

GRADIENT ESTIMATES FOR QUASILINEAR ELLIPTIC NEUMANN PROBLEMS WITH UNBOUNDED FIRST-ORDER TERMS

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ABSTRACT. This paper studies global a priori gradient estimates for divergence-type equations patterned over the p -Laplacian with first-order terms having power-growth with respect to the gradient under suitable integrability assumptions on the source term of the equation. The results apply to elliptic problems with unbounded data in Lebesgue spaces complemented with Neumann boundary conditions posed on convex domains of the Euclidean space. As a byproduct, we also obtain new second order estimates for these equations under different summability assumptions of the right-hand side.

1. INTRODUCTION

A well-known result in the theory of linear elliptic equations states that any strong solution to the Poisson equation $-\Delta u = f \in L^q$ posed on a bounded open set Ω of \mathbb{R}^N with enough regular boundary satisfies the so-called maximal L^q -regularity estimate, i.e. an estimate on $\|D^2u\|_{L^q}$ holds in terms of $\|f\|_{L^q}$, with linear dependence. Then, optimal gradient estimates follow by Sobolev embeddings, depending on the range of the exponent q with respect to the dimension N of the ambient space.

The aim of this manuscript is to provide a quasilinear counterpart of these maximal L^q -regularity properties for a class of quasilinear elliptic boundary-value problems with diffusion in divergence form and lower-order terms with power growth in the gradient. The class of diffusions we are able to encompass is patterned over the p -Laplacian, the main model being

$$\lambda u - \operatorname{div}(|Du|^{p-2}Du) + |Du|^\gamma = f(x) \quad \text{in } \Omega,$$

for $p \in (1, \infty)$, $\gamma > p - 1$, $f \in L^q(\Omega)$ for some $q > 1$ and $\lambda \geq 0$. For suitable solutions $u : \Omega \rightarrow \mathbb{R}$ (obtained by approximation) to Neumann boundary-value problems posed on

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convex C^2 domains, our main results give, for any $\gamma > p - 1$, the following a priori estimates

$$(1.1) \quad p > 1, \quad f \in L^N(\Omega) \implies Du \in L^r(\Omega), \quad 1 \leq r < \infty,$$

and

$$(1.2) \quad p \geq 2, \quad f \in L^q(\Omega), \quad q \geq \frac{N(\gamma - (p - 1))}{\gamma} \text{ and } q > 2 \implies |Du|^\gamma \in L^q(\Omega).$$

In the case $p = 2$, estimate (1.1) was proved in [57], whilst an estimate of the form (1.2) was conjectured by P.-L. Lions [59, 58] and has been the focus of a recent intensive research, especially in connection with the analysis of Mean Field Games systems: the recent work [25] addressed the conjecture of maximal regularity for the viscous problem in the periodic case, the later developments in [43, 39] treated global regularity for boundary-value problems with Neumann and Dirichlet boundary conditions respectively, while interior estimates in the superquadratic regime were the subject of the paper [26] through a different approach based on a blow-up argument. Along this line, we mention the analysis of the parabolic problem carried out in [24] by means of a rather different (nonlinear) duality method, even combined with the Bernstein technique [23]. The recent paper [22] addresses time-dependent problems with superquadratic nonlinearity via blow-up and duality methods. Finally, the work [41] contains interior estimates for stationary and parabolic equations with quadratic growth and diffusion in nondivergence form.

A peculiar feature of our results is that they hold for a degenerate/singular diffusion in the so-called supernatural growth regime of the first-order term, i.e. when $\gamma > p$. Though the sublinear and the subnatural growth, respectively $\gamma < p - 1$ and $\gamma < p$, have been widely analyzed for many years, see for example [66, 67, 68, 71, 38, 54, 50, 3, 60] and the references therein, the literature is to our knowledge poor in the case of gradient terms with supernatural growth. In particular, few results are available for PDEs with first-order terms having power-like growth in the gradient: Lipschitz bounds were obtained for fully nonlinear singular equations in [9] when $f \in W^{1,\infty}$ through viscosity solutions' methods and by pointwise Bernstein arguments in [52], while Hölder estimates for distributional semi-solutions were obtained for L^q or more general Morrey right-hand sides in [40]. These results have their roots in the earlier research carried out for the case $p = 2$ in [56, 14, 27]. Nonetheless, we mention that some of the results we obtain, especially concerning (1.2), are new even in the regime $p - 1 < \gamma < p$ and/or $q \leq N$.

An additional distinctive feature with respect to the previous works on the subject is that our arguments cover all the range of exponents q up to the endpoint threshold $q = \frac{N(\gamma - (p - 1))}{\gamma}$, for any $\gamma > p - 1$. In such a limiting case, we also enlighten the role of zero-th order term. This was first observed in [24] for parabolic problems and then in [39] for elliptic equations equipped with Dirichlet boundary conditions in the case $p = 2$, when the first-order term has subquadratic growth. The parabolic superquadratic case with linear diffusion has been recently addressed in [22], up to the endpoint threshold. We emphasize that such a lower bound on the summability exponent q is in general necessary for the validity of the maximal regularity property, see [25] for a counterexample in the linear case $p = 2$, and even for existence issues [46].

Finally, we emphasize that in contrast to compactness-based methods [37, 26], our approach is, in some cases, able to provide quantitative bounds, see e.g. the estimate in the next Theorem 2.1. This is related with the validity of the so-called “strong” maximal regularity for such nonlinear PDEs, which is at this stage widely open unless $p = 2$ and $n = 2$, cf. [59]. These

kind of stronger higher regularity properties were studied in Theorem 4.3 of [16] for $-\Delta_p u = f$.

The approach used in our main results (Theorem 2.1 and Theorem 3.1) is based on the so-called integral Bernstein method, see [57] and the later papers [51, 5] together with the recent developments [25, 43, 23].

In particular, the basic idea in Theorem 2.1 relies on using a \hat{p} -Laplacian of u as a test function in the weak formulation of the problem, with a suitably large \hat{p} . Theorem 3.1 exploits a delicate argument that still revolves around testing by a similar function, but also exploits a continuity argument that hinges on the integration on super-level sets of the gradient. This latter technique has been inspired by [25], and the previous work on integral estimates for solutions to quasilinear elliptic problems in the subnatural regime [44]. Unfortunately, to apply such a technique we need to add the hypothesis $p \geq 2$. Both the results exploit the coercivity of the gradient term via a weighted Bochner identity. Roughly speaking, for the p -Laplacian diffusion in nondivergence form

$$|Du|^{p-2} \left(\Delta u + (p-2) \frac{\Delta_\infty u}{|Du|^2} \right) =: |Du|^{p-2} \mathcal{A}(D^2 u)$$

we have the following identity solved by $w = |Du|^2$ (considering the leading operator $|Du|^{p-2} \Delta u$ and $|Du|^{p-2}$ as a ‘‘coefficient’’)

$$|Du|^{p-2} \Delta w = 2|Du|^{p-2} |D^2 u|^2 + 2|Du|^{p-2} Du \cdot D \Delta u.$$

From this, we use a ‘‘generalized’’ Cauchy-Schwarz inequality

$$|D^2 u| \geq c(N, p) \mathcal{A}(D^2 u),$$

plug the equation and exploit the lower bound

$$|Du|^{p-2} |D^2 u|^2 \geq \left(c_1 \frac{|Du|^{2\gamma}}{2} - c_2 (f - \lambda u)^2 \right) |Du|^{2-p}.$$

The last inequality shows that (part) of the second order term grants additional coercivity, crucial to conclude the higher-regularity properties for this class of equations. This term is also fundamental to deduce a second order estimate and, notably, obtain in the limit $\gamma = 0$ some known properties for the p -Poisson equation via a different approach, cf. Remark 4.2. We believe this is a neat difference with respect to classical references dealing with the integral Bernstein method for quasilinear equations, cf. [18, 29, 30, 42]. The derivation of this chain of inequalities will be discussed in detail in Lemma 2.6.

As already mentioned, this p -Bochner identity also grants an additional second order integral estimate. Keeping part of the Hessian term $|D^2 u|^2$, which is typically thrown away when using classical Bernstein arguments, we are able to derive some H^1 estimates in terms of $f \in L^q(\Omega)$ on the field

$$V = |Du|^\iota Du$$

for a suitable ι . Remarkably, this recovers in the limit some known estimates for the case $\gamma = 0$ known only in the case $f \in L^2(\Omega)$, see e.g. [21] and Remark 3.5.

Since solutions to our problems, even without gradient dependent terms, are in general no more regular than $C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$, we use an approximation procedure considering the uniformly elliptic problem

$$\lambda u_\varepsilon - \operatorname{div}((\varepsilon + |Du_\varepsilon|^2)^{\frac{p-2}{2}} Du_\varepsilon) + H(Du_\varepsilon) = f_\varepsilon(x) \quad \text{in } \Omega$$

where $\varepsilon > 0$, $\lambda \geq 0$ and f_ε is a smooth approximation of f , which admits a smooth solution u_ε (see [52]), and prove estimates independent of ε .

As far as gradient estimates for quasilinear elliptic equations are concerned, this classical problem has been extensively analyzed, especially when the equation is driven by the sole p -Laplacian. Classical works on the subject typically consider problems up to the natural growth $\gamma = p$, and include for example the papers [70, 10, 28, 31, 47, 53, 36, 54], that also study higher regularity properties at the level of $C^{1,\alpha}$ spaces. More recent papers [61, 18, 16, 19, 4, 6, 12] focused on gradient regularity estimates with unbounded right-hand side at the level of Lebesgue and Lorentz classes, treating both global and local bounds. Some recent works have been also concerned with the optimal second-order regularity for such problems, see [13, 21, 20, 7, 32], or even to study the gradient regularity of solutions driven by the mixed operator $-\Delta_1 - \Delta_p$, see [37, 69].

Regarding the assumptions on the integrability of f in the model case of the p -Poisson equation (possibly perturbed with gradient terms having sublinear growth), interior Lipschitz bounds have been studied in [28, Theorem 1 and the subsequent Remark] and [30, Remark 7.4] under the assumption that $q > \frac{Np}{p-1}$. Interior estimates in $W^{1,\infty}$ have been then obtained in [54] under the weaker integrability assumption $q > N$, while the work [55] assumes f controlled in the Morrey class $\mathcal{L}^{1,s}$, $s > N$. Optimal gradient regularity estimates for boundary-value problems of the p -Poisson equation can be found in [18, 16].

We further mention that the case of slowly increasing first-order terms, e.g. when $\gamma \leq p - 1$, has been already treated via techniques from the theory of nonlinear potentials, see [48] and the references therein, when the right-hand side datum $f \in L^\infty$ and $p \geq 2$. In the case $p < 2$, further restrictions on the growth γ have been imposed, see again [48], at least in the parabolic framework. We refer to [33, 34, 35, 61, 49, 62] for more details on the literature of nonlinear potential theory. Earlier results for parabolic problems driven by the p -Laplacian with first-order gradient terms growing at most as $|Du|^{p-1}$ and unbounded source terms in Lebesgue spaces can be found in [29, Chapter VIII, Section 1-(ii)], under the restriction $f \in L^q_{x,t}$, $q > N + 2$.

Preliminary forms of the integral Bernstein method appeared in [29, 30, 70], but we point out that they adapt at most for sublinear powers γ of the gradient. Our refinement of the Bernstein technique, instead, allows to handle problems with coercive, in fact supernatural, gradient terms.

More recently, gradient estimates in the framework of renormalized/approximated solutions in terms of right-hand sides in Marcinkiewicz spaces have been studied in [2, 8], see also the related work [63] and the references therein.

We conclude by saying that the convexity assumption on Ω in Theorems 2.1 and 3.1 allows us to give a sign on the boundary integrals coming from the diffusion term, since the second fundamental form on the boundary of a convex set is semidefinite, see Lemma 2.4. We believe that this constraint can be removed using some test function argument as in [52, 64], see Remark 2.11. The literature sometimes encompasses domains less regular than C^2 , cf. e.g. [45], but we do not pursue this direction, referring to Remark 4.1 for further references and discussions. Still, we do not know whether the restriction $p \geq 2$ in deriving an estimate like (1.2) is really necessary or a drawback of our method.

Plan of the paper. Section 2 will be devoted to the proof of gradient estimates in the full range $p > 1$, and integrability of f close to N , while Section 3 will address the full range of integrability of f , but under the restriction $p \geq 2$. Note that the second case develops the arguments which are used in the first case, so it might be useful to start with Section 2 and then proceed with Section 3 in order to get acquainted with the technicalities. Moreover, having the two sections different developments in a few steps, two slightly different sets of assumptions will be used (though they will both include the model problem described above).

2. GRADIENT ESTIMATES FOR $p > 1$ AND q CLOSE TO N

We consider the following problem

$$(2.1) \quad \begin{cases} -\operatorname{div}(a(|Du|^2)Du) + H(Du) = f(x) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ν denotes the outward unit vector on $\partial\Omega$ and $a : [0, \infty) \rightarrow [0, \infty)$ is of class $C^1(0, \infty)$. We also assume that there exist $p > 1$ and constants $\bar{c}_a, \bar{C}_a, > 0$, $C_a \in \mathbb{R}$ such that

$$(A1) \quad -1 < \inf_{t>0} \frac{2ta'(t)}{a(t)} \leq \sup_{t>0} \frac{2ta'(t)}{a(t)} \leq C_a < \infty,$$

and

$$(A2) \quad \bar{c}_a t^{\frac{p-2}{2}} \leq a(t) \leq \bar{C}_a t^{\frac{p-2}{2}}, \quad \forall t > 0$$

Notice that (A1) implies (in particular where $a'(t) < 0$) the existence of a constant $\tilde{c}_a > 0$ such that

$$(A3) \quad 2ta'(t) + a(t) \geq \tilde{c}_a a(t), \quad \forall t > 0$$

and

$$(A4) \quad t \left| \frac{a'(t)}{a(t)} \right| \leq \frac{C_a}{2}, \quad \forall t > 0.$$

As far as the lower order term is concerned, we assume that $H = H(\xi) \in C^2(\mathbb{R}^N \setminus \{0\}) \cap C^0(\mathbb{R}^N)$ and H is radial, i.e. $H(\xi) = h(|\xi|)$ for some $h \in C^2(\mathbb{R}^+) \cap C^0[0, +\infty)$ and that there exists constants $c_H, C_H > 0$, $\gamma > 1$, such that

$$(H1) \quad c_H |\xi|^\gamma \leq H(\xi), \quad \forall \xi \in \mathbb{R}^N,$$

and

$$(H2) \quad |H_{\xi\xi}(\xi)| \leq C_H |\xi|^{\gamma-2}, \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$

Finally we consider the right-hand side datum in the following way:

$$(F) \quad f \in L^q(\Omega) \quad \text{for some } q \geq N.$$

The model of nonlinear equation that we have in mind is the following:

$$-\operatorname{div}(|Du|^{p-2}Du) + |Du|^\gamma = f(x) \quad \text{in } \Omega.$$

Observe that in particular, the p -Laplacian satisfies the previous hypotheses with $a(t) = t^{\frac{p-2}{2}}$, so that $\inf_{t>0} \frac{2ta'(t)}{a(t)} = \sup_{t>0} \frac{2ta'(t)}{a(t)} = p - 2$, $\tilde{c}_a = p - 1$ and $\bar{c}_a = \bar{C}_a = 1$.

As already mentioned in the introduction, we cannot expect solutions to (2.1) to be more regular than $C^{1,\alpha}(\bar{\Omega})$, so that we need to argue by approximation. Thus we consider the approximated problem driven by a uniformly elliptic operator $a = a(t)$ with $t = |Du|^2 + \varepsilon$,

$\varepsilon > 0$, and a regularized right-hand side $f_\varepsilon \in C^\infty(\Omega)$ (for example, a regularization of f by a convolution with a smoothing kernel), namely

$$(2.2) \quad \begin{cases} -\operatorname{div}(a(|Du|^2 + \varepsilon)Du) + H(Du) = f_\varepsilon(x) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

Our goal is to prove estimates that are independent from ε , so that we can inherit them in the limit.

Before stating our first result, let us observe that for any $0 \leq \varepsilon \leq 1$, we have

$$\frac{c_H}{2}(\varepsilon + |\xi|^2)^{\frac{\gamma}{2}} - c_\varepsilon \leq H(\xi) \quad \forall \xi \in \mathbb{R}^N \setminus \{0\},$$

for some c_ε which vanishes as $\varepsilon \rightarrow 0$; hence without loss of generalization we can subtract such a constant in both sides of (2.2) and turn (H1) into

$$(2.3) \quad \frac{c_H}{2}(\varepsilon + |\xi|^2)^{\frac{\gamma}{2}} \leq H(\xi), \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$

We now state the main result of this section.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a C^2 convex domain and assume that (A1), (A2), (H1), (H2), (F) hold true with $\gamma > p - 1$. Then, for any $\eta > 1$ large enough, there exist two positive constants $C_{\eta,1}, C_{\eta,2}$ depending on $p, N, c_H, C_H, \bar{c}_a, \bar{C}_a, \tilde{c}_a, C_a, |\Omega|, \eta$, but not on ε , such that any smooth solution u_ε to (2.2) satisfies*

$$(2.4) \quad \|Du_\varepsilon\|_{L^\eta(\Omega)} \leq C_{\eta,1} + C_{\eta,2} \|f_\varepsilon\|_{L^{\frac{1}{q_\eta}}(\Omega)}, \quad \text{with } q_\eta \nearrow N \quad \text{as } \eta \rightarrow +\infty.$$

Moreover there exists $\iota_0 > 0$ such that

$$(2.5) \quad \||Du_\varepsilon|^\iota Du_\varepsilon\|_{H^1(\Omega)} \leq C(\|f_\varepsilon\|_{L^N(\Omega)}), \quad \forall \iota \in (\iota_0, \infty).$$

From now on we drop the subscript ε for brevity.

Remark 2.2. Due to the inclusion among Lebesgue spaces, as a byproduct of estimate (2.4) we get (1.1).

Remark 2.3. We remark that Theorem 2.1 leads to a (nonlinear) bound with sublinear dependence on the right-hand side datum. This is in line with the results for the p -Poisson equation found in [16, Theorem 4.3-(ii)].

We premise the following standard result that will allow to handle the boundary integrals, referring for the proof to [64, Lemma 2.3].

Lemma 2.4. *Let $u \in C^2(\bar{\Omega})$ be such that $\partial_\nu u = 0$ on $\partial\Omega$. If Ω is convex, then $\partial_\nu |Du|^2 \leq 0$ on $\partial\Omega$.*

The proof of the a priori estimate will be accomplished through some preliminary lemmas. In the sequel we will assume all the hypotheses listed in Theorem 2.1.

Let us denote by

$$w = |Du|^2 + \varepsilon$$

and let us consider the equation in (2.2). Multiplying it by a smooth enough function φ and integrating by parts, we deduce, since u satisfies the Neumann boundary condition, that u solves the following identity:

$$(2.6) \quad \int_{\Omega} a(w) Du \cdot D\varphi = \int_{\Omega} (f(x) - H(Du(x)))\varphi.$$

The main idea in order to prove Theorem 2.1 is to choose

$$\varphi = -2\operatorname{div}(Du(\varepsilon + |Du|^2)^\beta) = -2 \sum_{j=1}^N \partial_{x_j} (\partial_{x_j} u w^\beta),$$

with $\beta > 1$ to be specified later: this essentially amounts to test against a multiple of a regularized (2β) -Laplacian.

We first deal with the diffusive term and obtain the following bound from below.

Lemma 2.5. *There exist constants $\zeta_1, \zeta_2 > 0$, depending on \tilde{c}_a, \bar{c}_a , such that*

$$(2.7) \quad \int_{\Omega} a(w) Du \cdot D\varphi dx \geq \zeta_1 \int_{\Omega} a(w) |D^2 u|^2 w^\beta dx + \beta \zeta_2 \int_{\Omega} |Dw|^2 w^{\beta-1 + \frac{\beta-2}{2}} dx.$$

Proof. We use that $\partial_\nu |Du|^2 \leq 0$ on $\partial\Omega$ by Lemma 2.4, and after integrating by parts to get

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} a(w) \partial_{x_i} u \partial_{x_i} \varphi dx &= -2 \sum_{i,j=1}^N \int_{\Omega} a(w) \partial_{x_i} u \partial_{x_i} (\partial_{x_j} (\partial_{x_j} u w^\beta)) dx \\ &\geq 2 \sum_{i,j=1}^N \int_{\Omega} \partial_{x_j} (a(w) \partial_{x_i} u) \partial_{x_i} (\partial_{x_j} u w^\beta) dx \\ &= 2 \sum_{i,j=1}^N \int_{\Omega} [a'(w) \partial_{x_j} w \partial_{x_i} u + a(w) \partial_{x_i x_j} u] \cdot [\partial_{x_i x_j} u w^\beta + \beta \partial_{x_j} u \partial_{x_i} w w^{\beta-1}] dx \\ &= 2 \int_{\Omega} a(w) |D^2 u|^2 w^\beta dx + 2\beta \int_{\Omega} (Dw \cdot Du)^2 a'(w) w^{\beta-1} dx \\ &\quad + \int_{\Omega} \beta a(w) w^{\beta-1} |Dw|^2 dx + \int_{\Omega} a'(w) w^\beta |Dw|^2 dx \quad := J. \end{aligned}$$

It is easy to observe that when $a' \geq 0$ then

$$J \geq 2 \int_{\Omega} a(w) |D^2 u|^2 w^\beta dx + \beta \int_{\Omega} |Dw|^2 a(w) w^{\beta-1} dx.$$

On the contrary, when $a'(t) < 0$ we have, since $|Dw|^2 \leq 4w|D^2 u|^2$ and by (A3)

$$\begin{aligned} J &\geq \beta \int_{\Omega} [a(w) + 2wa'(w)] |Dw|^2 w^{\beta-1} dx + 2 \int_{\Omega} a(w) |D^2 u|^2 w^\beta dx + 4 \int_{\Omega} a'(w) w |D^2 u|^2 w^\beta dx \\ &\geq \beta \tilde{c}_a \int_{\Omega} |Dw|^2 w^{\beta-1} dx + 2\tilde{c}_a \int_{\Omega} a(w) |D^2 u|^2 w^\beta dx. \end{aligned}$$

Using now (A2), we deduce that there exist $\zeta_1, \zeta_2 > 0$, depending on \tilde{c}_a, \bar{c}_a , such that (2.7) holds true. \square

We now elaborate the first term on the right-hand side of the inequality in Lemma 2.5 using the non-variational formulation of the approximated problem.

Lemma 2.6. *There exists c_1 depending on $\tilde{c}_a, C_a, N, p, c_H, C_H$ such that*

$$(2.8) \quad \zeta_1 \int_{\Omega} a(w) |D^2 u|^2 w^{\beta} dx \geq \frac{\zeta_1}{2} \int_{\Omega} |D^2 u|^2 w^{\beta + \frac{p-2}{2}} dx + c_1 \int_{\Omega} \left(\frac{w^{\gamma}}{4} - 2f^2 \right) w^{\beta + \frac{2-p}{2}}.$$

Proof. First, we observe that the Cauchy-Schwarz inequality implies

$$(2.9) \quad \frac{1}{\sqrt{N}} |\Delta u| \leq |D^2 u| \quad \text{and} \quad \left| \frac{\Delta_{\infty} u}{\varepsilon + |Du|^2} \right| \leq |D^2 u|,$$

where $\Delta_{\infty} u = D^2 u Du Du$. This allows us to write the diffusion operator in non-divergence form as

$$\operatorname{div}(a(w) Du) = a(w) \Delta u + 2a'(w) D^2 u Du \cdot Du = a(w) \underbrace{\left[\Delta u + \frac{2a'(w)}{a(w)} \Delta_{\infty} u \right]}_{:= \mathcal{A}(D^2 u)}.$$

Hence by (2.9) and (A4) we get

$$|\mathcal{A}(D^2 u)| = \left| \Delta u + \frac{2a'(w)}{a(w)} D^2 u Du Du \right| \leq \sqrt{N} |D^2 u| + C_a \frac{|D^2 u| |Du|^2}{\varepsilon + |Du|^2} \leq (\sqrt{N} + C_a) |D^2 u|,$$

and setting $\nu = \frac{1}{\sqrt{N} + C_a}$, the above inequality yields to

$$|D^2 u| \geq \nu |\mathcal{A}(D^2 u)|.$$

We now exploit the non-divergence formulation of the equation, i.e. we rewrite the equation as

$$a(w) \mathcal{A}(D^2 u) = H(Du) - f(x) \quad \text{in } \Omega,$$

so that we can plug the equation back in the term $a(w) |D^2 u|^2$. Indeed, using the algebraic inequality $(A - B)^2 \geq \frac{A^2}{2} - 2B^2$, $A, B \in \mathbb{R}$, together with (2.3) and (A2), we deduce that

$$\begin{aligned} a(w) |D^2 u|^2 &\geq \nu^2 \frac{[a(w) \mathcal{A}(D^2 u)]^2}{a(w)} \geq \frac{\nu^2}{C_a} [H(Du) - f]^2 w^{\frac{2-p}{2}} \\ &\geq \frac{\nu^2}{C_a} \left[\frac{H^2(Du)}{2} - 2|f|^2 \right] w^{\frac{2-p}{2}} = \frac{\nu^2 c_H^2}{C_a} \left\{ \frac{w^{\gamma}}{8} - 2|f|^2 \right\} w^{\frac{2-p}{2}}, \end{aligned}$$

so that (2.8) holds true. \square

We now focus on the second term of the right-hand side appearing in (2.7).

Lemma 2.7. *There exists $\zeta_3, \zeta_4 > 0$ depending on $\Omega, \zeta_2, \beta, p$ such that*

$$\beta \zeta_2 \int_{\Omega} |Dw|^2 w^{\beta + \frac{p}{2} - 2} dx \geq \zeta_3 \left(\int_{\Omega} w^{(\beta + \frac{p}{2}) \frac{N}{N-2}} dx \right)^{\frac{N-2}{N}} - \zeta_4 \int_{\Omega} w^{\beta + \frac{p}{2}} dx.$$

Proof. We have by the Sobolev inequality

$$\begin{aligned} \beta \zeta_2 \int_{\Omega} |Dw|^2 w^{\beta + \frac{p}{2} - 2} dx &= \frac{4\beta \zeta_2}{(\beta + \frac{p}{2})^2} \int_{\Omega} |Dw^{\frac{\beta + \frac{p}{2}}{2}}|^2 dx \\ &\geq C_S \left[\frac{4\beta \zeta_2}{(\beta + \frac{p}{2})^2} \left(\int_{\Omega} w^{(\beta + \frac{p}{2}) \frac{N}{N-2}} dx \right)^{\frac{N-2}{N}} - \frac{4\beta \zeta_2}{(\beta + \frac{p}{2})^2} \int_{\Omega} w^{\beta + \frac{p}{2}} dx \right], \end{aligned}$$

where C_S is the constant of the Sobolev embedding. \square

We now handle the terms in the right-hand side of (2.6). We start with the one involving the source f of the equation.

Lemma 2.8. *For any $\delta_1 > 0$ there exists $c_2 > 0$ depending on β, N such that*

$$(2.10) \quad -2 \int_{\Omega} f \operatorname{div}(Du w^\beta) dx \leq \delta_1 \int_{\Omega} |D^2 u|^2 w^{\beta + \frac{p-2}{2}} dx + \frac{c_2}{\delta_1} \int_{\Omega} |f|^2 w^{\beta + \frac{2-p}{2}} dx.$$

Proof. Using (2.9) and then the weighted Young's inequality we deduce that for any $\delta_1 > 0$

$$\begin{aligned} -2 \int_{\Omega} f \operatorname{div}(Duu^\beta) dx &= -2 \int_{\Omega} f \Delta u w^\beta dx - 2\beta \int_{\Omega} f Du \cdot Dw w^{\beta-1} dx \\ &\leq 2\sqrt{N} \int_{\Omega} |f| |D^2 u| w^\beta dx + 4\beta \int_{\Omega} |f| |D^2 u| w^\beta dx \\ &= (4\beta + 2\sqrt{N}) \int_{\Omega} |f| |D^2 u| w^{\frac{\beta}{2} + \frac{p-2}{4}} w^{\frac{\beta}{2} + \frac{2-p}{4}} dx \end{aligned}$$

and (2.10) is a consequence of Young's inequality. \square

We now discuss the integral term involving H .

Lemma 2.9. *There exist constants $c_3, c_4 > 0$ depending on C_H such that*

$$- \int_{\Omega} H(Du) \varphi dx \leq \frac{c_3}{\beta+1} \int_{\Omega} w^{\beta+\gamma+\frac{2-p}{2}} + \frac{c_4}{\beta+1} \int_{\Omega} |D^2 u|^2 w^{\beta+\frac{p-2}{2}} dx .$$

Proof. We integrate by parts and use the boundary condition, together with the assumptions on H , to conclude

$$\begin{aligned} &2 \sum_{j=1}^N \int_{\Omega} H(Du) \partial_{x_j} (\partial_{x_j} u w^\beta) dx \\ &= -2 \sum_{j=1}^N \int_{\Omega} \partial_{x_j} H(Du) \partial_{x_j} u w^\beta dx + 2 \int_{\partial\Omega} w^\beta H(Du) \partial_\nu u dS \\ &= -2 \sum_{j=1}^N \int_{\Omega} \partial_{x_j} H(Du) \partial_{x_j} u w^\beta dx. \end{aligned}$$

Using once more the boundary condition $\partial_\nu u = 0$ and noticing that $H_\xi(Du) \cdot \nu = \frac{h'(|\xi|)}{|\xi|} \xi \cdot \nu$, combined with the Cauchy-Schwarz and Young inequalities, we get

$$\begin{aligned} -2 \sum_{j=1}^N \int_{\Omega} \partial_{x_j} H(Du) \partial_{x_j} u w^\beta dx &= -2 \int_{\Omega} H_\xi(Du) \cdot Dw w^\beta dx = -2 \int_{\Omega} H_\xi(Du) \cdot D \left(\frac{w^{\beta+1}}{\beta+1} \right) dx \\ &= \frac{2}{\beta+1} \int_{\Omega} \operatorname{div}(H_\xi(Du)) w^{\beta+1} dx - \frac{2}{\beta+1} \int_{\partial\Omega} w^{\beta+1} H_\xi(Du) \cdot \nu dS \\ &\leq \frac{2}{\beta+1} \int_{\Omega} |H_{\xi\xi}(Du)| |D^2 u| w^{\beta+1} dx \leq \frac{2C_H}{\beta+1} \int_{\Omega} |D^2 u| w^{\frac{\beta}{2} + \beta} dx \\ &\leq \frac{c_3}{\beta+1} \int_{\Omega} w^{\beta+\gamma+\frac{2-p}{2}} dx + \frac{c_4}{\beta+1} \int_{\Omega} |D^2 u|^2 w^{\beta+\frac{p-2}{2}} dx . \end{aligned}$$

\square

We plug the estimates in Lemmas 2.5, 2.6, 2.7, 2.8, 2.9 and choose $\delta_1 = \frac{\zeta_1}{4}$ to deduce the following estimate.

Corollary 2.10. *Under the hypotheses of Theorem 2.1 we have that the following inequality holds true:*

$$\begin{aligned} \zeta_3 \left(\int_{\Omega} w^{(\beta+\frac{p}{2})\frac{N}{N-2}} dx \right)^{\frac{N-2}{N}} + c_5 \int_{\Omega} w^{\beta+\gamma+\frac{2-p}{2}} dx + \frac{\zeta_1}{4} \int_{\Omega} |D^2u|^2 w^{\beta+\frac{p-2}{2}} dx \\ \leq \frac{c_3}{\beta+1} \int_{\Omega} w^{\beta+\gamma+\frac{2-p}{2}} dx + \frac{c_4}{\beta+1} \int_{\Omega} |D^2u|^2 w^{\beta+\frac{p-2}{2}} dx \\ + c_6 \int_{\Omega} f^2 w^{\beta+\frac{2-p}{2}} dx + \zeta_4 \int_{\Omega} w^{\beta+\frac{p}{2}} dx, \end{aligned}$$

where the constants that appear depend on the data of the problem, but not on ε .

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We first choose β sufficiently large to ensure the validity of the inequality

$$\begin{aligned} \zeta_3 \left(\int_{\Omega} w^{(\beta+\frac{p}{2})\frac{N}{N-2}} dx \right)^{\frac{N-2}{N}} + \frac{c_5}{2} \int_{\Omega} w^{\beta+\gamma+\frac{2-p}{2}} dx + \frac{\zeta_1}{4} \int_{\Omega} |D^2u|^2 w^{\beta+\frac{p-2}{2}} dx \\ \leq c_6 \int_{\Omega} f^2 w^{\beta+\frac{2-p}{2}} dx + \zeta_4 \int_{\Omega} w^{\beta+\frac{p}{2}} dx. \end{aligned}$$

We apply the Hölder's inequality and then the weighted Young's inequality (exploiting that $\gamma > p-1$) to the last term to find for any $\delta_2 > 0$

$$\zeta_4 \int_{\Omega} w^{\beta+\frac{p}{2}} dx \leq \delta_2 \int_{\Omega} w^{\beta+\gamma+\frac{2-p}{2}} dx + c_7,$$

where c_7 depends on $\delta_2, \beta, \gamma, p, |\Omega|$ and blows-up as $\delta_2 \rightarrow 0$. Taking $\delta_2 = \frac{1}{2}C_5$ we find through the Hölder's inequality

$$\begin{aligned} (2.11) \quad \zeta_3 \left(\int_{\Omega} w^{(\beta+\frac{p}{2})\frac{N}{N-2}} dx \right)^{\frac{N-2}{N}} + \frac{\zeta_1}{4} \int_{\Omega} |D^2u|^2 w^{\beta+\frac{p-2}{2}} dx \\ \leq c_6 \int_{\Omega} |f|^2 w^{\beta+\frac{2-p}{2}} dx + c_7 \leq c_6 \|f^2\|_{L^\alpha(\Omega)} \left(\int_{\Omega} w^{(\beta+\frac{p}{2})\frac{N}{N-2}} dx \right)^{\frac{(2\beta+2-p)(N-2)}{(2\beta+p)N}} + c_7, \end{aligned}$$

where

$$\alpha' = \frac{(\beta+\frac{p}{2})\frac{N}{N-2}}{\beta+\frac{2-p}{2}} = \frac{(2\beta+p)N}{(2\beta+2-p)(N-2)} \quad \text{and} \quad \alpha = \frac{N(2\beta+p)}{4\beta+2(p-1)N-2(p-2)}.$$

Hence, for any β large enough we have that

$$\|w\|_{L^{(\beta+\frac{p}{2})\frac{N}{N-2}}(\Omega)}^{\beta+\frac{p}{2}} \leq c_8 \left(\|f\|_{L^{2\alpha}(\Omega)}^2 \|w\|_{L^{(\beta+\frac{p}{2})\frac{N}{N-2}}(\Omega)}^{\beta+1-\frac{p}{2}} + 1 \right),$$

and we observe that as $\beta \rightarrow \infty$, $\alpha' \rightarrow \frac{N}{N-2}$ and $\alpha \nearrow \frac{N}{2}$. Since $\beta+1-\frac{p}{2} < \beta+\frac{p}{2}$, we can further apply the weighted Young inequality and find

$$\|w\|_{L^{(\beta+\frac{p}{2})\frac{N}{N-2}}(\Omega)}^{\beta+\frac{p}{2}} \leq c_9 \left(\|f\|_{L^{2\alpha}(\Omega)}^{\frac{2\beta+p}{p-1}} + 1 \right),$$

deducing (2.4) with $\eta = (\beta + \frac{p}{2})\frac{N}{N-2}$ and $q_\eta = 2\alpha$.

Finally we plug the above estimate into (2.11) and we get

$$\frac{\zeta_1}{4} \int_{\Omega} |D^2 u|^2 w^{\beta + \frac{p-2}{2}} dx \leq c_6 \left(1 + \|f^2\|_{L^\alpha(\Omega)} \left(\int_{\Omega} w^{(\beta + \frac{p}{2})\frac{N}{N-2}} \right)^{\frac{(2\beta+2-p)(N-2)}{(2\beta+p)N}} \right),$$

that yields to (2.5), choosing β arbitrarily large. \square

Remark 2.11 (On the convexity of the domain). Arguing as in [52], the convexity assumption in Theorem 2.1 can be removed by considering the equation satisfied by $z(x) = |Du(x)|^2 e^{\eta d(x)}$ (instead of just $w = |Du|^2$), where $d(x)$ is a C^2 positive function in Ω that coincides with the distance function in a neighborhood of the boundary. Indeed, if η is chosen such that $\eta \geq \|(D^2 d)_+\|_{L^\infty(\partial\Omega)}$, one has $\partial_\nu z \leq 0$ on $\partial\Omega$, see [64, Lemma 2.3]. We briefly show how to handle the extra terms in the simple case $a(t) \equiv 1$, i.e. for

$$\begin{cases} -\Delta u + |Du|^\gamma = f(x) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

First, we note that since $w = |Du|^2$ solves

$$-\Delta w + 2|D^2 u|^2 + \gamma|Du|^{\gamma-2} Du \cdot Dw = 2Df \cdot Du \quad \text{in } \Omega,$$

then z satisfies

$$\begin{aligned} -\Delta z + 2e^{\eta d(x)} |D^2 u|^2 + \gamma|Du|^{\gamma-2} Du \cdot Dz \\ = 2e^{\eta d(x)} Df \cdot Du + \gamma\eta|Du|^{\gamma-2} (Du \cdot Dd)z - 2\eta Dz \cdot Dd + c_\eta z \quad \text{in } \Omega. \end{aligned}$$

Observe that new terms are the last three in the right-hand side above. Having at hand the proof of Theorem 2.1, it follows that such terms can be, roughly speaking, absorbed by the superlinear term coming from $|D^2 u|^2$, after one has plugged in the equation for u and has exploited the growth condition (H1).

Remark 2.12. If $q > N$, one can get better estimates when the diffusion is driven by the p -Laplacian Δ_p , $p > 1$ (and for more general quasi-linear equations modelled on these operators). Indeed, if $f \in L^q(\Omega)$, $q > N$, it follows that Du is controlled in L^r for all finite $r > 1$, and hence for r sufficiently large, we have $|Du|^\gamma \in L^q$, $q > N$, so that the equation can be regarded as $-\Delta_p u = -|Du|^\gamma + f$, with right-hand side bounded in L^q , $q > N$. This leads to Lipschitz estimates through the results for the p -Poisson equation in e.g. [16, Theorem 4.3], [17, Theorem 3.1], see also [18, 27]. Actually, by bootstrapping one gets also $C^{1,\alpha}$ bounds, see e.g. [28, 29].

Remark 2.13. The assumption $f \in L^N$ is in general not sufficient to obtain gradient boundedness, neither for the Poisson equation. Sharp assumptions in Lorentz classes have been found in various works, see [6] and the references therein, while we refer to [15] for questions related to the optimality of such a condition.

Remark 2.14 (Low dimensions). When $N = 1, 2$ the result can be obtained in the same way, as one can exploit the continuous embedding of $W^{1,2}$ into L^s for any finite $s > 1$.

Remark 2.15. The result of Theorem 2.1 is still valid if one adds a zero-th order term λu , with $\lambda > 0$, in the equation. In this case the estimate will depend on $\|f - \lambda u\|_q$ instead of $\|f\|_q$.

3. MAXIMAL REGULARITY ESTIMATES FOR $p \geq 2$

Let us now consider a function $\tilde{a} : [0, \infty) \rightarrow [0, \infty)$ of class $C^1(0, \infty)$, and the problem

$$\begin{cases} \lambda u - \operatorname{div}(\tilde{a}(|Du|)Du) + H(Du) = f(x) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

which is the same as the one of the previous section, but written for convenience with the slightly different notation $\tilde{a}(t) = a(t^2)$. Here, we assume that there exist $p \geq 2$ and constants $\bar{c}_a, \bar{C}_a, C_a > 0$ such that

$$(\widetilde{A1}) \quad 0 \leq \inf_{t>0} \frac{t\tilde{a}'(t)}{\tilde{a}(t)} \leq \sup_{t>0} \frac{t\tilde{a}'(t)}{\tilde{a}(t)} \leq C_a < \infty,$$

and

$$(\widetilde{A2}) \quad \bar{c}_a t^{p-2} \leq \tilde{a}(t) \leq \bar{C}_a t^{p-2}, \quad \forall t > 0.$$

Note that the above hypotheses are nothing but (A1) and (A2), with constants possibly differing by a factor of two, and with the additional constraint $p \geq 2$.

As far as H is concerned, we assume here that

$$(\widetilde{H1}) \quad H \in C^0(\mathbb{R}^N \setminus \{0\})$$

and the existence of $k_1 > 0$, $c_H, C_H > 0$ and $\gamma > 1$ such that

$$(\widetilde{H2}) \quad c_H |\xi|^\gamma \leq H(\xi) \quad \text{and} \quad |H_\xi(\xi)| \leq C_H |\xi|^{\gamma-1} \quad \forall \xi \in \mathbb{R}^N : |\xi| \geq k_1.$$

Note that for any $0 \leq \varepsilon \leq 1$, we have as a consequence that

$$(3.1) \quad \frac{c_H}{2}(\varepsilon + |\xi|^2)^{\frac{\gamma}{2}} - c_\varepsilon \leq H(\xi) \quad \forall \xi \in \mathbb{R}^N : |\xi| \geq k_1.$$

for some c_ε that vanishes as $\varepsilon \rightarrow 0$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a C^2 convex domain and assume that $(\widetilde{A1}), (\widetilde{A2}), (\widetilde{H1}), (\widetilde{H2})$ are in force. Suppose that $\gamma > p - 1$, $p \geq 2$, $\lambda \geq 0$, and*

$$f \in L^q(\Omega) \quad \text{with} \quad q > \max \left\{ \frac{N(\gamma - (p - 1))}{\gamma}, 2 \right\},$$

then there exists K depending on $\|Du\|_{L^1(\Omega)}$, $\|f - \lambda u\|_{L^q(\Omega)}$, q, N, p, γ and the constants in the standing assumptions such that any smooth solution u_ε to

$$\begin{cases} \lambda u - \operatorname{div}(\tilde{a}(\sqrt{|Du|^2 + \varepsilon}) Du) + H(Du) = f_\varepsilon(x) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

satisfies

$$(3.2) \quad \|Du_\varepsilon\|_{L^{q\gamma}(\Omega)} \leq K.$$

Moreover if

$$q = \frac{N(\gamma - (p - 1))}{\gamma} \quad \text{and} \quad \gamma > \frac{N(p - 1)}{N - 2}$$

then, (3.2) holds true

- (i) *provided that $\|f\|_{L^q(\Omega)}$ is small enough if $\lambda = 0$;*
- (ii) *if $\lambda > 0$, with K depending also on $\lambda, \|u\|_{L^q(\Omega)}$ and it remains bounded whenever f varies in a set of $L^q(\Omega)$ -uniformly integrable functions.*

Furthermore

$$(3.3) \quad \| |Du_\varepsilon|^\iota Du_\varepsilon \|_{H^1(\Omega)} \leq C(\|f_\varepsilon\|_{L^q(\Omega)}), \quad \text{where} \quad \iota = \frac{p-1}{2} \frac{q(N-2)}{N-q} - 1.$$

Remark 3.2. The linear case $\tilde{a}(t) \equiv \tilde{a} > 0$ has been already covered in [25] in the periodic setting and later in [43] for Neumann problems when $\lambda = 0$. Here, we are also able to deal with the borderline integrability exponent $q = \frac{N(\gamma-1)}{\gamma}$. For such value of q the maximal regularity result for linear diffusions is new when $\gamma \geq 2$. The case (ii) in the regime $\gamma < 2$ is contained in [39].

Remark 3.3. Note that $N \frac{\gamma-(p-1)}{\gamma} > 1$ whenever $\gamma > \frac{N(p-1)}{N-1}$, so one has always to restrict at least to this range of growth for the first-order term.

Remark 3.4. Once $|Du|^\gamma$ is controlled in L^q by means of Theorem 3.1, one can regard the equation as $\lambda u - \operatorname{div}(\tilde{a}(|Du|)Du) = -|Du|^\gamma + f$, hence Du is bounded in $L^{\frac{Nq(p-1)}{N-q}}$ by the results in [16]. However, it is to our knowledge an open problem the validity of the stronger bound $|Du|^{p-2} Du \in W^{1,q}$. This has been shown to hold in the case $q = 2$ in the recent work [20], which by the way is not covered by Theorem 3.1.

Remark 3.5. Observe that as $q \rightarrow 2^+$ then the estimate in (3.3) coincides with the one in [21, Theorem 2.3]. Note that in the case $0 \leq \gamma \leq p-1$ one does not need to plug back the equation to conclude the estimates, cf. [42].

Proof. As in the previous case, since our method needs to deal with smooth solutions, we need to first approximate our problem. Let us consider the sequence of solutions to

$$(3.4) \quad \begin{cases} \lambda u_\varepsilon - \operatorname{div}(\tilde{a}(v_\varepsilon)Du_\varepsilon) + H(Du_\varepsilon) = f_\varepsilon(x) & \text{in } \Omega, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

where f_ε is a suitable regularization of f and $v_\varepsilon = \sqrt{|Du_\varepsilon|^2 + \varepsilon}$.

Let us now set $v_k = v_{\varepsilon,k} = (\sqrt{|Du_\varepsilon|^2 + \varepsilon} - k)^+$, for any $k \geq k_1$. Moreover, we denote by $\Omega_k = \{x \in \Omega : v_\varepsilon > k\}$ and observe that

$$\partial\Omega_k = \{x \in \Omega : v_\varepsilon = k\} \cup (\partial\Omega \cap \bar{\Omega}_k).$$

We also preliminary note that

$$Dv_{\varepsilon,k} = Dv_\varepsilon \text{ on } \Omega_k, \quad Dv_{\varepsilon,k} = 0 \text{ on } \Omega \setminus \Omega_k, \quad \text{and} \quad v_{\varepsilon,k} = 0 \text{ on } \partial\Omega_k \cap \Omega.$$

Moreover, by Lemma 2.4 we have, thanks to the convexity of Ω ,

$$(3.5) \quad \partial_\nu v_\varepsilon = \frac{\partial_\nu |Du_\varepsilon|^2}{2v_\varepsilon} \leq 0 \quad \text{on} \quad \partial\Omega.$$

We may also add c_ε to both sides of the equation, so that, by (3.1), H satisfies

$$(3.6) \quad \frac{cH}{2}(\varepsilon + |\xi|^2)^{\frac{\gamma}{2}} \leq H(\xi), \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$

The result will be a consequence of the following property, that will be shown below: there exist $k_0, c, \omega > 0$ depending on the data and the constants appearing in the assumptions such that ω is small enough so that the inequality

$$Z^{\frac{N-2}{N}} < \omega + cZ$$

is true if and only if $Z \in [0, Z^-) \cup (Z^+, +\infty)$. Our aim is to apply such an inequality to $Z = \|v_k\|_{L^{q\gamma}(\Omega)}$, i.e. we are done if we are able to prove that

$$(3.7) \quad \exists k_0 > 0 : \quad \forall k \geq k_0 \quad \left(\int_{\Omega} ((v_{\varepsilon} - k)^+)^{r\gamma} dx \right)^{\frac{N-2}{N}} \leq \omega_k + c \int_{\Omega} ((v_{\varepsilon} - k)^+)^{r\gamma} dx,$$

with $\omega_k \rightarrow 0$ as k diverges.

Indeed, once (3.7) is established, one can conclude as follows (see [25, Section 2, p.1524-1525]): since $k \mapsto Z(k) := \int_{\Omega} ((v_{\varepsilon} - k)^+)^{r\gamma}$ is continuous and it vanishes as $k \rightarrow \infty$, then $Z(k) \leq Z^-$ for all $k \geq k_0$, and therefore

$$\|v_{\varepsilon}\|_{L^{r\gamma}(\Omega)}^{r\gamma} \leq \|(v_{\varepsilon} - k_0)^+\|_{L^{r\gamma}(\Omega)}^{r\gamma} + \|k_0\|_{L^{r\gamma}(\Omega)}^{r\gamma} \leq Z^- + |\Omega|k_0^{r\gamma}.$$

From now on we drop the subscript ε for brevity, and look of course for estimates which are independent from ε .

To get (3.7), we start by testing the equation in (3.4) by

$$\varphi = \operatorname{div} \left(Du \frac{v_k^{\beta}}{v} \right) \quad \text{for some } \beta > 1,$$

(β to be determined later) and integrating by parts. Exploiting that u satisfies the homogeneous Neumann boundary condition, we have

$$(3.8) \quad - \int_{\Omega} \tilde{a}(v) Du \cdot D\varphi dx = \int_{\Omega} (H(Du) + \lambda u - f(x)) \varphi dx.$$

We first elaborate on the left-hand side of the previous equality.

Lemma 3.6. *We have the following inequality*

$$- \int_{\Omega} \tilde{a}(v) Du \cdot D\varphi dx \geq \int_{\Omega_k} \tilde{a}(v) \frac{v_k^{\beta}}{v} |D^2 u|^2 dx + \bar{c}_{\tilde{a}}(\beta - 1) \int_{\Omega_k} v^{p-2} v_k^{\beta-1} |Dv_k|^2 dx,$$

where $\bar{c}_{\tilde{a}}$ is the constant appearing in $(\widetilde{A2})$.

Proof. Using the boundary condition $\partial_{\nu} u = 0$ on $\partial\Omega$, we deduce that the left-hand side of the above inequality is equal to

$$\begin{aligned} - \int_{\Omega} \tilde{a}(v) Du \cdot D\varphi dx &= - \sum_{i,j=1}^N \int_{\Omega} \tilde{a}(v) \partial_{x_i} u \partial_{x_i} \left(\partial_{x_j} \left(\partial_{x_j} u \frac{v_k^{\beta}}{v} \right) \right) dx \\ &= - \sum_{i,j=1}^N \int_{\Omega} \tilde{a}(v) \partial_{x_i} u \partial_{x_j} \left(\partial_{x_i} \left(\partial_{x_j} u \frac{v_k^{\beta}}{v} \right) \right) dx \\ &= \sum_{i,j=1}^N \int_{\Omega} \partial_{x_j} (\tilde{a}(v) \partial_{x_i} u) \partial_{x_i} \left(\partial_{x_j} u \frac{v_k^{\beta}}{v} \right) dx - \sum_{i,j=1}^N \int_{\partial\Omega} \tilde{a}(v) \partial_{x_i} u \partial_{x_i} \left(\partial_{x_j} u \frac{v_k^{\beta}}{v} \right) \nu_j dS \\ &:= I_1 + I_2. \end{aligned}$$

We first handle the boundary terms as follows

$$\begin{aligned} I_2 &= - \sum_{i,j=1}^N \int_{\partial\Omega} \tilde{a}(v) \partial_{x_i} u \partial_{x_i} \left(\frac{v_k^\beta}{v} \right) (\partial_{x_j} u \nu_j) dS - \sum_{i,j=1}^N \int_{\partial\Omega} \tilde{a}(v) \frac{v_k^\beta}{v} \left(\partial_{x_j x_i} u \partial_{x_i} u \nu_j \right) dS \\ &= - \int_{\partial\Omega} \tilde{a}(v) Du \cdot D \left(\frac{v_k^\beta}{v} \right) \partial_\nu u dS - \int_{\partial\Omega} \tilde{a}(v) v_k^\beta \partial_\nu v dS \geq 0, \end{aligned}$$

where we have used both the boundary condition on u and inequality (3.5).

Then, since v_k vanishes outside Ω_k , we get

$$\begin{aligned} I_1 &= \sum_{i,j=1}^N \int_{\Omega_k} \partial_{x_j} (\tilde{a}(v) \partial_{x_i} u) \partial_{x_i} \left(\partial_{x_j} u \frac{v_k^\beta}{v} \right) dx \\ &= \sum_{i,j=1}^N \int_{\Omega_k} (\tilde{a}'(v) \partial_{x_j} v \partial_{x_i} u + \tilde{a}(v) \partial_{x_i x_j} u) \left(\frac{v_k^\beta}{v} \partial_{x_i x_j} u + \frac{\beta v v_k^{\beta-1} \partial_{x_i} v_k - v_k^\beta \partial_{x_i} v}{v^2} \partial_{x_j} u \right) dx. \end{aligned}$$

As $v \geq v_k$, we have in particular that

$$\beta v^{-1} v_k^{\beta-1} - v^{-2} v_k^\beta \geq (\beta - 1) \frac{v_k^{\beta-1}}{v},$$

and therefore, owing to the fact that $a'(s) \geq 0$ (see $(\widetilde{A1})$) and since $\beta > 1$, we get

$$\sum_{i,j=1}^N \int_{\Omega_k} \tilde{a}'(v) \frac{\beta v v_k^{\beta-1} - v_k^\beta}{v^2} \partial_{x_j} v \partial_{x_i} u \partial_{x_j} u \partial_{x_i} v_k dx \geq (\beta - 1) \int_{\Omega_k} \tilde{a}'(v) \frac{v_k^{\beta-1}}{v} (Dv \cdot Du)^2 dx \geq 0,$$

and also

$$\sum_{i,j=1}^N \int_{\Omega_k} \tilde{a}'(v) \frac{v_k^\beta}{v} \partial_{x_j} v \partial_{x_i} u \partial_{x_i x_j} u dx = \sum_{i,j=1}^N \int_{\Omega_k} \tilde{a}'(v) v_k^\beta |Dv|^2 dx \geq 0.$$

Thus, we are left with

$$\begin{aligned} I_1 &\geq \sum_{i,j=1}^N \int_{\Omega_k} \tilde{a}(v) \partial_{x_i x_j} u \left(\partial_{x_i x_j} u \frac{v_k^\beta}{v} + \partial_{x_j} u \frac{\beta v v_k^{\beta-1} \partial_{x_i} v_k - \partial_{x_i} v v_k^\beta}{v^2} \right) dx \\ &\geq \sum_{i,j=1}^N \int_{\Omega_k} \tilde{a}(v) \partial_{x_i x_j} u \left(\partial_{x_i x_j} u \frac{v_k^\beta}{v} + (\beta - 1) \partial_{x_j} u \partial_{x_i} v \frac{v_k^{\beta-1}}{v} \right) dx \\ &= \int_{\Omega_k} \tilde{a}(v) |D^2 u|^2 \frac{v_k^\beta}{v} dx + (\beta - 1) \int_{\Omega_k} \tilde{a}(v) v_k^{\beta-1} |Dv_k|^2 dx \\ &\geq \int_{\Omega_k} \tilde{a}(v) |D^2 u|^2 \frac{v_k^\beta}{v} dx + \bar{c}_{\tilde{a}} (\beta - 1) \int_{\Omega_k} v^{p-2} v_k^{\beta-1} |Dv_k|^2 dx, \end{aligned}$$

where we used $(\widetilde{A2})$ in the last inequality. \square

Lemma 3.7. *There exist constants $c_{10}, c_{11} > 0$ depending on $\bar{c}_{\tilde{a}}, \bar{C}_{\tilde{a}}, c_H, N, C_{\tilde{a}}, \lambda$ such that the following inequality holds*

$$(3.9) \quad \int_{\Omega_k} \tilde{a}(v) |D^2 u|^2 \frac{v_k^\beta}{v} dx \geq c_{10} \int_{\Omega_k} v^{2\gamma+1-p} v_k^\beta dx - c_{11} \int_{\Omega_k} (\lambda u - f)^2 v_k^\beta v^{1-p} dx$$

Proof. Note first that

$$\operatorname{div}(\tilde{a}(v)Du) = \tilde{a}(v)\Delta u + \tilde{a}'(v)\frac{D^2uDu}{v} \cdot Du = \tilde{a}(v)\Delta u + \tilde{a}'(v)\frac{\Delta_\infty u}{v} = \tilde{a}(v) \underbrace{\left[\Delta u + \frac{\tilde{a}'(v)}{v\tilde{a}(v)}\Delta_\infty u \right]}_{:=\tilde{\mathcal{A}}(D^2u)}.$$

Therefore, as before we define $\nu = \sqrt{N} + C_{\tilde{a}}$ and we have

$$|\tilde{\mathcal{A}}(D^2u)| = \left| \Delta u + \frac{\tilde{a}'(v)}{v\tilde{a}(v)}D^2uDu \right| \leq \sqrt{N}|D^2u| + C_{\tilde{a}}\frac{|D^2u||Du|^2}{\varepsilon + |Du|^2} \leq \nu|D^2u|.$$

We exploit, as in Theorem 2.1, the non-divergence formulation of the equation, i.e. we consider

$$\tilde{a}(v)\tilde{\mathcal{A}}(D^2u) = H(Du) + (\lambda u - f) \quad \text{in } \Omega,$$

and plugging the equation back in the term $\tilde{a}(v)|D^2u|^2$ by $(\tilde{\mathcal{A}}1)$ and (3.6), we get

$$\begin{aligned} \tilde{a}(v)|D^2u|^2 &\geq \frac{[\tilde{a}(v)\tilde{\mathcal{A}}(D^2u)]^2}{\nu^2\tilde{a}(v)} \\ &\geq \frac{1}{\nu^2C_2} [H(Du) + \lambda u - f]^2 v^{2-p} \geq \frac{1}{\nu^2C_2} \left[\frac{H^2(Du)}{2} - 2(\lambda u - f)^2 \right] v^{2-p} \\ &= \frac{c_H^2}{\nu^2C_2} \left\{ \frac{v^{2\gamma}}{8} - 2(\lambda u - f)^2 \right\} v^{2-p}, \end{aligned}$$

and thus (3.9) holds. \square

Lemma 3.8. *There exist constants $c_{12}, c_{13} > 0$ depending on $\bar{c}_{\tilde{a}}, \bar{C}_{\tilde{a}}, c_H, C_H, N, C_{\tilde{a}}, \lambda, \beta, p$ such that the following inequality holds true:*

$$\begin{aligned} - \int_{\Omega} \tilde{a}(v)Du \cdot D\varphi \, dx &\geq c_{12} \int_{\Omega_k} v^{p-3}v_k^\beta |D^2u|^2 \, dx + c_{12} \left(\int_{\Omega_k} v_k^{(p+\beta-1)\frac{N-2}{N}} \right)^{\frac{N-2}{N}} \\ &+ c_{12} \int_{\Omega_k} v^{p-2}v_k^{\beta-1} |Dv_k|^2 \, dx + c_{12} \int_{\Omega_k} v^{2\gamma+1-p}v_k^\beta \, dx - c_{13} \int_{\Omega_k} 2(\lambda u - f)^2 v_k^\beta v^{1-p} \, dx. \end{aligned}$$

Proof. First, we exploit the Sobolev inequality on the term $\int_{\Omega_k} v^{p-2}v_k^{\beta-1}|Dv_k|^2 \, dx$, and the fact that $v_k \leq v$ (and $p \geq 2$) to get

$$\begin{aligned} \int_{\Omega_k} v^{p-2}v_k^{\beta-1}|Dv_k|^2 \, dx &\geq \int_{\Omega_k} v_k^{\beta+p-3}|Dv_k|^2 \, dx = \frac{4}{(p+\beta-1)^2} \int_{\Omega_k} |Dv_k^{\frac{p+\beta-1}{2}}|^2 \\ &\geq \frac{4C_S}{(p+\beta-1)^2} \left(\int_{\Omega_k} v_k^{(p+\beta-1)\frac{N}{N-2}} \right)^{\frac{N-2}{N}} - \frac{4}{(p+\beta-1)^2} \int_{\Omega_k} v_k^{p+\beta-1} \, dx, \end{aligned}$$

where C_S is the constant of the Sobolev embedding. To conclude, it now suffices to use Lemma 3.6 and Lemma 3.7. \square

We now work on the right-hand side of (3.8), that is,

$$\begin{aligned} & \int_{\Omega_k} (H(Du) + \lambda u - f) \operatorname{div} \left(Du \frac{v_k^\beta}{v} \right) dx \\ &= \underbrace{\int_{\Omega_k} H(Du) \operatorname{div} \left(Du \frac{v_k^\beta}{v} \right)}_{:=I_3} + \lambda \underbrace{\int_{\Omega_k} u \operatorname{div} \left(Du \frac{v_k^\beta}{v} \right)}_{:=I_4} - \underbrace{\int_{\Omega_k} f \operatorname{div} \left(Du \frac{v_k^\beta}{v} \right)}_{:=I_5}. \end{aligned}$$

Lemma 3.9. *For any $\delta > 0$, there exists $c_{14} > 0$ depending on $\delta, c_H, C_H, \gamma, p, N$ such that*

$$\begin{aligned} I_3 + I_4 - I_5 &\leq -\lambda \int_{\Omega_k} |Du|^2 \frac{v_k^\beta}{v} dx + \delta \int_{\Omega_k} v^{p-2} v_k^{\beta-1} |Dv_k|^2 dx + \delta \int_{\Omega_k} v^{2\gamma+1-p} v_k^\beta dx \\ &+ \delta \int_{\Omega_k} |D^2u|^2 v^{p-3} v_k^\beta dx + \delta \int_{\Omega_k} v_k^{p+\beta-3} |Dv_k|^2 dx + c_{14} \int_{\Omega_k} v_k^{\beta+2\gamma+1-p} dx + c_{14} \int_{\Omega_k} |f|^2 v_k^{\beta+1-p} dx. \end{aligned}$$

Proof. We start with I_3 above, and use the divergence theorem, the boundary condition $\partial_\nu u = 0$ at $\partial\Omega$, ($\widetilde{H2}$) to find that

$$\begin{aligned} & \int_{\Omega} H(Du) \operatorname{div} \left(Du \frac{v_k^\beta}{v} \right) dx = - \int_{\Omega} \partial_{x_j} (H(Du)) \partial_{x_j} u \frac{v_k^\beta}{v} dx + \int_{\partial\Omega} H(Du) \frac{v_k^\beta}{v} \partial_\nu u dS \\ &= - \int_{\Omega} H_\xi(Du) D^2u Du \frac{v_k^\beta}{v} dx \leq \int_{\Omega_k} |H_\xi(Du)| |Dv| v_k^\beta dx \leq C_H \int_{\Omega_k} v^{\gamma-1} |Dv| v_k^\beta dx. \end{aligned}$$

We now set

$$\eta = 2\gamma - p + 1$$

and observe that $\eta > 1$ since $p \geq 2$ and $\gamma > p - 1$. Thus, for any $\delta > 0$ we have

$$C_H \int_{\Omega_k} v^{\gamma-1} |Dv| v_k^\beta dx \leq \delta \int_{\Omega_k} v^{p-2} |Dv|^2 v_k^{\beta-1} dx + \frac{C_H^2}{4\delta} \int_{\Omega_k} v^{2\gamma-p} v_k^{\beta+1} dx.$$

Moreover, applying the weighted Young's inequality to the last term, we have that

$$\frac{C_H^2}{4\delta} \int_{\Omega_k} v^{\eta-1} v_k^{\frac{\beta}{\eta}} v_k^{\frac{\beta}{\eta}+1} dx \leq \delta \int_{\Omega_k} v^\eta v_k^\beta dx + C(C_H, \gamma, p, N, \delta) \int_{\Omega_k} v_k^{\beta+\eta} dx,$$

and thus

$$I_3 \leq \delta \int_{\Omega_k} v^{p-2} |Dv|^2 v_k^{\beta-1} dx + \delta \int_{\Omega_k} v^\eta v_k^\beta dx + C(c_H, \gamma, p, N, \delta) \int_{\Omega_k} v_k^{\beta+\eta} dx.$$

In order to handle I_4 we apply the divergence theorem and we exploit the Neumann boundary condition, so that

$$I_4 = \lambda \int_{\Omega} u \operatorname{div} \left(Du \frac{v_k^\beta}{v} \right) = -\lambda \int_{\Omega} |Du|^2 \frac{v_k^\beta}{v} dx + \lambda \int_{\partial\Omega} \frac{uv_k^\beta}{v} \partial_\nu u dS = -\lambda \int_{\Omega_k} |Du|^2 \frac{v_k^\beta}{v} dx.$$

We finally consider the term I_5 involving the source f of the equation, which becomes, after integration by parts (recall also that $v_k \leq v$)

$$|I_5| = \left| \int_{\Omega} f \operatorname{div} \left(\frac{v_k^\beta}{v} Du \right) dx \right| \leq \int_{\Omega_k} |f| [v^{-1} v_k^\beta \sqrt{N} |D^2u| + (\beta + 1) |Dv_k| v_k^{\beta-1}] dx.$$

Then, the Young's inequality gives for any positive δ

$$\int_{\Omega_k} |f| v^{-1} v_k^\beta \sqrt{N} |D^2 u| dx \leq \delta \int_{\Omega_k} |D^2 u|^2 v^{p-3} v_k^\beta dx + \frac{C}{\delta} \int_{\Omega_k} |f|^2 v^{1-p} v_k^\beta dx,$$

and moreover

$$(\beta + 1) \int_{\Omega_k} |f| |D v_k| v_k^{\beta-1} dx \leq \delta \int_{\Omega_k} v_k^{p+\beta-3} |D v_k|^2 dx + \frac{C}{\delta} \int_{\Omega_k} |f|^2 v_k^{1-p+\beta} dx.$$

Since $v \geq v_k$ and $p > 1$, it follows that $v^{1-p} \leq v_k^{1-p}$, and collecting all the above estimates we are done. \square

Lemma 3.10. *There exists $c_{15} > 0$ depending on $\bar{c}_a, \bar{C}_a, C_H, c_H, N, C_{\bar{a}}, \lambda, \beta, p, \gamma$ such that, for any $r > 2$*

$$(3.10) \quad c_{15} \left(\int_{\Omega_k} v_k^{(p+\beta-1)\frac{N}{N-2}} dx \right)^{\frac{N-2}{N}} + \lambda c_{15} \int_{\Omega_k} |Du|^2 \frac{v_k^\beta}{v} dx \\ \leq \int_{\Omega_k} |f|^r dx + \int_{\Omega_k} v_k^{(\beta+1-p)\frac{r}{r-2}} dx + \lambda^2 \int_{\Omega_k} |u|^2 v_k^\beta v^{1-p} dx + \int_{\Omega_k} v_k^{p+\beta-1} dx + \int_{\Omega_k} v_k^{\beta+2\gamma-p+1} dx.$$

Proof. Combining Lemma 3.8 and 3.9 with δ small enough (with respect to c_4) we get

$$(3.11) \quad c_{12} \left(\int_{\Omega_k} v_k^{(p+\beta-1)\frac{N}{N-2}} dx \right)^{\frac{N-2}{N}} + \lambda \int_{\Omega_k} |Du|^2 \frac{v_k^\beta}{v} dx \leq c_{14} \int_{\Omega_k} v_k^{\beta+2\gamma-p+1} dx \\ + c_{14} \int_{\Omega_k} |f|^2 v_k^{1-p+\beta} dx + c_{12} \int_{\Omega_k} v_k^{p+\beta-1} dx + c_{14} \int_{\Omega_k} 2(\lambda u - f)^2 v_k^\beta v^{1-p} dx.$$

To handle the terms involving the datum f , we exploit that $v_k \leq v$, the inequality $(a+b)^2 \leq 2(a^2 + b^2)$, $a, b \geq 0$, and we finally employ Young's inequality, so that

$$\int_{\Omega_k} (\lambda u - f)^2 v_k^\beta v^{1-p} dx + \int_{\Omega_k} |f|^2 v_k^{1-p+\beta} dx \\ \leq c_{16} \left(\int_{\Omega_k} |f|^r dx + \int_{\Omega_k} v_k^{(1-p+\beta)\frac{r}{r-2}} dx + \lambda^2 \int_{\Omega_k} |u|^2 v_k^\beta v^{1-p} dx \right),$$

and (3.10) follows. \square

Now we are ready to prove Theorem 3.1.

Proof Theorem 3.1. In view of the discussion at the beginning of this section, we show (3.7).

- We start supposing that

$$q > \frac{N(\gamma - (p-1))}{\gamma} \quad \text{and} \quad \lambda = 0,$$

so that (3.10) reduces to

$$(3.12) \quad c_{15} \left(\int_{\Omega_k} v_k^{(p+\beta-1)\frac{N}{N-2}} dx \right)^{\frac{N-2}{N}} \leq \int_{\Omega_k} |f|^r dx + \int_{\Omega_k} v_k^{(1-p+\beta)\frac{r}{r-2}} dx \\ + \int_{\Omega_k} v_k^{\beta+2\gamma-p+1} dx + \int_{\Omega_k} v_k^{p+\beta-1} dx.$$

First, note that by Hölder inequality we get

$$(3.13) \quad \int_{\Omega_k} v_k^{p+\beta-1} \leq |\Omega_k|^{\frac{2}{N}} \left(\int_{\Omega_k} v_k^{(p+\beta-1)\frac{N}{N-2}} dx \right)^{\frac{N-2}{N}},$$

and notice that, $|\Omega_k| \leq \frac{1}{k} \|\sqrt{|Du_\varepsilon|^2 + 1}\|_{L^1(\Omega)}$. Thus choosing k sufficiently large in order to have $|\Omega_k|^{\frac{2}{N}} \leq \frac{c_6}{2}$, we can absorb the last integral in (3.12) in the left hand side.

Now we choose the parameters r and β in the following way

$$r = \frac{2}{N} \frac{N(\gamma - (p-1))}{\gamma} + \frac{N-2}{N} q \quad \text{and} \quad \beta = (r-2)\gamma + p-1.$$

It is immediate to verify that

$$\frac{r}{r-2}(\beta - p + 1) = r\gamma = \beta + \eta \quad \text{and} \quad (\beta + p - 1) \frac{N}{N-2} = q\gamma.$$

Finally we have, since $\frac{N(\gamma-(p-1))}{\gamma} < r < q$,

$$(3.14) \quad \int_{\Omega_k} |f|^r dx \leq \|f\|_{L^q(\Omega)}^r |\Omega_k|^{\frac{q-r}{q}},$$

and so plugging (3.13) and (3.14) into (3.12), by the choices of the parameters r and β and since $|\Omega_k| \rightarrow 0$, we end up with

$$c_{17} \left(\int_{\Omega_k} v_k^{r\gamma} \right)^{\frac{N-2}{N}} \leq \int_{\Omega_k} v_k^{r\gamma} + \omega(|\Omega_k|), \quad \text{where } \omega(|\Omega_k|) \rightarrow 0 \quad \text{as } |\Omega_k| \rightarrow 0.$$

Hence (3.7) holds.

- If $\lambda > 0$, then one concludes in the very same way replacing f by $f - \lambda u$.
- Suppose now

$$q = \frac{N(\gamma - (p-1))}{\gamma} \quad \text{and} \quad \lambda = 0,$$

and choose now $r = q$. We can repeat the same proof of the previous case, with the only difference that we require $\|f\|_q$ to be small, in order to deduce (3.7).

- When $\lambda > 0$, we can further exploit the zero-th other terms to avoid the smallness assumption on the datum f . Recalling (3.11) we have that

$$(3.15) \quad c_{18} \left(\int_{\Omega_k} v_k^{r\gamma} dx \right)^{\frac{N-2}{N}} + \lambda c_{18} \int_{\Omega_k} |Du|^2 \frac{v_k^\beta}{v} dx \leq \int_{\Omega_k} |f|^2 v_k^{1-p+\beta} dx + \\ + \lambda^2 \int_{\Omega_k} |u|^2 v_k^\beta v^{1-p} dx + \int_{\Omega_k} v_k^{p+\beta-1} + 2 \int_{\Omega_k} v_k^{q\gamma} dx.$$

On the left-hand side, since $v \geq v_k$,

$$\lambda c_{18} \int_{\Omega_k} |Du|^2 \frac{v_k^\beta}{v} dx = \lambda c_{18} \int_{\Omega_k} v v_k^\beta dx - \lambda c_{18} \varepsilon \int_{\Omega_k} \frac{v_k^\beta}{v} dx \geq \lambda c_{18} k \int_{\Omega_k} v_k^\beta dx - \lambda c_{18} \varepsilon \int_{\Omega_k} v_k^{\beta-1} dx.$$

Note that the last term can be absorbed into the leftmost term of (3.15) through the Young's inequality. On the right-hand side of (3.15), first, for any $\omega > 0$ (recall that $\gamma q = (\beta + 1 - p)\frac{q}{q-2}$)

$$\lambda^2 \int_{\Omega_k} |u|^2 v_k^\beta v^{1-p} dx \leq \lambda^2 \int_{\Omega_k} |u|^2 v_k^{\beta+1-p} dx \leq c_{19}(\lambda) \int_{\Omega_k} |u|^q dx + c_{20} \int_{\Omega_k} v_k^{\gamma q} dx.$$

Then, we need to show that $\int_{\Omega_k} |f|^2 v_k^{\beta-(p-1)}$ can be made small for k large. We then write

$$\begin{aligned} \int_{\Omega_k} |f|^2 v_k^{\beta-(p-1)} dx &= \int_{\Omega_k \cap \left\{ |f|^2 \leq k^{\frac{\beta-(p-1)}{\beta}} \right\}} |f|^2 v_k^{\beta-(p-1)} dx + \int_{\Omega_k \cap \left\{ |f|^2 \geq k^{\frac{\beta-(p-1)}{\beta}} \right\}} |f|^2 v_k^{\beta-(p-1)} dx \\ &\leq \int_{\Omega_k} k^{\frac{\beta-(p-1)}{\beta}} v_k^{\beta-(p-1)} dx + \int_{\Omega_k \cap \left\{ |f|^2 \geq k^{\frac{\beta-(p-1)}{\beta}} \right\}} |f|^2 v_k^{\beta-(p-1)} dx \\ &\leq \lambda c_{18} k \int_{\Omega_k} v_k^\beta dx + c_{21} |\Omega_k| + \int_{\Omega_k \cap \left\{ |f|^2 \geq k^{\frac{\beta-(p-1)}{\beta}} \right\}} |f|^q dx + \int_{\Omega_k} v_k^{\frac{q}{q-2}(\beta-(p-1))} dx \\ &= \lambda c_{18} k \int_{\Omega_k} v_k^\beta dx + \int_{\Omega_k} v_k^{\gamma q} dx + \omega'(|\Omega_k|), \end{aligned}$$

where $\omega'(|\Omega_k|)$ as $|\Omega_k| \rightarrow 0$ (note that ω' depends on f). The conclusion now follows as before, but the estimate depends in addition on $\lambda, \|u\|_{L^q(\Omega)}$.

Finally, let us observe that if we combine (again) Lemmas 3.8 and 3.9 with $k = 0$ we deduce that

$$\begin{aligned} \lambda \int_{\Omega_k} |Du|^2 \frac{v_k^\beta}{v} dx + \|v\|_{L^{(p+\beta-1)\frac{N}{N-2}}(\Omega)}^{p+\beta-1} + \int_{\Omega} |D^2 u|^2 v^{\beta+p-3} dx \\ \leq c_{16} \left(1 + \int_{\Omega} g^2 v^{\beta+1-p} dx \right) \leq c_{16} \left(1 + \|g\|_{L^q(\Omega)}^2 \|v\|_{L^{\frac{q(\beta+1-p)}{q-2}}(\Omega)}^{\beta+1-p} \right), \end{aligned}$$

where $(\lambda u - f)^2 + w^{2\gamma} := g^2(x)$. Since $g \in L^{\frac{q}{2}}(\Omega)$ and choosing $\beta = (p-1)\frac{N(q-1)-q}{N-q}$ (i.e. such that $\frac{q(\beta+1-p)}{q-2} = (p+\beta-1)\frac{N}{N-2}$) we deduce that

$$\int_{\Omega} |D^2 u|^2 v^{\frac{(p-1)q(N-2)}{N-q}-2} dx \leq C.$$

□

4. FINAL REMARKS

We list here some final comments and open problems.

Remark 4.1. We do not pursue here the direction of weakening the regularity of the domain. We just point out that it can be considerably weakened, as soon as the Sobolev inequality and the divergence theorem hold. These facts were thoroughly discussed for the Dirichlet and Neumann problem for second order elliptic equations in Chapter 3 of [45], and recently considered in [17, 21] for the p -Poisson equation.

Remark 4.2. An inspection throughout the proofs suggest that our integral Bernstein argument provides some new second order integral estimates. These have been discussed in [42] for sublinear problems, and lead to completely new properties for stationary quasilinear elliptic equations with superlinear gradient terms. In particular, the techniques of the present manuscript allow to retrieve a result of [16], see Remark 5.8 in [42].

Remark 4.3. We underline that when $\lambda > 0$ the estimates in Theorem 3.1 may depend in addition on $\|\lambda u\|_r$. This quantity can be a priori bounded in many different cases (though results appeared mostly for problems with Dirichlet boundary conditions): test function arguments can be used when $\gamma < p$, see [44]. Moreover, L^∞ -estimates can be achieved for problems having critical gradient growth $\gamma = p$ (and $r > N/p$), see [1, 65], whilst better estimates at the level of Hölder spaces can be obtained in the supernatural growth case $\gamma > p$, see [27, 40]. Finally, one can add a (linear) first-order term $b \cdot Du$ with $b \in L^s(\Omega)$ for s suitably large in Theorems 2.1 and 3.1 without substantial modifications in the proofs, as done in e.g. [5, 43].

Remark 4.4. It is still unclear whether the result in Theorem 3.1 continues to hold in the subquadratic case $p \in (1, 2)$, and this remains at this stage an open problem.

Remark 4.5. While the estimate in Theorem 2.1 leads to an explicit bound with respect to the L^q -norm of f , which agrees with the ones for the p -Poisson equation, the dependence with respect to f in Theorem 3.1 is much more implicit. It is still unclear if the maximal regularity bound could be proved in the stronger form of Theorem 2.1, even in the linear case $\bar{a}(t) \equiv \tilde{a} > 0$.

Remark 4.6. The techniques of the present manuscript do not apply to general parabolic quasilinear problems (unless $\partial_t f$ is proven to be regular enough). However, some advances for the evolutive counterpart of (2.1) with sublinear growth can be found in the recent paper [42].

Remark 4.7. Let us observe that, for f_ε smooth enough, any solution u_ε to

$$(4.1) \quad \begin{cases} \lambda u_\varepsilon - \operatorname{div}((\varepsilon + |Du_\varepsilon|^2)^{\frac{p-2}{2}} Du_\varepsilon) + |Du_\varepsilon|^\gamma = f_\varepsilon(x) & \text{in } \Omega, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

is smooth, for $\lambda > 0$, $p > 1$ and $\gamma > p - 1$. Indeed by [52, Proposition 7.1] we deduce that $u_\varepsilon \in W^{1,\infty}(\Omega)$. This implies in particular that the lower order term is bounded, so that standard regularity results (see [54]) imply that $u_\varepsilon \in C^{1,\alpha}$. Arguing by bootstrap we get that $-\operatorname{div}((\varepsilon + |Du_\varepsilon|^2)^{\frac{p-2}{2}} Du_\varepsilon)$ is Hölder continuous, and so by Schauder estimates for uniformly elliptic problems u_ε is C^3 up to the boundary of Ω . We stress that the first (delicate) step, i.e. the Lipschitz estimate, in principle holds only for the p -Laplacian, with $p > 2$. Anyway it is not hard to extend the results contained in [52] to a larger class of operators of the form $-\operatorname{div}(a(|Du|)Du)$, where a satisfies (A1)–(A2) and a technical assumption that involves the behavior of a'' with respect to a' (the presence of this additional assumption is not surprising since the method used in [52] is based on the derivation of the equation) that is trivially satisfied by the p -Laplacian.

Remark 4.8 (Existence of solutions). The estimates proved in our main results yield to the proof of the existence of (at least) a solution to

$$(4.2) \quad \begin{cases} \lambda u - \operatorname{div}(|Du|^{p-2} Du) + |Du|^\gamma = f(x) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume indeed that $\lambda > 0$, $\gamma > p-1$, $f \in L^q(\Omega)$, and either $p > 2$, and $q \geq \max\left\{\frac{N(\gamma-(p-1))}{\gamma}, 2\right\}$ or $p > 1$, and $q > N$. Then there exists at least a weak solution to (4.2) that belongs to $W^{1,\gamma q}(\Omega)$. Indeed arguing by approximation, we consider a sequence of (smooth) solutions u_ε to (4.1). Thus applying Theorem 2.1 combined with Remark 2.15 (or Theorem 3.1), we get the bound of Du_ε in $L^{q\gamma}(\Omega)$, and we also easily deduce that

$$\lambda \|u_\varepsilon\|_{L^1(\Omega)} \leq \|f_\varepsilon\|_{L^1(\Omega)} + \| |Du_\varepsilon|^\gamma \|_{L^1(\Omega)}.$$

Since $-\Delta_p u_\varepsilon$ is bounded in $L^1(\Omega)$, then, up to a (not relabeled) subsequence, Du_ε a.e. converges to Du by Lemma 1 in [11], and thanks to the bound given by Theorem 2.1 we deduce the compactness of u_ε in $W^{1,r}(\Omega)$ for any $1 \leq r < \gamma q$. This is enough to pass to the limit in the weak formulation of the approximating problem and get a solution to (4.2).

REFERENCES

- [1] B. Abdellaoui, A. Dall'Aglio, and I. Peral. Some remarks on elliptic problems with critical growth in the gradient. *J. Differential Equations*, 222(1):21–62, 2006.
- [2] A. Alvino, V. Ferone, and A. Mercaldo. Sharp a priori estimates for a class of nonlinear elliptic equations with lower order terms. *Ann. Mat. Pura Appl. (4)*, 194(4):1169–1201, 2015.
- [3] H. Amann and M. G. Crandall. On some existence theorems for semi-linear elliptic equations. *Indiana Univ. Math. J.*, 27(5):779–790, 1978.
- [4] A. Banerjee and I. H. Munive. Gradient continuity estimates for the normalized p -Poisson equation. *Commun. Contemp. Math.*, 22(8):1950069, 24, 2020.
- [5] M. Bardi and B. Perthame. Uniform estimates for some degenerating quasilinear elliptic equations and a bound on the Harnack constant for linear equations. *Asymptotic Anal.*, 4(1):1–16, 1991.
- [6] L. Beck and G. Mingione. Lipschitz bounds and nonuniform ellipticity. *Comm. Pure Appl. Math.*, 73(5):944–1034, 2020.
- [7] H. Beirão da Veiga and F. Crispo. On the global $W^{2,q}$ regularity for nonlinear N -systems of the p -Laplacian type in n space variables. *Nonlinear Anal.*, 75(11):4346–4354, 2012.
- [8] M. F. Betta, R. Di Nardo, A. Mercaldo, and A. Perrotta. Gradient estimates and comparison principle for some nonlinear elliptic equations. *Commun. Pure Appl. Anal.*, 14(3):897–922, 2015.
- [9] I. Birindelli, F. Demengel, and F. Leoni. Ergodic pairs for singular or degenerate fully nonlinear operators. *ESAIM Control Optim. Calc. Var.*, 25:Paper No. 75, 28, 2019.
- [10] L. Boccardo and T. Gallouët. Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, 87(1):149–169, 1989.
- [11] L. Boccardo and T. Gallouët. Nonlinear elliptic equations with right-hand side measures. *Comm. Partial Differential Equations*, 17(3-4):641–655, 1992.
- [12] L. Brasco. Global L^∞ gradient estimates for solutions to a certain degenerate elliptic equation. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 74(2):516–531, 2011.
- [13] L. Brasco and F. Santambrogio. A sharp estimate à la Calderón-Zygmund for the p -Laplacian. *Commun. Contemp. Math.*, 20(3):1750030, 24, 2018.
- [14] I. Capuzzo Dolcetta, F. Leoni, and A. Porretta. Hölder estimates for degenerate elliptic equations with coercive Hamiltonians. *Trans. Amer. Math. Soc.*, 362(9):4511–4536, 2010.
- [15] A. Cianchi. Maximizing the L^∞ norm of the gradient of solutions to the Poisson equation. *J. Geom. Anal.*, 2(6):499–515, 1992.
- [16] A. Cianchi and V. Maz'ya. Gradient regularity via rearrangements for p -Laplacian type elliptic boundary value problems. *J. Eur. Math. Soc. (JEMS)*, 16(3):571–595, 2014.
- [17] A. Cianchi and V. Maz'ya. Global gradient estimates in elliptic problems under minimal data and domain regularity. *Commun. Pure Appl. Anal.*, 14(1):285–311, 2015.
- [18] A. Cianchi and V. G. Maz'ya. Global Lipschitz regularity for a class of quasilinear elliptic equations. *Comm. Partial Differential Equations*, 36(1):100–133, 2011.
- [19] A. Cianchi and V. G. Maz'ya. Global boundedness of the gradient for a class of nonlinear elliptic systems. *Arch. Ration. Mech. Anal.*, 212(1):129–177, 2014.

- [20] A. Cianchi and V. G. Maz'ya. Second-order two-sided estimates in nonlinear elliptic problems. *Arch. Ration. Mech. Anal.*, 229(2):569–599, 2018.
- [21] A. Cianchi and V. G. Maz'ya. Optimal second-order regularity for the p -Laplace system. *J. Math. Pures Appl. (9)*, 132:41–78, 2019.
- [22] M. Cirant. On the improvement of Hölder seminorms in superquadratic Hamilton-Jacobi equations. *J. Funct. Anal.*, 288 (2025) 110692.
- [23] M. Cirant and A. Goffi. Lipschitz regularity for viscous Hamilton-Jacobi equations with L^p terms. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 37(4):757–784, 2020.
- [24] M. Cirant and A. Goffi. Maximal L^q -regularity for parabolic Hamilton-Jacobi equations and applications to Mean Field Games. *Ann. PDE*, 7(2):Paper No. 19, 40, 2021.
- [25] M. Cirant and A. Goffi. On the problem of maximal L^q -regularity for viscous Hamilton-Jacobi equations. *Arch. Ration. Mech. Anal.*, 240(3):1521–1534, 2021.
- [26] M. Cirant and G. Verzini. Local Hölder and maximal regularity of solutions of elliptic equations with superquadratic gradient terms. *Adv. Math.*, 409:108700, 16, 2022.
- [27] A. Dall'Aglio and A. Porretta. Local and global regularity of weak solutions of elliptic equations with superquadratic Hamiltonian. *Trans. Amer. Math. Soc.*, 367(5):3017–3039, 2015.
- [28] E. DiBenedetto. $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. *Nonlinear Anal.*, 7(8):827–850, 1983.
- [29] E. DiBenedetto. *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, 1993.
- [30] E. DiBenedetto and A. Friedman. Regularity of solutions of nonlinear degenerate parabolic systems. *J. Reine Angew. Math.*, 349:83–128, 1984.
- [31] E. DiBenedetto and J. Manfredi. On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems. *Amer. J. Math.*, 115(5):1107–1134, 1993.
- [32] H. Dong, F. Peng, Y. R.-Y. Zhang, and Y. Zhou. Hessian estimates for equations involving p -Laplacian via a fundamental inequality. *Adv. Math.*, 370:107212, 40, 2020.
- [33] F. Duzaar and G. Mingione. Partial differential equations—gradient estimates in non-linear potential theory. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 20(2):179–190, 2009.
- [34] F. Duzaar and G. Mingione. Gradient continuity estimates. *Calc. Var. Partial Differential Equations*, 39(3-4):379–418, 2010.
- [35] F. Duzaar and G. Mingione. Gradient estimates via non-linear potentials. *Amer. J. Math.*, 133(4):1093–1149, 2011.
- [36] L. C. Evans. A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e. *J. Differential Equations*, 45(3):356–373, 1982.
- [37] Y. Giga and S. Tsubouchi. Continuity of derivatives of a convex solution to a perturbed one-Laplace equation by p -Laplacian. *Arch. Ration. Mech. Anal.*, 244(2):253–292, 2022.
- [38] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [39] A. Goffi. On the optimal L^q -regularity for viscous Hamilton-Jacobi equations with sub-quadratic growth in the gradient. *Comm. Cont. Math.* 26 (2024), no. 6, 2350019.
- [40] A. Goffi. Hölder regularity and Liouville properties for nonlinear elliptic inequalities with power-growth gradient terms. *Proc. Roy. Soc. Edinburgh Sect. A*, 183 (2023), 1833–1857.
- [41] A. Goffi. Interior Hölder and Calderón-Zygmund estimates for fully nonlinear equations with natural gradient growth. *J. Funct. Anal.*, 288 (2025) 110800.
- [42] A. Goffi and T. Leonori. On maximal regularity estimates for quasilinear evolution equations via the integral Bernstein method. arXiv: 2310.15949 (2023), to appear on Indiana Univ. Math. J..
- [43] A. Goffi and F. Pediconi. *Sobolev regularity for nonlinear Poisson equations with Neumann boundary conditions on Riemannian manifolds*, Forum Math. 35 (2023), no. 2, 431–456
- [44] N. Grenon, F. Murat, and A. Porretta. A priori estimates and existence for elliptic equations with gradient dependent terms. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 13(1):137–205, 2014.
- [45] P. Grisvard. *Elliptic problems in nonsmooth domains*. Monographs and Studies in Mathematics, 24. Pitman, Boston, MA, 1985.
- [46] K. Hansson, V. G. Maz'ya, and I. E. Verbitsky. Criteria of solvability for multidimensional Riccati equations. *Ark. Mat.*, 37(1):87–120, 1999.
- [47] T. Iwaniec. Projections onto gradient fields and L^p -estimates for degenerated elliptic operators. *Studia Math.*, 75(3):293–312, 1983.

- [48] T. Kuusi and G. Mingione. Potential estimates and gradient boundedness for nonlinear parabolic systems. *Rev. Mat. Iberoam.*, 28(2):535–576, 2012.
- [49] T. Kuusi and G. Mingione. Guide to nonlinear potential estimates. *Bull. Math. Sci.*, 4(1):1–82, 2014.
- [50] O. A. Ladyzhenskaya and N. N. Ural'tseva. *Linear and quasilinear elliptic equations*. Academic Press, New York-London, 1968.
- [51] J.-M. Lasry and P.-L. Lions. Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem. *Math. Ann.*, 283(4):583–630, 1989.
- [52] T. Leonori and A. Porretta. Large solutions and gradient bounds for quasilinear elliptic equations. *Comm. Partial Differential Equations*, 41(6):952–998, 2016.
- [53] J. L. Lewis. Regularity of the derivatives of solutions to certain degenerate elliptic equations. *Indiana Univ. Math. J.*, 32(6):849–858, 1983.
- [54] G. M. Lieberman. The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations. *Comm. Partial Differential Equations*, 16(2-3):311–361, 1991.
- [55] G. M. Lieberman. Sharp forms of estimates for subsolutions and supersolutions of quasilinear elliptic equations involving measures. *Comm. Partial Differential Equations*, 18(7-8):1191–1212, 1993.
- [56] P.-L. Lions. Résolution de problèmes elliptiques quasilineaires. *Arch. Rational Mech. Anal.*, 74(4):335–353, 1980.
- [57] P.-L. Lions. Quelques remarques sur les problèmes elliptiques quasilineaires du second ordre. *J. Analyse Math.*, 45:234–254, 1985.
- [58] P.-L. Lions. Recorded video of Séminaire de Mathématiques appliquées at Collège de France, available at <https://www.college-de-france.fr/site/pierre-louis-lions/seminar-2014-11-14-11h15.htm>, November 14, 2014.
- [59] P.-L. Lions. On Mean Field Games. Seminar at the conference “Topics in Elliptic and Parabolic PDEs”, Napoli, September 11-12, 2014.
- [60] A. Maugeri, D. K. Palagachev, and L. G. Softova. *Elliptic and parabolic equations with discontinuous coefficients*, volume 109 of *Mathematical Research*. Wiley-VCH Verlag Berlin GmbH, Berlin, 2000.
- [61] G. Mingione. Gradient potential estimates. *J. Eur. Math. Soc. (JEMS)*, 13(2):459–486, 2011.
- [62] G. Mingione and G. Palatucci. Developments and perspectives in nonlinear potential theory. *Nonlinear Anal.*, 194:111452, 17, 2020.
- [63] N. C. Phuc. Nonlinear Muckenhoupt-Wheeden type bounds on reifenberg flat domains, with applications to quasilinear Riccati type equations. *Adv. Math.*, 250:387 – 419, 2014.
- [64] A. Porretta. On the regularity of the total variation minimizers. *Commun. Contemp. Math.*, 23(1):Paper No. 1950082, 17, 2021.
- [65] A. Porretta and S. Segura de León. Nonlinear elliptic equations having a gradient term with natural growth. *J. Math. Pures Appl. (9)*, 85(3):465–492, 2006.
- [66] J. Serrin. The problem of dirichlet for quasilinear elliptic differential equations with many independent variables. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 264(1153):413–496, 1969.
- [67] J. Serrin. Gradient estimates for solutions of nonlinear elliptic and parabolic equations. In *Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971)*, pages 565–601. Academic Press, New York, 1971.
- [68] L. Simon. Interior gradient bounds for non-uniformly elliptic equations. *Indiana Univ. Math. J.*, 25(9):821–855, 1976.
- [69] S. Tsubouchi. Local Lipschitz bounds for solutions to certain singular elliptic equations involving the one-Laplacian. *Calc. Var. Partial Differential Equations*, 60(1):Paper No. 33, 35, 2021.
- [70] K. Uhlenbeck. Regularity for a class of non-linear elliptic systems. *Acta Math.*, 138(3-4):219–240, 1977.
- [71] X.-J. Wang. Interior gradient estimates for mean curvature equations. *Math. Z.*, 228(1):73–81, 1998.

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