THE CLASSIFICATION OF CRITICAL POINTS OF HEAT KERNEL ON TORI AND APPLICATION TO ELLIPTIC EQUATION AND MUELLER-HO CONJECTURE

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Abstract. In this paper, we study the critical points of the heat kernel on two-dimensional flat tori. By using methods related to theta functions, we determine that the heat kernel exhibits four and six critical points on rectangular and hexagonal tori, respectively. Furthermore, on a rhombic torus, the number of critical points of the heat kernel depends on the geometry of torus. We have also established a connection between the heat kernel, linear elliptic equations with singularity, and particle energy. This connection allows us to recover partial results of the Green function in [15] and provides a positive answer to the conjecture regarding Mueller-Ho Conjecture in [20]. An intriguing finding of our study is that all three functions exhibit uniform critical points on rectangular and hexagonal tori.

1. Introduction and statement of main results

1.1. **The critical points of the heat kernel on tori.** A function that is defined on a n-dimensional torus can also be considered as a function on $\mathbb{R}^n$ with periodicity. The study of geometric or analytic problems on tori is a common theme in mathematics and physics, with examples including Weierstrass' elliptic functions [14], the minima of physical energy, such as the Thomas-Fermi model [4, 5], and periodic particles energy [19, 26]. The characteristics of these functions are closely tied to the geometric structure of tori.

In this paper, we consider two-dimensional tori. Geometrically, these tori are diffeomorphic to linear transformations. From this perspective, the geometry and differential equations on tori "may be" similar across different tori. In fact, in reality, mathematical and physical problems on tori frequently concentrate on rectangular tori. Interestingly, Lin and Wang [15] presented a surprising finding that differential equations on different tori can exhibit essential differences. Their research revealed that the Green function on a torus must possess either three or five critical points based on the torus' geometry. Besides, the solvability of the *mean field equation* is closely related to the critical points of the Green function.

This article is inspired by the research presented in [15] and [16]. In this paper, we demonstrate that the critical points of the heat kernel on a torus depend crucially on the geometry of the torus, which is similar to the result in [15]. Additionally,

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we present two applications about elliptic equation and Mueller-Ho Conjecture in Section 1.2 and Section 1.3.

Consider the flat torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$, where $\omega_1, \omega_2 \in \mathbb{C} \cong \mathbb{R}^2$ with $\omega_1, \omega_2 \neq 0$ and $\omega_1/\omega_2 \notin \mathbb{R}$ (as the complex division). Throughout this article, we treat $\mathbb{R}^2$ and $\mathbb{C}$ as equivalent. The heat kernel on a flat torus $\mathbb{T}$ is the unique function on $\mathbb{T} \times (0, +\infty)$ which satisfies

$$\begin{cases} \partial_t u(z,t) = \triangle_z u(z,t), \ (z,t) \in \mathbb{T} \times (0, +\infty), \\ u(z,0) = \delta_0(z), \ z \in \mathbb{T}. \end{cases} \tag{1.1}$$

Here, the initial condition means $\lim_{t \to 0^+} u(z,t) = \delta_0(z)$ in weak sense. That is, for any $v \in C^\infty(\mathbb{T})$,

$$\lim_{t \to 0^+} \int_{\mathbb{T}} u(z,t) v(z) dz = v(0), \tag{1.2}$$

where the differential dz will be considered the standard Lebesgue measure on $\mathbb{R}^2$ throughout this article.

For any $\omega_1, \omega_2 \in \mathbb{C} \cong \mathbb{R}^2$, it is well-known that (1.1) has an unique solution

$$p(z,t) = \frac{1}{4\pi t} \sum_{n,m \in \mathbb{Z}} e^{-\frac{1}{4t}|m\omega_1 + n\omega_2 + z|^2}, \ z \in \mathbb{T}, t > 0. \tag{1.3}$$

In order to help readers understand the heat kernel, we provide a brief verification in Section 2. The heat kernel is a crucial component in mathematics and physics. For an introduction to heat kernels on manifolds we refer to the textbook of Grigor'yan [10].

For any fixed $t > 0$, we consider the critical points of the heat kernel $p(z,t)$ concerning z. Since $p(z,t)$ is even and has two periods ω_1 and ω_2 with respect to z, it is elementary to verify that $0, \frac{1}{2}\omega_1, \frac{1}{2}\omega_2$, and $\frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$ are the four critical points. Furthermore, other critical points must appear in pairs. These four critical points are present for any even function on the torus, so they are considered as trivial critical points. The question that arises is whether $p(z,t)$ has non-trivial critical points, and if so, how many are there?

To the best of our knowledge, the specific critical points of the heat kernel on the torus have not been studied before. Previous works have primarily focused on extremal problems related to the heat kernel, such as those discussed in [1, 2, 9]. References [1] and [9] have demonstrated that the maximum point of the heat kernel is always 0, and the minimum points are $\frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$ and $\frac{1}{3}\omega_1 + \frac{1}{3}\omega_2$ for rectangular and hexagonal tori, respectively. Baernstein has also showed in [1] that the minimum point on general tori might be near the barycenter of a fundamental triangle.

In this article, we employ a novel technique utilizing theta functions to determine all critical points of the heat kernel $p(z,t)$ on rectangular, hexagonal, and rhombic tori. The connections between heat kernel and theta functions have long been known and are mentioned in previous articles/textbooks, like [2, 9, 29]. While, the technique for handling theta functions in this article is original, which can locate all critical points of the heat kernel on the three types of tori mentioned above. Surprisingly, The results are similar to those obtained for the Green function, as

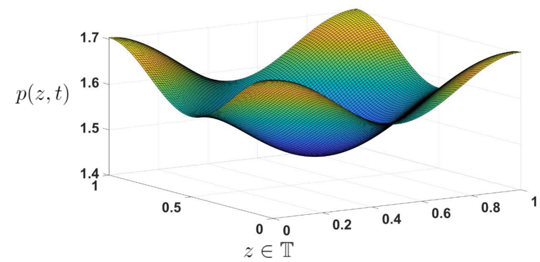


FIGURE 1. The graph of the heat kernel $p(z,t)$ on the rectangular torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$ with $\omega_1 = 1, \omega_2 = i$ and $t = 0.1$.

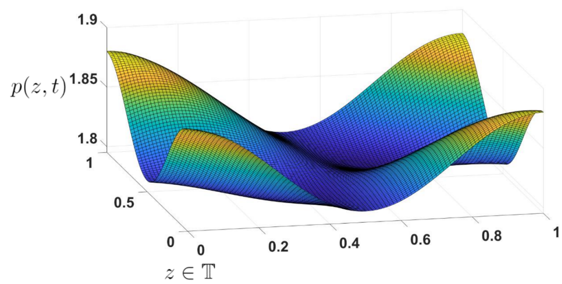


FIGURE 2. The graph of the heat kernel $p(z,t)$ on the hexagonal torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$ with $\omega_1 = 1, \omega_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $t = 0.1$.

reported in [15]. For general tori, we forecast that the results will resemble those for the special tori discussed, and we formulate that question in Section 7.

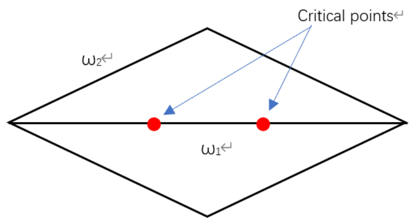
Theorem 1.1. *Consider the rectangular torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$, $\omega_1, \omega_2 \in \mathbb{C}$ with $\omega_2/\omega_1 = bi, b > 0$. For any $t > 0$, the heat kernel $p(z,t)$ defined in (1.3) has only four trivial critical points $0, \frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$ on $\mathbb{T}$.*

Moreover, 0 is the maximal point and $\frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$ is the minimal point. The remaining two critical points $\frac{1}{2}\omega_1, \frac{1}{2}\omega_2$ are saddle points, as illustrated in Figure 1.

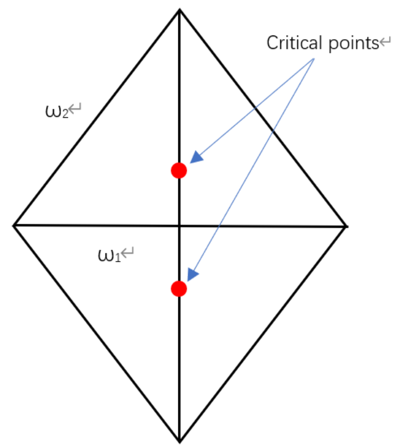
Theorem 1.2. *Consider the hexagonal torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$, $\omega_1, \omega_2 \in \mathbb{C}$ with $\omega_2/\omega_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{i\frac{\pi}{3}}$. For any $t > 0$, the heat kernel $p(z,t)$ defined in (1.3) has exact six critical points, including the trivial critical points $0, \frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$ and a pair of non-trivial critical points $\frac{1}{3}\omega_1 + \frac{1}{3}\omega_2, \frac{2}{3}\omega_1 + \frac{2}{3}\omega_2$ on $\mathbb{T}$.*

Moreover, 0 is the maximal point and $\frac{1}{3}\omega_1 + \frac{1}{3}\omega_2$, $\frac{2}{3}\omega_1 + \frac{2}{3}\omega_2$ are minimal points with the same values, as illustrated in Figure 2. It is worth noting that these critical points are all independent of t.

Remark 1.3. (1) It is indeed surprising that the critical points of the heat kernel $p(z,t)$ remains unchanged for all $t > 0$ on rectangular and hexagonal tori. It is desirable that the critical points of $p(z,t)$ are always independent of t. However, for general tori, this might not hold, such as the conditions in Theorem 1.4. This raises the question of that under what conditions,



(A) $b \in (0, b_1(t))$ case.



(B) $b \in (b_2(t), +\infty)$ case.

FIGURE 3. The non-trivial critical points of $p(z,t)$ on rhombic tori.

the number of the critical points is independent of $t > 0$. We leave it as a problem in Section 7.

(2) The invariance of the critical points on rectangular and hexagonal tori may indicate a more general result. In fact, in Section 1.2, we find that the solutions of certain elliptic equations on tori also exhibit these critical points.

For a general flat torus, we extend our results to the case of a rhombic torus, i.e., $\omega_2/\omega_1 = \frac{1}{2} + bi, b > 0$. We prove that the number of critical points of $p(z,t)$ varies with b, and $p(z,t)$ has at most a pair of non-trivial critical points.

Theorem 1.4. *Consider the rhombic torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$, $\omega_1, \omega_2 \in \mathbb{C}$ with $\omega_2/\omega_1 = \frac{1}{2} + bi, b > 0$. For any $t > 0$, there exists $\frac{\sqrt{3}}{6} < b_1(t) < \frac{1}{2} < b_2(t) < \frac{\sqrt{3}}{2}$ which depend on t such that*

(1) *if $b \in [b_1(t), b_2(t)]$, the heat kernel $p(z,t)$ defined in (1.3) has only four trivial critical points $0, \frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$ on $\mathbb{T}$.*

(2) *if $b \in (0, b_1(t)) \cup (b_2(t), +\infty)$, except for four trivial critical points, $p(z,t)$ has another pair of non-trivial critical points which are located on the long diagonal of rhombic $\mathbb{T}$.*

In fact, for $b \in (0, b_1(t))$, the non-trivial critical points are given by $\frac{\bar{\mu}(b,t)}{2}\omega_1$ and $\frac{2-\bar{\mu}(b,t)}{2}\omega_1$, where $\bar{\mu}(b,t) \in (\frac{1}{2}, 1)$ depends on b and t. When $b \in (b_2(t), +\infty)$, the non-trivial critical points are $\frac{1-\bar{\nu}(b,t)}{2}\omega_1 + \bar{\nu}(b,t)\omega_2$ and $\frac{1+\bar{\nu}(b,t)}{2}\omega_1 + (-\bar{\nu}(b,t))\omega_2$, where $\bar{\nu}(b,t) \in (0, \frac{1}{2})$ depends on b and t, as illustrated in Figure 3. The specific values of $\bar{\mu}(b,t)$ and $\bar{\nu}(b,t)$ are derived from Theorem 4.6.

Moreover, $b_1(t)$ and $b_2(t)$ are implicitly determined by

$$\begin{cases} b_1(t) \text{ is the unique zero point of } \dfrac{\vartheta_2'(0;16\pi t i)}{\vartheta_3'(0;16\pi t i)} - \dfrac{\vartheta_3(0;\frac{4\pi t}{b^2}i)}{\vartheta_2(0;\frac{4\pi t}{b^2}i)}, b \in (0,+\infty), \\ b_2(t) \text{ is the unique zero point of } \dfrac{\vartheta_2'(0;\frac{4\pi t}{b^2}i)}{\vartheta_3'(0;\frac{4\pi t}{b^2}i)} - \dfrac{\vartheta_3(0;16\pi t i)}{\vartheta_2(0;16\pi t i)}, b \in (0,+\infty), \end{cases}$$

where, ϑ_2, ϑ_3 are the theta functions defined in (3.1) and $\frac{\vartheta_2'(0;\tau)}{\vartheta_3'(0;\tau)}$ is the limit of $\frac{\vartheta_2'}{\vartheta_3'}$ at zero.

Additionally, the critical values $b_1(t), b_2(t)$ change with respect to $t > 0$, but they always satisfy the equality $b_1(t) \cdot b_2(t) = \frac{1}{4}, \forall t > 0$.

Remark 1.5. Theorem 1.4 presents the results for rhombic tori and includes a partial overview of Theorems 1.1 and 1.2. Specifically, when $b = \frac{1}{2}$, the torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$ with respect to $\omega_2/\omega_1 = \frac{1}{2} + \frac{1}{2}i$ corresponds to the square torus. Moreover, when $b = \frac{\sqrt{3}}{2}$, $\mathbb{T}$ becomes the hexagonal torus. Furthermore, all of the rectangular, hexagonal and rhombic tori have the same results like Theorems 1.1, 1.2 and 1.4. In fact, there are some rotation and scaling invariances of heat kernel which can be seen in Section 2.

The main proofs of Theorems 1.1 to Theorem 1.4 rely on the relationship between the heat kernel and theta functions. This is not surprising since theta functions are fundamental functions on tori and have been extensively utilized in related studies, as referenced in [6, 9, 15, 16]. We believe that our methods can be applied to all general tori by utilizing a general expression of the heat kernel.

1.2. **The critical points of the solution for linear elliptic equation with singularity.** In this subsection, we study a special function arising from the heat kernel and explore its critical points. Suppose the torus $\mathbb{T} = \mathbb{C}/\left(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2\right)$ and $p(z,t)$ is the heat kernel on $\mathbb{T}$. For any $\lambda \geqslant 0$, consider

$$F_\lambda(z) = \int_0^\infty e^{-\lambda t}\left(p(z,t) - \frac{1}{|\mathbb{T}|}\right)dt, z \in \mathbb{T}, \tag{1.4}$$

where $|\mathbb{T}| = |\det(\omega_1,\omega_2)| = |\omega_1 \times \omega_2|$ is the volume of torus $\mathbb{T}$. By direct verification, the function $F_\lambda(z)$ is well-defined in $\mathbb{T} \setminus \{0\}$ and exhibits a singularity at the origin. Moreover, $F_\lambda(z)$ is the unique solution of elliptic equation

$$-\triangle u(z) = -\lambda u(z) + \delta_0(z) - \frac{1}{|\mathbb{T}|},\ z \in \mathbb{T} \tag{1.5}$$

with $\int_{\mathbb{T}} u(z)dz = 0$. The detailed derivation can be found in Proposition 5.2. Here, δ_0 is the standard Dirac measure at zero.

We study the critical points of $F_\lambda(z)$ on rectangular, hexagonal, and rhombic tori. The method also relies on the properties of theta functions and bears resemblance to the approach utilized in the heat kernel case.

Theorem 1.6. *Given any fixed parameter $\lambda \geqslant 0$.*

(1) *On the rectangular torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$, $\omega_1,\omega_2 \in \mathbb{C}$ with $\omega_2/\omega_1 = bi, b > 0$, the function $F_\lambda(z)$ defined in (1.4) has only three trivial critical points $\frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$ on $\mathbb{T}$. Moreover, $\frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$ is the minimal point and the other two critical points are all saddle points.*

(2) *On the hexagonal torus* $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$, $\omega_1, \omega_2 \in \mathbb{C}$ *with* $\omega_2/\omega_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{i\frac{\pi}{3}}$*, except for three trivial critical points,* $\frac{1}{3}\omega_1 + \frac{1}{3}\omega_2$ *and* $\frac{2}{3}\omega_1 + \frac{2}{3}\omega_2$ *are the non-trivial critical points of* $F_\lambda(z)$*. Moreover, they are the minimal points with the same values.*

It's worth noting that the critical points of $F_\lambda(z)$ are one fewer than those of $p(z,t)$ since 0 is a singularity of $F_\lambda(z)$ while it remains a critical point of $p(z,t)$.

Theorem 1.6 demonstrates that $F_\lambda(z)$ exhibits three trivial critical points and a pair of non-trivial critical points on rectangular and hexagonal tori, resembling the behavior observed in the heat kernel. This raises an intriguing question: Does $F_\lambda(z)$ possess at most a pair of non-trivial critical points for all tori?

For the rhombic torus, this assertion may hold true. We nearly prove this result by using a method akin to that employed in Theorem 1.4, with the exception of two monotonic gaps (5.21) and (5.23). Although numerical computations suggest the validity of these two gaps, direct verification like Theorem 1.4 might require new techniques. Consequently, we present it as a conjecture.

Conjecture 1.7. On the rhombic torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$, $\omega_1, \omega_2 \in \mathbb{C}$ with $\omega_2/\omega_1 = \frac{1}{2} + bi, b > 0$. For any $\lambda \geqslant 0$, there exists $\frac{\sqrt{3}}{6} < B_1(\lambda) < \frac{1}{2} < B_2(\lambda) < \frac{\sqrt{3}}{2}$ which satisfy $B_1(\lambda) \cdot B_2(\lambda) = \frac{1}{4}$ such that

(1) if $b \in [B_1(\lambda), B_2(\lambda)]$, then $F_\lambda(z)$ has only three trivial critical points $\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_2$, $\frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$ on $\mathbb{T}$.

(2) if $b \in (0, B_1(\lambda)) \cup (B_2(\lambda), +\infty)$, except for three trivial critical points, $F_\lambda(z)$ has another pair of non-trivial critical points which are located on the long diagonal of rhombic $\mathbb{T}$.

Remark 1.8. If Conjecture 1.7 holds true, it suggests that both the heat kernel $p(z,t)$ and function $F_\lambda(z)$ have at most a pair of non-trivial critical points on rectangular, hexagonal and rhombic tori. This implies the existence of a fundamental principle capable of explaining this property across a wide range of functions.

Recall that the Green function $G(z)$ on flat torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$ is the unique function on $\mathbb{T}$ which satisfies

$$-\triangle G(z) = \delta_0(z) - \frac{1}{|\mathbb{T}|} \tag{1.6}$$

and $\int_{\mathbb{T}} G(z)dz = 0$. Lin and Wang [15] gave an interesting result of the critical points of Green function:

Theorem A([15]). *The Green function has three or five critical points with respect to the geometry of tori. Specially, the rectangular torus has three critical points and the hexagonal one has five.*

The formula (1.5) clarifies that $F_\lambda(z)$ corresponds to the Green function on tori when $\lambda = 0$. This observation allows us to recover a partial result from [15].

Corollary 1.9. (1) The Green function has only three trivial critical points $\frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$ on rectangular tori and $\frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$ is the minimal point.

(2) Except for three trivial critical points $\frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$, the Green function has another two critical points $\frac{1}{3}\omega_1 + \frac{1}{3}\omega_2, \frac{2}{3}\omega_1 + \frac{2}{3}\omega_2$ on hexagonal tori and they are the minimal points with the same values.

1.3. **The application to Mueller-Ho Conjecture.** In this subsection, we explore an application of the heat kernel to Mueller-Ho Conjecture, which is a conjecture for competing systems of Bose-Einstein condensates (BEC) [24]. In [22], Matthews al first observed that periodic vortices appeared in rotating two-component BEC. Since then, Mueller and Ho [24] observed experimentally that these vortices presented hexagonal-rhombic-square-rectangular lattice distribution depending on rotational velocity of the condensate. Following the pioneering work of Mueller and Ho, many authors have studied the lattice shape in two component BEC, for instance [11, 12, 13].

The mathematical expression of Mueller-Ho Conjecture is related to the lattice energy under the Gaussian potential. Suppose lattice $\Lambda_\tau = \sqrt{\frac{1}{\Im(\tau)}}\left(\mathbb{Z}\oplus\mathbb{Z}\tau\right), \tau \in \mathbb{H} := \{\tau = a+bi \in \mathbb{C} : b > 0\}$ and $z \in \mathbb{C}$. The lattice energy under the Gaussian energy $f(|\cdot|^2) = e^{-\pi|\cdot|^2}$ is

$$\theta(z;\tau) := \sum_{\omega\in\Lambda_\tau} f(|\omega+z|^2) = \sum_{n,m\in\mathbb{Z}} e^{-\frac{\pi}{\Im(\tau)}\left|m+n\tau+z\sqrt{\Im(\tau)}\right|^2}. \tag{1.7}$$

In Mueller and Ho [24], they have reduced the minimization problems on lattices to the minimization problems about $\vartheta(z;\tau)$.

Mueller-Ho Conjecture [24]**:** Consider $\alpha \in [-1,1]$ and $z = \nu\sqrt{\frac{1}{\Im(\tau)}} + \mu\sqrt{\frac{1}{\Im(\tau)}}\tau$ with $\nu,\mu \in [0,1]$. For a two-component Bose gas, the most favorable lattice minimizing $\min_{\tau\in\mathbb{H},z\in\mathbb{C}} \theta(0;\tau) + \alpha\theta(z;\tau)$ are

- **(a)**$\alpha < 0$**:** The vortices of the two components coincide with each other ($\nu = \mu = 0$) to form a triangular lattice ($\tau = e^{i\frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$).
- **(b)**$0 < \alpha < 0.172$**:** The vortex lattice in each component remains triangular ($\tau = e^{i\frac{\pi}{3}}$). However one lattice is displaced to the center of the triangle of the other ($\nu = \mu = \frac{1}{3}$).
- **(c)**$0.172 < \alpha < 0.373$**:** (ν,μ) jumps from the center of the triangle (i.e., half of the unit cell) to the center of the rhombic unit cell ($\nu = \mu = \frac{1}{2}$). The angle jumps from 60^o to 67.95^o at $\alpha = 0.172$, and increases continuously to 90^o as α increases to 0.372.
- **(d)**$0.373 < \alpha < 0.926$**:** The two lattices are "modelocked" into a centered square structure throughout the entire interval ($\tau = i, \nu = \mu = \frac{1}{2}$).
- **(e)**$0.926 < \alpha < 1$**:** The lattice type again varies continuously with interaction α. Each component's vortex lattice has a rectangular unit cell ($\tau = bi$) whose aspect ratio $|\tau|$ increases with α. At $\alpha = 1$, the aspect ratio is $\sqrt{3}$.

Luo and Wei [20] have made substantial advancements in the Mueller-Ho Conjecture (as $\nu = \mu = 0, \frac{1}{2}, \frac{1}{3}$). A natural approach to addressing the Mueller-Ho Conjecture involves initially studying the minimum of $\vartheta(z;\tau)$ with respect to z for a fixed τ, and subsequently determining the minimum $\theta(0;\tau)+\alpha\theta(z;\tau)$ with respect to τ. In fact, in [20], a key conjecture is proposed regarding $\vartheta(z;\tau)$.

Conjecture 1.10 ([20])**.** Suppose $\Lambda_\tau = \sqrt{\frac{1}{\Im(\tau)}}\left(\mathbb{Z}\oplus\mathbb{Z}\tau\right), \tau \in \mathbb{H}$ and $\theta(z;\tau)$ is defined in (1.7). There holds

(1) Alternative: The function $\theta(z;\tau)$ concerning the $z \in \mathbb{T}_\tau := \mathbb{R}^2/\Lambda_\tau$ has either four or six critical points depending on τ, i.e., it has four trivial critical points and at most a pair of non-trivial critical points.
(2) The rectangular torus has only four critical points and the hexagonal one has six.
(3) Invariance: If $\theta(z;\tau)$ has four critical points on $\mathbb{T}_\tau$, then $\theta(z;\Gamma(\tau))$ has four critical points on $\mathbb{T}_{\Gamma(\tau)}$. On the other hand, the six critical points cases are the same. Here the modular group is

$$\Gamma \in SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, a, b, c, d \in \mathbb{Z} \right\} \tag{1.8}$$

and $\Gamma(\tau) = \frac{a\tau+b}{c\tau+d}$.

As mentioned in [20], if Conjecture 1.10 holds true, by using the methods in [20], Mueller-Ho Conjecture will likely be proven completely. Based on the results regarding the critical points of the heat kernel, we make progress towards addressing Conjecture 1.10 and provide a positive result as follows.

Theorem 1.11. *The conjectures (2) and (3) in Conjecture1.10 are correct.*

The paper is structured as follows: In Section 2, we present the derivation and basic properties of the heat kernel. Section 3 contains properties of classical theta functions, which are used in the proofs of Theorem 1.1 to Theorem 1.6. Moving on to Section 4, we complete the proofs of Theorem 1.1 to Theorem 1.4. In Sections 5 and 6, we provide applications of our methods and establish the proofs of the theorems in Section 1.2 and Section 1.3. Lastly, in Section 7, we explore generalizations and pose open questions.

2. The heat kernel on tori

We start with the derivation and some basic properties of the heat kernel. For any $\omega_1, \omega_2 \in \mathbb{C} \cong \mathbb{R}^2$, suppose flat torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$. Let the function

$$p(z,t) = \frac{1}{4\pi t} \sum_{n,m\in\mathbb{Z}} e^{-\frac{1}{4t}|m\omega_1+n\omega_2+z|^2}, \ z \in \mathbb{C}, t > 0. \tag{2.1}$$

Note that this summation is convergent for all $z \in \mathbb{C}$ and $t > 0$, so (2.1) is well-defined.

From the definition (2.1), for any $z \in \mathbb{C}$ and $t > 0$, $p(z+\omega_1,t) = p(z+\omega_2,t) = p(z,t)$. So $p(z,t)$ can be regarded as a function on the torus $\mathbb{T}$. Moreover, $p(z,t)$ is a smooth function by the fast decay of $e^{-|\cdot|^2}$. Now we provide a brief verification that $p(z,t)$ is the heat kernel on $\mathbb{T}$.

Lemma 2.1. *The function $p(z,t)$ defined in (2.1) satisfies some properties at following.*

(1) *For any $(z,t) \in \mathbb{T} \times (0,+\infty)$, $\partial_t p(z,t) = \triangle_z p(z,t)$.*
(2) *For all $t > 0$,*

$$\int_{\mathbb{T}} p(z,t)dz = 1.$$

(3) *Suppose $v \in C^{\infty}(\mathbb{T})$ is a smooth function on the torus $\mathbb{T}$, it has*

$$\lim_{t \to 0^+} \int_{\mathbb{T}} p(z,t)v(z)dz = v(0).$$

Proof. The property (1) can be verified directly, so we omit that.

For the property (2), by using the Levi's Theorem,

$$\begin{aligned} \int_{\mathbb{T}} p(z,t)dz &= \frac{1}{4\pi t} \sum_{n,m \in \mathbb{Z}} \int_{\mathbb{T}} e^{-\frac{1}{4t}|m\omega_1 + n\omega_2 + z|^2} dz \\ &= \frac{1}{4\pi t} \int_{\mathbb{R}^2} e^{-\frac{1}{4t}|z|^2} dz = 1. \end{aligned} \tag{2.2}$$

Next, suppose $v \in C^{\infty}(\mathbb{T})$, for any $\delta > 0$, there exists $\varepsilon_1 > 0$, such that $|v(z) - v(0)| < \delta, \forall z \in B_{\varepsilon_1}$, where $B_{\varepsilon_1} = \{z \in \mathbb{C} : |z| < \varepsilon_1\}$ is a ball with center at origin and radius of ε_1.

To verify (3), we can suppose $\mathbb{T} = [-\frac{1}{2}, \frac{1}{2}]\omega_1 \times [-\frac{1}{2}, \frac{1}{2}]\omega_2$ since $p(\cdot, t)$ and $v(\cdot)$ are doubly-periodic functions. Choose $0 < \varepsilon < \min\{\varepsilon_1, \frac{1}{2}\min\{|m\omega_1 + n\omega_2| : (n,m) \in \mathbb{Z}^2 \setminus \{(0,0)\}\}\ \}$, by using formula (2.2),

$$\int_{\mathbb{T}} p(z,t)v(z)dz - v(0) = \left(\int_{\mathbb{T} \setminus B_\varepsilon} + \int_{B_\varepsilon} \right) \frac{1}{4\pi t} \sum_{n,m \in \mathbb{Z}} e^{-\frac{1}{4t}|m\omega_1 + n\omega_2 + z|^2} (v(z) - v(0)) dz. \tag{2.3}$$

Note that for all $n, m \in \mathbb{Z}$ and $z \in \mathbb{T} \setminus B_\varepsilon$, $|m\omega_1 + n\omega_2 + z| > \varepsilon$ has a positive lower bound. Thus, $\frac{1}{4\pi t} e^{-\frac{1}{4t}|m\omega_1 + n\omega_2 + z|^2}$ decreases to zero uniformly as t decreases to 0. Therefore, by using the Lebesgue's Dominated Convergence Theorem,

$$\lim_{t \to 0^+} \int_{\mathbb{T} \setminus B_\varepsilon} \frac{1}{4\pi t} \sum_{n,m \in \mathbb{Z}} e^{-\frac{1}{4t}|m\omega_1 + n\omega_2 + z|^2} |v(z) - v(0)| dz = 0. \tag{2.4}$$

On the other hand,

$$\int_{B_\varepsilon} p(z,t)|v(z) - v(0)|dz < \delta \int_{B_\varepsilon} p(z,t) \leqslant \delta. \tag{2.5}$$

Combining (2.3), (2.4) and (2.5),

$$\lim_{t \to 0^+} \left| \int_{\mathbb{T}} p(z,t)v(z)dz - v(0) \right| < \delta.$$

The result follows from $\delta \to 0^+$. □

From Lemma 2.1, $p(z,t)$ is the heat kernel which satisfies (1.1). Besides, the uniqueness of $p(z,t)$ follows from the uniqueness of the standard linear parabolic equation [10].

Let $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 = \{m\omega_1 + n\omega_2 : n, m \in \mathbb{Z}\}$ be a lattice on $\mathbb{R}^2 \cong \mathbb{C}$, then $\mathbb{T} = \mathbb{C}/\Lambda \cong [-\frac{1}{2}, \frac{1}{2}]\omega_1 \times [-\frac{1}{2}, \frac{1}{2}]\omega_2 \cong [0,1]\omega_1 \times [0,1]\omega_2$ can be regarded as a single domain of lattice Λ. Please note that the above expressions of $\mathbb{T}$ do not affect the critical points of $p(z,t)$ under the double periods. Suppose $z = \nu\omega_1 + \mu\omega_2$, with $\nu, \mu \in [0,1]$, then $p(z,t)$ can be rewritten by

$$p(z,t) = p(\nu, \mu, t) = \frac{1}{4\pi t} \sum_{n,m \in \mathbb{Z}} e^{-\frac{1}{4t}|(m+\nu)\omega_1 + (n+\mu)\omega_2|^2}. \tag{2.6}$$

In this viewpoint, to find the critical point $\nabla_z p(z,t)=0$, we only need to solve $\partial_\nu p=\partial_\mu p=0$ in (2.6).

If we focus on (ν,μ) instead of z, then formula (2.6) is invariant under the rotation $(\omega_1,\omega_2)\to(\omega_1 e^{i\eta},\omega_2 e^{i\eta}),\forall\eta\in[0,2\pi]$. Therefore, $z=\nu\omega_1+\mu\omega_2$ is a critical point of $p(z,t)$ on $\mathbb{T}=\mathbb{C}/(\mathbb{Z}\omega_1\oplus\mathbb{Z}\omega_2)$ if and only if $z=\nu\omega_1 e^{i\eta}+\mu\omega_2 e^{i\eta}$ is a critical point on $\mathbb{T}=\mathbb{C}/(\mathbb{Z}\omega_1 e^{i\eta}\oplus\mathbb{Z}\omega_2 e^{i\eta}),\eta\in[0,2\pi]$.

Similarly, under the scaling $(\omega_1,\omega_2)\to(k\omega_1,k\omega_2)$ for some $k>0$, point $(\nu\omega_1+\mu\omega_2,t)$ is a critical point of $p(z,t)$ on $\mathbb{T}=\mathbb{C}/(\mathbb{Z}\omega_1\oplus\mathbb{Z}\omega_2)$ if and only if $(\nu k\omega_1+\mu k\omega_2,k^2t)$ is a critical point on $\mathbb{T}=\mathbb{C}/(\mathbb{Z}k\omega_1\oplus\mathbb{Z}k\omega_2)$.

Based on the above discussion and the symmetry of $\{\omega_1,\omega_2\}$, without loss of generality, we suppose $\omega_1=1,\omega_2=\tau\in\mathbb{H}$. In this case, lattice $\Lambda=\mathbb{Z}\oplus\mathbb{Z}\tau=\{m+n\tau:n,m\in\mathbb{Z}\}$ and the heat kernel

$$p(z,t)=\frac{1}{4\pi t}\sum_{n,m\in\mathbb{Z}}e^{-\frac{1}{4t}|m+n\tau+z|^2},\ z\in\mathbb{T},t>0. \tag{2.7}$$

We recall the following well-known Poisson summation formula on $\mathbb{R}^2$

$$\sum_{y\in\Lambda}f(x+y)=\frac{1}{|\Lambda|}\sum_{y\in\Lambda^\perp}\hat{f}(y)e^{2\pi ix\cdot y},\ \forall f\in\mathcal{S}(\mathbb{R}^2),\ x\in\mathbb{R}^2, \tag{2.8}$$

where $|\Lambda|=vol(\mathbb{R}^2/\Lambda)=|\mathbb{T}|$ is called the volume of lattice Λ, $S(\mathbb{R}^2)$ is the standard Schwarz space on $\mathbb{R}^2$, the Fourier transform

$$\hat{f}(y)=\int_{\mathbb{R}^2}f(x)e^{-2\pi ix\cdot y}dx,$$

and the dual lattice $\Lambda^\perp$ of lattice Λ is

$$\Lambda^\perp:=\{z\in\mathbb{R}^2:z\cdot\omega\in\mathbb{Z},\forall\omega\in\Lambda\}. \tag{2.9}$$

Here $\cdot$ is the standard inner product in $\mathbb{R}^2$. The right-hand side of formula (2.8) can be verified directly to be the Fourier series of the left-hand side. Also, we give the reference [7] and [25].

For the $\Lambda=\mathbb{Z}\oplus\mathbb{Z}\tau$ generated by $\{1,\tau=a+bi\}$, the dual lattice $\Lambda^\perp=\mathbb{Z}\frac{-i\tau}{\Im(\tau)}\oplus\mathbb{Z}\frac{i}{\Im(\tau)}$ is generated by $\{\frac{-i\tau}{\Im(\tau)}=\frac{b-ai}{b},\frac{i}{\Im(\tau)}=\frac{i}{b}\}$.

By the Poisson Summation Formula (2.8), the heat kernel on torus $\mathbb{T}=\mathbb{C}/\Lambda=\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$ can be rewritten by

$$\begin{aligned}p(z,t)&=\frac{1}{4\pi t}\sum_{\omega\in\Lambda}e^{-\frac{1}{4t}|\omega+z|^2}=\frac{1}{\Im(\tau)}\sum_{\omega\in\Lambda^\perp}e^{-4\pi^2t|\omega|^2}e^{2\pi iz\cdot\omega}\\&=\frac{1}{b}\sum_{n,m\in\mathbb{Z}}e^{-\frac{4\pi^2t}{b^2}\left(n^2b^2+(m-na)^2\right)}e^{2\pi i(m\mu+n\nu)},\end{aligned} \tag{2.10}$$

where $z=\nu+\mu\tau$, with $\nu,\mu\in[0,1]$.

3. Theta functions and some properties

In this section, we present some properties of theta functions. These properties are used to prove Theorem 1.1 to Theorem 1.6.

Let $\tau = a + bi, b > 0$ and $q = e^{\pi i \tau}$ with $|q| = e^{-\pi b} < 1$. For any $z \in \mathbb{C}$, the theta functions are the exponentially convergent series:

$$\begin{aligned} \vartheta_1(z;\tau) &= -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)\pi i z}, \\ \vartheta_2(z;\tau) &= \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e^{(2n+1)\pi i z} = \vartheta_1\left(z + \frac{1}{2}; \tau\right), \\ \vartheta_3(z;\tau) &= \sum_{n=-\infty}^{\infty} q^{n^2} e^{2n\pi i z}, \\ \vartheta_4(z;\tau) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2n\pi i z} = \vartheta_3\left(z + \frac{1}{2}; \tau\right). \end{aligned} \tag{3.1}$$

These four theta functions are holomorphic function with respect to z. First, we outline some basic properties of theta functions.

Proposition 3.1. For any $\tau = a + bi, b > 0$ and $z \in \mathbb{C}$,

(1) $\vartheta_2(z;\tau) = e^{\pi i z + \frac{\pi i \tau}{4}} \vartheta_3(z + \frac{1}{2}\tau; \tau)$.
(2) $\vartheta_2(z;\tau) = \vartheta_2(-z;\tau) = -\vartheta_2(z+1;\tau)$.
(3) $\vartheta_3(z;\tau) = \vartheta_3(-z;\tau) = \vartheta_3(z+1;\tau)$.
(4) $\vartheta_2(z+\tau;\tau) = q^{-1}e^{-2\pi i z}\vartheta_2(z;\tau), \vartheta_3(z+\tau;\tau) = q^{-1}e^{-2\pi i z}\vartheta_3(z;\tau)$.
(5) each theta function has only one simple zero inside cell $\mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$. It follows that the zeros of $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ with respect to z are the points congruent respectively to $0, \frac{1}{2}, \frac{1}{2} + \frac{1}{2}\tau, \frac{1}{2}\tau$.

These properties can be referenced in ([29] Section 21). Please note that the symbols in [29] may slightly differ from ours. For the further reference of theta functions we refer to the textbooks [8, 29].

Next, we recall the well-known Jacobi triple product formula

$$\prod_{m=1}^{\infty} (1 - x^{2m})(1 + x^{2m-1}y)\left(1 + x^{2m-1}y^{-1}\right) = \sum_{n=-\infty}^{\infty} x^{n^2} y^n \tag{3.2}$$

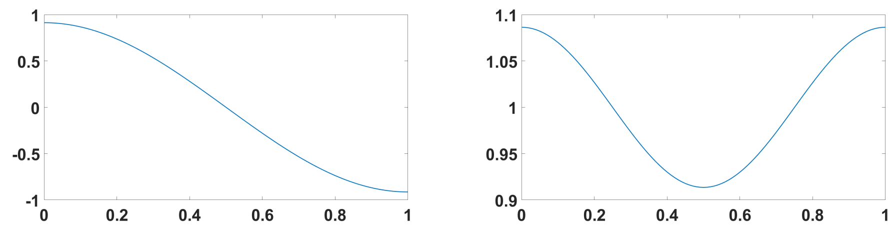
for any $x, y \in \mathbb{C}$ with $|x| < 1, y \neq 0$.

From the Jacobi triple product formula and Proposition 3.1, theta functions ϑ_2, ϑ_3 can be rewritten by product expression

$$\vartheta_3(z;\tau) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n-1}\cos(2\pi z) + q^{4n-2}), \tag{3.3}$$

$$\begin{aligned} \vartheta_2(z;\tau) &= e^{\pi i z + \frac{\pi i \tau}{4}} \vartheta_3\left(z + \frac{1}{2}\tau; \tau\right) \\ &= e^{\pi i z + \frac{\pi i \tau}{4}} \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n}e^{2\pi i z})(1 + q^{2n-2}e^{-2\pi i z}) \\ &= 2q^{\frac{1}{4}} \cos(\pi z) \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n}\cos(2\pi z) + q^{4n}). \end{aligned} \tag{3.4}$$

These expression (3.3) and (3.4) will be used frequently in Section 3 and Section 4.



(A) The behavior of $\vartheta_2(x;i)$ (B) The behavior of $\vartheta_3(x;i)$

Figure 4. The graphs of $\vartheta_2(x;Ti)$ and $\vartheta_3(x;Ti)$ with respect to $T=1, x\in[0,1]$.

The next lemma shows the graph of ϑ_2, ϑ_3 on real axis $z = x \in \mathbb{R}$ with $\tau = Ti$. Among them, results (1)(3) bear similar conclusions to that found in reference [23].

Lemma 3.2. *For any $\tau = Ti, T > 0$, the theta functions ϑ_2, ϑ_3 are smooth with respect to $z = x \in [0,1]$ and satisfy*

(1) $\vartheta_3(x;Ti) > 0, \forall x \in [0,1]$.
(2) $\vartheta_2(x;Ti) > 0, x \in [0,\frac{1}{2})$ *and* $\vartheta_2(x;Ti) < 0, x \in (\frac{1}{2},1]$.
(3) $\vartheta_3'(x;Ti) < 0, x \in (0,\frac{1}{2})$ *and* $\vartheta_3'(x;Ti) > 0, x \in (\frac{1}{2},1)$.
(4) $\vartheta_2'(x;Ti) < 0, \forall x \in (0,1)$.

The precise graphs of $\vartheta_2(x;Ti)$ and $\vartheta_3(x;Ti)$ can be seen in Figure 4.

Proof. The key point is using the expression (3.3) and (3.4). Note that when $\tau = Ti$, there is $q = e^{\pi i\tau} = e^{-\pi T} \in (0,1)$. Therefore,

$$1 + 2q^k cos(2\pi x) + q^{2k} > 0,\ \forall x \in [0,1], k \in \mathbb{Z}_+.$$

The conclusion (1) and (2) follow directly from (3.3) and (3.4).

Moreover, note that $\frac{d}{dx}\cos(2\pi x)$ and $\frac{d}{dx}\cos(\pi x)$ are negative on $x \in (0,\frac{1}{2})$. Thus, $\vartheta_3'(x;Ti) < 0$ on $x \in (0,\frac{1}{2})$ and $\vartheta_2'(x;Ti) < 0$ on $x \in (0,\frac{1}{2}]$. The conclusion (3) and (4) follow directly from the symmetries $\vartheta_2(1-x;Ti) = -\vartheta_2(x;Ti)$ and $\vartheta_3(1-x;Ti) = \vartheta_3(x;Ti)$.

□

In the remaining part of this section, we prove some equalities and inequalities related to theta functions. The equalities in Proposition 3.3 and Proposition 3.4 will be used to find the unique pair of non-trivial critical points of the heat kernel.

Proposition 3.3. For any $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$,

$$\begin{aligned} \frac{\vartheta_3(z;\tau)\vartheta_3(z-\frac{1}{3};\tau)\vartheta_3(z+\frac{1}{3};\tau)}{\vartheta_3(3z;3\tau)} &= \frac{\vartheta_2(z;\tau)\vartheta_2(z-\frac{1}{3};\tau)\vartheta_2(z+\frac{1}{3};\tau)}{\vartheta_2(3z;3\tau)} \\ &= \frac{\prod_{n=1}^{\infty}(1-q^{2n})^3}{\prod_{n=1}^{\infty}(1-q^{6n})} \end{aligned} \tag{3.5}$$

are independent on z, where $q = e^{\pi i \tau}$.

Proof. From Proposition 3.1(3)(4)(5), the zeros of the theta function ϑ_3 are all simple zeros at points $z = \frac{1}{2} + \frac{1}{2}\tau + n + m\tau$, with $n, m \in \mathbb{Z}$. Therefore $\vartheta_3(3z; 3\tau)$ has only simple zero at $\frac{1}{6} + \frac{1}{2}\tau + \frac{1}{3}n + m\tau$ and $\vartheta_3(z;\tau)\vartheta_3(z - \frac{1}{3};\tau)\vartheta_3(z + \frac{1}{3};\tau)$ has the same zeros. For this reason, if we fixed $\tau \in \mathbb{H}$,

$$F(z;\tau) := \frac{\vartheta_3(z;\tau)\vartheta_3(z - \frac{1}{3};\tau)\vartheta_3(z + \frac{1}{3};\tau)}{\vartheta_3(3z;3\tau)} \tag{3.6}$$

has no poles with respect to $z \in \mathbb{C}$.

Besides, associated with the periods in Proposition 3.1(3)(4), $F(z;\tau)$ is a doubly-periodic function with periods 1 and τ, i.e., elliptic function. Because a non-constant elliptic function must have at least two poles (including multiplicity) [14, 29], $F(z;\tau)$ is a constant function independent of z.

The value of this constant can be obtained by putting $z = 0$. By using (3.3), we obtain

$$\frac{\vartheta_3(0;\tau)\vartheta_3(-\frac{1}{3};\tau)\vartheta_3(\frac{1}{3};\tau)}{\vartheta_3(0;3\tau)} = \frac{\prod_{n=1}^{\infty}(1 - q^{2n})^3}{\prod_{n=1}^{\infty}(1 - q^{6n})}. \tag{3.7}$$

The other part of the formula (3.5) about ϑ_2 is similar to ϑ_3 by using the formula (3.4) and Proposition 3.1. □

Proposition 3.4. Fixed $\tau \in \mathbb{H}$, for any $u, v \in \mathbb{C}$, we have

$$\begin{aligned} &\vartheta_3(u+v)\vartheta_3(u-v)\vartheta_3(u)\vartheta_3(-u) - \vartheta_3(2u)\vartheta_3(0)\vartheta_3(-v)\vartheta_3(v) \\ =& -\vartheta_2(u+v)\vartheta_2(u-v)\vartheta_2(u)\vartheta_2(-u) + \vartheta_2(2u)\vartheta_2(0)\vartheta_2(-v)\vartheta_2(v). \end{aligned} \tag{3.8}$$

Here, we omit the fixed τ without any ambiguity.

Proof. First, we need a result in reference ([29] Section 21). Consider x', y', z', w' be defined in terms of $x, y, z, w \in \mathbb{C}$ by the set of equations

$$\begin{cases} 2x' = -x + y + z + w, \\ 2y' = x - y + z + w, \\ 2z' = x + y - z + w, \\ 2w' = x + y + z - w. \end{cases} \tag{3.9}$$

For brevity, we write

$$[r] = \vartheta_r(x)\vartheta_r(y)\vartheta_r(z)\vartheta_r(w), \ [r]' = \vartheta_r(x')\vartheta_r(y')\vartheta_r(z')\vartheta_r(w') \tag{3.10}$$

for $r \in \{1, 2, 3, 4\}$.

Under these symbols, E.T.Whittaker and G.N.Watson, in [29] Section 21, showed two formulas between $[r]$ and $[r]'$:

$$2[2] = [1]' + [2]' + [3]' - [4]'. \tag{3.11}$$

$$2[3] = -[1]' + [2]' + [3]' + [4]'. \tag{3.12}$$

The proof of the above two formulas is nearly the same as Proposition 3.3. Precisely, let the right-hand side of the equation divide by the left-hand side of the equation. Their quotient is a doubly-periodic function with, at most, a single simple pole in cell $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$. By using the property of elliptic, their quotient is a constant. The complete proof can be seen in ([29] Section 21).

Adding (3.11) and (3.12), we have

$$[2] + [3] = [2]' + [3]'. \tag{3.13}$$

By choosing $(x, y, z, w) = (u + v, u - v, u, -u)$, we obtain Proposition 3.4 immediately.

□

Finally, we prove some inequalities in Proposition 3.5, Proposition 3.6, and Proposition 3.7, which will be used to demonstrate that there are no other nontrivial critical points of the heat kernel.

Proposition 3.5. Suppose $x \in \mathbb{R}$ and $\tau = Ti, T > 0$, theta functions ϑ_2, ϑ_3 satisfy

$$\frac{d}{dx}\left(\vartheta_3^2(x;\tau) - \vartheta_2^2(x;\tau)\right) > 0,\ \forall x \in \left(k, k+\frac{1}{2}\right), k \in \mathbb{Z}, \tag{3.14}$$

$$\frac{d}{dx}\left(\vartheta_3^2(x;\tau) - \vartheta_2^2(x;\tau)\right) < 0,\ \forall x \in \left(k+\frac{1}{2}, k+1\right), k \in \mathbb{Z}. \tag{3.15}$$

Proof.

(3.16)

$$\begin{aligned}\vartheta_3(x;Ti) - \vartheta_2(x;Ti) &= \sum_{n=-\infty}^{\infty} e^{-\frac{\pi T}{4}(2n)^2} e^{2n\pi i x} - \sum_{n=-\infty}^{\infty} e^{-\frac{\pi T}{4}(2n+1)^2} e^{(2n+1)\pi i x} \\ &= \sum_{m=-\infty}^{\infty} (-1)^m e^{-\frac{\pi T}{4}m^2} e^{m\pi i x} \\ &= \sum_{m=-\infty}^{\infty} e^{-\frac{\pi T}{4}m^2} e^{2m\pi i \frac{x+1}{2}} = \vartheta_3\left(\frac{x+1}{2}; \frac{Ti}{4}\right).\end{aligned}$$

Similarly,

(3.17)

$$\begin{aligned}\vartheta_3(x;Ti) + \vartheta_2(x;Ti) &= \sum_{n=-\infty}^{\infty} e^{-\frac{\pi T}{4}(2n)^2} e^{2n\pi i x} + \sum_{n=-\infty}^{\infty} e^{-\frac{\pi T}{4}(2n+1)^2} e^{(2n+1)\pi i x} \\ &= \sum_{m=-\infty}^{\infty} e^{-\frac{\pi T}{4}m^2} e^{2m\pi i \frac{x}{2}} = \vartheta_3\left(\frac{x}{2}; \frac{Ti}{4}\right).\end{aligned}$$

Combine with the formula (3.3), we attain

$$\begin{aligned}&\vartheta_3^2(x;Ti) - \vartheta_2^2(x;Ti) = \vartheta_3\left(\frac{x+1}{2}; \frac{Ti}{4}\right)\vartheta_3\left(\frac{x}{2}; \frac{Ti}{4}\right) \\ &= \prod_{n=1}^{\infty}(1-q^{2n})^2(1-2q^{2n-1}\cos(\pi x)+q^{4n-2})(1+2q^{2n-1}\cos(\pi x)+q^{4n-2})\end{aligned} \tag{3.18}$$

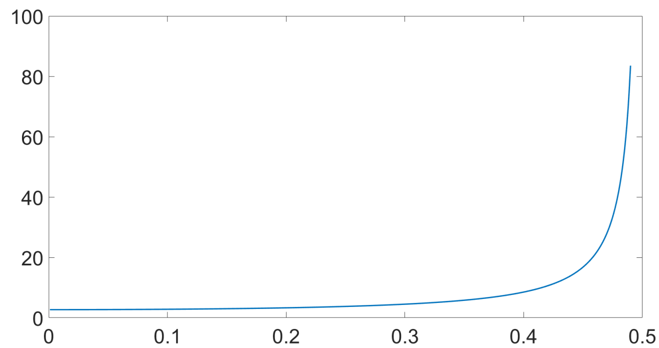


FIGURE 5. The graph of $\frac{\vartheta_2'(x;Ti)}{\vartheta_3'(x;Ti)}$ with respect to $T=1, x\in\left(0,\frac{1}{2}\right)$.

where $q = e^{\pi i\frac{Ti}{4}} = e^{-\frac{\pi T}{4}} \in (0,1)$. From the above formula (3.18), we obtain $\vartheta_3^2(x;Ti)-\vartheta_2^2(x;Ti)>0$ and

$$\begin{aligned}
&\frac{d}{dx}\ln\left(\vartheta_3^2(x;Ti)-\vartheta_2^2(x;Ti)\right)\\
&=\sum_{n=1}^{\infty}\left(\frac{2\pi q^{2n-1}\sin(\pi x)}{1-2q^{2n-1}\cos(\pi x)+q^{4n-2}}+\frac{-2\pi q^{2n-1}\sin(\pi x)}{1+2q^{2n-1}\cos(\pi x)+q^{4n-2}}\right)\\
&=\sum_{n=1}^{\infty}\frac{4\pi q^{4n-2}\sin(2\pi x)}{(1-2q^{2n-1}\cos(\pi x)+q^{4n-2})(1+2q^{2n-1}\cos(\pi x)+q^{4n-2})}.
\end{aligned}\tag{3.19}$$

Thus, we get the formulas (3.14) and (3.15) based on the positivity and negativity of $\sin(2\pi x)$. □

Proposition 3.6. Suppose $\tau = Ti, T>0$, theta functions ϑ_2, ϑ_3 satisfy

$$\frac{\vartheta_3(0;\tau)}{\vartheta_2(0;\tau)}<\frac{\vartheta_3(0;3\tau)}{\vartheta_2(0;3\tau)}.\tag{3.20}$$

Proof. By using the formula (3.16) and Lemma 3.2, we obtain $\vartheta_3(x;Ti)>\vartheta_2(x;Ti)>0$, $\forall x\in(-\frac{1}{2},\frac{1}{2})$. According to Proposition 3.3, we have

$$\frac{\vartheta_3(0;3\tau)}{\vartheta_2(0;3\tau)}=\frac{\vartheta_3(0;\tau)\vartheta_3(\frac{1}{3};\tau)\vartheta_3(-\frac{1}{3};\tau)}{\vartheta_2(0;\tau)\vartheta_2(\frac{1}{3};\tau)\vartheta_2(-\frac{1}{3};\tau)}>\frac{\vartheta_3(0;\tau)}{\vartheta_2(0;\tau)}.\tag{3.21}$$

□

Proposition 3.7. For any $\tau = Ti, T>0$ and $x\in(0,\frac{1}{2})$, there is

$$\frac{d}{dx}\left(\frac{\vartheta_2'(x;Ti)}{\vartheta_3'(x;Ti)}\right)>0.\tag{3.22}$$

Where $\vartheta_j', j=2,3$ is the derivative of ϑ_j with respect to x. The graph of $\frac{\vartheta_2'(x;Ti)}{\vartheta_3'(x;Ti)}$ is shown in Figure 5.

Proof. We divide the value range $(0,\frac{1}{2})$ into two cases: $(0,\frac{1}{4})$ and $[\frac{1}{4},\frac{1}{2})$.

Case 1: $x \in [\frac{1}{4}, \frac{1}{2})$. By direct calculation

$$\frac{d}{dx}\left(\frac{\vartheta_2'(x)}{\vartheta_3'(x)}\right) = \frac{\vartheta_2''(x)\vartheta_3'(x) - \vartheta_2'(x)\vartheta_3''(x)}{(\vartheta_3'(x))^2}, \tag{3.23}$$

where in this proof we omit the fixed $\tau = Ti$ without any ambiguity. We will judge the positivity of (3.23). For any $T > 0$ and $n \in \mathbb{Z}_+$, suppose

$$f_n(x) = f_n(x;T) := 1 + 2q^n \cos(2\pi x) + q^{2n}, \forall x \in \mathbb{R}, \tag{3.24}$$

where $q = e^{-\pi T} \in (0,1)$. Under this symbol and formula (3.3),

$$\vartheta_3(x) = \prod_{k=1}^{\infty}(1 - q^{2k}) \cdot \prod_{n=1}^{\infty} f_{2n-1}(x). \tag{3.25}$$

Note that, for any $x \in [\frac{1}{4}, \frac{1}{2})$,

$$f_{2n-1}'(x) = -4\pi q^{2n-1}\sin(2\pi x) < 0 \tag{3.26}$$

and

$$f_{2n-1}''(x) = -8\pi^2 q^{2n-1}\cos(2\pi x) \geqslant 0. \tag{3.27}$$

Therefore, from (3.24)-(3.27), we get $\vartheta_3''(x) > 0, \forall x \in [\frac{1}{4}, \frac{1}{2})$. combining with $\vartheta_3(1-x) = \vartheta_3(x)$, there is

$$\vartheta_3''(x) > 0, \forall x \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{3.28}$$

In addition, by using the formula (3.16), we obtain that for any $x \in [\frac{1}{4}, \frac{1}{2})$,

$$\vartheta_3'(x;Ti) - \vartheta_2'(x;Ti) = \frac{1}{2}\vartheta_3'\left(\frac{x+1}{2};Ti\right) > 0 \tag{3.29}$$

and

$$\vartheta_3''(x;Ti) - \vartheta_2''(x;Ti) = \frac{1}{4}\vartheta_3''\left(\frac{x+1}{2};Ti\right) > 0, \tag{3.30}$$

where we used Lemma 3.2 and (3.28). Therefore, combining with (3.28)-(3.30) and Lemma 3.2, we obtain that for any $x \in [\frac{1}{4}, \frac{1}{2})$,

$$\vartheta_2''(x)\vartheta_3'(x) - \vartheta_2'(x)\vartheta_3''(x) = (\vartheta_2''(x) - \vartheta_3''(x))\vartheta_3'(x) + \vartheta_3''(x)(\vartheta_3'(x) - \vartheta_2'(x)) > 0. \tag{3.31}$$

Case 2: $x \in (0, \frac{1}{4})$. Take the derivative of (3.8) with respect to v and let $u = \frac{1}{2}, v = x \in (0, \frac{1}{4})$, we obtain equality

$$\begin{aligned}
&\vartheta_3'\left(\frac{1}{2}+x\right)\vartheta_3\left(\frac{1}{2}-x\right)\vartheta_3^2\left(\frac{1}{2}\right) - \vartheta_3\left(\frac{1}{2}+x\right)\vartheta_3'\left(\frac{1}{2}-x\right)\vartheta_3^2\left(\frac{1}{2}\right)\\
&-2\vartheta_3(1)\vartheta_3(0)\vartheta_3(x)\vartheta_3'(x)\\
&= -\vartheta_2'\left(\frac{1}{2}+x\right)\vartheta_2\left(\frac{1}{2}-x\right)\vartheta_2^2\left(\frac{1}{2}\right) + \vartheta_2\left(\frac{1}{2}+x\right)\vartheta_2'\left(\frac{1}{2}-x\right)\vartheta_2^2\left(\frac{1}{2}\right)\\
&+2\vartheta_2(1)\vartheta_2(0)\vartheta_2(x)\vartheta_2'(x),
\end{aligned} \tag{3.32}$$

where we used $\vartheta_3(-z) = \vartheta_3(z)$, $\vartheta_2(-z) = \vartheta_2(z)$. Besides, by using $\vartheta_3\left(\frac{1}{2}+x;Ti\right) = \vartheta_3\left(\frac{1}{2}-x;Ti\right)$ and $\vartheta_2\left(\frac{1}{2};Ti\right) = 0$, the formula (3.32) can be rewritten by

$$2\vartheta_3'\left(\frac{1}{2}+x\right)\vartheta_3\left(\frac{1}{2}-x\right)\vartheta_3^2\left(\frac{1}{2}\right) - 2\vartheta_3^2(0)\vartheta_3(x)\vartheta_3'(x) = -2\vartheta_2^2(0)\vartheta_2(x)\vartheta_2'(x). \tag{3.33}$$

That is,

$$\begin{aligned}\frac{\vartheta_2'(x)}{\vartheta_3'(x)} &= \frac{\vartheta_3^2(0)}{\vartheta_2^2(0)}\frac{\vartheta_3(x)}{\vartheta_2(x)} - \frac{\vartheta_3^2\left(\frac{1}{2}\right)}{\vartheta_2^2(0)}\frac{\vartheta_3'\left(\frac{1}{2}+x\right)\vartheta_3\left(\frac{1}{2}-x\right)}{\vartheta_3'(x)\vartheta_2(x)} \\ &:= \frac{\vartheta_3^2(0)}{\vartheta_2^2(0)}I_1(x) - \frac{\vartheta_3^2\left(\frac{1}{2}\right)}{\vartheta_2^2(0)}I_2(x).\end{aligned} \tag{3.34}$$

Next, we claim that for any $x \in (0, \frac{1}{4})$,

$$\frac{d}{dx}I_1(x) > 0, \ \frac{d}{dx}I_2(x) < 0. \tag{3.35}$$

Then, for any $x \in (0, \frac{1}{4})$, (3.22) follows directly from (3.34) and (3.35).

At last, our target is to prove claim (3.35). Note that, when $x \in (0, \frac{1}{2})$, $\vartheta_3'\left(\frac{1}{2}+x\right) > 0, \vartheta_3'(x) < 0$ and ϑ_3 is always positive. So based on formula (3.34),

$$\frac{\vartheta_2'(x)}{\vartheta_3'(x)} > \frac{\vartheta_3^2(0)}{\vartheta_2^2(0)}\frac{\vartheta_3(x)}{\vartheta_2(x)} \Longrightarrow \vartheta_2'(x) < \frac{\vartheta_3^2(0)}{\vartheta_2^2(0)}\frac{\vartheta_3(x)}{\vartheta_2(x)}\vartheta_3'(x). \tag{3.36}$$

Thus,

$$\begin{aligned}\frac{d}{dx}I_1(x) &= \frac{\vartheta_3'(x)\vartheta_2(x) - \vartheta_3(x)\vartheta_2'(x)}{(\vartheta_2'(x))^2} \\ &> \frac{\vartheta_3'(x)\left(\vartheta_2(x) - \frac{\vartheta_3^2(0)}{\vartheta_2^2(0)}\frac{\vartheta_3^2(x)}{\vartheta_2(x)}\right)}{(\vartheta_2'(x))^2} > 0, \forall x \in (0, \frac{1}{2}),\end{aligned} \tag{3.37}$$

where the last inequality comes from Lemma 3.2 and $\vartheta_3 > \vartheta_2$ which comes from (3.16). On the other hand,

$$\begin{aligned}\frac{d}{dx}I_2(x) =& \frac{1}{(\vartheta_3'(x)\vartheta_2(x))^2}\left(\vartheta_3''\left(\frac{1}{2}+x\right)\vartheta_3\left(\frac{1}{2}-x\right)\vartheta_3'(x)\vartheta_2(x)\right. \\ &- \vartheta_3'\left(\frac{1}{2}+x\right)\vartheta_3'\left(\frac{1}{2}-x\right)\vartheta_3'(x)\vartheta_2(x) \\ &- \vartheta_3'\left(\frac{1}{2}+x\right)\vartheta_3\left(\frac{1}{2}-x\right)\vartheta_3''(x)\vartheta_2(x) \\ &\left.-\vartheta_3'\left(\frac{1}{2}+x\right)\vartheta_3\left(\frac{1}{2}-x\right)\vartheta_3'(x)\vartheta_2'(x)\right).\end{aligned} \tag{3.38}$$

By using Lemma 3.2, the terms $-\vartheta_3'\left(\frac{1}{2}+x\right)\vartheta_3'\left(\frac{1}{2}-x\right)\vartheta_3'(x)\vartheta_2(x)$ and $-\vartheta_3'\left(\frac{1}{2}+x\right)$ $\vartheta_3\left(\frac{1}{2}-x\right)\vartheta_3'(x)\vartheta_2'(x)$ are negative for all $x \in (0, \frac{1}{4})$. For the other two terms of

(3.38), let $f_n(x)$ be defined in (3.24), by using the formula (3.25),
(3.39)

$$\frac{\vartheta_3''\left(\frac{1}{2}+x\right)\vartheta_3'(x)-\vartheta_3'\left(\frac{1}{2}+x\right)\vartheta_3''(x)}{\vartheta_3\left(\frac{1}{2}+x\right)\vartheta_3(x)}=\frac{\vartheta_3''\left(\frac{1}{2}+x\right)}{\vartheta_3\left(\frac{1}{2}+x\right)}\frac{\vartheta_3'(x)}{\vartheta_3(x)}-\frac{\vartheta_3'\left(\frac{1}{2}+x\right)}{\vartheta_3\left(\frac{1}{2}+x\right)}\frac{\vartheta_3''(x)}{\vartheta_3(x)}$$

$$=\left(\sum_{n=1}^{\infty}\frac{f_{2n-1}''\left(\frac{1}{2}+x\right)}{f_{2n-1}\left(\frac{1}{2}+x\right)}+\sum_{\substack{n,m=1\\n\neq m}}^{\infty}\frac{f_{2n-1}'\left(\frac{1}{2}+x\right)f_{2m-1}'\left(\frac{1}{2}+x\right)}{f_{2n-1}\left(\frac{1}{2}+x\right)f_{2m-1}\left(\frac{1}{2}+x\right)}\right)\cdot\sum_{k=1}^{\infty}\frac{f_{2k-1}'(x)}{f_{2k-1}(x)}$$

$$-\left(\sum_{n=1}^{\infty}\frac{f_{2n-1}''(x)}{f_{2n-1}(x)}+\sum_{\substack{n,m=1\\n\neq m}}^{\infty}\frac{f_{2n-1}'(x)f_{2m-1}'(x)}{f_{2n-1}(x)f_{2m-1}(x)}\right)\cdot\sum_{k=1}^{\infty}\frac{f_{2k-1}'\left(\frac{1}{2}+x\right)}{f_{2k-1}\left(\frac{1}{2}+x\right)}.$$

Note that for any $n,k\in\mathbb{Z}_+$ and $x\in\mathbb{R}\setminus\frac{1}{2}\mathbb{Z}$,

$$f_n'\left(\frac{1}{2}+x\right)=-f_n'(x),\ f_n''\left(\frac{1}{2}+x\right)=-f_n''(x), \tag{3.40}$$

and

$$\frac{f_n''(x)}{f_k''(x)}=\frac{f_n'(x)}{f_k'(x)}=e^{-(n-k)\pi T}. \tag{3.41}$$

Therefore, from the formulas (3.40) and (3.41), we obtain

$$\begin{aligned}&\frac{\vartheta_3''\left(\frac{1}{2}+x\right)\vartheta_3'(x)-\vartheta_3'\left(\frac{1}{2}+x\right)\vartheta_3''(x)}{\vartheta_3\left(\frac{1}{2}+x\right)\vartheta_3(x)}\\&=\sum_{\substack{n,m=1\\n\neq m}}^{\infty}\sum_{k=1}^{\infty}\left(\frac{f_{2n-1}'\left(\frac{1}{2}+x\right)f_{2m-1}'\left(\frac{1}{2}+x\right)}{f_{2n-1}\left(\frac{1}{2}+x\right)f_{2m-1}\left(\frac{1}{2}+x\right)}\cdot\frac{f_{2k-1}'(x)}{f_{2k-1}(x)}\right)\\&\quad-\sum_{\substack{n,m=1\\n\neq m}}^{\infty}\sum_{k=1}^{\infty}\left(\frac{f_{2n-1}'(x)f_{2m-1}'(x)}{f_{2n-1}(x)f_{2m-1}(x)}\cdot\frac{f_{2k-1}'\left(\frac{1}{2}+x\right)}{f_{2k-1}\left(\frac{1}{2}+x\right)}\right)\\&<\sum_{\substack{n,m=1\\n\neq m}}^{\infty}\sum_{k=1}^{\infty}0-\sum_{\substack{n,m=1\\n\neq m}}^{\infty}\sum_{k=1}^{\infty}0=0,\ \forall x\in\left(0,\frac{1}{4}\right).\end{aligned} \tag{3.42}$$

Where the last inequality comes from the positivity and negativity of f_n and f_n'. Thus, for any $x\in(0,\frac{1}{4})$, from (3.42) and Lemma 3.2,

$$\left(\vartheta_3''\left(\frac{1}{2}+x\right)\vartheta_3'(x)-\vartheta_3'\left(\frac{1}{2}+x\right)\vartheta_3''(x)\right)\vartheta_3\left(\frac{1}{2}-x\right)\vartheta_2(x)<0. \tag{3.43}$$

To sum up, from the formula (3.38), $\frac{d}{dx}I_2(x)<0, x\in(0,\frac{1}{4})$ and we verify the claim (3.35).

□

4. The proof of Theorem 1.1 to Theorem 1.4

In this section, we study the critical points of the heat kernel $p(z,t)$ on the rectangular, hexagonal, and rhombic tori, and prove Theorem 1.1, Theorem 1.2 and Theorem 1.4. The main idea is that $p(z,t)$ can be explicitly expressed in terms of theta functions.

4.1. **Rectangular torus.** The case of the rectangular torus is trivial, but it gives us preliminary inspiration to express the heat kernel with theta functions. We consider the torus $\mathbb{T} = \mathbb{C}/\Lambda$ generated by rectangular lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ with $\tau = bi, b > 0$. Let $T = \frac{4\pi t}{b^2} \in (0, +\infty)$ and $z = \nu + \mu\tau$, where $\nu, \mu \in [0,1]$, then the heat kernel in (2.10) can be rewritten by

$$\begin{aligned} bp(z,t) &= \sum_{n,m\in\mathbb{Z}} e^{-\pi T\left(n^2b^2+m^2\right)} e^{2\pi i(m\mu+n\nu)} \\ &= \vartheta_3(\mu; Ti)\vartheta_3(\nu; b^2Ti). \end{aligned} \tag{4.1}$$

Proof of Theorem 1.1. From the formula (4.1), the critical point satisfies $\nabla p = 0$ if and only if $\vartheta_3'(\mu; Ti) = \vartheta_3'(\nu; b^2Ti) = 0$.

By using the Lemma 3.2(3) and $\vartheta_3(z;\tau) = \vartheta_3(z+1;\tau) = \vartheta_3(-z;\tau)$, we obtain that $\vartheta_3'(\mu; Ti) = 0$ if and only if $\mu = 0, \frac{1}{2}, 1$. Note that for $\mu = 0$ and $\mu = 1$, $z = \nu + \mu\tau$ are the same points on torus $\mathbb{T}$, so we only need to consider $\mu = 0, \frac{1}{2}$. Moreover, $\vartheta_3(0; Ti) = \vartheta_3(1; Ti)$ are the maximal points and $\vartheta_3(\frac{1}{2}; Ti)$ is the minimal point of $\vartheta_3(\mu; Ti)$. The another part $\vartheta_3(\nu; b^2Ti)$ is similar. Therefore, we attain Theorem 1.1 immediately. □

Remark 4.1. Based on the expression (4.1), the heat kernel on a rectangular torus is essentially the same as a one-dimensional heat kernel. Therefore, it can be regarded as a direct corollary of the one-dimensional results of heat kernel/theta functions, such as [23]. This is true for rectangular torus for any dimension.

4.2. **Hexagonal torus.** The case of the hexagonal torus is much different from the case of the rectangular torus. In this instance, besides the 4 trivial critical points, $p(z,t)$ also has a pair of fixed non-trivial critical points at $z = \frac{1}{3} + \frac{1}{3}\tau$ and $z = \frac{2}{3} + \frac{2}{3}\tau$ for all $t > 0$.

We consider the torus $\mathbb{T} = \mathbb{C}/\Lambda$ generated by hexagonal lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ with $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Let $T = \frac{16\pi t}{3} \in (0, +\infty)$ and $z = \nu + \mu\tau$, where $\nu, \mu \in [0,1]$, then the heat kernel in (2.10) can be rewritten by

$$\begin{aligned} \frac{\sqrt{3}}{2}p(z,t) &= \sum_{n,m\in\mathbb{Z}} e^{-\pi T\left(n^2+m^2-mn\right)} e^{2\pi i(m\mu+n\nu)} \\ &= \left(\sum_{m=2k} + \sum_{m=2k+1}\right) \sum_{n\in\mathbb{Z}} e^{-\pi T(n-\frac{m}{2})^2} e^{2\pi i(n-\frac{m}{2})\nu} e^{-\pi T\frac{3}{4}m^2} e^{2\pi i m(\mu+\frac{\nu}{2})} \\ &= \sum_{k\in\mathbb{Z}} \vartheta_3(\nu; Ti) e^{-\pi T3k^2} e^{2k\pi i(2\mu+\nu)} \\ &\quad + \sum_{k\in\mathbb{Z}} \vartheta_2(\nu; Ti) e^{-\pi T3(k+\frac{1}{2})^2} e^{(2k+1)\pi i(2\mu+\nu)} \\ &= \vartheta_3(\nu; Ti)\vartheta_3(2\mu+\nu; 3Ti) + \vartheta_2(\nu; Ti)\vartheta_2(2\mu+\nu; 3Ti). \end{aligned} \tag{4.2}$$

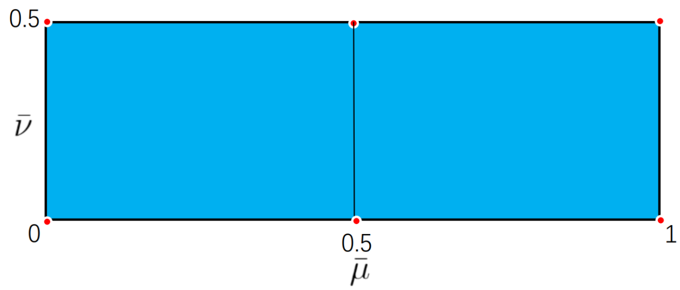


FIGURE 6. The domain of variables $(\bar{\nu}, \bar{\mu})$.

The special case $z = 0$ of the formula (4.2) has been found in the textbook [8].

Let $\bar{\mu} = 2\mu + \nu$ and $\bar{\nu} = \nu$, then

$$\frac{\sqrt{3}}{2} p(z,t) = \vartheta_3(\bar{\nu}; Ti)\vartheta_3(\bar{\mu}; 3Ti) + \vartheta_2(\bar{\nu}; Ti)\vartheta_2(\bar{\mu}; 3Ti). \tag{4.3}$$

Because $\bar{\nu}, \bar{\mu}$ are linear independent, the critical point satisfies $\nabla_z p(z,t) = 0$ if and only if $\partial_{\bar{\nu}} p = \partial_{\bar{\mu}} p = 0$. That is, we only need to find $(\bar{\nu}, \bar{\mu})$ such that

$$\begin{cases} \vartheta_3'(\bar{\nu}; Ti)\vartheta_3(\bar{\mu}; 3Ti) + \vartheta_2'(\bar{\nu}; Ti)\vartheta_2(\bar{\mu}; 3Ti) = 0, (*) \\ \vartheta_3(\bar{\nu}; Ti)\vartheta_3'(\bar{\mu}; 3Ti) + \vartheta_2(\bar{\nu}; Ti)\vartheta_2'(\bar{\mu}; 3Ti) = 0. (**) \end{cases} \tag{4.4}$$

To give a complete discussion, we solve (4.4) for all $\bar{\nu}, \bar{\mu} \in \mathbb{R}$ and $T > 0$. Note that for any $z \in \mathbb{C}, \tau \in \mathbb{H}$, $\vartheta_2(z + 2; \tau) = \vartheta_2(z; \tau)$ and $\vartheta_3(z + 2; \tau) = \vartheta_3(z; \tau)$. Because of these periodicities, we only need to solve equation (4.4) on $\bar{\nu}, \bar{\mu} \in [-1, 1]$.

By using Proposition 3.1(2)(3), if $(\bar{\nu}, \bar{\mu})$ is a solution of (4.4), then $(-\bar{\nu}, \bar{\mu})$, $(\bar{\nu}, -\bar{\mu})$, $(-\bar{\nu}, -\bar{\mu})$ are all solutions of (4.4). Thus, we could consider $\bar{\nu}, \bar{\mu} \in [0, 1]$.

Further more, for any $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$, also from Proposition 3.1(2)(3),

$$\vartheta_3(1 - z; \tau) = \vartheta_3(-z; \tau) = \vartheta_3(z; \tau)$$

and $\vartheta_3'(1 - z; \tau) = -\vartheta_3'(z; \tau)$. Similarly, $\vartheta_2(1 - z; \tau) = -\vartheta_2(z; \tau)$ and $\vartheta_2'(1 - z; \tau) = \vartheta_2'(z; \tau)$. Therefore, $(\bar{\nu}, \bar{\mu})$ is a solution of (4.4) if and only if $(1 - \bar{\nu}, 1 - \bar{\mu})$ is a solution.

To summarize, without loss of generality, we consider $\bar{\nu} \in [0, \frac{1}{2}]$ and $\bar{\mu} \in [0, 1]$ to solve equation (4.4). We categorize the variables $(\bar{\nu}, \bar{\mu})$ into three cases: half-integer points, boundary part, and inner part (the red, black, and blue parts in Figure 6 respectively). These three cases correspond to three lemmas as follows.

Lemma 4.2 aims to find critical points at half-integers, which are the trivial critical points of the heat kernel.

Lemma 4.2 (Half-integer points)**.** *Suppose $T > 0$, when $\bar{\nu} \in \{0, \frac{1}{2}\}$ and $\bar{\mu} \in \{0, \frac{1}{2}, 1\}$, the solutions $(\bar{\nu}, \bar{\mu})$ of (4.4) are* $(0,0), (0,1), (\frac{1}{2}, \frac{1}{2})$.

Proof. By using Lemma 3.2, for any $\tau = Ti$ on the imaginary positive half axis, we obtain $\vartheta_3'(0; \tau) = \vartheta_3'\left(\frac{1}{2}; \tau\right) = \vartheta_3'(1; \tau) = \vartheta_2'(0; \tau) = \vartheta_2'(1; \tau) = \vartheta_2\left(\frac{1}{2}; \tau\right) = 0$ and $\vartheta_2'\left(\frac{1}{2}; \tau\right), \vartheta_2(0; \tau), \vartheta_2(1; \tau) \neq 0$.

Substitute these properties into (4.4), we obtain directly that half-integer solutions are $(0,0), (0,1), (\frac{1}{2}, \frac{1}{2})$. □

Lemma 4.3 is to find critical points on the boundary part, which is the unique pair of non-trivial critical points of the heat kernel.

Lemma 4.3 (Boundary part)**.** *Suppose $T>0$ and*

$$(\bar{\nu},\bar{\mu})\in\left(\left(0,\frac{1}{2}\right)\times\left\{0,\frac{1}{2},1\right\}\right)\bigcup\left(\left\{0,\frac{1}{2}\right\}\times\left(0,\frac{1}{2}\right)\right)\bigcup\left(\left\{0,\frac{1}{2}\right\}\times\left(\frac{1}{2},1\right)\right), \tag{4.5}$$

then the unique solution $(\bar{\nu},\bar{\mu})$ of (4.4) in (4.5) is $(\bar{\nu},\bar{\mu})=(\frac{1}{3},1)$.

Proof. We prove this lemma in three steps.

Step 1: if $(\bar{\nu},\bar{\mu})\in\left(0,\frac{1}{2}\right)\times\left\{0,\frac{1}{2},1\right\}$ is the solution of (4.4), we utilize the positivity and negativity of theta functions to determine the necessary conditions for $(\bar{\nu},\bar{\mu})$. By using Lemma 3.2,

$$\vartheta_3'(\bar{\nu};Ti),\vartheta_2'(\bar{\nu};Ti)<0,\ \text{and}\ \vartheta_3(\bar{\mu};3Ti)>0.$$

Therefore, if (4.4)($*$) is correct, $\vartheta_2(\bar{\mu};3Ti)$ must be negative. So, we obtain $\bar{\mu}=1$. Now, we claim that $(\bar{\nu},\bar{\mu})=(\frac{1}{3},1)$ is the unique solution in $\left(0,\frac{1}{2}\right)\times\left\{0,\frac{1}{2},1\right\}$.

By using Proposition 3.3, the equation (4.4)($*$) can be rewritten by

$$\begin{aligned}&\vartheta_3'(\bar{\nu};Ti)\vartheta_3\left(\frac{\bar{\mu}}{3};Ti\right)\vartheta_3\left(\frac{\bar{\mu}-1}{3};Ti\right)\vartheta_3\left(\frac{\bar{\mu}+1}{3};Ti\right)\\ &+\vartheta_2'(\bar{\nu};Ti)\vartheta_2\left(\frac{\bar{\mu}}{3};Ti\right)\vartheta_2\left(\frac{\bar{\mu}-1}{3};Ti\right)\vartheta_2\left(\frac{\bar{\mu}+1}{3};Ti\right)=0.\end{aligned} \tag{4.6}$$

Take the derivative of (3.8) with respect to v and let $u=v=\frac{1}{3}$, by using $\vartheta_3'(\frac{1}{3};Ti)=-\vartheta_3'(-\frac{1}{3};Ti)=-\vartheta_3'(\frac{2}{3};Ti)$ and $\vartheta_2'(\frac{1}{3};Ti)=-\vartheta_2'(-\frac{1}{3};Ti)=\vartheta_2'(\frac{2}{3};Ti)$, we have

$$\begin{aligned}&-3\vartheta_3'\left(\frac{1}{3};Ti\right)\vartheta_3(0;Ti)\vartheta_3\left(-\frac{1}{3};Ti\right)\vartheta_3\left(\frac{1}{3};Ti\right)\\ =&-3\vartheta_2'\left(\frac{1}{3};Ti\right)\vartheta_2(0;Ti)\vartheta_2\left(-\frac{1}{3};Ti\right)\vartheta_2\left(\frac{1}{3};Ti\right).\end{aligned} \tag{4.7}$$

Combine (4.6) and (4.7), $(\bar{\nu},\bar{\mu})=(\frac{1}{3},1)$ is the solution of (4.4)($*$). Besides, there is also the solution of (4.4)($**$) by $\vartheta_3'(1;3Ti)=\vartheta_2'(1;3Ti)=0$. Therefore, $(\bar{\nu},\bar{\mu})=(\frac{1}{3},1)$ is the solution of (4.4). In addition, by using Proposition 3.7, as the necessary condition $\bar{\mu}=1$, the equation (4.4)($*$)

$$\frac{\vartheta_2'(\bar{\nu};Ti)}{\vartheta_3'(\bar{\nu};Ti)}=-\frac{\vartheta_3(1;3Ti)}{\vartheta_2(1;3Ti)}$$

has at most one solution $\bar{\nu}\in(0,\frac{1}{2})$. Therefore, $(\bar{\nu},\bar{\mu})=(\frac{1}{3},1)$ is unique in this step.

Step 2: if $(\bar{\nu},\bar{\mu})\in\left\{0,\frac{1}{2}\right\}\times\left(0,\frac{1}{2}\right)$ is the solution of (4.4). By using Lemma 3.2,

$$\vartheta_3'(\bar{\mu};3Ti),\vartheta_2'(\bar{\mu};3Ti)<0,\ \text{and}\ \vartheta_3(\bar{\nu};Ti)>0.$$

Therefore, similar to step 1, if (4.4)($**$) is correct, $\vartheta_2(\bar{\nu};Ti)$ must be negative. While this is impossible for $\bar{\nu}\in\{0,\frac{1}{2}\}$. Thus, there is no solution in this step.

Step 3: if $(\bar{\nu},\bar{\mu})\in\left\{0,\frac{1}{2}\right\}\times\left(\frac{1}{2},1\right)$ is the solution of (4.4). Because

$$\vartheta_3'(\bar{\mu};3Ti)>0,\ \vartheta_2'(\bar{\mu};3Ti)<0,\ \text{and}\ \vartheta_3(\bar{\nu};Ti)>0,$$

the solution $(\bar{\nu},\bar{\mu})$ satisfies (4.4)$(**)$ implies that $\vartheta_2(\bar{\nu};Ti)>0$. Then $\bar{\nu}$ must be zero. However, when $\bar{\nu}=0$, we have the following contradiction

$$
\begin{aligned}
\frac{\vartheta_3(0;Ti)}{\vartheta_2(0;Ti)} &= -\frac{\vartheta_2'(\bar{\mu};3Ti)}{\vartheta_3'(\bar{\mu};3Ti)} = \frac{\vartheta_2'(1-\bar{\mu};3Ti)}{\vartheta_3'(1-\bar{\mu};3Ti)} \\
&> \frac{\vartheta_3(1-\bar{\mu};3Ti)}{\vartheta_2(1-\bar{\mu};3Ti)} > \frac{\vartheta_3(0;3Ti)}{\vartheta_2(0;3Ti)} > \frac{\vartheta_3(0;Ti)}{\vartheta_2(0;Ti)}.
\end{aligned} \tag{4.8}
$$

Here, we utilized sequentially (4.4) $(**)$, Proposition 3.5, formula (3.37), and Proposition 3.6. Consequently, there is no solution in step 3.

To sum up, $(\bar{\nu},\bar{\mu})=(\frac{1}{3},1)$ is the unique solution in (4.5).

□

Lemma 4.4 aims to prove that the heat kernel does not have any more critical points in the inner part.

Lemma 4.4 (Inner part)**.** *Suppose* $T>0$ *and*

$$
(\bar{\nu},\bar{\mu}) \in \left(\left(0,\frac{1}{2}\right)\times\left(0,\frac{1}{2}\right)\right)\bigcup\left(\left(0,\frac{1}{2}\right)\times\left(\frac{1}{2},1\right)\right), \tag{4.9}
$$

then the solution $(\bar{\nu},\bar{\mu})$ *of (4.4) in (4.9) is not existent.*

Proof. We prove this lemma in two steps.

Step 1: for the case $(\bar{\nu},\bar{\mu})\in\big((0,\frac{1}{2})\times(0,\frac{1}{2})\big)$, by using Lemma(3.2), we have

$$
\vartheta_3'(\bar{\nu};Ti)\vartheta_3(\bar{\mu};3Ti)+\vartheta_2'(\bar{\nu};Ti)\vartheta_2(\bar{\mu};3Ti)<0+0=0,\ \forall T>0.
$$

Therefore, $(\bar{\nu},\bar{\mu})$ can not be the solution of (4.4).

Step 2: for the case $(\bar{\nu},\bar{\mu})\in\big((0,\frac{1}{2})\times(\frac{1}{2},1)\big)$, if $(\bar{\nu},\bar{\mu})$ is the solution of (4.4), then

$$
\frac{\vartheta_3(\bar{\nu};Ti)}{\vartheta_2(\bar{\nu};Ti)} = -\frac{\vartheta_2'(\bar{\mu};3Ti)}{\vartheta_3'(\bar{\mu};3Ti)} > -\frac{\vartheta_3(\bar{\mu};3Ti)}{\vartheta_2(\bar{\mu};3Ti)} = \frac{\vartheta_2'(\bar{\nu};Ti)}{\vartheta_3'(\bar{\nu};Ti)} > \frac{\vartheta_3(\bar{\nu};Ti)}{\vartheta_2(\bar{\nu};Ti)}, \tag{4.10}
$$

where the equalities are based on (4.4) and the inequalities are based on Proposition 3.5 and Lemma 3.2. Because (4.10) is a contradiction, there is no solution $(\bar{\nu},\bar{\mu})$ of (4.4) in (4.9). □

Lemma 4.2 to Lemma 4.4 study the solutions of (4.4) completely. Based on these three lemmas, we find the critical points of the heat kernel on the hexagonal torus.

Proof of Theorem 1.2. From the discussion in Section 2, we only need to consider the torus $\mathbb{T}=\mathbb{C}/\Lambda$ generated by hexagonal lattice $\Lambda=\mathbb{Z}\oplus\mathbb{Z}\tau$ with $\tau=\frac{1}{2}+\frac{\sqrt{3}}{2}i$. From Lemma 4.2, Lemma 4.3 and Lemma 4.4, we obtain that the solutions $(\bar{\nu},\bar{\mu})\in[0,\frac{1}{2}]\times[0,1]$ of equation (4.4) are $(\bar{\nu},\bar{\mu})=(0,0),(0,1),(\frac{1}{2},\frac{1}{2}),(\frac{1}{3},1)$. By using the periodicities and symmetries of theta functions (see the discussion above Lemma 4.2), for any $T>0$, all solutions of (4.4) are

$$
\begin{aligned}
(\bar{\nu},\bar{\mu})\in\left\{(0,0),(0,\pm1),\left(\pm\frac{1}{2},\pm\frac{1}{2}\right),\left(\pm\frac{1}{3},\pm1\right),(\pm1,\pm1),(\pm1,0),\left(\pm\frac{2}{3},0\right)\right\}& \\
+2\mathbb{Z}\oplus2\mathbb{Z}.&
\end{aligned} \tag{4.11}
$$

Because $\bar{\nu} = \nu, \bar{\mu} = 2\mu + \nu$ and $z = \nu + \mu\tau$, from the expression (4.2), the critical points of $p(z, t)$ on the hexagonal torus $\mathbb{T}$ are

$$z \in \left\{0, \frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2} + \frac{1}{2}\tau, \frac{1}{3} + \frac{1}{3}\tau, \frac{2}{3} + \frac{2}{3}\tau\right\} + \mathbb{Z} \oplus \mathbb{Z}\tau. \tag{4.12}$$

Finally, we verify the extremum properties of these critical points. At first, by using the formula (2.10), for any $z \in \mathbb{T}$ and $t > 0$,

$$\begin{aligned} |p(z,t)| &= \left| \frac{1}{\Im(\tau)} \sum_{\omega \in \Lambda^{\perp}} e^{-4\pi^2 t |\omega|^2} e^{2\pi i z \cdot \omega} \right| \\ &\leqslant \frac{1}{\Im(\tau)} \sum_{\omega \in \Lambda^{\perp}} \left| e^{-4\pi^2 t |\omega|^2} e^{2\pi i z \cdot \omega} \right| = p(0,t). \end{aligned} \tag{4.13}$$

Therefore, 0 is always the maximal point of the heat kernel. In fact, this formula (4.13) holds true for any torus $\mathbb{T}$. Similar formulas compared to formula (4.13) were derived by Bétermin and Faulhuber [3, 9].

On the other hand, in the previous articles [1] and [9], they studied the minimal point of the heat kernel and found that the point $\frac{1}{3} + \frac{1}{3}\tau$ is the minimal point of $p(z, t)$ on hexagonal torus. Besides, from the property

$$p(z, t) = p(-z, t) = p(1 + \tau - z, t),$$

$\frac{1}{3} + \frac{1}{3}\tau$ and $\frac{2}{3} + \frac{2}{3}\tau$ are both the minimal points with the same values.

□

4.3. **Rhombic torus.** The case of the rhombic torus is a generalization of the situation for the hexagonal torus. We consider the torus $\mathbb{T} = \mathbb{C}/\Lambda$ generated by rhombic lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ with $\tau = \frac{1}{2} + bi, b > 0$. Let $T = \frac{4\pi t}{b^2} \in (0, +\infty)$ and $z = \nu + \mu\tau$, where $\nu, \mu \in [0, 1]$. Similar to the formula (4.2), the heat kernel in (2.10) can be rewritten by

$$\begin{aligned} bp(z,t) &= \left(\sum_{n=2k} + \sum_{n=2k+1} \right) \sum_{m \in \mathbb{Z}} e^{-\pi T (m - \frac{n}{2})^2} e^{2\pi i (m - \frac{n}{2})\mu} e^{-\pi b^2 T n^2} e^{2\pi i n (\nu + \frac{\mu}{2})} \\ &= \sum_{k \in \mathbb{Z}} \vartheta_3(\mu; Ti) e^{-\pi 4 b^2 T k^2} e^{2k\pi i (2\nu + \mu)} \\ &\quad + \sum_{k \in \mathbb{Z}} \vartheta_2(\mu; Ti) e^{-\pi 4 b^2 T (k + \frac{1}{2})^2} e^{(2k+1)\pi i (2\nu + \mu)} \\ &= \vartheta_3(\mu; Ti)\vartheta_3(2\nu + \mu; 4b^2 Ti) + \vartheta_2(\mu; Ti)\vartheta_2(2\nu + \mu; 4b^2 Ti). \end{aligned} \tag{4.14}$$

Let $\bar{\nu} = \mu$ and $\bar{\mu} = 2\nu + \mu$, then

$$bp(z,t) = \vartheta_3(\bar{\nu}; Ti)\vartheta_3(\bar{\mu}; 4b^2 Ti) + \vartheta_2(\bar{\nu}; Ti)\vartheta_2(\bar{\mu}; 4b^2 Ti). \tag{4.15}$$

As the same derivation in Section 4.2, to find the critical points of $p(z, t)$, we only need to solve $(\bar{\nu}, \bar{\mu}) \in [0, \frac{1}{2}] \times [0, 1]$ satisfy

$$\begin{cases} \vartheta_3'(\bar{\nu}; Ti)\vartheta_3(\bar{\mu}; 4b^2 Ti) + \vartheta_2'(\bar{\nu}; Ti)\vartheta_2(\bar{\mu}; 4b^2 Ti) = 0, (*) \\ \vartheta_3(\bar{\nu}; Ti)\vartheta_3'(\bar{\mu}; 4b^2 Ti) + \vartheta_2(\bar{\nu}; Ti)\vartheta_2'(\bar{\mu}; 4b^2 Ti) = 0. (**) \end{cases} \tag{4.16}$$

By employing the same discussion as in Lemma 4.2, Lemma 4.3 and Lemma 4.4, except for the trivial critical points $(\bar{\nu},\bar{\mu}) \in \{(0,0),(0,1),(\frac{1}{2},\frac{1}{2})\}$, the other solutions of (4.16) can only appear on two boundaries

$$(\bar{\nu},\bar{\mu}) \in \left(\left(0,\frac{1}{2}\right)\times\{1\}\right)\bigcup\left(\{0\}\times\left(\frac{1}{2},1\right)\right). \tag{4.17}$$

Let's consider these two situations separately.

Case 1: When $\bar{\mu}=1$ and $\bar{\nu}\in(0,\frac{1}{2})$, the equation (4.16)(∗∗) is always correct since $\vartheta_2'(1)=\vartheta_3'(1)=0$. Therefore, we attribute the problem to solve

$$\frac{\vartheta_2'(\bar{\nu};Ti)}{\vartheta_3'(\bar{\nu};Ti)} = -\frac{\vartheta_3(1;4b^2Ti)}{\vartheta_2(1;4b^2Ti)} = \frac{\vartheta_3(0;4b^2Ti)}{\vartheta_2(0;4b^2Ti)} \tag{4.18}$$

on $\bar{\nu}\in(0,\frac{1}{2})$.

Case 2: When $\bar{\nu}=0$ and $\bar{\mu}\in(\frac{1}{2},1)$, the equation (4.16)(∗) is always correct since $\vartheta_2'(0)=\vartheta_3'(0)=0$. Therefore, we attribute the problem to solve

$$\frac{\vartheta_3(0;Ti)}{\vartheta_2(0;Ti)} = -\frac{\vartheta_2'(\bar{\mu};4b^2Ti)}{\vartheta_3'(\bar{\mu};4b^2Ti)} = \frac{\vartheta_2'(1-\bar{\mu};4b^2Ti)}{\vartheta_3'(1-\bar{\mu};4b^2Ti)}. \tag{4.19}$$

on $\bar{\mu}\in(\frac{1}{2},1)$. There is a sufficient and necessary condition for the existence of solution in (4.18) and (4.19).

Lemma 4.5. *The equations (4.18) and (4.19) both have at most one solution. Moreover, (4.18) has a solution $\bar{\nu}\in(0,\frac{1}{2})$ if and only if*

$$\frac{\vartheta_2'(0;Ti)}{\vartheta_3'(0;Ti)} < \frac{\vartheta_3(0;4b^2Ti)}{\vartheta_2(0;4b^2Ti)}, \tag{4.20}$$

and (4.19) has a solution $\bar{\mu}\in(\frac{1}{2},1)$ if and only if

$$\frac{\vartheta_2'(0;4b^2Ti)}{\vartheta_3'(0;4b^2Ti)} < \frac{\vartheta_3(0;Ti)}{\vartheta_2(0;Ti)}. \tag{4.21}$$

Here the meaning of the left hand sides of (4.20) and (4.21) are the limit of $\dfrac{\vartheta_2'}{\vartheta_3'}$ at zero.

Proof. By using Lemma 3.2, there is a direct observation that

$$\lim_{\bar{\nu}\to\frac{1}{2}^-}\frac{\vartheta_2'(\bar{\nu};Ti)}{\vartheta_3'(\bar{\nu};Ti)} = +\infty.$$

In addition, $\frac{\vartheta_2'}{\vartheta_3'}$ is strictly increasing based on the Proposition 3.7. Therefore, if

$$\frac{\vartheta_2'(0;Ti)}{\vartheta_3'(0;Ti)} < \frac{\vartheta_3(0;4b^2Ti)}{\vartheta_2(0;4b^2Ti)},$$

(4.18) has a unique solution $\bar{\nu}\in(0,\frac{1}{2})$. Conversely, no solution. The proof of another equation (4.19) is similar.

□

Now, we give a complete result of the solutions of (4.18) and (4.19).

Theorem 4.6. *For any $t>0$, let $T=\frac{4\pi t}{b^2}$,there exists $\frac{\sqrt{3}}{6}<b_1(t)<\frac{1}{2}<b_2(t)<\frac{\sqrt{3}}{2}$ depend on t such that*

(1) *the equation (4.19) has one solution* $\bar{\mu}(b,t) \in (\frac{1}{2}, 1)$ *if* $b \in (0, b_1(t))$, *and no solution* $\bar{\mu}$ *as* $b \in [b_1(t), +\infty)$.
(2) *the equation (4.18) has one solution* $\bar{\nu}(b,t) \in (0, \frac{1}{2})$ *if* $b \in (b_2(t), +\infty)$, *and no solution* $\bar{\nu}$ *as* $b \in (0, b_2(t)]$.

Proof. To solve the equation (4.19), we define a function

$$F(x;\alpha) = \frac{\vartheta_2'(0;\alpha i)}{\vartheta_3'(0;\alpha i)} - \frac{\vartheta_3(0;x\alpha i)}{\vartheta_2(0;x\alpha i)}, \text{ with } x, \alpha \in (0,+\infty). \tag{4.22}$$

Note that from Proposition 3.5,

$$\frac{\vartheta_2'(0;\alpha i)}{\vartheta_3'(0;\alpha i)} > \frac{\vartheta_3(0;\alpha i)}{\vartheta_2(0;\alpha i)}, \forall \alpha \in (0,+\infty)$$

Thus $F(1;\alpha) > 0, \forall \alpha > 0$. Moreover, from Proposition 3.7 and step 1 of the proof of Lemma 4.3, for any $\alpha > 0$,

$$\frac{\vartheta_2'(0;\alpha i)}{\vartheta_3'(0;\alpha i)} < \frac{\vartheta_2'(\frac{1}{3};\alpha i)}{\vartheta_3'(\frac{1}{3};\alpha i)} = -\frac{\vartheta_3(1;3\alpha i)}{\vartheta_2(1;3\alpha i)} = \frac{\vartheta_3(0;3\alpha i)}{\vartheta_2(0;3\alpha i)}.$$

So, we have $F(3;\alpha) < 0, \forall \alpha > 0$. Next, we will prove that $F(x;\alpha)$ is strictly decreasing with respect to x.

By using the equation $4\pi i \partial\tau \vartheta_j(z;\tau) = \partial z \partial z \vartheta_j(z;\tau), \forall j = 2,3$, we obtain

$$\begin{aligned}\frac{\vartheta_2(0;x\alpha i)}{\vartheta_3(0;x\alpha i)} \frac{\partial F(x;\lambda)}{\partial x} &= -\frac{\partial}{\partial x} \log\left(\frac{\vartheta_3(0;x\alpha i)}{\vartheta_2(0;x\alpha i)}\right) \\ &= -\alpha i \left(\frac{\partial\tau\vartheta_3(0;x\alpha i)}{\vartheta_3(0;x\alpha i)} - \frac{\partial\tau\vartheta_2(0;x\alpha i)}{\vartheta_2(0;x\alpha i)}\right) \\ &= -\frac{\alpha}{4\pi}\left(\frac{\vartheta_3''(0;x\alpha i)}{\vartheta_3(0;x\alpha i)} - \frac{\vartheta_2''(0;x\alpha i)}{\vartheta_2(0;x\alpha i)}\right).\end{aligned} \tag{4.23}$$

According to the formulas (3.16) and (3.28), we have the estimations

$$\vartheta_3''(0;x\alpha i) - \vartheta_2''(0;x\alpha i) > 0 \text{ and } \vartheta_2(0;x\alpha i) - \vartheta_3(0;x\alpha i) < 0. \tag{4.24}$$

Note that $\vartheta_2''(0;x\alpha i) < 0$ from the formula (3.4). Therefore,

$$\begin{aligned}&\vartheta_3''(0;x\alpha i)\vartheta_2(0;x\alpha i) - \vartheta_2''(0;x\alpha i)\vartheta_3(0;x\alpha i) \\ =&(\vartheta_3''(0;x\alpha i) - \vartheta_2''(0;x\alpha i))\vartheta_2(0;x\alpha i) + \vartheta_2''(0;x\alpha i)(\vartheta_2(0;x\alpha i) - \vartheta_3(0;x\alpha i)) > 0.\end{aligned} \tag{4.25}$$

Based on (4.23) and (4.25), for any $\alpha > 0$, function $F(x;\alpha)$ is strictly decreasing with respect to $x \in (0,+\infty)$. Therefore, $F(x;\alpha)$ exists a unique zero point $x_1(\alpha) \in (1,3)$ which depends on $\alpha > 0$. Moreover, $F(x;\alpha) \geqslant 0, \forall x \in (0, x_1(\alpha)]$ and $F(x;\alpha) < 0, \forall x \in (x_1(\alpha), +\infty)$.

From the Lemma 4.5, the equation (4.19) has one solution $\bar{\mu} \in (\frac{1}{2}, 1)$ if and only if $F(\frac{1}{4b^2}; 16\pi t) < 0$, i.e., $b < \sqrt{\frac{1}{4x_1(\alpha)}}$ with $\alpha = 16\pi t$. Choose $b_1(t) = \sqrt{\frac{1}{4x_1(\alpha)}} \in (\frac{\sqrt{3}}{6}, \frac{1}{2})$ with $\alpha = 16\pi t$, then we obtain result (1).

On the other hand, to solve the equation (4.18), we similarly define a function

$$H(x;\alpha) = \frac{\vartheta_2'(0;x\alpha i)}{\vartheta_3'(0;x\alpha i)} - \frac{\vartheta_3(0;\alpha i)}{\vartheta_2(0;\alpha i)}, \text{ with } x, \alpha \in (0,+\infty). \tag{4.26}$$

Similar to the proof in case (1), for any $\alpha > 0$, the function $H(x;\alpha)$ is strictly increasing on $(0,+\infty)$ with $H(\frac{1}{3};\alpha) < 0$ and $H(1;\alpha) > 0$. Therefore, $H(x;\alpha)$

exists a unique zero point $x_2(\alpha) \in (\frac{1}{3}, 1)$ which depends on $\alpha > 0$. Moreover, $H(x;\alpha) < 0, \forall x \in (0, x_2(\alpha))$ and $H(x;\alpha) \geqslant 0, \forall x \in [x_2(\alpha), +\infty)$.

From the Lemma 4.5, the equation (4.18) has one solution $\bar{\nu} \in (0, \frac{1}{2})$ if and only if $H(\frac{1}{4b^2}; 16\pi t) < 0$, i.e., $b > \sqrt{\frac{1}{4x_2(\alpha)}}$ with $\alpha = 16\pi t$. Choose $b_2(t) = \sqrt{\frac{1}{4x_2(\alpha)}} \in (\frac{1}{2}, \frac{\sqrt{3}}{2})$ with $\alpha = 16\pi t$, then we obtain result (2).

□

At the end of this section, we prove Theorem 1.4 by using Theorem 4.6.

Proof of Theorem 1.4. From the discussion in Section 2, we only need to consider the torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ with $\tau = \frac{1}{2} + bi, b > 0$. For any $t > 0$, let $b_1(t)$ and $b_2(t)$ be defined as in Theorem 4.6.

Case 1: If $b \in [b_1(t), b_2(t)]$, by using the Theorem 4.6 and the discussion above Lemma 4.5, the equation (4.16) only has trivial solutions $(\bar{\nu}, \bar{\mu}) \in \{(0,0), (0,1), (\frac{1}{2}, \frac{1}{2})\}$. Thus, by using the periodicities and symmetries of theta functions, as in the proof of Theorem 1.2, the critical points of the heat kernel $p(z,t)$ on the torus $\mathbb{T}$ are

$$z \in \left\{0, \frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2} + \frac{1}{2}\tau\right\} + \mathbb{Z} \oplus \mathbb{Z}\tau. \tag{4.27}$$

Case 2: If $b \in (0, b_1(t))$, from Theorem 4.6, except for trivial solutions, i.e., half-period solutions, (4.16) has one non-trivial solution $(\bar{\nu}(b,t), \bar{\mu}(b,t)) \in \{0\} \times (\frac{1}{2}, 1)$. Therefore, also following the same steps as the proof of Theorem 1.2, the critical points of the heat kernel $p(z,t)$ on the torus $\mathbb{T}$ are

$$z \in \left\{0, \frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2} + \frac{1}{2}\tau, \frac{\bar{\mu}(b,t)}{2}, 1 - \frac{\bar{\mu}(b,t)}{2}\right\} + \mathbb{Z} \oplus \mathbb{Z}\tau. \tag{4.28}$$

Besides, the non-trivial critical points $\frac{\bar{\mu}(b,t)}{2}$ and $1 - \frac{\bar{\mu}(b,t)}{2}$ are located on the long diagonal $\Im(z) = 0$ of rhombic $\mathbb{T}$.

Case 3: If $b \in (b_2(t), +\infty)$, this case is similar to the case 2, and (4.16) has one non-trivial solution $(\bar{\nu}(b,t), \bar{\mu}(b,t)) \in (0, \frac{1}{2}) \times \{1\}$. The same as above, the critical points of $p(z,t)$ are

$$z \in \left\{0, \frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2} + \frac{1}{2}\tau, \frac{1 - \bar{\nu}(b,t)}{2} + \bar{\nu}(b,t)\tau, \frac{1 + \bar{\nu}(b,t)}{2} + (1 - \bar{\nu}(b,t))\tau\right\} + \mathbb{Z} \oplus \mathbb{Z}\tau. \tag{4.29}$$

Note that $\frac{1+\bar{\nu}(b,t)}{2} + (1 - \bar{\nu}(b,t))\tau$ and $\frac{1+\bar{\nu}(b,t)}{2} + (-\bar{\nu}(b,t))\tau$ are the same points on $\mathbb{T}$. Therefore, the non-trivial critical points $\frac{1-\bar{\nu}(b,t)}{2} + \bar{\nu}(b,t)\tau$ and $\frac{1+\bar{\nu}(b,t)}{2} + (-\bar{\nu}(b,t))\tau$ are located on the long diagonal $\Re(z) = \frac{1}{2}$ of rhombic $\mathbb{T}$.

Finally, we prove $b_1(t) \cdot b_2(t) = \frac{1}{4}, \forall t > 0$. In Section 2, we have shown that $p(z,t)$ has the same number of critical points under the rotation and the scaling of the torus $\mathbb{T}$. For any $b > 0$, note that the torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus (\frac{1}{2} + bi)\mathbb{Z})$ is equivalent to $\mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus (\frac{1}{2} + \frac{1}{4b}i)\mathbb{Z})$ up to rotation and scaling, so $p(z,t)$ has the same number of critical points on them. Therefore, $b_1(t) \cdot b_2(t) = \frac{1}{4}, \forall t > 0$ is obtained directly.

□

5. The proof of Theorem 1.6

Suppose the lattice is $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and the torus is $\mathbb{T} = \mathbb{C}/\Lambda$. To verify that $F_\lambda(z)$ is well-defined in (1.4) and to find the equation it satisfies, we need the following infinite estimation of $F_\lambda(z)$.

Lemma 5.1. *Suppose*

$$l := \min\{|\omega| : \omega \in \Lambda^\perp \setminus \{0\}\} > 0 \tag{5.1}$$

is the minimal length of lattice $\Lambda^\perp$. For any $z \in \mathbb{T}$, the heat kernel $p(z,t)$ has the following uniform estimation on infinity

$$p(z,t) = \frac{1}{|\mathbb{T}|} + O(e^{-4\pi^2 l^2 t}), \text{ as } t \to +\infty. \tag{5.2}$$

Proof. By using the Poisson Summation Formula (2.8)

$$\begin{aligned} p(z,t) &= \frac{1}{4\pi t}\sum_{\omega\in\Lambda} e^{-\frac{1}{4t}|\omega+z|^2} = \frac{1}{|\Lambda|}\sum_{\omega\in\Lambda^\perp} e^{-4\pi^2 t|\omega|^2} e^{2\pi i z\cdot\omega} \\ &= \frac{1}{|\mathbb{T}|} + \frac{1}{|\mathbb{T}|}\sum_{\omega\in\Lambda^\perp\setminus\{0\}} e^{-4\pi^2 t|\omega|^2} e^{2\pi i z\cdot\omega}. \end{aligned} \tag{5.3}$$

For any $\omega \in \Lambda^\perp$ and $|\omega| > l$, the function $|e^{4\pi^2 l^2 t}(e^{-4\pi^2 t|\omega|^2} e^{2\pi i z\cdot\omega})|$ exponentially decreases as $t \to +\infty$. Note that there are only finite points $\omega \in \Lambda^\perp$ such that $|\omega| = l$. Thus,

$$\begin{aligned} \lim_{t\to+\infty}(|\mathbb{T}|p(z,t) - 1)e^{4\pi^2 l^2 t} &\leqslant \limsup_{t\to+\infty} \sum_{\omega\in\Lambda^\perp\setminus\{0\}} |e^{4\pi^2 t(l^2-|\omega|^2)} e^{2\pi i z\cdot\omega}| \\ &= \lim_{t\to+\infty} \sum_{\omega\in\Lambda^\perp\setminus\{0\}} e^{4\pi^2 t(l^2-|\omega|^2)} \leqslant C. \end{aligned} \tag{5.4}$$

Here, this constant C is independent of z.

□

By utilizing the infinite estimation presented in Lemma 5.1, we verify that $F_\lambda(z)$ satisfies the equation (1.5).

Proposition 5.2. For any $\lambda \geqslant 0$, the function $F_\lambda(z)$ defined by (1.4) is well defined and is smooth in $\mathbb{T} \setminus \{0\}$. Besides, it is the unique solution of

$$-\triangle u(z) = -\lambda u(z) + \delta_0(z) - \frac{1}{|\mathbb{T}|} \tag{5.5}$$

in weak sense with

$$\int_{\mathbb{T}} u(z)dz = 0. \tag{5.6}$$

Proof. Because $p(z,t)$ is a doubly-periodic function with periods 1 and τ, $F_\lambda(z)$ defined by (1.4) has periods 1 and τ. From the estimation (5.2) and the property

$$\lim_{t\to 0^+} p(z,t) = \delta_0(z) \tag{5.7}$$

of heat kernel, $F_\lambda(z)$ is well defined and smooth in $\mathbb{T} \setminus \{0\}$ and has a singularity at zero.

By using the Lemma 5.1, there exists $M > 0$ and constant $C_1 > 0$ such that $\left|p(z,t) - \frac{1}{|\mathbb{T}|}\right| \leqslant C_1 e^{-4\pi^2 l^2 t}, \forall t > M, z \in \mathbb{T}$. Therefore,

(5.8)

$$\int_{\mathbb{T}}\int_0^\infty e^{-\lambda t}\left|p(z,t) - \frac{1}{|\mathbb{T}|}\right| dtdz = \int_{\mathbb{T}}\left(\int_0^M + \int_M^\infty\right) e^{-\lambda t}\left|p(z,t) - \frac{1}{|\mathbb{T}|}\right| dtdz$$
$$\leqslant \int_{\mathbb{T}}\int_0^M e^{-\lambda t}\left|p(z,t) - \frac{1}{|\mathbb{T}|}\right| dtdz + \int_{\mathbb{T}}\int_M^\infty e^{-\lambda t} C_1 e^{-4\pi^2 l^2 t} dtdz$$
$$\leqslant \int_{\mathbb{T}}\int_0^M e^{-\lambda t} p(z,t) dtdz + C < \infty.$$

Where, the last inequality comes from $\int_{\mathbb{T}} p(z,t)dz = 1$. Therefore, $F_\lambda(z)$ is absolutely integrable on $\mathbb{T}$. By using the Fubini's Theorem and $\int_{\mathbb{T}} p(z,t)dz = 1$ again,

$$\begin{aligned}\int_{\mathbb{T}} F_\lambda(z)dz &= \int_{\mathbb{T}}\int_0^\infty e^{-\lambda t}\left(p(z,t) - \frac{1}{|\mathbb{T}|}\right) dtdz \\ &= \int_0^\infty \int_{\mathbb{T}} e^{-\lambda t}\left(p(z,t) - \frac{1}{|\mathbb{T}|}\right) dtdz = 0.\end{aligned} \tag{5.9}$$

For any $g(z) \in C^\infty(\mathbb{T})$, by using the Fubini's Theorem,

(5.10)

$$\begin{aligned}\int_{\mathbb{T}} -\triangle F_\lambda(z) g(z)dz &= \int_{\mathbb{T}} -F_\lambda(z)\triangle g(z)dz \\
&= -\int_{\mathbb{T}}\int_0^\infty e^{-\lambda t}\left(p(z,t) - \frac{1}{|\mathbb{T}|}\right)\triangle g(z)dtdz \\
&= -\int_0^\infty\int_{\mathbb{T}} e^{-\lambda t}\left(p(z,t) - \frac{1}{|\mathbb{T}|}\right)\triangle g(z)dzdt \\
&= -\int_0^\infty\int_{\mathbb{T}} e^{-\lambda t}\triangle p(z,t) g(z)dzdt \\
&= -\int_0^\infty\int_{\mathbb{T}} e^{-\lambda t}\partial_t p(z,t) g(z)dzdt \\
&= -\int_0^\infty\int_{\mathbb{T}} \partial_t\left(\left(p(z,t) - \frac{1}{|\mathbb{T}|}\right) e^{-\lambda t} g(z)\right) \\
&\qquad + \lambda e^{-\lambda t}\left(p(z,t) - \frac{1}{|\mathbb{T}|}\right) g(z)dzdt \\
&= -\int_{\mathbb{T}}\left(\left(p(z,t) - \frac{1}{|\mathbb{T}|}\right) e^{-\lambda t} g(z)\right)\Bigg|_{t=0}^\infty - \int_{\mathbb{T}} \lambda F_\lambda(z) g(z)dz \\
&= g(0) - \int_{\mathbb{T}}\left(\lambda F_\lambda(z) + \frac{1}{|\mathbb{T}|}\right) g(z)dz.\end{aligned}$$

Therefore, the function $F_\lambda(z)$ satisfies the equation (5.5) in weak sense.

At last, we prove the uniqueness of $F_\lambda(z)$. Consider the characteristic equation on torus $\mathbb{T} = \mathbb{C}/\Lambda$

$$-\triangle u(z) = \lambda u(z),\ z \in \mathbb{T}. \tag{5.11}$$

The eigenvalues λ_ω and eigenfunctions $e_\omega(z)$ with respect to (5.11) are given by

$$\lambda_\omega = 4\pi^2|\omega|^2,\ e_\omega(z) = \frac{1}{\sqrt{|\Lambda|}} e^{2\pi i z\cdot\omega},\ \forall \omega \in \Lambda^\perp. \tag{5.12}$$

Moreover, from the formula (2.10), we have a new expression of the heat kernel by eigenfunctions

$$p(z,t) = \sum_{\omega\in\Lambda^\perp} e^{-\lambda_\omega t} e_\omega(z) e_\omega(0). \tag{5.13}$$

For an introduction to the relationship between the heat kernel and the eigenfunctions we refer to the textbook of Grigor'yan [10].

Suppose $g(z)$ is the other function satisfies (5.5) and $\int_{\mathbb{T}} g(z)dz = 0$. We obtain that

$$-\triangle(F_\lambda(z) - g(z)) = -\lambda(F_\lambda(z) - g(z)). \tag{5.14}$$

That is, $F_\lambda(z) - g(z)$ is the eigenfunction with respect to eigenvalue $-\lambda$.

If $\lambda > 0$, then the characteristic equation (5.11) has a negative eigenvalue $-\lambda$, which is a contradiction with (5.12).

If $\lambda = 0$, from the expression (5.12), the eigenfunction with regard to eigenvalue 0 has only $\frac{1}{\sqrt{|\Lambda|}}$. So, eigenfunction $F_\lambda(z) - g(z)$ of (5.11) must be a constant. Note that,

$$\int_{\mathbb{T}} F_\lambda(z) - g(z)dz = 0.$$

Therefore, $F_\lambda(z) - g(z) = 0$. To sum up, the solution $F_\lambda(z)$ is unique. □

Remark 5.3. In $\mathbb{R}^2$, the relationship between the heat kernel $p_{\mathbb{R}^2}(z,t)$ and the Green function $G_{\mathbb{R}^2}(z)$ is

$$G_{\mathbb{R}^2}(z) = \int_0^\infty p_{\mathbb{R}^2}(z,t)dt. \tag{5.15}$$

This is a bit distinct from (1.4), as the coefficient $\frac{1}{|\mathbb{T}|}$ is absent in (5.15). If we regard $\mathbb{R}^2$ as a torus with "infinite volume", then an analogous expression arises for formulas (1.4) and (5.15).

Next, we discuss the critical points of $F_\lambda(z)$ on rectangular, hexagonal and rhombic tori.

Proof of Theorem 1.6. Similar to the discussion of (2.6) regarding $p(z,t)$ in Section 2, to find the critical points of $F_\lambda(z)$, we only need to consider the torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$ with $\tau\in\mathbb{H}$.

Step 1, suppose the rectangular torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$ with $\tau = bi, b > 0$. From the formulas (1.4) and (4.1)

$$F_\lambda(z) = \int_0^\infty e^{-\lambda t}\left(\vartheta_3(\mu; Ti)\vartheta_3(\nu; b^2Ti) - \frac{1}{|\mathbb{T}|}\right)dt, \tag{5.16}$$

where $z = \nu + \mu\tau$ and $T = \frac{4\pi t}{b^2}$. The critical points $\nabla_z F_\lambda = 0$ if and only if $\partial_v F_\lambda = \partial_\mu F_\lambda = 0$. Therefore, from Lemma 3.2(3) and zero is a singularity, F_λ has only three trivial critical points $\frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2}+\frac{1}{2}\tau$ on $\mathbb{T}$.

In addition, because $p(z,t)$ has the uniform minimal point $\frac{1}{2}+\frac{1}{2}\tau$ for all $t>0$, $\frac{1}{2}+\frac{1}{2}\tau$ is also the minimal point of $F_\lambda(z)$ for any $\lambda\geqslant 0$. Besides, from Lemma 3.2, the critical points $\frac{1}{2},\frac{1}{2}\tau$ are both saddle points.

Step 2, suppose the hexagonal torus $\mathbb{T}=\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$ with $\tau=\frac{1}{2}+\frac{\sqrt{3}}{2}i$. From Theorem 1.2, the heat kernel $p(z,t)$ exhibits uniform non-trivial critical points at $\frac{1}{3}+\frac{1}{3}\tau$ and $\frac{2}{3}+\frac{2}{3}\tau$, which are the minimal points with the same values on $\mathbb{T}$. Consequently, for any $\lambda\geqslant 0$, there are also the minimal points of $F_\lambda(z)$ with the same values.

Unfortunately, we are unable to prove the nonexistence of other non-trivial critical points here. This limitation arises due to monotonic gaps in our method, as indicated in Conjecture 1.7. □

At the end of this Section, we present a partial proof of Conjecture 1.7 which is the same as Theorem 1.4 completely except for two monotonic gaps. Thus,we outline the proof framework and omit redundant details.

Partial proof of Conjecture 1.7. Same as before, we also only need to consider the rhombic torus $\mathbb{T}=\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$ with $\tau=\frac{1}{2}+bi, b>0$. Based on the formulas (1.4) and (4.14), let $z=\nu+\mu\tau$, we have

$$F_\lambda(z)=\int_0^\infty e^{-\lambda t}\left(\vartheta_3\left(\mu;\frac{4\pi t}{b^2}i\right)\vartheta_3(2\nu+\mu;16\pi ti)+\vartheta_2\left(\mu;\frac{4\pi t}{b^2}i\right)\vartheta_2(2\nu+\mu;16\pi ti)-\frac{1}{|\mathbb{T}|}\right)dt. \tag{5.17}$$

Let $\bar\nu=\mu$ and $\bar\mu=2\nu+\mu$. To find the critical points of $F_\lambda(z)$, we just need to solve $(\bar\nu,\bar\mu)$ satisfy

$$\begin{cases}\int_0^\infty e^{-\lambda t}\left(\vartheta_3'\left(\bar\nu;\frac{4\pi t}{b^2}i\right)\vartheta_3(\bar\mu;16\pi ti)+\vartheta_2'\left(\bar\nu;\frac{4\pi t}{b^2}i\right)\vartheta_2(\bar\mu;16\pi ti)\right)dt=0,(*)\\ \int_0^\infty e^{-\lambda t}\left(\vartheta_3\left(\bar\nu;\frac{4\pi t}{b^2}i\right)\vartheta_3'(\bar\mu;16\pi ti)+\vartheta_2\left(\bar\nu;\frac{4\pi t}{b^2}i\right)\vartheta_2'(\bar\mu;16\pi ti)\right)dt=0.(**)\end{cases} \tag{5.18}$$

Similar to the discussion in Section 4.3, the non-trivial solution $(\bar\nu,\bar\mu)$ can only exist within $(\bar\nu,\bar\mu)\in(0,\frac{1}{2})\times\{1\}$ and $(\bar\nu,\bar\mu)\in\{0\}\times(\frac{1}{2},1)$. Therefore, we focus on solving the following two equations which are similar to (4.18) and (4.19).

$$\int_0^\infty e^{-\lambda t}\left(\vartheta_3'\left(\bar\nu;\frac{4\pi t}{b^2}i\right)\vartheta_3(0;16\pi ti)-\vartheta_2'\left(\bar\nu;\frac{4\pi t}{b^2}i\right)\vartheta_2(0;16\pi ti)\right)dt=0 \tag{5.19}$$

on $\bar\nu\in(0,\frac{1}{2})$. And

$$\int_0^\infty e^{-\lambda t}\left(\vartheta_3\left(0;\frac{4\pi t}{b^2}i\right)\vartheta_3'(1-\bar\mu;16\pi ti)-\vartheta_2\left(0;\frac{4\pi t}{b^2}i\right)\vartheta_2'(1-\bar\mu;16\pi ti)\right)dt=0 \tag{5.20}$$

on $\bar\mu\in(\frac{1}{2},1)$.

We begin by considering equation (5.19). To repeat the methods shown in Section 4.3, we rely on the following monotonic property:

$$\frac{d}{d\bar\nu}\left(\frac{\int_0^\infty e^{-\lambda t}\left(\vartheta_2'\left(\bar\nu;\frac{4\pi t}{b^2}i\right)\vartheta_2(0;16\pi ti)\right)dt}{\int_0^\infty e^{-\lambda t}\left(\vartheta_3'\left(\bar\nu;\frac{4\pi t}{b^2}i\right)\vartheta_3(0;16\pi ti)\right)dt}\right)>0, \tag{5.21}$$

for all $\bar{\nu} \in (0, \frac{1}{2}), b > 0$ and $\lambda \geqslant 0$. We believe that (5.21) is consistent with (3.22), while it might not be verified directly from (3.22). Therefore, we leave it as a gap.

If formula (5.21) is correct, we can replicate the procedure in Lemma 4.5 and obtain that equation (5.19) has at most one solution. Moreover, there exists unique solution $\bar{\nu} \in (0, \frac{1}{2})$ if and only if

$$\frac{\int_0^\infty e^{-\lambda t} \left(\vartheta_2' \left(0; \frac{4\pi t}{b^2} i\right) \vartheta_2(0; 16\pi t i)\right) dt}{\int_0^\infty e^{-\lambda t} \left(\vartheta_3' \left(0; \frac{4\pi t}{b^2} i\right) \vartheta_3(0; 16\pi t i)\right) dt} < 1. \tag{5.22}$$

Now we require another monotonic property which is similar to (4.26). For any $x \in (0, +\infty)$ and $\lambda \geqslant 0$,

$$\frac{\partial}{\partial x} \tilde{H}(x; \lambda) > 0, \text{ where } \tilde{H}(x; \lambda) = \frac{\int_0^\infty e^{-\lambda t} \left(\vartheta_2' \left(0; x 16\pi t i\right) \vartheta_2(0; 16\pi t i)\right) dt}{\int_0^\infty e^{-\lambda t} \left(\vartheta_3' \left(0; x 16\pi t i\right) \vartheta_3(0; 16\pi t i)\right) dt} - 1. \tag{5.23}$$

We are also unable to verify (5.23) directly by using (4.26), so we leave it as the second gap. If (5.23) is confirmed, following the process in Theorem 4.6, equation (5.19) possesses a solution $\bar{\nu} \in (0, \frac{1}{2})$ if and only if $b > B_2(\lambda) = \sqrt{\frac{1}{4\tilde{x}_2(\lambda)}}$. Where $\tilde{x}_2(\lambda) \in (\frac{1}{3}, 1)$ is the unique zero point of $\tilde{H}(x; \lambda)$.

The solution to equation (5.20) follows a similar procedure, yielding $B_1(\lambda) \in (\frac{\sqrt{3}}{6}, \frac{1}{2})$ such that (5.20) possesses a unique solution $\bar{\mu} \in (\frac{1}{2}, 1)$ if and only if $b \in (0, B_1(\lambda))$. At last, the remaining proof is the same as Theorem 1.4's proof in Section 4.3. We omit these repetitive proof details. □

Formulas (5.21) and (5.23) are so similar to (3.22) and (4.26) that we believe that they are correct. We sincerely hope that someone can resolve these two gaps.

6. The proof of Theorem 1.11

For any $\tau \in \mathbb{H}$, suppose lattice $\Lambda_\tau = \sqrt{\frac{1}{\Im(\tau)}} (\mathbb{Z} \oplus \mathbb{Z}\tau)$. Compare the definition (1.3) with (1.7), we have

$$\theta(z; \tau) = p\left(z, \frac{1}{4\pi}\right). \tag{6.1}$$

The formula (6.1) is also expressed in [2]. In this viewpoint, the energy $\theta(z; \tau)$ and the heat kernel $p(z, t)$ have similar properties.

Proof of Theorem 1.11. By employing formula (6.1) along with Theorem 1.1 and Theorem 1.2, we establish directly that $\theta(z; \tau)$ possesses four critical points on the rectangular torus, and the hexagonal one exhibits six critical points.

Next, we proceed to prove the invariance of the number of critical points. Suppose $z = \nu\omega_1 + \mu\omega_2$ with $\nu, \mu \in [0, 1]$, we consider the general $\theta(z; \Lambda)$

$$\theta(z; \Lambda) = \sum_{\omega \in \Lambda} e^{-\pi|\omega + z|^2} = \sum_{n, m \in \mathbb{Z}} e^{-\pi|(m+\nu)\omega_1 + (n+\mu)\omega_2|^2}, \tag{6.2}$$

with $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. Similar to the discussion of (2.6) in Section 2, the number of critical points of $\theta(z; \Lambda)$ is invariant up to the rotation and scaling of (ω_1, ω_2).

It is easy to check that the two-dimensional lattice $\Lambda_\tau = \sqrt{\frac{1}{\Im(\tau)}}(\mathbb{Z} \oplus \mathbb{Z}\tau)$ has some important invariance:

$$\Lambda_\tau = \Lambda_{\tau+1} = \Lambda_{-\frac{1}{\tau}} = \Lambda_{-\bar{\tau}},$$

up to the rotation and reflection, as referenced in [18, 20, 21, 23, 28]. Therefore, under the modular transform

$$\Gamma \in SL_2(\mathbb{Z}) = \text{span}\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},$$

as $\tau \to \Gamma(\tau)$, lattice $\Lambda_\tau = \sqrt{\frac{1}{\Im(\tau)}}(\mathbb{Z} \oplus \mathbb{Z}\tau)$ is invariant up to the rotation and reflection. Thus, the number of the critical points of $\theta(z;\tau)$ is invariant under the transform $\tau \to \Gamma(\tau), \forall \Gamma \in SL_2(\mathbb{Z})$.

□

For the further reference of modular form and modular properties we refer to the textbook of Serre [27] (Chap. VII, Modular Forms).

7. Generalizations and open problems

In Theorem 1.1 through Theorem 1.4, we observe that the heat kernel $p(z,t)$ possesses at most a pair of non-trivial critical points on rectangular, hexagonal, and rhombic tori. An intriguing question arises: does the heat kernel $p(z,t)$ exhibit this property for all general tori $\mathbb{T}$? Additionally, we note that the number of critical points between the Green function [15] and the heat kernel is equivalent on rectangular and hexagonal tori. This observation leads us to expect that the number of critical points between them remains the same on other types of tori as well.

Open problem 7.1. Suppose the general torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$, $\omega_1, \omega_2 \in \mathbb{C}$ and let the heat kernel $p(z,t)$ be defined in (1.3).

(1) whether $p(z,t)$ has at most one pair of non-trivial critical points for any $t > 0$ on two-dimensional $\mathbb{T}$? And under what conditions, the number of critical points of $p(z,t)$ is independent on $t > 0$.
(2) Suppose that the pair of non-trivial critical points of $p(z,t)$ exist, whether they are the minimal points with the same values?
(3) For which tori $\mathbb{T}$ is the number of critical points between the heat kernel and the Green function the same?

In Sections 1.2 and 1.3, we delve into the critical points of $F_\lambda(z), \lambda \geqslant 0$ and $\theta(z,\alpha;\Lambda), \alpha > 0$. It is noteworthy that their critical points bear resemblance to those of the heat kernel. Thus, we are inclined to believe that there exists a fundamental conclusion to elucidate this properties for extensive functions.

Open problem 7.2. Suppose the torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$ with $\omega_1, \omega_2 \in \mathbb{C}$, and f is an even function defined on $\mathbb{T}$. Under what conditions does f have no or have one pair of non-trivial critical points? Additionally, what type of function f has more critical points?

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