CONVERGENCE OF THE VOLUME PRESERVING FRACTIONAL MEAN CURVATURE FLOW FOR CONVEX SETS

VESA JULIN AND DOMENICO ANGELO LA MANNA

Abstract. We prove that the volume preserving fractional mean curvature flow starting from a convex set does not develop singularities along the flow. By the recent result of Cesaroni-Novaga [6] this then implies that the flow converges to a ball exponentially fast. In the proof we show that the a priori estimates due to Cinti-Sinestrari-Valdinoci [10] imply the $C^{1+\alpha}$-regularity of the flow and then provide a regularity argument which improves this to $C^{2+\alpha}$-regularity of the flow. The regularity step from $C^{1+\alpha}$ into $C^{2+\alpha}$ does not rely on convexity and can be adopted to more general setting.

1. Introduction

We say that a given set $E_0 \subset \mathbb{R}^{n+1}$ evolves under the volume preserving fractional mean curvature flow if there exists a flow $(E_t)_{t\in[0,T)}$ of sets starting from E_0, which satisfies the equation

$$V_t = -(H^s_{E_t} - \bar{H}^s_{E_t}) \qquad \text{on } \partial E_t, \tag{1.1}$$

where V_t is the normal velocity, H^s_E with $s \in (0,1)$ is the fractional mean curvature defined as

$$H^s_E(x) = \int_{\mathbb{R}^{n+1}\setminus E} \frac{dy}{|y-x|^{n+1+s}} - \int_E \frac{dy}{|y-x|^{n+1+s}} \tag{1.2}$$

and $\bar{H}^s_{E_t}$ is its integral average. The flow (1.1) bears many similarities to the classical volume preserving mean curvature flow, in particular, it can be seen as a gradient flow of the fractional perimeter. The flow (1.1) can also be seen as a perturbation of the fractional mean curvature flow, with the difference that (1.1) preserves the volume and can thus be interpreted as the evolutionary counterpart of the fractional isoperimetric inequality.

The study of the fractional perimeter problem goes back to [3], where the authors prove so called ε-regularity result, i.e., partial regularity of the perimeter minimizers. As with minimal surfaces, also in the fractional case the perimeter minimizers may have singularities in higher dimensions. The question of optimal regularity has been studied extensively and we merely refer to [25], which is the best known result on the size of the singular set at the moment. We also refer to [1], where the authors provide a bootstrap argument to obtain higher order regularity outside the singular set. The related isoperimetric problem, which states that the ball minimizes the fractional perimeter under the volume constraint, follows from standard symmetrization argument and the sharp quantitative version is proven in [15] (see also [16]). Moreover, the fractional version of the Alexandrov theorem, which states that the only regular set with constant fractional mean curvature is the ball, is proven independently in [2, 11]. This is important as it essentially implies that, if the flow (1.1) is well defined for all times, then it converges to the ball.

In [19] we prove the short time existence of the classical solution of (1.1) for $C^{1,1}$-regular initial sets. This result also applies to the fractional mean curvature flow, in which case the

existence of the level set solution is proven in [18] and another weak solution is constructed in [7]. As usual with geometric flows, the fractional mean curvature flow may develop singularities in finite time such as neckpinching [9] and fattening [4]. However, at the moment there is no systematic classification of the possible types of singularities in the fractional setting similar to the classical mean curvature flow [21]. We are also not aware of a construction of a weak solution for (1.1).

A natural problem is whether we may avoid the formation of singularities, if we constraint ourselves to special type of initial sets. In [6] the authors prove, that if the initial set is close enough to the ball, then the flow (1.1) remains smooth and converges exponentially fast to a translation of the ball. Another special case is when the initial set is a subgraph of a Lipschitz function. Note that then the set is unbounded and therefore, instead of (1.1), it is natural to study the standard fractional mean curvature flow. In this setting the authors [24] prove important a priori estimates, which essentially imply that if the flow starts from Lipschitz graph with bounded fractional curvature, these quantities are preserved along the flow. This problem is studied further in [5], where the authors show that in the above setting the fractional mean curvature flow does not develop singularities and converges to the flat surface exponentially fast. This is the fractional counterpart of the Ecker-Huisken result [13] for the mean curvature flow with the only difference that the initial set is assumed to have bounded fractional curvature. Due to the results [22, 26] one might expect the result to hold without the curvature assumption, but to the best of our knowledge this is not known at the moment.

As we already mentioned, here we consider the case of convex initial sets, which are bounded and $C^{1,1}$-regular. In the case of the classical volume preserving mean curvature flow the result by Huisken [17] implies that the flow remains smooth and converges expontially fast to the ball. In the fractional setting, this result is proven in [6] and [8] under the additional assumption that the flow remains $C^{2+\alpha}$-regular. Here we show that this additional regularity assumption is conditional, or in other words, this regularity assumption can proven for convex initial sets, and thus we obtain the Huisken result in the fractional setting.

Main Theorem. *Let $E_0 \subset \mathbb{R}^{n+1}$ be $C^{1,1}$-regular convex set with $|E_0| = |B_1|$. Then the classical solution of (1.1) exists for all times $(E_t)_{t\geq 0}$ and converges to the ball $B_1(x_0)$, for some $x_0 \subset \mathbb{R}^{n+1}$, in C^k for every $k \in \mathbb{N}$ exponentially fast. To be more precise, there is $T \geq 1$ depending on the initial set E_0 and on the dimension, such that we may write the moving boundaries by the height functions over the unit sphere $h(\cdot,t) : \mathbb{S}^n \to \mathbb{R}$ as*

$$\partial E_t = \{h(x,t)x + x_0 : x \in \mathbb{S}^n\}$$

for $x_0 \in \mathbb{R}^{n+1}$ when $t \geq T$ and it holds $\|h(\cdot,t) - 1\|_{C^k(\mathbb{S}^n)} \leq C_k e^{-c_k t}$, where the constants $c_k > 0$ and $C_k \geq 1$ depend on n, k, T and on the initial set E_0.

Let us comment the proof of the Main Theorem. First, we remark that the $C^{1,1}$ assumption on E_0 is needed only to guarantee the existence of the flow, as our proof relies on the short time existence result in [19]. As we mentioned above, the proof relies on the a priori estimates proven in [6, 8, 10]. Indeed, by the result in [8] the sets E_t along the flow remain convex, while the estimates in [10] imply that their fractional mean curvature remains uniformly bounded, i.e.

$$\sup_{t\in[0,T)} H^s_{E_t}(x) \leq C$$

and there are radii $0 < r < 1 < R$ such that $B_r(x_t) \subset E_t \subset B_R(x_t)$ as long as the flow is classically defined. These important estimates are our starting point and we first prove in Proposition 2.3 that for convex sets the above fractional curvature bound implies uniform C^{1+s}-Hölder regularity. We then recall that the authors in [6] (see also [10]) show that, if we would be able to improve the C^{1+s}-regularity into uniform $C^{2+\alpha}$-regularity, then the flow (1.1) does not develop singularities and converges exponentially fast to a ball. Our main contribution here is to prove this regularity step. Roughly speaking, the idea is that geometric flows do not develop singularities under uniform $C^{1+\alpha}$-regularity estimate, unless the singularity is due to change of topology. This is due to the fact that one may parametrize the equation such that it becomes a quasilinear PDE. We use this idea in [20] in a more complicated equation, and since we know that in our case the flow (1.1) remains convex, and thus the topology does not change, we may adopt it in this setting.

The first technical issue in the proof is to find a good parametrization of the flow. The idea is to differentiate the equation (1.1) with respect to space and parametrize the equation

$$\nabla_{X(t)} V_t = -\nabla_{X(t)} H_{E_t^s} \qquad \text{on } \partial E_t,$$

where $X(t)$ is a vector field on the moving boundary ∂E_t. By doing this we obtain a quasilinear PDE for the derivative of the height function, where all the error terms are lower order. We point out that we could also first parametrize the original equation (1.1) by the height function and differentiate that, but this leads to much more complicated calculations. We then improve some of the methods developed in [19] and prove the $C^{2+\alpha}$ regularity of the flow in Section 5. The main difference to [19] is that here we need sharp estimates for the error terms in order to show that they do not blow up in finite time. Another issue is that here our equation is not a small parturbation of the fractional heat equation with respect to the sphere, and we need Schauder estimates for general linear parabolic equations defined on the unit sphere (see Section 4). Similar Schauder estimates are proven in [22] in the case of evolving periodic graphs.

The paper is organized as follows. In Section 2 we recall some results in the existing literature and prove the C^{1+s}-estimate of the flow. Section 3 is mostly devoted to the calculations of the parametrization of the equation. In Section 4 we generalize the Schauder estimates for general linear parabolic equations in the Euclidean setting proven in [23] (see Theorem 2.2) to the case of the sphere (see Theorem 4.1). Since the result in the Euclidian setting is scattered in literature, we decide to give a self-contained proof in the Appendix. Finally in Section 5 we prove that C^{1+s}-regularity of the flow implies uniform $C^{2+s+\alpha}$-regularity, which concludes the proof of the Main Theorem.

2. Notation and preliminary results

Throughout the paper $C \geq 1$ and $c > 0$ stand for generic constants which may change from line to line. If needed, we specify their dependence on relevant parameters. We denote the open ball with radius r centered at x by $B_r(x)$ and by B_r if it is centered at the origin. The notation does not specify the dimension of the ball, but in order to avoid possible confusion, we write $B_r(x) \subset \mathbb{R}^k$ to point out that the ball is k-dimensional. We denote the inner product of $x, y \in \mathbb{R}^{n+1}$ by $\langle x, y \rangle$.

For a given smooth and bounded set $E \subset \mathbb{R}^{n+1}$ the fractional mean curvature of order $s \in (0,1)$ at $x \in \partial E$ is defined as (1.2) in the principal valued sense. In order to define the normal velocity we say that a family of smooth sets $(E_t)_{t\in[0,T)}$ is a smooth flow starting from

E_0, if there is a family of diffeomorphisms $\Phi_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $t \mapsto \Phi_t(x)$ is smooth, $\Phi_t(E_0) = E_t$ and $\Phi_0(x) = x$. The normal velocity at $y = \Phi_t(x) \in \partial E_t$ is then defined as $V_t(y) = \langle \frac{\partial}{\partial t}\Phi_t(x), \nu_{E_t}(y)\rangle$. Finally we say that a smooth flow $(E_t)_{t\in[0,T)}$ starting from E_0 is a solution to (1.1), if it satisfies the associated equation point-wise.

We recall that if the initial set $E_0 \subset \mathbb{R}^{n+1}$ is bounded and $C^{1,1}$-regular, i.e., its boundary is $C^{1,1}$-regular hypersurface, then by [19] the equation (1.1) has a solution for a short time $(E_t)_{t\in[0,T)}$ and the sets E_t for $t > 0$ are smooth (C^∞ -regular hypersurfaces). In this paper we restrict to convex initial sets E_0, which are $C^{1,1}$-regular. Then by the results in [8, 10] we know that the sets E_t remain convex, there are $r > 0, R > 1$ such that $B_r(x_t) \subset E_t \subset B_R(x_t)$ for some points $x_t \in \mathbb{R}^{n+1}$ and it holds

$$\sup_{t\in(0,T)} \sup_{x\in\partial E_t} H^s_{E_t}(x) \le C,$$

where the constant is independent of the time of existence T. We remark that by these results, we may parametrize the flow via the height function over the unit sphere $\mathbb{S}^n$, which means that for every $t \in [0,T)$ there is a function $h(\cdot,t) : \mathbb{S}^n \to \mathbb{R}$ such that

$$\partial E_t = \{h(x,t)x + x_t : x \in \mathbb{S}^n\} \tag{2.1}$$

for some $x_t \in \mathbb{R}^{n+1}$. Moreover there are $c > 0$ and $C \ge 1$ such that $c \le h(x,t) \le C$ for every $x \in \mathbb{S}^n$. We show later in Proposition 2.3, that the above estimate on the fractional mean curvature implies that the functions $h(\cdot,t)$ are uniformly C^{1+s}-regular.

Recall that our aim is to improve the C^{1+s}-regularity into $C^{2+\alpha}$-regularity. In other words, we show that the height functions $h(\cdot,t)$ defined in (2.1) are in fact, uniformly $C^{2+s+\alpha}$ regular for small $\alpha > 0$.

2.1. **Hölder norms and Schauder estimate.** The notation $\nabla^k F$ stands for the k:th order differential of a vector field $F : \mathbb{R}^n \to \mathbb{R}^m$. For matrix $A \in \mathbb{R}^{m\times k}$ we denote by $|A|$ its Frobenius norm. We define the usual Hölder norms of $F : \mathbb{R}^n \to \mathbb{R}$ by

$$\|F\|_{C^0(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} |F(x)| \quad \text{and} \quad \|F\|_{C^\beta(\mathbb{R}^n)} := \sup_{x\neq y\in\mathbb{R}^n} \frac{|F(y)-F(x)|}{|y-x|^\beta} + \|F\|_{C^0(\mathbb{R}^n)}$$

for $\beta \in (0,1]$. We extend this to C^l-norm, for $l = k+\beta$, with $k \in \mathbb{N}$ and $\beta \in [0,1)$ as

$$\|F\|_{C^l(\mathbb{R}^n)} := \sum_{j=0}^{k} \|\nabla^j F\|_{C^0(\mathbb{R}^n)} + \|\nabla^k F\|_{C^\beta(\mathbb{R}^n)}.$$

We then have the standard interpolation inequality [27, Section 2.7] i.e., assume l_1 and l_2 are positive numbers, $\theta \in (0,1)$ and denote

$$l = \theta l_1 + (1-\theta) l_2.$$

Then there is a constant $C \ge 1$ such that for every smooth and bounded function $u : \mathbb{R}^n \to \mathbb{R}$ it holds

$$\|u\|_{C^l(\mathbb{R}^n)} \le C\|u\|^\theta_{C^{l_1}(\mathbb{R}^n)} \|u\|^{1-\theta}_{C^{l_2}(\mathbb{R}^n)}. \tag{2.2}$$

We will mostly deal with functions defined on the unit sphere $\mathbb{S}^n$, which we interpret as the boundary of the unit ball $\partial B_1 \subset \mathbb{R}^{n+1}$. Since $\mathbb{S}^n$ is embedded in $\mathbb{R}^{n+1}$, we may extend every continuously differentiable vector field $F : \mathbb{S}^n \to \mathbb{R}^k$ to $\mathbb{R}^{n+1}$ and denote the extension still by F. We define the tangential differential of F at $x \in \mathbb{S}^n$ as

$$\nabla_\tau F(x) := \nabla F(x) - (\nabla F(x)x) \otimes x$$

and note that it does not depend on the chosen extension. For real valued functions $u : \mathbb{S}^n \to \mathbb{R}$ we define the tangential gradient $\nabla_\tau u$ as the transpose of the tangential differential, and define the tangential Hessian of $u : \mathbb{S}^n \to \mathbb{R}$ as $\nabla_\tau^2 u(x) = \nabla_\tau(\nabla_\tau u)(x)$. If e_i is a basis vector for $\mathbb{R}^{n+1}$ we denote

$$\nabla_i u = \langle \nabla_\tau u, e_i \rangle$$

for the tangential partial derivative and $\nabla_j \nabla_i u = \langle \nabla_\tau^2 u\, e_j, e_i \rangle$. We may then define the third order tangential derivatives as $\nabla_\tau \nabla_j \nabla_i u$ and iteratively for every $k = 3, 4, \dots$. We note that such a definition of $\nabla_\tau^k u$ does not agree with the standard definition of a covariant derivative.

We define the Hölder norms on $\mathbb{S}^n$ as in the Euclidian setting. That is for $u : \mathbb{S}^n \to \mathbb{R}$ we define

$$\|u\|_{C^0(\mathbb{S}^n)} := \sup_{x \in \mathbb{S}^n} |u(x)| \quad \text{and} \quad \|u\|_{C^\beta(\mathbb{S}^n)} := \sup_{x \neq y \in \mathbb{S}^n} \frac{|u(y) - u(x)|}{|y - x|^\beta} + \|u\|_{C^0(\mathbb{S}^n)}$$

for $\beta \in (0, 1]$ and for C^l-norm, with $l = k + \beta$, $k \in \mathbb{N}$ and $\beta \in [0, 1)$, as

$$\|u\|_{C^l(\mathbb{S}^n)} := \sum_{j=0}^{k} \|\nabla_\tau^j u\|_{C^0(\mathbb{S}^n)} + \|\nabla_\tau^k u\|_{C^\beta(\mathbb{S}^n)}.$$

The interpolation inequality (2.2) extends immediately to $\mathbb{S}^n$.

Proposition 2.1. *Assume l_1 and l_2 are positive numbers, $\theta \in (0, 1)$ and denote*

$$l = \theta l_1 + (1 - \theta) l_2.$$

Then there is a constant $C \geq 1$ such that for every smooth function $u : \mathbb{S}^n \to \mathbb{R}$ it holds

$$\|u\|_{C^l(\mathbb{S}^n)} \leq C \|u\|^\theta_{C^{l_1}(\mathbb{S}^n)} \|u\|^{1-\theta}_{C^{l_2}(\mathbb{S}^n)}.$$

Let us return to functions defined in $\mathbb{R}^n$. We recall the definition of the fractional Laplacian of order $\gamma \in (0, 2)$

$$\Delta^\gamma u(x) = \int_{\mathbb{R}^n} \frac{u(y + x) - u(x)}{|y|^{n+\gamma}}\, dy,$$

which has to be interpreted in the principal valued sense. We need parabolic Schauder estimates for equations with general linear fractional-type operator of order $\gamma = 1 + s$. To this aim we say that $A(\cdot) : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is *symmetric and elliptic* matrix field, if there are constants $\lambda > 0$ and $\Lambda \geq 1$ such that for every $x \in \mathbb{R}^n$ the matrix $A(x)$ is symmetric and satisfies

$$\lambda |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda |\xi|^2 \qquad \text{for all } x, \xi \in \mathbb{R}^n. \tag{2.3}$$

For a symmetric and elliptic matrix field $A(\cdot)$ we define linear operator as

$$L_A[u](x) := \int_{\mathbb{R}^n} \frac{u(y + x) - u(x)}{\langle A(x)y, y \rangle^{\frac{n+1+s}{2}}}\, dy. \tag{2.4}$$

Our aim is to give a simple proof of Schauder estimate for the Cauchy problem

$$\begin{cases} \partial_t u(x, t) = L_A[u] + f(x, t) \\ u(x, 0) = 0, \end{cases} \tag{2.5}$$

where $A(\cdot, t)$ is symmetric, elliptic and uniformly Hölder continuous for every $t \in [0, T)$.

The following Schauder estimate can be found for instance in [23]. For the sake of completeness and for the reader's convenience, we provide a proof based purely on PDE methods in the Appendix.

Theorem 2.2. *Assume $\alpha > 0$ is such that $\alpha < \min\{s, 1-s\}$ and fix $T > 0$. Assume that for every $t \in [0,T)$, $A(\cdot,t)$ is symmetric and elliptic, and assume that $\sup_{t<T} \|A(\cdot,t)\|_{C^\alpha(\mathbb{R}^n)} \le C_0$. Let $f : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ be such that $\sup_{t<T} \|f(\cdot,t)\|_{C^\alpha(\mathbb{R}^n)} < \infty$ and assume u is the solution of (2.5) such that $\lim_{|x|\to\infty} u(x,t) \le C$. Then there exists a constant C_T, depending on $\alpha, \lambda, \Lambda, C_0, \gamma, T$ and on the dimension, such that*

$$\sup_{t<T} \|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)} \le C_T \sup_{t<T} \|f(\cdot,t)\|_{C^\alpha(\mathbb{R}^n)}.$$

We will prove later, in Theorem 4.1, that the analogous result holds on the sphere.

2.2. **Preliminary results.** As we already mentioned, we know that the evolving sets E_t remain convex and satisfy the a priori estimates from [10] stated at the beginning of this section. Next we show that these a priori estimates imply that the sets E_t are uniformly C^{1+s}-regular.

Proposition 2.3. *Assume that $E \subset \mathbb{R}^{n+1}$ is a convex set such that $B_r \subset E \subset B_R$ and*

$$\sup_{x\in\partial E} H^s_E(x) \le C.$$

Then E is uniformly C^{1+s}-regular and the boundary can be written as

$$\partial E = \{h(x)x : x \in \mathbb{S}^n\}$$

with $\|h\|_{C^{1+s}(\mathbb{S}^n)} \le C$.

Proof. The assumptions on E imply that there is $r_0 > 0$ such that for every $x \in \partial E$ we may write the boundary locally as a graph of a bounded convex function, i.e., $\partial E \cap B_{r_0}(x)$ is contained on graph of a convex function which is uniformly bounded. Our goal is to prove

$$\sup_{y\neq x\in\partial E} \frac{\langle x-y, \nu_E(x)\rangle}{|y-x|^{1+s}} \le C. \tag{2.6}$$

The estimate (2.6) will then imply the claim by the following argument. Indeed, let us fix a point $x_0 \in \partial E$ and without loss of generality we may assume $x_0 = 0$ and $\nu_E(x_0) = -e_{n+1}$. As we mentioned above, we may write

$$\partial E \cap B_{r_0} \subset \{(x', u(x')) \in \mathbb{R}^{n+1} : x' \in B_{r_0} \subset \mathbb{R}^n\}$$

for a bounded convex function $u : B_{r_0} \subset \mathbb{R}^n \to \mathbb{R}$. Let us choose $x', y' \in B_{r_0/2} \subset \mathbb{R}^n$. Note that since u is convex and bounded in B_{r_0}, it is uniformly Lipschitz continuous in $B_{r_0/2}$. Then (2.6) implies

$$\begin{aligned}&\langle (y'-x', u(y')-u(x')), (-\nabla u(x'), 1)\rangle \\ &\qquad \le C\sqrt{1+|\nabla u(x')|^2}(|y'-x'|^2 + (u(y')-u(x'))^2)^{\frac{1+s}{2}} \le C|y'-x'|^{1+s}.\end{aligned}$$

In other words

$$\langle u(y') - u(x') - \nabla u(x'), y'-x'\rangle \le C|y'-x'|^{1+s} \quad \text{for every } x', y' \in B_{r_0/2}.$$

Since u is convex, then $u(y') - u(x') \ge \langle \nabla u(x'), y'-x'\rangle$. The above then implies that $\|u\|_{C^{1+s}(B_{r_0/4})} \le C$ (see e.g. **(H4)** in [14, Appendix A]). Hence, we need to prove (2.6).

We argue by contradiction and assume that there are sets E_k which satisfy the assumptions and points $y_k \neq x_k \in \partial E_k$ such that $|y_k - x_k| = r_k \le 2^{-4n} r_0$, with $r_k \to 0$, and

$$\frac{\langle x_k - y_k, \nu_{E_k}(x_k)\rangle}{|y_k - x_k|^{1+s}} \ge k. \tag{2.7}$$

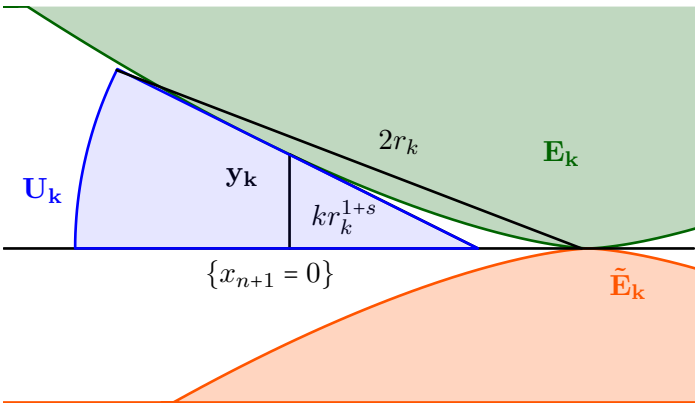

FIGURE 2.1. A picture of the argument used in the proof of Proposition 2.3. In green we have the set E_k, in orange $\tilde{E}_k$, i.e. the reflection of E_k over the plane $\{x_{n+1} = 0\}$ and in blue the set U_k

Again we may assume that $x_k = 0$, $\nu_{E_k}(0) = -e_{n+1}$. Since E_k is convex then

$$E_k \subset \{y \in \mathbb{R}^{n+1} : \langle y - y_k, \nu_{E_k}(y_k)\rangle \le 0\}.$$

Then the following wedge-type region

$$U_k := \{y \in B_{2r_k} : y_{n+1} > 0,\ \langle y - y_k, \nu_{E_k}(y_k)\rangle > 0\}$$

is contained in the complement of E_k, i.e., $U_k \cap E_k = \varnothing$. Note that (2.7) implies that

$$\langle y_k, e_{n+1}\rangle \ge kr_k^{1+s}.$$

Therefore we deduce that there is a uniform constant $c > 0$ such that

$$|U_k| \ge c\,k\,r_k^{n+1+s}. \tag{2.8}$$

Recall the definition of the fractional mean curvature

$$H^s_{E_k}(0) = \int_{\mathbb{R}^{n+1}\setminus E_k} \frac{dy}{|y|^{n+1+s}} - \int_{E_k} \frac{dy}{|y|^{n+1+s}} \le C.$$

By convexity and by $\nu_{E_k}(0) = -e_{n+1}$ it holds $E_k \subset \{y : y_{n+1} \ge 0\}$. Therefore, denoting by $\tilde{E}_k$ the reflection of E_k with respect to the hyperplane $\{x_{n+1} = 0\}$, we obtain that $\tilde{E}_k \subset E_k^c$ and $U_k \subset (E_k \cup \tilde{E}_k)^c$ which implies

$$\int_{U_k} \frac{dy}{|y|^{n+1+s}}\,dy \le \int_{\mathbb{R}^{n+1}\setminus(E_k\cup\tilde{E}_k)} \frac{dy}{|y|^{n+1+s}} = \int_{\mathbb{R}^{n+1}\setminus E_k} \frac{dy}{|y|^{n+1+s}} - \int_{E_k} \frac{dy}{|y|^{n+1+s}} \le C.$$

By the definition of the set U_k it holds $U_k \subset \{y \in \mathbb{R}^{n+1} : y_{n+1} > 0\}\setminus E_k$. Moreover, the definition of U_k implies that $|y| \le 2r_k$ for all $y \in U_k$. Therefore by (2.8) and by the above inequality we have

$$C \ge \int_{U_k} \frac{dy}{|y|^{n+1+s}} \ge c\frac{|U_k|}{r_k^{n+1+s}} \ge c\,k,$$

where the constant $c > 0$ does not depend on k. Hence, we obtain a contradiction when $k \to \infty$. □

At the end of the section we prove continuity estimates for functions defined via convolution over kernels. We define elliptic and symmetric matrix field on $\mathbb{S}^n$, $A(\cdot):\mathbb{S}^n\to\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ as in (2.3), i.e., there are $0<\lambda<1<\Lambda$ such that

$$\lambda|\xi|^2\le\langle A(x)\xi,\xi\rangle\le\Lambda|\xi|^2\quad\text{for all }\ \xi\in\mathbb{R}^{n+1}\text{ and }\ x\in\mathbb{S}^n.$$

We consider kernels $K:\mathbb{S}^n\times\mathbb{S}^n\setminus\{(x,x):x\in\mathbb{S}^n\}\to\mathbb{R}$ of the type

$$K_A(y,x)=\frac{1}{\langle A(x)(y-x),(y-x)\rangle^{\frac{n+1+s}{2}}},\tag{2.9}$$

where $A(\cdot)$ is symmetric, elliptic and $\|A\|_{C^\alpha(\mathbb{S}^n)}\le C$. In order to simplify the notation, we denote the induced norm by

$$\|\xi\|^2_{A(x)}:=\langle A(x)\xi,\xi\rangle\qquad\text{for }\ \xi\in\mathbb{R}^{n+1}.\tag{2.10}$$

We may then denote K defined in (2.9) as $K(y,x)=\|y-x\|^{-n-1-s}_{A(x)}$.

The following lemma is easy to prove and thus we omit it.

Lemma 2.4. *Assume that $A(\cdot)$ is symmetric, elliptic and $\|A\|_{C^\alpha(\mathbb{S}^n)}\le C$ for $0<\alpha<\min\{s,1-s\}$. Then the kernel K_A defined in (2.9) satisfies the following two conditions:*

(i) K_A is continuous on $\mathbb{S}^n\times\mathbb{S}^n\setminus\{(x,x):x\in\mathbb{S}^n\}$ and satisfies

$$|K_A(y,x)|\le\frac{C}{|y-x|^{n+1+s}}.$$

(ii) For all $x,y,z\in\mathbb{S}^n$ with $0<|z-x|\le\frac12|y-x|$ it holds

$$|K_A(y,z)-K_A(y,x)|\le C\frac{|z-x|}{|y-x|^{n+2+s}}+C\frac{|z-x|^\alpha}{|y-x|^{n+1+s}}.$$

We note that the above conditions agree with the assumptions in [19, Definition 4.1] with the difference that the second condition is slightly weaker. The following lemma is almost the same as [19, Lemma 4.3].

Lemma 2.5. *Assume that $A(\cdot)$ is symmetric, elliptic and $\|A\|_{C^\alpha(\mathbb{S}^n)}\le C$ for $0<\alpha<\min\{s,1-s\}$, and the kernel K_A is defined in (2.9). Assume that the function $F\in C(\mathbb{S}^n\times\mathbb{S}^n)$ satisfies the following conditions.*

(1) For all $x,y\in\mathbb{S}^n$ it holds

$$|F(y,x)|\le\kappa_0|y-x|^{1+s+\alpha}.$$

(2) For all $x,y,z\in\mathbb{S}^n$ with $|z-x|\le\frac12|y-x|$ it holds

$$|F(y,z)-F(y,x)|\le\kappa_0\big(|z-x|^\alpha|y-x|^{1+s+\alpha/2}+|z-x|^{s+\alpha}|y-x|\big).$$

Then the function

$$\psi(x)=\int_{\mathbb{S}^n}F(y,x)K_A(y,x)\,d\mathcal{H}^n_y$$

is Hölder continuous and $\|\psi\|_{C^\alpha(\mathbb{S}^n)}\le C\kappa_0$.

Proof. The proof is almost the same as in [19, Lemma 4.3] but we write it for the reader's convenience since the assumptions are slightly different. The condition (i) in Lemma 2.4 and the assumption (1) for F immediately implies that $\|\psi\|_{C^0}\le C\kappa_0$. To show the Hölder continuity we fix $z,x\in\mathbb{S}^n$ and divide the sphere into two sets as $S_-=\{y\in\mathbb{S}^n:|y-x|\le2|z-x|\}$

and $S_+ = \{y \in \mathbb{S}^n : |y-x| > 2|z-x|\}$. Note that for $y \in S_-$ it holds $|y-z| \le |y-x|+|z-x| \le 3|z-x|$. We use the condition (i) in Lemma 2.4 and the assumption (1) for F to deduce

$$\int_{S_-} |F(y,z)K_A(y,z)|\, d\mathcal{H}^n_y \le C\kappa_0 \int_{S_-} \frac{1}{|y-z|^{n-\alpha}}\, d\mathcal{H}^n_y \le C\kappa_0 \int_0^{3|z-x|} \rho^{\alpha-1}\, d\rho = C\kappa_0|z-x|^\alpha.$$

Similarly it holds $\int_{S_-} |F(y,x)K_A(y,x)|\, d\mathcal{H}^n_y \le C\kappa_0|z-x|^\alpha$.

Note that for $y \in S_+$ it holds $|y-z| \ge |y-x|-|z-x| \ge \frac{1}{2}|y-x|$. We use the conditions (i) and (ii) in Lemma 2.4 and the assumption (1) and (2) for F to estimate

$$\begin{aligned}
&\int_{S_+} |F(y,z)K_A(y,z) - F(y,x)K_A(y,x)|\, d\mathcal{H}^n_y \\
&\quad \le \int_{S_+} |F(y,z)-F(y,x)||K_A(y,z)|\, d\mathcal{H}^n_y + \int_{S_-} |F(y,x)||K_A(y,z)-K_A(y,x)|\, d\mathcal{H}^n_y \\
&\quad \le C\kappa_0|z-x|^\alpha \int_{S_+} \frac{1}{|y-x|^{n-\alpha/2}}\, d\mathcal{H}^n_y + C\kappa_0|z-x|^{s+\alpha} \int_{S_+} \frac{1}{|y-x|^{n+s}}\, d\mathcal{H}^n_y \\
&\qquad + C\kappa_0|z-x| \int_{S_+} \frac{1}{|y-x|^{n+1-\alpha}}\, d\mathcal{H}^n_y.
\end{aligned}$$

Since $\frac{1}{|y-x|^{n-\alpha/2}}$ is not singular, we obtain $\int_{S_+} \frac{1}{|y-x|^{n-\alpha/2}}\, d\mathcal{H}^n_y \le C$. Since $|z-x| \le |y-x|$ we estimate the last term by

$$\frac{|z-x|}{|y-x|^{n+1-\alpha}} \le \frac{|z-x|^{s+\alpha}}{|y-x|^{n+s}}$$

which then implies

$$|z-x| \int_{S_+} \frac{1}{|y-x|^{n+1-\alpha}}\, d\mathcal{H}^n_y \le |z-x|^{s+\alpha} \int_{S_+} \frac{1}{|y-x|^{n+s}}\, d\mathcal{H}^n_y \le C|z-x|^{s+\alpha} \int_{|z-x|}^2 \rho^{-s-1} \le C|z-x|^\alpha.$$

Therefore we have

$$\int_{S_+} |F(y,z)K_A(y,z) - F(y,x)K_A(y,x)|\, d\mathcal{H}^n_y \le C\kappa_0|z-x|^\alpha$$

and the claim follows. □

Remark 2.6. The above lemma holds also in $\mathbb{R}^n$ for a kernel defined as in (2.9), if we change the assumptions on $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as follows.

(1) For all $x \in \mathbb{R}^n$ it holds

$$|F(y,x)| \le \kappa_0 \min\{|y-x|, 1\}^{1+s+\alpha}.$$

(2) For all $x, y, z \in \mathbb{R}^n$ with $|z-x| \le \frac{1}{2}|y-x|$ it holds

$$|F(y,z) - F(y,x)| \le \kappa_0\big(|z-x|^\alpha \min\{|y-x|, 1\}^{1+s+\alpha/2} + |z-x|^{s+\alpha} \min\{|y-x|, 1\}\big).$$

3. Equation for the height function

As we mentioned in the introduction, instead of the original equation (1.1), we will differentiate it with respect to space and parametrize the equation

$$\nabla_{X(t)} V_t = -\nabla_{X(t)} H_{E_t^s} \qquad \text{on } \partial E_t, \tag{3.1}$$

where $X(t)$ is a vector field on the moving boundary ∂E_t, which we will define shortly, and $\nabla_{X(t)}$ is the derivative in the direction of $X(t)$. Note that the derivative of the integral average $\bar{H}^s_{E_t}$ vanishes.

We parametrize the flow using the height function similarly as in (2.1), with the difference that we assume we may write locally in time

$$\partial E_t = \{h(x,t)x + x_0 : x \in \mathbb{S}^n\},$$

for some fixed $x_0 \in \mathbb{R}^{n+1}$ and for $t \in [0,T)$. We denote

$$\eta(x,t) := h(x,t)x.$$

We choose the vector field $X(t)$ as follows. Fix a basis vector e_i, $i = 1,\dots,n+1$ of the ambient space $\mathbb{R}^{n+1}$. We obtain a tangent field on $\mathbb{S}^n$ as

$$\tau_i(x) = e_i - x_i\, x, \qquad \text{for } x \in \mathbb{S}^n,$$

where $x_i = \langle x, e_i\rangle$. This induces a tangent field on the moving boundary ∂E_t, which we denote by $X(t) = X(\cdot,t)$, defined as

$$X(\eta(x,t),t) := D\eta(x,t)\tau_i(x) = h(x,t)\tau_i(x) + \nabla_i h(x,t)x \qquad \text{for } x \in \mathbb{S}^n.$$

We may then write the derivative $\nabla_{X(t)}f$ of any given function $f \in C^1(\partial E_t)$ at the point $\eta(x,t) \in \partial E_t$ for $x \in \mathbb{S}^n$ as

$$(\nabla_{X(t)}f)(\eta(x,t)) = \nabla_i(f \circ \eta(\cdot,t))(x).$$

We will assume throughout the section that the height function satisfies the uniform bound $\|h(\cdot,t)\|_{C^{1+s}} \le C$ and $c \le h(x,t) \le C$ for all t and $x \in \mathbb{S}^n$.

Let us first deal the LHS of (3.1). This is rather easy as we may write the normal as

$$\nu_{E_t}(\eta(x,t)) = \frac{h(x,t)x - \nabla_\tau h(x,t)}{\sqrt{h^2(x,t) + |\nabla_\tau h(x,t)|^2}}$$

and the normal velocity then is

$$V_t(\eta(x,t)) = \partial_t h(x,t)\,\frac{h(x,t)}{\sqrt{h^2(x,t) + |\nabla_\tau h(x,t)|^2}}. \tag{3.2}$$

Hence, it holds

$$\begin{aligned}(\nabla_{X(t)}V_t)(\eta(x,t)) &= \nabla_i\left(\partial_t h(x,t)\,\frac{h(x,t)}{\sqrt{h^2(x,t) + |\nabla_\tau h(x,t)|^2}}\right)\\ &= \partial_t\nabla_i h(x,t)\,\frac{h(x,t)}{\sqrt{h^2(x,t) + |\nabla_\tau h(x,t)|^2}}\\ &\quad + \partial_t h(x,t)\nabla_i\left(\frac{h(x,t)}{\sqrt{h^2(x,t) + |\nabla_\tau h(x,t)|^2}}\right).\end{aligned} \tag{3.3}$$

Let us then focus on the RHS of (3.1). We recall that following the argument in [15, 24] we may write

$$\nabla_{X(t)}H^s_{E_t} = 2\int_{\partial E_t} \frac{\langle X(t), \nu_{E_t}(y)\rangle}{|y-x|^{n+1+s}}\, d\mathcal{H}^n_y. \tag{3.4}$$

Indeed, by regularizing the kernel, say by $K_\delta(t) = (t^2+\delta)^{-\frac{n+1+s}{2}}$, and differentiating in x_i direction (see [15, Formula (6.17)]) we have

$$\begin{aligned}
&\nabla_{x_i}\left(\int_{E_t^c} K_\delta(|y-x|)\,dy - \int_{E_t} K_\delta(|y-x|)\,dy\right)\\
&= -\int_{E_t^c} K_\delta'(|y-x|)\frac{\langle y-x, e_i\rangle}{|y-x|}\,dy + \int_{E_t} K_\delta'(|y-x|)\frac{\langle y-x, e_i\rangle}{|y-x|}\,dy\\
&= -\int_{E_t^c} \operatorname{div}_y(K_\delta(|y-x|)e_i)\,dy + \int_{E_t} \operatorname{div}_y(K_\delta(|y-x|)e_i)\,dy\\
&= \int_{\partial E_t} 2K_\delta(|y-x|)\langle e_i, \nu_{E_t}(y)\rangle\,d\mathcal{H}^n_y,
\end{aligned}$$

where the last equality follows from divergence theorem. Letting $\delta \to 0$ yields (3.4).

We parametrize (3.4) on $\mathbb{S}^n$, by recalling the formula for the normal $\nu_{E_t}(\eta(x,t))$ above and the Jacobian determinant of η as

$$J_{\eta(\cdot,t)}(x) = h(x,t)^{n-1}\sqrt{h^2(x,t) + |\nabla_\tau h(x,t)|^2}.$$

For simplicity we omit the time-dependence for a moment and write $h(x) = h(x,t)$. Then we may write

$$\nabla_{X(t)} H^s_{E_t} = 2\int_{\mathbb{S}^n} h(y)^{n-1}\frac{\langle h(x)\tau_i(x) + \nabla_i h(x)x, h(y)y - \nabla_\tau h(y)\rangle}{|h(y)y - h(x)x|^{n+1+s}}\,d\mathcal{H}^n_y$$

We continue and write the terms as

$$\begin{aligned}
h(x)\tau_i(x) + \nabla_i h(x)x =& h(y)\tau_i(y) + \nabla_i h(x)y\\
&+ \big(h(x)\tau_i(x) - h(y)\tau_i(y)\big) + (x-y)\nabla_i h(x).
\end{aligned}$$

This gives us the following expression

$$\nabla_{X(t)} H^s_{E_t} = 2\int_{\mathbb{S}^n} h(y)^{n-1}\frac{\langle h(y)\tau_i(y) + \nabla_i h(x)y, h(y)y - \nabla_\tau h(y)\rangle}{|h(y)y - h(x)x|^{n+1+s}}\,d\mathcal{H}^n_y + R_1$$

where the error terms are

$$\begin{aligned}
R_1 =& 2\int_{\mathbb{S}^n} \frac{\langle h(x)\tau_i(x) - h(y)\tau_i(y), h(y)y - \nabla_\tau h(y)\rangle}{|h(y)y - h(x)x|^{n+1+s}} h(y)^{n-1}\,d\mathcal{H}^n_y\\
&+ 2\int_{\mathbb{S}^n} \frac{\langle x-y, h(y)y - \nabla_\tau h(y)\rangle}{|h(y)y - h(x)x|^{n+1+s}}\nabla_i h(x) h(y)^{n-1}\,d\mathcal{H}^n_y.
\end{aligned}$$

Note that we may write the error terms as

$$R_1 = \sum_k a_k(x,h,\nabla_\tau h)\int_{\mathbb{S}^n} \frac{F_k(y,h(y)) - F_k(x,h(x))}{|h(y)y - h(x)x|^{n+1+s}} b_k(y,h,\nabla_\tau h)\,d\mathcal{H}^n_y$$

for smooth functions F_k, a_k, b_k. But since

$$\langle h(y)\tau_i(y) + \nabla_i h(x)y, h(y)y - \nabla_\tau h(y)\rangle = \big(\nabla_i h(x) - \nabla_i h(y)\big)h(y)$$

we obtain

$$-\nabla_{X(t)} H^s_{E_t} = 2\int_{\mathbb{S}^n} \frac{\nabla_i h(y) - \nabla_i h(x)}{|h(y)y - h(x)x|^{n+1+s}} h(y)^n\,d\mathcal{H}^n_y + R_1. \tag{3.5}$$

We further linearize the first term on the RHS of (3.5), which is the leading order term in the equation. We write using $\langle x-y, y\rangle = -\frac{1}{2}|x-y|^2$

$$\begin{aligned}|h(y)y - h(x)x|^2 &= |h(x)(x-y) + (h(x)-h(y))y|^2\\ &= h^2(x)|y-x|^2 + h(x)(h(y)-h(x))|x-y|^2 + |h(y)-h(x)|^2.\end{aligned}$$

We write the Taylor expansion of h at x as

$$T_x[h](y) = h(y) - h(x) - \langle \nabla_\tau h(x), y-x\rangle$$

and note that for $\beta \in [0,1]$ it holds

$$|T_x[h](y)| \le \|h\|_{C^{1+\beta}}|y-x|^{1+\beta}. \tag{3.6}$$

Therefore we deduce that we may write

$$\begin{aligned}|h(y)y - h(x)x|^2 &= h^2(x)|y-x|^2 + \langle \nabla_\tau h(x), y-x\rangle^2 + g_h(y,x)\\ &= \|x-y\|^2_{A(x)} + g_h(y,x)\end{aligned} \tag{3.7}$$

where the matrix is given by tensor product $A(x) = h^2(x)I + \nabla_\tau h(x)\otimes\nabla_\tau h(x)$, the norm $\|\xi\|_{A(x)}$ is defined in (2.10), and the error term is

$$g_h(y,x) = h(x)(h(y)-h(x))|x-y|^2 + 2(h(y)-h(x))T_x[h](y) - |T_x[h](y)|^2.$$

The precise form of g_h is not relevant. Instead, it is important to notice that it satisfies the following two conditions. For all $x, y \in \mathbb{S}^n$ and $0 < \alpha < \min\{s, 1-s\}$ it holds

$$|g_h(y,x)| \le C\|h\|_{C^{1+s+\alpha}}|y-x|^{2+s+\alpha} \tag{3.8}$$

and for all $x, y, z \in \mathbb{S}^n$ with $|z-x| \le \frac{1}{2}|y-x|$ it holds

$$|g_h(y,z) - g_h(y,x)| \le C\|h\|_{C^{1+s+\alpha}}|z-x|^{s+\alpha}|y-x|^2. \tag{3.9}$$

Indeed, the estimate (3.8) follows by using (3.6) for $\beta = 0$ and $\beta = s+\alpha$, and recalling that $\|h\|_{C^{1+s}} \le C$. The estimate (3.9) follows from the fact that for $|z-x| \le \frac{1}{2}|y-x|$ it holds

$$\begin{aligned}\big|T_z[h](y) - T_x[h](y)\big| &= \big|\big(h(y)-h(z)-\langle\nabla_\tau h(z), y-z\rangle\big) - \big(h(y)-h(x)-\langle\nabla_\tau h(x), y-x\rangle\big)\big|\\ &= \big|\big(h(x)-h(z)-\langle\nabla_\tau h(z), x-z\rangle\big) + \big(\langle\nabla_\tau h(x)-\nabla_\tau h(z), y-x\rangle\big)\big|\\ &\le \|h\|_{C^{1+s+\alpha}}|z-x|^{1+s+\alpha} + \|h\|_{C^{1+s+\alpha}}|z-x|^{s+\alpha}|y-x|\\ &\le 2\|h\|_{C^{1+s+\alpha}}|z-x|^{s+\alpha}|y-x|.\end{aligned} \tag{3.10}$$

Therefore we use the notation for $g_h(y,x)$, which is defined in (3.7), and the norm $\|\xi\|_{A(x)}$, which is defined in (2.10), and write

$$\begin{aligned}&\int_{\mathbb{S}^n}\frac{\nabla_i h(y)-\nabla_i h(x)}{|h(y)y-h(x)x|^{n+1+s}}h(y)^n\,d\mathcal{H}^n_y\\ &\quad= \int_{\mathbb{S}^n}\frac{\nabla_i h(y)-\nabla_i h(x)}{\|y-x\|^{n+1+s}_{A(x)}}h(y)^n\,d\mathcal{H}^n_y\\ &\quad+ \int_{\mathbb{S}^n}\frac{\nabla_i h(y)-\nabla_i h(x)}{\|y-x\|^{n+1+s}_{A(x)}}\int_0^1\frac{d}{d\mu}\left(\frac{\|y-x\|^2_{A(x)}}{\|y-x\|^2_{A(x)}+\mu g_h(y,x)}\right)^{\frac{n+1+s}{2}}h(y)^n\,d\mu\,d\mathcal{H}^n_y,\end{aligned} \tag{3.11}$$

where $A(x) = h^2(x)I + \nabla_\tau h(x)\otimes\nabla_\tau h(x)$.

Recall again the a priori estimates for the height function, i.e.,

$$\sup_{t\in[0,T)} \|h(\cdot,t))\|_{C^{1+s}} \le C \quad \text{and} \quad h(x,t) \ge c.$$

We may thus finally write the equation (3.1) by (3.3), (3.5) and (3.11) as

$$\partial_t \nabla_i h = L_A[\nabla_i h] + R_0(x,t) + R_1(x,t) + \partial_t h\, B(x,h,\nabla_\tau h) : \nabla_\tau^2 h. \tag{3.12}$$

Here B is a smooth matrix-field and the linear operator is defined as

$$L_A[u](x) := a_0(x,t) \int_{\mathbb{S}^n} \frac{u(y)-u(x)}{\|y-x\|^{n+1+s}_{A(x,t)}} b_0(y,t)\, d\mathcal{H}^n_y, \tag{3.13}$$

for symmetric elliptic matrix field

$$A(x,t) = h^2(x,t)I + \nabla_\tau h(x,t) \otimes \nabla_\tau h(x,t)$$

and the coefficients $a_0, b_0 \ge c_0$ satisfy

$$\sup_{t<T} \|a_0(\cdot,t)\|_{C^s(\mathbb{S}^n)} \le C \quad \text{and} \quad \sup_{t<T} \|b_0(\cdot,t)\|_{C^1(\mathbb{S}^n)} \le C. \tag{3.14}$$

The error terms are

$$R_0(x,t) = a_0(x,t) \int_{\mathbb{S}^n} \int_0^1 \frac{\nabla_i h(y,t) - \nabla_i h(x,t)}{\|y-x\|^{n+1+s}_{A(x,t)}} \frac{d}{d\mu} \left(\frac{\|y-x\|^2_{A(x,t)}}{\|y-x\|^2_{A(x,t)} + \mu g_h(y,x)} \right)^{\frac{n+1+s}{2}} b_0(y,t)\, d\mu d\mathcal{H}^n_y \tag{3.15}$$

and

$$R_1(x,t) = \sum_{k=1}^N a_k(x,h,\nabla_\tau h) \int_{\mathbb{S}^n} \frac{F_k(y,h(y,t)) - F_k(x,h(x,t))}{|h(y,t)y - h(x,t)x|^{n+1+s}} b_k(y,h,\nabla_\tau h)\, d\mathcal{H}^n_y, \tag{3.16}$$

where a_k, b_k and F_k for $k \ge 1$ are smooth functions.

Next we state the following technical lemma, which we use later to bound the error term R_0 defined in (3.15).

Lemma 3.1. *Assume that $A(\cdot)$ is symmetric elliptic matrix field as defined in (2.3) such that $\|A(\cdot)\|_{C^\alpha} \le C$ for $0 < \alpha < \min\{s, 1-s\}$ and assume that $g : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ satisfies the following two conditions. For all $x, y \in \mathbb{S}^n$ it holds $\|y-x\|^2_{A(x)} + g(y,x) \ge c|y-x|^2$, for some $c > 0$, and*

$$|g(y,x)| \le \kappa |y-x|^{2+s+\alpha},$$

and for all $x, y, z \in \mathbb{S}^n$ with $|z-x| \le \frac{1}{2}|y-x|$ it holds

$$|g(y,z) - g(y,x)| \le \kappa(|z-x|^{s+\alpha}|y-x|^2 + |z-x|^\alpha |y-x|^{2+s+\alpha}).$$

Denote $G(y,x) = \frac{d}{d\mu}\left(\frac{\|y-x\|^2_{A(x)}}{\|y-x\|^2_{A(x)} + \mu g(y,x)}\right)^q$, for $\mu \in (0,1), q \ge 1$, and assume $v_1 \in C^1(\mathbb{S}^n)$, $v_2 \in C(\mathbb{S}^n)$ and $v_3 \in C^\alpha(\mathbb{S}^n)$. Then the function

$$\psi(x) = \int_{\mathbb{S}^n} \big(v_1(y) - v_1(x)\big) v_2(y) v_3(x) G(y,x) K_A(y,x)\, d\mathcal{H}^n_y$$

is Hölder continuous and

$$\|\psi\|_{C^\alpha(\mathbb{S}^n)} \le C\kappa \|v_1\|_{C^1(\mathbb{S}^n)} \|v_2\|_{C^0(\mathbb{S}^n)} \|v_3\|_{C^\alpha(\mathbb{S}^n)}.$$

Proof. Let us first prove that it holds for all $y, x \in \mathbb{S}^n, y \neq x$

$$|G(y,x)| \leq C\kappa|y-x|^{s+\alpha} \tag{3.17}$$

and for all $x, y, z \in \mathbb{S}^n$ with $0 < |z-x| \leq \frac{1}{2}|y-x|$

$$|G(y,z) - G(y,x)| \leq C\kappa\big(|z-x|^{\alpha}|y-x|^{s+\alpha} + |z-x|^{s+\alpha}\big). \tag{3.18}$$

To this aim we denote $\Phi(t) = (1+t)^{-q}$,

$$\tilde{G}(y,x) = \frac{g(x,y)}{\|y-x\|^2_{A(x)}}$$

and notice that we may write $G(y,x) = \Phi'(\mu\tilde{G}(y,x))\tilde{G}(y,x)$. Since Φ is smooth we deduce that it is enough to show that $\tilde{G}$ satisfies the conditions (3.17) and (3.18). We obtain (3.17) immediately from the ellipticity of A and from the first assumption on g as

$$|\tilde{G}(y,x)| \leq C\kappa\frac{|y-x|^{2+s+\alpha}}{|y-x|^2} \leq C\kappa|y-x|^{s+\alpha}.$$

To prove (3.18) we fix $x, y, z \in \mathbb{S}^n$ with $0 < |z-x| \leq \frac{1}{2}|y-x|$. Note that then $\frac{1}{2}|y-x| \leq |y-z| \leq 2|y-x|$. We then write

$$|\tilde{G}(y,z) - \tilde{G}(y,x)| \leq |g(y,z)|\big|\|y-z\|^{-2}_{A(z)} - \|y-x\|^{-2}_{A(x)}\big| + \frac{|g(x,z)-g(y,z)|}{\|y-x\|^2_{A(x)}}.$$

We may estimate the last term by the second condition on g as

$$\frac{|g(x,z)-g(y,z)|}{\|y-x\|^2_{A(x)}} \leq C\kappa\frac{|z-x|^{s+\alpha}|y-x|^2 + |z-x|^{\alpha}|y-x|^{2+s+\alpha}}{|y-x|^2} \leq C\kappa(|z-x|^{s+\alpha} + |z-x|^{\alpha}|y-x|^{s+\alpha}).$$

For the first term we observe that

$$\begin{aligned}
&\big|\|y-z\|^2_{A(z)} - \|y-x\|^2_{A(x)}\big| \\
&\qquad = \big|\langle A(z)(y-z),(y-z)\rangle - \langle A(x)(y-x),(y-x)\rangle\big| \\
&\qquad \leq \big|\langle (A(z)-A(x))(y-z),(y-z)\rangle\big| + \big|\langle A(x)\big((y-z)+(y-x)\big),(x-z)\rangle\big| \\
&\qquad \leq C|z-x|^{\alpha}|y-x|^2 + C|y-x||z-x|.
\end{aligned}$$

We use the fact that $|z-x| \leq \frac{1}{2}|y-x|$ to deduce $|z-x| \leq |z-x|^{s+\alpha}|y-x|^{1-s-\alpha}$. Therefore it holds

$$\begin{aligned}
|g(y,z)|\big|\|y-z\|^{-2}_{A(z)} - \|y-x\|^{-2}_{A(x)}\big| &\leq \frac{|g(y,z)|}{|y-x|^4}\big|\|y-z\|^2_{A(z)} - \|y-x\|^2_{A(x)}\big| \\
&\leq \frac{C\kappa}{|y-x|^{2-s-\alpha}}(|z-x|^{\alpha}|y-x|^2 + |y-x||z-x|) \\
&\leq C\kappa\big(|z-x|^{\alpha}|y-x|^{s+\alpha} + |z-x|^{s+\alpha}\big)
\end{aligned}$$

and (3.18) follows.

We proceed by defining $F : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ as

$$F(y,x) := \big(v_1(y) - v_1(x)\big)v_2(y)v_3(x)G(y,x)$$

when $y \neq x$ and $F(x,x) = 0$. We prove that F satisfies the assumptions of Lemma 2.5 for

$$\tilde{\kappa} := \kappa\|v_1\|_{C^1}\|v_2\|_{C^0}\|v_3\|_{C^{\alpha}}.$$

This will then yield the claim. The first condition follows immediately from (3.17) as

$$|F(y,x)| = |v_1(y) - v_1(x)|\,|v_2(y)|\,|v_3(x)|\,|G(y,x)| \le C\tilde{\kappa}|y-x|^{1+s+\alpha}$$

for all $x \neq y \in \mathbb{S}^n$.

To show the second condition we fix $x, y, z \in \mathbb{S}^n$ with $0 < |z-x| \le \frac{1}{2}|y-x|$. We will in fact show that

$$|F(y,z) - F(y,x)| \le C\tilde{\kappa}\big(|z-x|^{\alpha}|y-x|^{1+s+\alpha} + |z-x|^{s+\alpha}|y-x|\big),$$

which is slightly stronger than what we need. To this aim we write

$$\begin{aligned}|F(y,z) - F(y,x)| &= |v_2(y)|\big|\big(v_1(y) - v_1(z)\big)v_3(z)G(y,z) - \big(v_1(y) - v_1(x)\big)v_3(x)G(y,x)| \\ &\le |v_2(y)||v_3(z)||G(y,z)||v_1(z) - v_1(x)| \\ &\quad + |v_2(y)||v_3(z)||v_1(y) - v_1(x)||G(y,z) - G(y,x)| \\ &\quad + |v_2(y)||v_1(y) - v_1(x)||G(y,x)||v_3(z) - v_3(x)|.\end{aligned}$$

We estimate the first term term on the RHS by (3.17) and by $|z-x| \le |y-x|$ as

$$|v_2(y)||v_3(z)||G(y,z)||v_1(z) - v_1(x)| \le C\tilde{\kappa}|y-x|^{s+\alpha}|z-x| \le C\tilde{\kappa}|y-x||z-x|^{s+\alpha}.$$

We estimate the last similarly

$$|v_2(y)||v_1(y) - v_1(x)||G(y,x)||v_3(z) - v_3(x)| \le C\tilde{\kappa}|y-x|^{1+s+\alpha}|z-x|^{\alpha}.$$

Finally, we estimate the second term by (3.18) as

$$|v_2(y)||v_3(z)||v_1(y) - v_1(x)||G(y,z) - G(y,x)| \le \tilde{\kappa}\big(|z-x|^{\alpha}|y-x|^{1+s+\alpha} + |z-x|^{s+\alpha}|y-x|\big).$$

We thus have the claim by Lemma 2.5. □

We conclude the section with a technical lemma, which is analogous to [19, Lemma 4.4].

Lemma 3.2. *Assume that $A(\cdot)$ is symmetric, elliptic and $\|A\|_{C^{\alpha}(\mathbb{S}^n)} \le C$ for $0 < \alpha < \min\{s, 1-s\}$, and the kernel K_A is defined in (2.9). Assume further $v_1 \in C^{1+s+\alpha}(\mathbb{S}^n)$, $v_2 \in C^{s+\alpha}(\mathbb{S}^n)$ and $v_3 \in C^{\alpha}(\mathbb{S}^n)$. Then the function*

$$\psi(x) = \int_{\mathbb{S}^n} \big(v_1(y) - v_1(x)\big)v_2(y)v_3(x)K_A(y,x)\,d\mathcal{H}^n_y$$

is Hölder continuous and

$$\|\psi\|_{C^{\alpha}(\mathbb{S}^n)} \le C\|v_1\|_{C^{1+s+\alpha}(\mathbb{S}^n)}\|v_2\|_{C^{s+\alpha}(\mathbb{S}^n)}\|v_3\|_{C^{\alpha}(\mathbb{S}^n)}.$$

Proof. The claim follows from [19, Lemma 4.4] once we show that the function

$$\psi(x) = \int_{\mathbb{S}^n} (y-x)K_A(y,x)\,d\mathcal{H}^n_y$$

is uniformly α-Hölder continuous. Indeed, even if the condition (ii) in Lemma 2.4 is slightly weaker than [19, Definition 4.1], the claim still follows from the proof of [19, Lemma 4.4]. Let us first prove the Hölder continuity of ψ under the additional assumption that at every point $x \in \mathbb{S}^n$ it holds

$$\langle A(x)x, \omega\rangle = 0 \qquad \text{for every } \omega \quad \text{orthogonal to } x. \tag{3.19}$$

To this aim fix x and notice that, by the symmetry of the sphere and by (3.19) for all vectors ω orthogonal to x, it holds

$$\int_{\mathbb{S}^n} \langle y-x, \omega\rangle K_A(y,x)\,d\mathcal{H}^n_y = 0.$$

Therefore

$$\psi(x) = \int_{\mathbb{S}^n} (y-x)K_A(y,x)\, d\mathcal{H}^n_y = x\int_{\mathbb{S}^n} \langle y-x,x\rangle K_A(y,x)\, d\mathcal{H}^n_y.$$

For all $y \in \mathbb{S}^n$ it holds $\langle y-x,x\rangle = -\frac{1}{2}|y-x|^2$. Therefore we have

$$\psi(x) = -\frac{x}{2}\int_{\mathbb{S}^n} |y-x|^2 K_A(y,x)\, d\mathcal{H}^n_y.$$

Obviously the function $F(y,x) = -\frac{x}{2}|y-x|^2$ satisfies the assumptions of Lemma 2.5 and therefore we deduce that $\|\psi\|_{C^\alpha(\mathbb{S}^n)} \le C$.

Let us then consider the general case. We write $A(x) = A_0(x) + A_1(x)$, where

$$A_0 = A - (Ax)\otimes x - x\otimes (Ax) + 2\langle Ax,x\rangle (x\otimes x)$$

and $A_1 = A - A_0$. Then $A_0(\cdot)$ is symmetric, elliptic and satisfies the condition (3.19). Moreover, recalling $\langle y-x,x\rangle = -\frac{1}{2}|y-x|^2$ and arguing as in (3.8) and (3.9) we observe that $g(y,x) = \langle A_1(x)(y-x), y-x\rangle$ satisfies the following two conditions: for $x,y \in \mathbb{S}^n$ it holds

$$|g(y,x)| \le C|y-x|^3 \tag{3.20}$$

and for all $x,y,z \in \mathbb{S}^n$ with $|z-x| \le \frac{1}{2}|y-x|$ it holds

$$|g(y,z) - g_\varphi(y,x)| \le C(|z-x||y-x|^2 + |z-x|^\alpha |y-x|^3). \tag{3.21}$$

Since the argument is similar as before, we leave the details for the reader. At the end, since we may write

$$\begin{aligned}\psi(x) &= \int_{\mathbb{S}^n} (y-x)K_A(y,x)\, d\mathcal{H}^n_y = \int_{\mathbb{S}^n} (y-x)K_{A_0}(y,x)\, d\mathcal{H}^n_y \\ &\quad + \int_{\mathbb{S}^n} (y-x)K_{A_0}(y,x)\int_0^1 \frac{d}{d\mu}\left(\frac{\|y-x\|^2_{A_0(x)}}{\|y-x\|^2_{A_0(x)} + \mu g(y,x)}\right)^{\frac{n+1+s}{2}} d\mu\, d\mathcal{H}^n_y\end{aligned}$$

the conclusion follows from Lemma 3.1. □

4. Schauder estimates for the fractional parabolic equation

In this section we extend Theorem 2.2 from $\mathbb{R}^n$ to the unit sphere $\mathbb{S}^n$. We consider linear operator of the form

$$L_A[u](x) := \int_{\mathbb{S}^n} \frac{u(y)-u(x)}{\|y-x\|^{n+1+s}_{A(x,t)}} a(x,t)b(y,t)\, d\mathcal{H}^n_y, \tag{4.1}$$

where the norm $\|\cdot\|_A$ is defined in (2.10), $A(\cdot,t)$ is symmetric elliptic matrix field for every $t \in [0,T)$ such that $\sup_{t<T}\|A(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)} \le C$ and the coefficients satisfy $a(x,t), b(x,t) \ge c > 0$ for all $x \in \mathbb{S}^n$ and $t \in [0,T)$ and

$$\sup_{t<T}\|a(\cdot,t)\|_{C^s(\mathbb{S}^n)} \le C \quad \text{and} \quad \sup_{t<T}\|b_0(\cdot,t)\|_{C^1(\mathbb{S}^n)} \le C.$$

We also assume without mentioning that all functions are continuous with respect to time.

We prove the following Schauder estimate on the sphere.

Theorem 4.1. *Let $T > 0$ and α such that $0 < \alpha < \min\{s, 1-s\}$. Assume that $A(\cdot,t)$ is symmetric and elliptic such that $\sup_{t<T} \|A(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)} \le C$ and let L_A be the operator defined in (4.1). Let $f : \mathbb{S}^n \times [0,T) \to \mathbb{R}$ be such that $\sup_{t<T} \|f(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)} < \infty$ and let $u : \mathbb{S}^n \times [0,T) \to \mathbb{R}$ be the solution of the problem*

$$\begin{cases} \partial_t u = L_A[u] + f(x,t) & (x,t) \in \mathbb{S}^n \times (0,T) \\ u(x,0) = 0 & x \in \mathbb{S}^n. \end{cases} \tag{4.2}$$

There exists a constant C_T such that

$$\sup_{t<T} \|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{S}^n)} \le C_T \sup_{t<T} \|f(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)}.$$

Proof. By approximation we may assume that all the functions in the assumptions are smooth. Moreover, we note that we may absorb the coefficient $a(x,t)$ in (4.1) into the matrix field $A(x,t)$ and thus obtain another symmetric elliptic matrix field which we still denote by $A(x,t)$. We also adopt the notation (2.9) for the kernel K_A, which means that

$$K_A(y,x) = \frac{1}{\|y-x\|_{A(x)}^{n+1+s}}.$$

First, it follows from the maximum principle that

$$\sup_{t<T} \|u(\cdot,t)\|_{C^0(\mathbb{S}^n)} \le T \sup_{t<T} \|f(\cdot,t)\|_{C^0(\mathbb{S}^n)}. \tag{4.3}$$

Indeed, fix a small $\varepsilon > 0$ and define $w_\varepsilon(x,t) = (t+\varepsilon)^{-1} u(x,t)$. Then w is smooth and $w(x,0) = 0$. Assume that w attains its maximum on $\mathbb{S}^n \times (0, T-\varepsilon]$ at (x_0,t_0) with $t_0 \in (0, T-\varepsilon]$. Then by the maximum principle it holds

$$0 \le \partial_t w(x_0,t_0) = \frac{\partial_t u(x_0,t_0)}{t_0+\varepsilon} - \frac{u(x_0,t_0)}{(t_0+\varepsilon)^2} \quad \text{and} \quad 0 \ge L_A[w](x_0,t_0) = \frac{L_A[u](x_0,t_0)}{t_0+\varepsilon}.$$

The equation (4.2) implies

$$\frac{u(x_0,t_0)}{t_0+\varepsilon} \le f(x_0,t_0).$$

By letting $\varepsilon \to 0$ and repeating the argument for $-u$ we obtain (4.3).

Let us fix $x_0 \in \mathbb{S}^n$ and by rotating the coordinates we may assume that it is the north pole, i.e., $x_0 = e_{n+1}$. Let us first localize the equation around x_0. To this aim fix a small $\delta > 0$ and choose a smooth cutoff function $\eta : \mathbb{R} \to [0,1]$ such that $\eta(r) = 1$ for $|r| < \delta/2$ and $\eta(r) = 0$ for $r \ge \delta$. In the following we write $x = (x', x_{n+1}) \in \mathbb{R}^{n+1}$ with $x' \in \mathbb{R}^n$. We denote

$$v(x,t) = \eta(4|x'|) u(x,t).$$

Then clearly $\|v(\cdot,t)\|_{C^\beta(\mathbb{S}^n)} \le C \|u(\cdot,t)\|_{C^\beta(\mathbb{S}^n \cap B_\delta(x_0))}$ for $\beta \in [0,2]$. In order to find an equation for v we multiply the equation (4.2) by $\eta(4|x'|)$ and obtain

$$\partial_t v(x,t) = \eta(4|x'|) L_A[u](x,t) + \eta(4|x'|) f(x,t).$$

We organize the terms as

$$\eta(4|x'|)\big(u(y,t) - u(x,t)\big) = v(y,t) - v(x,t) - u(y,t)(\eta(4|y'|) - \eta(4|x'|))$$

and write

$$\begin{aligned}\eta(4|x'|)L_A[u](x,t) &= \int_{\mathbb{S}^n} \eta(|y'|)\big(v(y,t)-v(x,t)\big)K_A(y,x)\,b(y,t)\,d\mathcal{H}^n_y \\ &\quad + \int_{\mathbb{S}^n} (1-\eta(|y'|))\big(v(y,t)-v(x,t)\big)K_A(y,x)\,b(y,t)\,d\mathcal{H}^n_y \\ &\quad - \int_{\mathbb{S}^n} u(y,t)\big(\eta(4|y'|)-\eta(4|x'|)\big)K_A(y,x)\,b(y,t)\,d\mathcal{H}^n_y.\end{aligned}$$

The last term is Hölder continuous due to Lemma 3.2. Since $v(x,t)=0$ for $|x'|\geq \delta/4$, then the Kernel $(1-\eta(|y'|))K_A(y,x)$ in the function

$$R_1(x,t) = \int_{\mathbb{S}^n} (1-\eta(|y'|))\big(v(y,t)-v(x,t)\big)K_A(y,x)\,b(y,t)\,d\mathcal{H}^n_y$$

is non-singular and therefore $\|R_1(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)} \leq C_\delta \|u(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)}$, where the constant depends on $\delta>0$. Therefore we deduce that it holds

$$\partial_t v(x,t) = \int_{\mathbb{S}^n} \eta(|y'|)\big(v(y,t)-v(x,t)\big)K_A(y,x)\,b(y,t)\,d\mathcal{H}^n_y + f_1(x,t) \tag{4.4}$$

and f_1 satisfies

$$\|f_1(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)} \leq C_\delta \|u(\cdot,t)\|_{C^{s+\alpha}(\mathbb{S}^n)} + C_\delta \|f(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)}. \tag{4.5}$$

We further manipulate the leading order term in (4.4) by writing

$$\begin{aligned}&\int_{\mathbb{S}^n} \eta(|y'|)\big(v(y,t)-v(x,t)\big)K_A(y,x)\,b(y,t)\,d\mathcal{H}^n_y \\ &\quad = \int_{\mathbb{S}^n} \eta(|y'|)\big(v(y,t)-v(x,t)\big)K_A(y,x)\,b(e_{n+1},t)\sqrt{1-|y'|^2}\,d\mathcal{H}^n_y + R_2(x,t),\end{aligned} \tag{4.6}$$

where

$$R_2(x,t) := \int_{\mathbb{S}^n} \eta(|y'|)\big(v(y,t)-v(x,t)\big)K_A(y,x)\,\big(b(y,t)-b(e_{n+1},t)\sqrt{1-|y'|^2}\big)\,d\mathcal{H}^n_y$$

Let us define $\psi(y,t)=\eta(|y'|)(b(y,t)-b(e_{n+1},t)\sqrt{1-|y'|^2})$. Then at the north pole it holds $\psi(e_{n+1},t)=0$. Therefore, since ψ is C^1-regular we have

$$\|\psi(\cdot,t)\|_{C^{s+\alpha}(\mathbb{S}^n)} \leq C\delta^{1-s-\alpha}.$$

Therefore again Lemma 3.2 yields

$$\|R_2(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)} \leq C\delta^{1-s-\alpha}\|u\|_{C^{1+s+\alpha}(\mathbb{S}^n\cap B_\delta(x_0))}. \tag{4.7}$$

Note that since $b(e_{n+1},t)$ depends only on time, we may absorb it into the matrix field $A(y,t)$ on the RHS of (4.6) and denote the new matrix field still by $A(y,t)$.

Let us then consider the RHS in (4.6). We write the unit sphere locally as a graph of the function $\varphi(x')=\sqrt{1-|x'|^2}$. Then we may write $y,x\in\mathbb{S}^n$ with $|y'|,|x'|<\delta$ as

$$y-x = \begin{pmatrix} y'-x' \\ \varphi(y')-\varphi(x')\end{pmatrix} = \begin{pmatrix} y'-x' \\ \langle \nabla\varphi(x'), y'-x'\rangle\end{pmatrix} + \begin{pmatrix} 0 \\ T_{x'}[\varphi](y')\end{pmatrix},$$

where $T_{x'}[\varphi](y') = \varphi(y') - \varphi(x') - \langle \nabla\varphi(x'), y' - x'\rangle$. Let us define the $(n+1\times n)$ - matrix field $J_\varphi(x')$ such that for $\xi' \in \mathbb{R}^n$ it holds

$$J_\varphi(x')\xi' = \begin{pmatrix} \xi' \\ \langle \nabla\varphi(x'), \xi'\rangle \end{pmatrix}.$$

Then we may write $y - x = J_\varphi(x')(y' - x') + T_{x'}[\varphi](y')e_{n+1}$. We may write

$$\langle A(x,t)(y-x),(y-x)\rangle = \langle \tilde{A}(x,t)(y'-x'),(y'-x')\rangle + g(y,x)$$

for

$$\tilde{A}(x,t) = J_\varphi(x')^T A(x,t) J_\varphi(x') \tag{4.8}$$

and

$$g(y,x) = 2T_{x'}[\varphi](y')\,\langle A(x,t)J_\varphi(x')(y'-x'), e_{n+1}\rangle + (T_{x'}[\varphi](y'))^2\langle A(x,t)e_{n+1}, e_{n+1}\rangle.$$

Using $|T_{x'}[\varphi](y')| \le C|y-x|^2$ and arguing as in (3.8) and (3.9), we deduce that g satisfies the conditions (3.20) and (3.21). We also note immediately that the new $(n\times n)$ - matrix field $\tilde{A}(\cdot,t)$ defined in (4.8) is symmetric and elliptic in $\mathbb{R}^n$, and also in this case, with a little abuse of notation, we write $\|\xi\|^2_{\tilde{A}(x,t)} = \langle \tilde{A}(x,t)\xi,\xi\rangle$ for $\xi \in \mathbb{R}^n$. We denote

$$G_\mu(y,x) = \frac{d}{d\mu}\left(\frac{\|y'-x'\|^2_{\tilde{A}(x,t)}}{\|y'-x'\|^2_{\tilde{A}(x,t)} - \mu g(y,x)}\right)^{\frac{n+1+s}{2}}$$

and write the RHS term in (4.6) as

$$\begin{aligned} &\int_{\mathbb{S}^n} \eta(|y'|)\big(v(y,t)-v(x,t)\big)K_A(y,x)\sqrt{1-|y'|^2}\,d\mathcal{H}^n_y \\ &\quad = \int_{\mathbb{S}^n} \eta(|y'|)\frac{v(y,t)-v(x,t)}{\|y'-x'\|^{n+1+s}_{\tilde{A}(x,t)}}\sqrt{1-|y'|^2}\,d\mathcal{H}^n_y + R_3(x,t), \end{aligned} \tag{4.9}$$

where

$$R_3(x,t) = \int_{\mathbb{S}^n}\int_0^1 \eta(|y'|)G_\mu(y,x)\frac{v(y,t)-v(x,t)}{\|y'-x'\|^{n+1+s}_{\tilde{A}(x,t)}}\sqrt{1-|y'|^2}\,d\mu\,d\mathcal{H}^n_y.$$

We may use Lemma 3.1 to deduce

$$\|R_3(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)} \le C\|u\|_{C^1(\mathbb{S}^n)}. \tag{4.10}$$

Finally we map the RHS term in (4.9) to $\mathbb{R}^n$, by slight abuse of notation we write $v(x',t) = v\big((x',\varphi(x')),t\big)$, $\tilde{A}(x',t) = \tilde{A}\big((x',\varphi(x')),t\big)$, and obtain

$$\begin{aligned} \int_{\mathbb{S}^n} \eta(|y'|)\frac{v(y,t)-v(x,t)}{\|y'-x'\|^{n+1+s}_{\tilde{A}(x,t)}}\sqrt{1-|y'|^2}\,d\mathcal{H}^n_y &= \int_{\mathbb{R}^n} \eta(|y'|)\frac{v(y',t)-v(x',t)}{\|y'-x'\|^{n+1+s}_{\tilde{A}(x',t)}}\,dy' \\ &= \int_{\mathbb{R}^n} \frac{v(y',t)-v(x',t)}{\|y'-x'\|^{n+1+s}_{\tilde{A}(x',t)}}\,dy' + R_4(x',t), \end{aligned}$$

where

$$R_4(x',t) = -\int_{\mathbb{R}^n}(1-\eta(|y'|))\frac{v(y',t)-v(x',t)}{\|y'-x'\|^{n+1+s}_{\tilde{A}(x',t)}}\,dy'.$$

Since v has compact support in $B_{\delta/4}$ and $\eta = 1$ in $B_{\delta/2}$, the integral in the definition of R_4 is non-singular and we deduce that

$$\|R_4(\cdot,t)\|_{C^\alpha(\mathbb{R}^n)} \le C_\delta \|v(\cdot,t)\|_{C^\alpha(\mathbb{R}^n)} \le C_\delta \|u(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)}.$$

We then use Theorem 2.2 and (4.4), (4.5), (4.6), (4.7), (4.9), (4.10) and the above to conclude that v is a solution of the equation

$$\partial_t v = L_{\tilde{A}}(v) + f_2$$

and satisfies

$$\begin{aligned}\sup_{t<T}\|v(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)} &\le C \sup_{t<T}\|f_2(\cdot,t)\|_{C^\alpha(\mathbb{R}^n)}\\ &\le \sup_{t<T}\big(C_\delta\|u(\cdot,t)\|_{C^1(\mathbb{S}^n)} + C\delta^{1-s-\alpha}\|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{S}^n\cap B_\delta(x_0))} + C_\delta\|f(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)}\big).\end{aligned}$$

We first observe that

$$\sup_{t<T}\|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{S}^n\cap B_{\delta/8}(x_0))} \le C\sup_{t<T}\|v(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)}.$$

We repeat the argument in balls $B_\delta(x_i)$ which cover the sphere, and use a standard covering argument to conclude that

$$\sup_{t<T}\|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{S}^n)} \le \sup_{t<T}\big(C_\delta\|u(\cdot,t)\|_{C^1(\mathbb{S}^n)} + C\delta^{1-s-\alpha}\|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{S}^n)} + C_\delta\|f(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)}\big).$$

Choosing $\delta > 0$ so small that $C\delta^{1-s-\alpha} \le \frac{1}{4}$ implies

$$\sup_{t<T}\|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{S}^n)} \le \sup_{t<T}\big(C_\delta\|u(\cdot,t)\|_{C^1(\mathbb{S}^n)} + \tfrac{1}{4}\|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{S}^n)} + C_\delta\|f(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)}\big).$$

We use the interpolation inequality (2.2) and Young's inequality

$$\|u(\cdot,t)\|_{C^1(\mathbb{S}^n)} \le C\|u(\cdot,t)\|^{\theta}_{C^{1+s+\alpha}(\mathbb{S}^n)}\|u(\cdot,t)\|^{1-\theta}_{C^0(\mathbb{S}^n)} \le \varepsilon\|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{S}^n)} + C_\varepsilon\|u(\cdot,t)\|_{C^0(\mathbb{S}^n)}.$$

This yields

$$\sup_{t<T}\|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{S}^n)} \le \sup_{t<T}\big((\varepsilon C_\delta + \tfrac{1}{4})\|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{S}^n)} + C_\varepsilon\|u(\cdot,t)\|_{C^0(\mathbb{S}^n)} + C_\delta\|f(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)}\big).$$

We then choose $\varepsilon > 0$ so small that $\varepsilon C_\delta \le \frac{1}{4}$ and have

$$\sup_{t<T}\|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{S}^n)} \le C_{\delta,\varepsilon}\sup_{t<T}\big(\|u(\cdot,t)\|_{C^0(\mathbb{S}^n)} + \|f(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)}\big).$$

The claim then follows from (4.3). □

5. $C^{2+\alpha}$-ESTIMATES FOR THE FLOW

Proof of the Main Theorem. By the result [19] the equation (1.1) has C^∞-solution for a time interval $(0,T_0)$ and we denote it by $(E_t)_{t\in(0,T_0)}$. We assume that T_0 is the maximal time of existence and our aim is to show that $T_0 = \infty$, i.e., the flow does not develop singularities. We argue by contradiction and assume $T_0 < \infty$.

By the result [8] the sets E_t are convex and by [10] they satisfy

$$\sup_{t<T_0}\|H^s_{E_t}\|_{L^\infty(\partial E_t)} \le C \tag{5.1}$$

where the constant is independent of T_0. Moreover again by [10] there are points $x_t \in \mathbb{R}^{n+1}$ such that

$$B_r(x_t) \subset E_t \subset B_R(x_t) \tag{5.2}$$

for $0 < r < R$, which are independent of T_0. By translation we may assume that $x_{T_0} = 0$. By (5.1) it holds that the normal velocity of the flow is bounded

$$\sup_{t<T_0} \|V_t\|_{L^\infty(\partial E_t)} \le C.$$

This implies that there is $t_0 \in (0, T_0)$ such that $B_{r/2} \subset E_t \subset B_{2R}$ for all $t \in [t_0, T_0]$. By restarting the flow at t_0 and by reparametrizing the time, we may assume that $t_0 = 0$ and denote the maximal time of existence by T.

From these results we deduce that we may parametrize the sets E_t via the height function over the unit sphere, i.e., there are $h(\cdot, t) : \mathbb{S}^n \to \mathbb{R}$ such that $h(x,t) \ge c_0 > 0$ and

$$\partial E_t = \{h(x,t)x : x \in \mathbb{S}^n\} \qquad \text{for all } t \in [0, T).$$

By (5.1) and Proposition 2.3 it holds

$$\sup_{t<T} \|h(\cdot,t)\|_{C^{1+s}(\mathbb{S}^n)} \le C. \tag{5.3}$$

As we mentioned in the introduction, our goal is to improve the estimate (5.3) to

$$\sup_{t<T} \|h(\cdot,t)\|_{C^{2+s+\alpha}(\mathbb{S}^n)} \le C, \tag{5.4}$$

where $\alpha > 0$ is such that $\alpha < \min\{\frac{s}{4}, 1-s\}$. The bound $\alpha < \frac{s}{4}$ is technical and the reason will be clear later. The authors in [6] (see also [10]) already point out that the estimate (5.4) implies that the maximal time of existence is infinity. Let us briefly recall why it is so. If we have (5.4) and if we assume $T < \infty$, then by continuity, the solution of the equation (1.1) exists up to the endpoint $[0, T]$. Since the set E_T is C^2-regular, by [19] we may restart the flow and obtain by the semigroup property that the solution exists for the time interval $[0, T+\delta]$ for some $\delta > 0$. This contradicts the fact that T is the maximal time of existence. Hence, once we have (5.4) we obtain that the flow exists globally in time. The fact that the flow $(E_t)_{t\ge 0}$ converges to some ball $B_1(x_0)$ exponentially fast follows from [6, Corollary 3.5]. Hence, we need to show (5.4).

By (3.12) we may write the equation (3.1) using the height functions as

$$\partial_t \nabla_i h = L_A[\nabla_i h] + R_0(x,t) + R_1(x,t) + \partial_t h\, B(x,h,\nabla_\tau h) : \nabla_\tau^2 h$$

with initial condition $h(\cdot, 0) = h_0$, where $h_0 \in C^\infty(\mathbb{S}^n)$. Here R_0 is defined in (3.15) and R_1 in (3.16). We define $u(x,t) := \nabla_i h(x,t) - \nabla_i h_0(x)$ and deduce that it is a solution of

$$\partial_t u = L_A[u] + f_i(x,t) + L_A[\nabla_i h_0]$$

with $u(x,0) = 0$ for

$$f_i(x,t) = R_0(x,t) + R_1(x,t) + \partial_t h\, B(x,h,\nabla_\tau h) : \nabla_\tau^2 h.$$

Theorem 4.1 implies

$$\sup_{t<T} \|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{S}^n)} \le C(\sup_{t<T} \|f_i(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)} + \|L_A[\nabla_i h_0]\|_{C^\alpha(\mathbb{S}^n)}). \tag{5.5}$$

We use this for every direction e_i and obtain

$$\sup_{t\le T_0} \|h(\cdot,t)\|_{C^{2+s+\alpha}(\mathbb{S}^n)} \le C \sup_{t\le T} \max_{i=1,\dots,n+1} \|f_i(\cdot,t)\|_{C^\alpha} + C(\|h_0\|_{C^{2+s+\alpha}} + \|L_A[\nabla_i h_0]\|_{C^\alpha}).$$

We are then left to estimate the RHS of (5.5), i.e., for every $i = 1, \dots, n+1$ we need to bound

$$\|L_A[\nabla_i h_0]\|_{C^\alpha(\mathbb{S}^n)} + \|R_0(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)} + \|R_1(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)} + \|\partial_t h\, B(x,h,\nabla_\tau h) : \nabla_\tau^2 h\|_{C^\alpha(\mathbb{S}^n)}. \tag{5.6}$$

The first term is easy to bound, since $h_0 \in C^\infty$. Indeed, by Lemma 3.2 it holds $\|L_A[\nabla_i h_0]\|_{C^\alpha(\mathbb{S}^n)} \le C\|h_0\|_{C^{2+s+\alpha}} \le C$.

For the second term on the RHS of (5.6) we recall that the term $R_0(\cdot,t)$ is given by (3.15). We denote

$$G_\mu(x,y) = \frac{d}{d\mu}\left(\frac{\|y-x\|^2_{A(x,t)}}{\|y-x\|^2_{A(x,t)} + \mu g_h(y,x)}\right)^{\frac{n+1+s}{2}} \tag{5.7}$$

and recall that by (3.8) and (3.9) the function g_h satisfies the conditions of Lemma 3.1 with $\kappa = C\|h\|_{C^{1+s+\alpha}}$. Since we may write

$$R_0(\cdot,t) = a_0(x,t)\int_{\mathbb{S}^n}\int_0^1 \frac{\nabla_i h(y,t) - \nabla_i h(x,t)}{\|y-x\|^{n+1+s}_{A(x,t)}} G_\mu(x,y) b_0(y,t)\, d\mu\, d\mathcal{H}^n_y$$

we may use Lemma 3.1 with $v_1 = \nabla_i h(\cdot,t), v_2 = b_0(\cdot,t)$ and $v_3 = a_0(\cdot,t)$ and recall that the coefficients a_0 and b_0 satisfy the conditions (3.14) to deduce

$$\sup_{t\le T}\|R_0(\cdot,t)\|_{C^\alpha} \le \sup_{t\le T} C\|h(\cdot,t)\|_{C^{1+s+\alpha}}\|h(\cdot,t)\|_{C^2}. \tag{5.8}$$

To treat the term $\|R_1(\cdot,t)\|_{C^\alpha(\mathbb{S}^n)}$ in (5.6) we recall that it is of the form (3.16). We fix t and write $F_k(x) = F(y,h(x,t)), a_k(x) = a_k(x,h,\nabla_\tau h)$ and $b_k(y) = b_k(y,h,\nabla_\tau h)$ for short, and note that then

$$\|F_k\|_{C^{k+\beta}} \le C\|h\|_{C^{k+\beta}}, \quad \|a_k\|_{C^{k+\beta}} \le C\|h\|_{C^{k+1+\beta}} \quad \text{and} \quad \|b_k\|_{C^{k+\beta}} \le C\|h\|_{C^{k+1+\beta}} \tag{5.9}$$

for all $k = 0,1,2,\dots$ and $\beta \in [0,1]$. We use the argument from (3.11) and write for $A(x,t) = h(x,t)^2 I + \nabla_\tau h(x,t)\otimes\nabla_\tau h(x,t)$

$$\begin{aligned}
&\int_{\mathbb{S}^n} a_k(x)b_k(y)\frac{F_k(y)-F_k(x)}{|h(y,t)y - h(x,t)x|^{n+1+s}}\, d\mathcal{H}^n_y \\
&= \overbrace{\int_{\mathbb{S}^n} a_k(x)b_k(y)\frac{F_k(y)-F_k(x)}{\|y-x\|^{n+1+s}_{A(x,t)}}\, d\mathcal{H}^n_y}^{=:\rho_1} \\
&+ \int_{\mathbb{S}^n} a_k(x)b_k(y)\frac{F_k(y)-F_k(x)}{\|y-x\|^{n+1+s}_{A(x,t)}}\int_0^1 \frac{d}{d\mu}\left(\frac{\|y-x\|^2_{A(x,t)}}{\|y-x\|^2_{A(x,t)} + \mu g_h(y,x)}\right)^{\frac{n+1+s}{2}} d\mu\, d\mathcal{H}^n_y = \rho_1(x) + \rho_2(x),
\end{aligned}$$

where g_h is given by (3.7). To estimate the Hölder norm of the first term ρ_1 on the RHS above, we use Lemma 3.2, (5.3),(5.9) and deduce that we may bound its C^α-norm by

$$\|\rho_1\|_{C^\alpha} \le C\|F_k\|_{C^{1+s+\alpha}}\|b_k\|_{C^{s+\alpha}}\|a_k\|_{C^\alpha} \le C\|h\|_{C^{1+s+\alpha}}\|h\|_{C^{1+s+\alpha}}\|h\|_{C^{1+\alpha}} \le C\|h\|^2_{C^{1+s+\alpha}}.$$

To estimate the Hölder norm of the second term ρ_2, we argue exactly as in (5.8), define G_μ as in (5.7) and recall that it satisfies the conditions of Lemma 3.1 with $\kappa = C\|h\|_{C^{1+s+\alpha}}$. We may then estimate $\|\rho_2\|_{C^\alpha}$ by Lemma 3.1 with $v_1 = F_k, v_2 = b_0$ and $v_3 = a_0$ and by (5.9) to deduce

$$\|\rho_2\|_{C^\alpha} \le C\|h\|_{C^{1+s+\alpha}}\|F_k\|_{C^1}\|b_0\|_{C^0}\|a_0\|_{C^\alpha} \le C\|h\|_{C^{1+s+\alpha}}\|h\|^2_{C^1}\|h\|_{C^{1+\alpha}} \le C\|h\|_{C^{1+s+\alpha}}.$$

In conclusion we have

$$\sup_{t\le T}\|R_1(\cdot,t)\|_{C^\alpha} \le \sup_{t\le T} C\|h(\cdot,t)\|^2_{C^{1+s+\alpha}} \le \sup_{t\le T} C\|h(\cdot,t)\|_{C^{1+s+\alpha}}\|h(\cdot,t)\|_{C^2}. \tag{5.10}$$

We are left with the last term in (5.6). We first use (5.3) and obtain

$$\|\partial_t h\, B(x,h,\nabla_\tau h) : \nabla_\tau^2 h\|_{C^\alpha} \le C\|\partial_t h \nabla_\tau^2 h\|_{C^\alpha} \le C\|\partial_t h\|_{C^\alpha}\|h\|_{C^{2+\alpha}}. \tag{5.11}$$

We claim that when $\alpha \le \frac{1}{2}$ it holds

$$\sup_{t\le T}\|\partial_t h\|_{C^\alpha} \le \sup_{t\le T} C(1+\|h\|_{C^{2+\alpha}}^{2\alpha}). \tag{5.12}$$

Note that the assumption $\alpha < \min\{\frac{s}{4}, 1-s\}$ implies $\alpha < \frac{1}{2}$.

To this aim we recall that the normal velocity of the flow is bounded, i.e.,

$$\sup_{t\le T}\|V_t\|_{L^\infty(\partial E_t)} \le C.$$

We then use the parametrization of the normal velocity (3.2), the a priori estimates $h \ge c_0$ and (5.3) to deduce that

$$\sup_{t\le T}\|\partial_t h\|_{C^0} \le C.$$

We use the interpolation inequality from Proposition 2.1 to estimate

$$\|\partial_t h\|_{C^\alpha} \le C\|\partial_t h\|_{C^1}^{\alpha}\|\partial_t h\|_{C^0}^{1-\alpha} \le C\|\partial_t h\|_{C^1}^{\alpha}. \tag{5.13}$$

Recall that $\nabla_i h$ is a solution of the equation

$$\partial_t \nabla_i h = L_A[\nabla_i h] + f_i(x,t).$$

Lemma 3.2 yields $\|L_A[\nabla_i h]\|_{C^\alpha} \le C\|h\|_{C^{2+s+\alpha}}$ and thus by Theorem 4.1 we have

$$\sup_{t\le T}\|\partial_t h\|_{C^1} \le \sup_{t\le T}\|\partial_t h\|_{C^{1+\alpha}} \le \sup_{t\le T}\max_{i=1,\dots,n+1} C(1+\|f_i(\cdot,t)\|_{C^\alpha}).$$

We use this, (5.8), (5.10), (5.11), (5.13) and Young's inequality to estimate

$$\begin{aligned}
\sup_{t\le T}\|\partial_t h\|_{C^1} &\le C\sup_{t\le T}\big(1+\|h\|_{C^{1+s+\alpha}}\|h\|_{C^2} + \|\partial_t h\|_{C^\alpha}\|h\|_{C^{2+\alpha}}\big)\\
&\le \sup_{t\le T}\big(C + C\|h\|_{C^2}^2 + C\varepsilon\|\partial_t h\|_{C^\alpha}^{\frac{1}{\alpha}} + C_\varepsilon\|h\|_{C^{2+\alpha}}^{\frac{1}{1-\alpha}}\big)\\
&\le \sup_{t\le T}\big(C + C\|h\|_{C^2}^2 + C\varepsilon\|\partial_t h\|_{C^1} + C_\varepsilon\|h\|_{C^{2+\alpha}}^{\frac{1}{1-\alpha}}\big).
\end{aligned}$$

By choosing ε so small that $C\varepsilon \le \frac{1}{2}$ we have

$$\sup_{t\le T}\|\partial_t h\|_{C^1} \le C\sup_{t\le T}\big(1+\|h\|_{C^2}^2 + \|h\|_{C^{2+\alpha}}^{\frac{1}{1-\alpha}}\big).$$

When $\alpha \le \frac{1}{2}$ it holds $\frac{1}{1-\alpha} \le 2$. Therefore we obtain (5.12) from (5.13) and from the above.

Now it follows from (5.11) and (5.12) that

$$\sup_{t\le T}\|\partial_t h\, B(x,h,\nabla_\tau h) : \nabla_\tau^2 h\|_{C^\alpha} \le \sup_{t\le T} C(1+\|h\|_{C^{2+\alpha}}^{1+2\alpha}).$$

Therefore we obtain by (5.5), (5.8), (5.10) and from the above that

$$\sup_{t\le T_0}\|h(\cdot,t)\|_{C^{2+s+\alpha}} \le \sup_{t\le T} C\big(1+\|h(\cdot,t)\|_{C^{1+s+\alpha}}\|h(\cdot,t)\|_{C^2} + \|h(\cdot,t)\|_{C^{2+\alpha}}^{1+2\alpha}\big).$$

We use the interpolation inequality from Proposition 2.1 to estimate first

$$\|h(\cdot,t)\|_{C^{1+s+\alpha}} \le C\|h(\cdot,t)\|_{C^{2+s+\alpha}}^{\frac{\alpha}{1+\alpha}}\|h(\cdot,t)\|_{C^{1+s}}^{\frac{1}{1+\alpha}}$$

and then

$$\|h(\cdot,t)\|_{C^2} \le \|h(\cdot,t)\|_{C^{2+\alpha}} \le C\|h(\cdot,t)\|_{C^{2+s+\alpha}}^{\frac{1-s+\alpha}{1+\alpha}}\|h(\cdot,t)\|_{C^{1+s}}^{\frac{s}{1+\alpha}}.$$

Recall that (5.5) implies $\sup_{t<T}\|h(\cdot,t)\|_{C^{1+s}} \le C$. Therefore we obtain by combining the above inequalities

$$\sup_{t<T}\|h(\cdot,t)\|_{C^{2+s+\alpha}} \le \sup_{t<T} C\Big(1+\|h(\cdot,t)\|_{C^{2+s+\alpha}}^{\frac{1-s+2\alpha}{1+\alpha}} + \|h(\cdot,t)\|_{C^{2+s+\alpha}}^{\frac{1-s+\alpha}{1+\alpha}(1+2\alpha)}\Big). \tag{5.14}$$

Recall that we choose $\alpha < \frac{s}{4}$. This implies

$$\frac{1-s+2\alpha}{1+\alpha} < \frac{1}{1+\alpha} < 1 \quad \text{and} \quad \frac{1-s+\alpha}{1+\alpha}(1+2\alpha) < \frac{1}{1+\alpha} < 1.$$

Thefore we obtain from (5.14) that

$$\sup_{t<T}\|h(\cdot,t)\|_{C^{2+s+\alpha}} \le C\Big(1+\sup_{t<T}\|h(\cdot,t)\|_{C^{2+s+\alpha}}^{\frac{1}{1+\alpha}}\Big)$$

which implies

$$\sup_{t<T}\|h(\cdot,t)\|_{C^{2+s+\alpha}} \le C$$

and the claim (5.4) follows. □

Appendix A.

In the appendix we give a self-contained proof of Theorem 2.2. Recall that we consider linear operator

$$L_A[u](x) = \int_{\mathbb{R}^n} \frac{u(y+x)-u(x)}{\langle A(x,t)y,y\rangle^{\frac{n+1+s}{2}}}dy = \int_{\mathbb{R}^n} \frac{u(y)-u(x)}{\langle A(x,t)(y-x),(y-x)\rangle^{\frac{n+1+s}{2}}}dy.$$

In the following, we denote by $\mathcal{F}[u](\xi)$ the Fourier transform of u evaluated at a point ξ

$$\mathcal{F}[u](\xi) = \int_{\mathbb{R}^n} u(x)e^{-2\pi i\langle\xi,x\rangle}\,dx$$

and denote by $\mathcal{F}^{-1}$ the inverse Fourier transform, i.e.,

$$\mathcal{F}^{-1}[v](x) = \int_{\mathbb{R}^n} v(\xi)e^{2\pi i\langle\xi,x\rangle}\,d\xi.$$

The proof of Theorem 2.2 follows from a small perturbation argument. We begin with an easy lemma.

Lemma A.1. *Let $A(t)$ be elliptic and symmetric matrix which is constant w.r.t x. Then*

$$\mathcal{F}\big(L_A[u]\big)(\xi) = -a(\xi,t)|\xi|^{1+s}\mathcal{F}[u](\xi) \tag{A.1}$$

where $a(\xi,t)$ is 0-homogeneous function w.r.t to ξ given by

$$a(\xi,t) = \int_{\mathbb{S}^{n-1}}\int_0^\infty \frac{1-\cos(2\pi r\langle\frac{\xi}{|\xi|},\omega\rangle)}{r^{2+s}\langle A(t)\omega,\omega\rangle^{\frac{n+1+s}{2}}}\,dr\,d\mathcal{H}^{n-1}_\omega.$$

Moreover, it holds

$$\frac{1}{C} \le a(\xi,t) \le C \quad \text{and} \quad |\partial^\beta a(\xi,t)| \le C_{|\beta|}\frac{1}{|\xi|^{|\beta|}} \qquad \text{for } \xi \ne 0 \text{ and } t\in(0,T). \tag{A.2}$$

Proof. Since $A(t)$ is constant with respect to the space and we are applying Fourier transform with respect to space, we drop the time dependence. Setting $u_y(x) := u(y+x)$, we have $\mathcal{F}[u_y](\xi) = e^{2\pi i\langle \xi, y\rangle}\mathcal{F}[u](\xi)$. By linearity of the Fourier transform, we get

$$\mathcal{F}[L_A u](\xi) = \mathcal{F}\left[\int_{\mathbb{R}^n} \frac{u(y+x)-u(x)}{\langle A\,y, y\rangle^{\frac{n+1+s}{2}}}\,dy\right](\xi) = \int_{\mathbb{R}^n} \frac{\mathcal{F}[u_y](\xi)-\mathcal{F}[u](\xi)}{\langle A\,y, y\rangle^{\frac{n+1+s}{2}}}\,dy$$

$$= -\mathcal{F}[u](\xi)\int_{\mathbb{R}^n} \frac{1-\cos(2\pi\langle \xi, y\rangle)}{\langle A\,y, y\rangle^{\frac{n+1+s}{2}}}\,dy$$

$$= -|\xi|^{1+s}\mathcal{F}[u](\xi)\int_{\mathbb{S}^{n-1}}\int_0^\infty \frac{1-\cos(2\pi r\langle \frac{\xi}{|\xi|}, \omega\rangle)}{r^{2+s}\langle A\omega, \omega\rangle^{\frac{n+1+s}{2}}}\,dr\,d\mathcal{H}^{n-1}_\omega.$$

From this formula and the fact that $a(\xi)$ is 0-homogeneous it is immediate to obtain the bounds (A.2) □

In order to prove Theorem 2.2, we first consider the case of coefficients which are constant in space. To that aim we recall that we may characterize the Hölder continuity by its Fourier transform using the Hardy-Littlewood decomposition. Let $\eta \in C_0^\infty(\mathbb{R}^n)$be such that $0 \le \eta \le 1$, $\operatorname{supp}\eta \subset \bar{B}_2$, and $\eta(\xi) = 1$ for $\xi \in B_1$. Define then $\delta(\xi) = \eta(\xi) - \eta(2\xi)$. Then the functions $\delta(2^{-j}\xi)$ form a partition of unity, i.e.

$$1 = \sum_{j=-\infty}^{\infty} \delta(2^{-j}\xi), \qquad \text{for } \xi \neq 0.$$

Next we define $\Psi : \mathbb{R}^n \to \mathbb{R}$ via its Fourier transform

$$\mathcal{F}[\Psi](\xi) = \delta(\xi) = \eta(\xi) - \eta(2\xi).$$

We write $\Psi_t(x) = t^{-n}\Psi(x/t)$. Then we have by scaling $\mathcal{F}[\Psi_{2^{-j}}](\xi) = \delta(2^{-j}\xi)$. Finally we define the operator Δ_j by convolution

$$\Delta_j(f) := f * \Psi_{2^{-j}}.$$

Note that since $\mathcal{F}[u * v] = \mathcal{F}[u]\cdot\mathcal{F}[v]$ we may write

$$\Delta_j(f) = \mathcal{F}^{-1}[\mathcal{F}[f]\cdot\delta(2^{-j}\xi)].$$

We recall that when $\gamma > 0$ is not an integer it holds (see e.g. [27])

$$\frac{1}{C}\|f\|_{C^\gamma(\mathbb{R}^n)} \le \sup_{j\ge 1} 2^{j\gamma}\|\Delta_j(f)\|_{L^\infty} \le C\|f\|_{C^\gamma(\mathbb{R}^n)}. \tag{A.3}$$

We assume that $A(t)$ is an elliptic and symmetric matrix field which is continuous w.r.t to time.

Theorem A.2. *Let $u \in C^{1+s+\alpha}(\mathbb{R}^n)$ be a solution of*

$$\begin{cases} \partial_t u = L_{A(t)}[u] + f(x,t), \\ u(x,0) = 0 \end{cases} \tag{A.4}$$

where the matrix field is symmetric, elliptic, continuous w.r.t. time and constant in space. For $\alpha < \min\{s, 1-s\}$ it holds

$$\sup_{t<T}\|u(\cdot,t)\|_{C^{1+s+\alpha}} \le C_T \sup_{t<T}\|f(\cdot,t)\|_{C^\alpha},$$

where the constant C depends only on α, s, n, T and on the ellipticity constants of $A(\cdot)$.

Proof. Applying Fourier transform to the equation (2.5) and using Lemma A.1 we obtain

$$\partial_t \mathcal{F}[u](\xi,t) = -a(\xi,t)|\xi|^{1+s}\mathcal{F}[u](\xi,t) + \mathcal{F}[f](\xi,t).$$

Multiplying the above by $e^{|\xi|^{1+s}\int_0^t a(\xi,\tau)\,d\tau}$ we have

$$\partial_t\big(\mathcal{F}[u](\xi,t)e^{g(\xi,t)|\xi|^{1+s}}\big) = \mathcal{F}[f](\xi,t)e^{|\xi|^{1+s}g(\xi,t)}$$

where we have set $g(\xi,t) = \int_0^t a(\xi,\tau)\,d\tau$. Integrate over $(0,t)$, recall that $u(x,0)=0$, and get

$$\mathcal{F}[u](\xi,t)e^{g(t,\xi)|\xi|^{1+s}} = \int_0^t \mathcal{F}[f](\xi,\tau)e^{g(\tau,\xi)|\xi|^{1+s}}\,d\tau.$$

Thus we may write the solution as

$$\mathcal{F}[u](\xi,t) = \int_0^t \mathcal{F}[f](\xi,\tau)e^{-(g(\xi,t)-g(\xi,\tau))|\xi|^{1+s}}\,d\tau.$$

We need to estimate $\|\Delta_j(u(\cdot,t))\|_{L^\infty}$. Since δ is a cutoff function it is enough to estimate

$$\|\mathcal{F}^{-1}\big(\mathcal{F}[u](\xi,t)\cdot\delta^2(2^{-j}\xi)\big)\|_{L^\infty} \le \int_0^t \|\mathcal{F}^{-1}\big(e^{-(g(\xi,t)-g(\xi,\tau))|\xi|^{1+s}}\mathcal{F}[f](\xi,\tau)\delta^2(2^{-j}\xi)\big)\|_{L^\infty}\,d\tau.$$

Since the product becomes a convolution we have

$$\begin{aligned}
&\|\mathcal{F}^{-1}\big(e^{-(g(\xi,t)-g(\xi,\tau))|\xi|^{1+s}}\mathcal{F}[f](\xi,\tau)\delta^2(2^{-j}\xi)\big)\|_{L^\infty}\\
&\quad= \|\mathcal{F}^{-1}\big(e^{-(g(\xi,t)-g(\xi,\tau))|\xi|^{1+s}}\delta(2^{-j}\xi)\big) * \mathcal{F}^{-1}\big(\mathcal{F}[f](\xi,\tau)\delta(2^{-j}\xi)\big)\|_{L^\infty}\\
&\quad\le \|\mathcal{F}^{-1}\big(e^{-(g(\xi,t)-g(\xi,\tau))|\xi|^{1+s}}\delta(2^{-j}\xi)\big)\|_{L^1}\,\|\mathcal{F}^{-1}\big(\mathcal{F}[f](\xi,\tau)\delta(2^{-j}\xi)\big)\|_{L^\infty}.
\end{aligned}$$

Since $f(\cdot,t)\in C^\alpha(\mathbb{R}^n)$ for every $\tau<t$ we have by (A.3)

$$\|\mathcal{F}^{-1}\big(\mathcal{F}[f](\xi,\tau)\delta(2^{-j}\xi)\big)\|_{L^\infty} \le C2^{-j\alpha}\sup_{\tau<t}\|f(\cdot,t)\|_{C^\alpha(\mathbb{R}^n)}.$$

We need therefore to show

$$\int_0^t \|\mathcal{F}^{-1}\big(e^{-(g(\xi,t)-g(\xi,\tau))|\xi|^{1+s}}\delta(2^{-j}\xi)\big)\|_{L^1}\,d\tau \le C2^{-j(1+s)}. \tag{A.5}$$

Obviously it is enough to prove (A.5) for $j\ge 2$. Recalling the definition of $g(\xi,t)$ and using (A.2) we have for $\tau<t$

$$g(\xi,t)-g(\xi,\tau)\ge c_0(t-\tau) \quad\text{and}\quad |\partial^\beta(g(\xi,t)-g(\xi,\tau))| \le C_{|\beta|}\frac{t-\tau}{|\xi|^{|\beta|}}.$$

Using these one may prove that, if $\gamma=(\gamma_1,\dots,\gamma_n)$ is a multi index of length n, then we have

$$\begin{aligned}
\big|\partial_\xi^\gamma e^{-(g(\xi,t)-g(\xi,\tau))|\xi|^{1+s}}\big| &\le C\sum_{k\le n}(1+(t-\tau)|\xi|^{1+s})^k|\xi|^{-n}e^{-(g(\xi,t)-g(\xi,\tau))|\xi|^{1+s}}\\
&\le C(1+(t-\tau)|\xi|^{1+s})^n|\xi|^{-n}e^{-c_0(t-\tau)|\xi|^{1+s}},
\end{aligned}$$

where the constant C depends on the index n. Therefore we have by the above and by $\delta(2^{-j}\xi)=0$ for $\xi\notin B_{2^{j+1}}\setminus B_{2^{j-1}}$ that

$$|\partial_\xi^\gamma(e^{-(g(\xi,t)-g(\xi,\tau))|\xi|^{1+s}}\delta(2^{-j}\xi))| \le C(1+(t-\tau)|\xi|^{1+s})^n|\xi|^{-n}e^{-c_0(t-\tau)|\xi|^{1+s}}\tilde{\delta}(2^{-j}\xi),$$

where $\tilde{\delta}$ is a smooth function such that $\tilde{\delta}(\xi) = 0$ for $\xi \notin B_{2^{j+1}} \setminus B_{2^{j-1}}$. Using Cauchy-Schwarz inequality, Plancherel's theorem and recalling that $\delta(2^{-j}\xi) = 0$ for $\xi \notin B_{2^{j+1}} \setminus B_{2^{j-1}}$ have that

$$\begin{aligned}
&\|\mathcal{F}^{-1}\big(e^{-(g(\xi,t)-g(\xi,\tau))|\xi|^{1+s}}\delta(2^{-j}\xi)\big)\|_{L^1} \\
&\quad\le \left(\int_{\mathbb{R}^n} \frac{1}{(1+|2^j x|^n)^2}dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} (1+|2^j x|^n)^2 |\mathcal{F}^{-1}\big(e^{-(g(\xi,t)-g(\xi,\tau))|\xi|^{1+s}}\delta(2^{-j}\xi)\big)|^2\,dx\right)^{\frac{1}{2}} \\
&\quad\le C2^{-\frac{jn}{2}} \left(\sum_{|\gamma|\le n}\int_{\mathbb{R}^n} \big|2^{jn}\partial_\xi^\gamma(e^{-(g(\xi,t)-g(\xi,\tau))|\xi|^{1+s}}\delta(2^{-j}\xi))\big|^2 d\xi\right)^{\frac{1}{2}} \\
&\quad\le C2^{-\frac{jn}{2}} \left(\int_{B_{2^{j+1}}\setminus B_{2^{j-1}}} |2^{jn}(1+(t-\tau)|\xi|^{1+s})^n|\xi|^{-n}e^{-c_0(t-\tau)|\xi|^{1+s}}|^2 dx\right)^{\frac{1}{2}} \\
&\quad\le C(1+(t-\tau)2^{(j+1)(1+s)})^n e^{-c_0(t-\tau)2^{(j-1)(1+s)}},
\end{aligned}$$

where in the last inequality we used that $\xi \in B_{2^{j+1}} \setminus B_{2^{j-1}}$ and the assumption $j \ge 2$. Finally we have

$$\begin{aligned}
\int_0^t \|\mathcal{F}^{-1}\big(e^{-(g(\xi,t)-g(\xi,\tau))|\xi|^{1+s}}\delta(2^{-j}\xi)\big)\|_{L^1}\,d\tau &\le C\int_0^t (1+(t-\tau)2^{(j+1)(1+s)})^n e^{-c_0(t-\tau)2^{(j-1)(1+s)}}d\tau \\
&\le C2^{-(j-1)(1+s)}\int_0^\infty (1+2^{2(1+s)}\mu)^n e^{-c_0\mu}\,d\mu \le C2^{-j(1+s)},
\end{aligned}$$

and we obtain (A.5). □

We are now ready prove Theorem 2.2. The proof follows from small perturbation argument, where we localize the equation and freeze the coefficients. The argument is similar to the proof of Theorem 4.1, with the difference that the Euclidean space is not compact. We may overcome this by using Remark 2.6 instead of Lemma 2.5.

Proof of Theorem 2.2. We begin by noticing that applying the maximum principle, as in the proof of Theorem 4.1, we have

$$\sup_{t<T}\|u(\cdot,t)\|_{C^0(\mathbb{R}^n)} \le C(1+T)\sup_{t<T}\|f(\cdot,t)\|_{C^0(\mathbb{R}^n)}. \tag{A.6}$$

For the Hölder continuity we localize the equation. First, fix $\delta \in (0,1)$ and choose $t_0 \in (0,T)$ and $x_0 \in \mathbb{R}^n$ such that

$$\sup_{t<T}\sup_{\substack{y\ne x\in\mathbb{R}^n\\ |y-x|<\delta/2}} \frac{|\nabla u(y,t)-\nabla u(x,t)|}{|y-x|^{s+\alpha}} \le 2\sup_{y\in B_{\delta/2}(x_0)} \frac{|\nabla u(y,t_0)-\nabla u(x_0,t_0)|}{|y-x_0|^{s+\alpha}} \tag{A.7}$$

Let $\eta:\mathbb{R}\to[0,1]$ be a smooth cutoff function such that $\eta(r) = 1$ for $|r| \le \frac{1}{2}$, $\eta(r) = 0$ for $|r| \ge 1$, denote

$$\eta_{x_0}(x) = \eta\left(\frac{|x-x_0|}{\delta}\right)$$

and let $v = u(x,t)\eta_{x_0}(x)$. Notice that

$$L_A[v] = \eta_{x_0}L_A[u] + \int_{\mathbb{R}^n} u(y,t)\frac{\eta_{x_0}(y)-\eta_{x_0}(x)}{\|y-x\|_{A(x,t)}^{n+1+s}}\,dy.$$

If u is a solution of the equation $\partial_t u = L_A[u] + f(x,t)$, then $v = u\eta_{x_0}$ solves

$$\partial_t v = L_{A_{x_0}}[v] + \tilde{f}(x,t),$$

where $A_{x_0} = A(x_0,t)$ and

$$\tilde{f} = \eta_{x_0} f + \overbrace{(L_A - L_{A_{x_0}})[v]}^{=I_1} - \int_{\mathbb{R}^n} u(y,t)\frac{\eta_{x_0}(y) - \eta_{x_0}(x)}{\|y-x\|^{n+1+s}_{A(x,t)}}\,dy = \eta_{x_0} f + I_1 - I_2.$$

Theorem A.2 implies

$$\sup_{(0,T)} \|v(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)} \le C \sup_{(0,T)} \|\tilde{f}(\cdot,t)\|_{C^{\alpha}(\mathbb{R}^n)}.$$

We claim that it holds

(A.8)
$$\sup_{(0,T)} (\|I_1\|_{C^\alpha(\mathbb{R}^n)} + \|I_2\|_{C^\alpha(\mathbb{R}^n)}) \le C\delta^\alpha \sup_{(0,T)} \|v(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)} + C_\delta \sup_{(0,T)} \|u(\cdot,t)\|_{C^{1+s+\alpha/2}(\mathbb{R}^n)}.$$

Since in the argument to prove (A.8) the time does not play any role, we drop it in order to simplify the notation. For $\mu \in [0,1]$ we denote $A_\mu(x) = (1-\mu)A(x_0) + \mu A(x)$ and write I_1 as

$$\begin{aligned}
(L_A - L_{A_{x_0}})[v](x) &= \int_{\mathbb{R}^n} (v(y) - v(x)) \int_0^1 \frac{d}{d\mu}\left(\frac{1}{\langle A_\mu(x)(y-x),(y-x)\rangle}\right)^{\frac{n+1+s}{2}} d\mu\, dy \\
&= -\tfrac{n+1+s}{2} \int_{\mathbb{R}^n}\int_0^1 \frac{(v(y)-v(x))}{\langle A_\mu(x)(y-x),(y-x)\rangle^{\frac{n+1+s}{2}}}\left(\frac{\langle (A(x)-A(x_0))(y-x),(y-x)\rangle}{\langle A_\mu(x)(y-x),(y-x)\rangle}\right) d\mu\, dy \\
&= -\tfrac{n+1+s}{2} \int_{\mathbb{R}^n}\int_0^1 \frac{(v(y)-v(x)-\eta(|y-x|)\langle \nabla v(x), y-x\rangle)}{\langle A_\mu(x)(y-x),(y-x)\rangle^{\frac{n+1+s}{2}}}\left(\frac{\langle (A(x)-A(x_0))(y-x),(y-x)\rangle}{\langle A_\mu(x)(y-x),(y-x)\rangle}\right) d\mu\, dy,
\end{aligned}$$

where the last equality follows from symmetry. Let us define $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$F(y,x) = (v(y) - v(x) - \eta(|y-x|)\langle \nabla v(x), y-x\rangle)\left(\frac{\langle (A(x)-A(x_0))(y-x),(y-x)\rangle}{\langle A_\mu(x)(y-x),(y-x)\rangle}\right),$$

for $y \neq x$ and $F(x,x) = 0$. Then one may check that F satisfies the conditions in Remark 2.6 with a constant

$$\kappa_0 = C\delta^\alpha \|v\|_{C^{1+s+\alpha}} + C_\delta \|v\|_{C^{1+s+\alpha/2}}.$$

We leave the details for the reader as it follows from (3.10) and using an argument similar to the one in the proof of Lemma 3.1 . We may thus use Remark 2.6 and obtain

$$\begin{aligned}
\|I_1(\cdot,t)\|_{C^\alpha(\mathbb{R}^n)} &\le C\delta^\alpha \|v(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)} + C_\delta \|v(\cdot,t)\|_{C^{1+s+\alpha/2}(\mathbb{R}^n)} \\
&\le C\delta^\alpha \|v(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)} + C_\delta \|u(\cdot,t)\|_{C^{1+s+\alpha/2}(\mathbb{R}^n)}.
\end{aligned}$$

The argument to bound the term I_2 is similar. We write it as

$$\begin{aligned}
I_2 = \int_{\mathbb{R}^n} u(y)\frac{(\eta_{x_0}(y) - \eta_{x_0}(x))}{\|y-x\|^{n+1+s}_{A(x)}}\,dy &= \int_{\mathbb{R}^n} \frac{(u(y)-u(x))(\eta_{x_0}(y) - \eta_{x_0}(x))}{\|y-x\|^{n+1+s}_{A(x)}}\,dy \\
&+ u(x)\int_{\mathbb{R}^n} \frac{\eta_{x_0}(y) - \eta_{x_0}(x) - \eta(|y-x|)\langle \nabla \eta_{x_0}(x), y-x\rangle}{\|y-x\|^{n+1+s}_{A(x)}}\,dy.
\end{aligned}$$

We apply Remark 2.6, by first choosing

$$F_1(y,x) = (u(y) - u(x))(\eta_{x_0}(y) - \eta_{x_0}(x))$$

and then

$$F_2(y,x) = \eta_{x_0}(y) - \eta_{x_0}(x) - \eta(|y-x|)\langle \nabla \eta_{x_0}(x), y - x\rangle$$

to infer

$$\|I_2(\cdot,t)\|_{C^\alpha(\mathbb{R}^n)} \le C_\delta \|u(\cdot,t)\|_{C^1(\mathbb{R}^n)}.$$

Hence we have (A.8).

Let us finally show how the claim follows from (A.8). First, by choosing δ small we have

$$\sup_{t<T} \|v(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)} \le C_\delta \sup_{t<T}(\|u(\cdot,t)\|_{C^{1+s+\alpha/2}(\mathbb{R}^n)} + \|f(\cdot,t)\|_{C^\alpha(\mathbb{R}^n)}).$$

By using $\|u(\cdot,t)\|_{C^{1+s+\alpha}(B_{\delta/2}(x_0))} \le \|v(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)}$ and by the choice of x_0 and t_0 in (A.7) we deduce

$$\sup_{t<T} \|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)} \le C_\delta \sup_{t<T}(\|u(\cdot,t)\|_{C^{1+s+\alpha/2}(\mathbb{R}^n)} + \|f(\cdot,t)\|_{C^\alpha(\mathbb{R}^n)}).$$

By interpolation (2.2) we have

$$\begin{aligned}\|u(\cdot,t)\|_{C^{1+s+\alpha/2}(\mathbb{R}^n)} &\le C\|u(\cdot,t)\|^{\theta}_{C^{1+s+\alpha}(\mathbb{R}^n)}\|u(\cdot,t)\|^{1-\theta}_{C^0(\mathbb{R}^n)}\\ &\le \varepsilon\|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)} + C_\varepsilon\|u(\cdot,t)\|_{C^0(\mathbb{R}^n)}.\end{aligned}$$

By choosing ε small we then have

$$\sup_{t<T} \|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)} \le C \sup_{t<T}(\|f(\cdot,t)\|_{C^\alpha(\mathbb{R}^n)} + \|u(\cdot,t)\|_{C^0(\mathbb{R}^n)}).$$

The claim then follows from (A.6). □

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References

[1] B. Barrios, A. Figalli & E. Valdinoci, *Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **13** (2014), 609–639.

[2] X. Cabré, M.M. Fall, J. Solá-Morales & T. Weth, *Curves and surfaces with constant nonlocal mean curvature: meeting Alexandrov and Delaunay.* J. Reine Angew. Math. **745** (2018), 253–280.

[3] L. Caffarelli, J.-M. Roquejoffre & O. Savin, *Nonlocal minimal surfaces.* Comm. Pure Appl. Math. **63** (2010), 1111–1144.

[4] A. Cesaroni, S. Dipierro, M. Novaga & E. Valdinoci, *Fattening and nonfattening phenomena for planar nonlocal curvature flows.* Math. Ann. **375** (2019), 687–736.

[5] A. Cesaroni & M. Novaga, *Fractional mean curvature flow of Lipschitz graphs.* Manuscripta Mathematica **170** (2021), 427–451.

[6] A. Cesaroni & M. Novaga, *Stability of the ball under volume preserving fractional mean curvature flow*, Advances in Calculus of Variations, (2022)

[7] A. Chambolle, M. Morini & M. Ponsiglione, *Nonlocal curvature flows.* Arch. Ration. Mech. Anal. **218** (2015), 1263–1329.

[8] A. Chambolle, M. Novaga, & B. Ruffini, *Some results on anisotropic fractional mean curvature flows.* Interfaces Free Bound. **19**(2017), 393–415.

[9] E. Cinti, C. Sinestrari & E. Valdinoci, *Neckpinch singularities in fractional mean curvature flows.* Proc. Amer. Math. Soc. **146** (2018), 2637–2646.

[10] E. Cinti, C. Sinestrari & E. Valdinoci, *Convex sets evolving by volume preserving fractional mean curvature flows.* Analysis and PDE, **13** (2020), 2149-2171.

[11] G. Ciraolo, A. Figalli, F. Maggi & M. Novaga, *Rigidity and sharp stability estimates for hypersurfaces with constant and almost-constant nonlocal mean curvature.* J. Reine Angew. Math. **741** (2018), 275–294.

[12] J. Dávila, M. del Pino & J. Wei, *Nonlocal s-minimal surfaces and Lawson cones.* J. Differential Geom. **109** (2018), 111–175.

[13] K. Ecker & G. Huisken, *Interior estimates for hypersurfaces moving by mean curvature.* Invent. Math. **105** (1991), 547–569.

[14] X. Fernández-Real, & X. Ros-Oton *Regularity Theory for Elliptic PDE.* Zurich Lectures in Advanced Mathematics. EMS books, (2022).

[15] A. Figalli, N. Fusco, F. Maggi, V. Millot & M. Morini, *Isoperimetry and stability properties of balls with respect to nonlocal energies.* Comm. Math. Phys. **336** (2015), 441–507.

[16] N. Fusco, V. Millot & M. Morini *A quantitative isoperimetric inequality for fractional perimeters*, J. Funct. Anal., **261** (2011), 697–715

[17] G. Huisken, *The volume preserving mean curvature flow.* J. Reine Angew. Math. **382** (1987), 35–48.

[18] C. Imbert, *Level set approach for fractional mean curvature flows.* Interfaces Free Bound. **11** (2009), 153–176.

[19] V. Julin & D.A. La Manna, *Short time existence of the classical solution to the fractional mean curvature flow.* Ann. Inst. H. Poincaré Anal. Non Linéaire **37** (2020), 983–1016.

[20] V. Julin & D. A. La Manna, *A priori estimates for the motion of charged liquid drop: A dynamic approach via free boundary Euler equations.* J. Math. Fluid Mech. **26** (2024), Paper No. 48.

[21] C. Mantegazza, *Lecture notes on Mean Curvature Flow.* Progress in Mathematics **290**. Birkhäuser/ Springer, Basel 2011.

[22] B-V. Matioc & C. Walker *The non-local mean curvature flow of periodic graphs.* Preprint 2022.

[23] R. Mikulevicius & H. Pragarauskas, *On Hölder solutions of the integro-differential Zakai equation.* Stoch. Process. Appl. **119** (2009), 3319–3355.

[24] M. Sáez & E. Valdinoci, *On the evolution by fractional mean curvature.* Comm. Anal. Geom. **27** (2019).

[25] O. Savin & E. Valdinoci, *Regularity of nonlocal minimal cones in dimension 2.* Calc. Var. Partial Differential Equations **48** (2013), 33–39.

[26] J. Serra, *Regularity for fully nonlinear nonlocal parabolic equations with rough kernels.* Calc. Var. Partial Differential Equations **54** (2015), 615–629.

[27] H. Triebel, *Interpolation theory, Function spaces, Differential operators.* North-Holland Publishing Company, 1978