

WORPITZKY-COMPATIBLE SETS AND THE FREENESS OF ARRANGEMENTS BETWEEN SHI AND CATALAN

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ABSTRACT. Given an irreducible root system, the Worpitzky-compatible subsets are defined by a geometric property of the alcoves inside the fundamental parallelepiped of the root system. This concept is motivated and mainly understood through a lattice point counting formula concerning the characteristic and Ehrhart quasi-polynomials. In this paper, we show that the Worpitzky-compatibility has a simple combinatorial characterization in terms of roots. As a byproduct, we obtain a complete characterization by means of Worpitzky-compatibility for the freeness of the arrangements interpolating between the extended Shi and Catalan arrangements. This is a completion of the earlier result by Yoshinaga in 2010 which was done for simply-laced root systems.

1. INTRODUCTION

1.1. Background and motivation. Let $V = \mathbb{R}^\ell$ with the standard inner product (\cdot, \cdot) . Let Φ be an irreducible (crystallographic) root system in V . Let $\Delta := \{\alpha_1, \dots, \alpha_\ell\}$ be a set of simple roots of Φ and Φ^+ the positive system associated to Δ . For $n \in \mathbb{Z}$ and $\alpha \in \Phi^+$, define an affine hyperplane $H_\alpha^n := \{x \in V \mid (\alpha, x) = n\}$ in V . For a hyperplane arrangement \mathcal{A} in V , denote by $c\mathcal{A}$ the *cone* of \mathcal{A} (see §2.1).

Definition 1.1. For a nonnegative integer $k \in \mathbb{Z}_{\geq 0}$ and a subset $\Sigma \subseteq \Phi^+$, define the following hyperplane arrangement in V :

$$\mathcal{S}_\Sigma^k = \mathcal{S}_\Sigma^k(\Phi) := \{H_\alpha^n \mid \alpha \in \Phi^+, 1 - k \leq n \leq k\} \cup \{H_\alpha^{-k} \mid \alpha \in \Sigma\}.$$

The subset Σ is called **Shi-free** (resp. **free**) if the cone $c\mathcal{S}_\Sigma^k$ is a free arrangement for every $k > 0$ (resp. for $k = 0$).

Free arrangements are defined formally in Definition 2.1. In brief, an arrangement is called **free** if its *module of logarithmic derivations* is a free module. The (Shi-)freeness of root systems has been a central topic in the study of free arrangements for decades. For simply-laced (type ADE) root systems, a characterization for the Shi-freeness is known due to Yoshinaga [27] (Theorem 1.2). The ultimate goal of this paper is to complete this characterization for all root systems (Theorem 1.8). First let us give more information about the freeness of $c\mathcal{S}_\Sigma^k$.

- (1) Let $k = 0$. When $\Sigma = \Phi^+$, the arrangement $\mathcal{A}_{\Phi^+} := \mathcal{S}_{\Phi^+}^0$ is known as the **Weyl arrangement** of Φ . For arbitrary Σ , $\mathcal{A}_\Sigma := \mathcal{S}_\Sigma^0$ is a subarrangement of \mathcal{A}_{Φ^+} . The Weyl arrangement is a well-known free arrangement, e.g. [18], [15, Theorem 6.60]. If the root system Φ is of type A , then \mathcal{A}_Σ can be identified with a *graphic arrangement* (see §5) whose freeness is completely characterized by *chordal graphs* [23, 8]. Apart from type A , the freeness of arbitrary Σ is unknown in general.

Date: April 10, 2025.

2010 Mathematics Subject Classification. Primary: 52C35, Secondary: 17B22.

Key words and phrases. Free arrangement, root system, extended Shi arrangement, extended Catalan arrangement, ideal, coclosed set, Worpitzky-compatible set.

- (2) Let $k > 0$. When $\Sigma = \emptyset$ and $\Sigma = \Phi^+$, the arrangements $\text{Shi}_\Phi^{[1-k, k]} := \mathcal{S}_\emptyset^k$ and $\text{Cat}_\Phi^{[-k, k]} := \mathcal{S}_{\Phi^+}^k$ are known as the **extended Shi arrangement** and **extended Catalan arrangement**, respectively. Thus the arrangement \mathcal{S}_Σ^k , when Σ varies, can be regarded as an interpolation between the extended Shi and Catalan arrangements. The freeness of $\text{cShi}_\Phi^{[1-k, k]}$ and $\text{cCat}_\Phi^{[-k, k]}$ had been conjectured by Edelman-Reiner [9] until they were affirmatively settled by Yoshinaga [26].
- (3) The most significant class for which the (Shi-)freeness is known to be true for any root system is that of the *ideals*. The **root poset** (Φ^+, \geq) is the poset with partial order defined by $\beta_1 \geq \beta_2$ if $\beta_1 - \beta_2 \in \sum_{i=1}^\ell \mathbb{Z}_{\geq 0} \alpha_i$. A subset $\Sigma \subseteq \Phi^+$ is called an **ideal** if for $\beta_1, \beta_2 \in \Phi^+$, $\beta_1 \geq \beta_2$, $\beta_1 \in \Sigma$ implies $\beta_2 \in \Sigma$. Then for any ideal Σ and $k \geq 0$, the cone $\text{c}\mathcal{S}_\Sigma^k$ is always free. The case $k = 0$ was first partially proved by Sommers-Tymoczko [22] and later completely settled by Abe-Barakat-Cuntz-Hoge-Terao [1]. The case $k > 0$ was done in a follow-up paper of Abe-Terao [3].
- (4) There is another arrangement closely related to \mathcal{S}_Σ^k . Define

$$\mathcal{S}_{-\Sigma}^k := \{H_\alpha^n \mid \alpha \in \Phi^+, 1 - k \leq n \leq k\} \setminus \{H_\alpha^k \mid \alpha \in \Sigma\}.$$

Abe-Terao [3] showed that if $k > 0$, then $\text{c}\mathcal{S}_\Sigma^k$ and $\text{c}\mathcal{S}_{-\Sigma}^k$ share the freeness, i.e. $\text{c}\mathcal{S}_\Sigma^k$ is free if and only if $\text{c}\mathcal{S}_{-\Sigma}^k$ is free (Theorem 5.1). If this occurs for some $k > 0$, then $\mathcal{A}_\Sigma = \mathcal{S}_\Sigma^0$ is also free. Thus the freeness of Σ is a necessary (but not sufficient) condition for its Shi-freeness.

- (5) Towards a search for a full characterization of the Shi-freeness, it is essential to extend the class of ideals. A subset $\Sigma \subseteq \Phi^+$ is called **coclosed** if for any $\alpha \in \Sigma$ and $\beta_1, \beta_2 \in \Phi^+$ such that $\alpha = d_1\beta_1 + d_2\beta_2$ with $d_1, d_2 \in \mathbb{Z}_{>0}$, either $\beta_1 \in \Sigma$ or $\beta_2 \in \Sigma$. It is easy to see that every ideal of a root system is coclosed. For simply-laced root systems, Yoshinaga showed that the coclosedness is the missing piece of a sufficient condition for the Shi-freeness.

Theorem 1.2. [27, Theorem 5.1] *Let Φ be an irreducible root system of type ADE and $\Sigma \subseteq \Phi^+$. Then Σ is Shi-free if and only if Σ is free and coclosed.*

However, the theorem above is not always true for doubly-laced root systems. In this paper, we complete the characterization for every root system by replacing the coclosed sets by a more general concept, the so-called *Worpitzky-compatible* sets due to Ashraf-Tran-Yoshinaga [6]. The appearance of the Worpitzky-compatibility here is interesting and unexpected as this concept has original motivation from a geometric property of alcoves of root system and a lattice point counting problem seemingly unrelated to the freeness.

1.2. The main results. To state the results formally, let us first recall the concept of compatibility. A connected component of $V \setminus \bigcup_{\alpha \in \Phi^+, n \in \mathbb{Z}} H_\alpha^n$ is called an **alcove**. The closure of an alcove is an ℓ -simplex. A **face** of a simplex is the convex hull of any subset of its vertices. A **facet** of an ℓ -simplex is a face that is an $(\ell - 1)$ -simplex. By abuse of notation, when we say a face of an alcove we mean a face of its closure. Let A be an alcove. A **wall** of A is a hyperplane that supports a facet of A . The **ceilings** of A are the walls which do not pass through the origin and have the origin on the same side as A . The **upper closure** A^\diamond of A is the union of A and its facets supported by the ceilings of A . We will often abuse notation and call A^\diamond an upper closed alcove (though it is not an alcove). Let $P^\diamond := \{x \in V \mid 0 < (\alpha_i, x) \leq 1 (1 \leq i \leq \ell)\}$ be the **fundamental parallelepiped**

(of the coweight lattice) of Φ . Then P^\diamond has the following partition:

$$P^\diamond = \bigsqcup_{A: \text{alcove}, A \subseteq P^\diamond} A^\diamond,$$

which is known as the **Worpitzky partition**, e.g. [29, Proposition 2.5], [11, Exercise 4.3].

Definition 1.3. [6, Definition 4.8] A subset $\Sigma \subseteq \Phi^+$ is called **Worpitzky-compatible** in Φ , or **compatible** for short, if for each alcove $A \subseteq P^\diamond$, the intersection $A^\diamond \cap H_\alpha^{n_\alpha}$ of its upper closure A^\diamond and any affine hyperplane $H_\alpha^{n_\alpha}$ for $\alpha \in \Sigma, n_\alpha \in \mathbb{Z}$ is either empty, or contained in a ceiling $H_\beta^{n_\beta}$ of A for some $\beta \in \Sigma, n_\beta \in \mathbb{Z}$. In short, every nonempty intersection can be lifted to a facet intersection.

In particular, the empty set \emptyset and the positive system Φ^+ itself are always compatible. The compatibility was originally defined in order to make a counting formula concerning the *characteristic* and *Ehrhart quasi-polynomials* valid (Theorem 2.4). It is proved that every coclosed subset is compatible [6, Proof of Theorem 4.16] (see also [25, Theorem 6]). Furthermore, when the root system is of type A , the converse of the previous fact is also true and these properties can be characterized by *cocomparability graphs* [25, Theorems 2 and 9].

The first main result in this paper is a characterization of the compatibility by a root theoretic argument and a local property of the compatibility itself. On the one hand, the root theoretic argument demonstrates a more direct combinatorial relationship of the compatibility and coclosedness. On the other hand, the local property gives a key reason why the compatibility appears in the Shi-freeness characterization.

We need a few more notations and definitions. For an arrangement \mathcal{A} in V , denote by $L(\mathcal{A})$ the *intersection poset* of \mathcal{A} . Set $L_p(\mathcal{A}) := \{X \in L(\mathcal{A}) \mid \text{codim}(X) = p\}$ for $0 \leq p \leq \ell$.

Definition & Notation 1.4. Let Φ be an irreducible root system and let $\mathcal{A} := \mathcal{A}_{\Phi^+}$ be the Weyl arrangement of Φ . A subset of Φ is a **root subsystem** if it is a root system in its own right. If $X \in L_p(\mathcal{A})$, then $\Phi_X := \Phi \cap X^\perp$ is a rank p root subsystem (not necessarily irreducible) of Φ . A positive system of Φ_X is taken to be $\Phi_X^+ := \Phi^+ \cap \Phi_X$. Let Δ_X be the set of simple roots of Φ_X associated to Φ_X^+ . For a subset $\Sigma \subseteq \Phi^+$, denote $\Sigma_X := \Sigma \cap \Phi_X^+$. We call Φ_X and Σ_X the **localizations** of Φ and Σ on X , respectively.

Definition 1.5. A subset $\Sigma \subseteq \Phi^+$ is called

- (a) **negatively coclosed** if for any $\alpha \in \Sigma$ and $\beta_1, \beta_2 \in \Phi^+$ such that $\alpha = d_1\beta_1 + d_2\beta_2$ with $d_1, d_2 \in \mathbb{Z}_{>0}$ and $(\beta_1, \beta_2) < 0$, either $\beta_1 \in \Sigma$ or $\beta_2 \in \Sigma$,
- (b) **2-locally compatible** if for any $X \in L_2(\mathcal{A})$ such that Φ_X is irreducible, the localization $\Sigma_X = \Sigma \cap \Phi_X^+$ is compatible in Φ_X ,
- (c) **2-locally simple** if for any $X \in L_2(\mathcal{A})$ such that Φ_X is irreducible, either Σ_X contains a simple root of Φ_X (i.e. $\Sigma_X \cap \Delta_X \neq \emptyset$) or $\Sigma_X = \emptyset$.

In the subsequent characterizations, we must distinguish some particular subsets of positive roots in a root system of type G_2 .

Definition 1.6. Given a root system $\Phi = G_2$ with $\Delta = \{\alpha_1, \alpha_2\}$ where α_2 is the unique long simple root, define the following subsets $\Sigma \subseteq \Phi^+$:

- (a) $\Sigma = \{\alpha_2\} \cup S$ with $\emptyset \neq S \subseteq \{2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$.
- (b) $\Sigma = \{\alpha_1, 3\alpha_1 + 2\alpha_2\} \cup S$ with $S \subseteq \{\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$.

We are ready to state our first main result connecting the compatibility, a geometric property of alcoves and the negative coclosedness, a combinatorial property of roots.

Theorem 1.7. *Let Φ be an irreducible root system and $\Sigma \subseteq \Phi^+$. The following are equivalent.*

- (1) Σ is compatible.
- (2) Σ is 2-locally compatible.
- (3) One of the following occurs:
 - (i) If $\Phi \neq G_2$, Σ is negatively coclosed.
 - (ii) If $\Phi = G_2$, Σ is negatively coclosed, or one of the seven exceptions in Definition 1.6(a).

Our second main result is a generalization of Theorem 1.2 to any root system.

Theorem 1.8. *Let Φ be an irreducible root system and $\Sigma \subseteq \Phi^+$. The following are equivalent.*

- (1) Σ is Shi-free.
- (2) Σ is free and 2-locally simple.
- (3) One of the following occurs:
 - (i) If $\Phi \neq G_2$, Σ is compatible and free.
 - (ii) If $\Phi = G_2$, Σ is compatible, or one of the four exceptions in Definition 1.6(b).

We emphasize that the proofs of Theorems 1.7 and 1.8 require only the classification of all rank 2 root systems ($A_1^2, A_2, B_2 = C_2, G_2$), and the fact that given a root system $\Phi \neq G_2$, any rank 2 irreducible root subsystem of Φ is of type A_2 or B_2 .

Remark 1.9. Given a root system Φ , denote by ID, CC, NC, CO, 2-LCO, 2-LS the set of all ideals, coclosed, negatively coclosed, compatible, 2-locally compatible, 2-locally simple sets in Φ , respectively. By the theorems above, we have the following relations between these concepts:

$$\text{ID} \subseteq \text{CC} \subseteq \text{NC} \subseteq \text{CO} = 2\text{-LCO} \subseteq 2\text{-LS}.$$

For any containment relation above, there exists an example that makes it strict. Let us add a few more combinatorial and geometric insights.

- (a) The containment $\text{CC} \subseteq \text{NC}$ (i.e. every coclosed subset is negatively coclosed) is clear from definition. If Φ is simply-laced, then $\text{CC} = \text{NC}$ since any rank 2 irreducible root subsystem of Φ is of type A_2 . Let $\Phi = B_2$ and suppose $\Delta = \{\alpha_1, \alpha_2\}$ with the long simple root α_2 . Then $\Sigma = \{2\alpha_1 + \alpha_2, \alpha_2\}$ is the unique subset of Φ^+ such that $\Sigma \in \text{NC} \setminus \text{CC}$. The reason is that although $2\alpha_1 + \alpha_2 = \alpha_1 + (\alpha_1 + \alpha_2)$, the negative coclosedness does not require α_1 or $\alpha_1 + \alpha_2$ to be in Σ since these roots are orthogonal.
- (b) Let $\Phi = A_2$ or B_2 with $\Delta = \{\alpha_1, \alpha_2\}$. Then

$$\text{NC} = \text{CO} = 2\text{-LCO} = 2\text{-LS}.$$

The second equality is obvious. The compatibility in these cases can be easily verified by two dimensional pictures. For a positive root $\alpha = \sum_{i=1}^{\ell} d_i \alpha_i \in \Phi^+$, the **height** of α is defined by $\text{ht}(\alpha) := \sum_{i=1}^{\ell} d_i$. Let $\{\varpi_1^\vee, \dots, \varpi_\ell^\vee\}$ be the dual basis of Δ , namely, $(\alpha_i, \varpi_j^\vee) = 1$ if $i = j$ and 0 otherwise. Then $H_\alpha^{\text{ht}(\alpha)} \cap P^\diamond = \left\{ \sum_{i=1}^{\ell} \varpi_i^\vee \right\}$ for any $\alpha \in \Phi^+$. In type A_2 or B_2 , an affine hyperplane orthogonal to a positive root intersects an upper closed alcove inside P^\diamond at a non-facet intersection only if the intersection is the point $v = \varpi_1^\vee + \varpi_2^\vee$. The point v is a vertex of the alcove “furthest away” from the origin, i.e. the alcove with ceilings $H_{\alpha_1}^1, H_{\alpha_2}^1$. By the preceding calculation, v is contained in every hyperplane of the form $H_\alpha^{\text{ht}(\alpha)}$ for $\alpha \in \Phi^+$. See Figure 1 for an illustration in type A_2 (and [29, Figure 2] for type B_2).

Thus a subset $\Sigma \subseteq \Phi^+$ is compatible if and only either Σ is empty or Σ contains a simple root, i.e. Σ is 2-locally simple. This property also characterizes the negative coclosedness. Hence these concepts must be the same when $\Phi = A_2$ or B_2 .

- (c) In general, given a subset $\Sigma \subseteq \Phi^+$, it is very difficult to check whether Σ is compatible or not by using the primary definition of compatibility or the characterization by quasi-polynomials in Theorem 2.4. The characterization of the compatibility by negative coclosedness in our Theorem 1.7, however, gives a very simple and effective way to do so. See 4.3 for an example.

2. PRELIMINARIES

2.1. Free arrangements. We begin by recalling some basic concepts and preliminary results of free arrangements. Our standard reference is [15]. Let \mathbb{K} be a field and let $V = \mathbb{K}^\ell$. A **hyperplane** in V is an affine subspace of codimension 1 of V . An **arrangement** is a finite collection of hyperplanes in V . An arrangement is called **central** if every hyperplane in it passes through the origin.

Let \mathcal{A} be an arrangement. Define the **intersection poset** $L(\mathcal{A})$ of \mathcal{A} by

$$L(\mathcal{A}) := \left\{ \bigcap_{H \in \mathcal{B}} H \neq \emptyset \mid \mathcal{B} \subseteq \mathcal{A} \right\},$$

where the partial order is given by reverse inclusion: $X \leq Y$ if $Y \subseteq X$ for $X, Y \in L(\mathcal{A})$. We agree that V is a unique minimal element in $L(\mathcal{A})$ as the intersection over the empty set.

For each $X \in L(\mathcal{A})$, define the **localization** of \mathcal{A} on X by

$$\mathcal{A}_X := \{K \in \mathcal{A} \mid X \subseteq K\} \subseteq \mathcal{A},$$

and the **restriction** \mathcal{A}^X of \mathcal{A} to X by

$$\mathcal{A}^X := \{K \cap X \neq \emptyset \mid K \in \mathcal{A} \setminus \mathcal{A}_X\}.$$

Let $\{x_1, \dots, x_\ell\}$ be a basis for the dual space V^* and let $S := \mathbb{K}[x_1, \dots, x_\ell]$. The **defining polynomial** $Q(\mathcal{A})$ of \mathcal{A} is given by

$$Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \in S,$$

where $\alpha_H = a_1x_1 + \dots + a_\ell x_\ell + d$ ($a_i, d \in \mathbb{K}$) satisfies $H = \ker \alpha_H$.

The **cone** $\mathbf{c}\mathcal{A}$ of \mathcal{A} is the central arrangement in $\mathbb{K}^{\ell+1}$ with the defining polynomial

$$Q(\mathbf{c}\mathcal{A}) := z \prod_{H \in \mathcal{A}} {}^h\alpha_H \in \mathbb{K}[x_1, \dots, x_\ell, z],$$

where ${}^h\alpha_H := a_1x_1 + \dots + a_\ell x_\ell + dz$ is the homogenization of α_H , and $z = 0$ is the hyperplane at infinity.

A \mathbb{K} -linear map $\theta : S \rightarrow S$ which satisfies $\theta(fg) = \theta(f)g + f\theta(g)$ is called a **derivation**. Let $\text{Der}(S)$ be the set of all derivations of S . It is a free S -module with a basis $\{\partial/\partial x_1, \dots, \partial/\partial x_\ell\}$ consisting of the usual partial derivatives.

Definition 2.1. [15, Definitions 4.5 and 4.15] Let \mathcal{A} be a central arrangement in $V = \mathbb{K}^\ell$. The **module** $D(\mathcal{A})$ of **logarithmic derivations** is defined by

$$D(\mathcal{A}) := \{\theta \in \text{Der}(S) \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A}\}.$$

We say that \mathcal{A} is **free** if the module $D(\mathcal{A})$ is a free S -module.

The freeness can be extended to a more general class of *multiarrangements*. A **multiarrangement** is a pair (\mathcal{A}, m) where \mathcal{A} is a central arrangement in \mathbb{K}^ℓ and m is a map $m : \mathcal{A} \rightarrow \mathbb{Z}_{\geq 0}$, called **multiplicity**. Let (\mathcal{A}, m) be a multiarrangement. The defining polynomial $Q(\mathcal{A}, m)$ of (\mathcal{A}, m) is given by

$$Q(\mathcal{A}, m) := \prod_{H \in \mathcal{A}} \alpha_H^{m(H)} \in S.$$

When $m(H) = 1$ for every $H \in \mathcal{A}$, (\mathcal{A}, m) is simply a hyperplane arrangement. The **module** $D(\mathcal{A}, m)$ of **logarithmic derivations** of (\mathcal{A}, m) is defined by

$$D(\mathcal{A}, m) := \{\theta \in \text{Der}(S) \mid \theta(\alpha_H) \in \alpha_H^{m(H)} S \text{ for all } H \in \mathcal{A}\}.$$

We say that (\mathcal{A}, m) is **free** if $D(\mathcal{A}, m)$ is a free S -module. It is known that (\mathcal{A}, m) is always free for $\ell \leq 2$ [30, Corollary 7].

Let $H \in \mathcal{A}$. The **Ziegler restriction** (\mathcal{A}^H, m^H) of \mathcal{A} onto H is a multiarrangement defined by

$$m^H(X) := |\mathcal{A}_X| - 1 \quad \text{for } X \in \mathcal{A}^H.$$

We say that \mathcal{A} is **3-locally free along** H if the localization \mathcal{A}_X is free for all $X \in L_3(\mathcal{A})$ with $X \subseteq H$.

Theorem 2.2. [26, Theorem 2.2], [5, Theorem 4.1], [30, Theorem 11] *Let \mathcal{A} be a central arrangement in \mathbb{K}^ℓ with $\ell \geq 3$ and let $H \in \mathcal{A}$. Then \mathcal{A} is free if and only if the Ziegler restriction (\mathcal{A}^H, m^H) is free and \mathcal{A} is 3-locally free along H .*

2.2. Characteristic quasi-polynomials and Worpitzky-compatibility. Next we recall the definition of the *characteristic quasi-polynomial* of an integral arrangement following [12, 13]. The main motivation in [6] for defining the Worpitzky-compatibility is to study this quasi-polynomial for arrangements arising from root systems.

A function $\varphi : \mathbb{Z} \rightarrow \mathbb{C}$ is called a **quasi-polynomial** if there exist a positive integer $\rho \in \mathbb{Z}_{>0}$ and polynomials $f^k(t) \in \mathbb{Q}[t]$ ($1 \leq k \leq \rho$) such that for any $q \in \mathbb{Z}_{>0}$ with $q \equiv k \pmod{\rho}$,

$$\varphi(q) = f^k(q).$$

The number ρ is called a **period**, and the polynomial $f^k(t)$ is called the k -**constituent** of the quasi-polynomial φ .

Let $\ell, n \in \mathbb{Z}_{>0}$ be positive integers. Denote by $\text{Mat}_{\ell \times n}(\mathbb{Z})$ the set of all $\ell \times n$ matrices with integer entries. Let $C = (c_1, \dots, c_n) \in \text{Mat}_{\ell \times n}(\mathbb{Z})$ with no zero columns and let $b = (b_1, \dots, b_n) \in \mathbb{Z}^n$. Set $A := \begin{pmatrix} C \\ b \end{pmatrix} \in \text{Mat}_{(\ell+1) \times n}(\mathbb{Z})$. The matrix A defines the following hyperplane arrangement in \mathbb{R}^ℓ , called **integral arrangement**

$$\mathcal{A} = \mathcal{A}(A) := \{H_j \mid 1 \leq j \leq n\}, \quad \text{where } H_j := \{x \in \mathbb{R}^\ell \mid xc_j = b_j\}.$$

Let $q \in \mathbb{Z}_{>0}$ and $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$. For $a \in \mathbb{Z}$, let $\bar{a} := a + q\mathbb{Z} \in \mathbb{Z}_q$ denote the q -reduction of a . For a matrix or vector A' with integral entries, denote by \bar{A}' the entry-wise q -reduction of A' . The q -**reduction** \mathcal{A}_q of \mathcal{A} is defined by

$$\mathcal{A}_q := \{H_{j,q} \mid 1 \leq j \leq n\}, \quad \text{where } H_{j,q} := \{z \in \mathbb{Z}_q^\ell \mid z\bar{c}_j = \bar{b}_j\}.$$

Denote $\mathbb{Z}_q^\times := \mathbb{Z}_q \setminus \{\bar{0}\}$. The **complement** $\mathcal{M}(\mathcal{A}_q)$ of \mathcal{A}_q is defined by

$$\mathcal{M}(\mathcal{A}_q) := \mathbb{Z}_q^\ell \setminus \bigcup_{j=1}^n H_{j,q} = \{z \in \mathbb{Z}_q^\ell \mid z\bar{C} - \bar{b} \in (\mathbb{Z}_q^\times)^n\}.$$

Theorem 2.3. [12, Theorem 2.4], [13, Theorem 3.1] *There exists a monic quasi-polynomial $\chi_{\mathcal{A}}^{\text{quasi}}(q)$ of degree ℓ such that for sufficiently large q ,*

$$|\mathcal{M}(\mathcal{A}_q)| = \chi_{\mathcal{A}}^{\text{quasi}}(q).$$

*This quasi-polynomial is called the **characteristic quasi-polynomial** of \mathcal{A} .*

The name ‘‘characteristic quasi-polynomial’’ is made by inspiration of the fact that the 1-constituent of $\chi_{\mathcal{A}}^{\text{quasi}}(q)$ coincides with the *characteristic polynomial* (of the intersection poset) of \mathcal{A} [13, Remark 3.3].

Now let Φ be an irreducible root system in \mathbb{R}^ℓ , with a fixed set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ and the associated positive system $\Phi^+ \subseteq \Phi$. For a subset $\Sigma \subseteq \Phi^+$, let C_Σ be the *coefficient matrix* of Σ with respect to Δ , i.e. $C_\Sigma = (C_{ij})$ is the $\ell \times |\Sigma|$ integral matrix that satisfies

$$\Sigma = \left\{ \sum_{i=1}^{\ell} C_{ij} \alpha_i \mid 1 \leq j \leq |\Sigma| \right\}.$$

The central integral arrangement $\mathcal{A}(C_\Sigma)$ defined by C_Σ is *linearly equivalent* to the Weyl subarrangement \mathcal{A}_Σ defined by Σ . Here the linear equivalence means there exists an invertible endomorphism of \mathbb{R}^ℓ that maps the hyperplanes of one onto the hyperplanes of the other.

It is natural to ask for which $\Sigma \subseteq \Phi^+$ the characteristic quasi-polynomial $\chi_{\Sigma}^{\text{quasi}}(q) := \chi_{\mathcal{A}(C_\Sigma)}^{\text{quasi}}(q)$ can be computed by means of some invariants of Φ . The study in [6] shows that a Worpitzky-compatible subset Σ is such an example.

Let $\tilde{\alpha} \in \Phi^+$ denote the highest root, i.e. the unique maximal element in the root poset (Φ^+, \geq) . Then $\tilde{\alpha}$ can be written uniquely as $\tilde{\alpha} = \sum_{i=1}^{\ell} c_i \alpha_i$ with all $c_i \in \mathbb{Z}_{>0}$. Denote $\alpha_0 := -\tilde{\alpha}$ and $c_0 := 1$. The sum $h := \sum_{i=0}^{\ell} c_i$ is called the **Coxeter number** of Φ . Recall that $\{\varpi_1^\vee, \dots, \varpi_\ell^\vee\}$ denotes the dual basis of Δ . The **coweight lattice** of Φ is defined by $Z(\Phi^\vee) := \bigoplus_{i=1}^{\ell} \mathbb{Z} \varpi_i^\vee \simeq \mathbb{Z}^\ell$. Then the fundamental parallelepiped of the coweight lattice is given by

$$P^\diamond = \sum_{i=1}^{\ell} (0, 1]_{\mathbb{R}} \varpi_i^\vee.$$

Let A° be the **fundamental alcove** of Φ , then its closure $\overline{A^\circ} = \text{conv} \left\{ 0, \frac{\varpi_1^\vee}{c_1}, \dots, \frac{\varpi_\ell^\vee}{c_\ell} \right\} \subseteq \overline{P^\diamond}$ can be regarded as a rational polytope in $\bigoplus_{i=1}^{\ell} \mathbb{R} \varpi_i^\vee \simeq \mathbb{R}^\ell$. The counting function

$$L_{\overline{A^\circ}}(q) := |q \overline{A^\circ} \cap Z(\Phi^\vee)|$$

is a quasi-polynomial in q , which is known as the **Ehrhart quasi-polynomial** of $\overline{A^\circ}$ w.r.t. the coweight lattice.

Let W be the *Weyl group* of Φ . For $\Sigma \subseteq \Phi^+$, set $\Sigma^c := \Phi^+ \setminus \Sigma$. The **descent** dsc_Σ **w.r.t.** Σ is a function $\text{dsc}_\Sigma : W \rightarrow \mathbb{Z}_{\geq 0}$ defined by

$$\text{dsc}_\Sigma(w) := \sum_{0 \leq i \leq \ell, w(\alpha_i) \in -\Sigma^c} c_i.$$

Let f be the **index of connection** of Φ . The **\mathcal{A} -Eulerian polynomial** $E_\Sigma(t)$ of Σ is defined by

$$E_\Sigma(t) := \frac{1}{f} \sum_{w \in W} t^{\text{h} - \text{dsc}_\Sigma(w)}.$$

It is proved in [6, Theorem 4.7] that $E_\Sigma(t)$ is a polynomial with all positive integer coefficients.

Theorem 2.4. [6, Theorems 4.11 and 4.24] *Let Φ be an irreducible root system and $\Sigma \subseteq \Phi^+$. Suppose $E_\Sigma(t) = \sum_{i=0}^n a_i t^i$. The following are equivalent.*

- (1) Σ is compatible.
- (2) For every $q \in \mathbb{Z}_{>0}$,

$$\chi_\Sigma^{\text{quasi}}(q) = \sum_{i=0}^n a_i L_{\overline{A^\circ}}(q - i).$$

- (3) The generating function of $\chi_\Sigma^{\text{quasi}}(q)$ is given by

$$\sum_{q \geq 1} \chi_\Sigma^{\text{quasi}}(q) t^q = \frac{E_\Sigma(t)}{\prod_{i=0}^{\ell} (1 - t^{c_i})}.$$

Example 2.5. Let $\Phi = A_2$ with $\Delta = \{\alpha_1, \alpha_2\}$ depicted in Figure 1. The Worpitzky partition of P^\diamond induces a partition of the q -dilation $qP^\diamond \cap Z(\Phi^\vee)$ intersected with the coweight lattice. Let $\Sigma_0 = \emptyset$, $\Sigma_1 = \{\alpha_1, \alpha_1 + \alpha_2\}$ and $\Sigma_2 = \{\alpha_1 + \alpha_2\}$. The empty set is always compatible. By Remark 1.9(b), Σ_1 is compatible while Σ_2 is not. By definition, for all $q > 0$

$$\begin{aligned} \chi_{\Sigma_0}^{\text{quasi}}(q) &= |\mathbb{Z}_q^2| = q^2, \\ \chi_{\Sigma_1}^{\text{quasi}}(q) &= |\{z \in \mathbb{Z}_q^2 \mid z_1, z_1 + z_2 \neq \bar{0}\}| = (q - 1)^2, \\ \chi_{\Sigma_2}^{\text{quasi}}(q) &= |\{z \in \mathbb{Z}_q^2 \mid z_1 + z_2 \neq \bar{0}\}| = q(q - 1). \end{aligned}$$

The fundamental alcove is given by $\overline{A^\circ} = \text{conv}\{0, \varpi_1^\vee, \varpi_2^\vee\}$. Hence its Ehrhart quasi-polynomial is given by $L_{\overline{A^\circ}}(q) = \frac{(q+1)(q+2)}{2}$. Moreover, one may compute the \mathcal{A} -Eulerian polynomials: $E_{\Sigma_0}(t) = t^2 + t$, $E_{\Sigma_1}(t) = t^3 + t^2$ and $E_{\Sigma_2}(t) = t^3 + t$ (by e.g. a graphical method in [25, §3]). Thus

$$\begin{aligned} \chi_{\Sigma_0}^{\text{quasi}}(q) &= L_{\overline{A^\circ}}(q - 2) + L_{\overline{A^\circ}}(q - 1), \\ \chi_{\Sigma_1}^{\text{quasi}}(q) &= L_{\overline{A^\circ}}(q - 3) + L_{\overline{A^\circ}}(q - 2), \\ \chi_{\Sigma_2}^{\text{quasi}}(q) &= L_{\overline{A^\circ}}(q - 3) + L_{\overline{A^\circ}}(q - 1) - 1. \end{aligned}$$

The calculation above is consistent with Theorem 2.4.

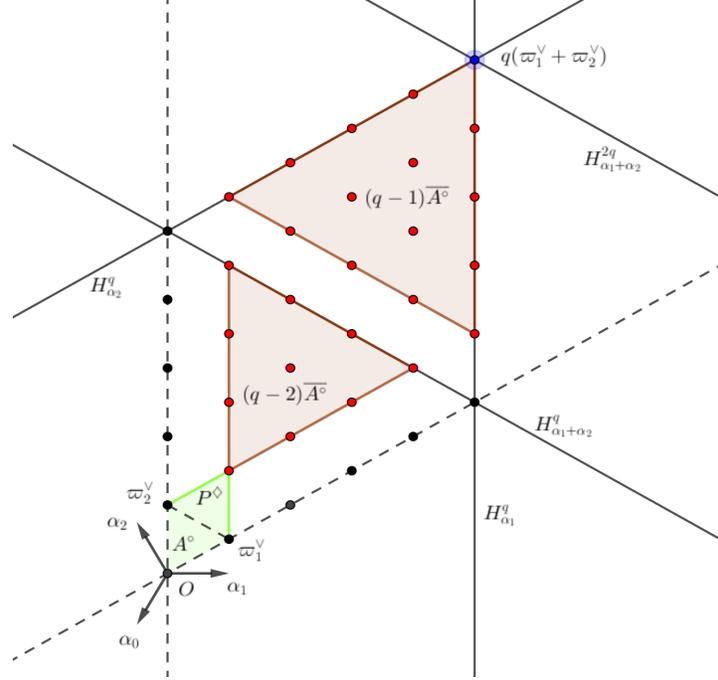
Remark 2.6. It follows from Theorem 2.4 that the compatibility is an essential geometric property for the characteristic quasi-polynomial of a Weyl subarrangement to be expressible in terms of the Ehrhart quasi-polynomial. Such an expression is also important for the study on the characteristic quasi-polynomials of deformed Weyl arrangements.

For instance, by using the Ehrhart theoretic method, Yoshinaga [29, Theorem 5.1] showed that the characteristic quasi-polynomial of the extended Shi arrangement is actually a polynomial. More precisely,

$$\chi_{\text{Shi}_\Phi^{[1-k, k]}}^{\text{quasi}}(q) = (q - kh)^\ell \quad \text{for every } q \in \mathbb{Z}_{>0}.$$

The formula above provides the first example for the *period collapse* phenomenon studied in [10].

The Ehrhart theoretic approach is also used to affirmatively settle the ‘‘Riemann hypothesis’’, a conjecture of Postnikov-Stanley [17] that all roots of the characteristic polynomial of the *extended Linial arrangement* have the same real part [29, 28, 24].


 FIGURE 1. Root system of type A_2 from Example 2.5.

3. PROOF OF THE FIRST MAIN RESULT: THEOREM 1.7

First we recall several known properties of roots.

Lemma 3.1. [16, Lemma 3.1] *Let $\beta \in \Phi^+$ and $\alpha, \alpha' \in \Delta$ with $\alpha \neq \alpha'$. If $\beta - \alpha \in \Phi^+$ and $\beta - \alpha' \in \Phi^+$, then either $\beta = \alpha + \alpha'$ or $\beta - \alpha - \alpha' \in \Phi^+$.*

Using a similar argument as in the proof of Lemma 3.1, one may show the following extension of it. (The case $\Phi = G_2$ should be treated separately.)

Lemma 3.2. *Suppose $\beta, \gamma_1, \gamma_2 \in \Phi^+$ with $\gamma_1 \neq \gamma_2$ and $\gamma_1 - \gamma_2 \notin \Phi$. If $\beta - \gamma_1 \in \Phi^+$ and $\beta - \gamma_2 \in \Phi^+$, then either $\beta = \gamma_1 + \gamma_2$ or $\beta - \gamma_1 - \gamma_2 \in \Phi$.*

Lemma 3.3. [14, Lemma 11.10] *Suppose $\beta_1, \beta_2, \beta_3 \in \Phi$ with $\beta_1 + \beta_2 + \beta_3 \in \Phi$ and $\beta_i + \beta_j \neq 0$ for $i \neq j$. Then at least two of the three partial sums $\beta_i + \beta_j$ belong to Φ .*

Lemma 3.4. [21, Lemma 3.2] *Let $\beta_1 \in \Phi \cup \{0\}$. Suppose that $\beta_i \in \Phi^+$ for $2 \leq i \leq k$ and $\sum_{i=1}^k \beta_i \in \Phi \cup \{0\}$. Then there exists a re-ordering of the β_i 's with $i \geq 2$ so that $\sum_{i=1}^j \beta_i \in \Phi \cup \{0\}$ for all $1 \leq j \leq k$.*

For any alcove A and $\gamma \in \Phi^+$, there exists a unique integer r with $r - 1 < (x, \gamma) < r$ for all $x \in A$. We denote this integer by $r(A, \gamma)$. The map $r(A, -) : \Phi^+ \rightarrow \mathbb{Z}$ is called the **address** of the alcove A .

Lemma 3.5. [19, Theorem 5.2], [7, Lemma 2.4] *Suppose that for each $\gamma \in \Phi^+$ we are given an integer $r(\gamma)$. The map $r : \Phi^+ \rightarrow \mathbb{Z}$ is the address of some alcove A if and only if*

$$r(\gamma) + r(\gamma') - 1 \leq r(\gamma + \gamma') \leq r(\gamma) + r(\gamma') \text{ whenever } \gamma, \gamma', \gamma + \gamma' \in \Phi^+.$$

For a subset $B \subseteq \Phi$, define $\Phi_B(\mathbb{Z}) := \Phi \cap \mathbb{Z}B$. Then $\Phi_B(\mathbb{Z})$ is a root subsystem of Φ . A positive system of $\Phi_B(\mathbb{Z})$ is taken to be $\Phi^+ \cap \Phi_B(\mathbb{Z})$.

Lemma 3.6. [6, Proof of Theorem 4.16] *Let $A \subseteq P^\diamond$ be an alcove inside the fundamental parallelepiped. Suppose there exist $\alpha \in \Phi^+$, $n_\alpha \in \mathbb{Z}$ such that the intersection $A^\diamond \cap H_\alpha^{n_\alpha}$ is a nonempty face of A^\diamond . Let $H_{\beta_1}^{n_{\beta_1}}, \dots, H_{\beta_m}^{n_{\beta_m}}$ for $m \geq 1$ be the (pairwise distinct) ceilings of A that define the intersection, i.e. $A^\diamond \cap H_\alpha^{n_\alpha} = \bigcap_{i=1}^m H_{\beta_i}^{n_{\beta_i}} \cap A^\diamond$. Then $B := \{\beta_i \mid 1 \leq i \leq m\}$ is a set of simple roots of $\Phi_B(\mathbb{Z})$, and $\alpha \in \Phi^+ \cap \Phi_B(\mathbb{Z})$.*

In order to make the proof of Theorem 1.7 more readable, we break it into three lemmas.

Proof of Theorem 1.7. We show (2) \Leftrightarrow (3), (2) \Rightarrow (1) and (1) \Rightarrow (2) in Lemmas 3.7, 3.8 and 3.13, respectively. The implication (1) \Rightarrow (2) is the most difficult part. \square

Lemma 3.7. *A subset $\Sigma \subseteq \Phi^+$ is 2-locally compatible if and only if it is negatively coclosed, or one of the seven exceptions in type G_2 in Definition 1.6(a).*

Proof. We call a subset $\Sigma \subseteq \Phi^+$ **2-locally negatively coclosed** if for any $X \in L_2(\mathcal{A})$, the localization $\Sigma_X = \Sigma \cap \Phi_X^+$ is negatively coclosed in Φ_X . First observe that Σ is negatively coclosed if and only if it is 2-locally negatively coclosed. This is easy to see since the angle between two roots does change after taking localization; hence $(\beta_1, \beta_2) < 0$ in Φ if and only if $(\beta_1, \beta_2) < 0$ in Φ_X .

Suppose $\Phi \neq G_2$. Then any rank 2 irreducible root subsystem Φ' of Φ is of type A_2 or B_2 . By Remark 1.9(b), $\text{NC} = \text{CO}$ in Φ' . Therefore, for any $\Sigma \subseteq \Phi^+$,

$$\Sigma \in \text{NC} \Leftrightarrow \Sigma \text{ is 2-locally negatively coclosed} \Leftrightarrow \Sigma \in \text{2-LCO}.$$

For $\Phi = G_2$, the assertion of Lemma 3.7 is proved by a direct check in Figure 2. The main reason why the exceptional cases exist is that for $\Sigma \in \text{NC}$, if a root of the form $\alpha = d_1\alpha_1 + d_2(\alpha_1 + \alpha_2)$ with $d_1, d_2 \geq 1$ is in Σ , then either α_1 or $\alpha_1 + \alpha_2$ is in Σ since $(\alpha_1, \alpha_1 + \alpha_2) < 0$. In particular, $\{\alpha_2, \alpha\} \notin \text{NC}$. However, $\{\alpha_2, \alpha\}$ is compatible since the hyperplane $H_{\alpha_2}^1$ prevents the other hyperplanes from having a non-facet intersection with an upper closed alcove. \square

Lemma 3.8. *If a subset $\Sigma \subseteq \Phi^+$ is 2-locally compatible, then it is compatible.*

Proof. Let $\Sigma \in \text{2-LCO}$. Suppose that there exist an alcove $A \subseteq P^\diamond$ and a hyperplane $H_\alpha^{n_\alpha}$ for $\alpha \in \Sigma$, $n_\alpha \in \mathbb{Z}$ such that the intersection $A^\diamond \cap H_\alpha^{n_\alpha}$ is nonempty and a non-facet of A^\diamond . Then we may write $A^\diamond \cap H_\alpha^{n_\alpha} = \bigcap_{i=1}^m H_{\beta_i}^{n_{\beta_i}} \cap A^\diamond$ for $m \geq 2$, $\beta_i \in \Phi^+$, $n_{\beta_i} \in \mathbb{Z}$ and $H_{\beta_i}^{n_{\beta_i}}$'s are the ceilings of A . The assertion is proved once we show that $\beta_i \in \Sigma$ for some i . Note that by Lemma 3.6, $B := \{\beta_i \mid 1 \leq i \leq m\}$ is a set of simple roots of $\Phi_B(\mathbb{Z})$, and $\alpha \in \Phi^+ \cap \Phi_B(\mathbb{Z})$. In particular, $\alpha = \sum_{i=1}^m d_i\beta_i$ with at least two $d_{i_1} \geq 1$, $d_{i_2} \geq 1$. Hence by Lemma 3.4, there exists $1 \leq k \leq m$ such that $\gamma := \alpha - \beta_k \in \Phi^+$.

For non-triviality we may assume that the rank of Φ is at least 3. Then any rank 2 irreducible root subsystem of Φ is of type A_2 or B_2 . Let $X := H_\gamma \cap H_{\beta_k} \in L_2(\mathcal{A})$. Then Φ_X is a rank 2 irreducible root subsystem of Φ of type A_2 or B_2 containing $\{\alpha, \beta_k, \gamma\}$. Since $\Sigma \in \text{2-LCO}$, the localization Σ_X is compatible in Φ_X . By Remark 1.9(b), Σ_X must contain a simple root of Φ_X , i.e. $\Sigma_X \cap \Delta_X \neq \emptyset$. Since $\alpha = \beta_k + \gamma$, either $\beta_k \in \Delta_X$ or $\gamma \in \Delta_X$. If $\beta_k \in \Sigma_X$, then $\beta_k \in \Sigma$ and we are done. If $\gamma \in \Sigma_X$, then we may conclude the proof by repeating the argument above for $\Sigma \ni \gamma = (d_k - 1)\beta_k + \sum_{i \neq k} d_i\beta_i$ in place of α . If $\beta_k, \gamma \notin \Sigma_X$, then it must happen that $\Phi_X = B_2$, α is the highest root in Φ_X^+ , and either $\alpha - 2\beta_k \in \Sigma_X$ or $\alpha - 2\gamma \in \Sigma_X$. The latter cannot happen because otherwise, $\alpha - 2\gamma = (2 - d_k)\beta_k - \sum_{i \neq k} d_i\beta_i \in \Phi^+ \cap \Phi_B(\mathbb{Z})$. This implies that $\alpha - 2\gamma$ is a positive root in $\Phi_B(\mathbb{Z})$ but this is a contradiction since there exists d_i with $i \neq k$ such that $d_i \geq 1$. Thus $\gamma' := \alpha - 2\beta_k \in \Sigma_X$, and we may conclude the proof by repeating the argument above for $\gamma' = (d_k - 2)\beta_k + \sum_{i \neq k} d_i\beta_i$ in place of α . \square

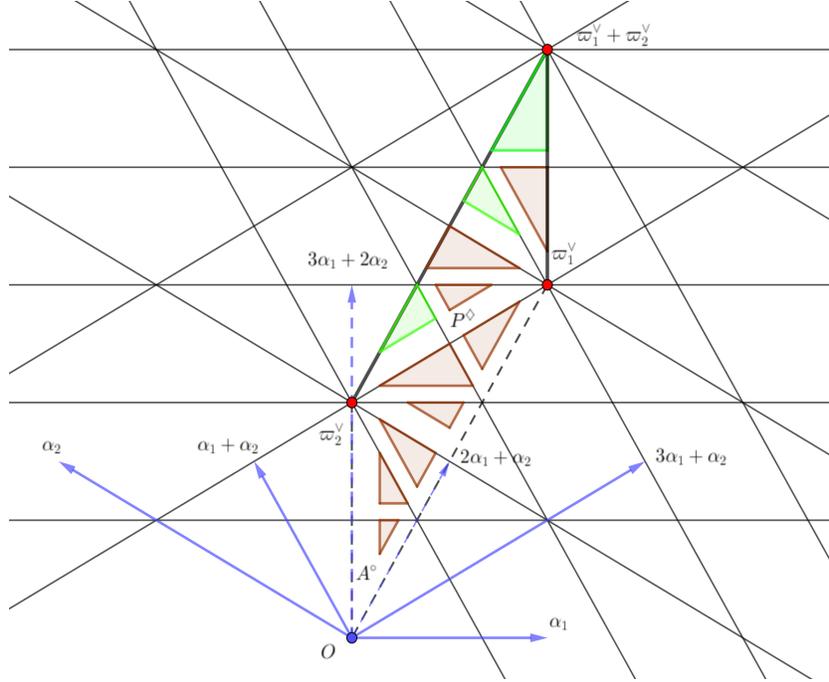


FIGURE 2. The Worpitzky partition of the fundamental parallelepiped P^\diamond in type G_2 . A non-facet intersection between an affine hyperplane and an upper closed alcove occurs only at the alcoves in green.

The following technical property of roots is the key ingredient in the proof of the implication (1) \Rightarrow (2) in Theorem 1.7.

Lemma 3.9. *Let Φ be an irreducible root system. Let $X \in L_2(\mathcal{A}_{\Phi^+})$ and suppose that the localization Φ_X is irreducible. Denote $\Delta_X = \{\gamma_1, \gamma_2\}$ for distinct $\gamma_1, \gamma_2 \in \Phi^+$. Define a map $r = r_X : \Phi^+ \rightarrow \mathbb{Z}$ inductively on height of positive roots as follows:*

- (i) $r(\beta) = 1$ if $\beta \in \Delta$ or $\beta \leq \gamma_1$ or $\beta \leq \gamma_2$.
- (ii) For $\beta \notin \Delta$, $\beta \not\leq \gamma_1$, and $\beta \not\leq \gamma_2$, $r(\beta) = r(\beta - \gamma_i) + 1$ if $\beta - \gamma_i \in \Phi^+$ for $i = 1$ or 2 .
- (iii) Otherwise, $r(\beta) = \max\{r(\beta - \alpha) \mid \alpha \in \Delta \text{ is a simple root such that } \beta - \alpha \in \Phi^+\}$.

Then $r : \Phi^+ \rightarrow \mathbb{Z}$ is the address of some alcove A .

Proof. First we show that the map r is indeed well-defined, i.e. $r(\beta)$ is uniquely determined for every $\beta \in \Phi^+$. We argue by an induction on the height $\text{ht}(\beta)$ of β . The case $\beta \in \Delta$ is clear. For $\beta \notin \Delta$, it suffices to show if $\beta - \gamma_1 \in \Phi^+$ and $\beta - \gamma_2 \in \Phi^+$, then $r(\beta - \gamma_1) = r(\beta - \gamma_2)$. By Lemma 3.2, $\delta := \beta - \gamma_1 - \gamma_2 \in \Phi \cup \{0\}$. If $\delta \in \Phi^- \cup \{0\}$, then $r(\beta - \gamma_1) = r(\beta - \gamma_2) = 1$ by condition 3.9(i). If $\delta \in \Phi^+$, then $r(\beta - \gamma_1) = r(\beta - \gamma_2) = r(\delta) + 1$ by condition 3.9(ii). Since $r(\delta)$, $r(\beta - \gamma_1)$, $r(\beta - \gamma_2)$ are uniquely determined by the induction hypothesis, the conclusion follows.

To show r is the address of an alcove, we use Lemma 3.5. It suffices to show for any $\beta \in \Phi^+$ and any choice of $\gamma, \gamma' \in \Phi^+$ such that $\beta = \gamma + \gamma'$, the following inequalities hold:

$$(3.1) \quad r(\gamma) + r(\gamma') - 1 \leq r(\beta) \leq r(\gamma) + r(\gamma').$$

We argue by an induction on $\text{ht}(\beta)$. The assertion is clear if $\beta \in \Delta$ or $\beta \leq \gamma_1$ or $\beta \leq \gamma_2$. Assume $\beta \notin \Delta$, $\beta \not\leq \gamma_1$, $\beta \not\leq \gamma_2$ and let $\gamma, \gamma' \in \Phi^+$ be such that $\beta = \gamma + \gamma'$. Our induction hypothesis is that (3.1) holds true for every $\delta \in \Phi^+$ with $\text{ht}(\delta) < \text{ht}(\beta)$.

Case 1. First consider the case $\beta - \gamma_i \in \Phi^+$ for $i = 1$ or 2 . Fix such i and write $\beta - \gamma_i = \gamma + \gamma' - \gamma_i$. If $\gamma = \gamma_i$ or $\gamma' = \gamma_i$, then (3.1) holds true trivially. We may assume $\gamma \neq \gamma_i$ and $\gamma' \neq \gamma_i$. By Lemma 3.3, either $\gamma - \gamma_i \in \Phi$ or $\gamma' - \gamma_i \in \Phi$. Without loss of generality, assume $\gamma - \gamma_i \in \Phi$. If $\gamma - \gamma_i$ is a positive root, then by applying the induction hypothesis to $\beta - \gamma_i = (\gamma - \gamma_i) + \gamma'$ we obtain

$$r(\gamma - \gamma_i) + r(\gamma') - 1 \leq r(\beta - \gamma_i) \leq r(\gamma - \gamma_i) + r(\gamma').$$

This is equivalent to (3.1) and we are done. If $\gamma - \gamma_i$ is a negative root, then $r(\gamma_i - \gamma) = r(\gamma) = 1$ since $\gamma, \gamma_i - \gamma \leq \gamma_i$. Now apply the induction hypothesis to $\gamma' = (\beta - \gamma_i) + (\gamma_i - \gamma)$ to obtain (3.1).

Before going to the next case, let us address an observation.

Observation 3.10. Let $\sigma \in \Phi^+$ be a positive root covered by β , i.e. $\beta - \sigma \in \Delta$. Then for any choice of $\gamma, \gamma' \in \Phi^+$ such that $\beta = \gamma + \gamma'$, we have the following estimation for $r(\sigma)$:

$$r(\gamma) + r(\gamma') - 2 \leq r(\sigma) \leq r(\gamma) + r(\gamma').$$

In particular, the estimation above holds true for $r(\beta)$ if $r(\beta) = r(\sigma)$ for some root σ covered by β .

Proof of Observation 3.10. Write $\alpha := \beta - \sigma \in \Delta$. If $\gamma = \alpha$ or $\gamma' = \alpha$, then the observation holds true trivially. We may assume $\gamma \neq \alpha$ and $\gamma' \neq \alpha$. Apply Lemma 3.3 to $\sigma = \gamma + \gamma' - \alpha \in \Phi^+$ to obtain either $\gamma - \alpha \in \Phi^+$ or $\gamma' - \alpha \in \Phi^+$. Without loss of generality, assume $\gamma - \alpha \in \Phi^+$. We apply the induction hypothesis to σ and γ for the expressions $\sigma = (\gamma - \alpha) + \gamma'$ and $\gamma = (\gamma - \alpha) + \alpha$, and we obtain the desired inequalities. \square

Case 2. It remains to consider $\beta - \gamma_i \notin \Phi^+$ for $i = 1$ and 2 . By condition 3.9(iii), $r(\sigma) \leq r(\beta)$ for any root σ covered by β , and we may choose $\alpha_1 \in \Delta$ such that $\sigma_1 := \beta - \alpha_1 \in \Phi^+$ with $r(\beta) = r(\sigma_1)$. By Observation 3.10,

$$r(\gamma) + r(\gamma') - 2 \leq r(\beta) = r(\sigma_1) \leq r(\gamma) + r(\gamma').$$

In particular, the upper bound of $r(\beta)$ in (3.1) follows.

Suppose to the contrary that the lower bound does not hold, i.e. $r(\beta) = r(\gamma) + r(\gamma') - 2$. Again by Observation 3.10, if σ is any root covered by β , then

$$r(\beta) = r(\sigma) = r(\gamma) + r(\gamma') - 2.$$

This implies that $\gamma \neq \alpha_1$ and $\gamma' \neq \alpha_1$. Apply Lemma 3.3 to $\sigma_1 = \gamma + \gamma' - \alpha_1 \in \Phi^+$ to obtain either $\gamma - \alpha_1 \in \Phi^+$ or $\gamma' - \alpha_1 \in \Phi^+$. Without loss of generality, assume $\gamma - \alpha_1 \in \Phi^+$. Now apply the induction hypothesis to $\sigma_1 = (\gamma - \alpha_1) + \gamma'$ and $\gamma = (\gamma - \alpha_1) + \alpha_1$ to obtain

$$r(\gamma) = r(\gamma - \alpha_1) + 1.$$

Let us recollect our assumptions and show the following claim. We will find a contradiction after applying the claim repeatedly.

Claim 3.11. Recall from the above that $r(\beta) = r(\sigma)$ for any root σ covered by β . An ordered pair $\{\gamma, \gamma'\}$ of positive roots is called **bad** for β if $\beta = \gamma + \gamma'$, $r(\beta) = r(\gamma) + r(\gamma') - 2$, and there is $\alpha_1 \in \Delta$ such that $\gamma - \alpha_1 \in \Phi^+$ with $r(\gamma) = r(\gamma - \alpha_1) + 1$. If there exists a bad ordered pair $\{\gamma, \gamma'\}$ for β , then there exists another bad ordered pair $\{\mu, \mu'\}$ for β with $\mu < \gamma$.

Proof of Claim 3.11. We consider two cases.

Subcase 1. $\gamma - \gamma_i \notin \Phi^+$ for $i = 1$ and 2 . By condition 3.9(iii), we may choose $\alpha_2 \in \Delta$ such that $\mu := \gamma - \alpha_2 \in \Phi^+$ and $r(\gamma) = r(\mu)$. In particular, $\alpha_1 \neq \alpha_2$ since $r(\gamma - \alpha_2) \neq r(\gamma - \alpha_1)$. It cannot happen that $\gamma = \alpha_1 + \alpha_2$; otherwise, $r(\gamma)$ has both values 1 and 2, which is absurd. By Lemma 3.1, $\mu - \alpha_1 = \gamma - \alpha_1 - \alpha_2 \in \Phi^+$. By the induction hypothesis,

$$r(\gamma) - 1 = r(\gamma - \alpha_1) \geq r(\mu - \alpha_1) \geq r(\mu) - 1.$$

This follows that

$$r(\mu - \alpha_1) = r(\mu) - 1.$$

See Figure 3 on the left for an illustration of these roots in the root poset.

Consider the expression $\beta = \mu + \gamma' + \alpha_2 \in \Phi^+$. If $\beta - \alpha_2 = \mu + \gamma' \in \Phi^+$, then by applying the induction hypothesis to $\beta - \alpha_2$ we have that

$$r(\beta) = r(\beta - \alpha_2) \geq r(\mu) + r(\gamma') - 1 = r(\gamma) + r(\gamma') - 1.$$

This is a contradiction. Hence $\beta - \alpha_2 \notin \Phi^+$, and Lemma 3.3 forces $\mu' := \gamma' + \alpha_2 \in \Phi^+$. Use the lower bound of $r(\beta)$ in Observation 3.10 for the expression $\beta = \mu + \mu'$ to obtain

$$r(\gamma) + r(\gamma') - 2 = r(\beta) \geq r(\mu) + r(\mu') - 2.$$

Thus $r(\gamma') \geq r(\mu')$. However, $r(\gamma') \leq r(\mu')$ by the induction hypothesis. Therefore, $r(\gamma') = r(\mu')$. This follows that $\{\mu, \mu'\}$ is a bad ordered pair for β with $\mu < \gamma$ we wanted to find.

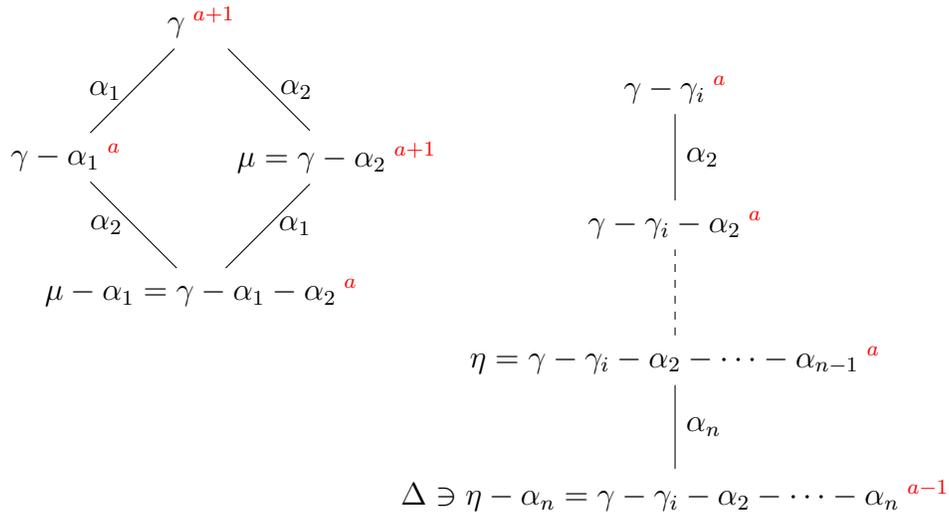


FIGURE 3. A pictorial illustration of the proof of Claim 3.11. Each root α in the root poset is written next to the evaluation $r(\alpha)$ of the map r . The illustration for **Subcase 1** is on the left, the illustration for **Subcase 2** is the entire figure.

Subcase 2. $\gamma - \gamma_i \in \Phi^+$ for $i = 1$ or 2 . Fix such i and write $\beta = (\gamma - \gamma_i) + \gamma' + \gamma_i$. Since $\beta - \gamma_i \notin \Phi^+$, by Lemma 3.3, $\gamma' + \gamma_i \in \Phi^+$. Suppose $\gamma - \gamma_i \in \Delta$. Then $r(\gamma) = 2$ and $r(\gamma') = r(\beta)$. However, $r(\beta) = r(\gamma' + \gamma_i) = r(\gamma') + 1$ since $\gamma' + \gamma_i$ is covered by β . This is a contradiction. Thus we may assume $\gamma - \gamma_i \in \Phi^+ \setminus \Delta$. In particular, $\text{ht}(\gamma) > 2$.

If there exists $\alpha \in \Delta$ such that $\gamma - \gamma_i - \alpha \in \Phi^+$ and $r(\gamma - \gamma_i - \alpha) = r(\gamma) - 2$, then we may choose $\{\gamma - \gamma_i, \gamma' + \gamma_i\}$ as a desired bad ordered pair for β . If not, by the induction hypothesis for any $\alpha_2 \in \Delta$ such that $\gamma - \gamma_i - \alpha_2 \in \Phi^+$, we must have $r(\gamma - \gamma_i - \alpha_2) = r(\gamma) - 1$. Fix such an $\alpha_2 \in \Delta$.

If $\beta - \alpha_2 \in \Phi^+$, then by applying the induction hypothesis to $\beta - \alpha_2 = (\gamma - \gamma_i - \alpha_2) + (\gamma' + \gamma_i)$ we obtain a contradiction. Hence $\beta - \alpha_2 \notin \Phi^+$, and Lemma 3.3 forces $\gamma' + \gamma_i + \alpha_2 \in \Phi^+$. Use the lower bound of $r(\beta)$ in Observation 3.10 for the expression $\beta = (\gamma - \gamma_i - \alpha_2) + (\gamma' + \gamma_i + \alpha_2)$ to obtain

$$r(\gamma' + \gamma_i + \alpha_2) = r(\gamma') + 1.$$

By a similar reason that $\gamma - \gamma_i \in \Phi^+ \setminus \Delta$ as above, we also have $\gamma - \gamma_i - \alpha_2 \in \Phi^+ \setminus \Delta$.

Furthermore, $\alpha_1 \neq \alpha_2$; otherwise, $r(\gamma - \gamma_i - \alpha_1) = r(\gamma - \alpha_1) - 1 = r(\gamma) - 2$, which is absurd. Apply Lemma 3.3 to $\gamma - \gamma_i - \alpha_2 \in \Phi^+$ to obtain either $\gamma - \alpha_2 \in \Phi^+$ or $\gamma_i + \alpha_2 \in \Phi^+$. If $\gamma - \alpha_2 \in \Phi^+$, then

$$r(\gamma - \alpha_2) = r(\gamma - \gamma_i - \alpha_2) + 1 = r(\gamma).$$

By Lemma 3.1, $\gamma - \alpha_1 - \alpha_2 \in \Phi^+$ since $\text{ht}(\gamma) > 2$. This leads us to the ‘‘diamond’’ poset on 4 elements on the left of Figure 3. By a similar argument as in **Subcase 1**, we obtain $\gamma' + \alpha_2 \in \Phi^+$ and we may choose $\{\gamma - \alpha_2, \gamma' + \alpha_2\}$ as a desired bad ordered pair for β .

Suppose $\gamma_i + \alpha_2 \in \Phi^+$ and consider the expression $\beta = (\gamma - (\gamma_i + \alpha_2)) + (\gamma' + (\gamma_i + \alpha_2))$. By repeating the arguments in the preceding three paragraphs with $\gamma_i + \alpha_2$ in place of γ_i , we can find either a bad ordered pair fulfilling the requirement in Claim 3.11, or a simple root $\alpha_3 \in \Delta \setminus \{\alpha_1\}$ such that $\beta - \alpha_3 \notin \Phi^+$, $\gamma - \gamma_i - \alpha_2 - \alpha_3 \in \Phi^+$ with $r(\gamma - \gamma_i - \alpha_2 - \alpha_3) = r(\gamma) - 1$, $\gamma' + \gamma_i + \alpha_2 + \alpha_3 \in \Phi^+$ with $r(\gamma' + \gamma_i + \alpha_2 + \alpha_3) = r(\gamma') + 1$, and $\gamma_i + \alpha_2 + \alpha_3 \in \Phi^+$. The proof for this fact runs along the lines of the preceding paragraphs. The only places we need further technique are to verify $\alpha_1 \neq \alpha_3$, and if $\gamma - \alpha_3 \in \Phi^+$ then $r(\gamma - \alpha_3) = r(\gamma)$. If $\alpha_1 = \alpha_3$, then by the induction hypothesis

$$r(\gamma - \gamma_i - \alpha_2 - \alpha_3) = r(\gamma - \gamma_i - \alpha_2 - \alpha_1) \leq r(\gamma - \alpha_1) - r(\gamma_i + \alpha_2) + 1 = r(\gamma) - 2,$$

which is absurd. If $\gamma - \alpha_3 \in \Phi^+$, then again by the induction hypothesis

$$r(\gamma) \geq r(\gamma - \alpha_3) \geq r(\gamma - \gamma_i - \alpha_2 - \alpha_3) + r(\gamma_i + \alpha_2) - 1 = r(\gamma).$$

Hence $r(\gamma - \alpha_3) = r(\gamma)$.

Repeat for $\gamma_i + \alpha_2 + \alpha_3$ in place of $\gamma_i + \alpha_2$ and so on. This process of finding bad ordered pairs will have to terminate until we find simple roots $\alpha_2, \dots, \alpha_{n-1} \in \Delta \setminus \{\alpha_1\}$ for $n \geq 2$ such that $\beta - \alpha_j \notin \Phi^+$ for $2 \leq j \leq n-1$, $\eta := \gamma - \gamma_i - \alpha_2 - \dots - \alpha_{n-1} \in \Phi^+$ with $\text{ht}(\eta) = 2$ and $r(\eta) = r(\gamma) - 1$, $\eta' := \gamma' + \gamma_i + \alpha_2 + \dots + \alpha_{n-1} \in \Phi^+$ with $r(\eta') = r(\gamma') + 1$. Moreover, there exists $\alpha_n \in \Delta$ such that $\eta - \alpha_n \in \Delta$ and $r(\eta - \alpha_n) = r(\gamma) - 2$. At each step of the process we always find a desired bad ordered pair for β , and $\{\eta, \eta'\}$ is the one in the final step. This concludes the proof of Claim 3.11. □

Now we return to the proof of **Case 2**. By the discussion before Claim 3.11, $\{\gamma, \gamma'\}$ is a bad ordered pair for β . Hence by Claim 3.11, there exist infinitely many mutually distinct bad ordered pairs for β . This is a contradiction since Φ^+ is finite. This completes the proof of Lemma 3.9. □

Corollary 3.12. *Suppose we are in the situation of Lemma 3.9. For $i = 1$ or 2 , define a map $r_i : \Phi^+ \rightarrow \mathbb{Z}$ by $r_i(\gamma_i) = 2$ and $r_i(\beta) = r(\beta)$ for all $\beta \neq \gamma_i$. Then $r_i : \Phi^+ \rightarrow \mathbb{Z}$ is the address of an alcove.*

Proof. We use Lemma 3.5. We need to show for any $\beta \in \Phi^+$ and any choice of $\gamma, \gamma' \in \Phi^+$ such that $\beta = \gamma + \gamma'$, the following holds

$$r_i(\gamma) + r_i(\gamma') - 1 \leq r_i(\beta) \leq r_i(\gamma) + r_i(\gamma').$$

It suffices to show the above when γ_i is involved, i.e. $\gamma_i = \beta$ or $\gamma_i = \gamma$ or $\gamma_i = \gamma'$. The rest is straightforward from the definition of the map r . \square

Lemma 3.13. *If a subset $\Sigma \subseteq \Phi^+$ is compatible, then it is 2-locally compatible.*

Proof. For non-triviality we may assume that the rank of Φ is at least 3. Then any rank 2 irreducible root subsystem of Φ is of type A_2 or B_2 . Let $X \in L_2(\mathcal{A}_{\Phi^+})$ and suppose that Φ_X is irreducible. Denote $\Delta_X = \{\gamma_1, \gamma_2\}$ for distinct $\gamma_1, \gamma_2 \in \Phi^+$. We need to show that the localization $\Sigma_X = \Sigma \cap \Phi_X^+$ is compatible in Φ_X . Since $\Phi_X = A_2$ or B_2 , by Remark 1.9(b), this is equivalent to showing that $\Sigma_X = \emptyset$ or $\Sigma_X \cap \Delta_X \neq \emptyset$. Suppose not, then there exists $\beta \in \Sigma_X$ such that $\beta = d_1\gamma_1 + d_2\gamma_2$ for $d_1, d_2 \in \mathbb{Z}_{>0}$.

Let $r = r_X : \Phi^+ \rightarrow \mathbb{Z}$ be the map defined by X from Lemma 3.9, and $r_i : \Phi^+ \rightarrow \mathbb{Z}$ for $i = 1, 2$ be the map from Corollary 3.12. Fix i . Let A, A_i be the alcoves with addresses r, r_i , respectively. It is easily seen that $A \subseteq P^\diamond$. Comparing the addresses of A and A_i implies that $H_{\gamma_i}^1$ is a wall of each alcove and separates these two. Hence $H_{\gamma_i}^1$ is a ceiling of A for each $i = 1, 2$. We claim that

$$A^\diamond \cap H_{\gamma_1}^1 \cap H_{\gamma_2}^1 = A^\diamond \cap H_\beta^{d_1+d_2}.$$

Since any two facets of a simplex are adjacent, the intersection $R := A^\diamond \cap H_{\gamma_1}^1 \cap H_{\gamma_2}^1$ is a nonempty face of A^\diamond . Moreover, it is contained in the face $Q := A^\diamond \cap H_\beta^{d_1+d_2}$ of A^\diamond by the definition of β . In particular, Q is not empty. Since any proper face of a polytope is the intersection of all facets containing it, if $Q = \bigcap_{\delta \in D} H_\delta^{n_\delta} \cap A^\diamond$ where each $H_\delta^{n_\delta}$ is a ceiling of A^\diamond and D is a set of positive roots, then $D \subseteq \{\gamma_1, \gamma_2\}$ and $n_\delta = 1$ for all $\delta \in D$. Applying Lemma 3.6 to the face Q implies that $D = \{\gamma_1, \gamma_2\}$ and $R = Q$.

Since $\beta \in \Sigma$, by the compatibility of Σ , either $\gamma_1 \in \Sigma$ or $\gamma_2 \in \Sigma$. Hence $\Sigma_X \cap \Delta_X \neq \emptyset$, a contradiction. This completes the proof. \square

4. PROOF OF THE SECOND MAIN RESULT: THEOREM 1.8

First we recall some freeness properties of Weyl arrangements.

Lemma 4.1. [4, Theorem 2] *Let $k \in \mathbb{Z}_{>0}$. Let \mathcal{A} be the Weyl arrangement of an irreducible root system Φ , and $m : \mathcal{A} \rightarrow \{0, 1\}$ be a multiplicity. Then there exists an isomorphism of S -modules*

$$D(\mathcal{A}, m) \rightarrow D(\mathcal{A}, 2k + m).$$

Here $2k + m$ means the multiplicity that sends any $H \in \mathcal{A}$ to $2k + m(H)$.

Lemma 4.2. [3, Claim in the proof of Theorem 1.6] *Let Φ be an irreducible root system of rank 2, and $\Sigma \subseteq \Phi^+$. Then Σ is Shi-free, i.e. the cone \mathbf{cS}_Σ^k is free for every $k > 0$ if and only if $\Sigma = \emptyset$ or $\Sigma \cap \Delta \neq \emptyset$, equivalently, Σ is 2-locally simple.*

We are ready to give the proof of our second main result.

Proof of Theorem 1.8. First we show (1) \Leftrightarrow (2). We need some notations. Let $H_\infty : z = 0$ denote the hyperplane at infinity. Let $\mathcal{A} := \mathcal{A}_{\Phi^+}$ be the Weyl arrangement of Φ . Define a multiplicity $m : \mathcal{A} \rightarrow \{0, 1\}$ by $m(H_\alpha) = 1$ if $\alpha \in \Sigma$ and $m(H_\alpha) = 0$ otherwise. Then the Weyl subarrangement \mathcal{A}_Σ can be identified with the multiarrangement (\mathcal{A}, m) . Moreover, the Ziegler restriction of \mathbf{cS}_Σ^k onto H_∞ can be identified with $D(\mathcal{A}, 2k + m)$. By Lemma 4.1, Σ is free, i.e. \mathcal{A}_Σ is free $\Leftrightarrow D(\mathcal{A}, 2k + m)$ is free.

By Theorem 2.2, it suffices to show that Σ is 2-locally simple $\Leftrightarrow \mathbf{cS}_\Sigma^k$ is 3-locally free along H_∞ , i.e. for any $X \in L_3(\mathbf{cS}_\Sigma^k)$ with $X \subseteq H_\infty$, the localization $(\mathbf{cS}_\Sigma^k)_X$ is free. By [2, Lemma 3.1] for

any such X , there exists $Y \in L_2(\mathcal{A})$ such that $X = Y \cap H_\infty$. Moreover, the relation $X = Y \cap H_\infty$ implies

$$(\mathbf{cS}_\Sigma^k)_X = \mathbf{cS}_{\Sigma_Y}^k(\Phi_Y) \times \emptyset_Y,$$

where \emptyset_Y denotes the empty arrangement in Y . Thus $(\mathbf{cS}_\Sigma^k)_X$ is free $\Leftrightarrow \mathbf{cS}_{\Sigma_Y}^k(\Phi_Y)$ is free.

When Φ_Y is reducible, it is easily seen that $\mathbf{cS}_{\Sigma_Y}^k(\Phi_Y)$ is always free. Otherwise, by Lemma 4.2, it is free if and only if $\Sigma_Y = \emptyset$ or $\Sigma_Y \cap \Delta_Y \neq \emptyset$. Hence if Σ is 2-locally simple, then \mathbf{cS}_Σ^k is 3-locally free along H_∞ . Conversely, suppose that \mathbf{cS}_Σ^k is 3-locally free along H_∞ but Σ is not 2-locally simple. Then there exists $Y \in L_2(\mathcal{A})$ such that Φ_Y is irreducible, $\Sigma_Y \neq \emptyset$ and $\Sigma_Y \cap \Delta_Y = \emptyset$. By Lemma 4.2, $\mathbf{cS}_{\Sigma_Y}^k(\Phi_Y)$ is not free. Define $X := Y \cap H_\infty$. Then by the preceding paragraph, $(\mathbf{cS}_\Sigma^k)_X$ is not free. This contradicts the 3-local freeness along H_∞ of \mathbf{cS}_Σ^k .

Next we show (2) \Leftrightarrow (3). Note that by Theorem 1.7, CO = 2-LCO for any irreducible root system Φ . Suppose $\Phi \neq G_2$. Then any rank 2 irreducible root subsystem Φ' of Φ is of type A_2 or B_2 . By Remark 1.9(b), 2-LCO = 2-LS in Φ' . Therefore, for any $\Sigma \subseteq \Phi^+$,

$$\Sigma \in \text{CO} \Leftrightarrow \Sigma \in \text{2-LCO} \Leftrightarrow \Sigma \in \text{2-LS}.$$

For $\Phi = G_2$, the equivalence (2) \Leftrightarrow (3) is proved by a direct check in Figure 2. The main reason why the exceptional cases exist is that the affine hyperplanes orthogonal to the highest root $3\alpha_1 + 2\alpha_2$ have non-facet intersections with three different upper closed alcoves (the alcoves in green in Figure 2). Moreover, only one of these non-facet intersections can be lifted to a facet intersection by an affine hyperplane orthogonal to a short root. More precisely, the non-facet intersection at the point $\varpi_1^\vee + \varpi_2^\vee$ of the alcove furthest away from the origin can be lifted to a facet supported by the ceiling $H_{\alpha_1}^1$. In particular, $\Sigma = \{\alpha_1, 3\alpha_1 + 2\alpha_2\}$ is not compatible. However, Σ is 2-locally simple since it contains the simple root α_1 . \square

We close this section by giving an example to illustrate the applicability of our Theorems 1.7 and 1.8.

Example 4.3. Let $\Phi = F_4$ with $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Suppose the Dynkin diagram of Φ is given by $\alpha_1 - \alpha_2 \xrightarrow{=} \alpha_3 - \alpha_4$ where α_1, α_2 are the long simple roots. Let $\Sigma_1 = \{\alpha_2, \alpha_2 + 2\alpha_3\}$ and $\Sigma_2 = \{\alpha_2, \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$. We will check the compatibility of Σ_1 and Σ_2 by using Theorem 1.7. In the type F_4 case, CO = NC so it suffices to check the negative coclosedness of these sets. The set Σ_1 is negatively coclosed by the same reason as in the type B_2 example in Remark 1.9(a). However, Σ_2 is not negatively coclosed since $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = (\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4)$ and $(\alpha_1 + \alpha_2, \alpha_3 + \alpha_4) = (\alpha_2, \alpha_3) < 0$ but neither $\alpha_1 + \alpha_2$ nor $\alpha_3 + \alpha_4$ belongs to Σ_2 . As a result, Σ_1 is Shi-free while Σ_2 is not by Theorem 1.8.

The compatibility of Σ_1 and Σ_2 is, however, very complicated to check by using Definition 1.3 or Theorem 2.4. The former requires the information of how the affine hyperplanes orthogonal to the roots in Σ_1 or Σ_2 intersect with the upper closed alcoves in P^\diamond . This is difficult to see in dimension 4 (or higher). The latter requires the calculation on the characteristic, Ehrhart quasi-polynomials and \mathcal{A} -Eulerian polynomial. Computing the \mathcal{A} -Eulerian polynomial is enormous: We need in principle to compute the descent of every Weyl group element (in this case, $|W(F_4)| = 1152$). For this reason, even if we know a subset $\Sigma \subseteq \Phi^+$ is compatible, it is still difficult to compute the characteristic quasi-polynomial of Σ by using the formulas in Theorem 2.4.

5. FURTHER COMMENTS AND REMARKS

In this section we address some further comments and remarks.

- (A) In Theorem 1.8, we discussed the freeness of \mathfrak{cS}_Σ^k and \mathcal{A}_Σ , and showed that they are closely related. For a given free arrangement, there is another important algebraic invariant called the *exponents*. More precisely, when an arrangement \mathcal{B} is free, we may choose a homogeneous basis $\{\theta_1, \dots, \theta_\ell\}$ for $D(\mathcal{B})$. Then the degrees of the θ_i 's are called the **exponents** of \mathcal{B} [15, Definition 4.25]. The exponents of \mathfrak{cS}_Σ^k and \mathcal{A}_Σ are also closely related via the following theorem of Abe-Terao [3].

Theorem 5.1. [3, Theorem 1.6] *Let h denote the Coxeter number of an irreducible root system Φ . Let $k \in \mathbb{Z}_{>0}$ and $\Sigma \subseteq \Phi^+$. Then, \mathfrak{cS}_Σ^k is free with exponents $(1, kh + e_1, \dots, kh + e_\ell)$ if and only if $\mathfrak{cS}_{-\Sigma}^k$ is free with exponents $(1, kh - e_1, \dots, kh - e_\ell)$. In this case, \mathcal{A}_Σ is also free with exponents (e_1, \dots, e_ℓ) .*

When Σ is an ideal of Φ^+ , there exists a nice combinatorial description of the exponents of \mathcal{A}_Σ in terms of the dual partition of the *height distribution* of Σ [1, Theorem 1.1]. It would be interesting to find a combinatorial description of the exponents of \mathcal{A}_Σ when Σ is both free and compatible.

- (B) Given a coclosed subset $\Sigma \subseteq \Phi^+$, Slofstra [20] showed that verifying the freeness of Σ amounts to verifying the freeness of all localizations on the flats of codimension at most 4.

Theorem 5.2. [20, Theorem 3.1] *Let $\Sigma \subseteq \Phi^+$ be a coclosed subset. Then \mathcal{A}_Σ is free if and only if the localization $(\mathcal{A}_\Sigma)_X$ is free for every $X \in L_p(\mathcal{A}_\Sigma)$ of codimension $p \leq 4$.*

We conjecture that the theorem above can be extended to compatible subsets.

Conjecture 5.3. *Let $\Sigma \subseteq \Phi^+$ be a compatible subset. Then \mathcal{A}_Σ is free if and only if the localization $(\mathcal{A}_\Sigma)_X$ is free for every $X \in L_p(\mathcal{A}_\Sigma)$ of codimension $p \leq 4$.*

Suppose Φ is a simply-laced root system. By Remark 1.9(a) and Theorem 1.7, $\text{CC} = \text{NC} = \text{CO}$. Then Conjecture 5.3 is equivalent to Theorem 5.2. For root systems of rank ≤ 4 , there is nothing to be done. Thus the conjecture remains open only when Φ is of type B_ℓ or C_ℓ for $\ell \geq 5$.

- (C) In the case of type A , the Shi-freeness of an arbitrary subset $\Sigma \subseteq \Phi^+$ can be characterized by combinatorial properties of graphs. Let $\{\epsilon_1, \dots, \epsilon_\ell\}$ be an orthonormal basis for $V = \mathbb{R}^\ell$. Define $U := \left\{ \sum_{i=1}^\ell x_i \epsilon_i \in V \mid \sum_{i=1}^\ell x_i = 0 \right\} \simeq \mathbb{R}^{\ell-1}$. The set

$$\Phi(A_{\ell-1}) = \{\pm(\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq \ell\}$$

is a root system of type $A_{\ell-1}$ in U with a positive system

$$\Phi^+(A_{\ell-1}) = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq \ell\}.$$

Let G be a simple graph (i.e. no loops and no multiple edges) with vertex set $V_G = [\ell] := \{1, 2, \dots, \ell\}$ and edge set E_G . Define

$$\Sigma(G) := \{\epsilon_i - \epsilon_j \mid \{i, j\} \in E_G (i < j)\} \subseteq \Phi^+(A_{\ell-1}).$$

Thus any subset of $\Phi^+(A_{\ell-1})$ is completely defined by a simple graph G . The corresponding Weyl subarrangement $\mathcal{A}_{\Sigma(G)}$ is known as the **graphic arrangement**. By Theorem 1.8 and [25, Corollary 15], we have the following graphic characterization for the Shi-freeness of $\Sigma(G)$.

Corollary 5.4. *Let $G = (V_G, E_G)$ be a graph with $|V_G| = \ell$. The following are equivalent.*

- (i) *G has a vertex-labeling using $[\ell]$ so that $\Sigma(G)$ is Shi-free.*
- (ii) *G has a vertex-labeling using $[\ell]$ so that $\Sigma(G)$ is compatible and free.*

(iii) G is an interval graph.

Interval graphs have several different characterizations. Among others, one of the relevant characterizations is that the graph has an ordering $v_1 < \cdots < v_\ell$ of its vertices such that if $i < k < j$ and $\{v_i, v_j\}$ is an edge, then $\{v_i, v_k\}$ is an edge.

Acknowledgements. The first author is partially supported by JSPS KAKENHI grant numbers JP18KK0389 and 23K17298.

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