ON THE TRACE OF $\dot{W}_a^{m+1,1}(\mathbb{R}_+^{n+1})$

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Abstract. In this paper we prove extension results for functions in Besov spaces. Our results are new in the homogeneous setting, while our technique applies equally in the inhomogeneous setting to obtain new proofs of classical results. While our results include $p > 1$, of principle interest is the case $p = 1$, where we show that

$$\int_{\mathbb{R}_+^{n+1}} t^a |\nabla^{m+1} u(x,t)| \; dtdx \lesssim |f|_{B^{m-a,1}(\mathbb{R}^n)}$$

for all $f \in \dot{B}^{m-a,1}(\mathbb{R}^n)$ (the homogeneous Besov space) where u is a suitably scaled heat extension of f. The proofs in this paper rely on the intrinsic seminorm for Besov spaces, thereby bypassing the need for harmonic analysis.

1. Introduction

In the classical paper [4], Gagliardo proved that when $1 < p < \infty$, the trace space of $W^{1,p}(\Omega)$ is the fractional Sobolev space $W^{1-1/p,p}(\partial\Omega)$ (see [11]). Here, Ω is an open bounded set of $\mathbb{R}^{n+1}$ with smooth boundary. An induction argument gives that the trace space of $W^{m+1,p}(\Omega)$ is $W^{m+1-1/p,p}(\partial\Omega)$ for $m \in \mathbb{N}$ and $1 < p < \infty$.

The history is somewhat more involved when $p = 1$. In the first order case, Gagliardo proved in the same paper that the trace space of $W^{1,1}(\Omega)$ is $L^1(\partial\Omega)$ (see also [14] or [9, Theorem 18.13] for a simpler proof due to Mironescu). However, in the higher order case the trace of $W^{m+1,1}(\Omega)$ for $m \in \mathbb{N}$ is not $W^{m,1}(\partial\Omega)$. Indeed, Uspenskiĭ [28] considered the homogeneous weighted Sobolev spaces $\dot{W}_a^{m+1,p}(\mathbb{R}_+^{n+1})$, defined as the space of all functions $u \in W_{\mathrm{loc}}^{m+1,p}(\mathbb{R}_+^{n+1})$ such that

$$|u|_{W_a^{m+1,p}(\mathbb{R}_+^{n+1})} := \left(\int_{\mathbb{R}_+^{n+1}} t^a |\nabla^{m+1} u(x,t)|^p dxdt \right)^{1/p} < \infty,$$

where

$$(x,t) \in \mathbb{R}^n \times (0,\infty) =: \mathbb{R}_+^{n+1},$$

$m \in \mathbb{N}_0$, $a > -1$, and $1 < p < \infty$, and where we use $\nabla^{m+1} u := \nabla^m(\nabla u)$ to denote the inductively defined higher order gradient in x and t. He proved that when $a - p(m+1) + 1 < 0$, the trace space of the *inhomogeneous* Sobolev space $\dot{W}_a^{m+1,p}(\mathbb{R}_+^{n+1}) \cap L^p(\mathbb{R}_+^{n+1})$ is given by the Besov space $B^{m+1-(a+1)/p,p}(\mathbb{R}^n)$, that is,

$$\operatorname{Tr}(\dot{W}_a^{m+1,p}(\mathbb{R}_+^{n+1}) \cap L^p(\mathbb{R}_+^{n+1})) = B^{m+1-(a+1)/p,p}(\mathbb{R}^n). \tag{1.1}$$

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As noted in the literature (see, e.g., [1, p.295]), [12, p.515], [15]), while Uspenskiĭ's result is only stated for $1 < p < \infty$, his proof extends[1] without modification to the case $p = 1$. In particular, Uspenskiĭ's trace theorem [28, Theorem 2] gives

$$\mathrm{Tr}(\dot{W}_a^{m+1,1}(\mathbb{R}_+^{n+1}) \cap L^1(\mathbb{R}_+^{n+1})) \subseteq B^{m-a,1}(\mathbb{R}^n) \tag{1.2}$$

(see also [2], [9, Theorem 18.57] and [15]), while his lifting theorem [28, Theorem 3] shows

$$B^{m-a,1}(\mathbb{R}^n) \subseteq \mathrm{Tr}(\dot{W}_a^{m+1,1}(\mathbb{R}_+^{n+1}) \cap L^1(\mathbb{R}_+^{n+1})). \tag{1.3}$$

Both directions of Uspenskiĭ's argument are a little tricky, though their presentation has been streamlined by Maz'ya in [12, Theorem 1 in Section 10.1]. As was observed by Mironescu and Russ [15], the lifting argument in [12] is missing the estimate for the cross term (second order mixed derivatives in the trace and normal variable). This is a natural motivation for their work [15], where utilizing Littlewood–Paley theory, they give a simple proof of the equality (1.1) that includes the case $p = 1$, see [15, Theorems 1.1 and 1.2]. Accordingly, their paper makes use of the Littlewood–Paley characterization of Besov spaces (see [9, Theorem 17.77]). We refer also to the paper [27] for a treatment of the trace/lifting problem for Sobolev spaces with Muckenhoupt weights.

In this paper, we are interested in the question of extension of functions in *homogeneous* Besov spaces, which arise naturally in the study of PDE on unbounded domains. Our approach is simple and has the benefit of being clearly well-defined in both the inhomogeneous and homogeneous setting. The main new idea in this paper is to replace Uspenskiĭ's use of the harmonic extension of f with

$$u(x,t) := (W_t * f)(x) = \int_{\mathbb{R}^n} W_t(x-y) f(y)\, dy, \tag{1.4}$$

where W is the Gaussian function

$$W(x) := \frac{\exp(-|x|^2/4)}{(4\pi)^{n/2}}, \quad W_t(x) := \frac{1}{t^n} W(xt^{-1}) = \frac{\exp(-|x|^2/(4t^2))}{(4\pi t^2)^{n/2}}. \tag{1.5}$$

While this rescaled Gauss-Weierstrass extension has been utilized previously in the literature (see, e.g. Taibleson [23, p. 458]), it is by no means obvious that it gives a relatively simple resolution of the extension question. A first clear gain is that the convolution (1.4) is always well-defined for functions $f \in \dot{B}^{m-a,1}(\mathbb{R}^n)$, in contrast to the harmonic extension utilized by Uspenskiĭ. The main point of interest is a simplification in the estimates which comes from a different underlying PDE. In particular, as $p(x,t) = W_{\sqrt{t}}(x)$ is the heat kernel, $\frac{\partial p}{\partial t} = \Delta p$. Hence, using the chain rule, or a direct computation, we have that

$$\frac{\partial W_t}{\partial t} = 2t \Delta W_t, \tag{1.6}$$

so that u satisfies the degenerate parabolic initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = 2t\Delta u & \text{in } \mathbb{R}_+^{n+1}, \\ u(x,0) = f(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{1.7}$$

We will see shortly the usefulness of this relation. We first state the main result of this paper in

[1] The reader should note here there is a slight inaccuracy in the assertion of how to handle the estimate for $r - 1$ odd in the case division on p. 137, but ultimately this case is not needed for the demonstration of his Theorem 3.

Theorem 1.1. *Let $m \in \mathbb{N}_0$ and $-1 < a < m$. Suppose that $f \in \dot{B}^{m-a,1}(\mathbb{R}^n)$ and let u be given by* (1.4)*. Then*

$$\int_{\mathbb{R}_+^{n+1}} t^a |\nabla^{m+1} u(x,t)| \, dtdx \lesssim |f|_{B^{m-a,1}(\mathbb{R}^n)} . \tag{1.8}$$

Note that when $a = 0$ this provides a lifting for $\dot{B}^{m,1}(\mathbb{R}^n)$ into $\dot{W}^{m+1,1}(\mathbb{R}_+^{n+1})$.

When a is an integer, the idea of the proof of Theorem 1.1 is based on Uspenskiĭ's trick of introducing second order differences and his use of repeated harmonic extension. In particular, replacing the Poisson kernel with the Gauss-Weierstrass kernel, Uspenskiĭ's ansatz [28, p. 137-138] reads

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = \frac{1}{2} t^{-n-2} \int_{\mathbb{R}^n} \frac{\partial^2 W_1}{\partial x_i \partial x_j}(ht^{-1})[f(x+h) + f(x-h) - 2f(x)] \, dh, \tag{1.9}$$

for $i, j = 1, \ldots, n$. This relies on the fact that $\frac{\partial^2 W_1}{\partial x_i \partial x_j}$ is even and has mean zero, and allows one to introduce the appropriate quantity on the right hand side when one has exactly two pure second order derivatives. The point is then that estimates for the entries of the tensor

$$t^a \nabla^{m+1} u$$

can be reduced to this case through the use of the identity (1.6) and the semi-group property of the Gaussian. The former allows one to directly trade derivatives in t for derivatives in the trace variable, up to a polynomial in the normal variable that is harmless, which in combination with the latter reduces the question of estimate for $t^a \nabla^{m+1} u$ to estimates for linear combinations of

$$t^{a+l} \frac{\partial^{\gamma'} W_{t/\sqrt{2}}}{\partial x^{\gamma'}} * \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_i \partial x_j} * \frac{\partial^\alpha f}{\partial x^\alpha}$$

for some $l \in \{0, \ldots, m+1\}$, and where the multi-indices $\alpha, \gamma \in \mathbb{N}_0^n$ satisfy $|\alpha| = m - k - 1$ and $|\gamma'| = l - k$, with k the integer part of $m - a$. The two pure second order derivatives in the trace variable are then amenable to the analog of Uspenskiĭ's ansatz (1.9), and the estimate follows, where one uses rapid decay of the Gaussian to ensure convergence of several rescaled integrals in the estimates.

In the proof of Theorem 1.1, we will show that a function $f \in \dot{B}^{m-a,1}(\mathbb{R}^n)$ has at most polynomial growth, so that in particular (1.4) is well-defined. This is no longer the case if we replace W_t with the Poisson kernel, as in Uspenskiĭ's paper. This is not an issue for the extensions utilized by Mironescu and Russ [15, Equation (4.1) on p. 362], however, to use their work, in addition to subtraction of a suitable polynomial for the applicability of the fundamental theorem of calculus in Lemma 4.1 on p. 362, one should establish a density result for $C_c^\infty(\mathbb{R}^n)$ in the homogeneous spaces, a question which is itself non-trivial, see e.g. [11, p. Theorem 6.107 on p. 251].

One has an analog of Theorem 1.1 for $1 < p < \infty$ by a similar argument.

Theorem 1.2. *Let $m \in \mathbb{N}_0$, $1 \leq p < \infty$, and $-1 < a < p(m+1) - 1$. Suppose that $f \in \dot{B}^{m+1-(a+1)/p,p}(\mathbb{R}^n)$ and let u be given by* (1.4)*. Then*

$$\int_{\mathbb{R}_+^{n+1}} t^a |\nabla^{m+1} u(x,t)|^p dxdt \lesssim |f|^p_{B^{m+1-(a+1)/p,p}(\mathbb{R}^n)} .$$

It is important to point out that the characterization of Besov spaces as initial values of evolution problems, as in (1.7), is classical. We refer to Triebel's book [25, Section 1.14.5] for the connection between interpolation theory and semigroups, and to [25, Section 2.5.2] for the special case of the heat equation. In this setting, Iwabuchi [7, Theorem 1.4] proved an estimate in a similar spirit of that in our Theorem 1.1 for the semigroup generated by the fractional-order Dirichlet Laplacian. The characterization by means of the heat kernel or the Gauss–Weierstrass semigroup can also be considered as a variants of the characterization of Besov spaces by local means, see Sawano's [19, Section 2.5.2.1], and Triebel's [24, Sections 1.6.5 and 1.8.1] books for more details.

In contrast to the references above, the proofs in this paper rely on the intrinsic seminorm for Besov spaces, thereby bypassing the need for harmonic analysis or interpolation theory. We believe that the proof of Theorem 3.1 below is accessible to an advanced undergraduate or beginning graduate student. Indeed, the original motivation of this paper was to complete the missing estimate in Maz'ya's book [12, Theorem 1 in Section 10.1] for the second order mixed derivatives in the trace and normal variable.

Next, we turn our attention to the inhomogeneous case. For $1 < p < \infty$, Triebel in [25, Theorem 2.9.1 on p. 214] considered the (inhomogeneous) weighted Sobolev space $W^{m+1,p}_a(\mathbb{R}^{n+1}_+)$ defined as the space of all functions $u \in W^{m+1,p}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ such that

$$\|u\|_{W^{m+1,p}_a(\mathbb{R}^{n+1}_+)} := \left(\int_{\mathbb{R}^{n+1}_+} t^a |u(x,t)|^p \, dx dt\right)^{1/p} + \sum_{j=1}^{m+1} |u|_{W^{j,p}_a(\mathbb{R}^{n+1}_+)} < \infty.$$

He proved that for $-1/p < a < m+1-1/p$, the mapping

$$u \mapsto \left(u(x,0), \frac{\partial u}{\partial t}(x,0), \dots, \frac{\partial^l u}{\partial t^l}(x,0)\right)$$

is a retraction from $W^{m+1,p}_a(\mathbb{R}^{n+1}_+)$ onto $\prod_{j=0}^{l} B^{m+1-j-a-1/p}(\mathbb{R}^n)$. Here we take $l = \lfloor m - a + 1/p \rfloor$, where $\lfloor s \rfloor$ is the floor of s. The lifting makes use of the harmonic extension (5.1).

Triebel also showed that if $a \geq m+1-1/p$, then

$$W^{m+1,p}_a(\mathbb{R}^{n+1}_+) = W^{m+1,p}_{0,a}(\mathbb{R}^{n+1}_+),$$

where $W^{m+1,p}_{0,a}(\mathbb{R}^{n+1}_+)$ is the completion of $C^\infty_c(\mathbb{R}^{n+1}_+)$ with respect to the norm in $W^{m+1,p}_a(\mathbb{R}^{n+1}_+)$. In particular, this implies that, in this case, the trace operator cannot be continuous since we can approximate a smooth function in $W^{m+1,p}_a(\mathbb{R}^{n+1}_+)$ with non zero trace with a sequence of functions in $C^\infty_c(\mathbb{R}^{n+1}_+)$. See also the paper [6] of Grisvard for the case $m = 0$.

We carry out this program in the case $p = 1$ in the following three theorems.

Theorem 1.3. *Let $m \in \mathbb{N}_0$ and $-1 < a \leq m$. If $a < m$, then for every $f \in B^{m-a,1}(\mathbb{R}^n)$, there exists $F \in W^{m+1,1}_a(\mathbb{R}^{n+1}_+)$ such that* $\mathrm{Tr}(F) = f$ *and*

$$\|F\|_{W^{m+1,1}_a(\mathbb{R}^{n+1}_+)} \lesssim \|f\|_{B^{m-a,1}(\mathbb{R}^n)}.$$

On the other hand, if $a = m$, then for every $f \in L^1(\mathbb{R}^n)$ there exists $F \in W^{m+1,1}_m(\mathbb{R}^{n+1}_+)$ such that $\mathrm{Tr}(F) = f$ *and*

$$\|F\|_{W^{m+1,1}_m(\mathbb{R}^{n+1}_+)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}. \tag{1.10}$$

The lifting in (1.10) was obtained by Mironescu and Russ [15, Proposition 1.14].

The following result is critical in reducing elliptic or parabolic boundary value problems with inhomogeneous boundary conditions to homogeneous ones (see, e.g., [17, Theorem 4.2.2 on p.218] or [13] for the case $p > 1$). See also the recent work of Gmeineder, Raita, and Van Schaftingen [5] for an application to boundary ellipticity.

Theorem 1.4. *Let $m \in \mathbb{N}$ and $-1 < a < m$. If $a = k \in \mathbb{N}_0$, suppose that $f_j \in B^{m-k-j,1}(\mathbb{R}^n)$ for $j = 0, \dots, m-k-1$, and $f_{m-k} \in L^1(\mathbb{R}^n)$. Then there exists $F \in W_a^{m+1,1}(\mathbb{R}_+^{n+1})$ such that $\mathrm{Tr}(F) = f_0$, $\mathrm{Tr}(\frac{\partial^j F}{\partial t^j}) = f_j$ for $j = 1, \dots, m-k-1$, and*

$$\|F\|_{W_k^{m+1,1}(\mathbb{R}_+^{n+1})} \lesssim \sum_{j=0}^{m-k-1} \|f_j\|_{B^{m-k-j,1}(\mathbb{R}^n)} + \|f_{m-k}\|_{L^1(\mathbb{R}^n)}. \tag{1.11}$$

On the other hand, if $a \notin \mathbb{N}_0$, suppose that $f_j \in B^{m-a-j,1}(\mathbb{R}^n)$ for $j = 0, \dots, l$, where $l := \lfloor m - a \rfloor$. Then there exists $F \in W_a^{m+1,1}(\mathbb{R}_+^{n+1})$ such that $\mathrm{Tr}(F) = f_0$, $\mathrm{Tr}(\frac{\partial^j F}{\partial t^j}) = f_j$ for $j = 1, \dots, l$, and

$$\|F\|_{W_a^{m+1,1}(\mathbb{R}_+^{n+1})} \lesssim \sum_{j=0}^{l} \|f_j\|_{B^{m-a-j,1}(\mathbb{R}^n)}.$$

Finally, we discuss the case $a > m$.

Theorem 1.5. *Let $m \in \mathbb{N}_0$ and $a > m$. Then $W_a^{m+1,1}(\mathbb{R}_+^{n+1}) = W_{0,a}^{m+1,1}(\mathbb{R}_+^{n+1})$.*

This paper is organized as follows. In Section 2, we discuss some basic properties of Besov spaces. In Section 3, we prove Theorems 1.1 and 1.2. Section 4 deals with the inhomogeneous case: We prove Theorems 1.3, 1.4, and 1.5. Finally, in Section 5, we prove several extension results via harmonic extension. Here we also show how Uspenskiĭ's characterization of the trace of $W^{2,1}(\mathbb{R}_+^{n+1})$ by harmonic extension yields a simple proof of the boundedness of the Riesz transforms on $\dot{B}^{1,1}(\mathbb{R}^n)$, thus giving a short proof of the latter fact which is a standard consequence of Littlewood–Paley theory (see, e.g., [20] or [26, Section 5.2.2]).

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2. Preliminaries

In this section, we present some basic properties of Besov spaces that we will use in the sequel. Throughout this paper, the expression

$$\mathcal{A} \lesssim \mathcal{B} \quad \text{means } \mathcal{A} \leq C\mathcal{B}$$

for some constant $C > 0$ that depends on the parameters quantified in the statement of the result (usually n and p), but not on the functions and their domain of integration.

Given $a \in \mathbb{R}$ and $1 \leq p < \infty$, we denote by $L^p_a(\mathbb{R}^{n+1}_+)$ the space all measurable functions $f : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ such that

$$\|f\|_{L^p_a(\mathbb{R}^{n+1}_+)} := \left(\int_{\mathbb{R}^{n+1}_+} t^a |f(x,t)|^p dxdt \right)^{1/p} < \infty. \tag{2.1}$$

Given a function $f \in W^{m,p}_{\text{loc}}(\Omega)$, where $\Omega \subseteq \mathbb{R}^{n+1}$, it will be convenient to have several different symbols to denote various derivatives beyond what we have introduced in Section 1, for $k = 1, \dots, m$,

- $\nabla^k f$ the inductively defined gradient jointly in (x,t) of order k;
- $\frac{\partial^\alpha f}{\partial x^\alpha}$ the kth order partial derivative of f in x given by the multi-index $\alpha \in \mathbb{N}^n_0$ with $|\alpha| = k$.

In particular, for $k = 1, \dots, m$, we also denote by

- $\nabla^k_x f$ the inductively defined gradient in x of order k;
- $\frac{\partial f}{\partial x_i}$ the first order partial derivative of f with respect to the trace variable x_i;
- $\frac{\partial^k f}{\partial t^k}$ the kth order partial derivative of f with respect to the extension variable;
- $\partial^\alpha f$ the partial derivative of f in (x,t) given by the multi-index $\alpha \in \mathbb{N}^k_0 \times \mathbb{N}$ with $|\alpha| = k$.

Definition 2.1. *Given an open set $\Omega \subseteq \mathbb{R}^{n+1}$, $m \in \mathbb{N}$, and $1 \leq p \leq \infty$, we say that a function $f \in W^{m,p}_{\text{loc}}(\Omega)$ belongs to the* homogeneous Sobolev space $\dot{W}^{m,p}(\Omega)$ *if $|\nabla^m f| \in L^p(\Omega)$.*

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and $x, h \in \mathbb{R}^n$, we write

$$\Delta_h f(x) := f(x+h) - f(x), \quad \Delta^{k+1}_h f(x) := \Delta^k_h(\Delta_h f(x)). \tag{2.2}$$

Observe that

$$\Delta^2_h f(x) = f(x+2h) - 2f(x+h) + f(x)$$

and

$$\Delta^2_h f(x-h) = f(x+h) - 2f(x) + f(x-h).$$

Definition 2.2. *Given* $1 \le p, q < \infty$ *and* $0 < s \le 1$, *we say that a function* $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ *belongs to the* homogeneous Besov space $\dot{B}^{s,p}_q(\mathbb{R}^n)$ *if* [2]

$$|f|_{B^{s,p}_q(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \|\Delta_h^{\lfloor s \rfloor +1} f\|^q_{L^p(\mathbb{R}^n)} \frac{dh}{|h|^{n+sq}} \right)^{1/q} < \infty,$$

where $\lfloor s \rfloor$ *is the integer part of* s. *The (inhomogeneous)* Besov space $B^{s,p}_q(\mathbb{R}^n)$ *is the space of all functions* $f \in L^p(\mathbb{R}^n) \cap \dot{B}^{s,p}_q(\mathbb{R}^n)$ *endowed with norm*

$$\|f\|_{B^{s,p}_q(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + |f|_{B^{s,p}_q(\mathbb{R}^n)}.$$

When $q = p$, we write $\dot{B}^{s,p}(\mathbb{R}^n)$ and $B^{s,p}(\mathbb{R}^n)$ for $\dot{B}^{s,p}_p(\mathbb{R}^n)$ and $B^{s,p}_p(\mathbb{R}^n)$, respectively.

In what follows, we will use the equivalent seminorm for $\dot{B}^{1,1}(\mathbb{R}^n)$:

$$|f|^*_{B^{1,1}(\mathbb{R}^n)} := \int_0^\infty \sup_{|h| \le r} \|\Delta_h^2 f\|_{L^1(\mathbb{R}^n)} \frac{dr}{r^2}$$

(see [9, Proposition 17.17]).

Definition 2.3. *Given* $1 \le p, q < \infty$ *and* $s > 1$, *we define the* homogeneous Besov space $\dot{B}^{s,p}_q(\mathbb{R}^n)$ *as the space of all functions* $f \in W^{\ell,p}_{\mathrm{loc}}(\mathbb{R}^n)$ *such that* $\frac{\partial^\alpha f}{\partial x^\alpha} \in \dot{B}^{s-\ell,p}_q(\mathbb{R}^n)$ *for all multi-indices* $\alpha \in \mathbb{N}_0^n$ *with* $|\alpha| = \ell$, *where* $\ell = \max\{m \in \mathbb{N} : m < s\}$. *The (inhomogeneous)* Besov space $B^{m,p}_q(\mathbb{R}^n)$ *is the space of all functions* $f \in W^{\ell,p}(\mathbb{R}^n) \cap \dot{B}^{s,p}_q(\mathbb{R}^n)$ *endowed with norm*

$$\|f\|_{B^{s,p}_q(\mathbb{R}^n)} := \|f\|_{W^{\ell,p}(\mathbb{R}^n)} + \sum_{|\alpha|=m-\ell} \left| \frac{\partial^\alpha f}{\partial x^\alpha} \right|_{B^{s-\ell,p}_q(\mathbb{R}^n)}.$$

Note that $\ell = \lfloor s \rfloor$ if $s \notin \mathbb{N}$ and $\ell = \lfloor s \rfloor - 1$ if $s \in \mathbb{N}$. As before, when $q = p$, we write $\dot{B}^{s,p}(\mathbb{R}^n)$ and $B^{s,p}(\mathbb{R}^n)$ for $\dot{B}^{s,p}_p(\mathbb{R}^n)$ and $B^{s,p}_p(\mathbb{R}^n)$, respectively.

It is important to remark that when $s \notin \mathbb{N}$, the Besov spaces $B^{s,p}(\mathbb{R}^n)$ coincide with the fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$, while for $s = k \in \mathbb{N}$,

$$B^{k,2}(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n), \quad B^{k,p}(\mathbb{R}^n) \subsetneq W^{k,p}(\mathbb{R}^n) \quad \text{for } p \neq 2.$$

For the continuous embedding, we refer to [9, Theorem 17.66]. The two spaces are not equivalent. In the case $p > 1$ this follows from [26, Theorems 2.3.9 and 2.5.6]. When $p = 1$, there is a simple example for $k = 1$: Assume by contradiction that

$$\|f\|_{B^{1,1}(\mathbb{R}^n)} \lesssim \|f\|_{W^{1,1}(\mathbb{R}^n)}$$

for all $f \in W^{1,1}(\mathbb{R}^n)$. It follows by a mollification argument that

$$\|f\|_{B^{1,1}(\mathbb{R}^n)} \lesssim \|f\|_{L^1(\mathbb{R}^n)} + |Df|(\mathbb{R}^n)$$

for all $f \in BV(\mathbb{R}^n)$. Take $f = \chi_{[0,1]^n} \in BV(\mathbb{R}^n)$. Given $x \in \mathbb{R}^n$, when $n \ge 2$ we write

$$x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

[2]The reader should be aware that the definitions of homogeneous and inhomogeneous Besov spaces often differ in the literature, see e.g. [24] for comparison.

Then, by the change of variables $h' = h_n z'$, we obtain

$$\begin{aligned}
&|f|_{B^{1,1}(\mathbb{R}^n)} \\
&\quad \geq \int_{[0,1]^{n-1}\times[0,1/2]} \int_{[0,1]^{n-1}\times[x_n,1/2]} \frac{|\Delta_h^2 \chi_{[0,1]^n}(x-h)|}{|h|^{n+1}} dh' dh_n dx' dx_n \\
&\quad = \int_{[0,1/2]} \int_{[x_n,1/2]} \int_{[0,1]^{n-1}} \frac{dh'}{(|h'|^2 + h_n^2)^{(n+1)/2}} dh_n dx_n \\
&\quad \geq \int_{[0,2]^{n-1}} \frac{dz'}{(|z'|^2+1)^{(n+1)/2}} \int_{[0,1/2]} \int_{[x_n,1/2]} \frac{1}{h_n^2} dh_n dx_n \\
&\quad = \infty.
\end{aligned}$$

The case $n = 1$ is simpler, where one makes a similar computation for $f = \chi_{[0,1]} \in BV(\mathbb{R})$.

When $k > p = 1$, this example can be modified as follows. Let $\varphi \in C_c^\infty([-1/2, 3/2]^n)$ be a function such that $\varphi \equiv 1$ on $[0,1]^n$. Define

$$\psi(x) := \varphi(x) \int_{-1}^{x_n} \int_{-1}^{s_{k-2}} \dots \int_{-1}^{s_2} \chi_{\{x_n>0\}}(s_1)\varphi(x', s_1)\, ds_1 \dots ds_{k-1}.$$

We claim $D(\nabla^{k-1}\psi) \in M_b(\mathbb{R}^n; \mathbb{R}^{n\times k})$. In fact, for every $\alpha \neq (0, \dots, k)$ such that $|\alpha| = k$

$$\frac{\partial^\alpha \psi}{\partial x^\alpha}$$

is a bounded, compactly supported function, so that it only remains to observe that

$$\left(D \frac{\partial^{k-1}\psi}{\partial x_n^{k-1}} \right)_k (x) = \varphi^2(x) \mathcal{H}^{n-1}|_{\{x_n=0\}} + \tilde{\psi}$$

for some bounded compactly supported function $\tilde{\psi}$ and the claim follows. On the other hand, $\psi \notin B^{k,1}(\mathbb{R}^n)$ since

$$\left\| \frac{\partial^{k-1}\psi}{\partial x_n^{k-1}} \right\|_{B^{1,1}(\mathbb{R}^n)} \geq \|\chi_{\{x_n>0\}}\varphi^2\|_{B^{1,1}(\mathbb{R}^n)} - \|\tilde{\psi}\|_{B^{1,1}(\mathbb{R}^n)},$$

and while

$$\|\tilde{\psi}\|_{B^{1,1}(\mathbb{R}^n)} < \infty,$$

a computation similar to the preceding shows

$$\|\chi_{\{x_n>0\}}\varphi^2\|_{B^{1,1}(\mathbb{R}^n)} = \infty.$$

The following result is well known (see, e.g., [9, Theorem 17.24] for a proof that uses abstract interpolation).

Proposition 2.4. *Let $0 < s \leq 1$. Then $W^{\lfloor s \rfloor+1,1}(\mathbb{R}^n)$ is continuously embedded in $B^{s,1}(\mathbb{R}^n)$.*

Proof. Assume that $s = 1$ and let $f \in W^{2,1}(\mathbb{R}^n)$. By the fundamental theorem of calculus,

$$\|\Delta_h^2 f\|_{L^1(\mathbb{R}^n)} \leq |h|^2 \|\nabla_x^2 f\|_{L^1(\mathbb{R}^n)},$$

and so

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \|\Delta_h^2 f\|_{L^1(\mathbb{R}^n)} \frac{dh}{|h|^{n+1}} \\
&\qquad = \int_{B(0,1)} \|\Delta_h^2 f\|_{L^1(\mathbb{R}^n)} \frac{dh}{|h|^{n+1}} + \int_{\mathbb{R}^n \setminus B(0,1)} \|\Delta_h^2 f\|_{L^1(\mathbb{R}^n)} \frac{dh}{|h|^{n+1}} \\
&\qquad \le \|\nabla_x^2 f\|_{L^1(\mathbb{R}^n)} \int_{B(0,1)} \frac{dh}{|h|^{n-1}} + 2^2 \|f\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B(0,1)} \frac{dh}{|h|^{n+1}} \\
&\qquad \lesssim \|\nabla_x^2 f\|_{L^1(\mathbb{R}^n)} + \|f\|_{L^1(\mathbb{R}^n)}.
\end{aligned}
$$

The case $0 < s < 1$ is similar. We omit the details. □

3. The Homogeneous Case

In this section, we prove Theorems 1.1 and 1.2. For simplicity of exposition, we present a version of Theorem 1.1 in the second order case.

Theorem 3.1. *Let $f \in B^{1,1}(\mathbb{R}^n)$ and let u be defined as in (1.4). Then*

$$\|\nabla^2 u\|_{L^1(\mathbb{R}^{n+1}_+)} \lesssim |f|_{B^{1,1}(\mathbb{R}^n)}. \tag{3.1}$$

Lemma 3.2. *Let $\alpha \in \mathbb{N}_0^n$ and $b \in \mathbb{R}$ be such that $n + |\alpha| - b - 1 > 0$. Then for every $x \in \mathbb{R}^n \setminus \{0\}$,*

$$\int_0^\infty t^b \left| \frac{\partial^\alpha W_t}{\partial x^\alpha}(x) \right| dt \lesssim \frac{1}{|x|^{n+|\alpha|-b-1}}$$

and

$$\int_{\mathbb{R}^n} \left| \frac{\partial^\alpha W_t}{\partial x^\alpha}(x) \right| dx \lesssim \frac{1}{t^{|\alpha|}}.$$

Proof. The change of variables $t = |x| r^{-1}$, $dt = -|x| r^{-2} dr$ yields

$$
\begin{aligned}
\int_0^\infty t^b \left| \frac{\partial^\alpha W_t}{\partial x^\alpha}(x) \right| dt &= \int_0^\infty \frac{t^b}{t^{n+|\alpha|}} \left| \frac{\partial^\alpha W}{\partial x^\alpha}(x t^{-1}) \right| dt \\
&= \frac{1}{|x|^{n+|\alpha|-b-1}} \int_0^\infty r^{n+|\alpha|-b-2} \left| \frac{\partial^\alpha W}{\partial x^\alpha}(r x/|x|) \right| dr.
\end{aligned}
$$

Since $n + |\alpha| - b - 2 > -1$ we have that $r^{n+|\alpha|-b-2}$ is integrable near 0, which together with the fact that W is a Gaussian, gives convergence of the integral on the right-hand side.

The facts that $\frac{\partial^\alpha W_t}{\partial x^\alpha}(x) = \frac{1}{t^{n+|\alpha|}} \frac{\partial^\alpha W}{\partial x^\alpha}(x t^{-1})$, that W is a Gaussian, and the change of variables $y = x t^{-1}$, $dy = t^{-n} dx$ imply

$$\int_{\mathbb{R}^n} \left| \frac{\partial^\alpha W_t}{\partial x^\alpha}(x) \right| dx = \frac{1}{t^{|\alpha|}} \int_{\mathbb{R}^n} \left| \frac{\partial^\alpha W}{\partial x^\alpha}(y) \right| dy \lesssim \frac{1}{t^{|\alpha|}}.$$

□

We turn to the proof of Theorem 3.1

Proof of Theorem 3.1. Given $f \in B^{1,1}(\mathbb{R}^n)$, let $u := W_t * f$, where W_t is defined in (1.5). **Step 1:** In this step we estimate the L^1 norms of $\frac{\partial^2 u}{\partial x_i \partial x_j}$ and $\frac{\partial^2 u}{\partial t^2}$. For any $i, j = 1, \ldots n$, one has

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = \int_{\mathbb{R}^n} \frac{\partial^2 W_t}{\partial x_i \partial x_j}(x-h) f(h)\, dh = \int_{\mathbb{R}^n} \frac{\partial^2 W_t}{\partial x_i \partial x_j}(h) f(x-h)\, dh.$$

Making use of the fact that

$$\frac{\partial^2 W_t}{\partial x_i \partial x_j}(-h) = \frac{\partial^2 W_t}{\partial x_i \partial x_j}(h)$$

(the second order partial derivatives purely in the trace variable of the Gaussian kernel, even mixed, are even functions), by a change of variables, one also has

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = \int_{\mathbb{R}^n} \frac{\partial^2 W_t}{\partial x_i \partial x_j}(h) f(x+h)\, dh,$$

Since

$$\int_{\mathbb{R}^n} \frac{\partial^2 W_t}{\partial x_i \partial x_j}(h) f(x)\, dh = 0,$$

this means one can write

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{\partial^2 W_t}{\partial x_i \partial x_j}(h)[f(x+h)+f(x-h)-2f(x)]\, dh. \tag{3.2}$$

In particular, we can estimate

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} \left| \frac{\partial^2 u}{\partial x_i \partial x_j}(x,t)\right| dxdt \\
&\quad \le \frac{1}{2}\int_0^\infty \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left| \frac{\partial^2 W_t}{\partial x_i \partial x_j}(h)\right| |f(x+h)+f(x-h)-2f(x)|\, dhdxdt \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^\infty \left| \frac{\partial^2 W_t}{\partial x_i \partial x_j}(h)\right| dt |f(x+h)+f(x-h)-2f(x)|\, dhdx.
\end{aligned}$$

By Lemma 3.2,

$$\int_0^\infty \left| \frac{\partial^2 W_t}{\partial x_i \partial x_j}(h)\right| dt \lesssim \frac{1}{|h|^{n+1}}. \tag{3.3}$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} \left| \frac{\partial^2 u}{\partial x_i \partial x_j}(x,t)\right| dxdt \\
&\qquad \lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x+h)+f(x-h)-2f(x)|}{|h|^{n+1}}\, dhdx.
\end{aligned}$$

Since $\frac{\partial^2 W_t}{\partial t^2}$ is even and integrates to zero, reasoning as before, we can write

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{\partial^2 W_t}{\partial t^2}(h)[f(x+h)+f(x-h)-2f(x)]\, dh.$$

Since $\frac{\partial^2 W_t}{\partial t^2}$ can be written as a linear combination of $t^{-2}W_t$, $t^{-1}(x_i t^{-1})\frac{\partial W_t}{\partial x_i}$, and $(x_i t^{-1})(x_j t^{-1})\frac{\partial^2 W_t}{\partial x_i x_j}$, reasoning as in the proof of Lemma 3.2, we have that

$$\int_0^\infty \left| \frac{\partial^2 W_t}{\partial t^2}(h)\right| dt \lesssim \frac{1}{|h|^{n+1}}. \tag{3.4}$$

We can now continue as before to obtain the estimate for this derivative.

Step 2: In this step we estimate the L^1 norm of $\frac{\partial^2 u}{\partial t \partial x_j}$. For the mixed derivatives involving t, one computes

$$\frac{\partial^2 u}{\partial t \partial x_j}(x,t) = \int_{\mathbb{R}^n} \frac{\partial W_t}{\partial t}(h) \frac{\partial f}{\partial x_j}(x-h)\, dh.$$

For fixed x, define $g(h) := f(x-h)$. Then

$$\frac{\partial f}{\partial x_j}(x-h) = -\frac{\partial g}{\partial h_j}(h),$$

an integration by parts yields

$$\frac{\partial^2 u}{\partial t \partial x_j}(x,t) = \int_{\mathbb{R}^n} \frac{\partial^2 W_t}{\partial t \partial x_j}(h) f(x-h)\, dh.$$

Here, $\frac{\partial^2 W_t}{\partial t \partial x_j}$ is not an even function (and, in fact, it is odd). Since

$$\frac{\partial W_t}{\partial t} = 2t\Delta W_t,$$

we can write

$$\frac{\partial^2 W_t}{\partial t \partial x_i} = 2t \sum_{j=1}^n \frac{\partial^3 W_t}{\partial x_i \partial^2 x_j}.$$

The semi-group property of the heat extension, which is just a manipulation of the Fourier transform (see (1.5)), leads to the identity[3]

$$\frac{\partial^3 W_t}{\partial x_i \partial^2 x_j} = \frac{\partial W_{t/\sqrt{2}}}{\partial x_i} * \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_j^2}. \tag{3.5}$$

Since $\frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_j^2}$ is even and has zero average in the trace variable x, we can write

$$\left(\frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_j^2} * f \right)(x) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_j^2}(h) \Delta_h^2 f(x-h)\, dh, \tag{3.6}$$

where $\Delta_h^2 f$ is as defined in (2.2). In turn,

$$\begin{aligned} &\left(\frac{\partial W_{t/\sqrt{2}}}{\partial x_i} * \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_j^2} * f \right)(x) \\ &\qquad = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\partial W_{t/\sqrt{2}}}{\partial x_i}(x-y) \int_{\mathbb{R}^n} \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_j^2}(h) \Delta_h^2 f(y-h)\, dh dy. \end{aligned} \tag{3.7}$$

In conclusion, we have

$$\frac{\partial^2 u}{\partial t \partial x_i}(x,t) = \sum_{j=1}^n \int_{\mathbb{R}^n} \frac{\partial W_{t/\sqrt{2}}}{\partial x_i}(x-y) \int_{\mathbb{R}^n} t \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_j^2}(h) \Delta_h^2 f(y-h)\, dh dy.$$

---

[3]We have

$$\mathcal{F}(W_t)(\xi) = e^{-4t^2\pi^2|\xi|^2} = e^{-4(t/\sqrt{2})^2\pi^2|\xi|^2} e^{-4(t/\sqrt{2})^2\pi^2|\xi|^2} = \mathcal{F}(W_{t/\sqrt{2}})\mathcal{F}(W_{t/\sqrt{2}}).$$

Integrating this quantity over $\mathbb{R}^{n+1}_+$ and using Fubini's theorem yields

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} \left| \frac{\partial^2 u}{\partial t \partial x_i}(x,t) \right| dxdt \\
&\quad \lesssim \sum_{j=1}^n \int_{\mathbb{R}^n} \left| \frac{\partial W_{t/\sqrt{2}}}{\partial x_i}(z) \right| dz \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^\infty t \left| \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_j^2}(h) \right| dt \left| \Delta_h^2 f(y-h) \right| dhdy \\
&\quad \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left| \Delta_h^2 f(y-h) \right|}{|h|^{n+1}} dhdy \\
&\quad = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y+h) - 2f(y) + f(y-h)|}{|h|^{n+1}} dhdy,
\end{aligned}$$

where in the last inequality we applied twice Lemma 3.2. □

Remark 3.3. *The proof in Step 1 is classical and follows Uspenskiĭ's ansatz* (5.4). *Note that in Step 1, we only used the fact that the kernel W_t is even and decays sufficiently fast at infinity for* (3.3) *and* (3.4) *to hold. In particular, in this step, we could replace the Gaussian function W with the Poisson kernel P or with an even mollifier $\varphi \in C_c^\infty(\mathbb{R}^n)$. In contrast, Step 2 uses the properties of the Gaussian kernel W_t in a crucial way.*

We next prove a preliminary version of Theorem 1.1 in the inhomogeneous case.

Theorem 3.4. *Let $m \in \mathbb{N}_0$ and $-1 < a < m$. Suppose that $f \in B^{m-a,1}(\mathbb{R}^n)$ and let u be given by* (1.4). *Then*

$$\int_{\mathbb{R}^{n+1}_+} t^a |\nabla^{m+1} u(x,t)| \, dtdx \lesssim |f|_{B^{m-a,1}(\mathbb{R}^n)} . \tag{3.8}$$

Proof of Theorem 3.4. Let $s := m - a > 0$ and $f \in B^{s,1}(\mathbb{R}^n)$.

Step 1: Assume that $s = k \in \mathbb{N}$. Then for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k-1$, we have that $\frac{\partial^\alpha f}{\partial x^\alpha} \in B^{1,1}(\mathbb{R}^n)$. The goal is then to show an L^1 bound for the entries of the tensor

$$t^a \nabla^{m+1} u,$$

where we recall that $u = W_t * f$. To reduce to the case where all the derivatives are in the trace variable x, consider a multi-index $(\beta, l) \in \mathbb{N}_0^n \times \mathbb{N}_0$, with $|\beta| + l = m+1$. Since $|\beta| + 2l = m + 1 + l > k - 1$, by applying l times formula (1.6) , we can write

$$t^a \left(\frac{\partial^\beta}{\partial x^\beta} \left(\frac{\partial^l W_t}{\partial t^l} \right) \right) * f$$

as linear combinations of

$$t^{a+l} \frac{\partial^\gamma W_t}{\partial x^\gamma} * \frac{\partial^\alpha f}{\partial x^\alpha}$$

for $l = 0, \dots, m+1$, and where the multi-indices $\alpha, \gamma \in \mathbb{N}_0^n$ satisfy $|\alpha| = k-1$ and $|\gamma| = m + l - k + 2$. As in (3.5), the semi-group property of the heat extension leads to the identity

$$\frac{\partial^\gamma W_t}{\partial x^\gamma} = \frac{\partial^{\gamma'} W_{t/\sqrt{2}}}{\partial x^{\gamma'}} * \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_i \partial x_j}$$

for some $i,j=1,\dots,n$ and $\gamma'\in\mathbb{N}_0^n$ such that $|\gamma'|=|\gamma|-2=m+l-k$. This shows that one actually estimates

$$t^{a+l}\frac{\partial^{\gamma'}W_{t/\sqrt{2}}}{\partial x^{\gamma'}}*\frac{\partial^2W_{t/\sqrt{2}}}{\partial x_i\partial x_j}*\frac{\partial^\alpha f}{\partial x^\alpha}$$

for some $l\in\{0,\dots,m+1\}$, and where the multi-indices $\alpha,\gamma\in\mathbb{N}_0^n$ satisfy $|\alpha|=k-1$ and $|\gamma'|=m+l-k$.

Since $\frac{\partial^2W_{t/\sqrt{2}}}{\partial x_i\partial x_j}$ is even and has zero average in the trace variable x, as in (3.7) we can write

$$\begin{aligned}t^{a+l}&\left(\frac{\partial^{\gamma'}W_{t/\sqrt{2}}}{\partial x^{\gamma'}}*\frac{\partial^2W_{t/\sqrt{2}}}{\partial x_i\partial x_j}*\frac{\partial^\alpha f}{\partial x^\alpha}\right)(x)\\&=\frac{t^{a+l}}{2}\int_{\mathbb{R}^n}\frac{\partial^{\gamma'}W_{t/\sqrt{2}}}{\partial x^{\gamma'}}(x-y)\int_{\mathbb{R}^n}\frac{\partial^2W_{t/\sqrt{2}}}{\partial x_i\partial x_j}(h)\Delta_h^2\frac{\partial^\alpha f}{\partial x^\alpha}(y-h)\,dhdy.\end{aligned}$$

The point is that when one integrates this quantity over $\mathbb{R}^{n+1}_+$, by Lemma 3.2,

$$\int_{\mathbb{R}^n}\left|\frac{\partial^{\gamma'}W_{t/\sqrt{2}}}{\partial x^{\gamma'}}(x)\right|dx\lesssim\frac{1}{t^{|\gamma'|}}$$

and therefore, by Tonelli's theorem, one has as an upper bound some universal constant times

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^\infty t^{a+l-|\gamma'|}\left|\frac{\partial^2W_{t/\sqrt{2}}}{\partial x_i\partial x_j}(h)\right|\left|\Delta_h^2\frac{\partial^\alpha f}{\partial x^\alpha}(w-h)\right|\,dtdhdw.$$

By Lemma 3.2 again,

$$\int_0^\infty t^{a+l-|\gamma'|}\left|\frac{\partial^2W_{t/\sqrt{2}}}{\partial x_i\partial x_j}(h)\right|\,dt\lesssim\frac{1}{|h|^{n+2-a-l+|\gamma'|-1}}=\frac{1}{|h|^{n+1}}.$$

Step 2: Assume that $s\notin\mathbb{N}$ and let $k:=\lfloor s\rfloor$. Then for every $\alpha\in\mathbb{N}_0^n$ with $|\alpha|=k$, we have that $\frac{\partial^\alpha f}{\partial x^\alpha}\in B^{s-k,1}(\mathbb{R}^n)$. As in Step 1, to estimate the L^1 norm of $t^a\nabla^{m+1}u$, it suffices to estimate the L^1 norm of

$$t^{a+l}\frac{\partial^\gamma W_t}{\partial x^\gamma}*\frac{\partial^\alpha f}{\partial x^\alpha}$$

for $l=0,\dots,m+1$, and where the multi-indices $\alpha,\gamma\in\mathbb{N}_0^n$ satisfy $|\alpha|=k$ and $|\gamma|=m+1+l-k$. Since $\int_{\mathbb{R}^n}\frac{\partial^\gamma W_t}{\partial x^\gamma}(h)\,dh=0$, we can write

$$\left(\frac{\partial^\gamma W_t}{\partial x^\gamma}*\frac{\partial^\alpha f}{\partial x^\alpha}\right)(x)=\int_{\mathbb{R}^n}\frac{\partial^\gamma W_t}{\partial x^\gamma}(h)\Delta_{-h}\frac{\partial^\alpha f}{\partial x^\alpha}(x)\,dh.\tag{3.9}$$

Multiplying both sides by t^{a+l} and integrating in (x,t) gives

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+}&t^{a+l}\left|\left(\frac{\partial^\gamma W_t}{\partial x^\gamma}*\frac{\partial^\alpha f}{\partial x^\alpha}\right)(x)\right|\,dxdt\\&\le\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^\infty t^{a+l}\left|\frac{\partial^\gamma W_t}{\partial x^\gamma}(h)\right|\left|\Delta_{-h}\frac{\partial^\alpha f}{\partial x^\alpha}(x)\right|\,dtdhdx,\end{aligned}$$

where we used Tonelli's theorem. By Lemma 3.2,

$$\int_0^\infty t^{a+l}\left|\frac{\partial^\gamma W_t}{\partial x^\gamma}(h)\right|\,dt\lesssim\frac{1}{|h|^{n+|\gamma|-a-l-1}}=\frac{1}{|h|^{n+s-k}}.$$

In turn,

$$\int_{\mathbb{R}^{n+1}_+} t^{a+l} \left| \left(\frac{\partial^\gamma W_t}{\partial x^\gamma} * \frac{\partial^\alpha f}{\partial x^\alpha} \right)(x) \right| dxdt \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \Delta_{-h} \frac{\partial^\alpha f}{\partial x^\alpha}(x) \right| \frac{dh}{|h|^{n+s-k}} dx.$$

□

We next prove a preliminary version of Theorem 1.2 in the inhomogeneous case.

Theorem 3.5. *Let $m \in \mathbb{N}_0$, $1 \leq p < \infty$, and $-1 < a < p(m+1) - 1$. Suppose that $f \in B^{m+1-(a+1)/p,p}(\mathbb{R}^n)$ and let u be given by (1.4). Then*

$$\int_{\mathbb{R}^{n+1}_+} t^a |\nabla^{m+1} u(x,t)|^p dxdt \lesssim |f|^p_{B^{m+1-(a+1)/p,p}(\mathbb{R}^n)}.$$

Proof of Theorem 3.5. Let $s := m+1-(a+1)/p > 0$ and $f \in B^{s,p}(\mathbb{R}^n)$. **Step 1:** Assume that $s = k \in \mathbb{N}$. Then for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k-1$, we have that $\frac{\partial^\alpha f}{\partial x^\alpha} \in B^{1,p}(\mathbb{R}^n)$. As in Step 1 of the proof of Theorem 1.1, to obtain an L^p bound for the entries of the tensor $t^{a/p}\nabla^{m+1}u$, it suffices to estimate the L^p norm of

$$t^{a/p+l} \frac{\partial^{\gamma'} W_{t/\sqrt{2}}}{\partial x^{\gamma'}} * \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_i \partial x_j} * \frac{\partial^\alpha f}{\partial x^\alpha}$$

for some $l \in \{0, \dots, m+1\}$, and where the multi-indices $\alpha, \gamma \in \mathbb{N}_0^n$ satisfy $|\alpha| = k-1$ and $|\gamma'| = m+l-k$.

Let

$$g(x) := \frac{1}{2} \int_{\mathbb{R}^n} \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_j^2}(h) \Delta_h^2 f(x-h)\, dh.$$

By Young's inequality for convolutions,

$$(3.10)\quad t^{a/p+l} \left\| \frac{\partial^{\gamma'} W_{t/\sqrt{2}}}{\partial x^{\gamma'}} * \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_i \partial x_j} * \frac{\partial^\alpha f}{\partial x^\alpha} \right\|_{L^p(\mathbb{R}^n)} = t^{a/p+l} \left\| \frac{\partial^{\gamma'} W_{t/\sqrt{2}}}{\partial x^{\gamma'}} * g \right\|_{L^p(\mathbb{R}^n)}$$

$$\leq t^{a/p+l} \left\| \frac{\partial^{\gamma'} W_{t/\sqrt{2}}}{\partial x^{\gamma'}} \right\|_{L^1(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)} \lesssim t^{a/p+l-|\gamma'|} \|g\|_{L^p(\mathbb{R}^n)},$$

where we used Lemma 3.2.

To estimate $\|g\|_{L^p(\mathbb{R}^n)}$, we write $\left| \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_i \partial x_j} \right| = \left| \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_i \partial x_j} \right|^{1/p'} \left| \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_i \partial x_j} \right|^{1/p}$. By Hölder's inequality

$$|g(x)| \leq \left(\int_{\mathbb{R}^n} \left| \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_i \partial x_j}(h) \right| dh \right)^{1/p'}$$

$$\times \left(\int_{\mathbb{R}^n} \left| \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_i \partial x_j}(h) \right| \left| \Delta_h^2 \frac{\partial^\alpha f}{\partial x^\alpha}(x-h) \right|^p dh \right)^{1/p}$$

$$\lesssim t^{-2/p'} \left(\int_{\mathbb{R}^n} \left| \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_i \partial x_j}(h) \right| \left| \Delta_h^2 \frac{\partial^\alpha f}{\partial x^\alpha}(x-h) \right|^p dh \right)^{1/p},$$

where we used again Lemma 3.2. Hence,

$$\int_{\mathbb{R}^n} |g(x)|^p dx \lesssim t^{-2(p-1)} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_i \partial x_j}(h)\right| \left|\Delta_h^2 \frac{\partial^\alpha f}{\partial x^\alpha}(x-h)\right|^p dh dx. \tag{3.11}$$

Raising both sides of (3.10) to power p, integrating in t, and using (3.11) gives

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+} t^{a+lp} \left|\left(\frac{\partial^{\gamma'} W_{t/\sqrt{2}}}{\partial x^{\gamma'}} * \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_i \partial x_j} * \frac{\partial^\alpha f}{\partial x^\alpha}\right)(x)\right|^p dx dt \\ &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^\infty t^{a+lp-|\gamma'|p-2(p-1)} \left|\frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_i \partial x_j}(h)\right| \left|\Delta_h^2 \frac{\partial^\alpha f}{\partial x^\alpha}(x-h)\right|^p dt dh dx.\end{aligned}$$

By Lemma 3.2,

$$\int_0^\infty t^{a+lp-|\gamma'|p-2(p-1)} \left|\frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_i \partial x_j}(h)\right| dt \lesssim \frac{1}{|h|^{n+2-a-lp+|\gamma'|p+2(p-1)-1}} = \frac{1}{|h|^{n+p}}.$$

In turn,

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+} t^{a+lp} \left|\left(\frac{\partial^{\gamma'} W_{t/\sqrt{2}}}{\partial x^{\gamma'}} * \frac{\partial^2 W_{t/\sqrt{2}}}{\partial x_i \partial x_j} * \frac{\partial^\alpha f}{\partial x^\alpha}\right)(x)\right|^p dx dt \\ &\qquad \lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\Delta_h^2 \frac{\partial^\alpha f}{\partial x^\alpha}(x-h)\right|^p \frac{dh}{|h|^{n+p}} dx.\end{aligned}$$

Step 2: Assume that $s \notin \mathbb{N}$ and let $k := \lfloor s \rfloor$. Then for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$, we have that $\frac{\partial^\alpha f}{\partial x^\alpha} \in B^{s-k,p}(\mathbb{R}^n)$. As in Step 2 of the proof of Theorem 1.1, to obtain an L^p bound for the entries of the tensor $t^{a/p}\nabla^{m+1}u$, it suffices to estimate

$$t^{a/p+l} \frac{\partial^\gamma W_t}{\partial x^\gamma} * \frac{\partial^\alpha f}{\partial x^\alpha}$$

for $l = 0, \dots, m+1$, and where the multi-indices $\alpha, \gamma \in \mathbb{N}_0^n$ satisfy $|\alpha| = k$ and $|\gamma| = m+1+l-k$. Writing $\left|\frac{\partial^\gamma W_t}{\partial x^\gamma}\right| = \left|\frac{\partial^\gamma W_t}{\partial x^\gamma}\right|^{1/p'} \left|\frac{\partial^\gamma W_t}{\partial x^\gamma}\right|^{1/p}$, by Hölder's inequality and (3.9)

$$\begin{aligned}&\left|\left(\frac{\partial^\gamma W_t}{\partial x^\gamma} * \frac{\partial^\alpha f}{\partial x^\alpha}\right)(x)\right| \\ &\le \left(\int_{\mathbb{R}^n} \left|\frac{\partial^\gamma W_t}{\partial x^\gamma}(h)\right| dh\right)^{1/p'} \left(\int_{\mathbb{R}^n} \left|\frac{\partial^\gamma W_t}{\partial x^\gamma}(h)\right| \left|\Delta_{-h} \frac{\partial^\alpha f}{\partial x^\alpha}(x)\right|^p dh\right)^{1/p} \\ &\lesssim t^{-|\gamma|/p'} \left(\int_{\mathbb{R}^n} \left|\frac{\partial^\gamma W_t}{\partial x^\gamma}(h)\right| \left|\Delta_{-h} \frac{\partial^\alpha f}{\partial x^\alpha}(x)\right|^p dh\right)^{1/p},\end{aligned}$$

where we used Lemma 3.2. Multiplying both sides by $t^{a/p+l}$, raising to power p, and integrating in (x,t) gives

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+} t^{a+lp} \left|\left(\frac{\partial^\gamma W_t}{\partial x^\gamma} * \frac{\partial^\alpha f}{\partial x^\alpha}\right)(x)\right|^p dx dt \\ &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^\infty t^{a+lp-(p-1)|\gamma|} \left|\frac{\partial^\gamma W_t}{\partial x^\gamma}(h)\right| \left|\Delta_{-h} \frac{\partial^\alpha f}{\partial x^\alpha}(x)\right|^p dt dh dx.\end{aligned}$$

By Lemma 3.2,

$$\int_0^\infty t^{a+lp-(p-1)|\gamma|} \left|\frac{\partial^\gamma W_t}{\partial x^\gamma}(h)\right| dt \lesssim \frac{1}{|h|^{n+|\gamma|-a-lp+(p-1)|\gamma|-1}} = \frac{1}{|h|^{n+(s-k)p}}.$$

In turn,

$$\int_{\mathbb{R}^{n+1}_+} t^{a+lp} \left|\left(\frac{\partial^\gamma W_t}{\partial x^\gamma} * \frac{\partial^\alpha f}{\partial x^\alpha}\right)(x)\right|^p dxdt \lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\Delta_{-h}\frac{\partial^\alpha f}{\partial x^\alpha}(x)\right|^p \frac{dh}{|h|^{n+(s-k)p}} dx.$$

□

We conclude this section with the proofs of Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2. Observe that the only place that the assumption $f \in B^{m+1-(a+1)/p,p}(\mathbb{R}^n)$ was used in the proofs of Theorems 1.1 and 1.2 was to ensure that $W_t * f$ is well-defined. The proofs of Theorems 1.1 and 1.2 will therefore be complete if we can show this convolution is well-defined for $f \in \dot{B}^{m+1-(a+1)/p,p}(\mathbb{R}^n)$.

By [9, Remark 17.27 on p. 556], for $f \in \dot{B}^{m+1-(a+1)/p,p}(\mathbb{R}^n)$ one can write

$$f = u + v,$$

where

$$u(x) := \frac{1}{t^{ln}} \int_{Q(0,t)} \cdots \int_{Q(0,t)} \Delta^l_{h_1+\cdots+h_l} u(x)\, dh_1 \cdots dh_l, \quad v(x) := f(x) - u(x),$$

with $u \in L^p(\mathbb{R}^n)$ and $v \in \dot{W}^{l,p}(\mathbb{R}^n)$ and we take $l \in \mathbb{N}$ so large that $lp > n$. As $W_t * u$ is well-defined for $u \in L^p(\mathbb{R}^n)$, it remains to show that $W_t * v$ is well-defined. This follows from the fact that v has polynomial growth by Theorem A.1 in the appendix below.

□

4. The Inhomogeneous Case

We now turn to the proof of Theorem 1.3. The proof of Step 2 is an adaptation of Mironescu's argument who studied the case $m = 0$ [14].

Proof of Theorem 1.3. **Step 1:** Assume that $a < m$. Let $\psi \in C^\infty([0,\infty))$ be a nonnegative decreasing function such that $\psi = 1$ in $[0,1]$, $\psi(t) = 0$ for $t \geq 2$ and define $F(x,t) = \psi(t)u(x,t)$, where $u = W_t * f$. By Tonelli's theorem and the change of variables $z = yt^{-1}$,

$$\int_{\mathbb{R}^n} |u(x,t)|\, dx \leq \int_{\mathbb{R}^n} |f(x)|\, dx \int_{\mathbb{R}^n} W(z)\, dz = \int_{\mathbb{R}^n} |f(x)|\, dx.$$

Since $0 \leq \psi \leq 1$ and $\psi(t) = 0$ for $t \geq 2$, it follows

(4.1)
$$\int_{\mathbb{R}^{n+1}_+} t^a |F(x,t)|\, dxdt = \int_0^\infty t^a \psi(t) \int_{\mathbb{R}^n} |u(x,t)|\, dxdt \leq \int_0^2 t^a dt \int_{\mathbb{R}^n} |f(x)|\, dx.$$

Observe that the first integral on the right-hand side is finite because $a > -1$.

Since $\psi = 1$ in $[0,1]$, $F(x,t) = u(x,t)$ for $t \leq 1$, and so $\mathrm{Tr}(F) = \mathrm{Tr}(u) = f$. It remains to estimate the derivatives of F. Consider a multi-index $(\beta, l) \in \mathbb{N}_0^n \times \mathbb{N}_0$, with $|\beta| + l = m + 1$. By the product rule

$$\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^l F}{\partial t^l}\right)$$

can be written as a linear combination of $\psi^{(l-j)}(t)\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^j u}{\partial t^j}\right)$ for $j = 0, \dots, l$. As in Step 1 of the proof of Theorem 1.1 we can use (1.6) to write $\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^j u}{\partial t^j}\right)$ as a linear combination of $t^i \frac{\partial^\gamma u}{\partial x^\gamma}$ with $i \in \{0, \dots, j\}$, where the multi-index $\gamma \in \mathbb{N}_0^n$ satisfies $|\gamma| = |\beta| + 2i = m + 1 - l + 2i$. There are now two cases. If $-l + 2i < 0$, we use the Gagliardo–Nirenberg interpolation inequality [9, Theorem 12.85] to estimate

$$\int_{\mathbb{R}^n} |\nabla_x^{m+1-l+2i} u(x,t)|\, dx \lesssim \int_{\mathbb{R}^n} |u(x,t)|\, dx + \int_{\mathbb{R}^n} |\nabla_x^{m+1} u(x,t)|\, dx.$$

In turn,

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}} t^{a+i} |\psi^{(l-j)}(t)| |\nabla_x^{m+1-l+2i} u(x,t)|\, dxdt \\
&\qquad \lesssim \int_0^2 t^{a+i} \int_{\mathbb{R}^n} |u(x,t)|\, dxdt + \int_0^2 t^{a+i} \int_{\mathbb{R}^n} |\nabla_x^{m+1} u(x,t)|\, dxdt \\
&\qquad \lesssim \int_0^2 t^{a} \int_{\mathbb{R}^n} |u(x,t)|\, dxdt + \int_0^2 t^{a} \int_{\mathbb{R}^n} |\nabla_x^{m+1} u(x,t)|\, dxdt \\
&\qquad \lesssim \int_{\mathbb{R}^n} |f(x)|\, dx + |f|_{B^{m-a,1}(\mathbb{R}^n)}
\end{aligned}$$

by (1.8) and (4.1).

If $-l + 2i \geq 0$, we use (1.8) with a replaced by $a + i$ and m by $m - l + 2i$ to obtain

$$\int_{\mathbb{R}_+^{n+1}} t^{a+i} |\nabla^{m+1-l+2i} u(x,t)|\, dtdx \lesssim |f|_{B^{m-a-l+i,1}(\mathbb{R}^n)}.$$

If $i = l$, we are done since $f \in B^{m-a,1}(\mathbb{R}^n)$. Otherwise, we use the fact that by Proposition 2.4,

$$|f|_{B^{m-a-l+i,1}(\mathbb{R}^n)} \lesssim \|f\|_{L^1(\mathbb{R}^n)} + \|\nabla_x^{\lfloor m-a-l+i \rfloor+1} f\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{B^{m-a,1}(\mathbb{R}^n)}.$$

Similar estimates hold when $1 \leq |\beta| + l < m + 1$. We omit the details.

Step 2: Assume that $a = m$ and let $f \in C_c^\infty(\mathbb{R}^n)$. Consider a function $\varphi \in C^\infty([0,\infty))$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ in $[0,1]$, and $\varphi = 0$ in $[2,\infty)$. For $l \in \mathbb{N}$ define $v_l(x,t) := f(x)\varphi(lt)$. By the properties of φ, $v_l(x,0) = f(x)$. The product rule implies that the entries of the tensor $\nabla^{m+1} v_l$ can be written as linear combinations of the functions

$$l^i \varphi^{(i)}(lt) \frac{\partial^\beta f}{\partial x^\beta}(x)$$

for multi-indices $\beta \in \mathbb{N}_0^n$ and $i = 0, \ldots, m+1-|\beta|$ if $\beta \neq 0$ or $i = 1, \ldots, m+1$ if $\beta = 0$. This, along with the change of variables $r = lt$ leads to the estimate

$$\int_{\mathbb{R}^{n+1}_+} t^m |\nabla^{m+1} v_l(x,t)|\, dxdt$$

$$\lesssim \sum_{|\beta|=1}^{m+1} \sum_{i=0}^{m+1-|\beta|} \frac{1}{l^{m-i+1}} \int_{\mathbb{R}^n} \left| \frac{\partial^\beta f}{\partial x^\beta}(x) \right| dx \int_0^\infty r^m |\varphi^{(i)}(r)|\, dr \tag{4.2}$$

$$+ \int_{\mathbb{R}^n} |f(x)|\, dx \int_0^\infty r^m |\varphi^{(m+1)}(r)|\, dr. \tag{4.3}$$

One observes that the quantity numbered by equation (4.2) tends to zero as $l \to \infty$. Thus, if $f \neq 0$, by taking l large enough, we can majorize the quantity (4.2) by $\|f\|_{L^1(\mathbb{R}^n)}$, while the quantity numbered by equation (4.3) is just a constant multiple of this norm. In particular, for such a sufficiently large l_0, $F = v_{l_0}$ satisfies the desired properties. If $f = 0$, we take $F = 0$. This proves (1.10) in the case $f \in C_c^\infty(\mathbb{R}^n)$. The general case follows from the density of $C_c^\infty(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$. We omit the details. □

Remark 4.1. *By taking $a = 0$ in (4.1), we have that $F \in \dot{W}^{m+1,1}(\mathbb{R}^{n+1}_+) \cap L^1(\mathbb{R}^{n+1}_+)$, with*

$$|F|_{W^{m+1,1}(\mathbb{R}^{n+1}_+)} + \|F\|_{L^1(\mathbb{R}^{n+1}_+)} \lesssim \|f\|_{B^{m-a,1}(\mathbb{R}^n)}.$$

Remark 4.2. *Using the fact that for $v \in W^{m+1,1}_m(\mathbb{R}_+) \cap C^\infty([0,\infty))$, we have*

$$v(0) = c \int_0^\infty t^m \frac{d^{(m+1)} v}{dt^{m+1}}(t)\, dt$$

for all $u \in W^{m+1,1}_m(\mathbb{R}^{n+1}_+) \cap C^\infty(\mathbb{R}^n \times [0,\infty))$, we can write

$$u(x,0) = c \int_0^\infty t^m \frac{\partial^{(m+1)} u}{\partial t^{m+1}}(x,t)\, dt,$$

and so,

$$\int_{\mathbb{R}^n} |u(x,0)|\, dx \lesssim \int_0^\infty \int_{\mathbb{R}^n} t^m \left| \frac{\partial^{(m+1)} u}{\partial t^{m+1}}(x,t) \right| dxdt.$$

By a reflection (see, e.g., [9, Exercise 13.3]*) and mollification argument, we have that for every function $u \in W^{m+1,1}_m(\mathbb{R}^{n+1}_+)$, the trace of u belongs to $L^1(\mathbb{R}^n)$. Together with the previous theorem, this shows that*

$$\mathrm{Tr}(W^{m+1,1}_m(\mathbb{R}^{n+1}_+)) = L^1(\mathbb{R}^n).$$

When $k = 0$, the proof of the following lemma is due to Gmeineder, Raita, and Van Schaftingen [5] and is an adaptation of Mironescu's argument in Step 2 of the proof of Theorem 1.3 above. See also the paper of Demengel [2] and [9, Theorem 18.43] for an alternative proof based on Gagliardo's original proof [4].

Lemma 4.3. *Let $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k < m$. Suppose that $g \in L^1(\mathbb{R}^n)$. Then there exists $G \in \dot{W}^{m+1,1}_k(\mathbb{R}^{n+1}_+)$ such that $\mathrm{Tr}(G) = 0$, $\mathrm{Tr}(\frac{\partial^j G}{\partial t^j}) = 0$ for $j = 1, \ldots, m-k-1$, $\mathrm{Tr}(\frac{\partial^{m-k} G}{\partial t^{m-k}}) = g$, and*

$$\int_{\mathbb{R}^{n+1}_+} t^k |\nabla^{m+1} G(x,t)|\, dxdt \lesssim \|g\|_{L^1(\mathbb{R}^n)}. \tag{4.4}$$

Proof. Assume first that $g \in C_c^\infty(\mathbb{R}^n)$ and let $\varphi \in C_c^\infty([0,\infty))$ be such that $\varphi(0) = 1$, $\varphi'(0) = \cdots = \varphi^{(m-k)}(0) = 1$. For $n \in \mathbb{N}$ define $v_l(x,t) := g(x)\frac{t^{m-k}}{(m-k)!}\varphi(lt)$. By the properties of φ, $v_l(x,0) = 0$, $\frac{\partial^j v_l}{\partial t^j}(x,0) = 0$ for $j = 1,\ldots,m-k-1$, and $\frac{\partial^{m-k} v_l}{\partial t^{m-k}}(x,0) = g(x)$. The product rule implies that the entries of the tensor $\nabla^{m+1}v_l$ can be written as linear combinations of the functions

$$l^i t^{(|\beta|-k-1+i)_+}\varphi^{(i)}(lt)\frac{\partial^\beta g}{\partial x^\beta}(x)$$

for multi-indices $\beta \in \mathbb{N}_0^n$ and $i = 0,\ldots,m+1-|\beta|$ if $\beta \neq 0$ or $i = 1,\ldots,m+1$ if $\beta = 0$. Here, s_+ is the positive part of s. This, along with the change of variables $r = lt$, leads to the estimate

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}} t^k |\nabla^{m+1}v_l(x,t)|\,dxdt \\
&\lesssim \sum_{|\beta|=1}^{m+1}\sum_{i=0}^{m+1-|\beta|}\frac{l^i}{l^{k+(|\beta|-k-1+i)_+ +1}}\int_{\mathbb{R}^n}\left|\frac{\partial^\beta g}{\partial x^\beta}(x)\right|dx \\
&\quad\times\int_0^\infty r^{k+(|\beta|-k-1+i)_+}|\varphi^{(i)}(r)|\,dr \qquad (4.5)\\
&\quad+\int_{\mathbb{R}^n}|g(x)|\,dx\sum_{i=1}^{m+1}\frac{l^i}{l^{k+(i-1-k)_+ +1}}\int_0^\infty r^{k+(i-1-k)_+}|\varphi^{(i)}(r)|\,dr. \qquad (4.6)
\end{aligned}$$

If $|\beta|-k-1+i \geq 0$, then $\frac{l^i}{l^{k+(|\beta|-k-1+i)_+ +1}} = \frac{1}{l^{|\beta|}} \to 0$ since $|\beta| \geq 1$, while if $|\beta|-k-1+i < 0$, then $\frac{l^i}{l^{k+(|\beta|-k-1+i)_+ +1}} = \frac{l^i}{l^{k+1}} \to 0$. Hence, the quantity numbered by equation (4.5) tends to zero for as $l \to \infty$. Thus, if $g \neq 0$, by taking l large enough, we can majorize the quantity (4.5) by $\|g\|_{L^1(\mathbb{R}^n)}$.

On the other hand, if $i-1-k \geq 0$, $\frac{l^i}{l^{k+(i-1-k)_+ +1}} = 1$, while if $i-1-k < 0$, then $\frac{l^i}{l^{k+(i-1-k)_+ +1}} \leq 1$. Hence, the quantity numbered by equation (4.6) is bounded from above by a constant multiple of $\|g\|_{L^1(\mathbb{R}^n)}$. If $g = 0$, we take $G = 0$. This proves (4.4) in the case $g \in C_c^\infty(\mathbb{R}^n)$. The general case follows from the density of $C_c^\infty(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$. We omit the details. ☐

Proof of Theorem 1.4. **Step 1:** If $a \in \mathbb{N}_0$ let $l := m-a-1$, while if $a \notin \mathbb{N}_0$ let $l := \lfloor m-a \rfloor$. Assume that $f_j \in B^{m-a-j,1}(\mathbb{R}^n)$ for $j = 1,\ldots,l$. We first prove that there exists a function $u_j \in \dot{W}_a^{m+1,1}(\mathbb{R}_+^{n+1})$ such that $\mathrm{Tr}(u_j) = 0$, $\mathrm{Tr}(\frac{\partial^i u_j}{\partial t^i}) = 0$ for all $i = 1,\ldots,j-1$, and $\mathrm{Tr}(\frac{\partial^j u_j}{\partial t^j}) = f_j$, with

$$\|\nabla^{m+1}u_j\|_{L_a^1(\mathbb{R}_+^{n+1})} \lesssim |f_j|_{B^{m-a-j,1}(\mathbb{R}^n)} \qquad (4.7)$$

(see (2.1)). Define

$$u_j(x,t) := \frac{t^j}{j!}(W_t * f_j)(x),$$

where W is the Gaussian function (1.5). The desired properties $\mathrm{Tr}(\frac{\partial^j u_j}{\partial t^j}) = f_j$, $\mathrm{Tr}(u_j) = 0$, $\mathrm{Tr}(\frac{\partial^i u_j}{\partial t^i}) = 0$ for all $i = 1,\ldots,j-1$ can be checked by the properties of W_t. Concerning the estimate (4.7), one observes that the product rule implies that the entries of the tensor $\nabla^{m+1}u_j$ are linear combinations of the entries of the tensor $t^{j-i}\nabla^{m+1-i}W_t * f_j$. Thus the estimate (4.7) is a consequence of (1.8) applied to the function $u = W_t * f_j$, with m replaced by $m-i$ and a by $a+j-i$, which asserts that one has the estimates

$$\int_{\mathbb{R}^{n+1}_+} t^{a+j-i} |\nabla^{m+1-i} (W_t * f_j)| \, dtdx \lesssim |f_j|_{B^{m-a-j,1}(\mathbb{R}^n)}$$

for $i = 0, \ldots, j$.

Step 2: We are now ready to prove the general case. Assume first that $a = k \in \mathbb{N}_0$ let $l := m-k-1$. We will use the fact that if $u \in W^{m+1,1}_k(\mathbb{R}^{n+1}_+)$, then $\operatorname{Tr}(u) \in B^{m-k,1}(\mathbb{R}^n)$, $\operatorname{Tr}(\frac{\partial^j u}{\partial t^j}) \in B^{m-k-j,1}(\mathbb{R}^n)$ for $j = 1, \ldots, l$, $\operatorname{Tr}(\frac{\partial^{m-k} u}{\partial t^{m-k}}) \in L^1(\mathbb{R}^n)$, with

$$\begin{aligned} |\operatorname{Tr}(u)|_{B^{m-k,1}(\mathbb{R}^n)} + \sum_{j=1}^{l} \left|\operatorname{Tr}\left(\frac{\partial^j u}{\partial t^j}\right)\right|_{B^{m-k-j,1}(\mathbb{R}^n)} + \left\|\operatorname{Tr}\left(\frac{\partial^{m-k} u}{\partial t^{m-k}}\right)\right\|_{L^1(\mathbb{R}^{n+1}_+)} \\ \lesssim \|\nabla^{m+1} u_m\|_{L^1_k(\mathbb{R}^{n+1}_+)} \end{aligned} \tag{4.8}$$

(see [9, Theorem 18.57]). By Theorem 1.3, there exists $v_0 \in \dot{W}^{m+1,1}_k(\mathbb{R}^{n+1}_+)$ such that $\operatorname{Tr}(v_0) = f_0$ and

$$\|\nabla^{m+1} v_0\|_{L^1_k(\mathbb{R}^{n+1}_+)} \lesssim |f_0|_{B^{m-k,1}(\mathbb{R}^n)}. \tag{4.9}$$

In turn, (4.8) holds for v_0. Hence, we can apply Step 1, with f_1 replaced by $f_1 - \operatorname{Tr}\left(\frac{\partial v_0}{\partial t}\right)$, to find a function $v_1 \in \dot{W}^{m+1,1}(\mathbb{R}^{n+1}_+)$ such that $\operatorname{Tr}(v_1) = 0$, $\operatorname{Tr}(\frac{\partial v_1}{\partial t}) = f_1 - \operatorname{Tr}\left(\frac{\partial v_0}{\partial t}\right)$, and

$$\begin{aligned} \|\nabla^{m+1} v_1\|_{L^1_k(\mathbb{R}^{n+1}_+)} &\lesssim |f_1|_{B^{m-1,1}(\mathbb{R}^n)} + \left|\operatorname{Tr}\left(\frac{\partial v_0}{\partial t}\right)\right|_{B^{m-1,1}(\mathbb{R}^n)} \\ &\lesssim |f_1|_{B^{m-1,1}(\mathbb{R}^n)} + |f_0|_{B^{m,1}(\mathbb{R}^n)}, \end{aligned}$$

where the last inequality follows from (4.8), with v_0 in place of u, and (4.9).

Inductively, assume that $v_j \in \dot{W}^{m+1,1}_k(\mathbb{R}^{n+1}_+)$ has been constructed with $\operatorname{Tr}(v_j) = 0$, $\operatorname{Tr}(\frac{\partial^i v_j}{\partial t^i}) = 0$ for all $i = 1, \ldots, j-1$,

$$\operatorname{Tr}\left(\frac{\partial^j v_j}{\partial t^j}\right) = f_j - \sum_{i=0}^{j-1} \operatorname{Tr}\left(\frac{\partial^j v_i}{\partial t^j}\right),$$

and

$$\|\nabla^{m+1} v_j\|_{L^1_k(\mathbb{R}^{n+1}_+)} \lesssim \sum_{i=0}^{j} |f_i|_{B^{m-k-i,1}(\mathbb{R}^n)}.$$

By (4.8), we can apply Step 1, with f_{j+1} replaced by $f_{j+1} - \sum_{i=0}^{j} \operatorname{Tr}\left(\frac{\partial^{j+1} v_i}{\partial t^{j+1}}\right)$, to find a function $v_{j+1} \in \dot{W}^{m+1,1}_k(\mathbb{R}^{n+1}_+)$ such that $\operatorname{Tr}(v_{j+1}) = 0$, $\operatorname{Tr}(\frac{\partial^i v_{j+1}}{\partial t^i}) = 0$ for all $i = 1, \ldots, j$, and $\operatorname{Tr}(\frac{\partial^{j+1} v_j}{\partial t^{j+1}}) = f_{j+1} - \sum_{i=0}^{j} \operatorname{Tr}\left(\frac{\partial^{j+1} v_i}{\partial t^{j+1}}\right)$, with

$$\begin{aligned} \|\nabla^{m+1} v_{j+1}\|_{L^1_k(\mathbb{R}^{n+1}_+)} &\lesssim |f_{j+1}|_{B^{m-k-j-1,1}(\mathbb{R}^n)} + \sum_{i=0}^{j} \left|\operatorname{Tr}\left(\frac{\partial^{j+1} v_i}{\partial t^{j+1}}\right)\right|_{B^{m-k-i,1}(\mathbb{R}^n)} \\ &\lesssim \sum_{i=0}^{j+1} |f_i|_{B^{m-k-i,1}(\mathbb{R}^n)}. \end{aligned}$$

This induction process gives functions $v_0, \ldots, v_l \in \dot W_k^{m+1,1}(\mathbb{R}^{n+1}_+)$. Again by (4.8), we can apply Lemma 4.3 to construct a function $v_{m-k} \in \dot W_k^{m+1,1}(\mathbb{R}^{n+1}_+)$ such that $\operatorname{Tr}(v_m) = 0$, $\operatorname{Tr}(\frac{\partial^i v_{m-k}}{\partial t^i}) = 0$ for all $i = 1, \ldots, l$, $\operatorname{Tr}(\frac{\partial^{m-k} v_{m-k}}{\partial t^{m-k}}) = f_{m-k} - \sum_{i=0}^l \operatorname{Tr}\left(\frac{\partial^{m-k} v_i}{\partial t^{m-k}}\right)$ with

$$\|\nabla^{m+1} v_{m-k}\|_{L^1_k(\mathbb{R}^{n+1}_+)} \lesssim \|f_{m-k}\|_{L^1(\mathbb{R}^n)} + \sum_{i=0}^l \left\|\operatorname{Tr}\left(\frac{\partial^{m-k} v_i}{\partial t^{m-k}}\right)\right\|_{L^1(\mathbb{R}^n)}$$
$$\lesssim \|f_{m-k}\|_{L^1(\mathbb{R}^n)} + \sum_{i=0}^l |f_i|_{B^{m-k-i,1}(\mathbb{R}^n)}.$$

We now define $u = v_0 + \cdots + v_m \in \dot W_k^{m+1,1}(\mathbb{R}^{n+1}_+)$. By construction $\operatorname{Tr}(u) = f_0$, $\operatorname{Tr}(\frac{\partial^j u}{\partial t^j}) = f_j$ for all $j = 1, \ldots, m$, and

$$\|\nabla^{m+1} u\|_{L^1_k(\mathbb{R}^{n+1}_+)} \lesssim \sum_{j=0}^l |f_j|_{B^{m-a-j,1}(\mathbb{R}^n)} + \|f_{m-k}\|_{L^1(\mathbb{R}^n)}.$$

To obtain a function in $W_k^{m+1}(\mathbb{R}^{n+1}_+)$ we proceed as in Theorem 1.3.

The case $a \notin \mathbb{N}_0$ is similar but simpler. We omit the details. □

Remark 4.4. *Note that when $a = 0$ we construct a function $u \in \dot W^{m+1,1}(\mathbb{R}^{n+1}_+)$ such that $\operatorname{Tr}(u) = f_0$, $\operatorname{Tr}(\frac{\partial^j u}{\partial t^j}) = f_j$ for $j = 1, \ldots, m$, and*

$$\|\nabla^{m+1} u\|_{L^1(\mathbb{R}^{n+1}_+)} \lesssim \sum_{j=0}^{m-1} |f_j|_{B^{m-j,1}(\mathbb{R}^n)} + \|f_m\|_{L^1(\mathbb{R}^n)}.$$

This estimate was used by Gmeineder, Raita, and Van Schaftingen [5].

Next, we prove Theorem 1.5. The proof follows the approach of Grisvard [6], who considered the case $p > 1$, $m = 0$ and $a \geq p - 1$.

Proof of Theorem 1.5. Let $a > m$ and let $u \in W_a^{m+1,1}(\mathbb{R}^{n+1}_+)$.

Step 1: Assume first that $u \in C^\infty(\mathbb{R}^{n+1}_+)$ with $u = 0$ outside $B_n(0, r) \times (0, r)$ for some large $r > 0$. Consider a function $\varphi \in C^\infty([0, \infty))$ such that $0 \leq \varphi \leq 1$, $\varphi = 0$ in $[0, 1]$ and $\varphi = 1$ in $[2, \infty)$. For $n \in \mathbb{N}$ define $v_j(x, t) := u(x, t)\varphi(jt)$. Given a multi-index $\alpha = (\beta, l) \in \mathbb{N}_0^n \times \mathbb{N}_0$, with $|\beta| + l = |\alpha| = m + 1$, the product rule implies that

$$\partial^\alpha v_j(x, t) = \sum_{i=0}^l \binom{l}{i} j^i \varphi^{(i)}(jt) \frac{\partial^{l-i}\partial^\beta u}{\partial t^{l-i}\partial x^\beta}(x, t).$$

If $i = 0$, we can use the Lebesgue dominated convergence theorem to show that

$$\varphi(jt)\partial^\alpha u(x, t) \to \partial^\alpha u(x, t) \text{ in } L^1_a(\mathbb{R}^{n+1}_+).$$

On the other hand, if $i \geq 1$, then by the change of variables $r = jt$, we have the estimate

$$j^i \int_{\mathbb{R}^{n+1}_+} t^a \left|\varphi^{(i)}(jt) \frac{\partial^{l-i}\partial^\beta u}{\partial t^{l-i}\partial x^\beta}(x, t)\right| dxdt$$
$$\lesssim \left\|\nabla^{l-i+|\beta|} u\right\|_\infty \mathcal{L}^n(B_n(0, r)) j^i \int_{1/j}^{2/j} t^a dt \lesssim \frac{1}{j^{a+1-i}} \to 0$$

since $a > m$ and $1 \le i \le m+1$.

Step 2: The general case $u \in W_a^{m+1,1}(\mathbb{R}_+^{n+1})$ can be obtained by a density argument. By a higher order reflection (see, e.g., [9, Exercise 13.3]) and mollifying u, we can assume that $u \in W_a^{m+1,1}(\mathbb{R}^{n+1}) \cap C^\infty(\mathbb{R}^{n+1})$.

Consider a cut-off function $\phi \in C_c^\infty(\mathbb{R}^{n+1})$ such that $\phi = 1$ in $B(0,1)$ and $\phi = 0$ outside $B(0,2)$. The function u_j, given by $u_j(x,t) := \phi(j^{-1}(x,t))u(x,t)$, satisfies the hypotheses of Step 1 and converges to u in $W_a^{1,1}(\mathbb{R}_+^{n+1})$ as $j \to \infty$. We omit the details (see [6, Lemma 1.2] for the case $m = 1$).

□

5. Harmonic Extension

The initial goal of this paper was to give a straightforward proof of the estimate for the missing cross terms in [12], where the following idea emerged. Following Uspenskiĭ, we introduce the harmonic extension of a function $f \in B^{1,1}(\mathbb{R}^n)$:

$$u(x,t) := (P_t * f)(x) = \int_{\mathbb{R}^n} P_t(x-y)f(y)\,dy, \tag{5.1}$$

where P_t is the Poisson kernel (cf [21, p. 61])

$$P(x) := \frac{c_n}{(|x|^2+1)^{(n+1)/2}}, \quad P_t(x) := \frac{1}{t^n}P(xt^{-1}) = \frac{c_n t}{(|x|^2+t^2)^{(n+1)/2}} \tag{5.2}$$

and

$$c_n = \frac{1}{\int_{\mathbb{R}^n} \frac{1}{(|x|^2+1)^{(n+1)/2}}\,dx} = \Gamma((n+1)/2)/\pi^{(n+1)/2}, \tag{5.3}$$

where Γ is the Gamma function.

As mentioned in the introduction, Uspenskiĭ's argument [28] on p. 137-138 shows that for $i,j = 1,\ldots,n$ one has

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = \frac{1}{2}t^{-n-2}\int_{\mathbb{R}^n} \frac{\partial^2 P_1}{\partial x_i \partial x_j}(ht^{-1})[f(x+h)+f(x-h)-2f(x)]\,dh, \tag{5.4}$$

which relies on the fact that $\frac{\partial^2 P_1}{\partial x_i \partial x_j}$ is even and has mean zero. This is sufficient to estimate the pure second order derivatives in the trace variable. Meanwhile, harmonicity allows one to reduce the pure second order derivatives in the normal variable to this case, as

$$\frac{\partial^2 u}{\partial t^2}(x,t) = -\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}(x,t).$$

We then observed that for the mixed case one can simply use the identities

$$R_i\left(\frac{\partial^2 P_t}{\partial t \partial x_j}\right) = \frac{\partial^2 P_t}{\partial x_i \partial x_j}, \quad P_t * f = \sum_{i=1}^n R_i(P_t) * R_i(f), \tag{5.5}$$

where R_i is the Riesz transform, to write

$$\frac{\partial^2 u}{\partial t \partial x_j}(x,t) = \frac{1}{2t^2}\sum_{i=1}^n \int_{\mathbb{R}^n} \frac{\partial^2 P_t}{\partial x_i \partial x_j}(h)[R_i(f)(x+h)+R_i(f)(x-h)-2R_i(f)(x)]\,dh.$$

The estimate for the pure second order derivatives can then be applied, using the fact that for every $f \in B^{1,1}(\mathbb{R}^n)$,

$$|R_j(f)|_{B^{1,1}(\mathbb{R}^n)} \lesssim |f|_{B^{1,1}(\mathbb{R}^n)}. \tag{5.6}$$

This estimate is well known and its classical proof makes use of the Littlewood–Paley theory (see, e.g., [20] or [26, Section 5.2.2]). We refer to [10] for a different proof that relies on the intrinsic seminorm of $\dot{B}^{1,1}(\mathbb{R}^n)$ and is based on an argument of Devore, Riemenschneider, Sharpley [3].

This argument yields a third proof of the following theorem[4].

Theorem 5.1. *Let $f \in B^{1,1}(\mathbb{R}^n)$ and let u be defined as in* (5.1)*. Then*

$$\|\nabla^2 u\|_{L^1(\mathbb{R}_+^{n+1})} \lesssim |f|_{B^{1,1}(\mathbb{R}^n)}. \tag{5.7}$$

After this paper was completed, Mironescu directed us to yet another approach to deal with the cross derivatives, which works under the additional assumption that $u(x,t) \to 0$ as $t \to \infty$ and relies on Taibleson's [23, Theorem 1 on p. 420] (see also [15, Lemma 4.1 and formula (5.8)]) to estimate the cross term via the pure trace derivatives:

$$\int_0^\infty \int_{\mathbb{R}^n} \left| \frac{\partial^2 u}{\partial x_i \partial t}(x,t) \right| \, dxdt \lesssim \max_{j=1,\ldots,n} \int_0^\infty \int_{\mathbb{R}^n} \left| \frac{\partial^2 u}{\partial x_i \partial x_j}(x,y) \right| \, dxdt$$

Actually, this is just a concise presentation of the original argument of Uspenskiĭ: a combination of Hardy's inequality ([28, Theorem 1] in his paper or [15, equation (2.3) in Proposition 2.1 on p. 358] in Mironescu and Russ's), the semi-group property of the Poisson kernel that allows one to express the lifting as a double convolution, and easy estimates of derivatives for the Poisson kernel.

The following is a more rigorous reiteration of the preceding discussion. To this end we recall some basic properties of the Riesz transform. Given $j \in \{1, \ldots, n\}$ and a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$, the *Riesz transform* of f is defined formally as

$$R_j(f)(x) = c_n \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} f(x-y) \frac{y_j}{|y|^{n+1}} \, dy, \tag{5.8}$$

provided the limit exists. The constant c_n here is the same as in (5.3).

Proposition 5.2. *Let P_t be the Poisson kernel* (5.2)*. Then*

$$R_j\left(\frac{\partial P_t}{\partial t}\right) = \frac{\partial P_t}{\partial x_j}.$$

[4][28, Theorem 3 on p. 135] is accomplished via the harmonic extension, [15, Theorem 1.9 on p. 356] is accomplished via Littlewood–Paley theory, while Burenkov [1, Theorem 3 in Section 5.4] gives a different proof of (1.1) that covers the case $p = 1$ with $a = 0$. However, a crucial point in his proof is the ability to factor the derivative of a mollifier as a linear combination of another integrable function, which in the context of the Poisson kernel in the second order case essentially amounts to showing the existence of an integrable function ν such that

$$\frac{\partial^2 P_1}{\partial t \partial x_j}(\xi) = 2\nu(\xi) - \frac{1}{2^n}\nu(\xi/2).$$

This is a step we have not been able to verify in the demonstration of Corollary 7 in Section 5.4 of [1, Theorem 3 in Section 5.4].

Proof. Taking the Fourier transform in the x variables gives (see [21, p. 125])

$$\left(R_j\left(\frac{\partial P_t}{\partial t}\right)\right)^{\wedge}(\xi) = i\frac{\xi_j}{|\xi|}\left(\frac{\partial P_t}{\partial t}\right)^{\wedge}(\xi) = i\frac{\xi_j}{|\xi|}\frac{\partial \hat{P}_t}{\partial t}(\xi).$$

As $\hat{P}_t(\xi) = e^{-2\pi|\xi|t}$, we have

$$\frac{\partial \hat{P}_t}{\partial t}(\xi) = -2\pi|\xi|e^{-2\pi|\xi|t},$$

and therefore

$$\left(R_j\left(\frac{\partial P_t}{\partial t}\right)\right)^{\wedge}(\xi) = i\frac{\xi_j}{|\xi|}\frac{\partial \hat{P}_t}{\partial t}(\xi) = -2\pi i\xi_j e^{-2\pi|\xi|t} = \left(\frac{\partial P_t}{\partial x_j}\right)^{\wedge}(\xi).$$

The claim follows by inverting the Fourier transform. □

Proposition 5.3. *Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$, where $1 < p < \infty$. Then*

$$\int_{\mathbb{R}^n} fg\,dx = \sum_{j=1}^n \int_{\mathbb{R}^n} R_j(f)R_j(g)\,dx.$$

Proof. Assume first that $f, g \in \mathcal{S}(\mathbb{R}^n)$. By making use of Parseval's identity and the facts that $\mathcal{F}(R_j(f))(\xi) = i\frac{\xi_j}{|\xi|}\mathcal{F}(f)(\xi)$ and $\mathcal{F}(f) \in L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} fg\,dx &= \int_{\mathbb{R}^n} \mathcal{F}(f)\overline{\mathcal{F}(g)}\,d\xi = \sum_{j=1}^n \int_{\mathbb{R}^n}\left(i\frac{\xi_j}{|\xi|}\right)\mathcal{F}(f)\overline{\left(i\frac{\xi_j}{|\xi|}\right)\mathcal{F}(g)}\,d\xi \\ &= \sum_{j=1}^n \int_{\mathbb{R}^n} \mathcal{F}(R_j(f))\overline{\mathcal{F}(R_j(g))}\,d\xi = \sum_{j=1}^n \int_{\mathbb{R}^n} R_j(f)R_j(g)\,dx.\end{aligned}$$

If $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$, we use a density argument and the fact that the Riesz transform is bounded in $L^q(\mathbb{R}^n)$ for all $1 < q < \infty$. □

First proof of Theorem 5.1. Assume that $f \in C_c^\infty(\mathbb{R}^n)$ and let $u = P_t * f$, where P_t is the Poisson kernel (5.2). Since $\left|\frac{\partial^2 P_1}{\partial x_i \partial x_j}(x)\right| \lesssim \frac{1}{|x|^{n+3}}$ for $|x| \geq 1$, the estimate (3.3) holds with W_t replaced by P_t. Hence, in view of Remark 3.3, we can estimate the L^1 norm of $\frac{\partial^2 u}{\partial x_i \partial x_j}$ as in Step 1 of the proof of Theorem 3.1. Since

$$\frac{\partial^2 P_t}{\partial t^2} + \sum_{i=1}^n \frac{\partial^2 P_t}{\partial x_i^2} = 0,$$

using (3.2), we can write

$$\frac{\partial^2 u}{\partial t^2}(x,t) = -\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}(x,t) = -\frac{1}{2}\sum_{i=1}^n \int_{\mathbb{R}^n} \frac{\partial^2 P_t}{\partial x_i^2}(h)[f(x+h)+f(x-h)-2f(x)]\,dh$$

and in the same way, argue the estimate for this derivative.

Step 2: To estimate the L^1 norm of $\frac{\partial^2 u}{\partial t \partial x_j}$, we use Taibleson's [23, Theorem 1 on p. 420] applied to the harmonic function $\frac{\partial^2 u}{\partial x_i \partial t}$:

$$\int_0^\infty \int_{\mathbb{R}^n} \left|\frac{\partial^2 u}{\partial x_i \partial t}(x,t)\right| dxdt \lesssim \sum_{j=1}^n \int_0^\infty \int_{\mathbb{R}^n} \left|\frac{\partial^2 u}{\partial x_i \partial x_j}(x,y)\right| dxdt,$$

which reduces the argument again the previous case.

Step 3: A standard density argument in $B^{1,1}(\mathbb{R}^n)$ allows one to remove the additional hypothesis that $f \in C_c^\infty(\mathbb{R}^n)$. We omit the details. $\square$

Interestingly, while the boundedness of the Riesz transforms gives a simple proof of the inclusion (1.3), i.e. the lifting estimate, the trace characterization of $W^{2,1}(\mathbb{R}^{n+1}_+)$ via harmonic extension itself yields a simple proof that Riesz transforms are bounded on the Besov space $B^{1,1}(\mathbb{R}^n)$. In particular, we next establish

Theorem 5.4. *For every $f \in B^{1,1}(\mathbb{R}^n)$,*

$$|R_j(f)|_{B^{1,1}(\mathbb{R}^n)} \lesssim |f|_{B^{1,1}(\mathbb{R}^n)}.$$

Remark 5.5. *We observe that if $f \in B^{1,1}(\mathbb{R}^n)$, then $f \in W^{1,1}(\mathbb{R}^n)$ (see* [9, Theorem 17.66]*). If $n = 1$, this implies that $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and in turn, $f \in L^p(\mathbb{R})$ for all $1 \le p \le \infty$. On the other hand, if $n \ge 2$, then by the Sobolev–Gagliardo–Nirenberg embedding theorem, we have $f \in L^{n/(n-1)}(\mathbb{R}^n)$. In both cases, the Riesz transform of f is well-defined.*

Theorem 5.4 is well known and its classical proof makes use of the Littlewood–Paley theory (see, e.g., [20] or [26, Section 5.2.2]). The simple proof proceeds as follows.

Proof of Theorem 5.4. Let $f_\epsilon := f * \rho_\epsilon$ for standard mollifiers ρ_ϵ. Then using the relation (which can be argued using Remark 5.5, for example)

$$R_j(f_\epsilon) \equiv \rho_\epsilon * R_j(f),$$

one observes that $R_j(f_\epsilon)$ is a smooth function. Therefore [9, Theorem 18.57 on p. 630] gives the inequality

$$|R_j(f_\epsilon)|_{B^{1,1}(\mathbb{R}^n)} \lesssim \int_{\mathbb{R}^{n+1}_+} |\nabla^2 P_t * R_j(f_\epsilon)|\, dxdt.$$

Next observe that for every $s \ge t > 0$, $\nabla^2 P_s * R_j(f_\epsilon)$ is a harmonic function such that

$$\int_{\mathbb{R}^n} |\nabla^2 P_s * R_j(f_\epsilon)|\, dx \le C_t.$$

A result Stein and Weiss [22, Theorem 2.6 on p. 51] gives the bound

$$\|\nabla^2 P_s * R_j(f_\epsilon)\|_{L^\infty(\mathbb{R}^n)} \le \frac{C_t'}{s^n}.$$

Thus one can apply the fundamental theorem of calculus to obtain

$$\nabla^2 P_t * R_j(f_\epsilon) = -\int_t^\infty \frac{\partial}{\partial t} \nabla^2 P_s * R_j(f_\epsilon)\, ds,$$

which in combination with Hardy's inequality [15, equation (2.3) in Proposition 2.1 on p. 358] and Proposition 5.2 yields

$$\begin{aligned} \int_{\mathbb{R}^{n+1}_+} |\nabla^2 P_t * R_j(f_\epsilon)|\, dxdt &\lesssim \int_{\mathbb{R}^{n+1}_+} t \left| \nabla^2 \frac{\partial P_t}{\partial t} * R_j(f_\epsilon) \right| dxdt \\ &= \int_{\mathbb{R}^{n+1}_+} t \left| \nabla^2 \frac{\partial P_t}{\partial x_j} * f_\epsilon \right| dxdt. \end{aligned}$$

Next Taibleson's [23, Lemma 4(b) on p. 419] and the argument presented in the introduction give

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} t\left|\nabla^2 \frac{\partial P_t}{\partial x_j} * f_\epsilon\right| dxdt &\lesssim \int_{\mathbb{R}^{n+1}_+} |\nabla^2 P_t * f_\epsilon|\, dxdt \\ &\lesssim |f_\epsilon|_{B^{1,1}(\mathbb{R}^n)}.\end{aligned}$$

These inequalities, the definition of the semi-norm on $B^{1,1}(\mathbb{R}^n)$, and two change of variables yields

$$\begin{aligned}|\rho_\epsilon * R_j(f)|_{B^{1,1}(\mathbb{R}^n)} &\lesssim |f_\epsilon|_{B^{1,1}(\mathbb{R}^n)} \\ &\leq |f|_{B^{1,1}(\mathbb{R}^n)},\end{aligned}$$

so that the claim follows from sending $\epsilon \to 0$ and using Fatou's lemma. □

Conversely, taking for granted that the Riesz transforms are bounded on the Besov spaces, in place of Taibleson's argument in Step 2 in the proof of Theorem 5.1, one has

Second proof of Theorem 5.1. **Step 2′:** To estimate the L^1 norm of $\frac{\partial^2 u}{\partial t \partial x_j}$, one can alternatively use Proposition 5.2. In particular, by differentiating the equality asserted in the proposition by x_j, we obtain

$$R_i\left(\frac{\partial^2 P_t}{\partial t \partial x_j}\right) = \frac{\partial^2 P_t}{\partial x_i \partial x_j}. \tag{5.9}$$

This relation, in combination with Proposition 5.3, yields the identity

$$\begin{aligned}\frac{\partial^2 u}{\partial t \partial x_j}(x,t) &= \int_{\mathbb{R}^n} \sum_i R_i\left(\frac{\partial^2 P_t}{\partial t \partial x_j}\right)(h) R_i(f)(x-h)\, dh \\ &= \int_{\mathbb{R}^n} \sum_i \frac{\partial^2 P_t}{\partial x_i \partial x_j}(h) R_i(f)(x-h)\, dh.\end{aligned}$$

An estimate for this mixed partial derivative of u can therefore be made by the same argument in Step 1 of the proof of Theorem 3.1, which results in the estimate

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+} \left|\frac{\partial^2 u}{\partial t \partial x_j}(x,t)\right| dxdt \\ &\qquad \lesssim \sum_i \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|R_i(f)(x+h) + R_i(f)(x-h) - 2R_i(f)(x)|}{|h|^{n+1}}\, dhdx.\end{aligned}$$

Finally, by Theorem 5.4, the right-hand side is bounded from above by $|f|_{B^{1,1}(\mathbb{R}^n)}$, up to a multiplicative constant. □

The following is the weighted, higher-order version of Theorem 5.6:

Theorem 5.6. *Let $m \in \mathbb{N}_0$ and $-1 < a < m$. Suppose that $f \in B^{m-a,1}(\mathbb{R}^n)$ and let u be given by (5.1). Then, one has*

$$\int_{\mathbb{R}^{n+1}_+} t^a |\nabla^{m+1} u(x,t)|\, dtdx \lesssim |f|_{B^{m-a,1}(\mathbb{R}^n)}.$$

Proof. Let $s := m - a > 0$ and $f \in B^{s,1}(\mathbb{R}^n)$.

Step 1: Assume that $s = k \in \mathbb{N}$ and that $f \in C_c^\infty(\mathbb{R}^n)$. Then for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k-1$, we have that $\frac{\partial^\alpha f}{\partial x^\alpha} \in C_c^\infty(\mathbb{R}^n)$. We claim that properties of harmonic functions and Taibleson's results allow the reduction to a single estimate which depends on the parity of $m-k+1$: When $m-k+1$ is even, we show that it suffices to prove that

$$\int_{\mathbb{R}_+^{n+1}} t^a \left| \frac{\partial}{\partial t} \frac{\partial^\beta u}{\partial x^\beta}(x,t) \right| \, dtdx \lesssim |f|_{B^{k,1}(\mathbb{R}^n)} \tag{5.10}$$

for any multi-index $\beta \in \mathbb{N}_0^n$ such that $|\beta| = m$, while when $m-k+1$ is odd we show instead it suffices to prove that

$$\int_{\mathbb{R}_+^{n+1}} t^a \left| \frac{\partial^{\beta'} u}{\partial x^{\beta'}}(x,t) \right| \, dtdx \lesssim |f|_{B^{k,1}(\mathbb{R}^n)} \tag{5.11}$$

for any multi-index $\beta' \in \mathbb{N}_0^n$ such that $|\beta'| = m+1$.

Indeed, any entry of the tensor $\nabla^{m+1}u(x,t)$ has either an odd or even number of derivatives in the normal variable t. Therefore iteration of the relation

$$\frac{\partial^2 u}{\partial t^2} = -\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$$

reduces the estimate to the case where there are either zero or one derivatives in t. In either case, Taibleson's Theorem [23, Theorem 1 on p. 420] allows to correct the final parity of the number of derivatives in the trace variable: For $m-k+1$ even, if there is no derivative in t one applies [23, Theorem 1 (a) on p. 420] to interchange a derivative in some x_j for a derivative in t, or leaves the quantity unchanged if there is one derivative in t, which reduces the estimate to the proof of the inequality (5.10); If $m-k+1$ is odd and there are no derivatives in t one leaves the quantity unchanged, or if there is one derivative in t, one applies [23, Theorem 1 (b) on p. 420] to interchange a derivative in some x_j for a derivative in t, which reduces the estimate to the proof of the inequality (5.11).

To show the estimate (5.10) for $m-k+1$ even, let $\beta = \gamma + \delta$ be a decomposition of the multi-index β with $|\gamma| = k-1 \geq 0$ and $|\delta| = m-k+1 = a+1 \geq 1$. Then

$$\frac{\partial}{\partial t} \frac{\partial^\beta u}{\partial x^\beta}(x,t) = \left(\frac{\partial}{\partial t} \frac{\partial^\delta P_t}{\partial x^\delta} * \frac{\partial^\gamma f}{\partial x^\gamma} \right)(x). \tag{5.12}$$

As $|\delta| = m-k+1$ is even, a repetition of the argument in Theorem 5.1 with the even function

$$\frac{\partial}{\partial t} \frac{\partial^\delta P_t}{\partial x^\delta}$$

in place of the mixed second partial derivatives of the Poisson kernel leads one to the desired bound, using the fact that (see the proof of Lemma 3.2)

$$\int_0^\infty t^a \left| \frac{\partial}{\partial t} \frac{\partial^\delta P_t}{\partial x^\delta}(h) \right| \, dt \lesssim \frac{1}{|h|^{n+1}}.$$

Similarly, for the case (5.11), let $\beta' = \gamma' + \delta'$ be a decomposition of the multi-index β' with $|\gamma'| = k-1 \geq 0$ and $|\delta'| = m-k+2 = a+2 \geq 2$,

$$\frac{\partial^{\beta'} u}{\partial x^{\beta'}}(x,t) = \left(\frac{\partial^{\delta'} P_t}{\partial x^{\delta'}} * \frac{\partial^{\gamma'} f}{\partial x^{\gamma'}} \right)(x). \tag{5.13}$$

As in this case $|\delta'| = m - k + 1 + 1$ is even, the argument is as before, where one uses

$$\int_0^\infty t^a \left| \frac{\partial^{\delta'} P_t}{\partial x^{\delta'}}(h) \right| \, dt \lesssim \frac{1}{|h|^{n+1}}.$$

A standard density argument in $B^{s,1}(\mathbb{R}^n)$ allows one to remove the additional hypothesis that $f \in C_c^\infty(\mathbb{R}^n)$. We omit the details.

Step 2: Assume that $s \notin \mathbb{N}$ and let $k := \lfloor s \rfloor$. Then for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$, we have that $\frac{\partial^\alpha f}{\partial x^\alpha} \in \dot{B}^{s-k,1}(\mathbb{R}^n)$. As in Step 1, to estimate the L^1 norm of $t^a \nabla^{m+1} u$, it suffices to prove the estimates (5.10) and (5.11). To show the estimate (5.10), we use (5.12) but now with $|\gamma| = k \geq 0$ and $|\delta| = m - k \geq 1$. Since $\int_{\mathbb{R}^n} \frac{\partial}{\partial t} \frac{\partial^\delta P_t}{\partial x^\delta}(h) \, dh = 0$, we can write

$$\left(\frac{\partial}{\partial t} \frac{\partial^\delta P_t}{\partial x^\delta} * \frac{\partial^\gamma f}{\partial x^\gamma} \right)(x) = \int_{\mathbb{R}^n} \frac{\partial}{\partial t} \frac{\partial^\delta P_t}{\partial x^\delta}(h) \Delta_{-h} \frac{\partial^\alpha f}{\partial x^\alpha}(x) \, dh.$$

Multiplying both sides by t^a and integrating in (x,t) gives

$$\begin{aligned} &\int_{\mathbb{R}^{n+1}_+} t^a \left| \left(\frac{\partial}{\partial t} \frac{\partial^\delta P_t}{\partial x^\delta} * \frac{\partial^\alpha f}{\partial x^\alpha} \right)(x) \right| dx dt \\ &\qquad \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^\infty t^a \left| \frac{\partial}{\partial t} \frac{\partial^\delta P_t}{\partial x^\delta}(h) \right| \left| \Delta_{-h} \frac{\partial^\alpha f}{\partial x^\alpha}(x) \right| dt dh dx, \end{aligned}$$

where we used Tonelli's theorem. As in the proof of Lemma 3.2, we have

$$\int_0^\infty t^a \left| \frac{\partial}{\partial t} \frac{\partial^\delta P_t}{\partial x^\delta}(h) \right| \, dt \lesssim \frac{1}{|h|^{n+|\delta|-a}} = \frac{1}{|h|^{n+s-k}}.$$

In turn,

$$\int_{\mathbb{R}^{n+1}_+} t^a \left| \left(\frac{\partial}{\partial t} \frac{\partial^\delta P_t}{\partial x^\delta} * \frac{\partial^\alpha f}{\partial x^\alpha} \right)(x) \right| dx dt \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \Delta_{-h} \frac{\partial^\alpha f}{\partial x^\alpha}(x) \right| \frac{dh}{|h|^{n+s-k}} dx.$$

Similarly, for the case (5.11), we use (5.13), with $|\gamma'| = k \geq 0$ and $|\delta'| = m+1-k \geq 2$, to write

$$\left(\frac{\partial^{\delta'} P_t}{\partial x^{\delta'}} * \frac{\partial^{\gamma'} f}{\partial x^{\gamma'}} \right)(x) = \int_{\mathbb{R}^n} \frac{\partial^{\delta'} P_t}{\partial x^{\delta'}}(h) \Delta_{-h} \frac{\partial^{\gamma'} f}{\partial x^{\gamma'}}(x) \, dh.$$

Since

$$\int_0^\infty t^a \left| \frac{\partial^{\delta'} P_t}{\partial x^{\delta'}}(h) \right| \, dt \lesssim \frac{1}{|h|^{n+|\delta'|-a-1}} = \frac{1}{|h|^{n+s-k}},$$

as before we have that

$$\int_{\mathbb{R}^{n+1}_+} t^a \left| \left(\frac{\partial^{\delta'} P_t}{\partial x^{\delta'}} * \frac{\partial^{\gamma'} f}{\partial x^{\gamma'}} \right)(x) \right| dx dt \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \Delta_{-h} \frac{\partial^{\gamma'} f}{\partial x^{\gamma'}}(x) \right| \frac{dh}{|h|^{n+s-k}} dx.$$

□

Remark 5.7. *It is possible to give a second proof of Theorem 5.6, which makes use of the boundedness of the Riesz transform in $B^{1,1}(\mathbb{R}^n)$, Theorem 5.4. The idea is similar to the second proof of Theorem 5.1.*

Appendix A. Homogeneous Sobolev Embeddings

In this appendix, we show that when $mp > n$ the homogeneous Sobolev space $\dot W^{m,p}(\mathbb{R}^n)$ is embedded into the homogeneous Besov space (or Zygmund space) $\dot B^{m-n/p,\infty}(\mathbb{R}^n)$. We also prove that a function in $\dot B^{m-n/p,\infty}(\mathbb{R}^n)$ has polynomial growth. While the latter result is probably known to experts, we have been unable to find a reference.

Theorem A.1. *Let $m \in \mathbb{N}$ and $1 < p < \infty$ be such that $mp > n$. Then*

$$|u|_{B^{m-n/p,\infty}(\mathbb{R}^n)} \lesssim \|\nabla_x^m u\|_{L^p(\mathbb{R}^n)} \tag{A.1}$$

for all $u \in \dot W^{m,p}(\mathbb{R}^n)$. Moreover, if $\bar u$ is a representative of u, then $\bar u$ has polynomial growth.

First proof. **Step 1:** Assume that $u \in C^\infty(\mathbb{R}^n)$. For every $x, h, y \in \mathbb{R}^n$ with $h \neq 0$, we use the identity

$$\Delta_h^m u(0) = \sum_{k=1}^m (-1)^{m-k}\tbinom{m}{k}\Delta_{(k/m)y}^m u((m-k)h) - (-1)^m \Delta_{h-(k/m)y}^m u(ky),$$

which can be proved using the binomial theorem (see the proof of Lemma 17.22 on p. 549 in [9]). Let $r = |h|$. Averaging in y over Q_r gives

$$\begin{aligned}|\Delta_h^m u(0)| &\le \sum_{k=1}^m \tbinom{m}{k}\frac{1}{r^n}\int_{Q_r} |\Delta_{(k/m)y}^m u((m-k)h)|\,dy \\ &\quad + \frac{1}{r^n}\int_{Q_r} |\Delta_{h-(k/m)y}^m u(ky)|\,dy := \sum_{k=1}^m \tbinom{m}{k}\mathcal{A}_k + \mathcal{B}.\end{aligned}$$

By the fundamental theorem of calculus and an induction argument,

$$\begin{aligned}&\Delta_{(k/m)y}^m u((m-k)h) \\ &\quad = \sum_{|\alpha|=m} ((k/m)y)^\alpha \int_{(0,1)^m} \frac{\partial^\alpha u}{\partial x^\alpha}((m-k)h + (t_1+\cdots+t_m)(k/m)y)\,dt.\end{aligned}$$

Hence,

$$\begin{aligned}|\mathcal{A}_k| &\lesssim \frac{1}{r^n}\int_{(0,1)^m}\int_{Q_r} |y|^m |\nabla_x^m u((m-k)h + (t_1+\cdots+t_m)(k/m)y)|\,dydt \\ &\lesssim \frac{1}{r^n}\int_{(0,1)^m} \frac{1}{(t_1+\cdots+t_m)^{n+m}} \\ &\qquad \times \int_{Q_{(t_1+\cdots+t_m)(k/m)r}} |z|^m |\nabla_x^m u((m-k)h+z)|\,dzdt \\ &\lesssim \frac{1}{r^n}\int_{B_m(0,\sqrt{m})} \frac{1}{|t|_m^{n+m}}\int_{Q_{|t|_m kr}} |z|^m |\nabla_x^m u((m-k)h+z)|\,dzdt\end{aligned}$$

where we made the change of variables $z = (t_1+\cdots+t_m)(k/m)y$, $dz = [(t_1 + \cdots + t_m)(k/m)^n dy$ and used the fact that $\sqrt{m}|t|_m \le t_1+\cdots+t_m \le m|t|_m$. Using Tonelli's theorem on the right-hand side, we obtain

$$|\mathcal{A}_k| \lesssim \frac{1}{r^n}\int_{Q_{\sqrt{m}kr}} |z|^m |\nabla_x^m u((m-k)h+z)| \left(\int_{E_z} \frac{1}{|t|_m^{n+m}}\,dt\right) dz,$$

where

$$E_z := \{t \in B_m(0, \sqrt{m}) : \, |z| < k\sqrt{n}|t|_m r\}.$$

By the change of variables $t = \frac{|z|}{r}\xi$, we have

$$\int_{E_z} \frac{1}{|t|_m^{n+m}} \, dt \leq \frac{r^n}{|z|^n} \int_{\mathbb{R}^m \setminus B_m(0,1/(k\sqrt{n}))} \frac{1}{|\xi|_m^{n+m}} \, d\xi \lesssim \frac{r^n}{|z|^n}.$$

Hence, also, by Hölder's inequality

$$\begin{aligned}
|\mathcal{A}_k| &\lesssim \int_{Q_{\sqrt{m}kr}} |z|^{m-n} |\nabla_x^m u((m-k)h + z)| \, dz \\
&\lesssim \left(\int_{Q_{\sqrt{m}kr}} |z|^{(m-n)p'} dz \right)^{1/p'} \left(\int_{Q_{\sqrt{m}kr}} |\nabla_x^m u((m-k)h + z)|^p dz \right)^{1/p} \\
&\lesssim r^{m-n/p} \|\nabla_x^m u\|_{L^p(Q_{cr})},
\end{aligned}$$

since

$$\begin{aligned}
\int_{\sqrt{m}kr} |z|^{(m-n)p'} dz \leq \int_{B(0,\sqrt{nm}r)} |z|^{(m-n)p'} dz = \beta_n \int_0^{\sqrt{nm}r} r^{n-1+(m-n)p'} dr \\
\lesssim r^{n+(m-n)p'}
\end{aligned}$$

and $n + (m-n)p' > 0$ because[5] $mp > n$.

The term $\mathcal{B}$ can be treated in a similar way. We omit the details. In conclusion, we have shown that

$$|\Delta_h^m u(0)| \lesssim |h|^{m-n/p} \|\nabla_x^m u\|_{L^p(\mathbb{R}^n)}.$$

By a translation, it follows that

$$|\Delta_h^m u(x)| \lesssim |h|^{m-n/p} \|\nabla_x^m u\|_{L^p(\mathbb{R}^n)}$$

for all $x, h \in \mathbb{R}^n$ with $h \neq 0$. This shows that

$$|u|_{B^{m-n/p,\infty}(\mathbb{R}^n)} \lesssim \|\nabla_x^m u\|_{L^p(\mathbb{R}^n)}.$$

The additional hypothesis that $u \in C^\infty(\mathbb{R}^n)$ can be removed using a mollification argument. Finally, observe that if $u \in \dot{W}^{m,p}(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$, then $u\varphi \in W^{m,p}(\mathbb{R}^n)$, and so by [9, Theorem 12.46, p. 378], $u\varphi \in L^\infty(\mathbb{R}^n)$. In particular, $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Hence, $u \in \dot{B}^{m-n/p,\infty}(\mathbb{R}^n)$.

Step 2: It remains to show that a function in $\dot{B}^{m-n/p,\infty}(\mathbb{R}^n)$ has polynomial growth. Without loss of generality, we may assume that $m - n/p \leq 1$. Indeed, if $m - n/p > 1$, let $\ell \in \mathbb{N}$ be such that $\ell \leq \max\{i \in \mathbb{N} : \, i < m - n/p\}$. Then by [9, Theorem 17.69, p. 575], u belongs to $\dot{B}^{m-n/p,\infty}(\mathbb{R}^n)$ if and only if for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = \ell$, the weak derivative $\partial^\alpha u$ belongs to $\dot{B}^{m-n/p-\ell,\infty}(\mathbb{R}^n)$. Moreover, if $\nabla^\ell u$ has polynomial growth, then so does u. Thus, in what follows we assume that $0 < m - n/p \leq 1$.

If $0 < m - n/p < 1$, then u has a representative $\bar{u}$ that is Hölder continuous with exponent $m - n/p$. In particular, $\bar{u}$ has polynomial growth.

Assume next that $m - n/p = 1$. Let $u \in \dot{B}^{1,\infty}(\mathbb{R}^n)$ and let $\bar{u}$ be a representative of u. Set

$$v(h) := \bar{u}(h) - \bar{u}(0), \quad h \in \mathbb{R}^n.$$

[5] $N(p-1) + (m-N)p = mp - N > 0$

Then for every $i \in \mathbb{Z}$ and every $h \in \mathbb{R}^n$,

$$|2^i v(2^{-i}h) - 2^{i-1}v(2^{-i+1}h)| = 2^{i-1}|\Delta_{2^{-i}h}\bar{u}(0)| \le |u|_{B^{1,\infty}(\mathbb{R}^n)}|h|.$$

It follows that for every $\ell \in \mathbb{Z} \setminus \{0\}$,

$$|v(h) - 2^\ell v(2^{-\ell}h)| \le \sum_{i=1}^{\ell} |2^i v(2^{-i}h) - 2^{i-1}v(2^{-i+1}h)| \le |u|_{B^{1,\infty}(\mathbb{R}^n)}\ell|h|$$

if $\ell > 0$, while

$$|v(h) - 2^\ell v(2^{-\ell}h)| \le \sum_{i=\ell-1}^{0} |2^i v(2^{-i}h) - 2^{i-1}v(2^{-i+1}h)| \le |u|_{B^{1,\infty}(\mathbb{R}^n)}|\ell||h|$$

if $\ell < 0$. Hence,

$$\begin{aligned} |v(2^{-\ell}h)| &\le |v(h)|2^{-\ell} + |v(h) - 2^\ell v(2^{-\ell}h)|2^{-\ell} \\ &\le |v(h)|2^{-\ell} + |u|_{B^{1,\infty}(\mathbb{R}^n)}|\ell|2^{-\ell}|h|. \end{aligned}$$

Given $x \in \mathbb{R}^n \setminus \{0\}$, let $\ell \in \mathbb{Z}$ be such that $2^{-\ell-1} \le |x| < 2^{-\ell}$. Taking $h = 2^\ell x$ gives

$$|v(x)| \le 2|v(2^\ell x)||x| + \frac{1}{\log 2}|u|_{B^{1,\infty}(\mathbb{R}^n)}|x|(1 + |\log|x||). \tag{A.2}$$

Hence,

$$\begin{aligned} |\bar{u}(x)| &\le |\bar{u}(0)| + |v(x)| \\ &\le |\bar{u}(0)| + \sup_{B(0,1)} |\bar{u} - \bar{u}(0)||x| + \frac{1}{\log 2}|u|_{B^{1,\infty}(\mathbb{R}^n)}|x|(1 + |\log|x||), \end{aligned}$$

where we used the fact that $|2^\ell x| < 1$. □

Remark A.2. *We refer to Peetre* [18, Theorem 8.2] *for the original proof of Step 1, which relies on interpolation theory and on an identity of the type* (A.3) *below. Step 2 is adapted from a paper of Krantz* [8, Lemma 2.8]. *Note that by* (A.2) *and a translation,*

$$|\bar{u}(x_0 + h) - \bar{u}(x_0)| \le \sup_{B(x_0,1)} |\bar{u} - \bar{u}(x_0)||h| + \frac{1}{\log 2}|u|_{B^{1,\infty}(\mathbb{R}^n)}|h|(1 + |\log|h||).$$

It is possible to provide a shorter proof of the fact that a function in $u \in \dot{W}^{m,p}(\mathbb{R}^n)$ has polynomial growth. This proof uses a representation result in Mizuta's book [16, Theorem 1.3 on p. 207], which makes use of the theory of singular integrals.

Second proof. By [16, Theorem 1.3 on p. 207], for $v \in \dot{W}^{m,p}(\mathbb{R}^n)$ there exists a polynomial P of degree at most $m-1$ such that one has for almost every $x \in \mathbb{R}^n$,

$$v(x) - P(x) = \sum_{|\lambda|=m} a_\lambda \int_{\mathbb{R}^n} k_{\lambda,l}(x,y) \frac{\partial^\lambda v}{\partial y^\lambda}(y)\, dy, \tag{A.3}$$

where $l \le m - n/p < l + 1$ and

$$k_{\lambda,l}(x,y) := \begin{cases} k_\lambda(x-y) & \text{if } |y| < 1, \\ k_\lambda(x-y) - \sum_{|\alpha|\le l} \frac{x^\alpha}{\alpha!}\frac{\partial^\alpha k_\lambda}{\partial y^\alpha}(-y) & \text{if } |y| \ge 1, \end{cases}$$

with $k_\lambda(x) := \frac{x^\lambda}{|x|^n}$. By [16, Lemma 1.2 on p. 207],

$$|k_{\lambda,l}(x,y)| \lesssim |x|^{l+1}|y|^{|\lambda|-n-l-1} \tag{A.4}$$

for $|y| \geq 1$ and $|y| > 2|x|$.

We claim that

$$|v(x) - P(x)| \leq |\tilde{P}(x)|$$

for some polynomial $\tilde{P}$. However, this is between the lines of Mizuta's argument [16, Proof of Theorem 1.3 on p. 207]. If $|x| \leq 1/2$, one splits the integral into two pieces, uses the bound for $k_\lambda(x-y)$ for the local piece and (A.4) for the global piece, and Hölder's inequality to obtain

$$\begin{aligned}
&|v(x) - P(x)| \\
&\lesssim \int_{B(0,1)} |k_{\lambda,l}(x,y)||\nabla_y^m v(y)| \; dy + \int_{\mathbb{R}^n\setminus B(0,1)} |k_{\lambda,l}(x,y)||\nabla_y^m v(y)| \; dy \\
&\lesssim \int_{B(0,1)} |x-y|^{|\lambda|-n}|\nabla_y^m v(y)| \; dy + |x|^{l+1}\int_{\mathbb{R}^n\setminus B(0,1)} |y|^{|\lambda|-n-l-1}|\nabla_y^m v(y)| \; dy \\
&\lesssim |P_1(x)|.
\end{aligned}$$

Here one uses that

$$(|\lambda| - n - l - 1)p' + n = (m - n - l - 1)p' + n < 0$$

because $m - n/p < l + 1$.

When $|x| > 1/2$ we use a similar splitting, along with Hölder's inequality

$$\begin{aligned}
&|v(x) - P(x)| \\
&\lesssim \int_{B(0,2|x|)} |k_{\lambda,l}(x,y)||\nabla_y^m v(y)| \; dy + \int_{\mathbb{R}^n\setminus B(0,2|x|)} |k_{\lambda,l}(x,y)||\nabla_y^m v(y)| \; dy \\
&\lesssim \int_{B(0,2|x|)} |k_\lambda(x-y)||\nabla_y^m v(y)| \; dy + |P_2(x)| \\
&\quad + |x|^{l+1}\int_{\mathbb{R}^n\setminus B(0,2|x|)} |y|^{|\lambda|-n-l-1}|\nabla_y^m v(y)| \; dy \lesssim |P_3(x)|,
\end{aligned}$$

where we have also made use of Mizuta's observation that for $f \in L^p(\mathbb{R}^n)$,

$$\int_{B(0,2|x|)} k_{\lambda,l}(x,y)f(y) \; dy = \int_{B(0,2|x|)} k_\lambda(x-y)f(y) \; dy + \text{ a polynomial.}$$

□

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