)^{p-1}[\log(L+z(x,t))]^{-pr}, \\
B(z(x,t)) &= \frac{z(x,t)}{\Psi^{-1}(z(x,t))} \le C[\log(L+z(x,t))]^r,
\end{aligned}
$$

hold for $(x,t) \in Q_\infty$. Then

$$
\begin{aligned}
&0 \le A(z(x,t)) \le C\|z(t)\|_{L^\infty(\Omega)}^{p-1-\epsilon}\|z(t)\|_{L^\infty(\Omega)}^{\epsilon}[\log(L+\|z(t)\|_{L^\infty(\Omega)})]^{-pr} \\
&\le C\gamma^{p-1-\epsilon}t_T^{-\frac{N+l}{\theta}(p-1)}|\log t_T|^{\left(r-\frac{N+l}{\theta}\right)(p-1)}|\log t_T|^{-pr} \\
&= C\gamma^{p-1-\epsilon}t_T^{-1}|\log t_T|^{-r-1}
\end{aligned}
$$

and

$$
0 \le B(z(x,t)) \le C[\log(L+\|z(t)\|_{L^\infty(\Omega)})]^r \le C|\log t_T|^r
$$

for all $(x,t) \in Q_T$, where C is independent of γ. Hence

$$
\begin{aligned}
&\|B(z(t))\|_{L^\infty(\Omega)} \int_0^t \|A(z(s))\|_{L^\infty(\Omega)}\,ds \\
&\le C\gamma^{p-1-\epsilon}|\log t_T|^r \int_0^t s^{-1}|\log s_T|^{-r-1}\,ds \\
&= C\gamma^{p-1-\epsilon}|\log t_T|^r \int_0^t \frac{2T}{s}\left[-\log\frac{s}{2T}\right]^{-r-1}\,ds \\
&= CT\gamma^{p-1-\epsilon}
\end{aligned}
$$

for all $t \in (0,T)$. Therefore, if $\gamma > 0$ is small enough, then we apply Theorem 4.2 to find a solution u of problem (SHE) in Q_T such that

$$
0 \le u(x,t) \le 2\Psi^{-1}(z(x,t)) \le C\Phi([\mathsf{G}(t)\Phi(f)](x))
$$

for almost all $(x,t) \in Q_T$. Thus, Theorem 4.4 follows. $\square$

5. Proofs of Theorems 1.2 and 1.3.

In this section by applying our necessary conditions and the sufficient conditions proved in previous sections, we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. We first prove assertion (i). Let $p < p_\theta(N,0)$ and $\nu \in \mathcal{M}$. Since

$$1 - \frac{N}{\theta}(p-1) > 0,$$

we have

$$\begin{aligned}
&\int_0^T s^{-\frac{N}{\theta}(p-1)} \left(\sup_{z\in\overline{\Omega}} \int_{B_\Omega(z,s^{\frac{1}{\theta}})} \frac{d\mu(y)}{d(y)^{\frac{\theta}{2}} + \sqrt{s}} \right)^{p-1} ds \\
&= \int_0^T s^{-\frac{N}{\theta}(p-1)} \left(\sup_{z\in\overline{\Omega}} \int_{B_\Omega(z,T^{\frac{1}{\theta}})} D(y,s)\, d\nu(y) \right)^{p-1} ds \\
&\le \left[\sup_{z\in\overline{\Omega}} \nu(B_\Omega(z,T^{\frac{1}{\theta}})) \right]^{p-1} \int_0^T s^{-\frac{N}{\theta}(p-1)}\, ds \\
&\le C \left[\sup_{z\in\overline{\Omega}} \nu(B_\Omega(z,T^{\frac{1}{\theta}})) \right]^{p-1} T^{1-\frac{N}{\theta}(p-1)}
\end{aligned}$$

for $T > 0$, where D is as in (2.4). Taking sufficient small $T > 0$ if necessary, we see that (4.1) holds. It follows from Theorem 4.1 that assertion (i) follows.

We prove assertion (ii). Let $z \in \Omega$, $\kappa > 0$, and $\mu = \kappa d(x)^{\theta/2}\varphi_z(x)$. If $p > p_\theta(N,0)$, then we find $q > 1$ such that

$$\sup_{x\in\overline{\Omega}} \int_{B_\Omega(x,\sigma)} D(y,\sigma^\theta)(\kappa\varphi_z(y))^q\, dy \le \kappa^q \int_{B(z,\sigma)} |y-z|^{-\frac{2q}{p-1}}\, dy \le C\kappa^q \sigma^{N-\frac{\theta q}{p-1}}$$

for all $\sigma \in (0,1)$. If $p = p_\theta(N,0)$, then for any $r \in (0, N/\theta)$, we have

$$\begin{aligned}
&\sup_{x\in\overline{\Omega}} \int_{B_\Omega(x,\sigma)} \kappa\varphi_z(y)[\log(e+\kappa\varphi_z(y))]^r\, dy \\
&\le C\kappa \int_{B(z,\sigma)} |y-z|^{-N} |\log|y-z||^{-\frac{N}{\theta}-1+r}\, dy \le C\kappa |\log\sigma|^{-\frac{N}{\theta}+r}
\end{aligned}$$

for all small enough $\sigma > 0$ and $\kappa \in (0,1)$. Then, if $\kappa > 0$ is small enough, by Theorem 4.3 with $p > p_\theta(N,0)$ and Theorem 4.4 with $l = 0$ we find a local-in-time solution of problem (SHE). Then we obtain the desired conclusion and assertion (ii) follows.

Finally, we prove assertion (iii). Assume that problem (SHE) possesses a local-in-time solution. By Theorem 1.1 we have

$$\begin{aligned}\kappa\int_{B_\Omega(z,\sigma)} d(y)^{\frac{\theta}{2}}\varphi_z(y)\,dy &\le C\sigma^{-\frac{2}{p-1}}\int_{B_\Omega(z,\sigma)} d(y)^{\frac{\theta}{2}}\,dy \\ &\le Cd(z)^{\frac{\theta}{2}}\sigma^{N-\frac{\theta}{p-1}}\end{aligned} \tag{5.1}$$

for all small enough $\sigma > 0$. Furthermore, if $p = p_\theta(N, 0)$, then

$$\kappa\int_{B_\Omega(z,\sigma)} d(y)^{\frac{\theta}{2}}\varphi_z(y)\,dy \le Cd(z)^{\frac{\theta}{2}}|\log\sigma|^{-\frac{N}{2}} \tag{5.2}$$

for all small $\sigma > 0$. On the other hand, it follows that

$$\int_{B_\Omega(z,\sigma)} d(y)^{\frac{\theta}{2}}\varphi_z(y)\,dy \ge \begin{cases} Cd(z)^{\frac{\theta}{2}}\sigma^{N-\frac{\theta}{p-1}}, & \text{if} \quad p > p_\theta(N,\theta/2), \\ Cd(z)^{\frac{\theta}{2}}|\log\sigma|^{-\frac{N}{\theta}}, & \text{if} \quad p = p_\theta(N,\theta/2). \end{cases}$$

This together with (5.1) and (5.2) implies that (1.10) and (1.11) do not hold for sufficiently large $\kappa > 0$ and κ_z is uniformly bounded on Ω. Thus, Theorem 1.2 follows and the proof is complete. □

Proof of Theorem 1.3. We first prove assertion (i). Let $p < p_\theta(N, \theta/2)$ and $\mu \in \mathcal{M}$. Since

$$1 - \frac{2N+\theta}{2\theta}(p-1) > 0,$$

we have

$$\begin{aligned}&\int_0^T s^{-\frac{N}{\theta}(p-1)}\left(\sup_{z\in\overline{\Omega}}\int_{B_\Omega(z,s^{\frac{1}{\theta}})}\frac{d\mu(y)}{d(y)^{\frac{\theta}{2}}+\sqrt{s}}\right)^{p-1} ds \\ &\le \left[\sup_{z\in\overline{\Omega}}\mu(z,T^{\frac{1}{\theta}})\right]^{p-1}\int_0^T s^{-\frac{2N+\theta}{2\theta}(p-1)}\,ds \\ &\le C\left[\sup_{z\in\overline{\Omega}}\mu(z,T^{\frac{1}{\theta}})\right]^{p-1} T^{1-\frac{2N+\theta}{2\theta}(p-1)}\end{aligned}$$

for $T > 0$. Taking sufficient small $T > 0$ if necessary, we see that (4.1) holds. It follows from Theorem 4.1 that assertion (i) follows.

We prove assertion (ii). Let $z \in \partial\Omega$, $\kappa > 0$, and $\mu = \kappa d(x)^{\theta/2}\varphi_z(x)$ in $\mathcal{M}$. If $p > p_\theta(N, \theta/2)$, then we find $q > 1$ such that

$$\begin{aligned}\int_{B_\Omega(x,\sigma)} D(y,\sigma^\theta)(\kappa\varphi_z(y))^q\,dy &\le \kappa^q\sigma^{-\frac{\theta}{2}}\int_{B_\Omega(z,3\sigma)} d(y)^{\frac{\theta}{2}}|y-z|^{-\frac{\theta q}{p-1}}\,dy \\ &\le C\kappa^q\sigma^{N-\frac{\theta q}{p-1}}\end{aligned}$$

for all $x \in B_\Omega(z, 2\sigma)$ and $\sigma \in (0,1)$, where D is as in (2.4). Furthermore, we have

$$\begin{aligned}\int_{B_\Omega(x,\sigma)} D(y,\sigma^\theta)(\kappa\varphi_z(y))^q\,dy &\le \kappa^q \int_{B(x,\sigma)} |y-z|^{-\frac{\theta q}{p-1}}\,dy\\ &\le C\kappa^q\sigma^N|x-z|^{-\frac{\theta q}{p-1}} \le C\kappa^q\sigma^{N-\frac{\theta q}{p-1}}\end{aligned}$$

for all $x \in \overline{\Omega} \setminus B_\Omega(z, 2\sigma)$ and $\sigma \in (0,1)$. These imply that

$$\sup_{x\in\overline{\Omega}} \int_{B_\Omega(x,\sigma)} D(y,\sigma^\theta)(\kappa\varphi_z(y))^q\,dy \le C\kappa^q\sigma^{N-\frac{\theta q}{p-1}}$$

for all $\sigma \in (0,1)$ if $p > p_\theta(N,\theta/2)$. If $p = p_\theta(N,\theta/2)$, then for any $r \in (0, N/\theta)$, we have

$$\begin{aligned}&\sup_{x\in\overline{\Omega}} \int_{B_\Omega(x,\sigma)} \kappa d(y)^{\frac{\theta}{2}}\varphi_z(y)[\log(e+\kappa\varphi_z(y))]^r\,dy\\ &\le C\kappa \int_{B(z,\sigma)} |y-z|^{-N}|\log|y-z||^{-\frac{N}{\theta}-1+r}\,dy \le C\kappa|\log\sigma|^{-\frac{N}{\theta}+r}\end{aligned}$$

for all small enough $\sigma > 0$ and $\kappa \in (0,1)$. Then, if $\kappa > 0$ is small enough, by Theorem 4.3 with $p > p_\theta(N,\theta/2)$ and Theorem 4.4 with $l = \theta/2$ we find a local-in-time solution of problem (SHE). Then we obtain the desired conclusion and assertion (ii) follows.

Finally, we prove assertion (iii). Assume that problem (SHE) possesses a local-in-time solution. By Theorem 1.1 we have

$$\begin{aligned}\kappa\int_{B_\Omega(z,\sigma)} d(y)^{\frac{\theta}{2}}\varphi_z(y)\,dy &\le C\sigma^{-\frac{2}{p-1}}\int_{B_\Omega(z,\sigma)} d(y)^{\frac{\theta}{2}}\,dy\\ &\le C\sigma^{N+\frac{\theta}{2}-\frac{\theta}{p-1}}\end{aligned} \tag{5.3}$$

for all small enough $\sigma > 0$. Furthermore, if $p = p_\theta(N,\theta/2)$, then

$$\kappa\int_{B_\Omega(z,\sigma)} d(y)^{\frac{\theta}{2}}\varphi_z(y)\,dy \le C|\log\sigma|^{-\frac{2N+\theta}{2\theta}} \tag{5.4}$$

for all small $\sigma > 0$. On the other hand, it follows that

$$\int_{B_\Omega(z,\sigma)} d(y)^{\frac{\theta}{2}}\varphi_z(y)\,dy \ge \begin{cases} C\sigma^{N+\frac{\theta}{2}-\frac{\theta}{p-1}}, & \text{if} \quad p > p_\theta(N,\theta/2),\\ C|\log\sigma|^{-\frac{2N+\theta}{2\theta}}, & \text{if} \quad p = p_\theta(N,\theta/2).\end{cases} \tag{5.5}$$

By (5.3), (5.4), and (5.5) we see that (1.10) and (1.11) do not hold for sufficiently large $\kappa > 0$ and $\kappa_z \le C$. Thus, Theorem 1.3 follows and the proof is complete. □

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