Invariant Gibbs measure for Anderson nonlinear wave equation

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Abstract

We study the Gaussian measure whose covariance is related to the Anderson Hamiltonian operator, proving that it admits a regular coupling to the (standard) Gaussian free field exploiting the stochastic optimal control formulation of Gibbs measures. Using this coupling, we define the renormalized powers of the Anderson free field and we prove that the associated quartic Gibbs measure is invariant under the flow of a nonlinear wave equation with renormalized cubic nonlinearity.

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1 Introduction

In this paper we want to study the invariant measure of the nonlinear Anderson wave equation, namely we consider the hyperbolic PDE

$$\partial_t^2 u(t,x) + \text{“}H^\omega u(t,x)\text{”} = -\text{“}u^3(t,x)\text{”}, \quad t \in \mathbb{R}, x \in \mathbb{T}^2, \omega \in \Omega \tag{1}$$

where H^ω is the operator usually called *Anderson Hamiltonian* (formally) defined as a singular Schrödinger operator

$$H^\omega u(t,x) \text{“=”} -\Delta u(t,x) + \xi(\omega,x)u(x,t)$$

where $\xi(\cdot,x) : \Omega \to \mathcal{S}'(\mathbb{T}^2)$ is a Gaussian white noise defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and we aim to prove that the (formal) measure

$$\exp\left(-\frac{1}{2}(\varphi, H^\omega(\varphi))\mathrm{d}x - \frac{1}{2}\int_{\mathbb{T}^2} |\partial_t\varphi|^2 \mathrm{d}x - \frac{1}{4}\int_{\mathbb{T}^2} \varphi^4 \mathrm{d}x\right)\mathcal{D}(\varphi, \partial_t\varphi), \tag{2}$$

where $\mathcal{D}(\varphi, \partial_t\varphi)$ is some (non-existent) Lebesgue measure on a function space, is invariant for the flow of equation (1). The above problem combines two different aspects of hyperbolic (stochastic) PDEs: the invariant measure of infinite dimensional dynamical systems, and the dispersive PDEs with singular stochastic potentials. Note that in the absence of the noise ξ,(2) would amount to the well-studied Φ^4_2 measure [8, 29, 34, 53, 62] in the φ integration variable and the white noise Gaussian measure in the $\partial_t\varphi$ variable.

The Φ^4_2 measure whose quadratic part is given by the Anderson Hamiltonian may have applications in the study of suitable continuum limits of Ising models with random bonds (see e.g. [23, 27, 28, 61], see also [34, 35, 52, 62] for the relation between Ising models with Φ^4 measures).

The study of invariant measures of Hamiltonian PDEs is at this point a classical problem in dispersive PDEs, having been studied intensively after the seminal papers by Bourgain from the 90's on nonlinear Schrödinger and KdV equations [15, 16] which itself came after [48] and [1]. The analogue of the current result was shown in [58], i.e. the invariance of the Gibbs measure under the flow of the Wick ordered cubic wave equation. Later this methodology was applied to closely related topics like *almost sure well-posedness* [19, 20] and *quasi-invariance* in settings when there is no invariant measure [59, 39] to name just a few.

Due to the low regularity of the support of the (formal) invariant measure, it is necessary to *Wick order* the cubic nonlinearity in (1) which naively means that we have to subtract a diverging linear counter term to make it well-defined (we can thus think of it as being akin to *renormalization*), hence writing the cube is just formal. This notion is similar to that appearing for example in [16] and [58], where the Wick ordering was done with respect to the Gaussian free field, as the invariant measure is mutually absolutely continuous with respect to it. Our case is somewhat analogous, except that our reference Gaussian measure is not the free field but the Gaussian measure with covariance (formally) given by $(H^\omega)^{-1}$.

The second renormalization needed in equation (1) is in the rigorous definition of the Anderson Hamiltonian operator H^ω. The issue is that the very low regularity of the noise ξ makes it necessary to subtract an infinite correction term to make sense of the product between u and ξ.

The parabolic equation whose linear part is given by the Anderson Hamiltonian, called *parabolic Anderson model or PAM,* has many applications, see e.g. [45] and the references therein. It was studied both in the linear case (see, e.g., [25, 41, 42, 49]) and in its nonlinear generalization (see, e.g., the gPAM equation [21, 24, 36, 40] or the rough super-Brownian motion equation [60]) in two and three dimensions. More recently wave and Schrödinger equations with Anderson Hamiltonian linear part were considered (see, e.g., [2] for the linear case and [32, 37, 55, 65, 64, 66, 70, 71] for the nonlinear one, see also [5, 26, 46, 47, 54, 50, 51] for problems relating to the spectral properties of the Anderson Hamiltonian).

The first paper to construct the Anderson Hamiltonian rigorously as a semi-bounded self-adjoint operator on $L^2(\mathbb{T}^2)$ together with an explicit description of its domain and a bound on the lowest eigenvalue was [2]. In particular, therein it was showed that one can rigorously define it as a limit (taken in the norm resolvent sense) of regularized operators via

$$\mathbb{H}^\omega := \lim_{\varepsilon\to 0}(-\Delta + a_\varepsilon * \xi(\omega) - c_\varepsilon), \tag{3}$$

where $a_\varepsilon(x) := \varepsilon^{-2}a(\varepsilon^{-1}x)$ is a smooth mollifier, and $c_\varepsilon \in \mathbb{R}_+$ is a suitable diverging sequence of real numbers (not depending on ω). This approach was reformulated and extended in [37] to 3 dimensions, where also some associated semilinear SPDEs were solved. This formulation is in fact the approach we follow in this work. Let us also mention the work [47] which deals with the construction of the operator with both periodic and Dirichlet boundary conditions and [54] where the operator is defined on compact surfaces and also a Weyl law is proved for its eigenvalues.

Let us emphasize the fact that for almost every realisation ω one can define $\mathbb{H}^\omega$ as a self-adjoint operator on L^2 which is bounded from below by a constant which of course also depends on ω, i.e.

$$(u, \mathbb{H}^\omega u) \geqslant -K(\omega)\|u\|_{L^2}^2,$$

and one can quantify the (measurable) dependence, indeed it can be chosen as a suitable norm of the enhanced noise, see Proposition 2.9 in [54] for an explicit expression of the constant. Often we will shift the operator by the constant $K(\omega)+1$ and define

$$\mathbb{H}^{\omega,K} := \mathbb{H}^\omega + K(\omega) + 1 \tag{4}$$

so as to make it uniformly positive and we can take fractional and negative powers with impunity.

We consider the regularized equation, the motivation for which will be given in Section 5.

$$\begin{aligned} \partial_t^2 u_\varepsilon(t,x) &= -\mathbb{H}^{\omega,K}u_\varepsilon(t,x) - \rho_\varepsilon * (\rho_\varepsilon * u_\varepsilon)^3(t,x) + (3a_\varepsilon + 1 + K(\omega))\rho_\varepsilon^{*2} * u_\varepsilon(t,x), \\ (u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} &= (u_0, u_1) \end{aligned} \tag{5}$$

where $t \in \mathbb{R}$, $x \in \mathbb{T}^2$, $\rho_\varepsilon : \mathbb{T}^2 \to \mathbb{R}_+$ is a symmetric mollifier and $a_\varepsilon = \sum_{k\in\mathbb{Z}^2} \frac{|\hat{\rho}_\varepsilon(k)|^2}{|k|^2+1} = \mathrm{Tr}_{\mathbb{T}^2}(\rho_\varepsilon * (-\Delta+1)^{-1} * \rho_\varepsilon)$, where the operator $\mathbb{H}^{\omega,K}$ (and the corresponding linear evolution on $L^2(\mathbb{T}^2)$) is defined as in equations (3) and (4).

First we prove a rigorous version of the heuristically defined measure (2).

Proposition 1 *Let $0 \leqslant K(\omega) < +\infty$ be such that $\mathbb{H}^{\omega,K} = \mathbb{H}^{\omega} + K(\omega) + 1$ is a (strictly) positive operator and let $\mu^{\mathbb{H}^{\omega,K}}$ be the law of the Gaussian field with covariance $(\mathbb{H}^{\omega,K})^{-1}$. Then, for almost every $\omega \in \Omega$, the (weak) limit of probability measures*

$$\begin{aligned}\nu^{\omega}(\mathrm{d}\varphi) &:= \lim_{\varepsilon\to 0} \nu^{\omega,\varepsilon}(\mathrm{d}\varphi)\\ &:= \lim_{\varepsilon\to 0} \frac{e^{-\frac{1}{4}\int_{\mathbb{T}^2} |\rho_\varepsilon * \varphi(x)|^4 \mathrm{d}x + \frac{(3a_\varepsilon + K(\omega)+1)}{2}\int_{\mathbb{T}^2} |\rho_\varepsilon * \varphi(x)|^2 \mathrm{d}x}}{\mathcal{Z}_\varepsilon} \mu^{\mathbb{H}^{\omega,K}}(\mathrm{d}\varphi) \otimes \mu^{\mathbb{I}_{L^2}}(\mathrm{d}\partial_t \varphi),\end{aligned}$$

where $\mathcal{Z}_\varepsilon$ is the normalization constant, exists in $\mathcal{P}(H^{-\kappa}(\mathbb{T}^2) \times H^{-1-\kappa}(\mathbb{T}^2))$ for $\kappa > 0$ and $\mu^{\mathbb{I}_{L^2}}$ is the white noise measure.

Having the candidate invariant measure ν^{ω}, we prove that it is invariant with respect to the limit of the flow defined by (5). More precisely:

Theorem 1 *Using the notation of Proposition 1, for almost all $\omega \in \Omega$, there is a set of full measure $\mathcal{A}^{\omega} \subset H^{-\kappa}(\mathbb{T}^2) \times H^{-1-\kappa}(\mathbb{T}^2)$ with respect to $\mu^{\mathbb{H}^{\omega,K}} \otimes \mu^{\mathbb{I}_{L^2}}$, for which equation (5) has a unique global in time solution, leaving the measure $\nu^{\omega,\varepsilon}$ invariant. Furthermore, for any $(u_0, \partial_t u_0) \in \mathcal{A}^{\omega}$, the solution u_ε (5) admits a unique limit defining a one parameter group of maps on $\mathcal{A}^{\omega}$. The measure μ^{ω} (defined as in Proposition 1) is an invariant measure with respect to that one parameter group of maps.*

It is important to note that the counter term a_ε used in the renormalization of the measure $\nu^{\omega,\varepsilon}$ and of the equation (5) is the same one appearing in the renormalization of classical Wick ordered wave equation (see [58]), and thus a_ε is independent of $\omega \in \Omega$. In other words we do not use the Wick renormalization with respect to the Gaussian measure $\mu^{\mathbb{H}^{\omega,K}}$ (as one might expect), whose counter term is

$$\tilde{a}_\varepsilon(\omega, x) := \int (\rho_\varepsilon * \varphi(x))^2 \mu^{\mathbb{H}^{\omega,K}}(\mathrm{d}\varphi),$$

which depends on both $x \in \mathbb{T}^2$ and $\omega \in \Omega$. The possibility of choosing a_ε constant is nontrivial since we are practically saying that we can use the (standard) free field Wick cube in order to define the nonlinearity "u^3" in equation (1), while the Wick power with respect to the free field is not a priori well defined on a set of measure one with respect to $\mu^{\mathbb{H}^{\omega,K}}$ since $\mu^{\mathbb{H}^{\omega,K}}$ are mutually singular for almost all ω.

The reason why we can nonetheless choose a_ε constant is due to the fact that, despite their mutual singularity, we can obtain the Gaussian field with law $\mu^{\mathbb{H}^{\omega,K}}$ as a Gaussian free field translated by a shift which has regularity just below that of the Cameron Martin space. More precisely,if we denote by $\mu^{-\Delta}$ the measure on $H^{-\varepsilon}(\mathbb{T}^2)$ of the Gaussian free field with covariance $(-\Delta + 1)^{-1}$, we obtain the following result.

Theorem 2 *For almost all $\omega \in \Omega$, there is a coupling $\eta^{\omega} \in \mathcal{P}(H^{-\varepsilon}(\mathbb{T}^2) \times H^{-\varepsilon}(\mathbb{T}^2))$ between the Gaussian measures $\mu^{\mathbb{H}^{\omega,K}} \in \mathcal{P}(H^{-\varepsilon}(\mathbb{T}^2))$ and $\mu^{-\Delta} \in \mathcal{P}(H^{-\varepsilon}(\mathbb{T}^2))$ such that if $(\varphi^{\mathbb{H}^{\omega,K}}, \varphi^{-\Delta}) \sim \eta^{\omega}$ then $\varphi^{\mathbb{H}^{\omega}_A} - \varphi^{-\Delta} \in H^{1-\delta}(\mathbb{T}^2)$ almost surely for any $\delta > 0$.*

The proof of Theorem 2 is based on the variational formulation of Gibbs measure first proved by Boué and Dupuis [14] (see also [67, 72]) and then applied first to quantum field theory in [8] and used in [6, 10, 9, 11, 12, 22, 57] for studying measures related to quantum field theory. This technique was already used in the context of invariant measures of PDEs in [17, 18, 56] where the invariant measure considered was singular with respect to the Gaussian free field. Thanks to the decomposition proved in Theorem 2, by writing $\varphi^{\mathbb{H}^{\omega,K}} - \varphi^{-\Delta} = h \in H^{1-\delta}$, we are able to prove that

$$(\varphi^{\mathbb{H}^{\omega,K}})^{\circ 3} := \lim_{\varepsilon\to 0}(\rho_\varepsilon * (\varphi^{\mathbb{H}^{\omega,K}}))^3 - 3\alpha_\varepsilon \rho_\varepsilon * \varphi^{\mathbb{H}^{\omega.K}} =: (\varphi^{-\Delta})^3 : +3 : (\varphi^{-\Delta})^2 : \cdot h + 3\varphi^{-\Delta}\cdot h^2 + h^3$$

holds, where $: (\varphi^{-\Delta})^3 :$ and $: (\varphi^{-\Delta})^2 :$ are the standard Wick powers of Gaussian free field $\varphi^{-\Delta}$, and the products $: (\varphi^{-\Delta})^2 : \cdot h$ and $\varphi^{-\Delta}\cdot h^2$ are well-defined and lie in $H^{-\varepsilon}$. This allows us to write

$$\begin{aligned} \partial_t^2 u(t,x) &= -\mathbb{H}^{\omega,K} u(t,x) - u^{\circ 3}(t,x) \\ (u, \partial_t u)|_{t=0} &= (u_0, u_1) \end{aligned} \tag{6}$$

as a rigorous version of (1) and the limit of (5). We think that Theorem 2 can be of independent interest, since it is useful to understand better the Gaussian measure with covariance $(\mathbb{H}^{\omega,K})^{-1}$ and the related Wick product (see also [5] where the Wick square of the Anderson Gaussian free field appears in the description of the polymer measure). As a byproduct of the proof of Theorem 2, we also obtain a different proof of the existence of the Anderson Hamiltonian operator using only the variational formulation of [8], see Remark 7.

Remark 1 We expect the coupling of Theorem 2 to have the "scale to scale property" which means that on large scales the coupling is independent from the underlying GFF on small scales. In particular $\Delta_i h$ and $\Delta_j \varphi^{-\Delta}$ are independent if $j > i$. In [12, 11] such a coupling was used, to establish that the recentred maximum of the associated log-correlated field (in this case $\mu^{\mathbb{H}^{\omega,K}}$) behaves similarly to the Gaussian free field [33].

The paper is organized as follows: In Section 2.1 we recall the rigorous construction of the operator $\mathbb{H}^\omega$ generally following [37] and collect some salient results about it. Next we provide some results on Gaussian measures on function spaces in Section 2.2. Section 3 is dedicated to the construction of the coupling between $\mu^{-\Delta}$ and $\mu^{\mathbb{H}^{\omega,K}}$, namely the proof of Theorem 2 which allows us to define the Wick powers. Section 4 details the local-in-time well-posedness theory of the Wick ordered Anderson wave equation (6) as well as the convergence of approximations. Finally Section 5 combines the local well-posedness and the Hamiltonian structure of the equation to prove invariance of the Gibbs measure via a Bourgain-type argument. Finally, in Appendix B we give an alternative proof of the coupling following ideas from [5] which works well for Gaussian measures (which is sufficient for our current setting) but is less general than the method from Section 3 which does not assume Gaussianity.

Notation: We frequently use function spaces which are either Lebesgue spaces denoted, as per usual, by L^p i.e. $u \in L^p$ if $\|u\|_{L^p}^p = \int_{\mathbb{T}^2} |u|^p < \infty$ with the usual modification for $p = \infty$ or Sobolev spaces H^σ whose norm is given by $\|v\|_{H^\sigma} = \left\|(1-\Delta)^{\frac{\sigma}{2}} v\right\|_{L^2}$. Moreover, we frequently employ the Besov spaces $B^s_{p,q}$ and the Hölder-Besov spaces $\mathcal{C}^s = B^s_{\infty\infty}$ whose definition together with related concepts is recalled for the reader's convenience in Appendix A. As we are exclusively working on the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ we sometimes write $H^\sigma = H^\sigma(\mathbb{T}^2)$ etc and occasionally, when we want to differentiate between space and time, write $L^p([0,T], H^\sigma)$ for the space of space-time functions $u(t,x)$ with finite norm $\|u\|_{L^p([0,T],H^\sigma)} := \left(\int_0^T \|u(t)\|_{H^\sigma}^p\right)^{\frac{1}{p}} < \infty$ and sometimes we use short-hand notations such as $L^p_t H^\sigma_x$ to differentiate between the variables.

Furthermore, we frequently use the notation $\lesssim$ to mean a bound up to an implicit constant that may change from line to line, relatively we use $\lesssim_\rho$ to mean a bound that may depend on ρ explicitly. Similarly $C, c > 0$ may frequently denote implicit constants which we allow to change from line to line and $C(\alpha)$ may denote a changing constant with a dependence on the quantity α.

As a general rule, we write $\mathbb{E}$ for an expectation and $\mathbb{P}$ for a probability. More concretely we have two different probability spaces, which we denote by $(\Omega, \mathcal{F}, \mathbb{P})$ for the definition of the Anderson Hamiltonian (which we construct for almost every $\omega \in \Omega$) and $(\Omega', \mathcal{F}', \mathbb{P}')$ for other randomnesses appearing after Section 2. On occasion we will use notations like $\mathbb{E}^\omega, \mathbb{E}^{\omega'}$ for expectations w.r.t. those probabilities and $\mathbb{E}_\mu$ as the expectation w.r.t. a probability measure μ etc.

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2 Preliminaries

2.1 Anderson Hamiltonian and Paracontrolled Calculus

In this section we briefly recall the salient properties of the *Anderson Hamiltonian* on $\mathbb{T}^2$ which will be required in the following sections. We will largely follow [37] where the interested reader can also find the details omitted here.

Formally we can see the (continuum) Anderson Hamiltonian on $\mathbb{T}^2$ as a Schrödinger operator with a spatial white noise potential, i.e.

$$H^\omega \text{``=''} - \Delta + \xi(\omega, \cdot), \tag{7}$$

where $\xi : \Omega \to \mathcal{S}'(\mathbb{T}^2)$ (where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathcal{S}'(\mathbb{T}^2)$ is the space of distributions on $\mathbb{T}^2$) is a random distribution satisfying the formal property $\mathbb{E}[\xi(\cdot, x)\xi(\cdot, y)] = \delta(x - y)$, see [37] for a rigorous definition. In particular, the spatial white noise can be written as the following random Fourier series (or Karhunen-Loève expansion)

$$\xi(\omega, x) = \sum_{k \in \mathbb{Z}^2} e_k(x)\xi_k(\omega), \tag{8}$$

where $e_k(x) = e^{2\pi i k \cdot x}$ and the $\xi_k = \overline{\hat{\xi}_{-k}} : \Omega \to \mathbb{R}$ are i.i.d. standard complex Gaussians. Note that the sum in (8) converges at best a.s. in $\mathcal{C}^{-1-\varepsilon}(\mathbb{T}^2)$ for any $\varepsilon > 0$ which means that $\xi(\omega, \cdot) \in \mathcal{C}^{-1-\varepsilon}(\mathbb{T}^2)$ for almost all $\omega \in \Omega$ (sometimes this is suggestively written as $\xi(\omega) \in \mathcal{C}^{-1-}$), see (90) in the appendix for the definition of the Hölder-Besov spaces $\mathcal{C}^\alpha(\mathbb{T}^d)$. Hereafter we sometimes use the notation $\xi(\omega) = \xi(\omega, \cdot) \in \mathcal{S}'(\mathbb{T}^2)$ or $\xi(x) = \xi(\cdot, x)$ when we need to stress the dependence of ξ on $\omega \in \Omega$ or $x \in \mathbb{T}^2$.

Due to the low regularity of ξ, one can not classically make sense of H^ω as an (unbounded) operator on L^2. However, for almost all $\omega \in \Omega$, it is possible to rigorously construct a *renormalized* version $\mathbb{H}^\omega$ of the formal operator H^ω as a self-adjoint, unbounded operator on $L^2(\mathbb{T}^2)$ which is bounded from below by a constant $-K(\omega)$ (which can be chosen $\mathcal{F}$-measurably as a random variable $K : \Omega \to \mathbb{R}_+$). In addition, one can give a domain and a form domain for this renormalized operator. Due to these properties, one can define a functional calculus for the positive self-adjoint operator $\mathbb{H}^{\omega,K} = \mathbb{H}^{\omega} + (K(\omega)+1)\mathbb{I}_{L^2}$, which allows us to define operators like

$$e^{it\mathbb{H}^{\omega,K}}, \sin\left(t\sqrt{\mathbb{H}^{\omega,K}}\right), \cos\left(t\sqrt{\mathbb{H}^{\omega,K}}\right) \text{ for } t \in \mathbb{R}, \tag{9}$$

as bounded operators on $L^2(\mathbb{T}^2)$ which are strongly continuous in time, see Section 3 of [37].

The functional calculus (9) allows us to solve linear wave-/ and Schrödinger equations whose linear part is given by $\mathbb{H}^{\omega}$ as was done in [37] which corresponds to solving the SPDE with a white noise potential (sometimes called *multiplicative stochastic wave/Schrödinger equations*).

In order to make rigorous sense of (7) in $L^2(\mathbb{T}^2)$, we introduce the final definition of the operator and the noise space and then motivate this by a formal derivation. We begin by recalling the correct notion of *noise space* which contains all the needed "higher-order" information on the noise term $\xi(\omega)$.

Definition 1 (and Lemma) *For $\alpha = 1+\varepsilon$ for very small $\varepsilon > 0$ we define the noise space*

$$\mathcal{X}^{\alpha} := \overline{\{(\psi, (1-\Delta)^{-1}\psi \circ \psi - a), \psi \in \mathcal{S}(\mathbb{T}^2), a \in \mathbb{R}\}}|_{\mathcal{C}^{-\alpha}\times\mathcal{C}^{2-2\alpha}}$$

i.e. the closure of tuples of the form $(\psi, (1-\Delta)^{-1}\psi \circ \psi - a)$ w.r.t. the $\mathcal{C}^{-\alpha}\times\mathcal{C}^{2-2\alpha}$-norm for smooth functions $\psi \in \mathcal{S}(\mathbb{T}^2)$ and constants $a \in \mathbb{R}$. See also equation (92) for the definition of the resonant product $\circ$.

For $\xi(\omega)$ the spatial noise as introduced in (8) and $\xi_\varepsilon(x,\omega) = \eta_\varepsilon * \xi(x,\omega)$ a smooth regularization one has that

$$\Xi_\varepsilon = (\xi_\varepsilon(\omega), (1-\Delta)^{-1}\xi_\varepsilon(\omega) \circ \xi_\varepsilon(\omega) - c_\varepsilon) \to \Xi = (\xi(\omega), \Xi^2(\omega)) \text{a.s. in } \mathcal{X}^{\alpha} \text{ as } \varepsilon \to 0 \tag{10}$$

where

$$c_\varepsilon := \mathbb{E}(\xi_\varepsilon(\omega) \circ (1-\Delta)^{-1}\xi_\varepsilon(\omega)) \sim \log\left(\frac{1}{\varepsilon}\right)$$

is a diverging sequence.

Often we drop the ω dependence for brevity when there is no confusion i.e. $\Xi^2 = \Xi^2(\omega)$ etc. It will be important in later sections to track this dependence but it is not pertinent to the discussion in this section.

Proof See Theorem 5.1 in [2]. □

Next we define the space of functions paracontrolled by the enhanced noise $\Xi \in \mathcal{X}^{\alpha}$ which is the limit from (10).

Definition 2 *Let α as above, then we define the paracontrolled (by Ξ) space $\mathcal{D}^{\alpha}_{\Xi} \subset L^2(\mathbb{T}^2)$ as the space of functions u of the form*

$$u - (1-\Delta)^{-1}((\xi + \Xi^2) \succ u + \xi \prec u) =: u^{\sharp} \in H^2. \tag{11}$$

On such functions we define the operator, called (renormalized) Anderson Hamiltonian

$$\mathbb{H}^{\omega}(u) := (1-\Delta)u^{\sharp} + u^{\sharp} \circ \xi + B(u, \Xi), \tag{12}$$

where

$$B(u,\Xi) := \Xi^2 \prec u + \Xi^2 \circ u + C(\xi,(1-\Delta)^{-1}\xi,u) + ((1-\Delta)^{-1}(\Xi^2 \succ u + \xi \prec u)) \circ \xi,$$

see Proposition 14 for the trilinear commutator C. Similarly we set

$$\mathbb{H}^{\omega}_{\varepsilon}(u_\varepsilon) := (1-\Delta)u^\sharp_\varepsilon + u^\sharp_\varepsilon \circ \xi_\varepsilon + B(u_\varepsilon,\Xi_\varepsilon), \tag{13}$$

where the noise Ξ_ε is as in (10) *and u_ε and $u^\sharp_\varepsilon$ are as in* (11) *with Ξ_ε instead of Ξ.*

Now we give a formal derivation of the form of the paracontrolled space $\mathcal{D}^\alpha_\Xi$ and the operator $\mathbb{H}^\omega$.

We formally start by decomposing the product $u \cdot \xi$ into para- and resonant products, see equation (92) and Lemma 28 from the appendix. For brevity we will write things like "$f \in H^{s-}$" meaning $f \in H^{s-\varepsilon}$ for any $\varepsilon > 0$ etc.

The aim is to construct a space of functions $u \in L^2(\mathbb{T}^2)$ s.t.

$$(1-\Delta)u + \xi u \in L^2(\mathbb{T}^2),$$

where $\xi \in \mathcal{C}^{-1-}$ a.s. This ansatz tells us that for this to be possible, one would need $u \in H^{1-}$ but not better. In order to proceed, we decompose the product via paraproducts, see equation (92) and Lemma 28,

$$\begin{aligned} (1-\Delta)u + \xi \cdot u &= (1-\Delta)u + \xi \prec u + \xi \circ u + \xi \succ u, \\ &\quad \text{where one expects} \\ \xi \succ u &\in H^{-1-}(\mathbb{T}^2) \\ \xi \circ u &\in H^{0-}(\mathbb{T}^2) \text{ but it is not defined!} \\ \xi \prec u &\in H^{0-}(\mathbb{T}^2) \\ (1-\Delta)u &\in H^{-1-}(\mathbb{T}^2). \end{aligned}$$

Now the point is to consider the functions u for which $(1-\Delta)u$ cancels the parts of the product $\xi \cdot u$ which are worse than L^2. In addition, we have to ensure that the resonant product $\xi \circ u$ can be defined somehow; this will actually lead to the necessity to renormalize.

This is where the theory of Paracontrolled Distributions (originally introduced in [36]) enters, which in this context was first used by Allez and Chouk in [2]. The idea is (we will have to refine this slightly) to consider functions $u \in L^2(\mathbb{T}^2)$ for which

$$u + (1-\Delta)^{-1}(\xi \succ u + \xi \prec u) =: u^\sharp \in H^2.$$

For such functions we have

$$\begin{aligned} (1-\Delta)u + \xi \cdot u &= (1-\Delta)u + \xi \prec u + \xi \circ u + \xi \succ u \\ &= (1-\Delta)u^\sharp + \xi \circ u \\ &= (1-\Delta)u^\sharp + \xi \circ u^\sharp - \xi \circ (1-\Delta)^{-1}(\xi \succ u + \xi \prec u), \end{aligned}$$

and we see that the situation is much improved since one has the regularities

$$\begin{aligned} (1-\Delta)u^\sharp &\in L^2 \\ \xi \circ u^\sharp &\in H^{1-} \\ \xi \circ (1-\Delta)^{-1}(\xi \prec u) &\in H^{1-}, \end{aligned}$$

with only the term $\xi \circ (1-\Delta)^{-1}(u \prec \xi)$ being problematic. Thanks to the commutator lemma from [36], see Proposition 14, one can however transform this term as follows

$$\xi \circ (1-\Delta)^{-1}(\xi \prec u) = C(\xi, (1-\Delta)^{-1}\xi, u) + u(\xi \circ (1-\Delta)^{-1}\xi), \tag{14}$$

where $C(\xi, (1-\Delta)^{-1}\xi, u) \in H^{1-}$.

The second term in (14) is now a classically defined product provided we can make sense of the purely stochastic term

$$\xi \circ (1-\Delta)^{-1}\xi \overset{!}{\in} \mathcal{C}^{0-}.$$

The issue here is that this object does not exist in any reasonable sense, unless we *renormalize* it, meaning –naively– that we replace it by an almost surely well-defined object

$$\xi \diamondsuit (1-\Delta)^{-1}\xi := \lim_{\varepsilon\to 0} \xi_\varepsilon \circ (1-\Delta)^{-1}\xi_\varepsilon - c_\varepsilon = \Xi^2 \in \mathcal{C}^{0-} \tag{15}$$

which is precisely the second component of the noise term Ξ from Definition 1 where the ξ_ε is the noise mollified by a standard test function and the constants c_ε satisfy

$$c_\varepsilon := \mathbb{E}(\xi_\varepsilon \circ (1-\Delta)^{-1}\xi_\varepsilon) \sim \log\left(\frac{1}{\varepsilon}\right) \tag{16}$$

i.e. they diverge logarithmically. This is intimately related to Wick ordering which will also appear in a different context later on in Section 2.2.

Thanks to (15), we can rigorously repeat the above computation with the regularized noise for smooth functions u setting

$$u - (1-\Delta)^{-1}((\xi_\varepsilon + \Xi^2_\varepsilon) \succ u + \xi_\varepsilon \prec u) =: u^\sharp \in H^2. \tag{17}$$

This yields

$$\begin{aligned}
(1-\Delta)u + \xi_\varepsilon \cdot u &= (1-\Delta)u^\sharp + \Xi^2_\varepsilon \succ u + u \circ \xi_\varepsilon \\
&= (1-\Delta)u^\sharp + \Xi^2_\varepsilon \succ u + u^\sharp \circ \xi_\varepsilon + ((1-\Delta)^{-1}\xi_\varepsilon \succ u) \circ \xi_\varepsilon + \tilde{B}(u, \Xi^2_\varepsilon) \\
&= (1-\Delta)u^\sharp + u^\sharp \circ \xi_\varepsilon + \Xi^2_\varepsilon u - u((1-\Delta)^{-1}\xi_\varepsilon \circ \xi_\varepsilon) + B(u, \Xi^2_\varepsilon) \\
&= (1-\Delta)u^\sharp + u^\sharp \circ \xi_\varepsilon - c_\varepsilon u + B(u, \Xi^2_\varepsilon)
\end{aligned}$$

which is rearranged to

$$\mathbb{H}^\omega_\varepsilon u := (1-\Delta)u + \xi_\varepsilon u + c_\varepsilon u = \Delta u^\sharp + u^\sharp \circ \xi_\varepsilon + B(u, \Xi^2_\varepsilon), \tag{18}$$

where $B, \tilde{B}$, given explicitly below, are bounded bilinear maps from $H^{1-} \times \mathcal{X}^\alpha \to H^{1-}$.

$$\begin{aligned}
\tilde{B}(u, \Xi_\varepsilon) &:= ((1-\Delta)^{-1}(\Xi^2_\varepsilon \succ u + \xi_\varepsilon \prec u)) \circ \xi_\varepsilon \\
B(u, \Xi_\varepsilon) &:= \Xi^2_\varepsilon \prec u + \Xi^2_\varepsilon \circ u + C(\xi_\varepsilon, (1-\Delta)^{-1}\xi_\varepsilon, u) + \tilde{B}(u, \Xi_\varepsilon).
\end{aligned}$$

These maps satisfy the following continuity property.

Lemma 1 *The bilinear maps*

$$\begin{aligned}
\tilde{B}(u, \Xi) &: H^\sigma \times \mathcal{X}^\alpha \to H^{2-2\alpha+\sigma} \\
B(u, \Xi) &: H^\sigma \times \mathcal{X}^\alpha \to H^{2-2\alpha+\sigma},
\end{aligned}$$

are bounded for $2\alpha - 2 < \sigma < 1$, *in particular this implies*

$$B(u_\varepsilon, \Xi_\varepsilon) \to B(u, \Xi) \text{ in } H^{2-2\alpha+\sigma} \text{ as } \varepsilon \to 0 \text{ for } u_\varepsilon \to u \text{ in } H^\sigma.$$

Proof This follows from the bounds on the paraproducts and the commutator from Lemmas 28 and Proposition 14. □

The point of this computation is that the right-hand side of (18) is now continuous w.r.t. $(\xi_\varepsilon, \Xi_\varepsilon^2)$ in the noise space $\mathcal{X}^\alpha$ so it allows us to pass to the limit in some sense. For now, we can rigorously define the operator

$$\begin{aligned} \mathbb{H}^\omega u &:= (1-\Delta)u^\sharp + u^\sharp \circ \xi + B(u,\Xi) && (19)\\ & \quad \text{for} && (20)\\ u \in L^2(\mathbb{T}^2) & \quad \text{s.t.} && \\ u - (1-\Delta)^{-1}((\xi+\Xi^2) \succ u + \xi \prec u) &=: u^\sharp \in H^2(\mathbb{T}^2) && (21) \end{aligned}$$

The ansatz (21) is of course the limit of the ansatz (17) and one has the following rigorous result.

Theorem 3 (Self-adjointness and (Form-)Domain of the Anderson Hamiltonian) *The operator $(\mathbb{H}^\omega, \mathcal{D}_\Xi^\alpha)$ is an unbounded self-adjoint semi-bounded operator on $L^2(\mathbb{T}^2)$. One has the norm equivalence*

$$\|\mathbb{H}^\omega u\|_{L^2} + \|u\|_{L^2} \approx \|u^\sharp\|_{H^2}.$$

Moreover, one has that if the remainder $u^\sharp$ in the paracontrolled ansatz is only in H^1, i.e. it satisfies the paracontrolled ansatz

$$u - (1-\Delta)^{-1}((\xi+\Xi^2) \succ u + \xi \prec u) =: u^\sharp \in H^1(\mathbb{T}^2),$$

such a paracontrolled function u is in the form domain of $\mathbb{H}^\omega$ meaning $|(u, \mathbb{H}^\omega u)| < \infty$. In fact one has the norm equivalence

$$|(u, \mathbb{H}^\omega u)| + \|u\|_{L^2}^2 \approx \|u^\sharp\|_{H^1}^2.$$

The operator $\mathbb{H}^\omega$ is bounded from below, meaning there exists a constant $K(\omega) > 0$ depending polynomially on the $\mathcal{X}$-norm of the enhanced noise Ξ s.t.

$$(\mathbb{H}^\omega u, u) \geqslant -K(\omega)\|u\|_{L^2}^2 \text{for all } u \in \mathcal{D}_\Xi^\alpha$$

and we define the shifted operators

$$\begin{aligned} \mathbb{H}^{\omega,K} &:= \mathbb{H}^\omega + K(\omega) + 1 && (22)\\ \mathbb{H}_\varepsilon^{\omega,K} &:= \mathbb{H}_\varepsilon^\omega + K(\omega) + 1 && (23) \end{aligned}$$

which is now uniformly positive and self-adjoint so one can define its square root and other fractional powers without issues.

Proof See Section 2.1 in [37]. □

Moreover we can quantify exactly in which way the regularized operators $\mathbb{H}_\varepsilon^\omega$ converge to $\mathbb{H}^\omega$.

Proposition 2 (Norm resolvent convergence of approximate operators) *For the operators $\mathbb{H}_\varepsilon^\omega$ and $\mathbb{H}^\omega$ as above we have that there exists a constant $K(\omega)$, which is a polynomial in the $\mathcal{X}^\alpha$ norm of Ξ, for which*

$$(K(\omega) + \mathbb{H}_\varepsilon^\omega)^{-1} \to (K(\omega) + \mathbb{H}^\omega)^{-1} \text{as } \varepsilon \to 0 \text{ in } \mathcal{L}(L^2(\mathbb{T}^2); L^2(\mathbb{T}^2)).$$

We may choose the constants $K(\omega)$as in the previous theorem.

Proof See Proposition 2.23 in [37]. □

For reasons which will become apparent later, we introduce a frequency cut-off,

$$\Pi_{>N} := \mathcal{F}^{-1}\mathbb{I}_{|\cdot|>2^N}\mathcal{F} \quad \text{for } N \in \mathbb{N},$$

as in [37] namely we define

$$\Phi_N(u) := u - \Pi_{>N}((1-\Delta)^{-1}((\xi+\Xi^2) \succ u + \xi \prec u)) \tag{24}$$

as a bounded operator on $L^2(\mathbb{T}^2)$ which admits an inverse for N large enough depending on the $\mathcal{X}-$norm of (ξ, Ξ^2) which we denote by

$$\Gamma v = v + \Pi_{>N}((1-\Delta)^{-1}((\xi+\Xi^2) \succ \Gamma v + \xi \prec \Gamma v)) \tag{25}$$

having omitted the N in the notation as was done in [37]. In precisely the same way, we can define Φ^ε and Γ^ε analogously to (24) and (25) respectively by replacing (ξ, Ξ^2) by $(\xi_\varepsilon, \Xi^2_\varepsilon)$. As was remarked in [37], one may choose the same N independently of ε (but of course depending measurably on ω).

We think of the Γ map in the following way:

$$\Gamma : \text{"Remainder"} \mapsto \text{"Paracontrolled function with that remainder"}$$

and it exactly parameterizes a paracontrolled space like the one in Definition 2, concretely one has

$$\mathcal{D}^\alpha_\Xi = \Gamma H^2.$$

With this notation in place, we collect some results on the maps Γ and Γ^ε as well as their convergence properties.

Proposition 3 *There is a choice of $N \in 2^{\mathbb{N}}$ for which the maps $\Gamma, \Gamma^\varepsilon : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ in (25) exists, i.e. as the inverse of the Φ_N defined in (24). One has the properties:*

Let $s \in [0,1)$, then Γ is a homeomorphism on the following spaces

$$\begin{aligned} \Gamma : H^s &\to H^s \\ \Gamma : H^1 &\to \mathcal{D}\left(\sqrt{\mathbb{H}^\omega}\right) \\ \Gamma : H^2 &\to \mathcal{D}(\mathbb{H}^\omega) \\ \Gamma : \mathcal{C}^s &\to \mathcal{C}^s. \end{aligned}$$

Γ^ε is also a homeomorphism on H^s and $\mathcal{C}^s$.

Furthermore we have $\Gamma^\varepsilon \to \Gamma$ for $\varepsilon \to 0$ as bounded operators on H^s or $\mathcal{C}^s$.

Proof See [37] but this follows from the paraproduct estimates and the fact that Γ was already defined as an inverse. □

Using these maps, we actually have the stronger convergence

$$\mathbb{H}^\omega_\varepsilon \Gamma_\varepsilon \to \mathbb{H}^\omega \Gamma \text{in } \mathcal{L}(H^2; L^2).$$

which implies Proposition 2, see Proposition 2.19 in [37].

We finish off the section by collecting some other properties of the Anderson Hamiltonian which we will need in the remainder of the paper.

Theorem 4 (Properties of the Anderson Hamiltonian) *With the notations as above, we have*

1. ***Embeddings:*** *For* $\mathcal{D}(\mathbb{H}^\omega) = \Gamma H^2$*, the domain of* $\mathbb{H}^\omega$*, we have*

$$\mathcal{D}(\mathbb{H}^\omega) \cap H^2 = \{0\}, \textit{ but } \mathcal{D}(\mathbb{H}^\omega) \hookrightarrow H^{1-\varepsilon} \textit{and } \mathcal{D}(\mathbb{H}^\omega) \hookrightarrow \mathcal{C}^{1-\varepsilon} \textit{ for any } \varepsilon > 0.$$

   For $\mathcal{D}\left(\sqrt{\mathbb{H}^\omega}\right) = \Gamma H^1$*, the form domain of* $\mathbb{H}^\omega$*,we have*

$$\mathcal{D}\left(\sqrt{\mathbb{H}^\omega}\right) \cap H^1 = \{0\}, \textit{ but } \mathcal{D}\left(\sqrt{\mathbb{H}^\omega}\right) \hookrightarrow H^{1-\varepsilon}\textit{for any } \varepsilon > 0.$$

2. ***Functional calculus:*** *For any bounded continuous function*

$$g : \mathbb{R}_+ \to \mathbb{R},$$

   one has (using the shifted operators from (22)*,* (23)*)*

$$g(\mathbb{H}_\varepsilon^{\omega,K}) \to g(\mathbb{H}^{\omega,K}) \textit{in } \mathcal{L}(L^2; L^2).$$

   In particular one has for all times $t \in \mathbb{R}$

$$\sin\left(t\sqrt{\mathbb{H}^{\omega,K}}\right), \cos\left(t\sqrt{\mathbb{H}^{\omega,K}}\right) \in \mathcal{L}(L^2; L^2) \tag{26}$$

$$\frac{\sin\left(t\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}} \in \mathcal{L}\left(L^2; \mathcal{D}\left(\sqrt{\mathbb{H}^{\omega,K}}\right)\right), \tag{27}$$

   and

$$\cos\left(t\sqrt{\mathbb{H}_\varepsilon^{\omega,K}}\right) \to \cos\left(t\sqrt{\mathbb{H}^{\omega,K}}\right) \quad \textit{in} \quad \mathcal{L}(L^2; L^2) \tag{28}$$

$$\frac{\sin\left(t\sqrt{\mathbb{H}_\varepsilon^{\omega,K}}\right)}{\sqrt{\mathbb{H}_\varepsilon^{\omega,K}}} \to \frac{\sin\left(t\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}} \quad \textit{in} \quad \mathcal{L}(L^2; H^{1-\varepsilon})\textit{for any } \varepsilon > 0. \tag{29}$$

   Moreover these operators are strongly continuous in t*.*

3. ***Eigenvalues and Weyl Law:*** $\mathbb{H}^{\omega,K}$ *has discrete spectrum and it has eigenvalues*

$$0 < \lambda_1(\omega) \leqslant \lambda_2(\omega) \leqslant \dots \lambda_n(\omega) \leqslant \cdots \to +\infty \textit{as } n \to \infty$$

   and L^2*normalized eigenfunctions* $f_n \in \mathcal{D}(\mathbb{H}^\omega)$

$$\mathbb{H}^{\omega,K} f_n = \lambda_n(\omega) f_n. \tag{30}$$

   Moreover, $\mathbb{H}^{w,K}$ *(and thus* $\mathbb{H}^\omega$*) satisfies a Weyl law, meaning almost surely*

$$\lim_{n\to\infty} \frac{\lambda_n(\omega)}{n} = C(\omega),$$

   i.e. the eigenvalues grow like the eigenvalues of the Laplacian.

4. ***Equivalence of fractional norms:*** *For* $s \in (-1, 1)$ *we have the norm equivalence*

$$\|u\|_{H^s} \approx \left\| \left(\mathbb{H}^{w,K}\right)^{\frac{s}{2}} u \right\|_{L^2}. \tag{31}$$

Proof The first two points are found in [37] the third point is from [54] and the last point was proved in Proposition 1.14 in [55]. □

Remark 2 A consequence of Theorem 4 Point 3 and Point 4 is that for almost every $\omega \in \Omega$, writing $P_{\leqslant N} : H^s(\mathbb{T}^2) \to H^s(\mathbb{T}^2)$, with $N \in \mathbb{N}$ and $s, s' \in (-1,1)$, $s \leqslant s'$, the orthogonal projection on the first N eigenvectors of $\mathbb{H}^\omega$, we have

$$\|(I - P_{\leqslant N})f\|_{H^s} \lesssim N^{s-s'} \|f\|_{H^{s'}}, \tag{32}$$

and similarly also

$$\|P_{\leqslant N} f\|_{H^{s'}} \lesssim N^{s'-s} \|f\|_{H^s} \tag{33}$$

see also Lemma 1.3 of [55].

2.2 Gaussian measures and Wick powers

A Gaussian measure μ on the space of tempered distribution $\mathcal{S}'(\mathbb{T}^d)$ on the $d \in \mathbb{N}$ dimensional torus, is a Radon measure μ on $\mathcal{S}'(\mathbb{T}^d)$ (with respect its strong topology) such that for any smooth function $f \in C^\infty(\mathbb{T}^d) =: \mathcal{S}(\mathbb{T}^d)$ the (real valued) random variable $x \mapsto \langle x, f \rangle_{\mathcal{S}',\mathcal{S}}$ (where $x \in \mathcal{S}'(\mathbb{T}^d)$) is a Gaussian random variable.

A Gaussian measure is completely characterized by its mean $m \in \mathcal{S}'(\mathbb{T}^d)$ and its covariance operator $\Sigma : \mathcal{S}(\mathbb{T}^d) \to \mathcal{S}'(\mathbb{T}^d)$ (which is a linear positive operator), which are the two unique objects appearing in the characteristic function $\hat{\mu} : \mathcal{S}(\mathbb{T}^d) \to \mathbb{C}$ of μ, which takes the form

$$\hat{\mu}(f) := \int_{\mathcal{S}'(\mathbb{T}^d)} e^{i\langle x,f \rangle_{\mathcal{S}',\mathcal{S}}} \mu(\mathrm{d}x) = \exp\left(i\langle m, f\rangle_{\mathcal{S}',\mathcal{S}} - \frac{1}{2}\langle \Sigma(f), f\rangle \right).$$

A consequence of Minlos-Sazonov theorem (see, e.g., Theorem 20.1 in [69]) is that for any $m \in \mathcal{S}'(\mathbb{T}^d)$ and any *continuous, linear, positive and symmetric operator* Σ there is a unique Gaussian measure μ on $\mathcal{S}'(\mathbb{T}^d)$ with mean m and covariance Σ. Hereafter we mainly focus on the case where $m = 0$ and we write $\mu^{\Sigma^{-1}}$ for the Gaussian measure on $\mathcal{S}'(\mathbb{T}^d)$ with variance Σ (the reason of the presence of -1 will become apparent later). We also often identify $\mathcal{S}(\mathbb{T}^d)$ with a subset of $\mathcal{S}'(\mathbb{T}^d)$ thanks to the $L^2(\mathbb{T}^d)$ scalar product. In this paper we are mainly interested in three cases of Gaussian measures:

1. when $\langle \Sigma(f), g\rangle := \int_{\mathbb{T}^d} f(x)g(x)\mathrm{d}x$, namely $\Sigma = I_{L^2}$, which is the *(Gaussian) white noise measure* on $\mathbb{T}^d$;
2. when $\langle \Sigma(f), g\rangle := \int_{\mathbb{T}^d} (-\Delta + K)^{-1}(f)(x)g(x)\mathrm{d}x$, where $K > 0$, namely $\Sigma = (-\Delta + K)^{-1}$, which is usually called *Gaussian free field* on $\mathbb{T}^d$ with mass K;
3. when $\langle \Sigma(f), g\rangle := \int_{\mathbb{T}^d} (\mathbb{H}^\omega + K)^{-1}(f)(x)g(x)\mathrm{d}x$, for $K > K(\omega)$ (see Section 2.1), namely $\Sigma = (\mathbb{H}^\omega + K)^{-1}$, which henceforth we will call *Anderson Gaussian free field* with mass K.

We also recall two convenient facts about Gaussian measures. Firstly the fact that convergence of the covariance operators implies weak convergence of the Gaussian measures and secondly that one has precise knowledge of whether two Gaussian measures are mutually singular or absolutely continuous.

Lemma 2 *a) If the operators*

$$\Sigma^\varepsilon \to \Sigma \text{ in } \mathcal{L}(L^2(\mathbb{T}^d); L^2(\mathbb{T}^d))$$

then we have

$$\mu^{(\Sigma^\varepsilon)^{-1}} \rightharpoonup \mu^{\Sigma^{-1}}$$

weakly in the sense of measures on $\mathcal{S}'(\mathbb{T}^d)$.

Consequently we have

$$\mu^{\mathbb{H}_\varepsilon^{\omega,K}} \rightharpoonup \mu^{\mathbb{H}^{\omega,K}} \text{ weakly in the sense of measures on } \mathcal{S}'(\mathbb{T}^d) \text{ as } \varepsilon \to 0, \quad (34)$$

see Proposition 2.

b) *If* $\sqrt{\Sigma}^{-1}(L^2) \neq \sqrt{\Lambda}^{-1}(L^2)$ *then the Gaussian measures* $\mu^{\Sigma^{-1}}$ *and* $\mu^{\Lambda^{-1}}$ *are mutually singular. In particular this implies that the Gaussian measure* $\mu^{\mathbb{H}^{\omega,K}}$ *is mutually singular with respect to* $\mu^{-\Delta+K}$ *and also* $\mu^{\mathbb{H}_\varepsilon^{\omega,K}}$.

Proof

a) The general statement is in Section 5 of [13]. The statement (34) then follows from Proposition 2.

b) The main statement is a consequence of the Feldman-Hajek theorem, see e.g. Theorem 2.23 in [30], the other statement follows from the fact that

$$\mathcal{D}\left(\sqrt{\mathbb{H}^{\omega,K}}\right) \neq H^1 = \mathcal{D}\left(\sqrt{\mathbb{H}_\varepsilon^{\omega,K}}\right) = \mathcal{D}\left(\sqrt{-\Delta+K}\right),$$

which is contained in Theorem 4.

□

Usually the support (not understood in a topological sense but more generically as a subset of full measure) of a Gaussian measure $\mu^{\Sigma^{-1}}$ on $\mathcal{S}'(\mathbb{T}^d)$ is not the whole space $\mathcal{S}'(\mathbb{T}^d)$ but there is a proper (Banach) subspace $W \subset \mathcal{S}'(\mathbb{T}^d)$ supporting $\mu^{\Sigma^{-1}}$. In the case of the white noise and Gaussian free field the support is well known.

Proposition 4 *For any* $\delta > 0$ *and* $1 \leqslant p \leqslant +\infty$ *we have*

$$\mu^{-\Delta+K}\left(B_{p,p}^{1-\frac{d}{2}-\delta}(\mathbb{T}^d)\right) = \mu^{I_{L^2}}\left(B_{p,p}^{-\frac{d}{2}-\delta}(\mathbb{T}^d)\right) = 1.$$

Proof See, e.g., Lemma 3.2 in [29]. □

From now on if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\psi : \Omega \to W \subset \mathcal{S}'(\mathbb{T}^d)$ (where W is a Banach space) is a measurable function, we say that ψ is a *Gaussian random field with covariance* Σ if the law of ψ is $\mu^{\Sigma^{-1}}$, written in symbols $\text{Law}(\psi) = \mu^{\Sigma^{-1}}$.

In the following we will use the notion of Wick products of Gaussian random variables: let $X_1, \ldots, X_n$, $n \in \mathbb{N}$, be a set of (jointly) Gaussian random variables with zero mean. We call the *Wick products* of $X_1, \ldots, X_n$ the random variable

$$: X_1 \cdots X_n ::= \left(\frac{\partial^n}{\partial t_1 \cdots \partial t_n}\left(\frac{\exp\left(\sum_{i=1}^n t_i X_i\right)}{\mathbb{E}\left[\exp\left(\sum_{i=1}^n t_i X_i\right)\right]}\right)\right)_{t_1=\cdots=t_n=0} \quad (35)$$

In the case where $X_1 = \cdots = X_n = Z$ then

$$: Z^n := (\text{var}(Z))^{\frac{n}{2}} H_n\left(\frac{Z}{\sqrt{\text{var}(Z)}}\right),$$

where H_n is the n-th Hermite polynomial.

Proposition 5 *Let $X_1, \ldots, X_n$, $Y_1, \ldots, Y_m$ be some jointly Gaussian random variables with zero mean then*

$$\mathbb{E}[: X_1 \cdots X_n :: Y_1 \cdots Y_m :] = \delta_{n,m} \sum_{\sigma \in \mathcal{S}_n} \prod_{j=1}^{m} \mathbb{E}[X_j Y_{\sigma(j)}]$$

where $\mathcal{S}_n$ is the set of permutations of $\{1, \ldots, n\}$.

Proof See Theorem 3.9 in [44]. □

Let $L^2(W, \mu^{\Sigma^{-1}})$ (where $W \subset \mathcal{S}'(\mathbb{T}^d)$ is a support of $\mu^{\Sigma^{-1}}$) be the set of L^2 random variables with defined on W with respect the Gaussian measure $\mu^{\Sigma^{-1}}$. For every $n \in \mathbb{N}$, We define

$$\Gamma_n = \overline{\mathrm{span}\{: \langle w, f_1 \rangle \cdots \langle w, f_n \rangle : | f_1, \ldots, f_n \in \mathcal{S}(\mathbb{T}^d)\}} \subset L^2(W, \mu^{\Sigma^{-1}})$$

where the closure is taken with respect the natural topology of $L^2(W, \mu^{\Sigma^{-1}})$. We write $\Gamma_0 = \mathbb{R}$. With these definition we have:

Proposition 6 (Chaos decomposition and hypercontractivity) *We have that*

$$L^2(W, \mu^{\Sigma^{-1}}) = \bigoplus_{n=0}^{\infty} \Gamma_n.$$

Furthermore for every $G \in \Gamma_n$ and $p \geqslant 2$ we have

$$\mathbb{E}[|G|^p] \leqslant (p-1)^{\frac{np}{2}} (\mathbb{E}[|G|^2])^{\frac{p}{2}}.$$

Proof For the first statement see, e.g., Theorem 4.1 in [44]. For the second statement see, e.g., Theorem 5.1 and Remark 5.11 in [44]. □

Let $\varphi : \Omega \to W \subset \mathcal{S}'(\mathbb{T}^d)$ be a Gaussian free field of mass m (namely $\mathrm{Law}(\varphi) = \mu^{-\Delta + m^2}$) and consider $a_\varepsilon(x)$ a smooth mollifier, where $a : \mathbb{T}^d \to \mathbb{R}_+$ is a smooth function with compact support such that $\int_{\mathbb{T}^d} a(x) \mathrm{d}x = 1$. We defined $\varphi_\varepsilon = a_\varepsilon * \varphi$, which is Gaussian random field taking values in $C^\infty(\mathbb{T}^d)$. Since φ_ε takes values in a space of functions, we can define $: \varphi_\varepsilon^n :$ as a smooth random function given by

$$: \varphi_\varepsilon^n : (x) =: (\varphi_\varepsilon(x))^n := \mathbb{E}[|\varphi_\varepsilon(x)|^2]^{\frac{n}{2}} H_n \left(\frac{\varphi_\varepsilon(x)}{\sqrt{\mathbb{E}[|\varphi_\varepsilon(x)|^2]}} \right).$$

Theorem 5 *In the above setting, taking $d = 2$, for any $1 \leqslant p < +\infty$ and $\delta > 0$ the sequence of random functions $: \varphi_\varepsilon^n :$ converges in*

$$L^p((\Omega, \mathcal{F}, \mathbb{P}), B_{p,p}^{-\delta}(\mathbb{T}^2))$$

to some random distribution $: \varphi^n :$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in $B_{p,p}^{-\delta}(\mathbb{T}^2)$. Furthermore the random distribution $: \varphi^n :$ does not depend on the mollifier a_ε.

Proof See, e.g., Lemma 3.2 in [29]. □

Remark 3 By Besov embedding, see Lemma 31, this implies that the Wick powers lie even in $L^p((\Omega, \mathcal{F}, \mathbb{P}), C^{-\delta}(\mathbb{T}^2))$ for any $p \geqslant 2$ and $\delta > 0$.

3 A (regular) coupling between GFF and AGFF

From Lemma 2 we know that the Gaussian measures $\mu^{-\Delta+K}$ and $\mu^{\mathbb{H}^\omega+K}$ are mutually singular for almost every ω and for any $K \geqslant K(\omega)$ (where $K(\omega)$ is the almost surely positive random variable in Section 2.1 which ensures that $\mathbb{H}^\omega + K$ is invertible and thus $\mu^{\mathbb{H}^\omega+K}$ is well-defined). In this section we prove a weaker but interesting result, namely the existence of a regular coupling between the measure $\mu^{-\Delta+K}$ and $\mu^{\mathbb{H}^\omega+K}$ for almost every $\omega \in \Omega$ and for any $K \geqslant K(\omega)$. This coupling permits us to extend the definition of the Wick product of the Gaussian free field to the support of the Anderson free field, see Section 3.6.

3.1 Variational formulation of the coupling problem

First we want to give a variational representation of the Gaussian measure $\mu^\varepsilon = \mu^{\mathbb{H}^{\omega,K}_\varepsilon}$ (or more generally μ^ε's Laplace transform) in a way similar to what was done in [6, 8, 10, 9]. The first step is to write the Radon-Nikodym derivatives of μ^ε with respect to $\mu^{-\Delta+K}$ and to prove that it has some good properties permitting the variational representation.

So, consider a Gaussian white noise $\xi : \Omega \to \mathcal{S}'(\mathbb{T}^2)$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define the measure

$$\frac{\mathrm{d}\mu^\varepsilon(\omega)}{\mathrm{d}\mu^{-\Delta+K}}(\varphi) = \frac{1}{\mathcal{Z}^\varepsilon(\omega)} \exp\left(- \int_{\mathbb{T}^2} (\xi_\varepsilon(x,\omega) - (\gamma^{(1)}(\omega) - \gamma^{(2)}_\varepsilon)) : \varphi^2 : (x) \mathrm{d}x \right) \tag{36}$$

where

$$\mathcal{Z}^\varepsilon(\omega) = \int \exp\left(- \int_{\mathbb{T}^2} (\xi_\varepsilon(x,\omega) - (\gamma^{(1)}(\omega) - \gamma^{(2)}_\varepsilon)) : \varphi^2 : (x) \mathrm{d}x \right) \mathrm{d}\mu^{-\Delta+K}(\varphi)$$

and where $\gamma^{(1)} : \Omega \to \mathbb{R}_+$ is a positive random variable (not depending on ε which plays a role analogous to the $K(\omega)$ from Proposition 2) and $\gamma^{(2)}_\varepsilon$ is a real number (depending on $\varepsilon > 0$ but not depending on ω), and $: \varphi^2 :$ denotes the Wick product of the random field φ with respect to the Gaussian measure $\mu^{-\Delta+K}$. As in prior sections, $\xi : \Omega \to \mathcal{S}'(\mathbb{T}^2)$ is a Gaussian white noise and ξ_ε is a smooth approximation of ξ (for example $\xi_\varepsilon = \rho_\varepsilon * \xi$ where $\rho_\varepsilon = \varepsilon^{-2}\rho(\cdot/\varepsilon)$ and $\rho \in C^\infty(\mathbb{T}^2)$ $\int_{\mathbb{T}^2} \rho(x)\mathrm{d}x = 1$).

The constant $\gamma^{(2)}_\varepsilon \to +\infty$ as $\varepsilon \to 0$ such that $-\Delta + \xi_\varepsilon - \gamma^{(2)}_\varepsilon \to \mathbb{H}^\omega$ as $\varepsilon \to 0$ in the norm resolvent sense. The constant $\gamma^{(1)}(\omega)$ is chosen in such a way, that there is $\gamma^{(1)}(\omega) \geqslant K(\omega)$ (where $K(\omega) > 0$ is the random variable defined in Theorem 3), for which, for every $\omega \in \Omega$, we have

$$-\Delta + \xi_\varepsilon(\omega) + \gamma^{(1)}(\omega) \geqslant \mathbb{I}_{L^2}$$

as self-adjoint operators on the domain of $-\Delta$ (i.e. we suppose that the lowest eigenvalue of $-\Delta + \xi_\varepsilon(\omega) + \gamma^{(1)}(\omega)$ is greater than 1 almost surely). We suppose also that for any $p \in \mathbb{R}_+$

$$\mathbb{E}[|\gamma^{(1)}|^p] < +\infty.$$

Under these notations and setting we prove the following lemma.

Lemma 3 *Let $g : \mathbb{T}^2 \to \mathbb{R}$ be a smooth function such that the operator $H_g = -\Delta + K + g(x)$ (on the domain of $-\Delta + K$) is positive, then if μ^{H_g} is the Gaussian measure with covariance $(-\Delta + K + g(x))^{-1}$ and $\mu^{-\Delta+K}$ is the measure of the Gaussian free field we have*

$$\log\left(\frac{\mathrm{d}\mu^{H_g}}{\mathrm{d}\mu^{-\Delta+K}}(\varphi) \right) = - \int_{\mathbb{T}^2} g(x) : \varphi^2 : (x)\mathrm{d}x + \log(\mathcal{Z}_{H_g}) \tag{37}$$

where $:\varphi^2:$ *denotes the Wick product of the field* φ *with respect to the Gaussian free field measure* $\mu^{-\Delta+K}$*, and*

$$\mathcal{Z}_{H_g} = \int \exp\left(-\int_{\mathbb{T}^2} g(x) :\varphi^2: (x)\mathrm{d}x\right)\mu^{-\Delta+K}(\mathrm{d}\varphi).$$

Proof First we prove that μ^{H_g} is absolutely continuous with respect to $\mu^{-\Delta+K}$. The measure μ^{H_g} can be obtained as the push-forward of the measure $\mu^{-\Delta+K}$ through the transformation

$$T(\varphi) = (-\Delta + K + g(x))^{-1/2}(-\Delta + K)^{1/2}(\varphi),$$

with inverse $S(w) = (-\Delta + K)^{-1/2}(-\Delta + K + g(x))^{1/2}(\varphi)$. We have that

$$\begin{aligned} v(\varphi) &= (-\Delta + K)^{-1/2}(-\Delta + K + g(x))^{1/2}(\varphi) - \varphi \\ &= ((I + (-\Delta + K)^{-1}g(x))^{1/2} - I)(\varphi) \end{aligned}$$

where I is the identity operator on $\mathcal{C}^{-\delta}(\mathbb{T}^2)$. The (linear) operator v restricted on H^1 (i.e. the Cameron-Martin space of the free field having law $\mu^{-\Delta+K}$) is a Hilbert-Schmidt operator. Indeed we have

$$\begin{aligned} v(\varphi) &= \left((I + (-\Delta + K)^{-1}g(x))^{1/2} - I - \frac{1}{2}(-\Delta + K)^{-1}g(x)\right)(\varphi) + \frac{1}{2}(-\Delta + K)^{-1}(g(x)\varphi) \\ &= \frac{1}{4}\left((I + (-\Delta + K)^{-1}g(x))^{1/2} + I + \frac{1}{2}(-\Delta + K)^{-1}g(x)\right)^{-1}((-\Delta + K)^{-1}g(x))^2(\varphi) + \\ &\quad + \frac{1}{2}(-\Delta + K)^{-1}(g(x)\varphi) \end{aligned}$$

which shows that $v(\varphi)$ is the sum of the Hilbert-Schmidt operator $\frac{1}{2}(-\Delta + m^2)^{-1}g(x)$ and of a trace class operator (being the remainder the product of a bounded operator and the square of a Hilbert-Schmidt operator). Since v is linear (and thus differentiable in $\mathcal{C}^{-\delta}(\mathbb{T}^2)$ with derivative equal to v), by Theorem 3.5.3 of [68], we get

$$\frac{\mathrm{d}\mu^{H_g}}{\mathrm{d}\mu^{-\Delta+K}} = \frac{\mathrm{d}T_*\mu^{-\Delta+K}}{\mathrm{d}\mu^{-\Delta+K}} = \det{}_2(I + v)\exp\left(-\delta(v(\varphi)) - \frac{1}{2}\|v(\varphi)\|_{H^1}^2\right) \tag{38}$$

where δ is the Skorokhod integral with respect to the Gaussian measure $\mu^{-\Delta+K}$ on $\mathcal{C}^{-\delta}(\mathbb{T}^2)$, and $\det_2$ is the regularized determinat (see, e.g., Chapter 9 of [63]). What remains to be shown is that the term in the exponential in (38) is (up to some finite additional constant) equal to $\int_{\mathbb{T}^2} g(x) \ :\varphi^2(x): dx$. In order to show this equality, we consider a (finite dimensional) approximation of v given by

$$v_N(\varphi) = ((I + (-\Delta + K)^{-1}\Pi_N g(x)\Pi_N)^{1/2} - I)(\varphi)$$

where Π_N is the $L^2(\mathbb{T}^2)$ projection onto the subspace of trigonometric polynomial of degree less or equal than N. We have that, by definition of the Skorokhod integral for functions having trace class derivatives in the Cameron-Martin space (see, e.g., Section B.4 of [68]), i.e. the formula

$$\delta(u(\varphi)) = \langle i^*(u(\varphi)), \varphi\rangle_{H^1(\mathbb{T}^2)} - \mathrm{Tr}_{H^1(\mathbb{T}^2)}(\nabla u(\varphi)),$$

the following holds

$$\delta(v_N(\varphi)) = \int(-\Delta + K)(v_N(\varphi))(x)\varphi(x) - \mathrm{Tr}_{L^2}((-\Delta + K)^{1/2}v_N(-\Delta + K)^{-1/2}).$$

We also obtain

$$
\begin{aligned}
& \mathrm{Tr}_{L^2}((-\Delta+K)^{1/2}v_N(-\Delta+K)^{-1/2})\\
= & \mathrm{Tr}_{L^2}(v_N)\\
= & \frac{1}{4}\mathrm{Tr}_{L^2}\left(\left((I+(-\Delta+K)^{-1}\Pi_N g(x)\Pi_N)^{1/2}+I+\frac{1}{2}(-\Delta+K)^{-1}\Pi_N g(x)\Pi_N\right)^{-1}\times\right.\\
& \left.((-\Delta+K)^{-1}\Pi_N g(x)\Pi_N)^2(\cdot)\right)+\frac{1}{2}\mathrm{Tr}_{L^2}((-\Delta+K)^{-1}(\Pi_N g(x)\Pi_N)).
\end{aligned}
$$

The first term in the previous sum is uniformly bounded in N, for the second term we get

$$
\begin{aligned}
\mathrm{Tr}_{L^2}((-\Delta+K)^{-1}(\Pi_N f(x))) & = \mathrm{Tr}_{L^2}(\Pi_N((-\Delta+K)^{-1}(\Pi_N f(x))))\\
& = \int_{\mathbb{T}^2}\mathcal{G}_N(x-x)f(x)\mathrm{d}x=\mathcal{G}_N(0)\int_{\mathbb{T}^2}f(x)\mathrm{d}x\\
& = \mathbb{E}[(\Pi_N\varphi(0))^2]\int_{\mathbb{T}^2}f(x)\mathrm{d}x
\end{aligned}
$$

where $\mathcal{G}_N$ is the integral kernel of the operator $\Pi_N(-\Delta+m^2)^{-1}\Pi_N$ and φ is a random distribution with law $\mu^{-\Delta+K}$. On the other hand, by an explicit computation, we get

$$
-\int(-\Delta+K)(v_N(\varphi))(x)\varphi(x)\mu^{-\Delta+K}(\mathrm{d}\varphi)-\frac{1}{2}\|v_N(\varphi)\|_{H^1}^2=\int_{\mathbb{T}^2}g(x)(\Pi_N\varphi)^2(x)\mathrm{d}x.
$$

Putting it all together, we get

$$
\begin{aligned}
-\delta(v_N(\varphi))-\frac{1}{2}\|v_N(\varphi)\|_{H^1}^2 & = \int_{\mathbb{T}^2}g(x)(\Pi_N\varphi)^2(x)\mathrm{d}x-\mathbb{E}[(\Pi_N\varphi(0))^2]\int_{\mathbb{T}^2}g(x)\mathrm{d}x+C_N\\
& = \int_{\mathbb{T}^2}g(x):(\Pi_N\varphi)^2:(x)\mathrm{d}x+C_N
\end{aligned}
$$

where C_N is a suitable constant converging to some $C\in\mathbb{R}$ as $N\to+\infty$. Taking the limit on both sides of the previous expression we get the thesis. □

Thus the previous lemma proves expression (36). We now introduce the following useful definition describing the key property for the variational representation of an exponential functional.

Definition 3 *Let $W\subset\mathcal{S}'(\mathbb{T}^2)$ be a Banach space supporting the law of Gaussian free field $\mu^{-\Delta+K}$. We say that a measurable function $G:W\to\mathbb{R}$ is tame (with respect to the law of the Gaussian free field $\mu^{-\Delta+K}$) if there are $p,q\geqslant 1$, $\frac{1}{p}+\frac{1}{q}=1$, such that*

$$
\int\exp(pG(\varphi))\mu^{-\Delta+K}(\mathrm{d}\varphi)+\int|G(\varphi)|^q\mu^{-\Delta+K}(\mathrm{d}\varphi)<+\infty.
$$

Under the previous hypotheses the Radon-Nikodym derivative $\frac{\mathrm{d}\mu^\varepsilon}{\mathrm{d}\mu^{-\Delta+K}}$ is a tame function. Indeed, we have the following result.

Lemma 4 *Suppose that g is smooth and that $(-\Delta+m^2+g(x))\geqslant R\mathbb{I}_{L^2}$ for some $R>0$,then the functional $\varphi\to\int_{\mathbb{T}^2}g(x):\varphi^2(x):\mathrm{d}x$ is tame.*

Proof If $(-\Delta + m^2 + g(x)) \geqslant R\mathbb{I}_{L^2}$ then there is $p > 1$ such that $(-\Delta + m^2 + pg(x)) > 0$. Indeed we have that

$$(-\Delta + m^2 + pg(x)) \geqslant (-\Delta + m^2 + g(x)) - (p-1)\|g\|_{L^\infty}\mathbb{I}_{L^2} \geqslant (R - (p-1)\|g\|_{L^\infty})\mathbb{I}_{L^2}$$

which is strictly positive whenever $p - 1 < \frac{R}{\|g\|_{L^\infty}}$. This means that we can apply Lemma 3 to the operator $(-\Delta + m^2 + pg(x))$, obtaining that

$$\exp\left(p \int_{\mathbb{T}^2} g(x) : \varphi^2(x) : \mathrm{d}x\right) \in L^1(\mu).$$

Since, by hypercontractivity, $\int_{\mathbb{T}^2} g(x) : \varphi^2(x) : \mathrm{d}x \in L^q(\mu)$ for any $1 \leqslant q < +\infty$ the thesis is proved. □

If we consider $W = \mathcal{C}^{-\delta}(\mathbb{T}^2) = B^{-\delta}_{\infty,\infty}(\mathbb{T}^2)$ (for some $\delta > 0$ small enough), under the previous conditions on $\gamma^{(1)}$ we will show that the functional

$$G^{\varepsilon,f}(\varphi,\omega) = f(\varphi) + \int_{\mathbb{T}^2} (\xi_\varepsilon(x,\omega) - (\gamma^{(1)}(\omega) - \gamma^{(2)}_\varepsilon)) : \varphi^2(x) : \mathrm{d}x$$

is tame, whenever $f(\cdot)$ is tame in the sense of Definition 3.

Notation 1 *In order to provide the variational representation of the formula* (36)*, we need to introduce a Gaussian white noise $X_t : \Omega' \times \mathbb{R}_+ \to \mathcal{S}'(\mathbb{T}^2)$ defined on the probability space $(\Omega', \mathcal{F}'_t, \mathbb{P}')$. We define also the product space $\bar{\Omega} = \Omega \times \Omega'$ with the product σ-algebra and equipped with the filtration $\overline{\mathcal{F}}_t = \mathcal{F} \vee \mathcal{F}_t$ and the product probability measure $\bar{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'$ (under this measure the white noise ξ and X_t are independent). We consider the operator $J_s : \mathcal{S}'(\mathbb{T}^2) \to C^\infty(\mathbb{T}^2)$ given by the expression*

$$J_s(f) = \mathcal{F}^{-1}\left(\frac{\sigma_s(|k|^2)}{\sqrt{m^2 + |k|^2}} \mathcal{F}(f)(k)\right)$$

where $\sigma_\cdot : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}_+$ is a smooth function with compact support such that $\int_0^t \sigma_s^2(k)\mathrm{d}s = \rho_t(k)$ where ρ_t is a smooth cut-off function of the ball of radius $t > 0$. In order to distinguish between the expectation with respect the $\omega \in \Omega$ variable and the probability law $\mathbb{P}$, with respect the $\omega' \in \Omega'$ variable and the probability law $\mathbb{P}'$, we will write $\mathbb{E}^\omega[\cdot]$ and $\mathbb{E}^{\omega'}[\cdot]$ respectively. We write also $\mathbb{E}$ for the expectation with respect to both the variables (namely on the probability space $(\bar{\Omega}, \overline{\mathcal{F}}_t, \bar{\mathbb{P}})$ described above).

Notation 2 *We use the following the notations*

$$W_t = \int_0^t J_s \mathrm{d}X_s \qquad Z_t(u) = \int_0^t J_s u_s \mathrm{d}s$$

where $u : \Omega' \times \mathbb{R}_+ \to L^2(\mathbb{T}^2)$ is a measurable function adapted with respect to the filtration $\mathcal{F}'_t$ (when it is clear from the context we drop the dependence on u in the random process Z_t).

The next theorem gives a variational representation of the Laplace transform of ν^ε.

Theorem 6 *Let $\delta > 0$. For every tame function $f : \mathcal{C}^{-\delta} \to \mathbb{R}$ and for every $\omega \in \Omega$ we have*

$$\begin{aligned} & -\log \textstyle\int \exp(-f(\varphi))\nu^{\varepsilon}(\omega, \mathrm{d}\varphi) + \log(\mathcal{Z}^{\varepsilon}(\omega)) \\ = \; & -\log \textstyle\int \exp(-G^{\varepsilon,f}(\varphi,\omega))\mu^{-\Delta+1}(\mathrm{d}\varphi) \\ = \; & \inf_{u\in\mathbb{H}_a} \mathbb{E}^{\omega'} \big[f(W_\infty + Z_\infty(u)) + \textstyle\int \xi_\varepsilon(\omega) W_\infty Z_\infty(u)\mathrm{d}x + \int \xi_\varepsilon(\omega) Z_\infty^2(u)\mathrm{d}x \\ & + \textstyle\int ((\gamma^{(1)}(\omega) + \gamma_\varepsilon^{(2)})) W_\infty Z_\infty(u) + \int \left(\left(\gamma^{(1)}(\omega) + \gamma_\varepsilon^{(2)} \right) \right) Z_\infty^2(u)\mathrm{d}x + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s \big] \\ = \; & \inf_{u\in\mathbb{H}_a} F^{\varepsilon}(f,\omega). \end{aligned} \tag{39}$$

Proof Fix an $\varepsilon > 0$, then $G^{\varepsilon,f}$ is a tame functional since is the sum of $f(\varphi)$ (which is tame by hypothesis) and

$$V(\varphi) = \int (\xi_\varepsilon(\omega, x) + (\gamma^{(1)}(\omega) + \gamma_\varepsilon^{(2)})) : \varphi^2(x) : \mathrm{d}x$$

which is tame by Lemma 4. Thus we can apply the main result of [67] and hence we have

$$\begin{aligned} & -\log \int \exp(-G^{\varepsilon,f}(\varphi,\omega))\mu^{-\Delta+1}(\mathrm{d}\varphi) \\ = \; & \inf_{u\in\mathbb{H}_a} \mathbb{E}^{\omega'} \left[f(W_\infty + Z_\infty(u)) + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s + V(W_\infty + Z_\infty(u)) \right] \\ = \; & \inf_{u\in\mathbb{H}_a} \mathbb{E}^{\omega'} \left[f(W_\infty + Z_\infty(u)) + \int \xi_\varepsilon(\omega) W_\infty Z_\infty(u)\mathrm{d}x + \int \xi_\varepsilon(\omega) Z_\infty^2(u)\mathrm{d}x \right. \\ & \left. + \int ((\gamma^{(1)}(\omega) + \gamma_\varepsilon^{(2)})) W_\infty Z_\infty(u) + \int \left(\left(\gamma^{(1)}(\omega) + \gamma_\varepsilon^{(2)} \right) \right) Z_\infty^2(u)\mathrm{d}x + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s \right] \end{aligned}$$

where we use the fact that the expectation of $: W_\infty^2 : (x)$ is zero, giving us the desired result. □

It is convenient to rewrite the functional $F^{\varepsilon}(\omega, u)$ in a more useful form for what follows.

Proposition 7 *Let $u : \bar{\Omega} \times \mathbb{R}_+ \to L^2(\mathbb{T}^2)$ be an adapted process.*

We have the following identity

$$F^{\varepsilon}(u,\omega) \;=\; \mathbb{E}^{\omega'} \left[\sum_{i=1}^{7} \Gamma_i + \mathfrak{G} + \frac{1}{2}\int_0^\infty \|l_t(u)\|_{L^2}^2 \mathrm{d}s \right] + \frac{1}{2}\mathbb{E}^{\omega'} \left[\int_0^\infty \|J_s \xi_\varepsilon W_s\|_{L^2}^2 \,\mathrm{d}s \right] \tag{40}$$

where

$$l_s(u) = J_s \xi_\varepsilon W_s + J_s(\xi_\varepsilon \succ Z_s) - u_s$$

and

$$\begin{aligned}
\Gamma_1 &= 2\int_0^\infty \int J_t(\xi_\varepsilon \preccurlyeq Z_t)u_t\mathrm{d}t\mathrm{d}x\\
\Gamma_2 &= 2\int_0^\infty \int \left(J_s\xi_\varepsilon W_s \circ J_s\xi_\varepsilon - \dot{\gamma}^{(2)}_{\varepsilon,s}W_s\right) Z_s\mathrm{d}x\mathrm{d}s\\
\Gamma_3 &= \int_0^\infty \int ((J_s\xi_\varepsilon \circ J_s\xi_\varepsilon) - \dot{\gamma}^{(2)}_{\varepsilon,s})Z_s^2\mathrm{d}x\mathrm{d}s\\
\Gamma_4 &= \int_0^\infty \int (J_s(\xi_\varepsilon \succ Z_s))^2\mathrm{d}x\mathrm{d}s - \int_0^\infty \int (J_s\xi_\varepsilon \circ J_s\xi_\varepsilon)Z_s^2\mathrm{d}x\mathrm{d}s\\
\Gamma_5 &= 2\int_0^\infty \int J_s(\xi_\varepsilon W_s)J_s(\xi_\varepsilon \succ Z_s)\mathrm{d}x\mathrm{d}s - 2\int_0^\infty \int (J_s\xi_\varepsilon W_s \circ J_s\xi_\varepsilon)\, Z_s\mathrm{d}x\mathrm{d}s\\
\Gamma_6 &= -2\int_0^\infty \int \gamma^{(2)}_{\varepsilon,t}W_tJ_tu_t\mathrm{d}x\mathrm{d}t - 2\int_0^\infty \int \gamma^{(2)}_{\varepsilon,t}Z_tJ_tu_t\mathrm{d}x\mathrm{d}t\\
\Gamma_7 &= 2\int_0^\infty \dot{\gamma}^{(1)}_t W_tZ_t\mathrm{d}t + 2\int_0^\infty \int \gamma^{(1)}_t W_tJ_tu_t\mathrm{d}x dt + 2\int_0^\infty \int \gamma^{(1)}_t Z_tJ_tu_t\mathrm{d}x\mathrm{d}t\\
\mathfrak{G} &= \int_0^\infty \int \dot{\gamma}^{(1)}_t(\omega)Z_t^2\mathrm{d}x\mathrm{d}t
\end{aligned}$$

where $\gamma^{(1)}_t : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ *and* $\gamma^{(2)}_{\varepsilon,t} : \mathbb{R}_+ \to \mathbb{R}$ *are* C^1 *functions such that* $\lim_{t\to+\infty}\gamma^{(1)}_t(\omega) = \gamma^{(1)}(\omega)$ *and* $\lim_{t\to+\infty}\gamma^{(2)}_{\varepsilon,t} = \gamma^{(2)}_\varepsilon$.

Proof Observe that by Ito's formula

$$\mathbb{E}\left[\int \xi_\varepsilon W_\infty Z_\infty \mathrm{d}x\right] = \mathbb{E}\left[\int_0^\infty \int \xi_\varepsilon W_tJ_tu_t\mathrm{d}t\mathrm{d}x\right] = \mathbb{E}\left[\int_0^\infty J_t(\xi_\varepsilon W_t)u_t\mathrm{d}t\right]$$

and

$$\int \xi_\varepsilon Z_\infty^2 \mathrm{d}x = 2\int_0^\infty \int J_t(\xi_\varepsilon Z_t)u_t\mathrm{d}t\mathrm{d}x = 2\int_0^\infty \int J_t(\xi_\varepsilon \succ Z_t)u_t\mathrm{d}t\mathrm{d}x + 2\int_0^\infty \int J_t(\xi_\varepsilon \preccurlyeq Z_t)u_t\mathrm{d}t\mathrm{d}x$$

now consider the ansatz

$$u_s = -J_s\xi_\varepsilon W_s - J_s(\xi_\varepsilon \succ Z_s(u)) + l_s$$

Then we compute

$$\begin{aligned}
\frac{1}{2}\int_0^\infty \|u_s\|^2_{L^2}\mathrm{d}s &= \frac{1}{2}\int_0^\infty \|J_s\xi_\varepsilon W_s + J_s(\xi_\varepsilon \succ Z_s)\|^2_{L^2}\mathrm{d}s\\
&\quad - \int_0^\infty (J_s\xi_\varepsilon W_s + J_s(\xi_\varepsilon \succ Z_s))\, l_s\mathrm{d}s\\
&\quad + \frac{1}{2}\int_0^\infty \|l_s\|^2_{L^2}\mathrm{d}s\\
&= -\frac{1}{2}\int_0^\infty \|J_s\xi_\varepsilon W_s + J_s(\xi_\varepsilon \succ Z_s)\|^2_{L^2}\mathrm{d}s\\
&\quad - \int_0^\infty (J_s\xi_\varepsilon W_s + J_s(\xi_\varepsilon \succ Z_s))\, u_s\mathrm{d}s\\
&\quad + \frac{1}{2}\int_0^\infty \|l_s\|^2_{L^2}\mathrm{d}s
\end{aligned}$$

where we have used that

$$
\begin{aligned}
& -\int_0^\infty \|(J_s\xi_\varepsilon W_s + J_s(\xi_\varepsilon \succ Z_s))\|_{L^2}^2 \,\mathrm{d}x - \int_0^\infty (J_s\xi_\varepsilon W_s + J_s(\xi_\varepsilon \succ Z_s))\, u_s \mathrm{d}s \\
= \quad & -\int_0^\infty (J_s\xi_\varepsilon W_s + J_s(\xi_\varepsilon \succ Z_s))\, l_s \mathrm{d}x.
\end{aligned}
$$

We compute that

$$
\begin{aligned}
& \int_0^\infty \|J_s\xi_\varepsilon W_s + J_s(\xi_\varepsilon \succ Z_s)\|_{L^2}^2 \,\mathrm{d}x\mathrm{d}s \\
= \quad & \int_0^\infty \int (J_s\xi_\varepsilon W_s)^2 + 2\,(J_s\xi_\varepsilon W_s)\,(J_s(\xi_\varepsilon \succ Z_s)) + (J_s(\xi_\varepsilon \succ Z_s))^2 \mathrm{d}x\mathrm{d}s
\end{aligned}
$$

Now the first term is the last term on the r.h.s of (40). We also have that

$$
\int_0^\infty \int 2\,(J_s\xi_\varepsilon W_s)\,(J_s(\xi_\varepsilon \succ Z_s))\mathrm{d}x\mathrm{d}s = 2\int_0^\infty \int (J_s\xi_\varepsilon W_s \circ J_s\xi_\varepsilon)\, Z_s \mathrm{d}x\mathrm{d}s + \Gamma_5
$$

and

$$
\int_0^\infty \int (J_s(\xi_\varepsilon \succ Z_s))^2 \mathrm{d}x\mathrm{d}s = \int_0^\infty \int (J_s\xi_\varepsilon \circ J_s\xi_\varepsilon) Z_s^2 \mathrm{d}x\mathrm{d}s + \Gamma_4.
$$

Recall that we also have the counter terms

$$
2(\gamma^{(1)}(\omega) - \gamma_\varepsilon^{(2)}) \int W_\infty Z_\infty \mathrm{d}x + (\gamma^{(1)}(\omega) - \gamma_\varepsilon^{(2)}) \int Z_\infty^2 \mathrm{d}x
$$

available. Writing $\gamma_{\varepsilon,\infty} = \gamma^{(1)}(\omega) - \gamma_\varepsilon^{(2)}$ and using that $\gamma_{\varepsilon,t} : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a C^1 function, by Ito's formula we get

$$
2\gamma_{\varepsilon,\infty} \int W_\infty Z_\infty \mathrm{d}x = 2\int_0^\infty \int \dot{\gamma}_{\varepsilon,t} W_t Z_t \mathrm{d}x + 2\int_0^\infty \int \gamma_{\varepsilon,t} W_t J_t u_t \mathrm{d}x + \text{martingale}
$$

Now the first term on the r.h.s. is put together with the first term on the r.h.s of (3.1) to form Γ_2. We have also

$$
\gamma_{\varepsilon,\infty} \int Z_\infty^2 \mathrm{d}x = \int_0^\infty \int \dot{\gamma}_{\varepsilon,t} Z_t^2 \mathrm{d}x + 2\int_0^\infty \int \gamma_{\varepsilon,t} Z_t J_t u_t \mathrm{d}x\mathrm{d}t
$$

and the first term on the r.h.s goes together with the first term on the r.h.s of (3.1) to form Γ_3. The respective remainders are collected in Γ_6, Γ_7 and $\mathfrak{G}$. □

3.2 Some stochastic estimates

The main reason for the formulation in Proposition 7 is that, up to a diverging constant, every term of Γ_i converges to something finite as $\varepsilon \to 0$. In order to guarantee this convergence we need to give some estimates for the stochastic terms involving ξ_ε and W_s in equation (40). The proof of this kind of estimates is the chief aim of the current section.

Lemma 5 *Consider*

$$
\gamma_{\varepsilon,t}^{(2)} := \int_0^t \mathbb{E}[J_s(\xi_\varepsilon) \circ J_s(\xi_\varepsilon)]\mathrm{d}s \tag{41}
$$

then for every for any $0<\kappa<\delta$, $\alpha>0$ such that $\kappa+\alpha<\delta$ small enough, and for every $p\geqslant 1$, we have that

$$\sup_{\varepsilon\in(0,1)}\mathbb{E}[\|J_s(\xi_\varepsilon)\circ J_s(\xi_\varepsilon)-\dot{\gamma}^{(2)}_{\varepsilon,t}\|^p_{B^{-\delta}_{p,p}}]\lesssim(1+s)^{-(1+\kappa)p} \tag{42}$$

$$\sup_{\varepsilon\in(0,1)}\mathbb{E}\left[\int_0^1\frac{\|J_s(\xi_\varepsilon)\circ J_s(\xi_\varepsilon)-\dot{\gamma}^{(2)}_{\varepsilon,s}-J_{s+\Delta s}(\xi_\varepsilon)\circ J_{s+\Delta s}(\xi_\varepsilon)-\dot{\gamma}^{(2)}_{\varepsilon,s+\Delta s}\|^p_{B^{-\delta}_{p,p}}}{(\Delta s)^{1+\alpha p/2}}\mathrm{d}\Delta s\right] \lesssim (1+s)^{(-1-\kappa+\alpha)p}. \tag{43}$$

Remark 4 From equation (41), defining the function $\gamma^{(2)}_{\varepsilon,t}$, we get the bound

$$\gamma^{(2)}_{\varepsilon,t}\lesssim\log(\min(2+t,\varepsilon^{-1})).$$

Proof of Lemma 5 We start the proof in the case that $p=2$. In order to simplify the notation we drop the upper index (2) from $\gamma^{(2)}_\varepsilon$. First we prove that

$$\sup_{\varepsilon\in(0,1)}\mathbb{E}[(\|J_s(\xi_\varepsilon)\circ J_s(\xi_\varepsilon)-\dot{\gamma}_{\varepsilon,s}\|_{H^{-\delta}})^2]\lesssim(1+s)^{-2-2\kappa}. \tag{44}$$

Indeed, if $K_j=\mathcal{F}^{-1}(\varphi_j)$ (where $\{\varphi_j\}_{j\geqslant-1}$ is the dyadic partition of unity in the definition of Besov space $B^\kappa_{p,p}$, see Appendix A) we have

$$\begin{aligned}&\mathbb{E}[|K_r*(J_s(\xi_\varepsilon)\circ J_s(\xi_\varepsilon)-\dot{\gamma}_{\varepsilon,s})|^2]\\ \lesssim&\sum_{j\sim\log(s),r\lesssim j}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}K_r(x)K_r(y)(\mathbb{E}[J_s(\Delta_j(\xi_\varepsilon))(x)J_s(\Delta_j(\xi_\varepsilon))(y)])^2\end{aligned}$$

Clearly

$$J_s(\xi_\varepsilon)\circ J_s(\xi_\varepsilon)-\dot{\gamma}_{\varepsilon,s}=\sum_{j\sim\log(s)}(J_s(\Delta_j(\xi_\varepsilon)))^2-\dot{\gamma}_{\varepsilon,s}=\sum_{j\sim\log(s)}:J_s(\Delta_j(\xi_\varepsilon))^2:$$

and, by the properties of Wick product (see Proposition 5),

$$\begin{aligned}&\mathbb{E}[J_s(\Delta_j\xi_\varepsilon)(x)J_s(\Delta_j\xi_\varepsilon)(y)]\\ \lesssim&\sum_{k\in\mathbb{Z}^2}\frac{\sigma_s^2(k)|\varphi_j(k)|^2}{(|k|^2+m^2)}\lesssim-\int_{R's}^{Rs}\frac{r}{s^2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\rho\right)\left(\frac{r}{s}\right)\frac{r}{r^2+m^2}\mathrm{d}r\\ \lesssim&\left(\frac{1}{s}\int_{R's}^{Rs}\rho\left(\frac{r}{s}\right)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{r^2+m^2}\right)\mathrm{d}r+\frac{R'^2s^2}{R'^2s^2+m^2}\right)\\ \lesssim&\frac{1}{s}\int_{R's}^{Rs}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{r^2+m^2}\right)\mathrm{d}r\lesssim\frac{1}{s}\left(\frac{R^2s^2}{R^2s^2+m^2}\right)\end{aligned} \tag{45}$$

and the bound is uniform in $0\leqslant\varepsilon<1$. Recalling that

$$\begin{aligned}\|J_s(\xi_\varepsilon)\circ J_s(\xi_\varepsilon)-\dot{\gamma}_{\varepsilon,s}\|^2_{H^{-\delta}}\sim&\sum_{j\sim\log(s)}2^{-2\delta j}\|K_j*(J_s(\xi_\varepsilon)\circ J_s(\xi_\varepsilon)-\dot{\gamma}_{\varepsilon,s})\|^2_{L^2}\\ \lesssim_{\kappa,\delta}&\frac{1}{s^{2+2\kappa}}\sum_{j\geqslant-1}2^{-2(\delta-\kappa)j},\end{aligned}$$

where $\kappa < \delta$, inequality (45) implies the bound (42). For the bound (43), it is sufficient to consider the case $\log(s) \sim \log(s+1)$ (which holds whenever s is big enough)

$$\begin{aligned} & \mathbb{E}[\|K_j * [(J_s(\xi_\varepsilon) \circ J_s(\xi_\varepsilon) - \gamma_{\varepsilon,s}) - (J_{s+\Delta s}(\xi_\varepsilon) \circ J_{s+\Delta s}(\xi_\varepsilon) - \gamma_{\varepsilon,s+\Delta s})]\|_{L^2}^2] \\ \lesssim & \sum_{j\sim\log(s)} \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} |K_j(x)||K_j(y)|(\mathbb{E}[\Delta_j(J_s(\xi_\varepsilon) - J_{s+\Delta s}(\xi_\varepsilon))^2])^{1/2} \times \\ & (\mathbb{E}[|\Delta_j J_s(\xi_\varepsilon)|^2] + \mathbb{E}[|\Delta_j J_{s+\Delta s}(\xi_\varepsilon)|^2])^{3/2} \end{aligned}$$

In order to estimate the term $\mathbb{E}[|\Delta_j(J_s(\xi_\varepsilon) - J_{s+\Delta s}(\xi_\varepsilon))|^2]$ we note that

$$\begin{aligned} \mathbb{E}[|\Delta_j(J_s(\xi_\varepsilon) - J_{s+\Delta s}(\xi_\varepsilon))|^2] &= \sum_{k\in\mathbb{Z}^2} \frac{|\sigma_s(k) - \sigma_{s+\Delta s}(k)|^2 \varphi_j(k)}{(|k|^2+m^2)} \\ &\lesssim \int_0^{R(s+\Delta s)} |\sigma_s(r) - \sigma_{s+\Delta s}(r)|^{2\alpha} \frac{r(|\sigma_s(r)|^2 + |\sigma_{s+\Delta s}(r)|^2)^{1-\alpha}}{(r^2+m^2)} \mathrm{d}r \\ &\lesssim |\Delta s|^\alpha (\log(s+\Delta s))^\alpha \\ &\quad \left(\int_0^{R(s+\Delta s)} \frac{r(|\sigma_s(r)|^2 + |\sigma_{s+\Delta s}(r)|^2)^{1-\alpha}}{(r^2+m^2)} \mathrm{d}r\right) \\ &\lesssim |\Delta s|^\alpha \frac{(\log(s+\Delta s))^\alpha}{(1+s)^{(1-\alpha)}} \lesssim |\Delta s|^\alpha \frac{1}{(1+s)^{1-2\alpha}}, \end{aligned} \tag{46}$$

where in the last step we do the same computation of (45), taking into account the loss of a power α. Furthermore we have

$$\begin{aligned} & : J_s(\xi_\varepsilon) \circ J_s(\xi_\varepsilon) - J_{s+\Delta s}(\xi_\varepsilon) \circ J_{s+\Delta s}(\xi_\varepsilon) : \\ = & : (J_s(\xi_\varepsilon) - J_{s+\Delta s}(\xi_\varepsilon)) \circ J_s(\xi_\varepsilon) : + : J_{s+\Delta s}(\xi_\varepsilon) \circ (J_s(\xi_\varepsilon) - J_{s+\Delta s}(\xi_\varepsilon)) : \end{aligned}$$

Thus, by the proof of the first part of the present lemma, inequality (45) and inequality (46),

$$\begin{aligned} & \mathbb{E}[\|K_j * [(J_s(\xi_\varepsilon) \circ J_s(\xi_\varepsilon) - \gamma_{\varepsilon,s}) - (J_{s+\Delta s}(\xi_\varepsilon) \circ J_{s+\Delta s}(\xi_\varepsilon) - \gamma_{\varepsilon,s+\Delta s})]\|_{L^2}^2] \\ \lesssim & \mathbb{E}[\|K_j * [: (J_s(\xi_\varepsilon) - J_{s+\Delta s}(\xi_\varepsilon)) \circ J_s(\xi_\varepsilon) :]\|_{L^2}^2] + \\ & + \mathbb{E}[\|K_j * [: J_{s+\Delta s}(\xi_\varepsilon) \circ (J_s(\xi_\varepsilon) - J_{s+\Delta s}(\xi_\varepsilon)) :]\|_{L^2}^2] \\ \lesssim & |\Delta s|^\alpha \frac{1}{(1+s)^{2-2\alpha}}. \end{aligned}$$

This concludes the proof for the case $p = 2$. The case of $p > 2$ can be obtained by the previous bounds and hypercontractivity applied to the second degree Gaussian polynomial $(J_s(\xi_\varepsilon) \circ J_s(\xi_\varepsilon) - \gamma_{\varepsilon,s})$. □

Remark 5 The result of Lemma 5 can be easily extended to the case where the resonant product $J_s(\xi_\varepsilon) \circ J_\varepsilon(\xi_\varepsilon)$ is replaced by the standard product $J_s(\xi_\varepsilon)J_s(\xi_\varepsilon)$, namely we have the following result: for every $\delta > 0, \ell < 1$, there is $c > 0$ such that for any $\delta > 0$, $p \geqslant 2$ and there is we have

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}[\|(1+s)^\ell \|J_s(\xi_\varepsilon)J_s(\xi_\varepsilon) - \gamma_{\varepsilon,s}^{(2)}\|_{B^{-\delta}_{p,p}}\|^p_{B^{c\delta}_{p,p}(\mathbb{R}_+)}] < +\infty$$

Lemma 6 *For any $\delta > 0$ there is $\kappa < \frac{1}{2}$ small enough such that we have*

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}\left[\|(J_s(\xi_\varepsilon W_s) J_s \xi_\varepsilon - J_s \xi_\varepsilon J_s \xi_\varepsilon W_s)\|^2_{H^{-\delta}(\mathbb{T}^2)}\right] \lesssim (1+s)^{-3+2\kappa}.$$

Proof We do the explicit computation in the case $\varepsilon = 0$. The case $\varepsilon < 1$ and the uniformity of the inequality with respect to $0 < \varepsilon < 1$ can be obtained with a similar method. We compute

$$\begin{aligned}
& \mathbb{E}[\|(J_s(\xi W_s) - (J_s\xi)W_s)J_s\xi\|^2_{H^{-\delta}}] \\
= & \sum_n \langle n\rangle^{-2\delta}\mathbb{E}\left(\sum_{n_1}\sum_{m_1}(J_s(n_1-n) - J_s(n_1-n-m_1))\xi(m_1-n_1-n)W_s(m_1)J_s(n_1)\xi(n_1)\right)^2 \\
= & \sum_n \langle n\rangle^{-2\delta}\mathbb{E}\left[\sum_{n_1,n_2}\sum_{m_1,m_2}(J_s(n_1-n) - J_s(n_1-n-m_1))(J_s(n_2-n) - J_s(n_2-n-m_2))\times\right. \\
& \left.\hat{\xi}(n_1-m_1-n)\hat{\xi}(n_2-m_2-n)\hat{W}_s(m_1)\hat{W}_s(m_2)J_s(n_1)\hat{\xi}(n_1)J_s(n_2)\hat{\xi}(n_2)\right] \\
= & \sum_n \langle n\rangle^{-2\delta}\sum_{n_1,n_2}\sum_{m_1,m_2}\Big((J_s(n_1-n) - J_s(n_1-n-m_1))(J_s(n_2-n) - J_s(n_2-n-m_2))\times \\
& J_s(n_1)J_s(n_2)\mathbb{E}\left[\hat{\xi}(n_1)\hat{\xi}(n_2)\hat{\xi}\left(n_1-m_1-n\right)\hat{\xi}(n_2-m_2-n)\right]\mathbb{E}[\hat{W}_s(m_1)\hat{W}_s(m_2)]\Big) \\
= & \sum_n \langle n\rangle^{-2\delta}\sum_{n_1}\sum_{m_1\leqslant s}(J_s(n_1-n) - J_s(n_1-m_1-n))^2\frac{1}{|m+m_1|^2}J_s^2(n_1) \\
& + \sum_{n\leqslant s}\langle n\rangle^{-2\delta}\sum_{n_1,n_2}\frac{1}{|m+n|^2}(J_s(n_1-n) - J_s(n_1))(J_s(n_2-n) - J_s(n_2))J_s(n_1)J_s(n_2) \\
& + \sum_{n\leqslant s}\langle n\rangle^{-2\delta}\sum_{n_1}\frac{1}{|m+n|^2}(J_s(n_1-n) - J_s(n_1))(J_s(n_2-n) - J_s(n_2))J_s(n_1)J_s(n_2) \\
= & \mathrm{I} + \mathrm{II} + \mathrm{III} \qquad (47)
\end{aligned}$$

Estimate on I:

Observe that J_s is supported in an annulus of radius s and we are restricting to $m_1 \leqslant s$. This means we can rewrite

$$\begin{aligned}
& \mathbb{1}_{\{|m_1|\leqslant s\}}(J_s(n_1-n) - J_s(n_1-m_1-n))^2 \\
= & \mathbb{1}_{\{|m_1|\leqslant 2|n-n_1|\}}(J_s(n_1-n) - J_s(n_1-m_1-n))^2 + \left(\mathbb{1}_{\{|n-n_1|\leqslant|m_1|/2\}}\right)J_s(n_1-n) - J_s(n_1-m_1-n) \\
\leqslant & \langle s\rangle^{-1-\kappa}\langle n_1-n\rangle^{-2+2\kappa}\langle m_1\rangle^{-\kappa}.
\end{aligned}$$

$$\begin{aligned}
& \mathbb{1}_{\{|m_1|\leqslant s\}}(J_s(n_1-n) - J_s(n_1-m_1-n))^2 \\
\leqslant & \mathbb{1}_{\{|m_1|\leqslant 2|n-n_1|\}}(J_s(n_1-n) - J_s(n_1-m_1-n))^2 + \left(\mathbb{1}_{\{|n-n_1|\leqslant|m_1|/2\}}\right)J_s^2(n_1-n-m_1) \\
\leqslant & \langle s\rangle^{-1-\kappa}\langle n_1-n\rangle^{-2+2\kappa}\langle m_1\rangle^{-\kappa}
\end{aligned}$$

for some $\kappa > 0$ small enough. Then plugging the previous inequality in (47) we get

$$\begin{aligned}
& \mathbb{E}[\|(J_s(\xi W_s) - (J_s\xi)W_s)J_s\xi\|^2_{H^{-\delta}}] \\
\leqslant & \sum_n \langle n\rangle^{-2\delta}\sum_{n_1}\sum_{m_1\leqslant s}\langle s\rangle^{-3-2\kappa}\langle n_1-n\rangle^{-2+2\kappa}\langle m_1\rangle^{-2-\kappa}\langle n_1\rangle^{-2} \\
\lesssim & \langle s\rangle^{-3+2\kappa}
\end{aligned}$$

Estimate on II: One can easily check that

$$\begin{aligned}
\nabla_k J_s(k) & \lesssim \mathbb{1}_{k\backsim s}(\langle s\rangle^{-1/2}\langle k\rangle^{-2} + \langle s\rangle^{-3/2}\langle k\rangle^{-1}) \\
& \lesssim \langle t\rangle^{-5/2}
\end{aligned}$$

So

$$|(J_s(n_1 - n) - J_s(n_1))| \lesssim \langle t\rangle^{-5/2}|n|.$$

and we have $J_s(k) \lesssim t^{-3/2}$.

Recall also that $|J_s(n_1)| \lesssim \mathbb{1}_{|n_1|\lesssim s}\langle s\rangle^{-1/2}\langle n_1\rangle$ so $\sum_{n_1} J_s(n_1) \lesssim \langle s\rangle^{1/2}$

Plugging this into the sum we get

$$\begin{aligned}
& \sum_{n\leqslant s}\langle n\rangle^{-2s}\sum_{n_1}\frac{1}{|m+n|^2}(J_s(n_1-n) - J_s(n_1)) \\
& \times(J_s(n_2-n) - J_s(n_2))J_s(n_1)J_s(n_2) \\
\lesssim & \sum_{n\leqslant s}\langle n\rangle^{-2s}\frac{1}{|m+n|^2}\langle s\rangle^{-8/2}|n|\sum_{n_1,n_2}|J_s(n_1)J_s(n_2)| \\
\lesssim & \sum_{n\leqslant s}\langle n\rangle^{-2s}\frac{1}{|m+n|^2}\langle s\rangle^{-4}\langle s\rangle|n| \\
\lesssim & \sum_{n\leqslant s}\langle n\rangle^{-2s}\frac{1}{|m+n|}\langle s\rangle^{-3} \\
\lesssim & \langle s\rangle^{-3}
\end{aligned}$$

So

$$\begin{aligned}
& \int_0^\infty \mathbb{E}[\|(J_s(\xi_\varepsilon W_s) - (J_s\xi_\varepsilon)W_s)J_s\xi_\varepsilon\|_{H^{-s}}]\mathrm{d}s \\
\leqslant & \int_0^\infty \mathbb{E}[\|(J_s(\xi_\varepsilon W_s) - (J_s\xi)W_s)J_s\xi_\varepsilon\|_{H^{-s}}^2]^{1/2}\mathrm{d}s \\
\lesssim & \int \langle s\rangle^{-3/2}\mathrm{d}s.
\end{aligned}$$

Finally the estimate on III is a simpler version of the estimate on II. □

Lemma 7 *For any $0 < \kappa < \delta$ and for any $p \geqslant 2$ we have*

$$\sup_{\varepsilon\in(0,1)}\mathbb{E}\left[\left\|\left(J_s(\xi_\varepsilon W_s)J_s\xi_\varepsilon - \dot{\gamma}^{(2)}_{\varepsilon,s}W_s\right)\right\|^p_{H^{-\delta}}\right] \lesssim (1+s)^{(-1-\kappa)p}.$$

Proof We have that

$$\begin{aligned}
& \left\|\left(J_s(\xi_\varepsilon W_s)J_s\xi_\varepsilon - \dot{\gamma}^{(2)}_{\varepsilon,s}W_s\right)\right\|_{H^{-\delta}} \\
\leqslant & \|(J_s(\xi_\varepsilon W_s)J_s\xi_\varepsilon - J_s\xi_\varepsilon J_s\xi_\varepsilon W_s)\|_{H^{-\delta}} + \left\|\left(J_s(\xi_\varepsilon)J_s\xi_\varepsilon - \dot{\gamma}^{(2)}_{\varepsilon,s}\right)W_s\right\|_{H^{-\delta}}.
\end{aligned}$$

The first term has been estimated in Lemma 6. To estimate the second part denote

$$f = \left(J_s(\xi_\varepsilon)J_s\xi_\varepsilon - \dot{\gamma}^{(2)}_{\varepsilon,s}\right).$$

Note that f is independent of $\mathcal{F}'$. We will show that for any $\kappa > 0$

$$\mathbb{E}_{w'}[\|fW_s\|^2_{H^{-\delta}}] \lesssim s^{\kappa/2+\delta}\|f\|^2_{H^{-\delta}}.$$

Indeed

$$
\begin{aligned}
& \mathbb{E}_{w'}[\|fW_s\|^2_{H^{-\delta}}] \\
= & \sum_{n\in\mathbb{Z}^2} \langle n\rangle^{-2\delta} \mathbb{E}\left(\sum_{k\in\mathbb{Z}^2} \hat{f}(n-k)\hat{W}_s(k)\right)^2 \\
\lesssim & \sum_{n\in\mathbb{Z}^2} \langle n\rangle^{-2\delta} \sum_{k\in\mathbb{Z}^2, |k|\lesssim s} \hat{f}(n-k)^2 \frac{1}{\langle k\rangle^2} \\
\lesssim & \; s^{\kappa} \sum_{n\in\mathbb{Z}^2}\sum_{k\in\mathbb{Z}^2} \langle n\rangle^{-2\delta} \frac{1}{\langle k\rangle^{2+\kappa}} \hat{f}(n-k)^2 \\
\lesssim & \sum_{n\in\mathbb{Z}^2}\sum_{k\in\mathbb{Z}^2} \frac{1}{\langle k\rangle^{2+\kappa-2\delta}} \frac{1}{\langle n-k\rangle^{2\delta}} \hat{f}(n-k)^2
\end{aligned}
$$

Now by Young's convolution inequality

$$
\left\| \sum_{k\in\mathbb{Z}^2}\sum_{k\in\mathbb{Z}^2} \frac{1}{\langle k\rangle^{2+\kappa-2\delta}} \frac{1}{\langle n-k\rangle^{2\delta}} \hat{f}(n-k)^2 \right\|_{l^1_n} \lesssim \left\| \frac{1}{\langle\cdot\rangle^{2\delta}} \hat{f}(\cdot)^2 \right\|_{l^1} \left\| \frac{1}{\langle\cdot\rangle^{2+\kappa-2\delta}} \right\|_{l^1}
$$

from which we can conclude.

In the case $p = 2$, the result then follows from Lemma 5 (see also Remark 5) and Lemma 6. The general case can be proved using the fact that $\left(J_s(\xi_\varepsilon)J_s\xi_\varepsilon - \dot{\gamma}^{(2)}_{\varepsilon,s}\right) W_s$ is a third degree polynomial and then applying hypercontractivity. □

Lemma 8 *We have that for any $\delta > 0$ and $p \geqslant 1$ there is $0 < \kappa \ll 1$ for which*

$$
\sup_{\varepsilon\in(0,1)} \mathbb{E}[\|\xi_\varepsilon W_s\|^p_{B^{-1-\delta}_{p,p}(\mathbb{T}^2)}] \lesssim (\log(1+s))^p,
$$

$$
\sup_{\varepsilon\in(0,1)} \mathbb{E}\left[\int_{-1}^{1} \frac{\|\xi_\varepsilon W_s - \xi_\varepsilon W_{s+\Delta s}\|^p_{B^{-1-\delta}_{p,p}(\mathbb{T}^2)}}{|\Delta s|^{1+\alpha p}} \mathrm{d}\Delta s\right] \lesssim (1+s)^{2\kappa p}.
$$

Proof The proof is similar to the one of Lemma 5, we report here only the main steps. First we note that, since W_s is independent of ξ_ε, we have $W_s\xi_\varepsilon =: W_s\xi_\varepsilon :$ this means that

$$
\begin{aligned}
\mathbb{E}[\|K_j * : W_s\xi_\varepsilon :\|^2_{L^2}] \;\lesssim\; & \int_0^s \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} (J^{*2}_\tau(0)) K_j(x)K_j(x)\mathrm{d}x\mathrm{d}\tau \\
\lesssim\; & \left(\int_0^s \|J^{*2}_\tau\|_{L^\infty}\mathrm{d}\tau\right)\left(\int_{\mathbb{T}^2} |K_j|^2\mathrm{d}x\right) \lesssim \left(\int_0^s \|J^{*2}_\tau\|_{L^\infty}\mathrm{d}\tau\right) 2^{2j}
\end{aligned}
$$

We also have

$$
\left(\int_0^s \|J^{*2}_\tau\|_{L^\infty}\mathrm{d}\tau\right) = \int_0^t \sum_{k\in\mathbb{Z}^2} \frac{|\sigma_\tau(k)|^2}{(|k|^2+1)} \lesssim \sum_{k\in\mathbb{Z}^2} \frac{\rho_t(k)}{(|k|^2+1)} \lesssim \log(1+s).
$$

The second bound and the generic case $p > 2$ can be obtained as in Lemma 5. □

3.3 Analytical estimates

We want now prove some estimates on the terms Γ_i, $i = 1, \ldots, 7$, appearing in the expansion (40) of F. The main aim is to prove some upper bounds depending on the sums involving either purely stochastic terms or the positive terms$\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s$ or $\int_0^\infty \frac{\|Z_s\|_{L^2}^2}{(1+s)^{1+\kappa}} \mathrm{d}s$ since this kind of term can be compensated by the positive parts of F (namely $\mathfrak{G} + \frac{1}{2}\int_0^\infty \|l_t(u)\|_{L^2}^2 \mathrm{d}s$).

Lemma 9 *For every* $0 < \delta$ *we have*

$$\sup_{t\in[0,\infty]} \|Z_t(u)\|_{H^{1-\delta}}^2 \leqslant \int_0^\infty \|u_s\|_{H^{-\delta}}^2 \mathrm{d}s$$

Proof We have that

$$\begin{aligned}
\|Z_t(u)\|_{H^{1-\delta}}^2 &= \sum_{k\in\mathbb{Z}^2} (m^2 + |k|^2)^{1-\delta} |\mathcal{F}(Z_t)(k)|^2 \\
&= \sum_{k\in\mathbb{Z}^2} (m^2 + |k|^2)^{1-\delta} \left| \int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}s}\rho_s(k)\right)^{1/2} \frac{\mathcal{F}(u_s)(k)}{\sqrt{(m^2+|k|^2)}} \mathrm{d}s \right|^2 \\
&= \sum_{k\in\mathbb{Z}^2} (m^2 + |k|^2)^{-\delta} \left(\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\rho_s(k)\mathrm{d}s\right) \left(\int_0^t |\mathcal{F}(u_s)(k)|^2 \mathrm{d}s\right) \\
&= \int_0^t \sum_{k\in\mathbb{Z}^2} |\rho_t(k)| (m^2 + |k|^2)^{-\delta} |\mathcal{F}(u_s)(k)|^2 \mathrm{d}s \\
&\leqslant \int_0^t \sum_{k\in\mathbb{Z}^2} (m^2 + |k|^2)^{-\delta} |\mathcal{F}(u_s)(k)|^2 \mathrm{d}s = \int_0^t \|u_s\|_{H^{-\delta}}^2 \mathrm{d}s \leqslant \int_0^\infty \|u_s\|_{H^{-\delta}}^2 \mathrm{d}s.
\end{aligned}$$

□

We collect the bounds of the Γ_i appearing in Proposition 7.

Lemma 10 *For any* $\kappa \in (0,1)$ *and for any* $0 < \delta \ll 1$ *there are* $\alpha > 0$ *and* $0 < \theta < 1$, $0 < \lambda < \tau < \delta \ll 1$, $\eta > 0$, *and* $0 < \ell \ll 1$ *such that we get*

$$\begin{aligned}
|\Gamma_1| &\lesssim \kappa \int_0^\infty \|u_s\|_{H^{-\delta}}^2 \mathrm{d}s + \frac{1}{\kappa^\alpha} \|\xi_\varepsilon\|_{\mathcal{C}^{-1-\delta}}^{\frac{2}{\delta}} \int_0^{+\infty} \frac{\|Z_t\|_{L^2}^2}{(1+t)^{3/\delta-8}} \mathrm{d}t \\
|\Gamma_2| &\lesssim \kappa \int_0^{+\infty} \|u_t\|_{H^{-\delta}}^2 \mathrm{d}t + \frac{1}{\kappa^\alpha} \left(\int_0^\infty \left\| \left(J_s\xi_\varepsilon W_s \circ J_s\xi_\varepsilon - \dot{\gamma}_{\varepsilon,s}^{(2)} W_s\right) \right\|_{H^{-\delta}} \mathrm{d}s\right)^2 \\
|\Gamma_3| &\lesssim \kappa \int_0^{+\infty} \|u_t\|_{H^{-\delta}}^2 \mathrm{d}t + \frac{1}{\kappa^\alpha} \int_0^\infty \frac{(\sup(1+s)^{1+\tau-\lambda} \|((J_s\xi_\varepsilon \circ J_s\xi_\varepsilon) - \dot{\gamma}_{\varepsilon,s}^{(2)})\|_{\mathcal{C}^{-\delta}})^{\frac{1}{1-\theta}}}{(1+s)^{1+\tau-\lambda}} \|Z_s\|_{L^2}^2 \mathrm{d}s \\
|\Gamma_4| &\lesssim \kappa \int_0^{+\infty} \|u_t\|_{H^{-\delta}}^2 \mathrm{d}t + \frac{1}{\kappa^\alpha} \int_0^{+\infty} \frac{\|\xi_\varepsilon\|_{\mathcal{C}^{-1-\delta}}^2 \|Z_s\|_{L^2}^2}{(1+t)^{1+\eta}} \mathrm{d}t
\end{aligned}$$

as well as

$$
\begin{aligned}
|\Gamma_5| &\lesssim \kappa\int_0^t \|u\|^2_{H^{-\delta}}\mathrm{d}t+\frac{1}{\kappa^\alpha}\|\xi\|^2_{\mathcal{C}^{-1-\delta}}(\sup_{s\in\mathbb{R}_+}(1+s)^\ell\|\xi W_s\|^2_{\mathcal{C}^{-1-\delta}})\\
|\Gamma_6| &\lesssim \kappa\int_0^\infty \|u_t\|^2_{H^{-\delta}}\mathrm{d}t+\frac{1}{\kappa^\alpha}\left(\int_0^\infty\frac{|\gamma^{(2)}_{\varepsilon,t}|^2}{(1+t)^{3/2-2\delta}}\|W_t\|^2_{\mathcal{C}^{-\delta}}\mathrm{d}t+\int_0^\infty\frac{|\gamma^{(2)}_{\varepsilon,t}|^{\frac{2}{1-\delta}}}{(1+t)^{3/2}}\|Z_t\|^2_{L^2}\mathrm{d}t\right)\\
|\Gamma_7| &\lesssim \kappa\int_0^{+\infty}\|u_t\|^2_{H^{-\delta}}\mathrm{d}t+\frac{1}{\kappa^\alpha}\left(\int_0^{+\infty}|\dot\gamma^{(1)}_t(\omega)|\|W_t\|_{\mathcal{C}^{-\delta}}\mathrm{d}t\right)^2\\
&\quad+\frac{1}{\kappa^\alpha}\int_0^{+\infty}|\gamma^{(1)}_t|^2\frac{\|W_t\|^2_{\mathcal{C}^{-\delta}}}{(1+t)^{3/2-2\delta}}\mathrm{d}t+\frac{1}{\kappa^\alpha}\int_0^\infty\frac{|\gamma^{(1)}_t|^{\frac{2}{1-\delta}}}{(1+t)^{3/2}}\|Z_t\|^2_{L^2}\mathrm{d}t.
\end{aligned}
$$

where the implied constants in the $\lesssim$ *do not depend on* $\kappa>0$*, where* $\Gamma_1,\dots,\Gamma_7$ *are defined as in Proposition 7.*

Proof The proof is essentially an application of the results in Section 3.2, Lemma 9, Besov embeddings, products properties and Young's inequality, see Appendix A. We report here only the main passages of the computations.

Γ_1 can be bounded as follows

$$
\begin{aligned}
|\Gamma_1| &\lesssim \int_0^\infty(1+t)^{-3/2+4\delta}\|\xi_\varepsilon\preccurlyeq Z_t\|_{H^{-3\delta}}\|u_t\|_{H^{-\delta}}\mathrm{d}t\\
&\lesssim \int_0^\infty(1+t)^{-3/2+4\delta}\|\xi_\varepsilon\|_{\mathcal{C}^{-1-\delta}}\|Z_t\|^{1-\delta}_{H^{1-\delta}}\|Z\|^{\delta}_{L^2}\|u_t\|_{H^{-\delta}}\mathrm{d}t\\
&\lesssim \kappa\int_0^\infty\|u_s\|^2_{H^{-\delta}}\mathrm{d}s+\frac{1}{\kappa^\alpha}\|\xi_\varepsilon\|^{\frac{2}{\delta}}_{\mathcal{C}^{-1-\delta}}\int_0^{+\infty}(1+t)^{-3/\delta+8}\|Z_t\|^2_{L^2}\mathrm{d}t.
\end{aligned}
$$

For Γ_2 we have

$$
\begin{aligned}
|\Gamma_2| &\lesssim \sup_{s\in\mathbb{R}_+}\|Z_s\|_{H^\delta}\int_0^\infty\left\|\left(J_s\xi_\varepsilon W_s\circ J_s\xi_\varepsilon-\dot\gamma^{(2)}_{\varepsilon,s}W_s\right)\right\|_{H^{-\delta}}\mathrm{d}s\\
&\lesssim \kappa\int_0^{+\infty}\|u_t\|^2_{H^{-\delta}}\mathrm{d}t+\frac{1}{\kappa^\alpha}\left(\int_0^\infty\left\|\left(J_s\xi_\varepsilon W_s\circ J_s\xi_\varepsilon-\dot\gamma^{(2)}_{\varepsilon,s}W_s\right)\right\|_{H^{-\delta}}\mathrm{d}s\right)^2.
\end{aligned}
$$

We bound Γ_3 as follows

$$
\begin{aligned}
|\Gamma_3| &\lesssim \int_0^\infty\|((J_s\xi_\varepsilon\circ J_s\xi_\varepsilon)-\dot\gamma^{(2)}_{\varepsilon,s})\|_{\mathcal{C}^{-\delta}}\|Z^2_s\|_{B^\delta_{1,1}}\mathrm{d}s\\
&\lesssim \int_0^\infty\frac{1}{(1+s)^{1+\tau-\lambda}}(\sup(1+s)^{1+\tau-\lambda}\|((J_s\xi_\varepsilon\circ J_s\xi_\varepsilon)-\dot\gamma^{(2)}_{\varepsilon,s})\|_{\mathcal{C}^{-\delta}})\times\\
&\quad\times\|Z_s\|^{2\theta}_{H^{1-\delta}}\|Z_s\|^{2(1-\theta)}_{L^2}\mathrm{d}s\lesssim\kappa\int_0^{+\infty}\|u_t\|^2_{H^{-\delta}}\mathrm{d}t+\\
&\quad+\frac{1}{\kappa^\alpha}\int_0^\infty\frac{(\sup(1+s)^{1+\tau-\lambda}\|((J_s\xi_\varepsilon\circ J_s\xi_\varepsilon)-\dot\gamma^{(2)}_{\varepsilon,s})\|_{\mathcal{C}^{-\delta}})^{\frac{1}{1-\theta}}}{(1+s)^{1+\tau-\lambda}}\|Z_s\|^2_{L^2}\mathrm{d}s.
\end{aligned}
$$

By Proposition 16 equation (95) we have for Γ_4 the bound

$$\begin{aligned}|\Gamma_4| &\lesssim \int_0^{+\infty}(1+t)^{-1+\delta}\|\xi_\varepsilon\|_{\mathcal{C}^{-1-\delta}}\|Z_t\|^2_{H^{1/2-\delta/2}}\mathrm{d}t\\ &\lesssim \int_0^{+\infty}(\sup_{s\in\mathbb{R}_+}\|Z_s\|_{H^{1-\delta}})(1+t)^{-1-\eta}\|\xi_\varepsilon\|_{\mathcal{C}^{-1-\delta}}\|Z_t\|_{L^2}\mathrm{d}t\\ &\lesssim \kappa\int_0^{+\infty}\|u_t\|^2_{H^{-\delta}}\mathrm{d}t+\frac{1}{\kappa}\int_0^{+\infty}(1+t)^{-1-\eta}\|\xi_\varepsilon\|^2_{\mathcal{C}^{-1-\delta}}\|Z_t\|^2_{L^2}\mathrm{d}t.\end{aligned}$$

Using (96) from Proposition 16 Γ_5 can be estimated as

$$\begin{aligned}|\Gamma_5| &\lesssim \int_0^{+\infty}(1+s)^{-\frac{3}{2}}\|Z\|_{H^{1-\delta}}\|\xi\|_{\mathcal{C}^{-1-\delta}}\|\xi W_s\|_{\mathcal{C}^{-1-\delta}}\mathrm{d}s\\ &\lesssim \kappa\int_0^t\|u\|^2_{H^{-\delta}}\mathrm{d}t+\frac{1}{\kappa}\|\xi_\varepsilon\|^2_{\mathcal{C}^{-1-\delta}}(\sup_{s\in\mathbb{R}_+}(1+s)^{-2\ell}\|\xi_\varepsilon W_s\|^2_{\mathcal{C}^{-1-\delta}}).\end{aligned}$$

For Γ_6 we bound

$$\begin{aligned}|\Gamma_6| &\lesssim \int_0^\infty\frac{\gamma^{(2)}_{\varepsilon,t}}{(1+t)^{3/2-2\delta}}\|W_t\|_{\mathcal{C}^{-\delta}}\|u_t\|_{H^{-\delta}}+\int_0^\infty\frac{\gamma^{(2)}_{\varepsilon,t}}{(1+t)^{3/2}}\|Z_t\|_{H^\delta}\|u_t\|_{H^{-\delta}}\mathrm{d}t\\ &\lesssim \kappa\int_0^\infty\|u_t\|^2_{H^{-\delta}}\mathrm{d}t+\frac{1}{\kappa^\alpha}\left(\int_0^\infty\frac{|\gamma^{(2)}_{\varepsilon,t}|^2}{(1+t)^{3/2-2\delta}}\|W_t\|^2_{\mathcal{C}^{-\delta}}\mathrm{d}t+\int_0^\infty\frac{|\gamma^{(2)}_{\varepsilon,t}|^{\frac{2}{1-\delta}}}{(1+t)^{3/2}}\|Z_t\|^2_{L^2}\mathrm{d}t\right).\end{aligned}$$

Lastly, for Γ_7 we estimate

$$\begin{aligned}|\Gamma_7| &\lesssim \int_0^{+\infty}|\dot\gamma^{(1)}_t(\omega)|\|W_t\|_{\mathcal{C}^{-\delta}}\|Z_t\|_{H^\delta}\mathrm{d}t+\int_0^\infty|\gamma^{(1)}_t|\frac{\|W_t\|_{\mathcal{C}^{-\delta}}}{(1+t)^{3/2-2\delta}}\|u_t\|_{H^{-\delta}}\mathrm{d}t+\\ &\quad+\int_0^\infty\frac{|\gamma^{(1)}_t|}{(1+t)^{3/2}}\|Z_t\|_{H^\delta}\|u_t\|_{H^{-\delta}}\mathrm{d}t\\ &\lesssim \kappa\int_0^{+\infty}\|u_t\|^2_{H^{-\delta}}\mathrm{d}t+\frac{1}{\kappa^\alpha}\left(\int_0^{+\infty}|\dot\gamma^{(1)}_t(\omega)|\|W_t\|_{\mathcal{C}^{-\delta}}\mathrm{d}t\right)^2\\ &\quad+\frac{1}{\kappa^\alpha}\int_0^{+\infty}|\gamma^{(1)}_t|^2\frac{\|W_t\|^2_{\mathcal{C}^{-\delta}}}{(1+t)^{3/2-2\delta}}\mathrm{d}t+\frac{1}{\kappa^\alpha}\int_0^\infty\frac{|\gamma^{(1)}_t|^{\frac{2}{1-\delta}}}{(1+t)^{3/2}}\|Z_t\|^2_{L^2}\mathrm{d}t.\end{aligned}$$

□

3.4 Bounds on F^ε

We start with a preliminary result that gives us the existence of a family of reference drifts depending on $\xi(\omega)$ which will be needed later on.

Lemma 11 *There exists a family of adapted processes u^ε such that*

$$\sup_{\varepsilon>0}\mathbb{E}\left[\int_0^\infty\|l^\varepsilon_s(u^\varepsilon)\|^2_{L^2}\mathrm{d}t\right]<\infty\quad\textit{and}\quad\sup_{\varepsilon>0}F^\varepsilon(u^\varepsilon,\omega)<\infty,$$

almost surely with respect to ω.

Proof Take u^ε to be a solution to the equation

$$u^\varepsilon_s = -\mathbb{1}_{s>T} J_s W_s \xi_\varepsilon - \mathbb{1}_{s>T} J_s (\xi_\varepsilon \succ Z_s(u)) \tag{48}$$

for some fixed $T > 0$ to be chosen later, independently of ε. Assume for the moment that this solution exists and satisfies

$$\sup_{\varepsilon>0} \mathbb{E}\left[\int_0^\infty \|u^\varepsilon_t\|^2_{H^{-\delta}} \mathrm{d}t\right] < \infty \tag{49}$$

for some small $\delta > 0$. Then

$$l^\varepsilon_s(u) = -\mathbb{1}_{s\leqslant T} J_s W_s \xi_\varepsilon - \mathbb{1}_{s\leqslant T} J_s (\xi_\varepsilon \succ Z_s(u))$$

so

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty \|l^\varepsilon_s(u^\varepsilon)\|^2_{L^2} \mathrm{d}t\right] &\lesssim \mathbb{E}\int_0^T \|J_s W_s \xi_\varepsilon\|^2_{L^2} \mathrm{d}s + \mathbb{E}\int_0^T \|J_s(\xi_\varepsilon \succ Z_s(u))\|^2_{L^2} \mathrm{d}s \\
&\lesssim C(T) + C(T)\|Z_s(u)\|^2_{H^{1-\delta}} \\
&\lesssim C(T)\left(1 + \sup_\varepsilon \mathbb{E}\left[\int_0^\infty \|u^\varepsilon_t\|^2_{H^{-\delta}} \mathrm{d}t\right]\right)
\end{aligned}$$

From Section 3.3 we have that

$$|F^\varepsilon(u^\varepsilon, \omega)| \lesssim 1 + \mathbb{E}\left[\int_0^\infty \|l^\varepsilon_s(u^\varepsilon)\|^2_{L^2} \mathrm{d}t\right].$$

This proves the assertion.

Now let us establish that (48) has a solution. Consider the map

$$\Phi(u) = -\mathbb{1}_{s>T} J_s W_s \xi_\varepsilon - \mathbb{1}_{s>T} J_s (\xi_\varepsilon \succ Z_s(u)).$$

We show that Φ is a contraction for T large enough. Indeed

$$\begin{aligned}
\mathbb{E}\|\Phi(u)\|^2_{H^{-\delta}} &\lesssim -\mathbb{1}_{s>T}\mathbb{E}[\|J_s W_s \xi_\varepsilon\|^2_{H^{-\delta}}] + \mathbb{E}\|J_s(\xi_\varepsilon \succ Z_s(u))\|^2_{H^{-\delta}} \\
&\lesssim T^{-\delta} \sup_s \frac{1}{s^{2+\delta}} \mathbb{E}[\|W_s \xi_\varepsilon\|^2_{H^{-1-\delta/2}}] + T^{-\delta} \frac{1}{s^{2+\delta}} \|\xi\|^2_{\mathcal{C}^{-1-\delta/2}} \mathbb{E}\left[\int_0^s \|u_t\|^2_{H^{-\delta}} \mathrm{d}t\right]
\end{aligned}$$

Using Lemma 8 it is not hard to see that $\Phi(u)$ is a contraction in a large enough ball $B(0, K) \subset L^2(\mathbb{P}', L^2(\mathbb{R}_+, H^{-\delta}))$ and thus also satisfies (49). □

With this in hand, we prove a uniform in ε coercivity result for $F^\varepsilon(\cdot, \omega)$ up to a renormalization constant (which in this particular case is simply $F^\varepsilon(u^\varepsilon, \omega)$ and corresponds to the diverging normalization constant $\log(\mathcal{Z}^\varepsilon(\omega))$ in Theorem 6).

Theorem 7 *There is a sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0, 1)$, $\varepsilon_n \to 0$, such that there exists a (non-negative) random variable $C : \Omega \to \mathbb{R}_+$ which is almost surely finite and such that for any $\omega \in \Omega$ we have*

$$F^{\varepsilon_n}(u, \omega) - F^{\varepsilon_n}(u^{\varepsilon_n}, \omega) \geqslant \frac{1}{4}\mathbb{E}\left[\int_0^\infty \|l^{\varepsilon_n}_s(u)\|^2_{L^2} \mathrm{d}s\right] - C(\omega) \geqslant -C(\omega), \tag{50}$$

where u^{ε_n} is as in Lemma 11.

We begin by proving the following useful lemmas.

Lemma 12 *Let $C_{\cdot,\varepsilon} : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a random process depending on $\varepsilon \in \mathcal{I} \subset [0,1)$ (where $\mathcal{I}$ has zero as an accumulation point) such that*

1. *For any $p \geqslant 1$ and s we have*
$$(\sup_{\varepsilon \in \mathcal{I}} \mathbb{E}[|C_{s,\varepsilon}|^p])^{1/p} \leqslant f_p(s),$$
for some integrable function $f_p : \mathbb{R}_+ \to \mathbb{R}_+$;

2. *$C_{s,\varepsilon}$ is continuous as $\varepsilon \to 0$ almost surely, i.e. $C_{s,\varepsilon} \to C_{s,0}$ almost surely.*

Then there exists a sequence $\varepsilon_n \subset \mathcal{I}$ such that, for any σ-algebra $\mathcal{G}$, the random variable $\sup_{n \in \mathbb{N}} \mathbb{E}\left[\int_0^\infty C_{s,\varepsilon_n} \mathrm{d}s|\mathcal{G}\right]$ is bounded almost surely.

Proof First we prove that $\mathbb{E}\left[\int_0^\infty C_{s,\varepsilon} \mathrm{d}s|\mathcal{G}\right]$ converges to $\mathbb{E}\left[\int_0^\infty C_{s,0} \mathrm{d}s|\mathcal{G}\right]$ in L^p. By Minkowski, we have that

$$\mathbb{E}\left[\left|\mathbb{E}\left[\int_0^\infty C_{s,0} \mathrm{d}s - \int_0^\infty C_{s,\varepsilon_n} \mathrm{d}s|\mathcal{G}\right]\right|^p\right] \leqslant \int_0^{+\infty} (\mathbb{E}[|C_{s,\varepsilon} - C_{s,0}|^p]^{1/p}) \mathrm{d}s.$$

On the other hand, for some $\kappa > 0$, we have $\sup_{\varepsilon \in \mathcal{I}} \mathbb{E}[|C_{s,\varepsilon}|^{p+\kappa}] < +\infty$, and thus the family $\{|C_{s,\varepsilon} - C_{s,0}|\}_{\varepsilon \in \mathcal{I}}$ of random variable is uniformly integrable. This means that, since $|C_{s,\varepsilon} - C_{s,0}|$ converges to 0 almost surely, then $|C_{s,\varepsilon} - C_{s,0}|$ converges to 0 in L^p. This implies that for every $s \in \mathbb{R}_+$ $\mathbb{E}[|C_{s,\varepsilon} - C_{s,0}|^p]^{1/p}$ converges to 0 and thus, since $\mathbb{E}[|C_{s,\varepsilon} - C_{s,0}|^p]^{1/p} \leqslant f_p$, by Lebesgue dominated convergence theorem that $\int_0^{+\infty} (\mathbb{E}[|C_{s,\varepsilon} - C_{s,0}|^p]^{1/p}) \mathrm{d}s \to 0$ as $\varepsilon \to 0$.

Since $\mathbb{E}\left[\int_0^\infty C_{s,\varepsilon} \mathrm{d}s|\mathcal{G}\right]$ goes to $\mathbb{E}\left[\int_0^\infty C_{s,0} \mathrm{d}s|\mathcal{G}\right]$ in L^p, there is a subsequence $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset \mathcal{I}$ with $\varepsilon_n \to 0$ such that $\mathbb{E}\left[\int_0^\infty C_{s,\varepsilon} \mathrm{d}s|\mathcal{G}\right]$ goes to $\mathbb{E}\left[\int_0^\infty C_{s,0} \mathrm{d}s|\mathcal{G}\right]$ almost surely. This concludes the proof. □

Remark 6 The expectation $\mathbb{E}^{\omega'}$ introduced in Notation 1, can be understood as a conditional expectation with respect to the σ-algebra generated by random field $\omega \mapsto \xi(\omega)$. In this way, we can exploit Lemma 12 for random fields of the form $\mathbb{E}^{\omega'}[\cdot]$.

A consequence of the previous result is the following lemma.

Lemma 13 *For every $0 < \kappa \ll 1$,we have that there is a sequence $\varepsilon_n \in (0,1)$ such that $\varepsilon_n \to 0$, as $n \to \infty$ for which*

$$\sup_{n \in \mathbb{N}} \mathbb{E}^{\omega'}\left[\int_0^\infty \left\| \left(J_s \xi_{\varepsilon_n} W_s \circ J_s \xi_{\varepsilon_n} - \dot{\gamma}^{(2)}_{\varepsilon_n,s} W_s \right) \right\|_{H^{-\delta}} \mathrm{d}s \right] \leqslant C_1(\omega)$$

$$\sup_{n \in \mathbb{N}} (\sup_{s \in \mathbb{R}_+} ((1+s)^{1+\kappa} \| J_s \xi_{\varepsilon_n}(\omega) \circ J_s \xi_{\varepsilon_n}(\omega) - \dot{\gamma}^{(2)}_{\varepsilon_n,s} \|_{\mathcal{C}^{-\delta}})) \leqslant C_2(\omega)$$

$$\sup_{n \in \mathbb{N}} \|\xi_{\varepsilon_n}(\omega)\|^2_{\mathcal{C}^{-1-\delta}} \leqslant C_3(\omega), \quad \sup_{n \in \mathbb{N}} (\mathbb{E}^{\omega'}[\sup_{s \in \mathbb{R}_+} (1+s)^{-2\ell} \|\xi_{\varepsilon_n} W_s\|^2_{\mathcal{C}^{-1-\delta}}]) \leqslant C_4(\omega)$$

for some (positive) random variables C_1, C_2, C_3, C_4 which are almost surely finite.

Proof The proof of this lemma is a straightforward application of Lemma 12 combined with the stochastic estimates from Lemma 5, Remark 5, Lemma 7 and Lemma 8 and the immersion properties of Besov spaces (namely the immersion of $B^{\kappa}_{p,p}(\mathbb{R}_+, B^s_{q,q})$ into $C^0\left(\mathbb{R}_+, \mathcal{C}^{s-\frac{2}{q}}\right)$ when $\kappa > \frac{1}{p}$). □

Lemma 14 *For every $\eta, C, P, K > 0$ and $\lambda > \frac{1}{2}$ there is a C^1 function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\lim_{t\to+\infty} f(t) = P, \quad \dot{f}(t) \geqslant \frac{K(f(t))^{2\lambda}}{(1+t)^{1+\eta}} + \frac{C}{(1+t)^{1+\eta}} \tag{51}$$

Proof Suppose that $\sup_{t\in\mathbb{R}_+} f(t) \leqslant L$, and f solves the equation

$$\dot{f}(t) = \frac{KL^{2\lambda-1}f(t)}{(1+t)^{1+\eta}} + \frac{C}{(1+t)^{1+\eta}}, \quad \lim_{t\to+\infty} f(t) = P, \tag{52}$$

then f satisfies the differential inequality (51). □

Proof of Theorem 7 If we apply the analytical estimates of Section 3.3 to the functional F^{ε} (in the form given by Proposition 7) we get

$$\begin{aligned}
F^{\varepsilon}(u) - F^{\varepsilon}(0) \geqslant\ & \frac{1}{2}\int_0^{\infty} \|l(u)\|^2_{L^2}\mathrm{d}t - 8K_1\kappa\int_0^{\infty}\|u\|^2_{H^{-\delta}}\mathrm{d}t + \\
& -K_2(\kappa)\left(\mathbb{E}^{\omega'}\left[\left(\int_0^{\infty}\left\|\left(J_t\xi_{\varepsilon}W_t \circ J_t\xi_{\varepsilon} - \dot{\gamma}^{(2)}_{\varepsilon,t}W_t\right)\right\|_{H^{-\delta}}\mathrm{d}t\right)^2\right] + \|\xi_{\varepsilon}\|^2_{\mathcal{C}^{-1-\delta}}\right) \\
& -K_2(\kappa)\|\xi\|^2_{\mathcal{C}^{-1-\delta}}\mathbb{E}^{\omega'}[(\sup_{s\in\mathbb{R}_+}(1+s)^{\ell}\|\xi W_s\|^2_{\mathcal{C}^{-1-\delta}})] \\
& -K_2(\kappa)\mathbb{E}^{\omega'}\left[\int_0^{\infty}\frac{|\gamma^{(2)}_{\varepsilon,t}|^2}{(1+t)^{3/2-2\delta}}\|W_t\|^2_{\mathcal{C}^{-\delta}}\mathrm{d}t + \left(\int_0^{+\infty}|\dot{\gamma}^{(1)}_t(\omega)|\|W_t\|_{\mathcal{C}^{-\delta}}\mathrm{d}t\right)^2\right] \\
& -K_2(\kappa)\mathbb{E}^{\omega'}\left[\int_0^{+\infty}|\gamma^{(1)}_t|^2\frac{\|W_t\|^2_{\mathcal{C}^{-\delta}}}{(1+t)^{3/2-2\delta}}\mathrm{d}t\right] - K_2(\kappa)\int_0^{\infty}\frac{|\gamma^{(2)}_{\varepsilon,t}|^{\frac{2}{1-\delta}}}{(1+t)^{3/2}}\|Z_t\|^2_{L^2}\mathrm{d}t \\
& -K_2(\kappa)\int_0^{\infty}\frac{\left(\sup_s(1+s)^{1+\tau-\lambda}\|((J_s\xi_{\varepsilon}\circ J_s\xi_{\varepsilon}) - \dot{\gamma}^{(2)}_{\varepsilon,s})\|_{\mathcal{C}^{-\delta}}\right)^{\frac{1}{1-\theta}}}{(1+t)^{1+\tau-\lambda}}\|Z_t\|^2_{L^2}\mathrm{d}t \\
& -K_2(\kappa)\int_0^{+\infty}\frac{\|\xi_{\varepsilon}\|^2_{\mathcal{C}^{-1-\delta}}}{(1+t)^{1+\eta}}\|Z_t\|^2_{L^2}\mathrm{d}t - K_2(\kappa)\int_0^{+\infty}\frac{|\gamma^{(1)}_t|^{\frac{2}{1-\delta}}}{(1+t)^{3/2}}\|Z_t\|^2_{L^2}\mathrm{d}t \\
& +K_3(\kappa)\int_0^{+\infty}\dot{\gamma}^{(1)}_t(\omega)\|Z_t\|^2_{L^2}\mathrm{d}t
\end{aligned}$$

Where $K_2, K_3 : \mathbb{R}_+\backslash\{0\} \to \mathbb{R}_+$ are continuous (decreasing) functions of κ and $K_1 > 0$ is a suitable constant. First we note that

$$
\begin{aligned}
\int_0^\infty \|u\|_{H^{-\delta}}^2 \mathrm{d}t &\lesssim \int_0^\infty \|J_t\xi_\varepsilon W_t\|_{H^{-\delta}}^2\,\mathrm{d}t + \int_0^\infty \|J_t(\xi_\varepsilon \succ Z_t(u))\|_{H^{-\delta}}^2 \mathrm{d}t + \int_0^\infty \|l(u_t)\|_{L^2}\mathrm{d}t\\
&\lesssim \left(\sup_{s\in\mathbb{R}_+}(1+s)^{\frac{\delta}{4}}\|\xi_\varepsilon W_s\|^2_{H^{-1-\frac{\delta}{2}}}\right)\int_0^\infty \frac{1}{(1+t)^{1+\frac{\delta}{4}}}\mathrm{d}t +\\
&\quad + \int_0^\infty \frac{1}{(1+t)^{1+\delta}}\|\xi_\varepsilon\|_{\mathcal{C}^{-1-\frac{\delta}{2}}}\|Z\|_{H^{1-\delta}}\mathrm{d}t + \int_0^\infty \|l(u_t)\|_{L^2}\mathrm{d}t\\
&\lesssim \kappa'\int_0^\infty \|u\|_{H^{-\delta}}^2\mathrm{d}t + \left(\sup_{s\in\mathbb{R}_+}(1+s)^{\frac{\delta}{4}}\|\xi_\varepsilon W_s\|^2_{H^{-1-\frac{\delta}{2}}}\right) + \|\xi_\varepsilon\|^2_{\mathcal{C}^{1-\frac{\delta}{2}}} + \int_0^\infty \|l(u_t)\|_{L^2}\mathrm{d}t.
\end{aligned}
$$

In other words there is $K_4 > 0$ for which

$$
\int_0^\infty \|u\|_{H^{-\delta}}^2 \mathrm{d}t \lesssim K_4 \int_0^\infty \|l(u_t)\|_{L^2}\mathrm{d}t + K_4\left(\left(\sup_{s\in\mathbb{R}_+}(1+s)^{\frac{\delta}{4}}\|\xi_\varepsilon W_s\|^2_{H^{-1-\frac{\delta}{2}}}\right) + \|\xi_\varepsilon\|^2_{\mathcal{C}^{1-\frac{\delta}{2}}}\right). \quad (53)
$$

This means that if we choose $\kappa < \frac{1}{16K_1K_4}$, then

$$
\begin{aligned}
\frac{1}{2}\int_0^\infty \|l(u)\|_{L^2}^2\mathrm{d}t - 8K_1\kappa\int_0^\infty \|u\|^2_{H^{-\delta}}\mathrm{d}t &\geqslant \frac{1}{4}\int_0^\infty \|l(u)\|^2_{L^2}\mathrm{d}t +\\
-\frac{1}{4}\left(\left(\sup_{s\in\mathbb{R}_+}(1+s)^{\frac{\delta}{4}}\|\xi_\varepsilon W_s\|^2_{H^{-1-\frac{\delta}{2}}}\right) + \|\xi_\varepsilon\|^2_{\mathcal{C}^{1-\frac{\delta}{2}}}\right).
\end{aligned}
$$

Now we fix $\kappa > 0$ such that $\kappa < \frac{1}{16K_1K_4}$. In this way $K_2(\kappa), K_3(\kappa)$ are (fixed) numbers (hereafter we drop the dependence on κ of $K_2(\kappa), K_3(\kappa)$). We focus on the parts depending on $\|Z\|^2_{L^2}$. If we take $\gamma_t^{(1)}$ such that

$$
\dot{\gamma}_t^{(1)}(\omega) \geqslant (K_2K_3+1)\frac{|\gamma_t^{(1)}(\omega)|^{\frac{2}{1-\delta}}}{(1+t)^{1+\eta'}} + \frac{(K_3H(\omega)+1)}{(1+t)^{1+\eta'}}, \quad \gamma_\infty^{(1)}(\omega) = \gamma^{(1)}(\omega) \quad (54)
$$

where and

$$
\begin{aligned}
H(\omega) &= \sup_{n\in\mathbb{N}, s\in\mathbb{R}_+}\Bigg((1+s)^{1+\tau-\lambda}\|((J_s\xi_{\varepsilon_n}\circ J_s\xi_{\varepsilon_n}) - \dot{\gamma}^{(2)}_{\varepsilon_n,s})\|^{\frac{1}{1-\theta}}_{\mathcal{C}^{-\delta}} +\\
&\quad + \frac{(\log(2+s))^{\frac{2}{1-\delta}}}{(1+s)^{1/2-\eta'}}((\log(2+s))^{-1}|\gamma^{(2)}_{\varepsilon_n,s}|)^{\frac{2}{1-\delta}} + \|\xi_{\varepsilon_n}\|^2_{\mathcal{C}^{-1-\delta}}\Bigg) \quad (55)
\end{aligned}
$$

which by Lemma 13 and Remark 4 is almost surely finite and $\eta' < \min\left(\eta, \frac{1}{2}, \delta, -\tau+\lambda\right)$. The existence of some solution to the differential inequality (54) is given by Lemma 14. Furthermore, again by Lemma 14, we can choose $\gamma_t^{(1)}$ which is bounded (for every fixed ω, not uniformly with respect to ω) and such that $\dot{\gamma}_t^{(1)}(\omega) \lesssim \frac{1}{(1+t)^{1+\eta'}}$. Fix now a solution $\gamma_t^{(1)}$ to the differential inequality (54) satisfying the previous two conditions.

For such a function $\gamma_t^{(1)}$, the sum of the terms involving $\|Z_t\|^2_{L^2}$ is strictly positive.

If we now took

$$\begin{aligned}C(\omega) &= K_2 \sup_{n\in\mathbb{N}} \left(\mathbb{E}^{\omega'} \left[\left(\int_0^\infty \left\| \left(J_t \xi_{\varepsilon_n} W_t \circ J_t \xi_{\varepsilon_n} - \dot{\gamma}^{(2)}_{\varepsilon_n,t} W_t \right) \right\|_{H^{-\delta}} \mathrm{d}t \right)^2 \right] + \|\xi_{\varepsilon_n}\|^2_{\mathcal{C}^{-1-\delta}} + \right. \\ &\quad + \|\xi_{\varepsilon_n}\|^2_{\mathcal{C}^{-1-\delta}} \mathbb{E}^{\omega'} [(\sup_{s\in\mathbb{R}_+} (1+s)^\ell \|\xi_{\varepsilon_n} W_s\|^2_{\mathcal{C}^{-1-\delta}})] + \mathbb{E}^{\omega'} \left[\int_0^{+\infty} |\gamma^{(1)}_t|^2 \frac{\|W_t\|^2_{\mathcal{C}^{-\delta}}}{(1+t)^{3/2-2\delta}} \mathrm{d}t \right] \\ &\quad \left. + \mathbb{E}^{\omega'} \left[\int_0^\infty \frac{|\gamma^{(2)}_{\varepsilon_n,t}|^2}{(1+t)^{3/2-2\delta}} \|W_t\|^2_{\mathcal{C}^{-\delta}} \mathrm{d}t + \left(\int_0^{+\infty} |\dot{\gamma}^{(1)}_t(\omega)| \|W_t\|_{\mathcal{C}^{-\delta}} \mathrm{d}t \right)^2 \right] \right) + \\ &\quad + \sup_{n\in\mathbb{N}} \frac{1}{4} \left(\left(\sup_{s\in\mathbb{R}_+} (1+s)^{\frac{\delta}{4}} \|\xi_{\varepsilon_n} W_s\|^2_{H^{-1-\frac{\delta}{2}}} \right) + \|\xi_{\varepsilon_n}\|^2_{\mathcal{C}^{-1-\frac{\delta}{2}}} \right)\end{aligned}$$

which is almost surely finite by Lemma 13 and thus we are finished. □

Remark 7 It is interesting to note that it is possible to use the proof of Theorem 7, in particular the expression of the constant (55), to deduce the existence of a limit operator $\mathbb{H}^{\omega,0} = \lim_{\varepsilon_k\to 0}(-\Delta + \xi_{\varepsilon_k} - \gamma^{(2)}_{\varepsilon_k,\infty})$ (as the covariance operator of obtained as the limit of a Gaussian measure convergent subsequence μ^{ε_k} defined in equation (36)). Furthermore, using Lemma 14, we get also that $\mathbb{H}^{\omega,0} \geqslant -(\lim_{\varepsilon_k\to 0} \gamma^{(1)}_{\varepsilon_k,\infty}(\omega))\mathbb{I}_{L^2}$. This means that the proof of Theorem 7 in the current section can be seen as providing an autonomous proof of the main part of Theorem 3 using the variational techniques of [8].

3.5 Construction of the coupling

In this last subsection we will prove the following theorem.

Theorem 8 *For any $\chi, \delta > 0$ $\chi > \delta$ and for almost every $\omega \in \Omega$, there is a probability measure $\overline{\nu_0}$ on $\mathcal{C}^{-\chi} \times H^{1-\chi}$ such that*

1. $P_{\mathcal{C}^{-\chi},*}(\bar{\nu}_0) = \mathrm{Law}(\mu^{-\Delta+1})$ *(where $P_{\mathcal{C}^{-\chi}} : \mathcal{C}^{-\chi} \times H^{1-\chi} \to \mathcal{C}^{-\chi}$ is the natural projection);*
2. $\int \|Z\|^2_{H^{1-\chi}} \bar{\nu}_0(\mathrm{d}\varphi, \mathrm{d}Z) < +\infty$ *(where $(\varphi, Z) \in \mathcal{C}^{-\chi} \times H^{1-\chi}$)*
3. $\mathrm{Law}_{\nu_0}(\varphi + Z) := (P_{\mathcal{C}^{-\chi}} + P_{H^{1-\chi}})_*(\bar{\nu}_0) = \mu^{\mathbb{H}^\omega + K(\omega)}$ *(where $\lim_{\varepsilon_n\to 0} \mu^{\varepsilon_n}(\omega) = \mu^{\mathbb{H}^\omega + K(\omega)}$ weakly and where μ^{ε_n} are the Gaussian measure introduced in equation (36)).*

Remark 8 Theorem 8 can be also reformulated in this way: for almost every $\omega \in \Omega$, there is a coupling $\widetilde{\nu_0}$ on $\mathcal{C}^{-\chi}(\mathbb{T}^2) \times \mathcal{C}^{-\chi}(\mathbb{T}^2)$ between the Gaussian measures $\mu^{-\Delta+1}$ and $\mu^{\mathbb{H}^\omega + K(\omega)}$ such that if we write $(X, Y) \sim \widetilde{\nu_0}$, and thus $X \sim \mu^{-\Delta+1}$ and $Y \sim \mu^{\mathbb{H}^\omega + K(\omega)}$, we have $X - Y \in H^{1-\chi}$ $\tilde{\nu}_0$-almost surely (see [31] for a formulation of this property using a Wasserstein-type distance between measures). In other words, there exists a $H^{1-\chi}$regular coupling between the standard free field and the Anderson free field.

In order to prove Theorem 8 we employ bounds on the functional F^ε proved in Theorem 7 to get tightness of the measures μ^ε. To achieve this aim, we need to extend the functionals F^ε to functionals depending on the laws of $(\mathbb{W}, u)$ (where $\mathbb{W}$ is the Gaussian process defined in Notation 2) so that we may obtain some compactness properties of the functionals which will allow us to apply the direct method of the calculus of variations.

Notation 3 *Let us consider the space of Radon measures*

$$\mathcal{X} \subset \mathcal{P}(C^0(\mathbb{R}_+, C^{-\chi}(\mathbb{T}^2)) \times L^2(\mathbb{R}_+ \times \mathbb{T}^2)) =: \mathcal{P}(\mathfrak{S} \times L^2(\mathbb{R}_+ \times \mathbb{T}^2)),$$

defined as follows: We say that the measure $\sigma \in \mathcal{X}$ if, writing $\mathfrak{S} \times L^2(\mathbb{R}_+ \times \mathbb{T}^2) \ni (\mathbb{W}, u) \sim \sigma$ for the random variable with law σ, we have that $\mathbb{W}$ is a Gaussian process with covariance as defined in Notation 2 and u can be written (almost surely) as a progressively measurable process of $\mathbb{W}$ and finally $\|u\|_{L^2(\mathbb{R}_+\times\mathbb{T}^2)} \in L^2(\sigma)$.

Remark 9 Usually the space $\mathfrak{S}$, on which the process $\mathbb{W}$ takes values, is the space on enhanced noise (i.e. containing also the processes $J_s(\xi_\varepsilon W_s)J_s\xi_\varepsilon - J_s\xi_\varepsilon J_s\xi_\varepsilon W_s$ etc. considered in Section 3.2). Here we define $\mathfrak{S}$ to be only $C^0(\mathbb{R}_+, C^{-\chi}(\mathbb{T}^2))$ (i.e. the space where W_s takes values) because we never consider directly the limit $\varepsilon \to 0$ but we ask merely for estimates of stochastic terms uniformly in ε. For this reason, when $\varepsilon > 0$, since the enhanced noise is a continuous function of W_s, we need only the space $\mathfrak{S} = C^0(\mathbb{R}_+, C^{-\chi}(\mathbb{T}^2))$.

In any case, since all the stochastic terms, considered in Section 3.2, by Lemma 12 and Lemma 13, converge (almost surely with respect to $\omega \in \Omega$), as $\varepsilon \to 0$, to some well defined adapted processes, our argument can be extended to the space of enhanced noise.

Hereafter we write $L^2_w(\mathbb{R}_+ \times \mathbb{T}^2)$ for the space $L^2(\mathbb{R}_+ \times \mathbb{T}^2)$ equipped with the weak topology.

Definition 4 *Let $\mathcal{X}$ be defined as in Notation 3. Consider the space*

$$\overline{\mathcal{X}} = \left\{ \sigma : \quad \exists \sigma_n \in \mathcal{X} : \sigma_n \to \sigma \text{ weakly on } \mathfrak{S} \times L^2_w(\mathbb{R}_+ \times \mathbb{T}^2), \text{ and } \sup_n \mathbb{E}^{\omega'}_{\sigma_n}[\|u\|^2_{L^2(\mathbb{R}_+\times\mathbb{T}^2)}] < \infty \right\}$$

We say that $\sigma_n \to \sigma$ in $\bar{\ }\mathcal{X}$ if $\sigma_n \to \sigma$ weakly and $\sup_n \mathbb{E}^{\omega'}_{\sigma_n}[\|u\|^2_{L^2(\mathbb{R}_+\times\mathbb{T}^2)}] < \infty$.

If $\sigma \in \overline{\mathcal{X}}$ we define

$$\bar{F}^\varepsilon(\omega, \sigma) = \mathbb{E}^{\omega'}_\sigma \left[\sum_{i=1}^7 \Gamma_i(\omega) + \mathfrak{G} + \frac{1}{2} \int_0^\infty \|l^\varepsilon_s(u)\|^2_{L^2} \mathrm{d}s \right] + \frac{1}{2} \mathbb{E}^{\omega'}_\sigma \left[\int_0^\infty \|J_s\xi_\varepsilon(\omega)W_s\|^2_{L^2} \,\mathrm{d}s \right].$$

We have the following statement that says that $\overline{F}^\varepsilon$ has the same minimum as F^ε.

Lemma 15 *For almost every $\omega \in \Omega$ and $\varepsilon_n > 0$ (where ε_n is in the sequence defined in Theorem 7) we have*

$$\inf_{u \in \mathbb{H}_a} F^{\varepsilon_n}(\omega, u) = \inf_{\sigma \in \mathcal{X}} \bar{F}^{\varepsilon_n}(\omega, \sigma) = \inf_{\sigma \in \overline{\mathcal{X}}} \bar{F}^{\varepsilon_n}(\omega, \sigma)$$

Proof The first equality is obvious from the definition of $\mathcal{X}$, $\bar{F}^{\varepsilon_n}$. The second inequality can be proved in the same way of Lemma 8 of [10]. □

We introduce here a class of functional which is important in what follows.

Definition 5 *We say that the functional $G : \mathcal{X} \to \mathbb{R}$ is admissible, if G is lower semicontinuous and*

$$\mathbb{E}_\sigma \left[\int_0^{+\infty} \|u_t\|^2_{L^2} \mathrm{d}t \right] \lesssim 1 + G(\sigma).$$

Lemma 16 *For every fixed $\varepsilon_n > 0$, the functional $\bar{F}^{\varepsilon_n}$ is an admissible functional.*

Proof For brevity we drop the n index. We consider the form of F^ε given in Theorem 6. In particular we have

$$\left|\int (\xi_\varepsilon(x)+\gamma^{(1)}+\gamma_\varepsilon^{(2)})Z_\infty(x)W_\infty(x)\mathrm{d}x\right| \lesssim (\|\xi_\varepsilon\|_{\mathcal{C}^\delta}+\gamma^{(1)}+\gamma_\varepsilon^{(2)})\|Z_\infty\|_{H^1}\|W_\infty\|_{\mathcal{C}^{-\delta}}.$$

Thus we get

$$F(\mu) \geqslant \frac{1}{2}\mathbb{E}_\mu\left[\int_0^\infty \|u_t\|_{L^2}^2\mathrm{d}t\right] - K(\|\xi_\varepsilon\|_{\mathcal{C}^\delta}+\gamma^{(1)}+\gamma_\varepsilon^{(2)})(\mathbb{E}_\mu[\|Z_\infty\|_{H^1}\|W_\infty\|_{\mathcal{C}^{-\delta}}+\|W_\infty\|_{\mathcal{C}^{-\delta}}^2]).$$

Thus, from Young's inequality, we get

$$\left(\frac{1}{2}-\kappa\right)\mathbb{E}_\mu\left[\int_0^\infty \|u_t\|_{L^2}^2\mathrm{d}t\right] \lesssim F^\varepsilon(\mu) + K^2(\|\xi_\varepsilon\|_{\mathcal{C}^\delta}+\gamma^{(1)}+\gamma_\varepsilon^{(2)}+1)^2\mathbb{E}_\mu[\|W_\infty\|_{\mathcal{C}^{-\delta}}^2].$$

Since for any $\varepsilon>0$ and δ, $\xi_\varepsilon\in\mathcal{C}^\delta(\mathbb{T}^2)$ this finishes the proof. The lower semicontinuity of F^ε can be proved as in Lemma 17 of [10]. □

Lemma 17 *For every $\varepsilon>0$ there is $\sigma^\varepsilon\in\overline{\mathcal{X}}$ such that $\bar{F}^\varepsilon=\inf_{\sigma\in\overline{\mathcal{X}}}\bar{F}^\varepsilon(\omega,\sigma)$. Furthermore, for each σ^ε as before we have*

$$\mu^{\mathbb{H}_\varepsilon^{\omega,K}}=\mu^\varepsilon=\mathrm{Law}_{\sigma^\varepsilon}(W_\infty+Z_\infty).$$

Proof Since $\bar{F}^\varepsilon$ is admissible in the sense of Definition 5 (see also Definition 6 of [10]) then the existence of a minimizer is guaranteed by Lemma 7 of [10]. The fact that $\mu^\varepsilon=\mathrm{Law}_{\sigma^\varepsilon}(W_\infty+Z_\infty)$ is proved in Theorem 11 of [10]. Finally the fact that μ^ε is a Gaussian free field related to the (regularized) Anderson Hamiltonian is proved in Lemma 3. □

Proof of Theorem 8 Consider $\varepsilon_n\in\mathbb{R}_+$, $\varepsilon_n\to 0$, and $C:\Omega\to\mathbb{R}_+$ as in Theorem 7, then we have

$$\sup_{n\in\mathbb{N}}\mathbb{E}_{\sigma^{\varepsilon_n}}\left[\int_0^\infty \|l_s^{\varepsilon_n}(u)\|_{L^2}^2\mathrm{d}s\right]\leqslant 8C(\omega) \tag{56}$$

for any μ^{ε_n} minimizer of $\bar{F}^{\varepsilon_n}$. Indeed, we know that , by Lemma 11 there exists a sequence of drifts $\bar{u}^{\varepsilon_n}$ such that

$$\sup_{n\in\mathbb{N}} F^{\varepsilon_n}(\omega,\bar{u}^{\varepsilon_n}) =: C(w)<\infty.$$

Then, with $\tilde{C}$ being a constant changing from line to line, we have

$$\begin{aligned}
0 &\geqslant (\inf_{u\in\mathbb{H}_a}(F^{\varepsilon_n}(\omega,u)))-F^{\varepsilon_n}(\omega,\bar{u}^{\varepsilon_n})\\
&\geqslant \frac{1}{4}\mathbb{E}_{\sigma^{\varepsilon_n}}\left[\int_0^\infty \|l_s^{\varepsilon_n}(u)\|_{L^2}^2\mathrm{d}s\right]-\sup_{n\in\mathbb{N}}F^{\varepsilon_n}(\omega,\bar{u}^{\varepsilon_n})-\tilde{C}(\omega)\\
&\geqslant \frac{1}{4}\mathbb{E}_{\sigma^{\varepsilon_n}}\left[\int_0^\infty \|l_s^{\varepsilon_n}(u)\|_{L^2}^2\mathrm{d}s\right]-\tilde{C}(\omega)
\end{aligned}$$

Thus, taking the sup over $n\in\mathbb{N}$, we get inequality (56). Consider

$$\bar{\nu}^{\varepsilon_n}:=\mathrm{Law}_{\sigma^{\varepsilon_n}}(W_\infty,Z_\infty)\in\mathcal{P}(\mathcal{C}^{-\delta}\times H^{1-\delta}).$$

We want to prove that $\bar{\nu}^{\varepsilon_n}$ is a family of tight measures in $\mathcal{P}(\mathcal{C}^{-\delta}\times H^{1-\delta})$. Since $\mathcal{C}^{-\delta'}\times H^{1-\delta'}$ compactly embeds in $\mathcal{C}^{-\delta}\times H^{1-\delta}$ (whenever $\delta>\delta'$), it is enough to prove that

$$\sup_{n\in\mathbb{N}}\mathbb{E}_{\bar{\nu}^{\varepsilon_n}}[\|W_\infty\|_{\mathcal{C}^{-\delta'}}^2+\|Z_\infty\|_{H^{1-\delta'}}^2]<+\infty.$$

Since the law of W_∞ is the Gaussian free field, obviously $\sup_{n\in\mathbb{N}} \mathbb{E}_{\bar{\nu}^{\varepsilon_n}}[\|W_\infty\|^2_{\mathcal{C}^{-\delta'}}] < +\infty$. On the other hand, by Lemma 9, inequality (53) in the proof of Theorem 7, and inequality (56), obtain

$$
\begin{aligned}
\mathbb{E}_{\bar{\nu}^{\varepsilon_n}}[\|Z_\infty\|^2_{H^{1-\delta'}}] &\leqslant \mathbb{E}_{\sigma^{\varepsilon_n}}\left[\int_0^{+\infty} \|u_s\|^2_{H^{1-\delta'}} \mathrm{d}s\right] \\
&\lesssim \mathbb{E}_{\sigma^{\varepsilon_n}}\left[\int_0^{\infty} \|l(u_s)\|^2_{L^2} \mathrm{d}s\right] + \mathbb{E}_{\sigma^{\varepsilon_n}}\left[\left(\sup_{s\in\mathbb{R}_+} (1+s)^{\frac{\delta'}{4}} \|\xi_\varepsilon W_s\|^2_{H^{-1-\frac{\delta'}{2}}}\right)\right] + \|\xi_\varepsilon\|^2_{\mathcal{C}^{-1-\frac{\delta'}{2}}} \\
&\lesssim 8\tilde{C}(\omega) + \tilde{C}(\omega).
\end{aligned}
$$

Since the previous bound is uniform in ε_n the tightness of $\bar{\nu}^{\varepsilon_n}$ follows. Considering any weak limit $\bar{\nu}_0 \in \mathcal{P}(\mathcal{C}^{-\delta} \times H^{1-\delta})$ of a suitable subsequence of $\bar{\nu}^{\varepsilon_n}$, we have that $\bar{\nu}_0$ satisfies the point 1. and 2. of Theorem 8.

The point 3. follows from the second part of Lemma 17 (namely that $\mu^{\mathbb{H}^{\omega,K}_{\varepsilon_n}} = \mathrm{Law}_{\bar{\nu}^{\varepsilon_n}}(W_\infty + Z_\infty)$) and the fact that $-\Delta + m^2 + \xi_{\varepsilon_n} + \gamma^{(2)}_{\varepsilon_n}$ converges to $\mathbb{H}^\omega$ in the norm resolvent sense (see Theorem 2.30 of [37]). □

Remark 10 Thanks to the existence of the coupling proved in Theorem 8, we can deduce some regularity properties of the Gaussian Anderson free field φ^A. Indeed, consider $\varphi^A = \varphi^G + h$, where φ^G is the (standard) Gaussian free field and $h \in H^{1-\delta}(\mathbb{T}^2)$ almost surely is a regular coupling between φ^A and φ^G. Then, since φ^G is supported on $\mathcal{C}^{-\delta'}(\mathbb{T}^2)$ and by Besov embedding (see Lemma 31), $\mathcal{C}^{-\delta'} \supset H^{1-\delta}$ for $\delta' > \delta > 0$, we have $\varphi^A \in \mathcal{C}^{-\delta'}$ almost surely. This implies by Fernique's theorem for Gaussian measures that $\varphi^A \in L^p(\Omega', \mathcal{C}^{-\delta'}(\mathbb{T}^2))$ and thus $h \in L^p(\Omega', \mathcal{C}^{-\delta'}(\mathbb{T}^2))$ for any $1 \leqslant p < +\infty$.

3.6 On the renormalization of the powers of AGFF

In this section we talk about the renormalization of powers of the AGFF. First we suppose that φ^A is a Gaussian random distribution with covariance $(\mathbb{H}^{\omega,K})^{-1}$. Then by Theorem 8 there is a Gaussian free field (with mass K) φ^G and a random field h taking values in $H^{1-\delta}(\mathbb{T}^2)$ (for any $\delta > 0$), with $\mathbb{E}[\|h\|^2_{H^{1-\delta}}] < +\infty$, such that $\varphi^A = \varphi^G + h$. Let ρ_ε be a mollifier and define $\varphi^A_\varepsilon := \rho_\varepsilon * \varphi^A$, $\varphi^G := \rho_\varepsilon * \varphi^G$ etc. For $M \in \mathbb{N}$, let $H_M : \mathbb{R} \to \mathbb{R}$ be the M-th Hermite polynomial and we define

$$
(\varphi^A_\varepsilon)^{\circ M} = c_\varepsilon^{\frac{M}{2}} H_M\left(\frac{\varphi^A_\varepsilon}{\sqrt{c_\varepsilon}}\right) \tag{57}
$$

where $c_\varepsilon = \left(\sum_{k\in\mathbb{Z}^2} \frac{|\hat{\rho}_\varepsilon(x)|^2}{(|k|^2+m^2)}\right)^{1/2} \sim \log\left(\frac{1}{\varepsilon}\right)$. By the properties of sums of Hermite polynomials and the fact that $c_\varepsilon = (\mathbb{E}[|\varphi^G_\varepsilon|^2])^{1/2}$, we get

$$
(\varphi^A_\varepsilon)^{\circ M} = \sum_{k=0}^{M} \left(\begin{array}{c} M \\ k \end{array}\right) : (\varphi^G_\varepsilon)^k : h_\varepsilon^{M-k}. \tag{58}
$$

Remark 11 Consider the operator $A_\varepsilon(f) := \rho_\varepsilon * f - f$, then there is $c > 0$ such that

$$
\|A_\varepsilon\|_{\mathcal{L}(B^{s+\kappa}_{p,q}, B^s_{p,q})} \leqslant c\varepsilon^\kappa,
$$

where the constants in the symbol $\lesssim$ are independent of ε.

Lemma 18 *We have that for every $\delta > 0$, $k \in \mathbb{N}$ and $p \geqslant 1$ there is $c > 0$ such that*

$$\mathbb{E}[\| :(\varphi^G_\varepsilon)^k: - :(\varphi^G)^k: \|^p_{\mathcal{C}^{-\delta}}] \lesssim \varepsilon^c.$$

Proof See, e.g., Theorem V.3 in [62] (see also Lemma 3.12 of [7]). □

Lemma 19 *For every $\delta > 0$ and $p \geqslant 1$, we have that, for any $M \in \mathbb{N}$, $(\varphi^A_\varepsilon)^{\circ M}$ is a Cauchy sequence in $L^p(\Omega', B^{-\delta}_{p,p}(\mathbb{T}^2))$ with a limit $(\varphi^A)^{\circ M}$. Furthermore we have*

$$(\varphi^A)^{\circ M} = \sum_{k=0}^{M} \binom{M}{k} :(\varphi^G)^k: h^{M-k}. \tag{59}$$

Remark 12 We should stress that the singular product $(\varphi^A)^{\circ M}$ (defined thanks to Lemma 19) is different from the Gaussian Wick product

$$:(\varphi^A)^M: := \lim_{\varepsilon \to 0} :(\varphi^A_\varepsilon)^M: \tag{60}$$

defined as in equation (35) and Theorem 5. Indeed, the renormalization procedure is done through a limit of a function of the random field $\varphi^A_\varepsilon(x)$ and the variable x, and not only on $\varphi^A_\varepsilon(x)$ as for the product $(\varphi^A)^{\circ M}$ (see Section 6 of [5] for a discussion on the product (60)).

However, it turns out that the difference of the renormalization functions should be (at least) an L^p function for any $p \geqslant 1$. Take for example the square case, where we get

$$\mathbb{E}[(\varphi^A)^2(\cdot) - (\varphi^G)^2(\cdot)] = 2\mathbb{E}[(\varphi^G h)(\cdot)] + \mathbb{E}[h^2(\cdot)].$$

Now the term $\mathbb{E}[h(x)^2]$ is in L^p, by Sobolev embedding, since $h \in H^{1-\delta}$. If we decompose the other term as

$$\mathbb{E}[\varphi^G h] = \mathbb{E}[\varphi^G \succ h] + \mathbb{E}[\varphi^G \preccurlyeq h],$$

we obtain that it is in $H^{1-2\delta}$. In fact $\mathbb{E}[\varphi^G \preccurlyeq h]$ is in $H^{1-2\delta}$ by the properties of h and of the paraproduct $\preccurlyeq$, see Appendix A. To study $\mathbb{E}[\varphi^G \succ h]$, we observe

$$\begin{aligned} \mathbb{E}[\varphi^G \succ h] &= \sum_{i \leqslant j-1} \mathbb{E}[\Delta_j \varphi^G \Delta_i h] \\ &= \sum_{i \leqslant j-1} \mathbb{E}[\Delta_j \varphi^G] \mathbb{E}[\Delta_i h] \\ &= 0 \end{aligned}$$

where we have used the "scale to scale" property of Remark 1.

This remark is an example of the renormalization of singular products through diverging constant (in space) functions of the singular field whose law is not invariant with respect to translation (see [4, 43] for some examples of this kind of phenomenon in the case of smooth Riemannian manifolds).

Proof of Lemma 19 We fix $p \geqslant 1$ and $\delta > 0$, we want to prove that

$$\lim_{\varepsilon \to 0} \mathbb{E}[\|(\varphi^A_\varepsilon)^{\circ M} - (\varphi^A)^{\circ M}\|^p_{B^{-\delta}_{p,p}}] = 0.$$

We have that

$$\mathbb{E}[\|(\varphi_\varepsilon^A)^{\circ M}-(\varphi^A)^{\circ M}\|^p_{B^{-\delta}_{p,p}}] \;\lesssim\; \sum_{k=0}^{M}\mathbb{E}[\|:(\varphi_\varepsilon^G)^k:h_\varepsilon^{M-k}-:(\varphi^G)^k:h^{M-k}\|^p_{B^{-\delta}_{p,p}}]. \qquad (61)$$

Let us focus on each separate term in the previous sum. We have that

$$\begin{aligned}
& \mathbb{E}[\|:(\varphi_\varepsilon^G)^k:h_\varepsilon^{M-k}-:(\varphi^G)^k:h^{M-k}\|^p_{B^{-\delta}_{p,p}}]\\
\lesssim\; & \mathbb{E}[\|:(\varphi_\varepsilon^G)^k:-:(\varphi^G)^k:\|^p_{\mathcal{C}^{-\delta}}\|h_\varepsilon\|^{p(M-k)}_{B^{2\delta}_{p(M-k),p(M-k)}}]+\\
& +\sum_{\ell=1}^{M-k-1}\mathbb{E}[\|:(\varphi^G)^k:\|^p_{\mathcal{C}^{-\delta}}\|h_\varepsilon-h\|^p_{B^{2\delta}_{p(M-k),p(M-k)}}\|h_\varepsilon\|^{p(M-k-\ell)}_{B^{2\delta}_{p(M-k),p(M-k)}}\|h\|^{p\ell}_{B^{2\delta}_{p(M-k),p(M-k)}}].
\end{aligned}$$

Fix $n\in\mathbb{N}$ and k, then there is $\delta',\delta''>0$, $p'\geqslant 1$ and $0\leqslant\theta\leqslant 1$ such that

$$-\theta\delta'+(1-\theta)(1-\delta'')\geqslant 3\delta,\qquad \frac{(1-\theta)}{2}\geqslant\frac{1}{p(M-k)},\qquad \frac{p'}{\theta}>p(M-k).$$

Since $h\in L^2(\Omega',H^{1-\delta''}(\mathbb{T}^2))$ and $h\in L^{p'}(\Omega',\mathcal{C}^{-\delta'})$, (see Remark 10) we get that

$$h\in L^{\frac{p'}{\theta}}(\Omega',B^{3\delta}_{p(M-k),p(M-k)}).$$

Putting all the previous observations together, we obtain

$$\begin{aligned}
& \mathbb{E}[\|:(\varphi_\varepsilon^G)^k:h_\varepsilon^{M-k}-:(\varphi^G)^k:h^{M-k}\|^2_{B^{-\delta}_{2,2}}]\\
\lesssim\; & (\mathbb{E}[\|:(\varphi_\varepsilon^G)^k:-:(\varphi^G)^k:\|^{2q'}_{\mathcal{C}^{-\delta}}])^{\frac{1}{q'}}(\mathbb{E}[\|h_\varepsilon\|^{p'}_{B^{2\delta}_{p(M-k),p(M-k)}}])^{\frac{p(M-k)}{p'}}\\
& +\sum_{\ell=1}^{2n-k-1}\|A_\varepsilon\|_{\mathcal{L}(B^{3\delta}_{p,q},B^{2\delta}_{p,q})}\mathbb{E}[\|:(\varphi^G)^k:\|^{2q'}_{\mathcal{C}^{-\delta}}]^{\frac{1}{q'}}\mathbb{E}[\|h_\varepsilon\|^{p'}_{B^{3\delta}_{p(M-k),p(M-k)}}]^{\frac{p(M-k-\ell)}{p'}}\times\\
& \times\mathbb{E}[\|h\|^{p'}_{B^{3\delta}_{p(M-k),p(M-k)}}]^{\frac{p(\ell+1)}{p'}}
\end{aligned}$$

where, as usual, $q'\geqslant 1$ such that $\frac{1}{q'}+\frac{1}{p'}=1$, and $A_\varepsilon(f)=\rho_\varepsilon*f-f$. Since by Lemma 18 $\mathbb{E}[\|:(\varphi_\varepsilon^G)^k:-:(\varphi^G)^k:\|^{2q'}_{\mathcal{C}^{-\delta}}]\to 0$, as $\varepsilon\to 0$ as $\mathbb{E}[\|h_\varepsilon\|^{p'}_{B^{2\delta}_{p(M-k),2(M-k)}}]$ is uniformly bounded when $0\leqslant\varepsilon\leqslant 1$ and since by Remark 11 $\lim_{\varepsilon\to 0}\|A_\varepsilon\|_{\mathcal{L}(B^{3\delta}_{p,q},B^{2\delta}_{p,q})}=0$, we get $\mathbb{E}[\|:(\varphi_\varepsilon^G)^k:h_\varepsilon^{2n-k}-:(\varphi^G)^k:h^{2n-k}\|^p_{B^{-\delta}_{p,p}}]\to 0$ as $\varepsilon\to 0$. By inequality (61) and since the previous proof can be repeated (with different $\delta',\delta''>0$, $p'\geqslant 1$ and $0\leqslant\theta\leqslant 1$) for any $n\geqslant k\in\mathbb{N}$, this implies the thesis. □

Remark 13 A consequence of the proof of Lemma 19 and of the estimates in Lemma 11 and Lemma 18 is that there exists a $c>0$ for which

$$\mathbb{E}[\|(\varphi_\varepsilon^A)^{\circ M}-(\varphi^A)^{\circ M}\|^p_{B^{-\delta}_{p,p}}]\lesssim\varepsilon^{c\wedge\delta}.$$

Let $P(x)=\sum_{k=0}^M c_k x^k$ be a polynomial then we write

$$P^\circ(\varphi^A)=\sum_{k=0}^{M}c_k(\varphi^A)^{\circ k}.$$

Theorem 9 *Let P be a polynomial of even degree with positive leading coefficient, then for any $p \geqslant 0$ we have*

$$\mathbb{E}\left[\exp\left(-p\int_{\mathbb{T}^2} P^{\circ}(\varphi^A)(x)\mathrm{d}x\right)\right] < +\infty.$$

Proof The proof is similar to the proof of exponential integrability of the even Wick products of Gaussian free field in 2d (see, e.g., Chapter V of [62]). For this reason, here we only provide a sketch of the proof in the case where $P(x) = x^{2n}$ for some $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$, then there is $K \geqslant 0$ such that $H_{2n}(x) \geqslant -K$. This means that

$$(\varphi^A_\varepsilon)^{\circ 2n} \geqslant -K c_\varepsilon^{2n}.$$

By recalling that $c_\varepsilon \gtrsim \sqrt{\log \frac{1}{\varepsilon}}$, and thus the previous inequality becomes

$$\int_{\mathbb{T}^2} (\varphi^A_\varepsilon)^{\circ 2n}\mathrm{d}x \geqslant -K'\left(\log\left(\frac{1}{\varepsilon}\right)\right)^n \geqslant 1 - 2K'\left(\log\left(\frac{1}{\varepsilon}\right)\right)^n \tag{62}$$

for $\varepsilon \ll 1$ and for some $K' \geqslant 0$. By hypercontractivity and the proof of Lemma 19 (see Remark 13) there is $c, K'' > 0$ such that, for every $p \geqslant 2$,

$$\begin{aligned}\mathbb{E}\left[\left|\int_{\mathbb{T}^2}(\varphi^A_\varepsilon)^{\circ 2n}\mathrm{d}x - \int_{\mathbb{T}^2}(\varphi^A)^{\circ 2n}\mathrm{d}x\right|^p\right] &\lesssim (p-1)^{np}(\mathbb{E}[\|(\varphi^A_\varepsilon)^{\circ 2n} - (\varphi^A)^{\circ 2n}\|^2_{H^{-\delta}}])^{\frac{p}{2}} \\ &\leqslant (K'')^p(p-1)^{np}\varepsilon^{\frac{cp}{2}}. \end{aligned}\tag{63}$$

Now, if $\int_{\mathbb{T}^2}(\varphi^A)^{\circ 2n}\mathrm{d}x \leqslant -2K'\left(\log\left(\frac{1}{\varepsilon}\right)\right)^n$ then, by inequality (62), $\left|\int_{\mathbb{T}^2}(\varphi^A)^{\circ 2n}\mathrm{d}x - \int_{\mathbb{T}^2}(\varphi^A_\varepsilon)^{\circ 2n}\mathrm{d}x\right| \geqslant 1$, and thus, by Markov inequality and inequality (63) we get

$$\begin{aligned} &\mathbb{P}\left(\int_{\mathbb{T}^2}(\varphi^A)^{\circ 2n}\mathrm{d}x \leqslant -2K'\left(\log\left(\frac{1}{\varepsilon}\right)\right)^n\right) \\ \leqslant\ & \mathbb{P}\left(\left|\int_{\mathbb{T}^2}(\varphi^A)^{\circ 2n}\mathrm{d}x - \int_{\mathbb{T}^2}(\varphi^A_\varepsilon)^{\circ 2n}\mathrm{d}x\right| \geqslant 1\right) \\ \leqslant\ & \mathbb{E}\left[\left|\int_{\mathbb{T}^2}(\varphi^A)^{\circ 2n}\mathrm{d}x - \int_{\mathbb{T}^2}(\varphi^A_\varepsilon)^{\circ 2n}\mathrm{d}x\right|^p\right] \lesssim (K'')^p(p-1)^n\varepsilon^{\frac{cp}{2}}. \end{aligned}$$

If we choose $p = \left(\frac{1}{\varepsilon}\right)^{\frac{c}{6n}}$ we get, for $\varepsilon \ll 1$, that

$$\mathbb{P}\left(\int_{\mathbb{T}^2}(\varphi^A)^{\circ 2n}\mathrm{d}x \leqslant -2K'\left(\log\left(\frac{1}{\varepsilon}\right)\right)^n\right) \leqslant e^{-\left(\frac{1}{\varepsilon}\right)^{\frac{c}{6n}}} \tag{64}$$

Since inequality (64) holds for any $\varepsilon > 0$ small enough, by standard methods (see Theorem V.7 [62]), inequality (64) implies the thesis. □

Thanks to the previous theorem we can define Anderson Φ^4_2 Gibbs measure (so the Φ^4_2 measure with respect to the AGFF). Indeed we introduce the following definition.

Definition 6 (Anderson Φ^4_2) *Consider $\omega \in \Omega$ and $\mu^{\mathbb{H}^{\omega},K}$ (defined in Section 2.2), we defined the Anderson Φ^4_2 measure as the measure ν^ω defined as*

$$\nu^\omega = (\mathcal{Z}(\omega))^{-1}\exp\left(-\int_{\mathbb{T}^2}(\varphi^A)^{\circ 4}\mathrm{d}x + K(\omega)\int_{\mathbb{T}^2}(\varphi^A)^{\circ 2}\right)\mathrm{d}\mu^{\mathbb{H}^{\omega},K}$$

where $\mathcal{Z}(\omega) = \int \exp\left(-\int_{\mathbb{T}^2}(\varphi^A)^{\circ 4}\mathrm{d}x + K(\omega)\int_{\mathbb{T}^2}(\varphi^A)^{\circ 2}\right)\mathrm{d}\mu^{\mathbb{H}^{\omega},K}(\varphi)$ which is well defined by Theorem 9.

Remark 14 From the definition of the Anderson Φ^4_2 measure and Theorem 9 we obtain readily that ν^ω is absolutely continuous with respect to the Gaussian AGFF measure.

4 Local-in-time solution

In this section we prove local-in-time well-posedness for a renormalised nonlinear wave equation with white noise potential with initial data supported in the Gibbs measure

$$(u_0, u_1) \sim \nu^\omega \otimes \mu^{I_{L^2}},$$

where ν^ω is the Anderson Φ^4_2 measure from Section 3.6 and $\mu^{I_{L^2}}$ is the white noise measure, defined in Section 2.2.

As we have seen in Remark 14, the Anderson Φ^4_2 measure is mutually absolutely continuous w.r.t. the Anderson Free Field $\mu^{\mathbb{H}^{\omega,K}}$ and so we can write the initial conditions via random series as

$$\begin{aligned} u_0(\omega') &= \sum_{n\in\mathbb{N}} \frac{\hat{g}_0(\omega')}{\lambda_n} f_n \in \mathcal{C}^{-\delta} \text{a.s.} && (65) \\ u_1(\omega') &= \sum_{n\in\mathbb{N}} \hat{g}_1(\omega') f_n, \in \mathcal{C}^{-1-\delta} \text{ a.s.} && (66) \end{aligned}$$

for any $\delta > 0$, where λ_n, f_n are eigenvalues and -function of $\mathbb{H}^\omega$ as in (30) and the $\hat{g}_0, \hat{g}_1$ are i.i.d. standard Gaussians, similar to the definition of ξ in (8). Note here that we distinguish the two probability spaces $\omega' \in \Omega'$ for the random initial data and $\omega \in \Omega$ for the randomness in the Anderson Hamiltonian. Since the operator $\mathbb{H}^{\omega,K}$ depends on $\omega \in \Omega$, the eigenvalues and -functions of course also depend on ω but we omit that dependence here for brevity.

We consider the random data SPDE formally given by

$$\begin{aligned} (\partial_t^2 + \mathbb{H}^\omega)u + u^3 &= 0 && (67) \\ (u, \partial_t u)|_{t=0} &= (u_0(\omega'), u_1(\omega')), \end{aligned}$$

where due to the irregularity of the initial data, we actually have to Wick-ordered the nonlinearity as was done in [58] in the classical case without the white noise potential.

In order to illustrate this point, we make the following ansatz for u (named after Da Prato Debussche in the SPDE literature and Bourgain/McKean in the dispersive PDE literature) and we use the shifted operator $\mathbb{H}^{\omega,K}$ as in (22) in order to avoid difficulties with taking square roots. This means we change the equation (67) by adding a linear term but we will see that one has to renormalize the cube by subtracting an infinite linear term in any case; We set

$$u := \theta + v, \text{ where } \begin{cases} (\partial_t^2 + \mathbb{H}^{\omega,K})\theta &= 0 \\ (\theta, \partial_t\theta)|_{t=0} &= (u_0(\omega'), u_1(\omega')) \end{cases}$$

and v formally satisfies the equation

$$\begin{cases} (\partial_t^2 + \mathbb{H}^{\omega,K})v &= -(\theta+v)^3 = -\theta^3 - 3v\theta^2 - 3v^2\theta - v^3 \\ (v, \partial_t v)|_{t=0} &= 0 \end{cases}. \tag{68}$$

Now from the mild formulation (as was derived in [37])

$$\theta(t) = \cos\left(t\sqrt{\mathbb{H}^{\omega,K}}\right) u_0(\omega') + \frac{\sin\left(t\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}} u_1(\omega') \tag{69}$$

and Theorem 4 we see that θ will have regularity $C(\mathbb{R}_+; H^{-\delta})$, using hypercontractivity it is even in $C(\mathbb{R}_+; \mathcal{C}^{-\delta})$ cf. Section 2.2 but crucially it has negative regularity. Thus it is not possible to classically define powers of θ appearing in (68), however we can replace them by Wick powers as introduced in Section 3.6 as was done in [16], [58] etc.

Proposition 8 (Anderson wave Wick polynomials) *Let* $(u_0, u_1) \sim \mu^{\mathbb{H}^{\omega,K}} \otimes \mu^{I_{L^2}}$*, i.e. as in* (65) *and* (66)*, then*

$$\theta(t) \sim u_0 \sim \mu^{\mathbb{H}^{\omega,K}} \text{ for all timest} \in \mathbb{R},$$

where θ *is defined in* (69)*. Moreover*

$$\theta^{\circ k}(t) \sim (u_0)^{\circ k} \text{ for all } k \in \mathbb{N} \text{and } t \in \mathbb{R}$$

i.e. in law the time dependent Wick ordering is the same as the Wick ordering of the initial condition w.r.t. the Anderson GFF from Section 3.6. Also one has the bound

$$\|\theta^{\circ k}\|_{L^p_{\Omega'} L^p_{[0,T]} \mathcal{C}^{-k\delta}} \leqslant T^{\frac{1}{p}} \|(u_0)^{\circ k}\|_{L^p_{\Omega'} \mathcal{C}^{-k\delta}} \text{for } 2 \leqslant p < \infty, \delta > 0 \text{and } k \in \mathbb{N}$$

and the norm on the right hand side is finite by Theorem 9 and one also has an exponential tail estimate of the form

$$\mathbb{P}'(\|\theta^{\circ k}\|_{L^p_{[0,T]} \mathcal{C}^{-k\delta}} > R) \lesssim e^{-CR} \tag{70}$$

for some $C > 0$ *and all* $R > 0$.

Proof *This is because the law of* $\theta(t)$ *is a rotated Gaussian, see Proposition 2.3 in [58].*

In fact for i.i.d. centered standard Gaussian random variables $g_0^i(\omega')$ *and* $g_1^j(\omega')$ *one can write*

$$u_0(\omega') = \sum_{i \in \mathbb{N}} \frac{g_0^i(\omega')}{\lambda_i} f_i \text{ and } u_1(\omega') = \sum_{j \in \mathbb{N}} \frac{g_1^j(\omega')}{\lambda_j} f_j, \tag{71}$$

where λ_n *and* f_n *are the eigenvalues and -functions of* $\mathbb{H}^{\omega,K}$ *respectively. Thus one has*

$$\theta(t) = \cos\left(t\sqrt{\mathbb{H}^{\omega,K}}\right) u_0(\omega') + \frac{\sin\left(t\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}} u_1(\omega') = \sum_i \frac{1}{\lambda_i} (\cos(t\lambda_i) g_0^i(\omega') + \sin(t\lambda_i) g_1^i(\omega')) f_i$$

and one sees that the term in the brackets is nothing but a rotated Gaussian with mean zero and Variance $\cos^2(t\lambda_i) + \sin^2(t\lambda_i) = 1$*. Thus* $\theta(t)$ *is in law equal to* u_0 *and the* L^p*bound follows readily. The exponential tail estimate follows from the previous bound and Theorem 9.* □

The corrected equation for v then reads

$$\begin{cases} (\partial_t^2 + \mathbb{H}^{\omega,K}) v &= -(\theta + v)^{\circ 3} = -\theta^{\circ 3} - 3v\theta^{\circ 2} - 3v^2\theta - v^3 \\ (v, \partial_t v)|_{t=0} &= 0 \end{cases}, \tag{72}$$

where $\theta^{\circ k} \in L^p_T \mathcal{C}^{-\varepsilon k}$ for any $\varepsilon > 0$ and $p < \infty$ is the Wick power from Proposition 8. This looks quite similar to the renormalized wave equation from [58] except that we have replaced the Laplace operator by the Anderson Hamiltonian and consequently we use a different Wick ordering.

However, due to the norm equivalence $\|u\|_{H^s} \approx \left\|(\mathbb{H}^{\omega,K})^{\frac{s}{2}} u\right\|_{L^2}$ and the regularizing property of $\frac{\sin\left(t\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}}$, see Theorem 4, we are able to solve (72) locally in time without much effort. To do this we prove a simple lemma which quantifies those two properties of $\mathbb{H}^{\omega,K}$.

Lemma 20 (Inhomogeneous estimate) *For $\sigma \in (0,1)$ and $p \geqslant 1$ we have*

$$\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}} f(s)ds \right\|_{H^{1-\sigma}} \lesssim t^{\frac{p-1}{p}} \|f\|_{L^p_{[0,t]}H^{-\sigma}}$$

for all times $t \geqslant 0$, with the obvious modification for $p=\infty$.
Proof *This is a simple consequence of Hölder's inequality in time and the aforementioned norm equivalence from Theorem 4. Indeed we may bound*

$$\begin{aligned}
\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}} f(s)ds \right\|_{H^{1-\sigma}} &\overset{(31)}{\approx} \left\| \sqrt{\mathbb{H}^{\omega,K}}^{1-\sigma} \int_0^t \frac{\sin\left((t-s)\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}} f(s)ds \right\|_{L^2} \\
&\lesssim \int_0^t \left\| \frac{\sin\left((t-s)\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}^{\sigma}} f(s) \right\|_{L^2} ds \\
&\overset{(26)}{\lesssim} \int_0^t \left\| \sqrt{\mathbb{H}^{\omega,K}}^{-\sigma} f(s) \right\|_{L^2} ds \\
&\overset{(31)}{\approx} \int_0^t \|f(s)\|_{H^{-\sigma}} ds \\
&\lesssim t^{\frac{p-1}{p}} \|f\|_{L^p_{[0,t]}H^{-\sigma}}
\end{aligned}$$

and thus we are done. □

This allows us to prove local well-posedness for the SPDE (72) via fixed point.

Theorem 10 (Local well-posedness) *Let $0 < \delta \ll 1$ and $p \gg 1$. With θ defined as above, there exists a time*

$$T \sim \left(\|\theta^{\circ 3}\|^{\frac{1}{3}}_{L^p_{[0,1]}H^{-\delta}} + \|\theta^{\circ 2}\|^{\frac{1}{2}}_{L^p_{[0,1]}\mathcal{C}^{-\delta}} + \|\theta\|_{L^\infty_{[0,1]}\mathcal{C}^{-\delta}} + 1 \right)^{-2\frac{p}{p-1}}$$

which is almost surely in $(0,1)$ so that there exists a unique solution

$$v \in C([0,T]; H^{1-\delta}) \cap C^1([0,T]; H^{-\delta})$$

to the equation

$$v(t) = \int_0^t \frac{\sin\left((t-s)\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}} (\theta^{\circ 3}(s) + 3v(s)\theta^{\circ 2}(s) + 3v^2(s)\theta(s) + v^3(s))ds, \tag{73}$$

which is the mild formulation of (72).

Proof We make a contraction argument for v in a ball in $L^\infty_T H^{1-\delta}$ and then show a posteriori that one in fact has continuity in time as well.

As usual, we define the map

$$\Psi(v) := \int_0^t \frac{\sin\left((t-s)\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}} (\theta^{\circ 3}(s) + 3v(s)\theta^{\circ 2}(s) + 3v^2(s)\theta(s) + v^3(s))ds$$

and we want to show that it is a contraction on a suitable ball. We make the following estimations which are valid for any time $t > 0$ using heavily Lemma 20

$$\begin{aligned}
\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}} \theta^{\circ 3}(s) ds \right\|_{H^{1-\delta}} &\lesssim t^{\frac{p-1}{p}} \|\theta^{\circ 3}\|_{L^p_{[0,t]} H^{-\delta}} \\
\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}} (3v(s)\theta^{\circ 2}(s)) ds \right\|_{H^{1-\delta}} &\lesssim t^{\frac{p-1}{p}} \|v\|_{L^\infty_{[0,t]} H^{2\delta}} \|\theta^{\circ 2}(s)\|_{L^p_{[0,t]} \mathcal{C}^{-\delta}} \\
\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}} (3v^2(s)\theta(s)) ds \right\|_{H^{1-\delta}} &\lesssim t \|\theta\|_{L^\infty_{[0,t]} \mathcal{C}^{-\delta}} \|v^2\|_{L^\infty H^{2\delta}} \\
&\lesssim t \|\theta\|_{L^\infty_{[0,t]} \mathcal{C}^{-\delta}} \|v\|^2_{L^\infty H^{\frac{1}{2}}} \\
\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}} v^3(s) ds \right\|_{H^{1-\delta}} &\lesssim t \|v\|^3_{L^\infty_{[0,t]} L^6} \\
&\lesssim t \|v\|^3_{L^\infty_{[0,t]} H^{\frac{2}{3}}}.
\end{aligned}$$

This allows us to bound (taking $t < 1$)

$$\begin{aligned}
& \|\Psi(v)\|_{L^\infty_{[0,t]} H^{1-\delta}} \\
\lesssim\ & t^{\frac{p-1}{p}} \left(\|\theta^{\circ 3}\|_{L^p_{[0,t]} H^{-\delta}} + \|v\|_{L^\infty_{[0,t]} H^{2\delta}} \|\theta^{\circ 2}\|_{L^p_{[0,t]} \mathcal{C}^{-\delta}} + \|\theta\|_{L^\infty_{[0,t]} \mathcal{C}^{-\delta}} \|v\|^2_{L^\infty_{[0,t]} H^{\frac{1}{2}}} + \|v\|^3_{L^\infty_{[0,t]} H^{\frac{2}{3}}} \right)
\end{aligned}$$

and similarly

$$\begin{aligned}
& \|\Psi(v) - \Psi(w)\|_{L^\infty_{[0,t]} H^{1-\delta}} \\
\lesssim\ & \left(\|\theta^{\circ 2}\|_{L^p_{[0,t]} \mathcal{C}^{-\delta}} + \|\theta\|_{L^p_{[0,t]} \mathcal{C}^{-\delta}} (\|v\|_{L^\infty_{[0,t]} H^{2\delta}} + \|w\|_{L^\infty_{[0,t]} H^{2\delta}}) + \|v\|^2_{L^\infty_{[0,t]} H^{\frac{2}{3}}} + \|w\|^2_{L^\infty_{[0,t]} H^{\frac{2}{3}}} \right) \times \\
& t^{\frac{p-1}{p}} \|v - w\|_{L^\infty_{[0,t]} H^{2\delta}}
\end{aligned}$$

Now we simply take $\|v\|_{L^\infty_{[0,T]} H^{1-\delta}}, \|w\|_{L^\infty_{[0,T]} H^{1-\delta}} \leqslant R$ with $R > 0$ and $0 < T \leqslant 1$ chosen s.t.

$$\begin{aligned}
T^{\frac{p-1}{p}} \left(\|\theta^{\circ 3}\|_{L^p_{[0,T]} H^{-\delta}} + R \|\theta^{\circ 2}\|_{L^p_{[0,T]\mathcal{C}^{-\delta}}} + \|\theta\|_{L^\infty_{[0,T]} \mathcal{C}^{-\delta}} R^2 + R^3 \right) &\leqslant R \\
\text{and} & \\
T^{\frac{p-1}{p}} \left(\|\theta^{\circ 2}\|_{L^p_{[0,T]\mathcal{C}^{-\delta}}} + \|\theta\|_{L^p_{[0,T]\mathcal{C}^{-\delta}}} 2R + 2R^2 \right) &\leqslant \frac{1}{2}
\end{aligned}$$

this can be achieved e.g. by setting

$$R = \|\theta^{\circ 3}\|^{\frac{1}{3}}_{L^p_{[0,1]} H^{-\delta}} + \|\theta^{\circ 2}\|^{\frac{1}{2}}_{L^p_{[0,1]\mathcal{C}^{-\delta}}} + \|\theta\|_{L^\infty_{[0,1]} \mathcal{C}^{-\delta}} + 1 \quad \text{and } T = \left(\frac{1}{10R^2} \right)^{\frac{p}{p-1}}.$$

Thus by Banach's fixed point theorem we have that there exists a unique solution v to (72).

Continuity in time follows as per usual from Stone's theorem, see e.g. the proof of Theorem 3.19 in [37]. □

We summarize this result as follows: The flow of the equation

$$\begin{cases} (\partial_t^2 + \mathbb{H}^{\omega,K})u &= -u^{\circ 3} \\ (u, \partial_t u)|_{t=0} &= (u_0(\omega'), u_1(\omega')) \end{cases} \tag{74}$$

which we denote by $\Phi^\omega(t)((u_0(\omega'), u_1(\omega'))) := u(t)$ is measurable as a map

$$\Phi^\omega : B(R) \to \theta + C([0,T(R)]; H^{1-\delta}) \cap C^1([0,T(R)]; H^{-\delta}), \tag{75}$$

where as before θ is as in (69), the linear propagator applied to the initial data which satisfy

$(u_0(\omega'), u_1(\omega')) \in B(R)$, where

$$B(R) := \left\{ (u_0, u_1) \in \operatorname{supp}\left(\mu^{\mathbb{H}^{w,K}} \otimes \mu^{\mathbb{I}_{L^2}} : \|\theta^{\circ 3}\|^{\frac{1}{3}}_{L^p_{[0,1]}H^{-\delta}} + \|\theta^{\circ 2}\|^{\frac{1}{2}}_{L^p_{[0,1]}\mathcal{C}^{-\delta}} + \|\theta\|_{L^\infty_{[0,1]}\mathcal{C}^{-\delta}} \leqslant R - 1 \right\}$$

for $R \gg 1$ and $0 < T < 1$ satisfying $T = \left(\frac{1}{10R^2}\right)^{\frac{p}{p-1}}$ where $p \gg 2$ is taken to be large.

Proposition 8 implies that the measure of $B(R)$ is large, i.e.

$$\mathbb{P}'(B(R)^c) \lesssim e^{-CR}. \tag{76}$$

This means one has

$$\|\Phi^\omega(u_0(\omega'), u_1(\omega')) - \theta\|_{L^\infty_{[0,T(R)]}H^{1-\delta}} \leqslant R \text{ and } \|\Phi^\omega(u_0(\omega'), u_1(\omega'))\|_{L^\infty_{[0,T(R)]}H^{-\delta}} \leqslant 2R$$

for such initial data.

Remark 15 As we have a Wick cube in the equation which is a cube renormalized by a logarithmically diverging constant times the function, see Section 3.6, one could consider also the *focusing* version of the equation (meaning to change the sign of the nonlinearity) with Gaussian initial data and tune the parameters in such a way that the divergence cancels with the logarithmically diverging constant, see (16), in the definition of the renormalized product of the Anderson Hamiltonian.

As the short-time well-posedness does not depend on the sign and is continuous in the noise and the initial data, one would obtain local-in-time dynamics for this focusing equation where the infinities cancel. Clearly the globalization and invariance arguments will fail, however.

4.1 Local-in-time convergence of approximations

Furthermore we need an analogous short-time well-posedness result for different approximations to the SPDE (72). We consider three different approximations and one approximation that combines two of those.

One natural approximation is to smoothen the noise ξ i.e. replacing the operator $\mathbb{H}^{\omega,K}$ by the operator $\mathbb{H}^{\omega,K}_\delta$ from (23) and to replace the Wick powers by regularized Wick powers as in Section 3.6. This would amount to solving the SPDE

$$\begin{aligned} (\partial_t^2 + \mathbb{H}^{\omega,K}_\delta)u_{(\delta)} + u_{(\delta)}^{\circ_\delta 3} &= 0 \\ (u_{(\delta)}, \partial_t u_{(\delta)})|_{t=0} &= (u_0^{(\delta)}(\omega'), u_1(\omega')), \end{aligned} \tag{77}$$

where one would define the Wick ordering $(\cdot)^{\circ_\delta}$ analogously to Section 3.6 by replacing $\mathbb{H}^{\omega,K}$ by $\mathbb{H}^{\omega,K}_\delta$ and $u_0^{(\delta)}$ is in the support of the Gaussian measure with covariance $(\mathbb{H}^{\omega,K}_\delta)^{-1}$. While this seems like a very natural approximation, it is actually not very useful if we want to approximate the dynamics (74) since one has not only a different operator in the equation but also, crucially, a different reference Gaussian measure $\mu^{\mathbb{H}_\delta^{\omega,K}}$ for the support of the initial condition which is mutually singular w.r.t. the reference Gaussian for (74) $\mu^{\mathbb{H}^{\omega,K}}$, namely the Anderson Free Field as was remarked in Lemma 2. This then of course also leads to a different Wick ordered nonlinearity for which one would have to prove strong enough convergence.

Another downside of the approximation (77) is that it does not behave well under finite dimensional projection which is crucial if one wants to prove invariance. This leads us to the next natural choice, namely is the finite dimensional Galerkin approximation, where we project the equation onto the span of eigenfunctions f_n of $\mathbb{H}^{\omega,K}$, see (30). To this end we define the projection

$$\begin{aligned} P_{\leqslant N} \quad &: \quad L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2) \\ P_{\leqslant N} g \quad &:= \quad \sum_{n \leqslant N} (g, f_n) f_n \end{aligned}$$

for $N \in \mathbb{N}$ and u_N as the solution to

$$\begin{aligned} (\partial_t^2 + \mathbb{H}^{\omega,K}) u_N + P_{\leqslant N}(u_N^{\circ 3}) \quad &= \quad 0 \\ (u_N, \partial_t u_N)|_{t=0} \quad &= \quad (P_{\leqslant N} u_0(\omega'), P_{\leqslant N} u_1(\omega')) \end{aligned} \tag{78}$$

and we denote by $\Phi^{\omega,N}$ its flow. This will be useful in Section 5 since this is just a finite dimensional Hamiltonian system whose Gibbs measure is invariant and approximates ν^ω. The finite dimensional projections are also compatible with the choice of initial conditions (see (71)) and the Wick product, see Section 3.6.

The third approximation we consider is a regularization of the nonlinearity in which we replace the cubic nonlinearity by a regularized cube in a way that we still get an invariant dynamics. The modified equation then reads

$$\begin{aligned} (\partial_t^2 + \mathbb{H}^{\omega,K}) u_\varepsilon + \rho_\varepsilon * ((u_\varepsilon * \rho_\varepsilon)^{\circ 3}) \quad &= \quad 0 \\ (u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} \quad &= \quad (u_0(\omega'), u_1(\omega')), \end{aligned} \tag{79}$$

where ρ_ε is the convolution with a standard symmetric mollifier. Again we define Φ_ε^ω as its flow.

This approximation has the upside that it has the (for now formally) invariant measure ν^ε which is mutually absolutely continuous w.r.t. the Anderson Free Field $\mu^{\mathbb{H}^{\omega,K}}$ with smooth density

$$\frac{d\nu_\varepsilon^\omega}{d\mu^{\mathbb{H}^{\omega,K}}}(v) = \int_{\mathbb{T}^2} (v * \rho_\varepsilon)^{\circ 4} dx.$$

By construction of the Wick ordering in Section 3.6, we have that this density converges strongly in L^p to

$$\frac{d\nu^\omega}{d\mu^{\mathbb{H}^{\omega,K}}}(v) = \int_{\mathbb{T}^2} v^{\circ 4} dx$$

implying convergence in total variation of the measures.

Due to this property, one may use the the same initial data when approximating the equation (74) by (79).

Finally we define an approximation that combines the last two, i.e. we define the Galerkin approximation of (79) so

$$\begin{aligned}(\partial_t^2+\mathbb{H}^{\omega,K})u_{N,\varepsilon}+P_{\leqslant N}(\rho_\varepsilon*((u_{N,\varepsilon}*\rho_\varepsilon)^{\circ 3})) &= 0 \\ (u_{N,\varepsilon},\partial_t u_{N,\varepsilon})|_{t=0} &= (P_{\leqslant N}u_0(\omega'),P_{\leqslant N}u_1(\omega')),\end{aligned} \tag{80}$$

writing $\Phi_\varepsilon^{\omega,N}$ for its flow.

By the Hamiltonian structure, as in [58], the equations (78) and (80) are globally well-posed and their flows leave the following finite dimensional Gibbs measures

$$\begin{aligned} d\nu_N^\omega &:= \exp\left(-\tfrac{1}{4}\textstyle\int_{\mathbb{T}^2}(P_{\leqslant N}\phi)^{\circ 4}\mathrm{d}x\right)\mathrm{d}\mu^{\mathbb{H}^{\omega,K}}(\phi)\otimes\mathrm{d}\mu^{\mathbb{I}_{L^2}}(\partial_t\phi) \\ &\text{and} \\ d\nu_{N,\varepsilon}^\omega &:= \exp\left(-\tfrac{1}{4}\textstyle\int_{\mathbb{T}^2}(\rho_\varepsilon*P_{\leqslant N}\phi)^{\circ 4}\mathrm{d}x\right)\mathrm{d}\mu^{\mathbb{H}^{\omega,K}}\otimes\mathrm{d}\mu^{\mathbb{I}_{L^2}}(\partial_t\phi)\end{aligned}$$

invariant respectively.

For (79) one has the same local well-posedness as for the limiting equation (74). Later we will globalize the solutions to these equations and prove that their flows are invariant.

We summarize these results in the following result.

Proposition 9 (Well-posedness of approximate equations) *Let* $(u_0,u_1)\in\operatorname{supp}\left(\mu^{\mathbb{H}^{\omega,K}}\otimes\mu^{\mathbb{I}_{L^2}}\right)$, *then the flows of the equations* (78) *and* (80) *called* $\Phi^{\omega,N}$ *and* $\Phi_\varepsilon^{\omega,N}$ *respectively, exist for all times for initial data* $(P_{\leqslant N}u_0,P_{\leqslant N}u_1)$ *and we have the following convergence*

$$\|\Phi_\varepsilon^{\omega,N}(P_{\leqslant N}u_0,P_{\leqslant N}u_1)-\Phi^{\omega,N}(P_{\leqslant N}u_0,P_{\leqslant N}u_1)\|_{L^\infty_{[0,T]}L^2(\mathbb{T}^2)}\to 0 \text{ as } \varepsilon\to 0 \text{ for all } T>0.$$

Now let $R\gg 1$, $p\gg 2$ *and* $T=\left(\frac{1}{10R^2}\right)^{\frac{p}{p-1}}$ *as well as* $\delta>0$ *small. If the initial data satisfy additionally*

$$\|u_0^{\circ 3}\|_{\mathcal{C}^{-\delta}}^{\frac{1}{3}}+\|u_0^{\circ 2}\|_{\mathcal{C}^{-\delta}}^{\frac{1}{2}}+\|u_0\|_{\mathcal{C}^{-\delta}}+\|u_1\|_{H^{-1-\delta}}+1\leqslant R$$

then the flow Φ_ε^ω *of the equation* (79) *exists up to the time* T *and we have*

$$\|\Phi^{\omega,\varepsilon}(u_0,u_1)\|_{L^\infty_{[0,T]}H^{-\delta}} \leqslant 2R \tag{81}$$

$$\lim_{\varepsilon\to 0}\|\Phi_\varepsilon^\omega(u_0,u_1)-\Phi^\omega(u_0,u_1)\|_{L^\infty_{[0,T]}H^{-\delta}} = 0 \tag{82}$$

$$\lim_{N\to\infty}\|\Phi_\varepsilon^\omega(u_0,u_1)-\Phi_\varepsilon^{\omega,N}(P_{\leqslant N}u_0,P_{\leqslant N}u_1)\|_{L^\infty_{[0,T]}H^{-\delta}} = 0, \tag{83}$$

where we recall that Φ^ω *is the flow of the full equation* (74) *which was shown to exist up to time* T *in Theorem 10 for such initial data.*

5 Globalization and Invariance

By the computations on the covariance, we have that the pair $\left(\mu^{\mathbb{H}^{\omega,K}},\mu^{\mathbb{I}_{L^2}}\right)$ is invariant under the flow of the linear equation

$$(\partial_t^2+\mathbb{H}^{\omega,K})u=0 \qquad (u,\partial_t u)|_{t=0}=(u_0,u_1). \tag{84}$$

Now, let ρ_ε be a standard mollifier. Let $P_{\leqslant N}$ be the projection on to the first N eigenfunctions of $\mathbb{H}^{\omega,K}$. We will denote by $u^{N,\varepsilon}$ the solution to the equation

$$\begin{aligned}(\partial_t^2+\mathbb{H}^{\omega,K})u^{N,\varepsilon} &= -P_{\leqslant N}(\rho_\varepsilon*(\rho_\varepsilon*P_{\leqslant N}u^{N,\varepsilon})^{\circ 3}) \\ (u,\partial_t u)|_{t=0} &= \varphi=(\varphi_0,\varphi_1).\end{aligned} \tag{85}$$

Note that here the Wick ordering is taken with respect to the covariance of $\rho_\varepsilon * W$ where W is an AGFF. We will consider $u^{N,\varepsilon}$ as a function of φ and when we want to stress this dependence, we will write $\Phi_t^{N,\varepsilon}\varphi = u^{N,\varepsilon}(t)$, where $\Phi^{N,\varepsilon}$ denotes the flow (we again drop the dependence on ω of this and related objects for brevity). We will also consider the solution u^ε equation

$$(\partial_t^2 + \mathbb{H}^{\omega,K})u^\varepsilon = -(\rho_\varepsilon * (\rho_\varepsilon * u^\varepsilon)^{\circ 3}) \qquad (u, \partial_t u)|_{t=0} \quad = \varphi = (\varphi_0, \varphi_1), \tag{86}$$

and the Wick ordering is the same as above. We write $\Phi_t^\varepsilon \varphi = u^\varepsilon(t)$. Finally we also consider

$$(\partial_t^2 + \mathbb{H}^{\omega,K})u = -u^{\circ 3} \qquad (u, \partial_t u)|_{t=0} \quad = \varphi = (\varphi_0, \varphi_1). \tag{87}$$

where the Wick ordering is now taken with respect to the Anderson free field, see Section 3.6. We will denote $\Phi_t\varphi = u(t)$. Our goal is to show that (87) has ν-almost surely global solutions and ν is invariant under the flow Φ_t. To do this we implement the well-known Bourgain argument.

5.1 Approximation by finite dimensional system

In this section we fix an $\varepsilon > 0$ and will prove that a smoothened version of our system can be approximated by a finite-dimensional one. Note that (85) splits into a finite dimensional Hamiltonian dynamical system and a linear equation. By Liouville's theorem in finite dimensions we know that

$$\nu^{N,\varepsilon} = \left(\exp\left(-\frac{1}{4}\int_{\mathbb{T}^2} (\rho_\varepsilon * P_{\leqslant N}\phi)^{\circ 4} \mathrm{d}x\right) \mathrm{d}\mu^{\mathbb{H}^{\omega,K}}, \mathrm{d}\mu^{\mathbb{I}_{L^2}}\right)$$

is invariant under the flow of (85). Observe that if we keep ε fixed $(\rho_\varepsilon * P_{\leqslant N}\phi)^{\circ 4}$ is bounded below uniformly in N since the renormalization function $\mathbb{E}(\rho_\varepsilon * \phi)(x)^2$ is bounded uniformly in N and x (but not ε). Furthermore, $(\rho_\varepsilon * P_{\leqslant N}\phi)^{\circ 4} \to (\rho_\varepsilon * \phi)^{\circ 4}$ in $L^1(\mathbb{T}^2)$ μ-almost surely. Combined with the lower bound, this gives us by dominated convergence

$$\nu^{N,\varepsilon} \quad \to \quad \nu^\varepsilon := \left(\exp\left(-\frac{1}{4}\int_{\mathbb{T}^2} (\rho_\varepsilon * \phi)^{\circ 4} \mathrm{d}x\right) \mathrm{d}\mu^{\mathbb{H}^{\omega,K}}, \mathrm{d}\mu^{\mathbb{I}_{L^2}}\right)$$

in total variation. Our first goal is to show that ν^ε is invariant under Φ^ε. To this end we first have to establish that u^ε exists ν^ε-almost surely for all times. We begin with the following local well-posedness statement:

Lemma 21 *For every $\delta > 0$ there exists an $L \in \mathbb{N}$ and $c > 0$ such that for $t \leqslant c(1 + \|\varphi\|_{H^{-\delta}})^{-L}$ such that*

$$\|\Phi_t^{N,\varepsilon}\varphi\|_{H^{-\delta}} \leqslant 2\|\varphi\|_{H^{-\delta}}.$$

Note that c, L can depend on ε and δ but not on N.

Proof This is similar to Proposition 9. □

Lemma 22 *With the notation of Lemma 21 we have*

$$\sup_{N \in \mathbb{N}} \nu^{N,\varepsilon}\Big(\sup_{0 \leqslant t \leqslant T} \|\Phi_t^{N,\varepsilon}\varphi\|_{H^{-\delta}} > 2R\Big) \to 0 \quad \text{as } R \to \infty.$$

Proof By Lemma 21 we have for $\tau = c(1+R)^{-L}$, using invariance in the 4th line of the following computation

$$\begin{aligned}
& \nu^{N,\varepsilon}(\sup_{0\leqslant t\leqslant T} \|\Phi_t^{N,\varepsilon}\varphi\|_{H^{-\delta}} > 2R) \\
\leqslant\ & \nu^{N,\varepsilon}(\sup_{n\leqslant T/\tau} \|\Phi_{n\tau}^{N,\varepsilon}\varphi\|_{H^{-\delta}} > R) \\
\leqslant\ & \sum_{n\leqslant T/\tau} \nu^{N,\varepsilon}(\|\Phi_{n\tau}^{N,\varepsilon}\varphi\|_{H^{-\delta}} > R) \\
=\ & (T/\tau)\nu^{N,\varepsilon}(\|\varphi\|_{H^{-\delta}} > R) \\
\leqslant\ & C(T)(1+R)^L \nu^{N,\varepsilon}(\|\varphi\|_{H^{-\delta}} > R) \\
\leqslant\ & C(T)(1+R)^L R^{-p}
\end{aligned}$$

for any $p < \infty$. Choosing $p > L$ and taking the supremum over N we can conclude. □

Lemma 23 *Let $u^\varepsilon, u_N^\varepsilon$ be solutions to equations* (86) *and* (85) *respectively and suppose that*

$$\sup_{0\leqslant t\leqslant T}(\|u^\varepsilon(t)\|_{H^{-\delta}} + \|u^{N,\varepsilon}(t)\|_{H^{-\delta}}) \leqslant R.$$

Then, for $0 < \delta' < \delta$

$$\sup_{0\leqslant t\leqslant T} \|(u^{N,\varepsilon} - u^\varepsilon)(t)\|_{H^{-\delta}} \lesssim_{\varepsilon,\delta,\delta'} R^2 N^{-\delta'}.$$

Proof Denote

$$f^{\varepsilon,N}(\varphi) = P_{\leqslant N}(\rho_\varepsilon * (\rho_\varepsilon * P_{\leqslant N}\varphi)^{\circ 3}) = P_{\leqslant N}(\rho_\varepsilon * ((\rho_\varepsilon * P_{\leqslant N}\varphi)^{\circ 3} - a_N^\varepsilon(\rho_\varepsilon * P_{\leqslant N}\varphi)))$$

$$f^\varepsilon(\varphi) = (\rho_\varepsilon * (\rho_\varepsilon * \varphi)^{\circ 3}) = (\rho_\varepsilon * ((\rho_\varepsilon * \varphi)^3 - a_N^\varepsilon(\rho_\varepsilon * \varphi)))$$

where

$$a_N^\varepsilon(x) = 3\mathbb{E}[(\rho_\varepsilon * P_{\leqslant N}W)^2(x)] \qquad a^\varepsilon(x) = 3\mathbb{E}[(\rho_\varepsilon * W)^2(x)] = a^\varepsilon(0)$$

Note that a^ε is translation invariant (hence a constant function) while a_N^ε is not. Furthermore $a_N^\varepsilon \to a^\varepsilon$ in $C_b^2(\mathbb{T}^2)$ as $N \to \infty$. By Remark 2 and the smoothing properties of $\rho_\varepsilon *$, we can estimate

$$\begin{aligned}
& \|f^{\varepsilon,N}(\varphi) - f^\varepsilon(\varphi)\|_{H^{-\delta}} \\
\leqslant\ & \|P_{>N}(\rho_\varepsilon * (\rho_\varepsilon * P_{\leqslant N}\varphi)^{\circ 3})\|_{H^{-\delta}} + \|(\rho_\varepsilon * (\rho_\varepsilon * P_{\leqslant N}\varphi)^{\circ 3}) - (\rho_\varepsilon * (\rho_\varepsilon * \varphi)^{\circ 3})\|_{H^{-\delta}} \\
\leqslant\ & N^{-1}\|(\rho_\varepsilon * (\rho_\varepsilon * P_{\leqslant N}\varphi)^{\circ 3})\|_{H^{1-\delta}} + \|(\rho_\varepsilon * (\rho_\varepsilon * P_{\leqslant N}\varphi)^3) - (\rho_\varepsilon * (\rho_\varepsilon * \varphi)^3)\|_{H^{-\delta}} \\
& + a^\varepsilon\|(\rho_\varepsilon * (\rho_\varepsilon * P_{>N}\varphi))\|_{H^{-\delta}} + \|(a_N^\varepsilon - a^\varepsilon)(\rho_\varepsilon * (\rho_\varepsilon * P_{\leqslant N}\varphi))\|_{H^{-\delta}} \\
\lesssim_\varepsilon\ & N^{-1}\| : (\rho_\varepsilon * (\rho_\varepsilon * P_{\leqslant N}\varphi)^3) : \|_{H^{1-\delta}} + \|\rho_\varepsilon * P_{>N}\varphi\|_{L^\infty}^3 + a^\varepsilon\|P_{>N}\varphi\|_{H^{-2\delta}} \\
& + \|(a_N^\varepsilon - a^\varepsilon)\|_{L^\infty}\|\rho_\varepsilon * P_{\leqslant N}\varphi\|_{L^\infty} \\
\lesssim_\varepsilon\ & N^{-\delta'}(1 + \|\varphi\|_{H^{-\delta}}^3).
\end{aligned}$$

It is not hard to see that f^ε is locally Lipschitz continuous on $H^{-\delta}$.

From the mild formulation, we get

$$
\begin{aligned}
& \|(u_N^\varepsilon - u^\varepsilon)(t)\|_{H^{-\delta}} \\
= \quad & \left\| \int_0^t \frac{\sin\left((t-s)\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}} f^{\varepsilon,N}(u^{N,\varepsilon}(s)) - f^\varepsilon(u^\varepsilon(s))\mathrm{d}s \right\|_{H^{-\delta}} \\
= \quad & \left\| \int_0^t \frac{\sin\left((t-s)\sqrt{\mathbb{H}^{\omega,K}}\right)}{\sqrt{\mathbb{H}^{\omega,K}}} \left(\left(f^{\varepsilon,N}(u_N^\varepsilon(s)) - f^\varepsilon\,(u_N^\varepsilon(s))\right) + \left(f^\varepsilon(u^{N,\varepsilon}(s)) - f^\varepsilon(u^\varepsilon(s))\right) \mathrm{d}s \right\|_{H^{-\delta}} \\
\lesssim_{\varepsilon,T} \quad & N^{-\delta'} \sup_{s\leqslant T}(1 + \|u^{N,\varepsilon}(s)\|^3_{H^{-\delta}}) + \int_0^T \|f^\varepsilon(u^{N,\varepsilon}(s)) - f^\varepsilon(u^\varepsilon(s))\|_{H^{-\delta}}\mathrm{d}s \\
\lesssim_{\varepsilon,T,R} \quad & N^{-\delta'} + \int_0^T \|(u^{N,\varepsilon}(s)) - u^\varepsilon(s))\|_{H^{-\delta}}\mathrm{d}s.
\end{aligned}
$$

So from Gronwall's Lemma we can conclude. □

Note that if $\sup_{0\leqslant t\leqslant T}(\|u^{N,\varepsilon}(t)\|_{H^{-\delta}}) \leqslant R$ then from (85) we have

$$
\|u^{N,\varepsilon}\|_{W_t^{\delta,\infty}H_x^{-2\delta}} \leqslant \|f^{\varepsilon,N}(u^{N,\varepsilon})\|_{L_t^\infty H_x^{-\delta}} + \|S(t)\varphi\|_{W_t^{\delta,\infty}H_x^{-2\delta}} \lesssim_\varepsilon 1 + \|u^{N,\varepsilon}\|^3_{L_t^\infty H_x^{-\delta}} \leqslant 1 + R^3,
$$

where $S(t)\varphi$ is a short-hand notation for the linear part of the equation. In particular $u^{N,\varepsilon}$ possesses a convergent subsequence in $L_t^\infty H_x^{-\delta}$ by the compact embedding $W_t^{\delta,\infty}H_x^{-2\delta} \hookrightarrow L_t^\infty H_x^{-4\delta}$. Any subsequential limit can be seen to satisfy (86) in the same way as in the proof of Lemma 23. In particular if we denote by

$$
\Sigma_R^\varepsilon = \{\varphi : \sup_{0\leqslant t\leqslant T} \|\Phi_t^\varepsilon\varphi\|_{H^{-\delta}} \leqslant R\}
$$

and

$$
\Sigma_R^{\varepsilon,N} = \{\varphi : \sup_{0\leqslant t\leqslant T} \|\Phi_t^{\varepsilon,N}\varphi\|_{H^{-\delta}} \leqslant R\},
$$

we have

$$
\Sigma_R^\varepsilon \supset \limsup_{N\to\infty} \Sigma_R^{\varepsilon,N} = \bigcap_{N=1}^\infty \bigcup_{N_1=N}^\infty \Sigma_R^{\varepsilon,N}.
$$

This implies by Fatou's lemma that

$$
\nu^\varepsilon(\Sigma_R^\varepsilon) \geqslant \limsup_{N\to\infty} \nu^\varepsilon(\Sigma_R^{\varepsilon,N}) = \limsup_{N\to\infty} \nu^{\varepsilon,N}(\Sigma_R^{\varepsilon,N}) \geqslant 1 - C(T)(1+R)^L R^{-p} \qquad (88)
$$

where the equality holds since $\nu^{\varepsilon,N} \to \nu^\varepsilon$ in total variation. In particular, Φ_t^ε is well defined for all t, ν^ε-almost surely. Note that since $\Phi_t^\varepsilon\varphi$ is well defined ν^ε-almost surely we can define the pushforward measure $(\Phi_t^\varepsilon)^*\nu^\varepsilon$ by

$$
\int f(\varphi)\mathrm{d}(\Phi_t^\varepsilon)^*\nu^\varepsilon = \int f(\Phi_t^\varepsilon\varphi)\mathrm{d}\nu^\varepsilon
$$

for any continuous and bounded function $f : H^{-\delta} \to \mathbb{R}$.

We now establish invariance.

Proposition 10 *For any* $0 < t < \infty$ *and* $\varepsilon > 0$*one has*

$$
(\Phi_t^\varepsilon)^*\nu^\varepsilon = \nu^\varepsilon.
$$

Proof We will show that for any Lipschitz-continuous and bounded function $f : H^{-\delta} \to \mathbb{R}$ with Lipschitz constant L, we have

$$
\int f(\Phi_t^\varepsilon(\varphi))\mathrm{d}\nu^\varepsilon = \int f(\varphi)\mathrm{d}\nu^\varepsilon.
$$

We already know that

$$\int f(\Phi_t^{\varepsilon,N}(\varphi))\mathrm{d}\nu^{\varepsilon,N} = \int f(\varphi)\mathrm{d}\nu^{\varepsilon,N},$$

and since $\nu^{\varepsilon,N} \to \nu^{\varepsilon}$ in total variation

$$\lim_{N\to\infty} \int f(\varphi)\mathrm{d}\nu^{\varepsilon,N} = \int f(\varphi)\mathrm{d}\nu^{\varepsilon}.$$

So it remains to show that

$$\lim_{N\to\infty} \int f(\Phi_t^{\varepsilon,N}(\varphi))\mathrm{d}\nu^{\varepsilon,N} = \int f(\Phi_t^{\varepsilon}(\varphi))\mathrm{d}\nu^{\varepsilon}$$

Now we bound

$$\begin{aligned} & \left|\int f(\Phi_t^{\varepsilon,N}(\varphi))\mathrm{d}\nu^{\varepsilon,N} - \int f(\Phi_t^{\varepsilon}(\varphi))\mathrm{d}\nu^{\varepsilon}\right| \\ \leqslant & \left|\int f(\Phi_t^{\varepsilon,N}(\varphi))\mathrm{d}\nu^{\varepsilon,N} - \int f(\Phi_t^{\varepsilon,N}(\varphi))\mathrm{d}\nu^{\varepsilon}\right| \\ & + \left|\int f(\Phi_t^{\varepsilon,N}(\varphi))\mathrm{d}\nu^{\varepsilon} - \int f(\Phi_t^{\varepsilon}(\varphi))\mathrm{d}\nu^{\varepsilon}\right| \end{aligned}$$

The first term goes to 0 since $\nu^{\varepsilon,N}$ converges in total variation. For the second term we have

$$\begin{aligned} & \left|\int f(\Phi_t^{\varepsilon,N}(\varphi)) - f(\Phi_t^{\varepsilon}(\varphi))\mathrm{d}\nu^{\varepsilon}\right| \\ \leqslant & \|f\|_{L^\infty}(\nu^{\varepsilon}(\sup_{t\leqslant T}\|\Phi_t^{\varepsilon,N}(\varphi)\|_{H^{-\delta}} \geqslant R) + \nu^{\varepsilon}(\sup_{t\leqslant T}\|\Phi_t^{\varepsilon}(\varphi)\|_{H^{-\delta}} \geqslant R)) \\ & + \left|\int \mathbb{1}_{\{\sup_t(\|\Phi_t^{\varepsilon,N}(\varphi)\|_{H^{-\delta}} + \|\Phi_t^{\varepsilon}(\varphi)\|_{H^{-\delta}})\leqslant 2R\}}(f(\Phi_t^{\varepsilon,N}(\varphi)) - f(\Phi_t^{\varepsilon}(\varphi)))\mathrm{d}\nu^{\varepsilon}\right| \end{aligned}$$

Now applying Lemma 23 we get by Lipschitz continuity of f

$$\begin{aligned} & \left|\int \mathbb{1}_{\{\sup_t(\|\Phi_t^{\varepsilon,N}(\varphi)\|_{H^{-\delta}} + \|\Phi_t^{\varepsilon}(\varphi)\|_{H^{-\delta}})\leqslant 2R\}}(f(\Phi_t^{\varepsilon,N}(\varphi)) - f(\Phi_t^{\varepsilon}(\varphi)))\mathrm{d}\nu^{\varepsilon}\right| \\ \leqslant & LR^2N^{-\delta'} \end{aligned}$$

Finally, by Lemma 22 and (88) we get

$$\sup_{N\in\mathbb{N}} \nu^{\varepsilon}(\sup_t\|\Phi_t^{\varepsilon,N}(\varphi)\|_{H_x^{-\delta}} \geqslant R) + \nu^{\varepsilon}(\sup_t\|\Phi_t^{\varepsilon}(\varphi)\|_{H_x^{-\delta}} \geqslant R) \quad \to \quad 0$$

as $R \to \infty$. From this we can conclude by taking $N \to \infty$ and then $R \to \infty$. □

5.2 Removal of the smoothing

We now want to show that (87) has global solutions ν-almost surely and that ν is invariant under Φ_t. Firstly we define $\theta(t) = U(t)\varphi_1 + S(t)\varphi_2$ to be the "linear part" of the solution, we also set $\theta^{\varepsilon} = \rho_{\varepsilon} * \theta$. We then have

$$\begin{aligned} \mathbb{E}_{\nu^{\varepsilon}}\left[\int_0^T \|\theta^{\varepsilon}(t)^{\circ i}\|_{B_{p,p}^{-\delta}}^p \mathrm{d}t\right] &= \int_0^T \mathbb{E}_{\nu^{\varepsilon}}[\|\theta^{\varepsilon}(0)^{\circ i}\|_{B_{p,p}^{-\delta}}^p]\mathrm{d}t \\ &\lesssim T\mathbb{E}_{\mu^{\mathbb{H}^{\omega},K}}[\|\theta^{\varepsilon}(0)^{\circ i}\|_{B_{p,p}^{-\delta}}^{2p}]^{1/2} \\ &\leqslant CT \end{aligned}$$

where, in the second line we used invariance of the AGFF with respect to the free field and that ν^ε is absolutely continuous with respect to $\mu^{\mathbb{H}^{\omega,K}}$ and the density is in $L^2(\mu)$ uniformly in ε. So in particular $(\theta^\varepsilon)^{\circ i}$ are in $L^p\mathcal{C}^{-\delta}$ almost surely with respect to both ν^ε, ν uniformly in ε and the same holds for $\theta^{\circ i}$. From now on we write $v^\varepsilon = u^\varepsilon - \theta^\varepsilon$.

Lemma 24 *There exists $\delta > 0$ such that, assuming*

$$\sup_{t \leqslant T}(\|v^\varepsilon(t)\|_{H^{1-\delta}}) \leqslant R$$

and

$$\sup_{i \leqslant 3}\sup_{t \leqslant T}(\|\theta^{\circ i}\|_{H^{-\delta}} + \|(\theta^\varepsilon)^{\circ i}\|_{H^{-\delta}}) \leqslant R,$$

we have

$$\left| v^\varepsilon(t) - \int_0^t S(t-s)(u^\varepsilon(s))^{\circ 3}\mathrm{d}s \right| \lesssim R^3 \mathcal{R}^\varepsilon,$$

where $\mathcal{R}^\varepsilon$ is a random variable such that $\|\mathcal{R}^\varepsilon\|_{L^p(\mu)} \to 0$ as $\varepsilon \to 0$.

Proof Recall that

$$0 \;=\; v^\varepsilon(t) - \int_0^t S(t-s)\rho_\varepsilon * (\rho_\varepsilon * u^\varepsilon(s))^3 \mathrm{d}s$$

and

$$\begin{aligned} & v^\varepsilon(t) - \int_0^t S(t-s)(u^\varepsilon(s))^{\circ 3}\mathrm{d}s \\ = \;& v^\varepsilon(t) - \int_0^t S(t-s)\rho_\varepsilon * (\rho_\varepsilon * u^\varepsilon(s))^{\circ 3}\mathrm{d}s \\ & + \int_0^t S(t-s)\rho_\varepsilon * (\rho_\varepsilon * u^\varepsilon(s))^{\circ 3}\mathrm{d}s - \int_0^t S(t-s)(u^\varepsilon(s))^{\circ 3}\mathrm{d}s \end{aligned}$$

so we have to estimate the last line. Recalling that $u^\varepsilon = \theta + v^\varepsilon$ that is equal to

$$\begin{aligned} & \int_0^t S(t-s)\rho_\varepsilon * (\rho_\varepsilon * u^\varepsilon(s))^{\circ 3}\mathrm{d}s - \int_0^t S(t-s)(u^\varepsilon(s))^{\circ 3}\mathrm{d}s \\ = \;& \sum_{i=0}^3 \int_0^t S(t-s)(((\rho_\varepsilon * \theta(s))^{\circ i}(\rho_\varepsilon * u^\varepsilon(s))^{i-j}) - (\theta(s))^{\circ i}(u^\varepsilon)^{i-j})\mathrm{d}s \\ & + \int_0^t S(t-s)((\rho_\varepsilon * u^\varepsilon(s))^{\circ 3} - \rho_\varepsilon * (\rho_\varepsilon * u^\varepsilon(s))^3)\mathrm{d}s \\ \leqslant \;& \sum_{i=0}^3 \int_0^t \|(((\rho_\varepsilon * \theta(s))^{\circ i}(\rho_\varepsilon * u^\varepsilon(s))^{i-j}) - (\theta(s))^{\circ i}(u^\varepsilon)^{i-j})\|_{H^{-\delta}}\mathrm{d}s \\ & + \int_0^t \|(\rho_\varepsilon * u^\varepsilon(s))^3 - \rho_\varepsilon * : (\rho_\varepsilon * u^\varepsilon(s))^3 : \|_{H^{-\delta}}\mathrm{d}s \\ =: \;& \mathrm{I} + \mathrm{II} \end{aligned}$$

Now to estimate I we have for p sufficiently large and $1/p + 1/p' = 1$:

$$\begin{aligned}
&|((\theta^\varepsilon(s))^{\circ i}(\rho_\varepsilon * u^\varepsilon(s))^{i-j}) - (\theta(s))^{\circ i}(u^\varepsilon)^{i-j}| \\
\leqslant & \|(\theta^\varepsilon)^{\circ i} - (\theta)^{\circ i}\|_{B^{-2\delta}_{p,p}}(\|u^\varepsilon(s)^{i-j}\|_{B^{2\delta}_{p',p'}} + \|(\rho_\varepsilon * u^\varepsilon(s))^{i-j}\|_{B^{2\delta}_{p',p'}}) \\
& + (\|(\theta^\varepsilon)^{\circ i}\|_{B^{-2\delta}_{p,p}} + \|(\theta)^{\circ i}\|_{B^{-2\delta}_{p,p}})\|u^\varepsilon(s)^{i-j} - (\rho_\varepsilon * u^\varepsilon(s))^{i-j}\|_{B^{2\delta}_{p',p'}} \\
\leqslant & \|(\theta^\varepsilon)^{\circ i} - (\theta)^{\circ i}\|_{B^{-2\delta}_{p,p}}(\|u^\varepsilon(s)\|^{i-j}_{H^{1-\delta}} + \|(\rho_\varepsilon * u^\varepsilon(s))\|^{i-j}_{H^{1-\delta}}) \\
& + (\|(\theta^\varepsilon)^{\circ i}\|_{B^{-2\delta}_{p,p}} + \|(\theta)^{\circ i}\|_{B^{-2\delta}_{p,p}}) \\
& \times \|u^\varepsilon(s) - (\rho_\varepsilon * u^\varepsilon(s))\|_{H^{1-\delta}}(\|u^\varepsilon(s)\|_{H^{1-\delta}} + \|(\rho_\varepsilon * u^\varepsilon(s))\|_{H^{1-\delta}})^{i-j-1} \\
= & \|(\theta^\varepsilon)^{\circ i} - (\theta)^{\circ i}\|_{B^{-2\delta}_{p,p}} R^3 + R^3(\|(\theta^\varepsilon)^{\circ i}\|_{B^{-2\delta}_{p,p}} + \|(\theta)^{\circ i}\|_{B^{-2\delta}_{p,p}})\|u^\varepsilon(s) - (\rho_\varepsilon * u^\varepsilon(s))\|_{H^{1-\delta}}
\end{aligned}$$

Now

$$\|u^\varepsilon(s) - (\rho_\varepsilon * u^\varepsilon(s))\|_{H^{1-\delta}} \lesssim \varepsilon^{\delta/2}\|u^\varepsilon(s)\|_{H^{1-\delta/2}} \lesssim \varepsilon^{\delta/2} R$$

so

$$\begin{aligned}
& \|(\theta^\varepsilon)^{\circ i}(s) - (\theta)^{\circ i}(s)\|_{B^{-2\delta}_{p,p}} R^3 \\
& + R^3(\|(\theta^\varepsilon)^{\circ i}(s)\|_{B^{-2\delta}_{p,p}} + \|\theta^{\circ i}(s)\|_{B^{-2\delta}_{p,p}})\|u^\varepsilon(s) - (\rho_\varepsilon * u^\varepsilon(s))\|_{H^{1-\delta}} \\
\leqslant \quad & \|(\theta^\varepsilon)^{\circ i}(s) - \theta^{\circ i}(s)\|_{B^{-2\delta}_{p,p}} R^3 + \varepsilon^{\delta/2} R^4(\|(\theta^\varepsilon)^{\circ i}(s)\|_{B^{-2\delta}_{p,p}} + \|(\theta)^{\circ i}(s)\|_{B^{-2\delta}_{p,p}})
\end{aligned}$$

so integrating in time, we have that

$$\mathrm{I} \lesssim R^3 \int_0^t \|(\theta^\varepsilon)^{\circ i}(s) - (\theta)^{\circ i}(s)\|_{B^{-2\delta}_{p,p}} \mathrm{d}s + R^4 \varepsilon^{\delta/2} \int_0^t (\|(\theta^\varepsilon)^{\circ i}(s)\|_{B^{-2\delta}_{p,p}} + \|\theta^{\circ i}(s)\|_{B^{-2\delta}_{p,p}}) \mathrm{d}s$$

and we recall that

$$\mathbb{E}_\mu \int_0^T \|(\theta^\varepsilon)^{\circ i}(s) - (\theta)^{\circ i}(s)\|^p_{B^{-2\delta}_{p,p}} \mathrm{d}s \to 0.$$

Now finally to estimate II we have

$$\int_0^t \|(\rho_\varepsilon * u^\varepsilon(s))^{\circ 3} - \rho_\varepsilon * (\rho_\varepsilon * u^\varepsilon(s))^{\circ 3}\|_{H^{-\delta}} \mathrm{d}s \lesssim \varepsilon^{\delta/2} \int_0^t \|(\rho_\varepsilon * u^\varepsilon(s))^{\circ 3}\|_{H^{-\delta/2}} \mathrm{d}s$$

and

$$\begin{aligned}
\|(\rho_\varepsilon * u^\varepsilon(s))^{\circ 3}\|_{H^{-\delta/2}} \quad & \leqslant \quad \sum_{i=0}^3 \|(\theta^\varepsilon)^{\circ i}(u^\varepsilon)^{3-i}\|_{H^{-\delta/2}} \\
& \leqslant \quad \sum_{i=0}^3 \|(\theta^\varepsilon)^{\circ i}\|_{\mathcal{C}^{-\delta/2}} \|(u^\varepsilon)^{3-i}\|_{H^\delta} \\
& \leqslant \quad \sum_{i=0}^3 \|(\theta^\varepsilon)^{\circ i}\|_{\mathcal{C}^{-\delta/2}} \|u^\varepsilon\|^{3-i}_{W^{\delta,6}} \\
& \leqslant \quad \sum_{i=0}^3 \|(\theta^\varepsilon)^{\circ i}\|_{\mathcal{C}^{-\delta/2}} \|u^\varepsilon\|^{3-i}_{H^{1-\delta}} \\
& \leqslant \quad (1+R)^3 \sum_{i=0}^3 \|(\theta^\varepsilon)^{\circ i}\|_{\mathcal{C}^{-\delta/2}}
\end{aligned}$$

and again integrating in time we can conclude. □

Let v be the solution (72). Similarly to the above we have the following statement:

Lemma 25 *There exists a $\delta > 0$ such that, assuming that*

$$\sup_{t\leqslant T}(\|v^{\varepsilon}(t)\|_{H^{1-\delta}}) \leqslant R,$$

and

$$\sup_{i\leqslant 3}\int_0^T (\|\theta^{\circ i}\|^p_{H^{-\delta}} + \|(\theta^{\varepsilon})^{\circ i}\|^p_{H^{-\delta}})\mathrm{d}t \leqslant R,$$

we get

$$\sup_{t\leqslant T}(\|v^{\varepsilon}(t) - v(t)\|_{H^{1-\delta}}) \lesssim R\bar{\mathcal{R}}^{\varepsilon},$$

where $\bar{\mathcal{R}}^{\varepsilon}$ is given by

$$\bar{\mathcal{R}}^{\varepsilon} := \sum_{0\leqslant i\leqslant 3}\int_0^T \|(\theta^{\varepsilon})^{\circ i}(s) - \theta^{\circ i}(s)\|^p_{B^{-2\delta}_{p,p}}\mathrm{d}s.$$

In particular$\|\bar{\mathcal{R}}^{\varepsilon}\|_{L^p(\mu)} \to 0$ as $\varepsilon \to 0$.

Proof Using the definition of v^{ε}, v we have

$$\begin{aligned}
& v(t) - v^{\varepsilon}(t) \\
= & \int_0^t S(t-s)\rho_{\varepsilon} * \left(\theta^{\circ 3}(s) - (\theta^{\varepsilon})^{\circ 3}(s) + 3\left(\theta^{\circ 2}(s) - (\theta^{\varepsilon})^{\circ 2}(s)\right)v(s) + 3(\theta^{\varepsilon})^{\circ 2}(v(s) - v^{\varepsilon}(s))\right)\mathrm{d}s \\
& +3\int_0^t S(t-s)\rho_{\varepsilon} * ((\theta(s) - \theta^{\varepsilon}(s))v^2(s) + (\theta^{\varepsilon}(s))(v^2(s) - (v^{\varepsilon})^2(s)))\mathrm{d}s \\
& +\int_0^t S(t-s)\rho_{\varepsilon} * (v^3(s) - (v^{\varepsilon})^3(s))\mathrm{d}s
\end{aligned}$$

Now with the same estimates as in the proof of Lemma 24 we obtain

$$\begin{aligned}
\|v(t) - v^{\varepsilon}(t)\|_{H^{1-2\delta}} \lesssim & R^3\int_0^t \|(\theta^{\varepsilon})^{\circ 2}(s)\|_{\mathcal{C}^{-\delta}}\|v(s) - v^{\varepsilon}(s)\|_{H^{1-2\delta}}\mathrm{d}s + CR\varepsilon^{\delta} \\
& + \sum_{0\leqslant i\leqslant 3}\int_0^T \|(\theta^{\varepsilon})^{\circ 2}(s) - \theta^{\circ 2}(s)\|^p_{B^{-2\delta}_{p,p}}\mathrm{d}s
\end{aligned}$$

and Gronwall's lemma gives

$$\begin{aligned}
& \|v(t) - v^{\varepsilon}(t)\|_{H^{1-2\delta}} \\
\lesssim & CR(\varepsilon^{\delta} + \bar{\mathcal{R}}^{\varepsilon})\exp\left(R^3\int_0^T \|\theta^{\circ 2}_{\varepsilon}(s)\|_{\mathcal{C}^{-\delta}}\mathrm{d}s\right) \\
\lesssim & \varepsilon^{\delta}(\varepsilon^{\delta} + \bar{\mathcal{R}}^{\varepsilon})\exp(CR^4)
\end{aligned}$$

from which we can conclude. □

Lemma 26 *Assume that*

$$\sup_{t \leqslant T} \|(\rho_\varepsilon * u^\varepsilon)^{\circ 3}\|_{H^{-\delta}} \leqslant R$$

Then for $\alpha \in (0,1)$ *and* $\delta > 0$

$$\|v^\varepsilon\|_{\mathcal{C}_t^\alpha H^{1-\delta-\alpha}} \lesssim R.$$

The analogous statement also holds for u, v.

Proof By definition of v^ε we have

$$v^\varepsilon(t) = \int_0^t S(t-s)\rho_\varepsilon * ((\rho_\varepsilon * u^\varepsilon)^{\circ 3})\mathrm{d}s$$

so by the properties of the Wave propagator

$$\|v^\varepsilon\|_{\mathcal{C}_t^\alpha([0,T],H^{1-\alpha-\delta})} \lesssim_T \|(u^\varepsilon)^{\circ 3}\|_{L^2([0,T],H^{-\delta})}.$$

which gives the statement. The proof for v is analogous. □

Proposition 11 *For* $K > 0$ *we have the bound*

$$\nu^\varepsilon(\|(\rho_\varepsilon * u^\varepsilon)^{\circ 3}\|_{L^p([0,T],H^{-\delta})} \leqslant R) \geqslant 1 - \frac{CT}{R^p}.$$

Proof By Markov's inequality

$$\begin{aligned} & \nu^\varepsilon(\|(\rho_\varepsilon * u^\varepsilon)^{\circ 3}\|_{L^p([0,T],H^{-\delta})} \geqslant R) \\ \leqslant & \frac{1}{R^p}\mathbb{E}_{\nu^\varepsilon}[\|(\rho_\varepsilon * u^\varepsilon)^{\circ 3}\|^p_{L^p([0,T],H^{-\delta})}] \\ \leqslant & \frac{1}{R^p}\int_0^T \mathbb{E}_{\nu^\varepsilon}[\|(\rho_\varepsilon * u^\varepsilon(t))^{\circ 3}\|^p_{H^{-\delta}}] \\ = & \frac{1}{R^p}\int_0^T \mathbb{E}_{\nu^\varepsilon}[\|(\rho_\varepsilon * u^\varepsilon(0))^{\circ 3}\|^p_{H^{-\delta}}] \\ = & \frac{1}{R^p}CT \end{aligned}$$

where in the last line we used invariance of ν^ε under the flow of u^ε . □

Corollary 1 *Recall that* $v = u - \theta$. *If* $\|v^\varepsilon(t)\|_{H^{1-\delta}} \leqslant R$, *and for* $i \leqslant 3$ $\|(\theta^\varepsilon)^{\circ i}\|_{H^{-\delta}} \leqslant R$, *then* $u^\varepsilon \to u$ *in* $\mathcal{C}_t^\alpha([0,T],H^{1-\delta-\alpha})$.

Corollary 2 *With the same notation as above*

$$\nu^\varepsilon(\|v^\varepsilon\|_{\mathcal{C}_t^\alpha H^{1-\delta-\alpha}} \leqslant R) \geqslant 1 - \frac{CT}{R^p}.$$

Proof This follows immediately from Lemmas 26 and 11. □

Next we show that having a sequence of uniformly bounded approximate solutions for the cut-off flow is sufficient to construct the solution for the limiting equation.

Lemma 27 *Assume that*

$$\sup_{\varepsilon>0}\sup_{0\leqslant i\leqslant 3}\|(\theta^{\varepsilon})^{\circ i}\|_{L^p([0,T],H^{-\delta})}\leqslant R$$

and that v^{ε} *solves the equation*

$$(\partial_t^2+\mathbb{H}^{\omega,K})v^{\varepsilon}-\sum_{i=0}^{3}(\theta^{\varepsilon})^{\circ i}(v^{\varepsilon})^{3-i}=0. \qquad v^{\varepsilon}(0)=0$$

and satisfies

$$\|v^{\varepsilon}\|_{L^{\infty}H^{1-\delta}}\leqslant L$$

Then v^{ε} *has a subsequence converging in* $L^{\infty}H^{1-2\delta}$ *and the limit* v *solves*

$$(\partial_t^2+\mathbb{H}^{\omega,K})v-\sum_{i=0}^{3}\theta^{\circ i}v^{3-i}=0 \quad (v,\partial_t v)|_{t=0}=0$$

and satisfies $\|v\|_{L^{\infty}H^{1-\delta}}\lesssim RL^3$.

Proof Note that to obtain a converging subsequence we only need to bound v^{ε} in $\mathcal{C}_t^{\alpha}H^{1-\alpha-\delta'}$ for small $\alpha,\delta'>0$. To this end we will apply Lemma 26. Recall that $u(s)=\theta(s)+v(s)$ so

$$\begin{aligned}\|v^{\varepsilon}\|_{H^{1-\delta}} &\leqslant \sum_{i=0}^{3}\|(\theta^{\varepsilon})^{\circ i}(\rho_{\varepsilon}*v^{\varepsilon})^{3-i}\|_{H^{-\delta}}\\ &\leqslant \sum_{i=0}^{3}\|\theta^{\circ i}\|_{H^{-\delta}}\|v\|_{H^{1-\delta}}^{3-i}\\ &\leqslant 3R(1+L)^3\end{aligned}$$

so applying Lemma 26 we get that

$$\|v^{\varepsilon}\|_{\mathcal{C}_t^{\alpha}H^{1-\alpha-\delta'}}\lesssim 3R(1+L)^3$$

and the compactness claim follows. Lastly, that v solves the limiting equation follows from Lemma 24. □

Proposition 12 *We have that*

$$\lim_{R\to\infty}\nu(\|v\|_{L^{\infty}H^{1-\delta}}\leqslant R)=1.$$

Proof Denote by $\Sigma_R^{\varepsilon}=\{\varphi:\|v^{\varepsilon}\|_{L^{\infty}H^{1-\delta}}\leqslant R\}$ and $\Sigma_R=\{\varphi:\|v\|_{L^{\infty}H^{1-\delta}}\leqslant R\}$. From Lemma 27 we have that

$$\Sigma_R\supset\limsup_{\varepsilon\to 0}\Sigma_R^{\varepsilon}=\bigcap_{\varepsilon>0}\bigcup_{\varepsilon'<\varepsilon}\Sigma_R^{\varepsilon}.$$

Note that $\sup_{\varepsilon>0}\sup_{0\leqslant i\leqslant 3}\|:(\rho_{\varepsilon}*w)^i:\|_{L^p([0,T],H^{-\delta})}<\infty$ ν,ν^{ε}-almost surely. Then by Fatou's Lemma

$$\nu(\Sigma_R)\geqslant\nu(\limsup_{\varepsilon\to 0}\Sigma_R^{\varepsilon})\geqslant\limsup_{\varepsilon\to 0}\nu(\Sigma_R^{\varepsilon})=\limsup_{\varepsilon\to 0}\nu^{\varepsilon}(\Sigma_R^{\varepsilon})$$

where the last equality is true since $\nu^{\varepsilon}\to\nu$ in total variation. Now an application of Proposition 11 yields the claim. □

Proposition 13 *The measure* ν *is invariant under the flow* Φ_t, *that is*

$$\Phi_t^*\nu=\nu.$$

Proof This follows in exactly the same way as Proposition 10. □

A Besov spaces and related concepts

We collect some basic definitions and elementary results about Besov spaces, paraproducts etc., see, e.g., [2, 3, 36] for more details. We work here in the case of Besov spaces defined on the d-dimensional torus

$$\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d.$$

First we define the Sobolev space $H^\alpha(\mathbb{T}^d)$ with index $\alpha \in \mathbb{R}$ which is the Banach space of distribution u such that $(1-\Delta)^{-\frac{\alpha}{2}}(u)$ is a function and

$$H^\alpha(\mathbb{T}^d) := \left\{ u \in \mathcal{S}'(\mathbb{T}^d) : \left\| (1-\Delta)^{\frac{\alpha}{2}} u \right\|_{L^2} < \infty \right\}. \tag{89}$$

Next, we recall the definition of Littlewood-Paley blocks. We denote by χ and ρ two non-negative smooth and compactly supported radial functions $\mathbb{R}^d \to \mathbb{C}$ such that

i. The support of χ is contained in a ball and the support of ρ is contained in an annulus $\{x \in \mathbb{R}^d : a \leqslant |x| \leqslant b\}$

ii. For all $\xi \in \mathbb{R}^d$, $\chi(\xi) + \sum_{j \geqslant 0} \rho(2^{-j}\xi) = 1$;

iii. For $j \geqslant 1$, $\chi(\cdot)\rho(2^{-j}\cdot) = 0$ and $\rho(2^{-j}\cdot)\rho(2^{-i}\cdot) = 0$ for $|i-j| > 1$.

The Littlewood-Paley blocks $(\Delta_j)_{j \geqslant -1}$ associated to $f \in \mathcal{S}'(\mathbb{T}^d)$ are defined by

$$\Delta_{-1} f := \mathcal{F}^{-1} \chi \mathcal{F} f \text{and } \Delta_j f := \mathcal{F}^{-1} \rho(2^{-j}\cdot) \mathcal{F} f \text{for} j \geqslant 0,$$

and we define the Littlewood-Paley function $K_j = \mathcal{F}^{-1}(\Delta_j)$, i.e. the function for which $K_j * f = \Delta_j f$. We also set, for $f \in \mathcal{S}'(\mathbb{T}^d)$ and $j \geqslant -1$

$$S_j f := \sum_{i=-1}^{j-1} \Delta_i f.$$

Then the Besov space with parameters $p \in [1, \infty], q \in [1, \infty), \alpha \in \mathbb{R}$ can now be defined as

$$B^\alpha_{p,q}(\mathbb{T}^d) := \{ u \in \mathcal{S}'(\mathbb{T}^d) : \|u\|_{B^\alpha_{p,q}} < \infty \},$$

where the norm is defined as

$$\|u\|_{B^\alpha_{p,q}} := \left(\sum_{k \geqslant -1} ((2^{\alpha k} \|\Delta_j u\|_{L^p})^q) \right)^{\frac{1}{q}}, \tag{90}$$

with the obvious modification for $q = \infty$. In the paper we often omit the dependence of $B^\alpha_{p,q}$ from the torus $\mathbb{T}^d$. There are two special cases of Besov spaces: the *Besov-Hölder* spaces for $p = q = \infty$, i.e.

$$\mathcal{C}^\alpha := B^\alpha_{\infty\infty}(\mathbb{T}^d) \tag{91}$$

and the Sobolev spaces $H^\alpha = B^\alpha_{2,2}(\mathbb{T}^d)$ (defined above) for $p = q = 2$.

Using this notation, we can formally decompose the product $f \cdot g$ of two distributions f and g as

$$f \cdot g = f \prec g + f \circ g + f \succ g,$$

where

$$f \prec g := \sum_{j \geqslant -1} S_{j-1} f \Delta_j g \quad \text{and} \quad f \succ g := \sum_{j \geqslant -1} \Delta_j f S_{j-1} g$$

are referred to as the *paraproducts*, whereas

$$f \circ g := \sum_{j \geqslant -1} \sum_{|i-j| \leqslant 1} \Delta_i f \Delta_j g \tag{92}$$

is called the *resonant product*. An important point is that the paraproduct terms are always well defined whatever the regularity of f and g. The resonant product, on the other hand, is a priori only well defined if the sum of their regularities is positive. We collect some results.

Lemma 28 *Let $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ and $p, p_1, p_2, q \in \{2, \infty\}$ be such that*

$$\alpha_1 \neq 0 \quad \alpha = (\alpha_1 \wedge 0) + \alpha_2, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \, and \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then we have the bound

$$\|f \prec g\|_{B^{\alpha}_{p,q}} \lesssim \|f\|_{B^{\alpha_1}_{p_1,q_1}} \|g\|_{B^{\alpha_2}_{p_2,q_2}}$$

and in the case where $\alpha_1 + \alpha_2 > 0$ we have the bound

$$\|f \circ g\|_{B^{\alpha_1+\alpha_2}_{p,q}} \lesssim \|f\|_{B^{\alpha_1}_{p_1,q_1}} \|g\|_{B^{\alpha_2}_{p_2,q_2}}.$$

Proof The proof can be found in [3] Theorem 2.47 and Theorem 2.52 for Besov spaces on $\mathbb{R}^d$. The proof for Besov spaces on $\mathbb{T}^d$ is similar. □

Lemma 29 (Bernstein's inequalities) *Let $\mathcal{A}$ be an annulus and $\mathcal{B}$ be a ball. For any $k \in \mathbb{N}, \lambda > 0,$and $1 \leqslant p \leqslant q \leqslant \infty$ we have*

1. *if $u \in L^p(\mathbb{R}^d)$is such that* $\operatorname{supp}(\mathcal{F}u) \subset \lambda\mathcal{B}$ *then*

$$\max_{\mu \in \mathbb{N}^d : |\mu| = k} \|\partial^\mu u\|_{L^q} \lesssim_k \lambda^{k + d\left(\frac{1}{p} - \frac{1}{q}\right)} \|u\|_{L^p}$$

2. *if $u \in L^p(\mathbb{R}^d)$is such that* $\operatorname{supp}(\mathcal{F}u) \subset \lambda\mathcal{A}$ *then*

$$\lambda^k \|u\|_{L^p} \lesssim_k \max_{\mu \in \mathbb{N}^d : |\mu| = k} \|\partial^\mu u\|_{L^p}.$$

Proof The proof can be found in [3] Lemma 2.1 □

Lemma 30 *Let $\sigma : \mathbb{Z}^d \to \mathbb{R}_+$ such that*

$$|\sigma(k)|^{\pm 1} \lesssim (|k| + 1)^{\pm\alpha}$$

for some $\alpha \in \mathbb{R}$ then, for every $p, q \in [1, \infty]$ and $s \in \mathbb{R}$, operator $\sigma(\nabla)$ with symbol σ is a linear homeomorphism from $B^s_{p,q}$ into $B^{s-\alpha}_{p,q}$.

Proof The proof is an easy application of Lemma 29 (see, e.g. [3] Chapter 2). □

Remark 16 The hypotheses of Lemma 30 are satisfied when $\sigma(k) = (|k|^2 + m^2)^\alpha$, for any $\alpha \in \mathbb{R}$ and $m > 0$, and thus $\sigma(\nabla) = (-\Delta + m^2)^\alpha$.

Lemma 31 (Besov embedding) *Let $\alpha < \beta \in \mathbb{R}$ and $p > r \in [1, \infty]$ be such that*

$$\beta \leqslant \alpha + d\left(\frac{1}{r} - \frac{1}{p}\right), \tag{93}$$

then we have the following bound

$$\|f\|_{B^\alpha_{p,q}(\mathbb{T}^d)} \lesssim \|f\|_{B^\beta_{r,q}(\mathbb{T}^d)}.$$

If the inequality (93) *is strict, the embedding is compact.*

Proof The proof can be found in Proposition 2.71 of [3]. □

Proposition 14 (Commutator Lemma) *Given $\alpha \in (0, 1)$, $\beta, \gamma \in \mathbb{R}$ such that $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$, the following trilinear operator C defined for any smooth functions f, g, h by*

$$C(f, g, h) := (f \prec g) \circ h - f(g \circ h)$$

can be extended continuously to the product space $H^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma$. Moreover, we have the following bound

$$||C(f, g, h)||_{H^{\alpha+\beta+\gamma-\delta}} \lesssim ||f||_{H^\alpha} ||g||_{\mathcal{C}^\beta} ||h||_{\mathcal{C}^\gamma}$$

for all $f \in H^\alpha$, $g \in \mathcal{C}^\beta$ and $h \in \mathcal{C}^\gamma$, and every $\delta > 0$.

The analogous bound is true if we replace the Sobolev space by a Hölder-Besov space, i.e.

$$||C(f, g, h)||_{\mathcal{C}^{\alpha+\beta+\gamma}} \lesssim ||f||_{\mathcal{C}^\alpha} ||g||_{\mathcal{C}^\beta} ||h||_{\mathcal{C}^\gamma},$$

as was shown in [36].

Proof The proof can be found in Proposition 4.3 of [2]. □

Lemma 32 (Fractional Leibniz rule) *Let $1 < p < \infty$ and p_1, p_2, p'_1, p'_2 such that*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p'_1} + \frac{1}{p'_2} = \frac{1}{p}.$$

Then for any $s, \alpha \geqslant 0$ there exists a constant s.t.

$$\|\langle\nabla\rangle^s (fg)\|_{L^p} \leqslant C \|\langle\nabla\rangle^{s+\alpha} f\|_{L^{p_2}} \|\nabla^{-\alpha} g\|_{L^{p_1}} + C \|\langle\nabla\rangle^{-\alpha} f\|_{L^{p'_2}} \|\nabla^{s+\alpha} g\|_{L^{p'_1}}.$$

Proof The proof can be found in Theorem 1.4 of [38]. □

In the rest of the appendix, we discuss some properties of the operator J_s introduced in Section 3, see Remark 1. Hereafter, we require a more explicit form of the operator J_s. Let $\rho : \mathbb{R}^2 \to \mathbb{R}_+$ be a bump function which is identically 1 in the ball of center 0 and radius 1/2 and it has compact support in the ball of center 0 and radius 1. We suppose also that ρ is radially symmetric of the form $\rho(|x|^2)$. We write, for $t \geqslant 0$, $\rho_t(x) = \rho\left(\frac{|x|^2}{t^2}\right)$. It is clear that the operator

$$J_t := \sigma_t(-\Delta)(-\Delta + m^2)^{-1/2} := \left(\frac{\Delta}{t^3}\rho'\left(\frac{-\Delta}{t^2}\right)\right)^{1/2} (-\Delta + m^2)^{-1/2} \tag{94}$$

satisfies the condition of Remark 1.

Proposition 15 *Let J_s be defined by equation* (94) *then for every $s \in \mathbb{R}_+$, $p, q \in [1, +\infty]$ and $0 < m \leqslant 1$*

$$\|J_s\|_{\mathcal{L}(B^s_{p,q}, B^{s+m}_{p,q})} \lesssim (1+s)^{-\frac{1}{2}-1+m}.$$

where the constant in the symbol $\lesssim$ do not depend on $s \in \mathbb{R}_+$.

Proof This is a standard application of the regularization properties of the Laplacian (see Lemma 29, Lemma 30 and Remark 16). □

A consequence of Proposition 15 is the following:

Proposition 16 *Suppose that $f_1 \in H^1$, $f_2, f_3 \in \mathcal{C}^{-1-\delta}$ (where $\delta < \frac{1}{6}$) then we have there is $0 < \eta(\delta) < 1$ such that*

$$\begin{aligned} & \left| \int_{\mathbb{T}^2} (J_s(f_1 \prec f_2) J_s(f_3 \prec f_4) - (J_s(f_2) \circ J_s(f_4)) f_1 f_3) \mathrm{d}x \right| \\ \lesssim & \;(1+s)^{-1-\eta(\delta)} \|f_1\|_{H^{1/2-\delta}} \|f_3\|_{H^{1/2-\delta}} \|f_2\|_{\mathcal{C}^{-1-\delta}} \|f_3\|_{\mathcal{C}^{-1-\delta}}; \end{aligned} \tag{95}$$

$$\begin{aligned} & \left| \int_{\mathbb{T}^2} (J_s(f_1 \prec f_2) J_s(f_3) - (J_s(f_2) \circ J_s(f_3)) f_1) \mathrm{d}x \right| \\ \lesssim & \;(1+s)^{-2+\eta(\delta)} \|f_1\|_{H^1} \|f_2\|_{\mathcal{C}^{-1-\delta}} \|f_3\|_{\mathcal{C}^{-1-\delta}} \end{aligned} \tag{96}$$

Proof Inequality (95) is proved in Proposition 11 of [6], meanwhile (96) can be proved in a similar way with slight modifications. □

B An alternative proof of the coupling existence

We give a shorter, alternative (but less general) proof of the existence of a coupling between the Gaussian measures $\mu^{\mathbb{H}^{\omega,K}}$ (i.e. the *Anderson free field*) and the Gaussian free field $\mu^{-\Delta+K}$ from Theorem 8, see Section 2 for the relevant definitions.

Proposition 17 *Consider the Gaussian measures $\mu^{\mathbb{H}^{\omega,K}}$ and $\mu^{-\Delta+K}$ from Section 2. Then we may write elements in the support of these measures as*

$$\begin{aligned} (\mathbb{H}^{\omega,K})^{-\frac{1}{2}}\psi &\in \mathrm{supp}(\mu^{\mathbb{H}^{\omega,K}}) \\ (-\Delta+K)^{-\frac{1}{2}}\psi &\in \mathrm{supp}(\mu^{-\Delta+K}) \\ \textit{where} & \\ \psi &\in \mathrm{supp}\left(\mu^{\mathbb{I}_{L^2}}\right), \end{aligned}$$

where $\mu^{\mathbb{I}_{L^2}}$ is the white noise measure defined in Section 2.2.

Then we have that for any $\delta > 0$

$$\int \left\| \left((\mathbb{H}^{\omega,K})^{-\frac{1}{2}} - (-\Delta+K)^{-\frac{1}{2}} \right) \psi \right\|^2_{H^{1-\delta}} d\mu^{\mathbb{I}_{L^2}}(\psi) < \infty.$$

In other words, we can see the Anderson free field $\mu^{\mathbb{H}^{\omega,K}}$ as a random shift (of regularity $H^{1-\delta}$) of the GFF.

Proof This is somewhat similar to what was done in Sections 4 and 5 of [5] but we adapt it to our notation and setting.

We collect a few properties from previous sections. Firstly, from (30), we have an orthonormal eigenbasis of $\mathbb{H}^{\omega,K}$ (suppressing the ω for readability)

$$\mathbb{H}^{\omega,K} f_n = \lambda_n f_n,\ f_n \in \mathcal{D}(\mathbb{H}^{\omega.K}), \lambda_n \sim n \text{ as } n \to \infty.$$

Thus we may write $\psi \in \operatorname{supp}\left(\mu^{\mathbb{I}_{L^2}}\right) \subset \mathcal{C}^{-1-\varepsilon}\ \forall \varepsilon > 0$ as a random series

$$\psi = \sum_{n\in\mathbb{N}} g_n f_n \text{where the } g_n \sim \mathcal{N}(0,1) \text{ are i.i.d Gaussians} \tag{97}$$

then $(\mathbb{H}^{\omega,K})^{-\frac{1}{2}}\psi \in \operatorname{supp}(\mu^{\mathbb{H}^{\omega,K}})$ and $(-\Delta + K)^{-\frac{1}{2}}\psi \in \operatorname{supp}(\mu^{-\Delta+K})$. We further recall the following formula from functional calculus

$$(\mathbb{H}^{\omega,K})^{-\frac{1}{2}} = c\int_0^\infty t^{-\frac{1}{2}} e^{-t\mathbb{H}^{\omega,K}} dt, \tag{98}$$

see Theorem 35 in [5] and the discussion thereafter. Moreover, from the definition of Γ in (25) and its inverse in (24) together with the paraproduct bounds from Lemma 28, one can readily check that the following regularizing property holds

$$(\Gamma^{\pm 1} - 1) : H^{-\sigma} \to H^{1-\sigma-\varepsilon} \,\text{bounded}\, \forall \sigma, \varepsilon > 0. \tag{99}$$

Moreover, we have using (19) and the paraproduct estimates

$$\|(\mathbb{H}^{\omega,K}\Gamma - (K-\Delta))v\|_{H^\kappa} \lesssim \|v\|_{H^{1+\kappa+\varepsilon}}\ \forall \kappa, \varepsilon > 0. \tag{100}$$

Lastly, we use that Theorem 4 implies that

$$(\mathbb{H}^{\omega,K})^{-\frac{1}{2}} : H^{-\sigma} \to H^{1-\sigma} \text{ for } \sigma \in (0,1). \tag{101}$$

We make a computation where we start with (98) and artificially insert the Γ in order to use the fact that $\mathbb{H}^{\omega,K}\Gamma$ is a lower order perturbation of the Laplacian. In order to isolate the problematic term, we adopt the brief notation $\mathcal{O}(1-)$ to mean a term which is bounded from $\mathcal{C}^{-1-\varepsilon} \to H^{1-\varepsilon-\kappa}$ for $\varepsilon, \kappa > 0$ i.e. for which one does not use the Gaussian property.

We compute

$$\begin{aligned}
(\mathbb{H}^{\omega,K})^{-\frac{1}{2}} &\overset{(99),(101)}{=} (\mathbb{H}^{\omega,K})^{-\frac{1}{2}}\Gamma + \underbrace{(\mathbb{H}^{\omega,K})^{-\frac{1}{2}}(1-\Gamma)}_{\mathcal{O}(1-)} \\
&\overset{(98)}{=} c\int_0^\infty t^{-\frac{1}{2}} e^{-t\mathbb{H}^{\omega,K}}\Gamma dt + \mathcal{O}(1-) \\
&= c\int_0^\infty t^{-\frac{1}{2}} e^{-t(K-\Delta)} dt\Gamma + c\int_0^\infty t^{-\frac{1}{2}}(e^{-t\mathbb{H}^{\omega,K}}\Gamma - e^{-t(K-\Delta)})dt + \mathcal{O}(1-) \\
&= c\int_0^\infty t^{-\frac{1}{2}} e^{-t(K-\Delta)} dt + \underbrace{c\int_0^\infty t^{-\frac{1}{2}} e^{-t(K-\Delta)} dt(\Gamma - 1)}_{=(K-\Delta)^{-\frac{1}{2}}(\Gamma-1)=\mathcal{O}(1-)} + \\
&\quad + c\int_0^\infty t^{-\frac{1}{2}}(e^{-t\mathbb{H}^{\omega,K}}\Gamma - e^{-t(K-\Delta)})dt + \mathcal{O}(1-) \\
&= (K-\Delta)^{-\frac{1}{2}} + c\int_0^\infty t^{-\frac{1}{2}}(e^{-t\mathbb{H}^{\omega,K}}\Gamma - e^{-t(K-\Delta)})\Gamma^{-1} dt + \mathcal{O}(1-).
\end{aligned}$$

Thus it remains to control the integral term for elements in the support of $\mu^{\mathbb{I}_{L^2}}$ of the form (97).

We begin by rewriting it as follows

$$\begin{aligned}
(e^{-t\mathbb{H}^{\omega,K}}\Gamma - e^{-t(K-\Delta)})\Gamma^{-1}\sum_n g_n f_n &= \sum_n g_n \int_0^t e^{-(t-s)(K-\Delta)}(\mathbb{H}^{\omega,K}\Gamma - (K-\Delta))e^{-s\mathbb{H}^{\omega,K}}\Gamma\Gamma^{-1} f_n \\
&= \sum_n g_n \int_0^t e^{-(t-s)(K-\Delta)}(\mathbb{H}^{\omega,K}\Gamma - (K-\Delta))e^{-s\mathbb{H}^{\omega,K}} f_n \\
&= \sum_n g_n \int_0^t e^{-s\lambda_n} e^{-(t-s)(K-\Delta)}(\mathbb{H}^{\omega,K}\Gamma - (K-\Delta)) f_n
\end{aligned}$$

now we may compute the $L^2_{\mathbb{P}}H^\alpha$ $\alpha < 1$ norm. Recall that $H^\sigma = \mathcal{D}\left((\mathbb{H}^{\omega,K})^{\frac{\sigma}{2}}\right)$ for $\sigma \in (-1,1))$ with equivalent norms, see Theorem 4. For a sequence of functions h_n, we can thus bound, using the independence of the Gaussians and the orthogonality of the eigenfunctions,

$$\begin{aligned}
\left\|\sum_n g_n h_n\right\|^2_{L^2_{\mathbb{P}}H^\alpha} &\approx \mathbb{E}\left(\sum_j \left(\sum_n g_n h_n, \sqrt{H}^\alpha f_j\right)^2\right) \\
&= \mathbb{E}\left(\sum_j \left(\sum_n g_n h_n, \sqrt{H}^\alpha f_j\right)\left(\sum_n g_m h_m, \sqrt{H}^\alpha f_j\right)\right) \\
&= \sum_j \sum_{n,m} \underbrace{\mathbb{E}(g_n, g_m)}_{=\delta_{n,m}} \left(h_n, \sqrt{H}^\alpha e_j\right)\left(h_m, \sqrt{H}^\alpha e_j\right) \\
&= \sum_j \sum_n \left(h_n, \sqrt{H}^\alpha e_j\right)^2 \\
&= \sum_n \left(\sqrt{H}^\alpha h_n, \sqrt{H}^\alpha h_n\right) \\
&= \sum_n \|h_n\|^2_{H^\alpha}.
\end{aligned}$$

Now if we apply this to $h_n = \int_0^\infty t^{-\frac{1}{2}} \int_0^t e^{-s\lambda_n} e^{-(t-s)(K-\Delta)}(\mathbb{H}^{\omega,K}\Gamma - (K-\Delta)) f_n ds dt$ and we bound its H^α norm for $\alpha < 1$ norm and show its square summability this implies the desired result.

We compute for some parameters $0 < \delta, \kappa \ll 1$ to be fixed later and $\bar{\lambda}_n := \lambda_n - K$

$$\begin{aligned}
\|h_n\|_{H^\alpha} &\lesssim \int_0^\infty t^{-\frac{1}{2}} \int_0^t e^{-s\lambda_n} \|e^{-(t-s)(K-\Delta)}(\mathbb{H}^{\omega,K}\Gamma - (K-\Delta))f_n\|_{H^\alpha} dsdt \\
&\overset{(103)}{\lesssim} \int_0^\infty t^{-\frac{1}{2}} \int_0^t e^{-s\lambda_n} |t-s|^{-\frac{1}{2}(\alpha-\delta)} e^{-(t-s)K} \|(\mathbb{H}^{\omega,K}\Gamma - (K-\Delta))f_n\|_{H^\delta} dsdt \\
&\overset{(99),(100)}{\lesssim} \int_0^\infty t^{-\frac{1}{2}} e^{-Kt} \int_0^t e^{-s\bar{\lambda}_n |t-s|^{-\frac{1}{2}(\alpha-\delta)}} (\|(\mathbb{H}^{\omega,K}\Gamma - (K-\Delta))\Gamma^{-1} f_n\|_{H^\delta} + \|f_n\|_{H^{\delta+\kappa}}) dsdt \\
&\overset{(102)}{\lesssim} \int_0^\infty t^{-\frac{1}{2}} e^{-Kt} \int_0^t e^{-s\bar{\lambda}_n} |t-s|^{-\frac{1}{2}(\alpha-\delta)} \underbrace{\|\Gamma^{-1} f_n\|_{H^{1+\delta+\kappa}}}_{\lesssim \lambda_n^{\frac{1}{2}+\frac{\delta}{2}+\frac{\kappa}{2}}} dsdt \\
&\underset{s\bar{\lambda}_n =: \sigma}{\overset{t\bar{\lambda}_n =: \tau}{\lesssim}} \bar{\lambda}_n^{-2} \lambda_n^{\frac{1}{2}+\frac{\delta}{2}+\frac{\kappa}{2}} \int_0^\infty \bar{\lambda}_n^{\frac{1}{2}} \tau^{-\frac{1}{2}} e^{-\frac{\tau}{\bar{\lambda}_n}} \int_0^\tau e^{-\sigma} |\tau-\sigma|^{-\frac{1}{2}(\alpha-\delta)} \lambda_n^{\frac{1}{2}(\alpha-\delta)} d\sigma d\tau \\
&\sim \bar{\lambda}_n^{-1+\frac{\alpha}{2}+\frac{\kappa}{2}} \int_0^\infty \tau^{-\frac{1}{2}} e^{-\frac{\tau}{\bar{\lambda}_n}} \int_0^\tau e^{-\sigma} |\tau-\sigma|^{-\frac{1}{2}(\alpha-\delta)} \\
&\overset{(104)}{\lesssim} \lambda_n^{-\frac{1}{2}(1+\kappa)} \text{choosing} \alpha = 1 - 2\kappa \text{ hence} \\
\sum_n \|h_n\|_{H^\alpha}^2 &\lesssim \sum_n \lambda_n^{-1-\kappa} \lesssim \infty
\end{aligned}$$

having used the Weyl asymptotic $\lambda_n \sim n$ for large n, the bound

$$\|\Gamma^{-1} f_n\|_{H^{1+\sigma}} \sim \left\| (\mathbb{H}^{\omega,K})^{\frac{1+\sigma}{2}} f_n \right\|_{L^2} \sim \lambda_n^{\frac{1+\sigma}{2}} \tag{102}$$

which is true for $\sigma \in \{0, 1\}$ by Theorem 4 and for $\sigma \in (0, 1)$ by interpolation, and the standard heat kernel bound

$$\|e^{t\Delta} g\|_{H^\beta} \lesssim t^{-\frac{\beta-\gamma}{2}} \|g\|_{H^\gamma} \text{for all } t > 0 \text{ and } \beta > \gamma. \tag{103}$$

Lastly we prove the finiteness of the integral that we used to conclude. We compute

$$\begin{aligned}
\int_0^\infty \tau^{-\frac{1}{2}} e^{-\frac{\tau}{\bar{\lambda}_n}} \int_0^\tau e^{-\sigma} |\tau-\sigma|^{-\frac{1}{2}(\alpha-\delta)} &\overset{\rho = \frac{\sigma}{\tau}}{=} \int_0^\infty \tau^{-\frac{1}{2}} e^{-\frac{\tau}{\bar{\lambda}_n}} \int_0^1 \tau e^{-\rho\tau} \tau^{-\frac{1}{2}(\alpha-\delta)} (1-\rho)^{-\frac{1}{2}(\alpha-\delta)} \quad (104) \\
&= \int_0^\infty \tau^{\frac{1}{2}(1-\alpha+\delta)} e^{-\frac{\tau}{\bar{\lambda}_n}} \int_0^1 e^{-\rho\tau} (1-\rho)^{-\frac{1}{2}(\alpha-\delta)} \\
&\overset{\nu = \rho\tau}{=} \int_0^\infty \nu^{\frac{1}{2}(1-\alpha+\delta)} e^{-\nu} \int_0^1 \underbrace{e^{-\frac{\nu}{\rho\bar{\lambda}_n}}}_{\leqslant 1} \frac{1}{\rho^{\frac{1}{2}(1-\alpha+\delta)}} (1-\rho)^{-\frac{1}{2}(\alpha-\delta)} \\
&\lesssim 1
\end{aligned}$$

having used that

$$\int_0^1 \frac{1}{x^a} \frac{1}{(1-x)^b} dx < \infty \text{ if } a, b < 1.$$

This finishes the proof. □

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