ON PLURISUBHARMONIC DEFINING FUNCTIONS FOR PSEUDOCONVEX DOMAINS IN $\mathbb{C}^2$

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Abstract. We investigate the question of existence of plurisubharmonic defining functions for smoothly bounded, pseudoconvex domains in $\mathbb{C}^2$. In particular, we construct a family of simple counterexamples to the existence of plurisubharmonic smooth local defining functions. Moreover, we give general criteria equivalent to the existence of plurisubharmonic smooth defining functions on or near the boundary of the domain. These equivalent characterizations are then explored for some classes of domains.

1. Introduction

In this paper, we investigate the question of existence of plurisubharmonic smooth defining functions for pseudoconvex domains with smooth boundary. This basic question has been long resolved for strictly pseudoconvex domains. However, there is a great lack of understanding in the case of weakly pseudoconvex domains, although the existence of plurisubharmonic defining functions is of relevance, e.g., for the classification problem of domains in $\mathbb{C}^n$.

It is a basic fact that a smoothly bounded domain in $\mathbb{C}^n$ has a plurisubharmonic smooth local defining function near a boundary point if the domain is strictly pseudoconvex or convex near that point; similarly, the domain admits a smooth plurisubharmonic defining function near the boundary if it is strictly pseudoconvex or convex at each boundary point. No other geometric conditions which are sufficient for the existence of plurisubharmonic smooth (local) defining functions are known. Moreover, the existence of local or global plurisubharmonic defining functions may fail on pseudoconvex domains with weakly pseudoconvex boundary points. For instance, the worm domain, constructed by Diederich–Fornæss in [3], does not admit a plurisubharmonic global defining function, although it does admit plurisubharmonic local defining functions near each boundary point. Further, Fornæss [6] constructs a smoothly bounded, pseudoconvex domain in $\mathbb{C}^3$ for which all $\mathcal{C}^2$-smooth local defining functions fail to be plurisubharmonic on the boundary near some boundary point. Behrens [1] gives an example of a pseudoconvex domain in $\mathbb{C}^2$ with real-analytic boundary which exhibits the same failure of plurisubharmonicity of $\mathcal{C}^6$-smooth local defining functions. Behrens' domain is of type 6 at the boundary point in question. No such examples are known for pseudoconvex domains which are of type 4 at the considered boundary point.

In the first part of this paper, we construct a family of counterexamples in the spirit of Behrens' example in [1]. Namely, we construct domains $\Omega_{2k} \subset \mathbb{C}^2$, $k \geq 3$,

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Date: March 7, 2025.

2020 *Mathematics Subject Classification.* 32Txx, 32U05, 32T25.

Key words and phrases. Plurisubharmonic defining functions, finite type.

such that Ω_{2k} is a domain with real-analytic boundary that is pseudoconvex and of type $2k$ at some boundary point p but any $\mathcal{C}^2$-smooth defining function of Ω near p fails to be plurisubharmonic on the boundary.

The second part of the paper is concerned with introducing new geometric conditions that are sufficient for the existence of plurisubharmonic smooth (local) defining functions. We first give equivalent, yet non-geometric, characterizations for the existence of plurisubharmonic smooth local defining functions, both on and near the boundary, see Proposition 5.2 and Proposition 6.8. These characterizations are then exploited to show the existence of plurisubharmonic defining functions under each of two newly introduced geometric conditions for smoothly bounded pseudoconvex domains.

The first condition pertains to a new class of weakly pseudconvex boundary points. We first show that if $\Omega \subset \mathbb{C}^2$ is a pseudoconvex domain with smooth boundary, then the Levi form λ of Ω satisfies

$$(L\bar{L}\lambda)(p) \geq |(LL\lambda)(p)| \tag{1.1}$$

at points $p \in b\Omega$ where Ω is weakly pseudoconvex, for all tangential $(1,0)$-vector fields L near p, see Theorem 4.10. Through (1.1), we may then classify weakly pseudoconvex boundary points p of Ω into two groups as follows: we call p *non-degenerate* if inequality (1.1) is strict, and we say that p is *degenerate* otherwise. This classification is easily seen to be invariant under biholomorphic coordinate changes, see Lemma 4.14. We then give a description of pseudoconvex domains near non-degenerate weakly pseudoconvex boundary points in suitable local holomorphic coordinates in Proposition 4.16. In this form, the notion of non-degenerate weakly pseudconvex boundary points has appeared, albeit implicitly, in the literature. Namely, Kolář showed in [12] that a pseudoconvex domain with real-analytic boundary is convexifiable near any non-degenerate weakly pseudoconvex boundary point. He also states in [12] that this result does not hold if the real-analyticity assumption is replaced by mere $\mathcal{C}^\infty$-smoothness of the boundary. A proof of this statement is not provided in [12].

We show that if $\Omega \subset \mathbb{C}^2$ is a smoothly bounded, pseudoconvex domain, then near any non-degenerate weakly pseudoconvex boundary point p there exists a smooth local defining function for Ω near p which is plurisubharmonic on $b\Omega$, see Theorem 5.1. This local defining function is constructed explicitly by solving the following first order partial differential equation on $b\Omega$ near p. For a given, smooth, complex-valued function F near p, find a smooth, real-valued function h near p such that

$$Lh = F + \mathcal{O}(\sqrt{\lambda}) \quad \text{on } b\Omega \text{ near } p.$$

We also prove that if $\Omega \subset \mathbb{C}^2$ admits a smooth defining function which is plurisubharmonic on $b\Omega$ near a type 4 boundary point p (note that every non-degenerate weakly pseudoconvex boundary point is of type 4), then Ω admits a plurisubharmonic smooth local defining function near p, see Theorem 6.19. To construct this defining function, we solve the following system of partial differential equations on $b\Omega$ near p. For a given, smooth, real-valued function F near p, find a smooth, real-valued function h near p such that

$$\begin{cases} Lh = \mathcal{O}(\sqrt{\lambda}) \\ L\bar{L}h = F + \mathcal{O}(\sqrt{\lambda}) \end{cases} \quad \text{on } b\Omega \text{ near } p.$$

While this method of proof in general fails for degenerate weakly pseudoconvex boundary points, parts of our analysis can be salvaged to give a simplified, short proof of the fact that the Diederich–Fornæss index and Steinness index of $\Omega \subset \mathbb{C}^2$ are 1 if Ω admits a smooth defining function that is plurisubharmonic on $b\Omega$, but does not have type 4 boundary points, see Corollary 6.24.

For the second application of our general characterizations of the existence of plurisubharmonic defining functions, see Propositions 5.2 and 6.8, we introduce the notion of sesquiconvexity, a new geometric condition for an open set in $\mathbb{C}^2$ which may be formulated independently of the choice of a defining function. Sesquiconvexity is sufficient for the existence of (local) defining functions that are plurisubharmonic on the boundary (and inside the domain), see Proposition 7.10 and Corollary 7.15, however, it is not a necessary condition, see Example 7.6.

2. Preliminaries

In this section, we detail our notations and list some known facts for later reference. We note that Section 3 requires almost no prerequisites. As such, readers familiar with basic notions from several complex variables may delay looking through the current section until the study of Section 4 and onward.

The generic term "smooth" always means $\mathcal{C}^\infty$-smooth. Domains and functions of finite smoothness classes will only be considered in Section 3.

2.1. **Some basic notions from almost complex geometry.** Let (M, J) be an almost complex manifold. For every $p \in M$, write $T_p(M)$ for the real tangent space to M at p. As usual, the complexification $T_p(M)^{\mathbb{C}} := T_p(M) \otimes \mathbb{C}$ decomposes as $T_p(M)^{\mathbb{C}} = T_p(M)^{1,0} \oplus T_p(M)^{0,1}$ into the $+i$ and $-i$ eigenspaces of J_p. Smooth sections over M in the corresponding bundles $T(M)$, $T(M)^{\mathbb{C}}$, $T(M)^{1,0}$, $T(M)^{0,1}$ are denoted by $\mathcal{V}(M)$, $\mathcal{V}(M)^{\mathbb{C}}$, $\mathcal{V}(M)^{1,0}$, $\mathcal{V}(M)^{0,1}$, and are called, real, complexified, $(1,0)$- and $(0,1)$-vector fields on M, respectively.

Let $T_p^*(M)$ denote the real cotangent space of M at p, and let $J_p^*\colon T_p^*(M) \to T_p^*(M)$ be the dual almost complex structure. The complexification $T_p^*(M)_{\mathbb{C}} := T_p^*(M) \otimes \mathbb{C}$ decomposes as $T_p^*(M)_{\mathbb{C}} = T_p^*(M)_{1,0} \oplus T_p^*(M)_{0,1}$ into the $+i$ and $-i$ eigenspaces of J_p^*. Smooth sections over M in the corresponding bundles $T^*(M)$, $T^*(M)_{\mathbb{C}}$, $T^*(M)_{1,0}$, $T^*(M)_{0,1}$ are denoted by $\Omega^1(M)$, $\Omega^1(M)_{\mathbb{C}}$, $\Omega^1(M)_{1,0}$, $\Omega^1(M)_{0,1}$, and are called, real, complexified, $(1,0)$- and $(0,1)$-forms on M, respectively.

For $\alpha \in \Omega^1(M)$, write $\alpha_{\mathbb{C}}$ to denote the pointwise $\mathbb{C}$-linear extension of α to $\Omega^1(M)_{\mathbb{C}}$. Then $\alpha_{1,0} := \alpha_{\mathbb{C}} - i(J^*\alpha)_{\mathbb{C}}$ and $\alpha_{0,1} := \alpha_{\mathbb{C}} + i(J^*\alpha)_{\mathbb{C}}$ define elements in $\Omega^1(M)_{1,0}$ and $\Omega^1(M)_{0,1}$, respectively. If $f\colon M \to \mathbb{C}$ is a smooth function, write $df \in \Omega^1(M)$ to denote the differential of f, and set $\partial f := \frac{1}{2}(df)_{1,0}$ and $\bar\partial f := \frac{1}{2}(df)_{0,1}$. Clearly, $(df)_{\mathbb{C}} = \partial f + \bar\partial f$ holds.

If M is an open subset in $\mathbb{C}^n$ with coordinates $(z^1, \dots, z^n)$, $z^j = x^j + iy^j$, $x^j, y^j \in \mathbb{R}$, and if J is the standard almost complex structure, then

$$\partial f = \sum_{j=1}^n \frac{\partial f}{\partial z^j} dz^j, \qquad \bar\partial f = \sum_{j=1}^n \frac{\partial f}{\partial \bar z^j} d\bar z^j,$$

where

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right), \qquad \frac{\partial}{\partial \bar z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right),$$

and

$$dz^j = dx^j + i dy^j, \qquad d\bar{z}^j = dx^j - i dy^j.$$

Here, and in what follows, we always suppress the index $\mathbb{C}$ and use the same symbol to denote the vector field $V \in \mathcal{V}(M)$ and its complexification $V^{\mathbb{C}} := V \otimes 1 \in \mathcal{V}(M)^{\mathbb{C}}$, and the 1-form $\alpha \in \Omega^1(M)$ and its complexification $\alpha_{\mathbb{C}} \in \Omega^1(M)_{\mathbb{C}}$, respectively. Euclidean inner products on $\mathcal{V}(M)$ and $\Omega^1(M)$ are introduced by declaring that $(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n})$ and $(dx^1, dy^1, \dots, dx^n, dy^n)$ are orthonormal bases for $\mathcal{V}(M)$ and $\Omega^1(M)$, respectively. Similarly, Hermitian inner products on $\mathcal{V}(M)^{\mathbb{C}}$ and $\Omega^1(M)_{\mathbb{C}}$ are introduced by declaring that $(\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n})$ is an orthogonal basis for $\mathcal{V}(M)^{\mathbb{C}}$ with vectors of constant length $1/\sqrt{2}$, and $(dz^1, \dots, dz^n, d\bar{z}^1, \dots, d\bar{z}^n)$ is an orthogonal basis for $\Omega^1(M)_{\mathbb{C}}$ with vectors of constant length $\sqrt{2}$. Note that the canonical inclusions $\mathcal{V}(M) \hookrightarrow \mathcal{V}(M)^{\mathbb{C}}$, $V \mapsto V^{\mathbb{C}}$, and $\Omega^1(M) \hookrightarrow \Omega^1(M)_{\mathbb{C}}$, $\alpha \mapsto \alpha_{\mathbb{C}}$, are isometries, so the notations $|V|$ and $|\alpha|$ for the corresponding norms are well-defined, even if the index $\mathbb{C}$ is suppressed in the notation.

2.2. **Some basic notions from differential geometry.** Let M be a smooth manifold. For every tensor field γ on M and every $p \in M$, let $\gamma_p = \gamma(p)$ denote the value of γ at p. If $\alpha \in \Omega^1(M)$ and $V \in \mathcal{V}(M)$, then $\langle \alpha, V \rangle := \alpha(V)$. Given smooth vector fields $V_1, \dots, V_k$ on M and a smooth function $f \colon M \to \mathbb{C}$, we write $V_k \dots V_1 f := V_k(\dots(V_1 f)\dots)$. Moreover, $[V_1, V_2]$ denotes the Lie bracket of V_1 and V_2, i.e., $[V_1, V_2]f = V_1 V_2 f - V_2 V_1 f$.

Let $M \subset \mathbb{R}^N$ be open and let $(t^1, \dots, t^N)$ be the standard Euclidean coordinates on M. For $V, W \in \mathcal{V}(M)$, $V = \sum_{\nu=1}^N V^\nu \frac{\partial}{\partial t^\nu}$, $W = \sum_{\nu=1}^N W^\nu \frac{\partial}{\partial t^\nu}$, we set

$$\nabla_V W = \sum_{\nu=1}^N \left(\sum_{\mu=1}^N V^\mu \frac{\partial W^\nu}{\partial t^\mu} \right) \frac{\partial}{\partial t^\nu}.$$

Note that $\nabla_V W$ is precisely the covariant derivative of W along V with respect to the Levi-Civita connection in $T(M)$ corresponding to the standard Riemannian metric $g = \sum_{\nu=1}^N dt^\nu \otimes dt^\nu$ on $T(M)$. If for $f \in \mathcal{C}^\infty(M, \mathbb{R})$ we write

$$Q_f^{\mathbb{R}}(V, W) = \sum_{\nu,\mu=1}^N \frac{\partial^2 f}{\partial t^\nu \partial t^\mu} V^\nu W^\mu,$$

then

$$VWf = Q_f^{\mathbb{R}}(V, W) + (\nabla_V W) f. \tag{2.1}$$

Let $I \subset \mathbb{R}$ be an open interval, and let $\gamma \colon I \to M$ be a smooth curve. Then $\dot{\gamma}, \ddot{\gamma} \colon I \to T(M)$ denote the vector fields along γ given by $\dot{\gamma}(\tau) := \sum_{\nu=1}^N \gamma_\nu'(\tau) (\frac{\partial}{\partial t_\nu})_{\gamma(\tau)}$ and $\ddot{\gamma}(\tau) := \sum_{\nu=1}^N \gamma_\nu''(\tau) (\frac{\partial}{\partial t_\nu})_{\gamma(\tau)}$. Observe that $\ddot{\gamma}$ is the covariant derivative of $\dot{\gamma}$ with respect to ∇.

Let $M \subset \mathbb{C}^n$ be open and let $(z^1, \ldots, z^n)$ be the standard Euclidean coordinates on M. For $V, W \in \mathcal{V}(M)^{1,0}$, $V = \sum_{j=1}^n V^j \frac{\partial}{\partial z^j}$, $W = \sum_{j=1}^n W^j \frac{\partial}{\partial z^j}$, we set

$$\nabla_V W = \sum_{j=1}^n \left(\sum_{k=1}^n V^k \frac{\partial W^j}{\partial z^k} \right) \frac{\partial}{\partial z^j}, \quad \nabla_{\bar{V}} \bar{W} := \overline{\nabla_V W},$$

$$\nabla_{\bar{V}} W = \sum_{j=1}^n \left(\sum_{k=1}^n \bar{V}^k \frac{\partial W^j}{\partial \bar{z}^k} \right) \frac{\partial}{\partial z^j}, \quad \nabla_V \bar{W} := \overline{\nabla_{\bar{V}} W}.$$

Note that $\nabla_V W$ and $\nabla_{\bar{V}} W$ are precisely the covariant derivatives of W along V and $\bar{V}$ with respect to the Chern connection in $T(M)^{1,0}$ corresponding to the standard Hermitian metric $h = \sum_{j=1}^n dz^j \otimes d\bar{z}^j$ on $T(M)^{1,0}$, respectively. If for $f \in \mathcal{C}^\infty(M, \mathbb{C})$ we write

$$Q_f^{\mathbb{C}}(V, W) = \sum_{j,k=1}^n \frac{\partial^2 f}{\partial z^j \partial z^k} V^j W^k,$$

$$H_f^{\mathbb{C}}(V, W) = \sum_{j,k=1}^n \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} V^j \bar{W}^k,$$

then

$$VWf = Q_f^{\mathbb{C}}(V, W) + (\nabla_V W) f, \tag{2.2}$$

$$V\bar{W}f = H_f^{\mathbb{C}}(V, W) + (\nabla_V \bar{W}) f. \tag{2.3}$$

Observe that the above formula for $H_f^{\mathbb{C}}(V, W)$ defines a map

$$H_f^{\mathbb{C}} \colon \mathcal{V}(M)^{1,0} \times \mathcal{V}(M)^{1,0} \to \mathbb{C}, \quad H_f^{\mathbb{C}}(V, W) = \partial\bar{\partial} f(V, \bar{W}).$$

If f is real-valued, then $H_f^{\mathbb{C}}$ is a sesquilinear form on the $\mathcal{C}^\infty(M, \mathbb{C})$-module $\mathcal{V}(M)^{1,0}$.

The above notations for the Levi-Civita connection and the Chern connection on an open set $M \subset \mathbb{R}^{2n} \simeq \mathbb{C}^n$ are unambiguous in the following sense. If $A, B, C, D \in \mathcal{V}(M)$ such that $C + iD \in \mathcal{V}(M)^{1,0}$, then

$$\nabla_{A+iB}(C + iD) = (\nabla_A C + i\nabla_A D) + i(\nabla_B C + i\nabla_B D),$$

where on the left-hand side ∇ denotes the Chern connection, and on the right-hand side ∇ denotes the Levi-Civita connection.

2.3. **Defining functions, pseudoconvexity, and finite type.** Let $\Omega \subset \mathbb{C}^n$ be a $\mathcal{C}^2$-smoothly bounded domain and let p_0 in $b\Omega$. We say that a $\mathcal{C}^2$-smooth function $r \colon U \to \mathbb{R}$ is a defining function for Ω, if

(i) U is an open neighborhood of $b\Omega$,
(ii) $\Omega \cap U = \{r < 0\}$,
(iii) $dr \neq 0$ on $b\Omega$.

Moreover, we say that a smooth function $r \colon U \to \mathbb{R}$ is a local defining function for Ω (near p_0), if U is an open neighborhood of p_0, and r satisfies (ii) and (iii).

For every $p \in b\Omega$, set $T_p(b\Omega)^{1,0} := T_p(\mathbb{C}^n)^{1,0} \cap T_p(b\Omega)^{\mathbb{C}}$. A vector field $L \in \mathcal{V}(U)^{1,0}$ is called tangential if $L_p \in T_p(b\Omega)^{1,0}$ for every $p \in b\Omega \cap U$. If $r \colon U \to \mathbb{R}$ is a local defining function for Ω, then $T_p(b\Omega)^{1,0} = \{L_p \in T_p(\mathbb{C}^n)^{1,0} : \langle \partial r, L \rangle(p) = 0\}$ for every $p \in b\Omega \cap U$, and L is tangential if and only if $Lr \equiv 0$ on $b\Omega \cap U$.

The domain Ω is called pseudoconvex at p if $H_r^{\mathbb{C}}(L, L)_{|_{b\Omega \cap U}} \geq 0$ near p for all tangential vector fields $L \in \mathcal{V}(U)^{1,0}$ near p. It is called strictly pseudoconvex at p

if in the previous condition the inequality is strict whenever L is nonvanishing. If Ω is pseudoconvex at p but not strictly pseudoconvex at p, then Ω is said to be weakly pseudoconvex at p.

Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded domain, and let $p \in b\Omega$. Let $r\colon U \to \mathbb{R}$ be a smooth local defining function for Ω near p, let L be a nonvanishing tangential $(1,0)$-vector field near p, and let $\lambda := H_r^{\mathbb{C}}(L,L)_{|b\Omega\cap U}$. Then the following conditions are equivalent, and independent of the choices of r and L.

(1) There exists $k \in \mathbb{N}$ such that
$$\exists\, L_1,\dots,L_k \in \{L,\bar{L}\} : \langle \partial r, [\dots[[L_1,L_2],L_3],\dots,L_k]\rangle\,(p) \neq 0,$$
and k is the smallest integer with this property.

(2) There exists $k \in \mathbb{N}$ such that
$$\exists\, L_1,\dots,L_{k-2} \in \{L,\bar{L}\} : (L_{k-2}\dots L_1\lambda)(p) \neq 0,$$
and k is the smallest integer with this property.

(3) There exists $k \in \mathbb{N}$ and local holomorphic coordinates z, w centered at p, such that
$$r(z,w) = \mathrm{Re}(w) + h(z,\bar{z}) + o(|z|^k, \mathrm{Im}(w)), \tag{2.4}$$
where h is a nonvanishing homogeneous polynomial of degree k without pure terms.

The original definition (1) is given in [11, Definition 2.3]. For the equivalence of (1) and (2), see [11, Proposition 2.8]. The third characterization is implicitly contained in the proof of [11, Lemma 3.16]; see also [2, Theorem 3.3].

If the above properties are satisfied, then $b\Omega$ is said to be of finite type at p, and the number $c_p := c_p(b\Omega) := k$ is called the type of $b\Omega$ at p. If Ω is pseudoconvex at p, then c_p is an even number, see [11, Theorem 3.1], Ω is strictly pseudoconvex at p if and only if $c_p = 2$, and Ω is weakly pseudoconvex at p if and only if $c_p \geq 4$.

Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain, and let $U \subset \mathbb{C}^n$ be open. A $\mathcal{C}^2$-smooth function $r\colon U \to \mathbb{R}$ is called plurisubharmonic if $H_r^{\mathbb{C}}(V,V) \geq 0$ for every $V \in \mathcal{V}(U)^{1,0}$, and it is called strictly plurisubharmonic if $H_r^{\mathbb{C}}(V,V) > 0$ for every $V \in \mathcal{V}(U)^{1,0}$, $V \neq 0$. Moreover, we say that r is plurisubharmonic on $b\Omega \cap U$ if $H_r^{\mathbb{C}}(V,V)_{|b\Omega\cap U} \geq 0$ for every $V \in \mathcal{V}(U)^{1,0}$, and we say that r is strictly plurisubharmonic on $b\Omega \cap U$ if $H_r^{\mathbb{C}}(V,V)_{|b\Omega\cap U} > 0$ for every $V \in \mathcal{V}(U)^{1,0}$, $V \neq 0$. Note that if $p \in b\Omega \cap U$ and r is plurisubharmonic on $b\Omega \cap U$, then in general it does not follow that r is plurisubharmonic on any open neighborhood $U' \subset U$ of p, even if r is a local defining function for Ω.

2.4. **Canonical vector fields in $\mathbb{C}^2$.** Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded domain, and let $r\colon U \to \mathbb{R}$ be a smooth local defining function for Ω. After possibly shrinking U, we may assume that $dr \neq 0$ on U. In this case, define vector fields $L_r, N_r \in \mathcal{V}^{1,0}(U)$ by
$$L_r = \frac{\sqrt{2}}{|\partial r|}\left(r_w \frac{\partial}{\partial z} - r_z \frac{\partial}{\partial w}\right), \tag{2.5}$$
$$N_r = \frac{\sqrt{2}}{|\partial r|}\left(r_{\bar{z}} \frac{\partial}{\partial z} + r_{\bar{w}} \frac{\partial}{\partial w}\right). \tag{2.6}$$

Then

(i) (L_r, N_r) is an orthogonal frame for $T(U)^{1,0}$ such that $|L_r| \equiv \frac{1}{\sqrt{2}} \equiv |N_r|$,

(ii) $L_r r \equiv 0$,
(iii) $N_r r = \frac{|\partial r|}{\sqrt{2}}$.

Note that, if ρ is some other smooth local defining function for Ω on U, then

$$L_\rho = L_r \quad \text{and} \quad N_\rho = N_r \quad \text{on } b\Omega \cap U. \tag{2.7}$$

We will always use the notations $L = L_r$ and $N = N_r$ without explicit reference to the choice of the defining function r, if we consider these vector fields only on $b\Omega$.

We will sometimes use the abbreviated notations L and N also for the vector fields on the whole open set U, if it is clear from the context which defining function we are referring to. As an example, given a fixed local defining function for Ω as above, we introduce $X, Y, T, \nu \in \mathcal{V}(U)$ to be the unique real vector fields such that

$$L = \tfrac{1}{2}(X + iY) \quad \text{and} \quad N = \tfrac{1}{2}(\nu + iT).$$

It follows from properties (ii) and (iii) above, that

$$Xr = Yr = Tr = 0 \quad \text{and} \quad \nu = \frac{\operatorname{grad} r}{|\operatorname{grad} r|}. \tag{2.8}$$

By property (i), and since $Y = -JX$ and $T = -J\nu$, the vector fields X, Y, T, ν are linearly independent at each point $q \in U$. In particular, it follows from (2.7) and (2.8) that the restrictions of X, Y, T to $b\Omega \cap U$ define a frame for $T(b\Omega \cap U)$, which is independent of the choice of r.

We write $\mathcal{H}_r^{\mathbb{C}}$ to denote the matrix associated with $H_r^{\mathbb{C}} \colon \mathcal{V}(U)^{1,0} \times \mathcal{V}(U)^{1,0} \to \mathbb{C}$ relative to the basis (L, N), i.e.,

$$\mathcal{H}_r^{\mathbb{C}} := \begin{pmatrix} H_r^{\mathbb{C}}(L,L) & H_r^{\mathbb{C}}(L,N) \\ H_r^{\mathbb{C}}(N,L) & H_r^{\mathbb{C}}(N,N) \end{pmatrix}.$$

The function r is plurisubharmonic if and only if $\mathcal{H}_r^{\mathbb{C}}(q)$ is positive semi-definite at every point $q \in U$.

2.5. **Landau symbols.** Let M be a smooth manifold, $p_0 \in M$, and $f, g \colon M \to \mathbb{R}$ be smooth functions. We use the usual notations

$$f = o(g) \text{ for } p \to p_0 \quad :\Leftrightarrow \quad \forall\, C > 0 \,:\, |f| \le C|g| \text{ in some neighborhood of } p_0,$$
$$f = \mathcal{O}(g) \text{ for } p \to p_0 \quad :\Leftrightarrow \quad \exists\, C > 0 : |f| \le C|g| \text{ in some neighborhood of } p_0.$$

If it is clear from the context, we usually drop the explicit reference to the point p_0. Moreover, we write for $U \subset M$ open

$$f = \mathcal{O}(g) \text{ on } U \quad :\Leftrightarrow \quad \forall\, K \Subset U \,\exists\, C > 0 : |f| \le C|g| \text{ on } K.$$

Roughly speaking, the condition "$f = \mathcal{O}(g)$ on U" means that f has at least the same order of vanishing on U as g, but it does not contain any information about the growth of f near the boundary of U. Similarly, we write

$$f \le \mathcal{O}(g) \text{ on } U \quad :\Leftrightarrow \quad \forall\, K \Subset U \,\exists\, C > 0 : f \le C|g| \text{ on } K.$$

Remark 2.9. Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain. If $r, \rho \colon U \to \mathbb{R}$ are two local defining functions for Ω, and if L, L' are two nonvanishing tangential $(1,0)$-vector fields on U, then there exists a nonvanishing smooth function $h \colon U \to \mathbb{R}$ such that $H_r^{\mathbb{C}}(L,L)_{|b\Omega\cap U} = hH_\rho^{\mathbb{C}}(L',L')_{|b\Omega\cap U}$. In particular, if we write $\lambda := H_r^{\mathbb{C}}(L,L)_{|b\Omega\cap U}$, then the class of smooth functions $f \colon b\Omega \cap U \to \mathbb{R}$ such that, for example,

$$f = \mathcal{O}(\lambda) \quad \text{on } b\Omega \cap U$$

is well-defined, i.e., independent of the choice of r and L.

2.6. **Miscellanea.** Let $\Omega \subset \mathbb{R}^N$ be a smoothly bounded domain. Let $d_{b\Omega}\colon \mathbb{R}^N \to [0,\infty)$,

$$d_{b\Omega}(q) := \inf_{p\in b\Omega} |q-p|,$$

denote the Euclidean distance to the boundary $b\Omega$, and let $\delta_{b\Omega}\colon \mathbb{R}^N \to \mathbb{R}$,

$$\delta_{b\Omega}(q) := \begin{cases} d_{b\Omega}(q), & \text{if } q \in \mathbb{R}^N \setminus \Omega \\ -d_{b\Omega}(q), & \text{if } q \in \Omega \end{cases},$$

be the associated signed distance function. Let $U \subset \mathbb{R}^N$ be an open neighborhood of $b\Omega$ such that there exists a smooth map $\pi\colon U \to b\Omega$ with $|q-\pi(q)| = d_{b\Omega}(q)$, see, e.g., [5, Lemma 4.11] for existence and [9, Lemma 1 in §15.5] for smoothness of the map π. Finally, let ν denote the outward unit normal vector field along $b\Omega$, and note that this notation is consistent with the one given in Section 2.4. If $f\colon U \to \mathbb{R}$ is smooth, then by Taylor's formula it follows that

$$f = f\circ\pi + \delta_{b\Omega}((\nu f)\circ\pi) + \mathcal{O}(d_{b\Omega}^2) \text{ on } U. \tag{2.10}$$

See also, e.g., (2.1) in [7, 8].

If $f\colon \mathbb{R} \to \mathbb{R}$ is a nonnegative $\mathcal{C}^2$-smooth function and $f(x_0) = 0$, then it follows readily from L'Hospital's rule that $|f'|^2 \le Cf$ near x_0 for some constant $C > 0$. We will repeatedly need the following generalizations of this simple fact.

Lemma 2.11. *The following assertions hold true.*

(1) *Let $U \subset \mathbb{R}^N$ be open, and let $f\colon U \to [0,\infty)$ be a $\mathcal{C}^2$-smooth function. Then for every $K \Subset U$ there exists a constant $C > 0$ such that*
$$|df|^2 \le Cf \text{ on } K. \tag{2.12}$$

(2) *Let $\Omega \subset \mathbb{R}^N$ be a smoothly bounded domain, let $U \subset \mathbb{R}^N$ be open, and let $f\colon b\Omega\cap U \to [0,\infty)$ be a $\mathcal{C}^2$-smooth function. Then*
$$Vf = \mathcal{O}(\sqrt{f}) \text{ on } b\Omega\cap U \tag{2.13}$$
for every $V \in \mathcal{V}(b\Omega\cap U)$.

Proof. For a proof of part (1) see, e.g., [8, Lemma 4.3]. In order to prove part (2), let $F\colon U \to [0,\infty)$ be a $\mathcal{C}^2$-smooth extension of f, and for given $K \Subset b\Omega\cap U$ let $C > 0$ be a constant such that $|dF|^2 \le CF$ on K, see (2.12). Then $|Vf|^2 = |\langle df, V\rangle|^2 = |\langle dF, V\rangle|^2 \le |dF|^2 \le CF = Cf$ on K for every $V \in \mathcal{V}(b\Omega\cap U)$ such that $|V| \le 1$, which implies (2.13). □

3. A counterexample

Consider $\mathbb{C}^2$ with coordinates (z,w), $z = x+iy$, $w = u+iv$.

Theorem 3.1. *For fixed $k \in \mathbb{N}$, $k \ge 3$, let*
$$r(z,w) := u + \tfrac{1}{k^2}|z|^{2k} - \tfrac{2}{(k-1)^2}|z|^{2k-2}v + \tfrac{1}{(k-2)^2}|z|^{2k-4}v^2 + |z|^{4k-2},$$
and set
$$\Omega := \big\{(z,w) \in \mathbb{C}^2 : r(z,w) < 0\big\}.$$
Then the following assertions hold true.

(i) *There exists an open neighborhood $U \subset \mathbb{C}^2$ of $0 \in b\Omega$, such that $\Omega \cap U$ is pseudoconvex, and such that the following holds. If $k = 3$, then $0 \in b\Omega$ is the only weakly pseudoconvex boundary point of Ω in $b\Omega \cap U$. If $k > 3$, then the set of weakly pseudoconvex boundary points of Ω in $b\Omega \cap U$ is $(b\Omega \cap U) \cap (\{0\} \times \mathbb{C})$. Moreover, $b\Omega$ is of finite type $c_0 = 2k$ at 0.*

(ii) *Let $V \subset \mathbb{C}^2$ be an open neighborhood of 0 and let $\rho\colon V \to \mathbb{R}$ be a $\mathcal{C}^2$-smooth local defining function for Ω. Then ρ is not plurisubharmonic on $b\Omega \cap V$.*

Proof. (i) In a slight deviation from Section 2, set $L = r_w \frac{\partial}{\partial z} - r_z \frac{\partial}{\partial w}$. Then

$$H_r^{\mathbb{C}}(L, L) = r_{z\bar{z}}|r_w|^2 - 2\,\mathrm{Re}[r_{z\bar{w}} r_w r_{\bar{z}}] + r_{w\bar{w}}|r_z|^2.$$

Computing the relevant terms, we obtain

$$\begin{aligned}
r_z(z,w) &= \tfrac{1}{k}\bar{z}|z|^{2k-2} - \tfrac{2}{k-1}\bar{z}|z|^{2k-4}v + \tfrac{1}{k-2}\bar{z}|z|^{2k-6}v^2 + (2k-1)\bar{z}|z|^{4k-4},\\
r_w(z,w) &= \tfrac{1}{2} + i\big(\tfrac{1}{(k-1)^2}|z|^{2k-2} - \tfrac{1}{(k-2)^2}|z|^{2k-4}v\big),\\
r_{z\bar{z}}(z,w) &= |z|^{2k-6}(|z|^2 - v)^2 + (2k-1)^2|z|^{4k-4},\\
r_{z\bar{w}}(z,w) &= -\tfrac{i}{k-1}\bar{z}|z|^{2k-4} + \tfrac{i}{k-2}\bar{z}|z|^{2k-6}v,\\
r_{w\bar{w}}(z,w) &= \tfrac{1}{2(k-2)^2}|z|^{2k-4}.
\end{aligned}$$

These equations lead straightforwardly to the estimates

$$\begin{aligned}
(r_{z\bar{z}}|r_w|^2)(z,w) &\geq \tfrac{1}{4}\big[|z|^{2k-6}(|z|^2 - v)^2 + |z|^{4k-4}\big],\\
-2\,\mathrm{Re}[r_{z\bar{w}} r_w r_{\bar{z}}](z,w) &= |z|^{6k-14}\sum_{j=0}^{4}\mathcal{O}(|z|^{8-2j}v^j),\\
(r_{w\bar{w}}|r_z|^2)(z,w) &\geq 0.
\end{aligned}$$

Setting $a := |z|^2 - v$, it follows that for $|z|$ and v sufficiently close to 0

$$\begin{aligned}
H_r^{\mathbb{C}}(L,L)(z,w) &\geq \tfrac{1}{4}(|z|^{2k-6}a^2 + |z|^{4k-4}) + |z|^{6k-14}\textstyle\sum_{j=0}^4 \mathcal{O}(|z|^{8-2j}a^j)\\
&= \tfrac{1}{4}|z|^{2k-6}\big[(a^2 + |z|^{2k+2}) + o(1)(a^2 + a|z|^{k+1} + |z|^{2k+2})\big]\\
&\geq \tfrac{1}{8}|z|^{2k-6}(a^2 + |z|^{2k+2}).
\end{aligned}$$

This shows that Ω is pseudoconvex near 0, and that the set of weakly pseudoconvex points of Ω has the form as described above. It is clear from (2.4) that $c_0 = 2k$.

(ii) Assume, in order to get a contradiction, that $\rho\colon V \to \mathbb{R}$ is a $\mathcal{C}^2$-smooth local defining function for Ω near $0 \in b\Omega$ such that ρ is plurisubharmonic on $b\Omega \cap V$. There exists a $\mathcal{C}^1$-smooth function $h\colon V \to \mathbb{R}$ such that $\rho = re^h$. Thus on $b\Omega \cap V$ one has (since $h * \delta_j \to h$ in the $\mathcal{C}^1$-norm for every Dirac sequence $\{\delta_j\}$)

$$\begin{aligned}
\rho_{z\bar{z}} &= \big(r_{z\bar{z}} + 2\,\mathrm{Re}[r_z h_{\bar{z}}]\big)e^h,\\
\rho_{z\bar{w}} &= \big(r_{z\bar{w}} + r_z h_{\bar{w}} + r_{\bar{w}} h_z\big)e^h.
\end{aligned} \tag{3.2}$$

For $\varepsilon > 0$ sufficiently small, let $f\colon \mathbb{D}(0,\varepsilon) \to b\Omega \cap V$ be the smooth map

$$f(\zeta) := (\zeta, u(|\zeta|) + i|\zeta|^2),\; u(|\zeta|) := -\big(\tfrac{1}{k^2} - \tfrac{2}{(k-1)^2} + \tfrac{1}{(k-2)^2}\big)|\zeta|^{2k} - |\zeta|^{4k-2}, \tag{3.3}$$

where $\mathbb{D}(0,\varepsilon) := \{\zeta \in \mathbb{C} : |\zeta| < \varepsilon\}$. We claim that

$$(h_z \circ f)(\zeta) = O(|\zeta|^{2k-3}). \tag{3.4}$$

Indeed, if not, then the number $m \in \mathbb{N} \cup \{0\}$ such that $(h_z \circ f)(\zeta) = O(|\zeta|^m)$ but $(h_z \circ f)(\zeta) \neq O(|\zeta|^{m+1})$ satisfies $m < 2k-3$. Hence, in view of (3.2) and the computations in part (i), we see that $(\rho_{z\bar{z}} \circ f)(\zeta) = O(|\zeta|^{2k-1+m})$ and $(\rho_{z\bar{w}} \circ f)(\zeta) \neq O(|\zeta|^{m+1})$. Thus $((\rho_{z\bar{z}}\rho_{w\bar{w}}) \circ f)(\zeta) = O(|\zeta|^{2k-1+m})$ and $(|\rho_{z\bar{w}}|^2 \circ f)(\zeta) \neq O(|\zeta|^{2m+2})$. In view of the inequality $2k-1+m > 2m+2$, this contradicts the fact that ρ is psh on $b\Omega$ near 0, since this implies that $\left(\rho_{z\bar{z}}\rho_{w\bar{w}} - |\rho_{z\bar{w}}|^2\right) \circ f \geq 0$.

From (3.2), (3.4) and the computations in part (i), we conclude that

$$\begin{aligned}
&(\rho_{z\bar{z}} \circ f)(\zeta) = O(|\zeta|^{4k-4}),\\
&(\rho_{z\bar{w}} \circ f)(\zeta) = i\mu_k\bar{\zeta}|\zeta|^{2k-4} + \tfrac{1}{2}(h_z \circ f)(\zeta) + O(|\zeta|^{2k-2}), \quad \mu_k := \tfrac{1}{(k-1)(k-2)}.
\end{aligned}$$

Since $\left(\rho_{z\bar{z}}\rho_{w\bar{w}} - |\rho_{z\bar{w}}|^2\right) \circ f \geq 0$, it follows that

$$(h_z \circ f)(\zeta) = -2i\mu_k\bar{\zeta}|\zeta|^{2k-4} + o(|\zeta|^{2k-3}). \tag{3.5}$$

For given $\sigma > 0$, define the curve $\gamma_\sigma = \gamma = (\gamma_1, \gamma_2)\colon [0, 2\pi] \to b\Omega$ by $\gamma(t) := f(\sigma e^{it})$. Then, for $\sigma > 0$ sufficiently small, it follows from (3.5) that

$$\begin{aligned}
\int_\gamma dh &= 2\,\mathrm{Re}\int_0^{2\pi} h_z(\gamma(t)) \cdot \dot{\gamma}_1(t)\, dt\\
&= 2\,\mathrm{Re}\int_0^{2\pi} (-2i\mu_k\sigma^{2k-3}e^{-it} + o(\sigma^{2k-3})) \cdot (i\sigma e^{it})\, dt\\
&= 4\mu_k\int_0^{2\pi} (\sigma^{2k-2} + o(\sigma^{2k-2}))\, dt.
\end{aligned}$$

Thus $\int_\gamma dh \neq 0$ whenever $\sigma > 0$ is sufficiently small. This is a contradiction. □

Theorem 3.6. *For fixed $k \in \mathbb{N}$, $k \geq 3$, let*

$$r(z, w) := u + \tfrac{1}{k^2}|z|^{2k} - \tfrac{2}{(k-1)^2}|z|^{2k-2}v + \tfrac{1}{(k-2)^2}|z|^{2k-4}v^2 + |z|^{4k-2} + |w|^2,$$

and set

$$\Omega := \{(z, w) \in \mathbb{C}^2 : r(z, w) < 0\}.$$

Then the following assertions hold true.

- *(i) Ω is a bounded domain with smooth real-analytic boundary.*
- *(ii) Ω is pseudoconvex. If $k = 3$, then $0 \in b\Omega$ is the only weakly pseudoconvex boundary point of Ω. If $k > 3$, the set of weakly pseudonconvex boundary points of Ω is $b\Omega \cap (\{0\} \times \mathbb{C})$. Moreover, $b\Omega$ is of finite type $c_0 = 2k$ at 0.*
- *(iii) Any $\mathcal{C}^2$-smooth local defining function for Ω near $0 \in b\Omega$ fails to be plurisubharmonic on $b\Omega$ near 0.*

Proof. In order to show (iii), one may proceed exactly as in the proof of part (ii) of Theorem 3.1, after noting that there are two choices for $u(|\zeta|)$ in (3.3) to be a solution to $r(\zeta, u(|\zeta|) + i|\zeta|^2) = 0$. While the proof of (i) is also straightforward, see below, the difficulty in proving Theorem 3.6 lies in showing that the introduction of the additional term $|w|^2$ in the defining function r turns Ω into a globally pseudoconvex domain, which is subject to the precise properties described in (i) and (ii).

In the proofs of (i) and (ii), we repeatedly use the following fact: if $A, B, C \in \mathbb{R}$ and $A, C \geq 0$, then

$$\exists\, \varepsilon > 0\; \forall\, (\sigma, \tau) \in \mathbb{R}^2 : A\sigma^2 + B\sigma\tau + C\tau^2 \geq \varepsilon(\sigma^2 + \tau^2) \quad \Leftrightarrow \quad 4AC - B^2 > 0. \tag{3.7}$$

(i) It follows from (3.7), with $\sigma = |z|^2$ and $\tau = v$, that

$$\tfrac{1}{k^2}|z|^{2k} - \tfrac{2}{(k-1)^2}|z|^{2k-2}v + \tfrac{1}{(k-2)^2}|z|^{2k-4}v^2 \geq 0.$$

Thus, for every $(z,w) \in \bar{\Omega}$, one has $0 \geq r(z,w) \geq u + u^2$, and hence $u \in [-1,0]$. In particular, $u + u^2 \in [-\frac{1}{4}, 0]$, so that

$$\begin{aligned} 0 &\geq r(z,w) \geq u + u^2 + |z|^{4k-2} \geq -\tfrac{1}{4} + |z|^{4k-2}, \\ 0 &\geq r(z,w) \geq u + u^2 + v^2 \geq -\tfrac{1}{4} + v^2. \end{aligned}$$

Hence, $|z|^2 \leq 4^{-\frac{1}{2k-1}}$ and $|v| \leq \frac{1}{2}$. This shows that Ω is bounded.

To see that $b\Omega$ is smooth, we compute

$$\begin{aligned} r_z(z,w) &= \bar{z}\left(\tfrac{1}{k}|z|^{2k-2} - \tfrac{2}{k-1}|z|^{2k-4}v + \tfrac{1}{k-2}|z|^{2k-6}v^2 + (2k-1)|z|^{4k-4}\right), \\ r_w(z,w) &= (\tfrac{1}{2} + u) + i\left(\tfrac{1}{(k-1)^2}|z|^{2k-2} - \tfrac{1}{(k-2)^2}|z|^{2k-4}v - v\right). \end{aligned}$$

It follows from (3.7) that $\frac{1}{k}|z|^{2k-2} - \frac{2}{k-1}|z|^{2k-4}v + \frac{1}{k-2}|z|^{2k-6}v^2 \geq 0$, so $r_z(z,w) \neq 0$ whenever $z \neq 0$. Moreover, if $(0,w) \in b\Omega$, then $u + u^2 + v^2 = 0$. In this case, either $v = 0$ and $u \in \{-1, 0\}$, and thus $r_u(0,w) \neq 0$, or $v \neq 0$, and thus $r_v(0,w) \neq 0$.

(ii) Let $(z,w) \in b\Omega$. As before, write $a := |z|^2 - v$. We consider two cases.

Case 1: $|z| < \frac{1}{10}$ and $|a| < \frac{1}{10}$. Since then $|v| < \frac{1}{5}$, it follows from $r(z,w) = 0$ that

$$u + u^2 = -\tfrac{1}{k^2}|z|^{2k} + \tfrac{2}{(k-1)^2}|z|^{2k-2}v - \tfrac{1}{(k-2)^2}|z|^{2k-4}v^2 - |z|^{4k-2} - v^2 > -\tfrac{3}{16}.$$

Hence $u \notin [-\frac{3}{4}, -\frac{1}{4}]$, and thus $|r_u(z,w)| > \frac{1}{4}$. In particular,

$$(r_{z\bar{z}}|r_w|^2)(z,w) \geq \tfrac{1}{16}|z|^{2k-6}a^2 + \tfrac{25}{16}|z|^{4k-4}. \tag{3.8}$$

Inserting the equation $v = |z|^2 - a$ into the formulas for r_z, r_w and $r_{z\bar{w}}$, we see that

$$\begin{aligned} r_z(z,w) &= \bar{z}|z|^{2k-6}\left(\tfrac{2}{k(k-1)(k-2)}|z|^4 - \tfrac{2}{(k-1)(k-2)}|z|^2a + \tfrac{1}{k-2}a^2 + (2k-1)|z|^{2k+2}\right), \\ r_w(z,w) &= (\tfrac{1}{2} + u) + i\left(-\tfrac{2k-3}{(k-1)^2(k-2)^2}|z|^{2k-2} + \tfrac{1}{(k-2)^2}|z|^{2k-4}a - |z|^2 + a\right), \\ r_{z\bar{w}}(z,w) &= i\bar{z}|z|^{2k-6}\left(\tfrac{1}{(k-1)(k-2)}|z|^2 - \tfrac{1}{k-2}a\right). \end{aligned}$$

Thus, since $|z| < 1$ for $(z,w) \in b\Omega$, we obtain that, for every $k \geq 3$,

$$\begin{aligned} |r_z(z,w)| &\leq |z|^{2k-5}\left((|z|^2 + |a|)^2 + (2k-1)|z|^{2k+2}\right), \\ |r_v(z,w)| &\leq 2(|z|^2 + |a|), \\ |r_{z\bar{w}}(z,w)| &\leq |z|^{2k-5}(|z|^2 + |a|). \end{aligned}$$

From $|2\,\mathrm{Re}[r_{z\bar{w}}r_w r_{\bar{z}}]| = |2\,\mathrm{Re}[r_{z\bar{w}}r_v r_{\bar{z}}]| \leq 2|r_{z\bar{w}}||r_v||r_z|$, it then follows that

$$\begin{aligned} |2\,\mathrm{Re}[r_{z\bar{w}}r_w r_{\bar{z}}]|\,(z,w) &\leq 4|z|^{4k-10}(|z|^2 + |a|)^4 + (8k-4)|z|^{6k-8}(|z|^2 + |a|)^2 \\ &= \left(4|z|^2 + (8k-4)|z|^{2k-4}(|z|^2 + |a|)^2\right)\cdot|z|^{4k-4} \\ &\quad + 16|z|^{k+1}\cdot|z|^{3k-5}|a| \\ &\quad + 4|z|^{2k-4}(6|z|^4 + 4|z|^2|a| + |a|^2)\cdot|z|^{2k-6}|a|^2. \end{aligned}$$

Since $|z| < \frac{1}{10}$ and $|a| < \frac{1}{10}$, this implies that, for every $k \geq 3$,

$$|2\,\mathrm{Re}[r_{z\bar{w}}r_w r_{\bar{z}}]|\,(z,w) \leq \tfrac{11}{250}|z|^{2k-6}|a|^2 + \tfrac{4}{25}|z|^{3k-5}|a| + \tfrac{1}{2}|z|^{4k-4}. \tag{3.9}$$

From $H_r^{\mathbb{C}}(L,L) \geq r_{z\bar{z}}|r_w|^2 - |2\operatorname{Re}[r_{z\bar{w}} r_w r_{\bar{z}}]|$, it follows with (3.8) and (3.9) that

$$H_r^{\mathbb{C}}(L,L)(z,w) \geq |z|^{2k-6}\left(\tfrac{37}{2000}|a|^2 - \tfrac{4}{25}|z|^{k+1}|a| + \tfrac{17}{16}|z|^{2k+2}\right).$$

Thus, since $4 \cdot \frac{37}{2000} \cdot \frac{17}{16} - (\frac{4}{25})^2 > 0$, an application of (3.7) shows that

$$H_r^{\mathbb{C}}(L,L)(z,w) \geq \varepsilon |z|^{2k-6}(|a|^2 + |z|^{2k+2})$$

for some constant $\varepsilon > 0$.

Case 2: $|z| > \frac{1}{10}$ or $|a| > \frac{1}{10}$. Set $D := r_{z\bar{z}} r_{w\bar{w}} - |r_{z\bar{w}}|^2$. Then

$$D(z,w) \geq |z|^{2k-6}|a|^2 + (2k-1)^2|z|^{4k-4} - \tfrac{|z|^{4k-10}}{(k-2)^2}|a|^2 - \tfrac{2|z|^{4k-8}}{(k-1)(k-2)^2}|a| - \tfrac{|z|^{4k-6}}{(k-1)^2(k-2)^2},$$

so $D(z,w) \geq |z|^{2k-6} d(z,w)$ with

$$d(z,w) := \left(1 - \tfrac{|z|^{2k-4}}{(k-2)^2}\right)|a|^2 - \tfrac{2|z|^{2k-2}}{(k-1)(k-2)^2}|a| + \left((2k-1)^2|z|^{2k+2} - \tfrac{|z|^{2k}}{(k-1)^2(k-2)^2}\right).$$

We will show that $d(z,w) > 0$. Since, in the currently considered case, we have $r_{z\bar{z}}(z,w) \geq c|z|^{2k-6}$ with some constant $c > 0$, this implies the claim.

Assume first that $|a| > \frac{1}{10}$ and $|z|^2 \leq \frac{1}{10}$. Then $d(z,w) \geq \frac{9}{10}a^2 - \frac{1}{100}a - \frac{1}{4000} > 0$. On the other hand, assume now that $|z|^2 > \frac{1}{10}$. Then $d(z,w) > 0$ provided that

$$4\left(1 - \tfrac{|z|^{2k-4}}{(k-2)^2}\right)\left((2k-1)^2|z|^{2k+2} - \tfrac{|z|^{2k}}{(k-1)^2(k-2)^2}\right) - \tfrac{4|z|^{4k-4}}{(k-1)^2(k-2)^4} > 0,$$

see (3.7), and this inequality is satisfied if and only if

$$|z|^{2k-2} - (k-2)^2|z|^2 + \tfrac{1}{(k-1)^2(2k-1)^2} < 0.$$

But the left-hand side is negative for $|z|^2 \in (\frac{1}{10}, 4^{-\frac{1}{2k-1}}] := I_k$, since the function

$$f_k(t) := t^{k-1} - (k-2)^2 t + \tfrac{1}{(k-1)^2(2k-1)^2}$$

is negative on I_k: Indeed, for $k = 3$ a straightforward computation shows that $f_3 < 0$ on $(\frac{1}{2} - \frac{\sqrt{6}}{5}, \frac{1}{2} + \frac{\sqrt{6}}{5}) \supset I_3$, and for $k \geq 4$ note that $f_k(\frac{1}{10}) \leq f_4(\frac{1}{10}) < 0$ and $f_k(1) < 0$, so that convexity of f_k on $(0,\infty)$ implies $f_k < 0$ on $(\frac{1}{10}, 1) \supset I_k$. □

Remark 3.10. In the case of $k > 3$, the defining function in Theorem 3.1 can be modified in such a way that the origin is the only weakly pseudoconvex boundary point of the modified domain near the origin while maintaining all other properties of (i)-(ii). Similarly, in Theorem 3.6, the defining function can be adapted in such a way that the origin is the only weakly pseudoconvex boundary point of the thereby obtained domain while keeping all other properties of (i)-(iii).

4. Non-degenerate weakly pseudoconvex boundary points

Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain, and $p_0 \in b\Omega$. Assume that Ω is weakly pseudoconvex at p_0. Let $r\colon U \to \mathbb{R}$ be a smooth local defining function for Ω on an open neighborhood $U \subset \mathbb{C}^2$ of p_0, let L be a nonvanishing tangential $(1,0)$-vector field on U, and set

$$\lambda := H_r^{\mathbb{C}}(L,L)_{|b\Omega\cap U}.$$

Then λ attains a local minimum at p_0. Since $L\bar{L}\lambda = \frac{1}{4}(XX + YY - i[X,Y])\lambda$, and since $[X,Y]$ is tangential to $b\Omega$, it follows that $(L\bar{L}\lambda)(p_0)$ is real. In this section we will show, in particular, that

$$(L\bar{L}\lambda)(p_0) \geq |(LL\lambda)(p_0)|. \tag{4.1}$$

Remark 4.2. Observe that (4.1) is independent of the choice of r. Indeed, let $\rho\colon U \to \mathbb{R}$ be another smooth local defining function for Ω. Then there exists a smooth function $h > 0$ on U such that $\rho = rh$, and one easily computes that $\lambda_\rho = h\lambda_r$, where $\lambda_* := H_*^{\mathbb{C}}(L, L)_{|b\Omega\cap U}$. Since λ_r attains a local minimum at p_0, all tangential derivatives of λ_r at p_0 vanish. It thus follows that all second order tangential derivatives of λ_r and λ_ρ at p_0 differ by the same constant factor $c := h(p_0) > 0$. In particular,

$$(L\bar{L}\lambda_\rho)(p_0) = c(L\bar{L}\lambda_r)(p_0), \tag{4.3}$$

$$(LL\lambda_\rho)(p_0) = c(LL\lambda_r)(p_0), \tag{4.4}$$

and thus (4.1) is independent of the local defining function r.

Note further that (4.1) is also independent of the choice of the tangential $(1, 0)$-vector field L. Namely, for L, L' nonvanishing tangential $(1, 0)$-vector fields on U, there exists a nonvanishing complex-valued function h such that $L' = hL$ on $b\Omega\cap U$. Let $\lambda := H_r^{\mathbb{C}}(L, L)_{|b\Omega\cap U}$ and $\lambda' := H_r^{\mathbb{C}}(L', L')_{|b\Omega\cap U}$. Then, by similar arguments as above, one obtains that

$$(L'\bar{L}'\lambda')(p_0) = |c|^4(L\bar{L}\lambda)(p_0), \tag{4.5}$$

$$(L'L'\lambda')(p_0) = c^2|c|^2(LL\lambda)(p_0), \tag{4.6}$$

where $c := h(p_0)$.

The following two Lemmata 4.7 and 4.9 are used to derive (4.1) in Theorem 4.10.

Lemma 4.7. *Let $\Omega \subset \mathbb{R}^N$ be a smoothly bounded domain, $p_0 \in b\Omega$, and U an open neighborhood of p_0. Let V be a smooth vector field along $b\Omega\cap U$. For $I \subset \mathbb{R}$ an open interval containing 0, let $\gamma : I \longrightarrow b\Omega \cap U$ be a smooth curve such that $\gamma(0) = p_0$ and $\dot\gamma(\tau) = V_{\gamma(\tau)}$ for every $\tau \in I$. Then*

$$(f \circ \gamma)''(0) = (V V f)(p_0). \tag{4.8}$$

for every smooth function $f\colon b\Omega \cap U \to \mathbb{R}$.

Proof. Choose local coordinates $\varphi = (t^1, \dots, t^{N-1})^\top$ for $b\Omega$ near p_0 such that $\varphi(p_0) = 0$ and $V = \frac{\partial}{\partial t^1}$. Then $(\varphi \circ \gamma)(\tau) = (\tau, 0, \dots, 0)^\top$ and $(D(f \circ \varphi^{-1})) \circ \varphi = (\frac{\partial f}{\partial t^1}, \dots, \frac{\partial f}{\partial t^{N-1}})$, where $D(f \circ \varphi^{-1})$ denotes the Jacobian matrix of $f \circ \varphi^{-1}$. Thus

$$(f \circ \gamma)' = [(D(f \circ \varphi^{-1})) \circ (\varphi \circ \gamma)] \cdot D(\varphi \circ \gamma) = \frac{\partial f}{\partial t^1} \circ \gamma = (V f) \circ \gamma,$$

and hence $(f \circ \gamma)'' = ((V f) \circ \gamma)' = (V V f) \circ \gamma$.

□

Lemma 4.9. *Let $\Omega \subset \mathbb{R}^N$ be a smoothly bounded domain, $p_0 \in b\Omega$, and U an open neighborhood of p_0. Let $f\colon b\Omega \cap U \to \mathbb{R}$ be smooth such that f attains a local minimum at p_0. Suppose that $V^1, \dots, V^{N-1}$ are smooth vector fields along $b\Omega \cap U$ such that $\left(V^1, \dots, V^{N-1}\right)_{p_0}$ is a basis for $T_{p_0}(b\Omega)$. Then the matrix*

$$\begin{pmatrix} V^1V^1f & \cdots & V^1V^{N-1}f \\ \vdots & & \vdots \\ V^{N-1}V^1f & \cdots & V^{N-1}V^{N-1}f \end{pmatrix}(p_0)$$

is symmetric and positive semi-definite.

Proof. The main point is to observe that a version of (2.1) implies that, with $M := b\Omega \cap U$, the map $\mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{C}^\infty(M)$, $(V, W) \mapsto VWf$, is tensorial at critical points of f, and thus that at critical points it does not matter if the Hessian of f is computed with respect to local coordinates or an arbitrary frame of vector fields (see, e.g., §2 in [13] for this standard fact). For the convenience of the reader, we include a full proof of the lemma.

Consider the function $q\colon T_{p_0}(M) \to \mathbb{R}$ given by

$$q(V_{p_0}) := (VVf)(p_0),$$

where V is any extension of V_{p_0} to a vector field along M such that $V_p \in T_p(M)$ for every $p \in M$. This is well-defined. To wit, if $\bar{\nabla}$ is any linear connection on M, then

$$VVf = \bar{Q}^{\mathbb{R}}_f(V, V) + \left(\bar{\nabla}_V V\right) f,$$

where $\bar{Q}^{\mathbb{R}}_f = \bar{\nabla}^2 f$ denotes the covariant Hessian. Since p_0 is a local minimum of f, the derivative $(\bar{\nabla}_V V)f$ vanishes at p_0 independently of the above choice of V. Furthermore, since $\bar{Q}^{\mathbb{R}}_f$ is a tensor, the value $\bar{Q}^{\mathbb{R}}_f(V, V)(p_0)$ depends only on V_{p_0}.

It follows from (4.8) and an application of the Picard–Lindelöf theorem, that $q \geq 0$. Thus the associated symmetric bilinear form $B\colon T_{p_0}(M) \times T_{p_0}(M) \to \mathbb{R}$,

$$B(V_{p_0}, W_{p_0}) := \tfrac{1}{2}\left(q(V_{p_0} + W_{p_0}) - q(V_{p_0}) - q(W_{p_0})\right),$$

is positive semi-definite. Moreover, note that

$$B\big(V^j_{p_0}, V^k_{p_0}\big) = \left(\tfrac{1}{2}V^jV^kf + \tfrac{1}{2}V^kV^jf\right)(p_0) = \left(V^jV^kf + \tfrac{1}{2}[V^k, V^j]f\right)(p_0).$$

Since the vector field $[V^k, V^j]$ is tangential to M, and since f attains a local minimum at p_0, it follows that $[V^k, V^j]f$ vanishes at p_0. Therefore,

$$B\left(V^j_{p_0}, V^k_{p_0}\right) = \left(V^jV^kf\right)(p_0),$$

which proves the claim. □

Theorem 4.10. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain. Assume that Ω is weakly pseudoconvex at $p_0 \in b\Omega$. Then*

$$\left(L\bar{L}\lambda\right)(p_0) \geq |\left(LL\lambda\right)(p_0)|. \tag{4.11}$$

Proof. Let X, Y and T be the real vector fields such that

$$L = \tfrac{1}{2}\left(X + iY\right) \quad \text{and} \quad N = \tfrac{1}{2}\left(\nu + iT\right),$$

and recall that X_p, Y_p, and T_p form a basis of $T_p(b\Omega)$ for all $p \in b\Omega$, see Section 2.4. Since λ attains a local minimum at p_0, it follows from Lemma 4.9 that

$$\begin{pmatrix} XX\lambda & XY\lambda & XT\lambda \\ YX\lambda & YY\lambda & YT\lambda \\ TX\lambda & TY\lambda & TT\lambda \end{pmatrix}(p_0)$$

is symmetric and positive semi-definite. In particular,

$$\begin{pmatrix} XX\lambda & XY\lambda \\ YX\lambda & YY\lambda \end{pmatrix}(p_0)$$

is symmetric and positive-semidefinite, i.e.,

$$\left(XX\lambda \cdot YY\lambda - (XY\lambda)^2\right)(p_0) \geq 0.$$

But a straightforward computation shows that

$$\begin{aligned}
&\left(L\bar{L}\lambda\right)^2 - |LL\lambda|^2 \\
&= \tfrac{1}{16}\left((X+iY)\,(X-iY)\,\lambda\right)^2 - \tfrac{1}{16}\left|(X+iY)(X+iY)\lambda\right|^2 \\
&= \tfrac{1}{16}\left(XX\lambda + YY\lambda\right)^2 - \tfrac{1}{16}\left|XX\lambda - YY\lambda + 2iXY\lambda\right|^2 \\
&= \tfrac{1}{4}\left(XX\lambda\cdot YY\lambda - (XY\lambda)^2\right).
\end{aligned}$$

This proves the claim. □

Definition 4.12. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain. A weakly pseudoconvex point $p_0 \in b\Omega$ is called non-degenerate, if*

$$\left(L\bar{L}\lambda\right)(p_0) > |(LL\lambda)\,(p_0)|. \tag{4.13}$$

If (4.13) *does not hold, then the weakly pseudoconvex point p_0 is called degenerate.*

From (4.3) and (4.4), we see that (4.13) is independent of the choice of a local defining function. Therefore, it describes a property of the domain Ω at p_0. In the next lemma, it is shown that this property is invariant under biholomorphic transformations.

Lemma 4.14. *Let $\Omega', \Omega \subset \mathbb{C}^2$ be smoothly bounded domains such that Ω' and Ω are pseudoconvex near $p_0' \in b\Omega'$ and $p_0 \in b\Omega$, respectively. Let $\Phi\colon U' \to U$ be a biholomorphic map from an open neighborhood $U' \subset \mathbb{C}^2$ of p_0' to an open neighborhood $U \subset \mathbb{C}^2$ of p_0 such that $\Phi(\Omega' \cap U') = \Omega \cap U$ and $\Phi(p_0') = p_0$. Then p_0' is a non-degenerate weakly pseudoconvex boundary point of Ω' if and only if p_0 is non-degenerate weakly pseudoconvex boundary point of Ω.*

Proof. Let $r\colon U \to \mathbb{R}$ be a smooth local defining function for Ω near p_0. Let L' be a nonvanishing tangential $(1,0)$-vector field on U', and let $L := \Phi_* L'$ be the pushforward of L'. Then define

$$\lambda' = H^{\mathbb{C}}_{r\circ\Phi}(L', L')_{|b\Omega'\cap U'} \quad \text{and} \quad \lambda = H^{\mathbb{C}}_{r}(L, L)_{|b\Omega\cap U},$$

and observe that, by the usual transformation law, one has $\lambda' = \lambda \circ \Phi$. Since, by definition, $L'(f\circ\Phi) = Lf$ for every smooth function $f\colon U \to \mathbb{C}$, it thus follows that

$$\left(L'\bar{L}'\lambda'\right)(p_0') = \left(L\bar{L}\lambda\right)(p_0) \quad \text{and} \quad \left(L'L'\lambda'\right)(p_0') = (LL\lambda)\,(p_0). \tag{4.15}$$

In view of Remark 4.2, this proves the claim. □

In view of Lemma 4.14, it is meaningful to look for local holomorphic coordinates around p_0, in which the condition (4.13) takes a particularly simple form. The next result shows how this can be achieved.

Proposition 4.16. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain. Assume that $0 \in b\Omega$ is a point of weak pseudoconvexity, and let r be a smooth local defining function for Ω near 0. If $r_z(0) = r_{zz}(0) = 0$, then*

$$\left(L\bar{L}\lambda\right)(0) > |\,(LL\lambda)\,(0)| \tag{4.17}$$

if and only if there exists a constant $\varepsilon > 0$ such that

$$H^{\mathbb{C}}_r(L, L)(z, 0) \geq \varepsilon|z|^2 + o(|z|^2). \tag{4.18}$$

Proof. Let $U \subset \mathbb{C}^2$ denote the domain of definition of r, and define $\Lambda\colon U \to \mathbb{R}$ by $\Lambda := H_r^{\mathbb{C}}(L,L)$. We will show that both (4.17) and (4.18) are equivalent to the condition that the matrix

$$A := \begin{pmatrix} \Lambda_{xx} & \Lambda_{xy} \\ \Lambda_{yx} & \Lambda_{yy} \end{pmatrix}(0)$$

is positive definite, where $z = x + iy$ with $x, y \in \mathbb{R}$. This proves the claim.

Observe first that $\Lambda_{|b\Omega\cap U} = \lambda$. In particular, $\Lambda(0) = 0$. Moreover, since λ attains a local minimum at 0, it follows that $d\Lambda$ vanishes on $T_0(b\Omega)$, and thus $\Lambda_x(0) = \Lambda_y(0) = 0$. Hence

$$\Lambda(z,0) = \tfrac{1}{2}\begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix} + o(|z|^2).$$

This shows that (4.18) holds true if and only if $A > 0$.

On the other hand, we claim that $r_z(0) = r_{zz}(0) = 0$ implies that

$$(L\bar{L}\lambda)(0) = \Lambda_{z\bar{z}}(0), \quad (LL\lambda)(0) = \Lambda_{zz}(0). \tag{4.19}$$

From this, we immediately obtain that

$$(L\bar{L}\lambda)(0) = \tfrac{1}{4}(\Lambda_{xx} + \Lambda_{yy})(0),$$
$$\left((L\bar{L}\Lambda)^2 - |LL\Lambda|^2\right)(0) = \tfrac{1}{4}\left(\Lambda_{xx}\Lambda_{yy} - \Lambda_{xy}^2\right)(0).$$

Since a real 2×2 matrix is positive definite if and only if both its trace and determinant are positive, it follows that (4.17) holds true if and only if $A > 0$. (An easy computation shows that the eigenvalues of A are given by $2(L\bar{L}\Lambda \pm |LL\Lambda|)(0)$, and thus that the largest possible value for ε in (4.18) is $(L\bar{L}\Lambda)(0) - |(LL\Lambda)(0)|$.)

In order to see (4.19) we first note that, in view of Remark 4.2, and after possibly shrinking U, we can assume without loss of generality that $L = L_r$. Recall that

$$L\bar{L}\lambda = H_\Lambda^{\mathbb{C}}(L,L) + (\nabla_L \bar{L})\Lambda,$$
$$LL\lambda = Q_\Lambda^{\mathbb{C}}(L,L) + (\nabla_L L)\Lambda.$$

Note that the vectors $(\nabla_L \bar{L})(0)$ and $(\nabla_L L)(0)$ are tangential to $b\Omega$. Indeed, straightforward computations show that, for $L = L_r$ and $N = N_r$,

$$\nabla_L \bar{L} = \frac{1}{|r_z|^2 + |r_w|^2}\Big(H_r^{\mathbb{C}}(L,N)\bar{L} - H_r^{\mathbb{C}}(L,L)\bar{N}\Big),$$
$$\nabla_L L = \frac{1}{|r_z|^2 + |r_w|^2}\Big(Q_r^{\mathbb{C}}(L,N)L - Q_r^{\mathbb{C}}(L,L)N\Big).$$

Moreover, $H_r^{\mathbb{C}}(L,L)(0) = 0$, since Ω is weakly pseudoconvex at 0, and $Q_r^{\mathbb{C}}(L,L)(0) = 0$, since $r_z(0) = r_{zz}(0) = 0$, which proves the claim. It follows that both derivatives $(\nabla_{\bar{L}} L)\Lambda$ and $(\nabla_L L)\Lambda$ vanish at 0, since λ attains a local minimum there. Since the condition $r_z(0) = 0$ implies that $L_0 = (\partial_z)_0$, the claim follows. (For an alternative proof of (4.19) under slightly stronger conditions, see [11, Lemma 3.23].) □

Remark 4.20. Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain, and let $p_0 \in b\Omega$ be a weakly pseudoconvex boundary point. If p_0 is non-degenerate, then $L\bar{L}\lambda(p_0) \neq 0$, and thus $b\Omega$ is of type 4 at p_0. Recall that Ω is said to be of strict type 4 at the weakly pseudoconvex point $p_0 \in b\Omega$, if

$$\tfrac{1}{3}\operatorname{Re}[(LL\lambda)(p_0)] + \tfrac{1}{4}(L\bar{L}\lambda)(p_0) > 0 \tag{4.21}$$

holds for all nonvanishing tangential $(1,0)$-vector fields L near p_0, see [11, Definition 2.16] in the case $m = 3$.

If $L' := hL$ for some smooth complex-valued function h near p_0, then one has $\frac{1}{3}\operatorname{Re}[(L'L'\lambda)(p_0)] + \frac{1}{4}(L'\bar{L}'\lambda)(p_0) = \frac{1}{3}\operatorname{Re}[h(p_0)^2(LL\lambda)(p_0)] + \frac{1}{4}|h(p_0)|^2(L\bar{L}\lambda)(p_0)$. From this, one easily sees that Ω is of strict type 4 at p_0 if and only if

$$\left(L\bar{L}\lambda\right)(p_0) > \tfrac{4}{3}|(LL\lambda)(p_0)| \tag{4.22}$$

for some, and then every, nonvanishing tangential $(1,0)$-vector field L near p_0. The condition (4.22) is invariant under biholomorphic transformations by (4.15).

Assume that coordinates are chosen in such a way that $p_0 = 0$ and $(\partial_z^j r)(0) = 0$ for $j \in \{1,\dots,4\}$. Then (4.22) holds true if and only if there exists a constant $\varepsilon > 0$ such that

$$r(z,0) \geq \varepsilon|z|^4 + o(|z|^4). \tag{4.23}$$

Indeed, since $(\partial_z^j\partial_{\bar z}^k r)(0) = 0$ for $j+k \leq 2$, it follows that $0 = (L\lambda)(0) = \lambda_z(0) = \Lambda_z(0) = (\partial_z^2\partial_{\bar z} r)(0)$, where $\Lambda := H_r^{\mathbb{C}}(L,L)$ and without loss of generality $L = L_r$. Thus $r(z,0) = \frac{1}{3}\operatorname{Re}[(\partial_z^3\partial_{\bar z} r)(0)z^3\bar z] + \frac{1}{4}(\partial_z^2\partial_{\bar z}^2 r)(0)|z|^4 + o(|z|)^4$. Since $(\partial_z^j\partial_{\bar z}^k r)(0) = 0$ for $j+k \leq 3$, it follows with (4.19) that $(LL\lambda)(0) = (\partial_z^3\partial_{\bar z} r)(0)$ and $(\bar{L}L\lambda)(0) = (\partial_z^2\partial_{\bar z}^2 r)(0)$. (The characterization of points of strict type 4 by means of (4.23) is already implicitly contained in Kohn's original paper, see formulas (3.8) and (3.12) in [11]. See also [2, Theorem 3.3].)

In view of (4.22), the definition of non-degenerate weakly pseudoconvex points given in Definition 4.12 is more general than the notion of strict type 4. In particular, if Ω is of strict type 4 at p_0, then p_0 is a non-degenerate weakly pseudoconvex point in the sense of Definition 4.12.

Lastly, consider the following example. Let $r\colon \mathbb{C}^2 \to \mathbb{R}$ be given by

$$r(z,w) := \operatorname{Re} w + \operatorname{Re}[az^3\bar z] + |z|^4.$$

Then $\Omega := \{r < 0\}$ is pseudoconvex at $0 \in b\Omega$ iff $|a| \leq \frac{4}{3}$. Moreover, by checking the conditions (4.18) and (4.23), one easily sees that 0 is a non-degenerate weakly pseudoconvex point in the sense of Definition 4.12 iff $|a| < \frac{4}{3}$, and 0 is of strict type 4 iff $|a| < 1$.

5. Plurisubharmonicity on the boundary

The main goal of this section is to prove the following theorem.

Theorem 5.1. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain, and let $p_0 \in b\Omega$ be a point of weak pseudoconvexity for Ω. If p_0 is non-degenerate, then Ω admits a smooth local defining function which is plurisubharmonic on $b\Omega$ near p_0.*

We use the following basic lemma to show Theorem 5.1.

Proposition 5.2. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain, $p_0 \in b\Omega$. Then Ω admits a smooth local defining function which is plurisubharmonic on $b\Omega$ near p_0 if and only if there exist an open neighborhood U of p_0 and a smooth local defining function ρ for Ω on U such that*

$$\left|H_\rho^{\mathbb{C}}(L,N)\right|^2 = \mathcal{O}(H_\rho^{\mathbb{C}}(L,L)) \text{ on } b\Omega \cap U. \tag{5.3}$$

In fact, (5.3) holds true for every smooth local defining function $\rho\colon U \to \mathbb{R}$ that is plurisubharmonic on $b\Omega \cap U$.

Proof. If $\rho\colon U \to \mathbb{R}$ is plurisubharmonic on $b\Omega \cap U$, then

$$\mathcal{H}^{\mathbb{C}}_{\rho} = \begin{pmatrix} H^{\mathbb{C}}_{\rho}(L,L) & H^{\mathbb{C}}_{\rho}(L,N) \\ H^{\mathbb{C}}_{\rho}(N,L) & H^{\mathbb{C}}_{\rho}(N,N) \end{pmatrix}$$

is positive semi-definite at every point $p \in b\Omega \cap U$. In particular, $\det \mathcal{H}^{\mathbb{C}}_{\rho} \geq 0$ on $b\Omega \cap U$, which implies (5.3).

On the other hand, suppose that (5.3) holds for some smooth local defining function $\rho\colon U \to \mathbb{R}$ of Ω near p_0. Let $\chi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\chi(0) = 0$, $\chi'(0) = 1$, and write $\chi''(0) =: C$. If $\hat{\rho} := \chi \circ \rho$, then we get

$$\mathcal{H}^{\mathbb{C}}_{\hat{\rho}} = \begin{pmatrix} H^{\mathbb{C}}_{\rho}(L,L) & H^{\mathbb{C}}_{\rho}(L,N) \\ H^{\mathbb{C}}_{\rho}(N,L) & H^{\mathbb{C}}_{\rho}(N,N) + C|N\rho|^2 \end{pmatrix} \quad \text{on } b\Omega \cap U.$$

Let $V \Subset U$ be another open neighborhood of p_0. Note that $|N\rho| = \frac{1}{2}|d\rho| > c > 0$ on $b\Omega \cap V$ for some $c \in \mathbb{R}$. Thus, if $C > 0$ is sufficiently large, it follows that $\operatorname{tr} \mathcal{H}^{\mathbb{C}}_{\hat{\rho}} > 0$ and $\det \mathcal{H}^{\mathbb{C}}_{\hat{\rho}} \geq 0$ on $b\Omega \cap V$, where for the second inequality we use assumption (5.3). This means that $\mathcal{H}^{\mathbb{C}}_{\hat{\rho}} \geq 0$ on $b\Omega \cap V$, i.e., $\hat{\rho}$ is plurisubharmonic on $b\Omega \cap V$. □

Remark 5.4. The proof of Proposition 5.2 shows, in particular, the following: if (5.3) holds true, then for every $V \Subset U$ there exists a smooth local defining function for Ω on U that is plurisubharmonic on $b\Omega \cap V$.

Note that (5.3) may be reformulated as

$$\left|H^{\mathbb{C}}_{\rho}(L,N)\right| = \mathcal{O}(\sqrt{\lambda}) \text{ on } b\Omega \cap U,$$

where $\lambda = H^{\mathbb{C}}_{r}(L,L)_{|b\Omega\cap U}$ for any smooth local defining function $r\colon U \to \mathbb{R}$ for Ω and any nonvanishing tangential $(1,0)$-vector field L on U, see Remark 2.9.

Proof of Theorem 5.1. Let $r\colon U \to \mathbb{R}$ be a smooth local defining function for Ω near p_0, and set $\lambda := H^{\mathbb{C}}_{r}(L,L)_{|b\Omega\cap U}$. Since p_0 is non-degenerate, after possibly shrinking U, we may assume that $B := |L\bar{L}\lambda|^2 - |LL\lambda|^2 > 0$ on U. We claim that, for every smooth function $F\colon U \to \mathbb{C}$, there exists a smooth function $h\colon U \to \mathbb{R}$ such that

$$Lh = F + \mathcal{O}(\sqrt{\lambda}) \quad \text{on } b\Omega \cap U. \tag{5.5}$$

Indeed, define smooth functions $A_1, A_2\colon U \to \mathbb{R}$ by

$$A_1 := 2\operatorname{Re}\left((\bar{L}L\lambda - LL\lambda)\bar{L}\lambda\right) \quad \text{and} \quad A_2 := 2\operatorname{Re}\left(i(\bar{L}L\lambda + LL\lambda)\bar{L}\lambda\right),$$

and set

$$h := \operatorname{Re}(F)\frac{A_1}{B} + \operatorname{Im}(F)\frac{A_2}{B}. \tag{5.6}$$

By Lemma 2.11, both $L\lambda$ and $\bar{L}\lambda$ are of class $\mathcal{O}(\sqrt{\lambda})$. Thus

$$Lh = \operatorname{Re}(F)\frac{LA_1}{B} + \operatorname{Im}(F)\frac{LA_2}{B} + \mathcal{O}(\sqrt{\lambda}) \quad \text{on } b\Omega \cap U,$$

and, again on $b\Omega \cap U$,

$$LA_1 = (\bar{L}L\lambda - LL\lambda)L\bar{L}\lambda + (L\bar{L}\lambda - \bar{L}\bar{L}\lambda)LL\lambda + \mathcal{O}(\sqrt{\lambda}) = B + \mathcal{O}(\sqrt{\lambda}),$$
$$LA_2 = i(\bar{L}L\lambda + LL\lambda)L\bar{L}\lambda - i(L\bar{L}\lambda + \bar{L}\bar{L}\lambda)LL\lambda + \mathcal{O}(\sqrt{\lambda}) = iB + \mathcal{O}(\sqrt{\lambda}).$$

Hence h is a solution to (5.5).

Now let $h\colon U \to \mathbb{R}$ be an arbitrary smooth function, and set $\rho := re^h$. Then

$$H^{\mathbb{C}}_{\rho}(L,N) = e^h\left(H^{\mathbb{C}}_{r}(L,N) + Lh \cdot \bar{N}r\right) \quad \text{on } b\Omega \cap U.$$

Since $\bar{N}r = \frac{1}{2}|dr| \neq 0$ on $b\Omega \cap U$, it follows that (5.3) is satisfied if h is the solution for (5.5) with $F := -\frac{2}{|dr|} H_r^{\mathbb{C}}(L, N)$. Thus, the claim follows from Proposition 5.2. □

Remark 5.7. The functions A_1, A_2, and B in the proof of Theorem 5.1 depend on the given smooth local defining function $r\colon U \to \mathbb{R}$. However, the function h defined in (5.6) solves (5.5) for any choice of r.

A global version of Theorem 5.1 easily follows.

Corollary 5.8. *Let $\Omega \Subset \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain. Assume that all weakly pseudoconvex boundary points of $b\Omega$ are non-degenerate. Then Ω admits a smooth defining function which is plurisubharmonic on $b\Omega$.*

Proof. Let $r\colon V \to \mathbb{R}$ be a smooth defining function for Ω. Let $\mathcal{W} \subset b\Omega$ denote the set of points at which Ω is weakly pseudoconvex. Then $\mathcal{W}$ is closed in $b\Omega$. Further, it follows from the hypothesis that $B = |L\bar{L}\lambda|^2 - |LL\lambda|^2$ is strictly positive on some open neighborhood $U \Subset V$ of $\mathcal{W}$. As in the proof of Theorem 5.1, we find a smooth function $h\colon U \to \mathbb{R}$ such that $\rho := re^h$ satisfies (5.3) on $b\Omega \cap U$.

Let $U' \Subset U$ be another open neighborhood of $\mathcal{W}$, and let χ be a real-valued, smooth function which is compactly supported in U and identically 1 on U'. Then $\tilde{\rho} := re^{\chi \cdot h}$ is a smooth defining function for Ω such that $\tilde{\rho} = \rho$ on U', hence it satisfies (5.3) on $b\Omega \cap U'$. Now note that $b\Omega \setminus U'$ is a compact set at whose points Ω is strictly pseudoconvex. Hence, (5.3) is satisfied on $b\Omega \setminus U'$ by any defining function for Ω, in particular by $\tilde{\rho}$. Thus, $\tilde{\rho}$ satisfies (5.3) on all of $b\Omega$. It now follows from Proposition 5.2 that Ω admits a defining function which is plurisubharmonic on $b\Omega$. □

In the following, we give two examples of smoothly bounded, pseudoconvex domains which admit a plurisubharmonic defining function on the boundary, although they have degenerate weakly pseudoconvex boundary points.

Example 5.9. For $(z, w) \in \mathbb{C}^2$, write $z = x + iy$ and $w = u + iv$. Then define $\Omega = \{(z, w) \in \mathbb{C}^2 : r(z, w) < 0\}$ with $r(z, w) := u + f(z)$ for some smooth, subharmonic function f. It follows that for all $\in \mathcal{V}(\mathbb{C}^2)^{1,0}$, $V = V^1 \frac{\partial}{\partial z} + V^2 \frac{\partial}{\partial w}$,

$$H_r^{\mathbb{C}}(V, V) = f_{z\bar{z}} |V^1|^2.$$

Hence, r is a plurisubharmonic defining function for Ω, independent of the type of $b\Omega$ at any of its boundary points. In particular, $b\Omega$ may have a degenerate weakly pseudoconvex boundary point p_0, e.g., if $f(z) = x^4$ and $p_0 = 0$.

Example 5.10. As in the previous example, write $z = x + iy$ and $w = u + iv$. Set

$$U = \{(z, w) \in \mathbb{C}^2 : |x| < \pi/2\},$$

and define $\Omega = \{(z, w) \in U : r(z, w) < 0\}$ for

$$r(z, w) = u - \tfrac{1}{2}(x - v)^2 - \ln(\cos(x)).$$

We compute that, for every $(z, w) \in U$,

$$\begin{aligned} r_z(z, w) &= \tfrac{1}{2}(\tan(x) - (x - v)), \\ r_w(z, w) &= \tfrac{1}{2}(1 - i(x - v)) \\ r_{z\bar{z}}(z, w) &= \tfrac{1}{4}\tan^2(x),\ r_{z\bar{w}}(z, w) = \tfrac{i}{4},\ r_{w\bar{w}}(z, w) = -\tfrac{1}{4}, \end{aligned}$$

so that

$$(r_{z\bar{z}}|r_w|^2)(z,w) = \tfrac{1}{16}\tan^2(x)(1+(x-v)^2),$$
$$(r_{w\bar{w}}|r_z|^2)(z,w) = -\tfrac{1}{16}\left(\tan(x)-(x-v)\right)^2,$$
$$-2\operatorname{Re}[r_{z\bar{w}}r_w r_{\bar{z}}](z,w) = -\tfrac{1}{8}(x-v)\tan(x) + \tfrac{1}{8}\left(x-v\right)^2.$$

Hence, for $L = r_w\frac{\partial}{\partial z} - r_z\frac{\partial}{\partial w}$, we obtain

$$H_r^{\mathbb{C}}(L,L)(z,w) = \tfrac{1}{16}(x-v)^2\sec^2(x) \geq 0,$$

that is, Ω is pseudoconvex. In fact, Ω is strictly pseudoconvex except at boundary points satisfying $x=v$. Moreover, since $-\ln(\cos(x)) = \frac{1}{2}x^2+\frac{1}{12}x^4+o(x^4)$, it follows from Proposition 4.16 that $0\in b\Omega$ is a degenerate weakly pseudoconvex boundary point. In particular, the function h constructed in the proof of Theorem 5.1, see (5.6), is not defined at the origin. However, a straightforward computation, see Example 7.6 in Section 7, shows that $\rho(z,w) := r(z,w)e^y\cos x$ satisfies (5.3), i.e., for every $V \Subset U$ there exists a function $\hat{\rho}\colon U\to\mathbb{R}$ such that $\hat{\rho}$ is plurisubharmonic on $b\Omega\cap V$, see Remark 5.4.

We show in the following that any smooth function h such that re^h is plurisubharmonic on $b\Omega$ near the origin must have nonvanishing derivative with respect to y at the origin, although r is independent of y. This is noteworthy because it shows that, in the case that the domain possesses degenerate weakly pseudoconvex boundary points, the multiplier function h, if it exists, can in general not be given as a combination of derivatives of r as in the non-degenerate case.

Now, suppose h is a positive, smooth function near the origin such that $h_y(0)=0$. A straightforward computation, with $V = -\frac{\partial}{\partial z} + is\frac{\partial}{\partial w}$ for $s>0$, yields

$$H_r^{\mathbb{C}}(V,V)(0) = -\tfrac{1}{2}s + \mathcal{O}(s^2).$$

Moreover,

$$(Vr)(0) = \tfrac{i}{2}s, \text{ and}$$
$$(Vh)(0) = -\tfrac{1}{2}h_x(0) + ish_w(0).$$

Therefore,

$$2\operatorname{Re}\left(Vh\cdot\bar{V}r\right)(0) = \mathcal{O}(s^2).$$

It then follows that

$$H_{e^h r}^{\mathbb{C}}(V,V)(0) = e^{h(0)}\Big(H_r^{\mathbb{C}}(V,V) + 2\operatorname{Re}\left(Vh\cdot\bar{V}r\right)\Big)(0) = -\tfrac{1}{2}s+\mathcal{O}(s^2) < 0$$

for all $s>0$ sufficiently close to 0.

6. Plurisubharmonicity near the boundary

In this section, we first consider smoothly bounded, pseudoconvex domains in $\mathbb{C}^2$ such that all weakly pseudoconvex boundary points are of type 4. In the case of bounded domains, we show that there exists a smooth, plurisubharmonic defining function for the domain whenever there is a smooth defining function which is plurisubharmonic on the boundary of the domain.

In the latter part of this section, we consider smoothly bounded, pseudoconvex domains in $\mathbb{C}^2$ that are at least of type 6 at their weakly pseudoconvex boundary points. In the case that such a domain admits a smooth defining function which is plurisubharmonic on the boundary, we give a simplified proof that both the Diederich–Fornæss index and the Steinness index are 1.

A lack of understanding of the notion of existence of a plurisubharmonic defining function is rooted in the lack of an equivalent condition which is checkable for *any* defining function. The following lemma yields a condition which is checkable on a class of defining functions strictly larger than the class of plurisubharmonic defining functions. A version of this lemma in the context of convex domains is given in [10, Proposition 6.17].

Lemma 6.1. *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded, pseudoconvex domain, $p_0 \in b\Omega$. Then Ω admits a smooth local defining function which is plurisubharmonic near p_0 if and only if there exists a smooth local defining function r for Ω near p_0 such that*

$$H_r^{\mathbb{C}}(\xi,\xi) \geq -C\left(r^2|\xi|^2 + |\langle \partial r, \xi\rangle|^2\right) \quad \forall\, \xi \in \mathbb{C}^n \tag{6.2}$$

for some constant $C > 0$.[1]

To put (6.2) in context, we recall that for any smoothly bounded, pseudoconvex domain $\Omega \Subset \mathbb{C}^n$ with smooth defining function r, there exist an open neighborhood U of the boundary of Ω and a constant $C > 0$ such that

$$H_r^{\mathbb{C}}(\xi,\xi) \geq -C\left(|r|\cdot|\xi|^2 + |\langle \partial r, \xi\rangle|\cdot|\xi|\right) \quad \forall \xi \in \mathbb{C}^n \tag{6.3}$$

on U. That (6.3) holds true on $\Omega \cap U$ is derived by Range, see (5) in [16], to reprove the result of Diederich–Fornæss [4, Theorem 1] on the existence of bounded, strictly plurisubharmonic exhaustion functions for smoothly bounded, pseudoconvex domains, see [16, Theorem 2]. Arguments similar to the ones in [16] yield (6.3) on U. We note that, if for every $\varepsilon > 0$, there exists a smooth defining function $r = r_\varepsilon$ such that (6.3) holds with $C = \varepsilon$, then both the Diederich–Fornæss index and the Steinness index are 1, see the proof of Corollary 1.6 in [8].

Proof of Lemma 6.1. Note first that if r is a smooth local defining function for Ω which is plurisubharmonic on an open neighborhood U of p_0, then (6.2) holds trivially for r on U.

Let $p_0 \in b\Omega$ and let $U \Subset \mathbb{C}^n$ be an open neighborhood of p_0. Now suppose that $r : U \longrightarrow \mathbb{R}$ is a smooth local defining function for Ω on U such that (6.2) holds. Consider $\rho := r + r^2\psi$ with $\psi(z) := K_1 + K_2|z|^2$ for fixed, positive constants K_1 and K_2 to be determined later. It follows from a straightforward computation that

$$H_\rho^{\mathbb{C}}(\xi,\xi) = (1+2r\psi)H_r^{\mathbb{C}}(\xi,\xi) + 2\psi|\langle \partial r,\xi\rangle|^2 + 4K_2 r\,\mathrm{Re}\left(\langle \partial r,\xi\rangle\langle z,\xi\rangle\right) + r^2K_2|\xi|^2$$

for $\xi \in \mathbb{C}^n$. Next, it follows from

$$2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2 \quad \text{for } a,b \geq 0 \text{ and } \varepsilon > 0, \tag{6.4}$$

with $a = |r||\xi|$, $b = |z||\langle \partial r,\xi\rangle|$ and $\varepsilon = \frac{1}{4}$, that

$$4K_2\left|r\,\mathrm{Re}\big(\langle \partial r,\xi\rangle\langle z,\xi\rangle\big)\right| \leq r^2\frac{K_2}{2}|\xi|^2 + 8K_2|z|^2|\langle \partial r,\xi\rangle|^2$$

holds. Therefore, we obtain

$$H_\rho^{\mathbb{C}}(\xi,\xi) \geq (1+2r\psi)H_r^{\mathbb{C}}(\xi,\xi) + 2\left|\langle \partial r,\xi\rangle\right|^2\left(K_1 - 3K_2|z|^2\right) + r^2\frac{K_2}{2}|\xi|^2.$$

[1]Here, and occasionally later on, we consider the complex Hessian form $H_r^{\mathbb{C}}$ at a point as a sesquilinear form on $\mathbb{C}^n$.

Next, for each pair $K := (K_1, K_2)$, there exists an open neighborhood $U_K \subset U$ of p_0 such that

$$1 + 2r(z)(K_1 + K_2|z|^2) \leq 3/2$$

holds for all $z \in U_K$. Note that U_K may be chosen such that $b\Omega \cap U = b\Omega \cap U_K$. Using (6.2), we then obtain on U_K

$$(1 + 2r\psi)H_r^{\mathbb{C}}(\xi, \xi) \geq -\frac{3C}{2}\left(r^2|\xi|^2 + |\langle \partial r, \xi\rangle|^2\right)$$

for all $\xi \in \mathbb{C}^n$. Therefore,

$$H_\rho^{\mathbb{C}}(\xi, \xi) \geq r^2|\xi|^2\left(\frac{K_2}{2} - \frac{3}{2}C\right) + 2\,|\langle \partial r, \xi\rangle|^2\left(K_1 - 3K_2|z|^2 - \frac{3}{4}C\right)$$

holds on U_K for all $\xi \in \mathbb{C}^n$. Fix K_2 such that $K_2 > 3C$ holds. Let D be the maximum of $|z|$ on U, then fix K_1 such that $K_1 > 3K_2D^2 + \frac{3}{4}C$ holds. It follows easily that there exists a positive contant $c > 0$ such that

$$H_\rho^{\mathbb{C}}(\xi, \xi) \geq c\left(\rho^2|\xi|^2 + |\langle \partial\rho, \xi\rangle|^2\right) \tag{6.5}$$

on U_K for all $\xi \in \mathbb{C}^n$, i.e., ρ is plurisubharmonic on U_K. □

Remark 6.6. Note that the function ρ constructed in the proof of Lemma 6.1 is strictly plurisubharmonic on $U_K \setminus b\Omega$, see (6.5). Moreover, the complex Hessian of ρ is positive definite at strictly pseudoconvex boundary points of Ω. To wit, the complex Hessian of ρ is strictly positive in non-zero complex tangential directions at these boundary points by definition, and it is strictly positive in all directions with a non-vanishing normal component to the boundary by (6.5).

We note that a global version of Lemma 6.1 holds if Ω is bounded and U is an open neighborhood of $b\Omega$. Moreover, by a result of Morrow–Rossi [14, Lemma 1.3], see also [15], any smoothly bounded, strictly pseudoconvex, bounded domain in $\mathbb{C}^n$ admits a smooth defining function which is strictly plurisubharmonic in an open neighborhood of the closure of the domain. The same argument as the one used in the proof of Lemma 1.3 in [14] yields the following.

Corollary 6.7. *Let $\Omega \Subset \mathbb{C}^n$ be a smoothly bounded domain. Assume that there exists a smooth defining function r for Ω such that*

$$H_r^{\mathbb{C}}(\xi, \xi) \geq -C\left(r^2|\xi|^2 + |\langle \partial r, \xi\rangle|^2\right) \quad \forall \xi \in \mathbb{C}^n$$

holds near $b\Omega$ for some constant $C > 0$. Then there exists a smooth defining function ρ for Ω on an open neighborhood of $\bar{\Omega}$ such that

$$H_\rho^{\mathbb{C}}(\xi, \xi) \geq c\left(\rho^2|\xi|^2 + |\langle \partial\rho, \xi\rangle|^2\right) \quad \forall \xi \in \mathbb{C}^n$$

holds for some $c > 0$.

Whether a given local defining function actually satisfies condition (6.2) in some open neighborhood of the boundary, can be detected from the behaviour of the complex Hessian of that defining function and its normal derivative *on the boundary of the domain* as follows.

Proposition 6.8. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain, $p_0 \in b\Omega$. Then Ω admits a smooth local defining function which is plurisubharmonic* *near p_0 if and only if there exist an open neighborhood U of p_0 and a smooth local defining function r for Ω on U such that*

$$|H^{\mathbb{C}}_r(L,N)|^2 = \mathcal{O}\left(H^{\mathbb{C}}_r(L,L)\right) \quad \text{on } b\Omega\cap U, \quad \text{and} \tag{6.9}$$

$$|\nu H^{\mathbb{C}}_r(L,L)|^2 = \mathcal{O}\left(H^{\mathbb{C}}_r(L,L)\right) \quad \text{on } b\Omega\cap U. \tag{6.10}$$

Proof. Let $r\colon U\to\mathbb{R}$ be a smooth local defining function for Ω near p_0, and let L be a nonvanishing tangential $(1,0)$-vector field on U. Note that the normal derivative $(\nu H^{\mathbb{C}}_r(L,L))_{|b\Omega\cap U}$ depends on L, while $H^{\mathbb{C}}_r(L,L)_{|b\Omega\cap U}$ depends only on $L_{|b\Omega\cap U}$. However, in case that (6.9) holds true, the condition (6.10) is in fact independent of the choice of L. To see this, let L' be another nonvanishing tangential $(1,0)$-vector field on U. Then there exist a smooth function $h\colon U\to\mathbb{C}$ and a vector field $E\in\mathcal{V}(U)^{1,0}$ such that $L'=hL+rE$. Thus

$$H^{\mathbb{C}}_r(L',L') = |h|^2H^{\mathbb{C}}_r(L,L) + 2r\operatorname{Re}\big(hH^{\mathbb{C}}_r(L,E)\big) + r^2H^{\mathbb{C}}_r(E,E),$$

so that

$$\nu H^{\mathbb{C}}_r(L',L') = |h|^2\nu H^{\mathbb{C}}_r(L,L) + 2|dr|\operatorname{Re}\big(hH^{\mathbb{C}}_r(L,E)\big) + \mathcal{O}(\lambda) \ \text{ on } b\Omega\cap U.$$

Since $E=aL+bN$ on $b\Omega\cap U$ for smooth functions a,b, it follows from (6.9) that $H^{\mathbb{C}}_r(L,E)=\mathcal{O}(\sqrt{\lambda})$ on $b\Omega\cap U$. In particular, $\nu H^{\mathbb{C}}_r(L',L') = |h|^2\nu H^{\mathbb{C}}_r(L,L) + \mathcal{O}(\sqrt{\lambda})$ on $b\Omega\cap U$, which shows that (6.10) is well-defined.

Now suppose first that Ω admits a plurisubharmonic, smooth local defining function $r\colon U\to\mathbb{R}$ near p_0. Then (6.9) holds by Proposition 5.2. Moreover, an application of the first part of Lemma 2.11, with $f=H^{\mathbb{C}}_r(L,L)$, shows that (6.10) is satisfied.

On the other hand, let $r\colon U\to\mathbb{R}$ be a smooth local defining function for Ω near p_0 such that (6.9) and (6.10) hold true. After possibly shrinking U in the direction normal to $b\Omega$, we may assume that there exists a smooth map $\pi\colon U\to b\Omega$ such $|q-\pi(q)|=d_{b\Omega}(q)$ (see Section 2.6 for the definition of $d_{b\Omega}$). Fix $q\in U$, and set $p:=\pi(q)$. Moreover, fix $\xi\in\mathbb{C}^2$, and write $\xi=aL(q)+bN(q)$ for some $a=a(q,\xi), b=b(q,\xi)\in\mathbb{C}$. Then

$$H^{\mathbb{C}}_r(\xi,\xi)(q) = |a|^2H^{\mathbb{C}}_r(L,L)(q) + 2\operatorname{Re}(a\bar{b}H^{\mathbb{C}}_r(L,N)(q)) + |b|^2H^{\mathbb{C}}_r(N,N)(q).$$

In view of (2.10), with $f=H^{\mathbb{C}}_r(L,L)$, the Taylor expansion at p in direction ν gives

$$\begin{aligned}H^{\mathbb{C}}_r(\xi,\xi)(q) = {}& |a|^2\left(H^{\mathbb{C}}_r(L,L)(p) + \delta_{b\Omega}(q)(\nu H^{\mathbb{C}}_r(L,L))(p) + \mathcal{O}(d^2_{b\Omega})(q)\right)\\ &+ 2\operatorname{Re}\left(a\bar{b}(H^{\mathbb{C}}_r(L,N)(p) + \mathcal{O}(d_{b\Omega})(q))\right)\\ &+ |b|^2\mathcal{O}(1)(q),\end{aligned}$$

where the $\mathcal{O}$-terms are functions that do not depend on q and ξ. Fix an open neighborhood $V\Subset U$ of p_0. Using (6.4), it follows from (6.10) and (6.9) that there exist constants $C_1,C_2>0$ such that on V

$$\tfrac{1}{2}H^{\mathbb{C}}_r(L,L)(p) + \delta_{b\Omega}(q)(\nu H^{\mathbb{C}}_r(L,L))(p) \geq -C_1d^2_{b\Omega}(q),$$

$$\tfrac{1}{2}|a|^2H^{\mathbb{C}}_r(L,L)(p) + 2\operatorname{Re}\left(a\bar{b}H^{\mathbb{C}}_r(L,N)(p)\right) \geq -C_2|b|^2.$$

Hence, there exists a constant $C_3>0$, which does not depend on q and ξ, such that

$$H^{\mathbb{C}}_r(\xi,\xi)(q) \geq -C_3(|a|^2d^2_{b\Omega}(q)+|b|^2) \quad \forall\,\xi\in\mathbb{C}^n \ \ \forall\,q\in V.$$

Since $|a|\leq|\xi|$, $|b|=\mathcal{O}(|\langle\partial r,\xi\rangle|)$ on U, and $d^2_{b\Omega}=\mathcal{O}(r^2)$ on U, it follows that $r_{|V}$ satisfies (6.2). The claim thus follows from Lemma 6.1. □

Remark 6.11. The proofs of Lemma 6.1 and Proposition 6.8 imply the following: if (6.9) and (6.10) hold true, then for every $K \Subset b\Omega \cap U$ there exist an open neighborhood $V \subset U$ of K and a smooth local defining function for Ω on U that is plurisubharmonic on V.

Remark 6.12. Versions of Proposition 5.2 and Proposition 6.8 can also be shown for domains in $\mathbb{C}^n$, $n > 2$. In this case, L has to be substituted by a frame $\{L_j\}_{j=1}^{n-1}$ for $T(b\Omega)^{1,0}$ near p_0.

Condition (6.10) may always be achieved near boundary points of type 4, independent of whether the smoothly bounded, pseudoconvex domain in consideration actually admits a smooth local defining function which is plurisubharmonic on the boundary.

Lemma 6.13. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain, $p_0 \in b\Omega$. If $b\Omega$ is of type 4 at p_0, then for every nonvanishing tangential $(1,0)$-vector field L near p_0 there exists a smooth local defining function $\rho\colon U \to \mathbb{R}$ for Ω near p_0 such that*

$$\left|\nu H_\rho^{\mathbb{C}}(L,L)\right|^2 = \mathcal{O}\left(H_\rho^{\mathbb{C}}(L,L)\right) \quad \text{on } b\Omega \cap U. \tag{6.14}$$

Proof. Let $r\colon U \to \mathbb{R}$ be a smooth local defining function for Ω near p_0, and let L be a nonvanishing tangential $(1,0)$-vector field near p_0. After possibly shrinking U, we may assume that $dr \neq 0$, and that L is defined on U. Set $\Lambda := H_r^{\mathbb{C}}(L,L)$ and $\lambda := \Lambda_{|b\Omega\cap U}$. We claim that, after possibly shrinking U, for every smooth function $F\colon U \to \mathbb{R}$, there exists a smooth function $h\colon U \to \mathbb{R}$ such that

$$\begin{cases} Lh = \mathcal{O}(\sqrt{\lambda}) \\ L\bar{L}h = F + \mathcal{O}(\sqrt{\lambda}) \end{cases} \quad \text{on } b\Omega \cap U. \tag{6.15}$$

Indeed, define smooth functions $A, B\colon U \to \mathbb{R}$ by

$$A := |L\Lambda|^2 \quad \text{and} \quad B := \Lambda^2 + |L\bar{L}\Lambda|^2 + |LL\Lambda|^2,$$

and set

$$h := F\frac{A}{B}. \tag{6.16}$$

Since Ω is of type 4 at p_0, after possibly shrinking U, we can assume that $B > 0$ on U. Thus, h is well-defined. Moreover, since $L\Lambda = \mathcal{O}(\sqrt{\lambda})$ on $b\Omega \cap U$ by (2.13), it follows that A and LA are of class $\mathcal{O}(\sqrt{\lambda})$ on $b\Omega \cap U$, and, in particular, that

$$Lh = \mathcal{O}(\sqrt{\lambda}) \quad \text{on } b\Omega \cap U. \tag{6.17}$$

Moreover,

$$L\bar{L}h = F\frac{L\bar{L}A}{B} + \mathcal{O}(\sqrt{\lambda}) = F\frac{|L\bar{L}\Lambda|^2 + |LL\Lambda|^2}{B} + \mathcal{O}(\sqrt{\lambda}) = F + \mathcal{O}(\sqrt{\lambda}) \quad \text{on } b\Omega \cap U.$$

Now let $h\colon U \to \mathbb{R}$ be an arbitrary smooth function, and set $\rho := re^h$. Then

$$H_\rho^{\mathbb{C}}(L,L) = e^h\left(H_r^{\mathbb{C}}(L,L) + r\left(H_h^{\mathbb{C}}(L,L) + |Lh|^2\right)\right),$$

and thus

$$\nu H_\rho^{\mathbb{C}}(L,L) = e^h\left(\nu H_r^{\mathbb{C}}(L,L) + |dr|\left(H_h^{\mathbb{C}}(L,L) + |Lh|^2\right)\right) + \mathcal{O}(\lambda) \quad \text{on } b\Omega \cap U.$$

Since $H^{\mathbb{C}}_h(L,L) = L\bar{L}h - (\nabla_L \bar{L})h$, and since on $b\Omega \cap U$ it follows from $H^{\mathbb{C}}_r(L,L) = -(\nabla_L \bar{L})r$ on $b\Omega \cap U$ that the component of $\nabla_L \bar{L}$ normal to $b\Omega$ is of the form $\mathcal{O}(\lambda)$, it follows from (6.17) that

$$\nu H^{\mathbb{C}}_\rho(L,L) = e^h\left(\nu H^{\mathbb{C}}_r(L,L) + |dr| L\bar{L}h\right) + \mathcal{O}(\sqrt{\lambda}) \quad \text{on } b\Omega \cap U.$$

Thus, if h is the solution for (6.15) with

$$F := -\frac{1}{|dr|}\nu H^{\mathbb{C}}_r(L,L), \tag{6.18}$$

then ρ satisfies (6.14). □

We now can prove the main result of this section.

Theorem 6.19. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain. Suppose $p_0 \in b\Omega$ is such that $c_{p_0} = 4$. If Ω admits a smooth local defining function near p_0 which is plurisubharmonic on $b\Omega$ near p_0, then Ω admits a smooth local defining function near p_0 which is plurisubharmonic.*

Proof. Let $r\colon U \to \mathbb{R}$ be a smooth local defining function for Ω near p_0 such that r is plurisubharmonic on $b\Omega \cap U$. After possibly shrinking U, we can assume that $c_p \leq 4$ for all $p \in b\Omega \cap U$. Set $\rho := re^h$, where h is the solution to (6.15) with F given as in (6.18). Then, as in the proof of Lemma 6.13, we see that ρ satisfies (6.10). On the other hand, since, by Proposition 5.2, r satisfies (6.9), and since

$$H^{\mathbb{C}}_\rho(L,N) = e^h\left(H^{\mathbb{C}}_r(L,N) + Lh \cdot \bar{N}r\right) \quad \text{on } b\Omega \cap U,$$

it follows with (6.17) that ρ satisfies (6.9). The claim thus follows from Proposition 6.8. □

Note that, if r is a smooth defining function for a smoothly bounded, pseudoconvex domain $\Omega \subset \mathbb{C}^2$ such that Ω is of type 4 at all its weakly pseudoconvex boundary points, then the function h defined in (6.16) and (6.18) is defined in an open neighborhood of $b\Omega$. Moreover, the function h solves (6.15) on $b\Omega$. In view of Remark 6.11, this implies the following global result.

Corollary 6.20. *Let $\Omega \Subset \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain. Suppose that Ω has a smooth defining function which is plurisubharmonic on $b\Omega$, and that $c_p \leq 4$ for all $p \in b\Omega$. Then Ω admits a smooth defining function which is plurisubharmonic in an open neighborhood of $b\Omega$.*

An analogon to Theorem 6.19 near higher order boundary points is not apparent, although condition (6.10) always holds *at* boundary points of type larger than 4, whenever r is a defining function that is plurisubharmonic on the boundary, as shown by the next Lemma.

Lemma 6.21. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain. Let $p_0 \in b\Omega$ such that $c_{p_0} \geq 6$. If r is a smooth local defining function of $b\Omega$ near p_0 which is plurisubharmonic on $b\Omega$ near p_0, and if L is a nonvanishing tangential $(1,0)$-vector field near p_0, then*

$$\nu H^{\mathbb{C}}_r(L,L)(p_0) = 0.$$

Proof. Let r be a smooth local defining function for Ω on some open neighborhood U of p_0 such that r is plurisubharmonic on $b\Omega \cap U$. After possibly shrinking U, we may assume that $N := N_r$ is defined on U, see (2.6). As usual, we write $L = \frac{1}{2}(X + iY)$ with $X, Y \in \mathcal{V}(U)$.

Consider the function $g\colon b\Omega \cap U \to \mathbb{R}$ given by

$$g = H_r^{\mathbb{C}}(L, L) H_r^{\mathbb{C}}(N, N) - |H_r^{\mathbb{C}}(L, N)|^2 \quad \text{on } b\Omega \cap U.$$

Since r is plurisubharmonic on $b\Omega \cap U$, it follows that $g \geq 0$, and since Ω is weakly pseudoconvex at p_0, it follows that $g(p_0) = 0$. Thus, $(XXg)(p_0) \geq 0$ and $(YYg)(p_0) \geq 0$, see Lemma 4.7. Since $L\bar{L}g = \frac{1}{4}(XXg + YYg) - \frac{i}{4}[X, Y]g$, and since the tangential derivative $[X, Y]g$ vanishes at p_0, it follows that $(L\bar{L}g)(p_0) \geq 0$. Moreover, the fact that $c_{p_0} > 4$ implies that for $\lambda := H_r^{\mathbb{C}}(L, L)_{|b\Omega \cap U}$ the functions $\lambda, L\lambda, \bar{L}\lambda, L\bar{L}\lambda$ all vanish at p_0. Hence, since $H_r^{\mathbb{C}}(L, N)(p_0) = 0$,

$$(L\bar{L}g)(p_0) = -\left|LH_r^{\mathbb{C}}(L, N)\right|^2(p_0) - \left|\bar{L}H_r^{\mathbb{C}}(L, N)\right|^2(p_0).$$

In particular, it follows that $\bar{L}H_r^{\mathbb{C}}(L, N)(p_0) = 0$.

However, if we denote coordinates in $\mathbb{C}^2$ by $z = (z_1, z_2)$, and if we write $L = \sum_{j=1}^2 L^j \frac{\partial}{\partial z_j}$, $N = \sum_{j=1}^2 N^j \frac{\partial}{\partial z_j}$, then

$$\bar{L}H_r^{\mathbb{C}}(L, N) = \sum_{j,k,\ell=1}^{2} \frac{\partial^3 r}{\partial \bar{z}_\ell \partial z_j \partial \bar{z}_k} \bar{L}^\ell L^j \bar{N}^k + H_r^{\mathbb{C}}(\nabla_{\bar{L}} L, N) + H_r^{\mathbb{C}}(L, \nabla_L N), \tag{6.22}$$

$$\bar{N}H_r^{\mathbb{C}}(L, L) = \sum_{j,k,\ell=1}^{2} \frac{\partial^3 r}{\partial \bar{z}_\ell \partial z_j \partial \bar{z}_k} \bar{N}^\ell L^j \bar{L}^k + H_r^{\mathbb{C}}(\nabla_{\bar{N}} L, L) + H_r^{\mathbb{C}}(L, \nabla_N L). \tag{6.23}$$

Since Ω is weakly pseudoconvex at p_0, one has $H_r^{\mathbb{C}}(L, L)(p_0) = H_r^{\mathbb{C}}(L, N)(p_0) = 0$, i.e., $H_r^{\mathbb{C}}(L, \cdot)(p_0)\colon \mathcal{V}(U)^{1,0} \to \mathbb{R}$ is identically zero. Moreover, since $0 \equiv L\bar{L}r = H_r^{\mathbb{C}}(L, L) + (\nabla_L \bar{L})r$, it follows that $((\nabla_L \bar{L})r)(p_0) = 0$, i.e., $(\nabla_L \bar{L})_{p_0} = cL_{p_0}$ for some constant $c \in \mathbb{C}$. Thus, the two rightmost terms in both (6.22) and (6.23) vanish at p_0, which proves that $\bar{L}H_r^{\mathbb{C}}(L, N)(p_0) = \bar{N}H_r^{\mathbb{C}}(L, L)(p_0)$. Since all tangential derivatives of $H_r^{\mathbb{C}}(L, L)$ vanish at p_0, it follows that $\nu H_r^{\mathbb{C}}(L, L)(p_0) = 2\bar{N}H_r^{\mathbb{C}}(L, L)(p_0) = 0$, which completes the proof. □

This weaker result for boundary points of type greater than 4 leads to a simplified proof of the Diederich–Fornæss index and the Steinness index being 1 for smoothly bounded, pseudoconvex domains which admit a smooth defining function that is plurisubharmonic on the boundary of the domain.

Corollary 6.24. *Let $\Omega \Subset \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain. Suppose that*

(i) $c_p \neq 4$ for all $p \in b\Omega$, and
(ii) Ω admits a smooth defining function which is plurisubharmonic on $b\Omega$.

Then for every $\eta \in (0, 1)$ there exist a constant $K > 0$ and an open neighborhood U of $b\Omega$ such that $-(-r - Kr^2)^\eta$ is plurisubharmonic on $\Omega \cap U$.

Similarly, for every $\mu > 1$ there exist a constant $K > 0$ and an open neighborhood U of $b\Omega$ such that $-(-r - Kr^2)^\mu$ is plurisubharmonic on $\Omega^c \cap U$.

We note that (ii) itself leads to the Diederich–Fornæss index being 1, see [7]. However, the additional condition (i) simplifies the construction in [7, Section 3] considerably.

Proof. Let r be a smooth defining function of Ω which is plurisubharmonic on $b\Omega$, and assume that $L := L_r$ and $N := N_r$ are defined on some open neighborhood U of $b\Omega$. After possibly shrinking U, we may use (2.10) for $f = H_r^{\mathbb{C}}(L, L)$ and $q \in U$ with $c_{\pi(q)} \geq 6$. It then follows from Lemma 6.21 that

$$H_r^{\mathbb{C}}(L, L)(q) = \mathcal{O}(r^2)(q).$$

Similarly, one obtains $H_r^{\mathbb{C}}(L, N)(q) = \mathcal{O}(r)(q)$ and $H_r^{\mathbb{C}}(N, N)(q) = \mathcal{O}(1)(q)$. It then follows

$$H_r^{\mathbb{C}}(\xi, \xi)(q) = \left(\mathcal{O}(r^2)|\xi|^2 + \mathcal{O}(|\langle \partial r, \xi\rangle|^2)\right)(q) \quad \forall\, \xi \in \mathbb{C}^n \;\; \forall\, q \in U \text{ with } c_{\pi(q)} \geq 6.$$

The arguments following (3.7) in [7] then prove the claim. □

7. On a special class of pseudoconvex domains

In this section, we derive a sufficient condition for the existence of local defining functions, which are plurisubharmonic on the boundary, in terms of real coordinates. While this condition, in contrast to the criterion given in Proposition 5.2, is not an equivalent characterization, it has the advantage of being independent of the choice of defining function, and thus is more easily checkable.

Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded domain, and let $r\colon U \to \mathbb{R}$ be a smooth local defining function for Ω near some point $p_0 \in b\Omega$. After possibly shrinking U, let L be a nonvanishing tangential $(1,0)$-vector field on U, and let $N = N_r$ be defined as in (2.6). Write

$$L = \tfrac{1}{2}(X + iY), \quad N = \tfrac{1}{2}(\nu + iT)$$

with $X, Y, T, \nu \in \mathcal{V}(U)$. The matrix associated with the real Hessian form $Q_r^{\mathbb{R}}\colon \mathcal{V}(U) \times \mathcal{V}(U) \to \mathbb{R}$ relative to the basis (X, Y, T, ν) will be denoted by $\mathcal{Q}_r^{\mathbb{R}}$, i.e.,

$$\mathcal{Q}_r^{\mathbb{R}} := \begin{pmatrix} Q_r^{\mathbb{R}}(X,X) & Q_r^{\mathbb{R}}(X,Y) & Q_r^{\mathbb{R}}(X,T) & Q_r^{\mathbb{R}}(X,\nu) \\ Q_r^{\mathbb{R}}(Y,X) & Q_r^{\mathbb{R}}(Y,Y) & Q_r^{\mathbb{R}}(Y,T) & Q_r^{\mathbb{R}}(Y,\nu) \\ Q_r^{\mathbb{R}}(T,X) & Q_r^{\mathbb{R}}(T,Y) & Q_r^{\mathbb{R}}(T,T) & Q_r^{\mathbb{R}}(T,\nu) \\ Q_r^{\mathbb{R}}(\nu,X) & Q_r^{\mathbb{R}}(\nu,Y) & Q_r^{\mathbb{R}}(\nu,T) & Q_r^{\mathbb{R}}(\nu,\nu) \end{pmatrix}.$$

We readily recognize that various convexity-like boundary conditions for Ω near p_0 may be expressed through conditions on entries of the leading principal 3×3 submatrix of $\mathcal{Q}_r^{\mathbb{R}}$ for $p \in b\Omega$ near p_0.

(i) Ω is convex near p_0 if the leading principal 3×3 submatrix of $\mathcal{Q}_r^{\mathbb{R}}(p)$ is positive semi-definite for all $p \in b\Omega$ near p_0.
(ii) Ω is $\mathbb{C}$-convex near p_0 if the leading principal 2×2 submatrix of $\mathcal{Q}_r^{\mathbb{R}}(p)$ is positive semi-definite for all $p \in b\Omega$ near p_0.
(iii) Ω is pseudoconvex near p_0 if the trace of the leading principal 2×2 submatrix of $\mathcal{Q}_r^{\mathbb{R}}(p)$ is non-negative for all $p \in b\Omega$ near p_0.

In order to see how to express plurisubharmonicity on the boundary of a smooth local defining function in real coordinates, we need to formulate condition (5.3) in real coordinates. Thus, we compute

$$\begin{aligned}4H_r^{\mathbb{C}}(L,N) &= 4L(\bar{N}r) - 4\left(\nabla_L \bar{N}\right) r \\ &= (X+iY)(\nu - iT)r - \left(\nabla_{X+iY}(\nu - iT)\right) r \\ &= X\nu r - (\nabla_X \nu)r + YTr - (\nabla_Y T)r \\ &\quad + i\big(Y\nu r - (\nabla_Y \nu)r - XTr + (\nabla_X T)r\big) \\ &= Q_r^{\mathbb{R}}(X,\nu) + Q_r^{\mathbb{R}}(Y,T) + i\big(Q_r^{\mathbb{R}}(Y,\nu) - Q_r^{\mathbb{R}}(X,T)\big).\end{aligned} \tag{7.1}$$

Proposition 5.2 may now be reformulated in terms of entries of $\mathcal{Q}_r^{\mathbb{R}}$ as follows.

Lemma 7.2. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain, $p_0 \in b\Omega$. Then Ω admits a smooth local defining function which is plurisubharmonic on $b\Omega$ near p_0 if and only if there exists a smooth local defining function $\rho\colon U \to \mathbb{R}$ for Ω on some open neighborhood U of p_0 such that*

$$\big(Q_\rho^{\mathbb{R}}(X,\nu) + Q_\rho^{\mathbb{R}}(Y,T)\big)^2 + \big(Q_\rho^{\mathbb{R}}(Y,\nu) - Q_\rho^{\mathbb{R}}(X,T)\big)^2 = \mathcal{O}\left(H_\rho^{\mathbb{C}}(L,L)\right) \tag{7.3}$$

on $b\Omega \cap U$.

In the proof of Theorem 5.1 it is shown that, given any smooth local defining function $r\colon U \to \mathbb{R}$ for Ω, then $\rho := re^h$ satisfies (7.3) if $h \in \mathcal{C}^\infty(U,\mathbb{R})$ solves the equation

$$Lh = -\frac{2}{|dr|} H_r^{\mathbb{C}}(L,N) + \mathcal{O}(\sqrt{\lambda}) \quad \text{on } b\Omega \cap U,$$

with $\lambda := H_r^{\mathbb{C}}(L,L)_{|b\Omega\cap U}$. This can be reformulated in real coordinates as follows,

$$Xh = \frac{-1}{|dr|}\big(Q_r^{\mathbb{R}}(X,\nu) + Q_r^{\mathbb{R}}(Y,T)\big) + \mathcal{O}(\sqrt{\lambda}) \quad \text{on } b\Omega \cap U, \tag{7.4}$$

$$Yh = \frac{1}{|dr|}\big(Q_r^{\mathbb{R}}(X,T) - Q_r^{\mathbb{R}}(Y,\nu)\big) + \mathcal{O}(\sqrt{\lambda}) \quad \text{on } b\Omega \cap U. \tag{7.5}$$

Example 7.6. Let us revisit Example 5.10. There, in $\mathbb{C}^2$ with coordinates $z = x+iy$ and $w = u+iv$, we consider $\Omega = \{(z,w) \in U : r(z,w) < 0\}$ with

$$U = \{(z,w) \in \mathbb{C}^2 : |x| < \pi/2\}, \quad r(z,w) = u - \tfrac{1}{2}(x-v)^2 - \ln(\cos(x)).$$

We already computed that, with $L = r_w \frac{\partial}{\partial z} - r_z \frac{\partial}{\partial w}$,

$$H_r^{\mathbb{C}}(L,L)(z,w) = \tfrac{1}{16}(x-v)^2 \sec^2(x).$$

In particular, note that the function $P(z,w) := x - v$ is of class $\mathcal{O}(\sqrt{\lambda})$ on $b\Omega$. Considering real vector fields on $\mathbb{C}^2$ as maps to $\mathbb{R}^4$, we can then compute further that

$$\begin{aligned}X &= \frac{1}{|dr|}(r_u, -r_v, -r_x, r_y) = \frac{1}{|dr|}\left(1, 0, -\tan(x), 0\right) + \mathcal{O}(\sqrt{\lambda}), \\ Y &= \frac{1}{|dr|}(-r_v, -r_u, r_y, r_x) = \frac{1}{|dr|}\left(0, -1, 0, \tan(x)\right) + \mathcal{O}(\sqrt{\lambda}), \\ T &= \frac{1}{|dr|}(r_y, -r_x, r_v, -r_u) = \frac{1}{|dr|}\left(0, -\tan(x), 0, -1\right) + \mathcal{O}(\sqrt{\lambda}), \\ \nu &= \frac{1}{|dr|}(r_x, r_y, r_u, r_v) = \frac{1}{|dr|}\left(\tan(x), 0, 1, 0\right) + \mathcal{O}(\sqrt{\lambda}),\end{aligned}$$

where, in slight deviation from previous notation, the terms $\mathcal{O}(\sqrt{\lambda})$ denote vector fields with coefficients that are of class $\mathcal{O}(\sqrt{\lambda})$ on $b\Omega$. From this, it follows readily that on $b\Omega$

$$Q_r^{\mathbb{R}}(X,\nu) = \frac{1}{|dr|^2}\tan^3(x) + \mathcal{O}(\sqrt{\lambda}), \qquad Q_r^{\mathbb{R}}(Y,T) = \frac{1}{|dr|^2}\tan(x) + \mathcal{O}(\sqrt{\lambda}),$$
$$Q_r^{\mathbb{R}}(X,T) = -\frac{1}{|dr|^2} + \mathcal{O}(\sqrt{\lambda}), \qquad Q_r^{\mathbb{R}}(Y,\nu) = \frac{1}{|dr|^2}\tan^2(x) + \mathcal{O}(\sqrt{\lambda}).$$

Since $|dr|^2 = 1 + \tan^2(x) + \mathcal{O}(\lambda)$ on $b\Omega$, it follows further that (7.4) and (7.5) are given by

$$h_x - \tan(x)h_u = -\tan(x) + \mathcal{O}(\sqrt{\lambda}) \quad \text{on } b\Omega,$$
$$-h_y + \tan(x)h_v = -1 + \mathcal{O}(\sqrt{\lambda}) \quad \text{on } b\Omega.$$

It is easy to see that, e.g., $h(z,w) = y + \ln(\cos(x))$ and $h(z,w) = y + u$ both satisfy these last two equations, so that re^h satisfies (7.3). Hence, if $\rho = re^h$, then for every $V \Subset U$ there exists a smooth defining function for Ω on U that is plurisubharmonic on $b\Omega \cap V$, see Remark 5.4. In view of Theorem 6.19, this means that Ω admits plurisubharmonic smooth local defining functions near each boundary point.

Definition 7.7. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded domain. We say that Ω is sesquiconvex at $p_0 \in b\Omega$ if Ω is pseudoconvex at p_0 and if there exists a smooth local defining function $\rho\colon U \to \mathbb{R}$ for Ω near p_0 such that*

$$Q_\rho^{\mathbb{R}}(Y,T)^2 + Q_\rho^{\mathbb{R}}(X,T)^2 = \mathcal{O}\left(H_\rho^{\mathbb{C}}(L,L)\right) \quad \text{on } b\Omega \cap U. \tag{7.8}$$

Remark 7.9. (1) Let $\rho\colon U \to \mathbb{R}$ be a smooth local defining function for Ω near p_0, and let $h\colon U \to \mathbb{R}$ be smooth. Then for any two tangential vector fields V, W near p_0 one has $Q_{\rho e^h}^{\mathbb{R}}(V,W) = -(\nabla_V W)(\rho e^h) = -e^h(\nabla_V W)\rho = e^h Q_\rho^{\mathbb{R}}(V,W)$ on $b\Omega \cap U$. In particular, this shows that condition (7.8) is independent of the choice of a local defining function ρ.

(2) Since $H_\rho^{\mathbb{C}}(L,L) = Q_\rho^{\mathbb{R}}(X,X) + Q_\rho^{\mathbb{R}}(Y,Y)$, one easily sees that every domain $\Omega \subset \mathbb{C}^2$ that is convex at $p_0 \in b\Omega$ is sesquiconvex at p_0. Moreover, it is clear from the definition that if Ω is strictly pseudoconvex at $p_0 \in b\Omega$, then Ω is sesquiconvex at p_0. On the other hand, the domain Ω considered in Example 5.10 and Example 7.6 is not sesquiconvex at 0.

In the following, we show that sesquiconvexity at a boundary point $p_0 \in b\Omega$ implies the existence of local defining functions which are plurisubharmonic on a one-sided neighborhood $\bar{\Omega} \cap U$ of p_0, i.e., $\mathcal{H}_\rho^{\mathbb{C}}(q) \geq 0$ for every $q \in \bar{\Omega} \cap U$.

Proposition 7.10. *If $\Omega \subset \mathbb{C}^2$ is sesquiconvex at $p_0 \in b\Omega$, then Ω admits a smooth local defining function $\rho : U \longrightarrow \mathbb{R}$ near p_0 which is plurisubharmonic on $\bar{\Omega} \cap U$.*

Proof. Let $r\colon U \to \mathbb{R}$ be a smooth local defining function for Ω near p_0, and assume that $dr \neq 0$ on U. We will show that $\rho := r/|dr|$ satisfies

$$\left|H_\rho^{\mathbb{C}}(L,N)\right| = \mathcal{O}(\sqrt{\lambda}) \ \text{ on } b\Omega \cap U, \text{ and} \tag{7.11}$$
$$\nu H_\rho^{\mathbb{C}}(L,L) \leq \mathcal{O}(\sqrt{\lambda}) \ \text{ on } b\Omega \cap U. \tag{7.12}$$

The claim then follows from a brief analysis of the proofs of Lemma 6.1 and Proposition 6.8.

Let L be a nonvanishing tangential $(1,0)$-vector field on U, and let $X, Y \in \mathcal{V}(U)$ such that $L = \frac{1}{2}(X + iY)$. A straightforward computation shows that

$$0 = L(|d\rho|^2) = Q^{\mathbb{R}}_\rho(X, \nu) + iQ^{\mathbb{R}}_\rho(Y, \nu) \quad \text{on } b\Omega \cap U. \tag{7.13}$$

Since Ω is sesquiconvex at p_0, it thus follows from (7.8) and the computations in (7.1) that (7.11) is true. The equation in (7.13) may be expressed in complex notation, with $N = 2\rho_{\bar{z}^1}\frac{\partial}{\partial z^1} + 2\rho_{\bar{z}^2}\frac{\partial}{\partial z^2}$, as

$$0 = L(|\partial\rho|^2) = Q^{\mathbb{C}}_\rho(L, N) + H^{\mathbb{C}}_\rho(L, N) \ \text{ on } b\Omega \cap U.$$

Since (7.11) holds, it then follows that

$$\left|Q^{\mathbb{C}}_\rho(L, N)\right| = \mathcal{O}(\sqrt{\lambda}) \ \text{ on } b\Omega \cap U. \tag{7.14}$$

Moreover, we compute on $b\Omega \cap U$

$$\begin{aligned} 0 = \bar{L}L\left(|\partial\rho|^2\right) &= \bar{L}\left(Q^{\mathbb{C}}_\rho(L, N)\right) + \bar{L}\left(H^{\mathbb{C}}_\rho(L, N)\right) \\ &= \sum_{j,k,\ell=1}^{2} \frac{\partial^3\rho}{\partial\bar{z}^\ell\partial z^j\partial z^k}\bar{L}^\ell L^j N^k + Q^{\mathbb{C}}_\rho(\nabla_{\bar{L}}L, N) + Q^{\mathbb{C}}_\rho(L, \nabla_{\bar{L}}N) \\ &+ \sum_{j,k,\ell=1}^{2} \frac{\partial^3\rho}{\partial\bar{z}^\ell\partial z^j\partial \bar{z}^k}\bar{L}^\ell L^j \bar{N}^k + H^{\mathbb{C}}_\rho(\nabla_{\bar{L}}L, N) + H^{\mathbb{C}}_\rho(L, \nabla_{L}N). \end{aligned}$$

In view of (6.23), it follows from (7.11) that on $b\Omega \cap U$

$$2\,\mathrm{Re}\left(\sum_{j,k,\ell=1}^{2} \frac{\partial^3\rho}{\partial\bar{z}^\ell\partial z^j\partial z^k}\bar{L}^\ell L^j N^k\right) = 2\,\mathrm{Re}(N)H^{\mathbb{C}}_\rho(L, L) + \mathcal{O}(\sqrt{\lambda}).$$

Moreover, it follows from (7.11) that

$$\left|H^{\mathbb{C}}_\rho(L, \nabla_L N)\right| = \mathcal{O}(\sqrt{\lambda}) \quad \text{on } b\Omega \cap U.$$

Furthermore, since $H^{\mathbb{C}}_\rho(L, L) = -(\nabla_{\bar{L}}L)\rho$ on $b\Omega \cap U$, it follows that the normal component of $\nabla_{\bar{L}}L$ is $\mathcal{O}(H^{\mathbb{C}}_\rho(L, L))$ on $b\Omega \cap U$. This, together with (7.11) and (7.14), implies that

$$\begin{aligned} \left|Q^{\mathbb{C}}_\rho(\nabla_{\bar{L}}L, N)\right| &= \mathcal{O}(\sqrt{\lambda}) \ \text{ on } b\Omega \cap U, \text{ and} \\ \left|H^{\mathbb{C}}_\rho(\nabla_{\bar{L}}L, N)\right| &= \mathcal{O}(\sqrt{\lambda}) \ \text{ on } b\Omega \cap U. \end{aligned}$$

Therefore we obtain

$$2\,\mathrm{Re}(N)H^{\mathbb{C}}_\rho(L, L) = -Q^{\mathbb{C}}_\rho(L, \nabla_{\bar{L}}N) + \mathcal{O}(\sqrt{\lambda}) \ \text{ on } b\Omega \cap U.$$

Since

$$\nabla_{\bar{L}}N = 2\sum_{j,k=1}^{2} \bar{L}^k \rho_{\bar{z}^j\bar{z}^k}\frac{\partial}{\partial z^j},$$

it follows easily that $Q^{\mathbb{C}}_\rho(L, \nabla_{\bar{L}}N) = |Q^{\mathbb{C}}_\rho(L, .)|^2$, where $Q^{\mathbb{C}}_\rho(L, .) \in \mathcal{V}(U)^{1,0}$ is defined via the condition $\langle Q^{\mathbb{C}}_\rho(L, .), V\rangle := Q^{\mathbb{C}}_\rho(L, V)$ for all $V \in \mathcal{V}(U)^{1,0}$. Hence

$$\nu H^{\mathbb{C}}_\rho(L, L) = 2\,\mathrm{Re}(N)H^{\mathbb{C}}_\rho(L, L) = -|Q^{\mathbb{C}}_\rho(L, .)|^2 + \mathcal{O}(\sqrt{\lambda}) \ \text{ on } b\Omega \cap U,$$

i.e., (7.12) holds and the claim follows. □

A global analog of Proposition 7.10 easily follows for sesquiconvex, bounded domains.

Corollary 7.15. *If $\Omega \Subset \mathbb{C}^2$ is sesquiconvex, then Ω admits a smooth global defining function $\rho : U \longrightarrow \mathbb{R}$ which is plurisubharmonic on $\bar{\Omega} \cap U$.*

Remark 7.16. Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded domain, let $N = N_\rho$ for some smooth local defining function $\rho\colon U \to \mathbb{R}$ for Ω, see (2.6), and let L be a nonvanishing tangential $(1,0)$-vector field on U. A straightforward computation shows that then

$$\nabla_{\bar{L}} N = \frac{\overline{Q_\rho^{\mathbb{C}}(L,L)}}{2|L|^2} L + \frac{\overline{Q_\rho^{\mathbb{C}}(L,N)}}{2|N|^2} N \text{ on } b\Omega \cap U,$$

and hence, in particular,

$$Q_\rho^{\mathbb{C}}(L, \nabla_{\bar{L}} N) = \frac{|Q_\rho^{\mathbb{C}}(L,L)|^2}{2|L|^2} + \frac{|Q_\rho^{\mathbb{C}}(L,N)|^2}{2|N|^2} \text{ on } b\Omega \cap U.$$

Moreover, it is easy to see that if Ω is $\mathbb{C}$-convex at $p_0 \in b\Omega \cap U$, then $|Q_\rho^{\mathbb{C}}(L,L)| = \mathcal{O}(\lambda)$ on $b\Omega$ near p_0. In view of (7.14), it thus follows from the arguments in the proof of Proposition 7.10, that Ω admits a plurisubharmonic smooth local defining function near p_0 whenever Ω is both sesquiconvex and $\mathbb{C}$-convex at p_0. Similarly, if $\Omega \Subset \mathbb{C}^2$ is sesquiconvex and $\mathbb{C}$-convex, then Ω admits a plurisubharmonic smooth defining function.

References

[1] M. Behrens, *Plurisubharmonic defining functions of weakly pseudoconvex domains in* $\mathbf{C}^2$, Math. Ann. **270** (1985), 285–296.

[2] T. Bloom, *Remarks on type conditions for real hypersurfaces in* $\mathbf{C}^n$, In: "Several complex variables (Cortona, 1976/1977)", J. J. Kohn and E. Vesentini (eds.), Scuola Norm. Sup. Pisa, Pisa, 1978, 14–24.

[3] K. Diederich and J. E. Fornæss, *Pseudoconvex domains: an example with nontrivial Nebenhülle*, Math. Ann. **225** (1977), 275–292.

[4] K. Diederich and J. E. Fornæss, *Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. **39** (1977), 129–141.

[5] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–491.

[6] J. E. Fornæss, *Plurisubharmonic defining functions*, Pacific J. Math. **80** (1979), 381–388.

[7] J. E. Fornæss and A.-K. Herbig, *A note on plurisubharmonic defining functions in* $\mathbb{C}^2$, Math. Z. **257** (2007), 769–781.

[8] J. E. Fornæss and A.-K. Herbig, *A note on plurisubharmonic defining functions in* $\mathbb{C}^n$, Math. Ann. **342** (2008), 749–772.

[9] D. Gilbarg and N. Trudinger, "Elliptic partial differential equations of second order, 2nd Ed.", Springer-Verlag, New York, 1983.

[10] A.-K. Herbig and J. D. McNeal, *Convex defining functions for convex domains*, J. Geom. Anal. **22** (2012), 433–454.

[11] J. J. Kohn, *Boundary behavior of* $\bar{\partial}$ *on weakly pseudo-convex manifolds of dimension two*, J. Differential Geom. **6** (1972), 523–542.

[12] M. Kolář, *On local convexifiability of type four domains in* $\mathbb{C}^2$, In: "Differential geometry and applications. Proceedings of the 7th international conference, DGA 98, and satellite conference of ICM in Berlin, Brno, Czech Republic, August 10–14, 1998", I. Kolář, O. Kowalski, D. Krupka and J. Slovák (eds.), Brno: Masaryk University, 1999, 361–370.

[13] J. Milnor, "Morse theory", Princeton University Press, Princeton, NJ, 1963.

[14] J. Morrow and H. Rossi, *Some theorems of algebraicity for complex spaces*, J. Math. Soc. Japan **27** (1975), 167–183.

[15] J. Morrow and H. Rossi, *Correction to: "Some theorems of algebraicity for complex spaces"*, J. Math. Soc. Japan **29** (1977), 783.

[16] R. Michael Range, *A remark on bounded strictly plurisubharmonic exhaustion functions*, Proc. Amer. Math. Soc. **81** (1981), 220-222.

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