NAPOLEONIC CONSTRUCTIONS IN THE HYPERBOLIC PLANE

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Abstract. In the Euclidean setting, Napoleon's Theorem states that if one constructs an equilateral triangle on either the outside or the inside of each side of a given triangle and then connects the barycenters of those three new triangles, the resulting triangle happens to be equilateral.

The case of spherical triangles has been recently shown to be different: on the sphere, besides equilateral triangles, a necessary and sufficient condition for a given triangle to enjoy the above Napoleonic property is that its congruence class should lie on a suitable surface (namely, an ellipsoid in suitable coordinates).

In this article we show that the hyperbolic case is significantly different from both the Euclidean and the spherical setting. Specifically, we establish here that the hyperbolic plane does not admit any Napoleonic triangle, except the equilateral ones. Furthermore, we prove that iterated Napoleonization of any triangle causes it to become smaller and smaller, more and more equilatera,l and converge to a single point in the limit.

1. Introduction

The *Napoleonic construction* deals with triangles on a surface and proceeds according to the following steps:

- A triangle $P_0P_1P_2$ in a two-dimensional surface is given.
- Three equilateral triangles are constructed on the sides of $P_0P_1P_2$: namely, one takes points Q_0, Q_1 and Q_2 on the surface such that the triangles $P_0P_1Q_2$, $P_1P_2Q_0$ and $P_0P_2Q_1$ are equilateral (two different constructions arise, according to the direction chosen for the points Q_0, Q_1 and Q_2).
- The centroids (i.e. barycenters) R_0, R_1 and R_2 of the equilateral triangles $P_1P_2Q_0$, $P_0P_2Q_1$ and $P_0P_1Q_2$ are considered and the triangle $R_0R_1R_2$ is called the *Napoleonization* of $P_0P_1P_2$.
- If the Napoleonization $R_0R_1R_2$ is an equilateral triangle, then the initial triangle $P_0P_1P_2$ is called *Napoleonic*.

The classical case of this construction occurs when the ambient surface is the *Euclidean plane*. In this situation, *all triangles are Napoleonic*: this is a famous result going under the name of Napoleon's Theorem: see e.g. [Grü12] and the references therein for the fascinating history of this result (see also Figures 1.1 and 1.2 for a sketch of the Napoleonic constructions in the Euclidean plane).

In the Euclidean case, Napoleon's Theorem attracted the attention of several first-rate mathematicians, including Fields Medallist Jesse Douglas; in fact, the question of extending

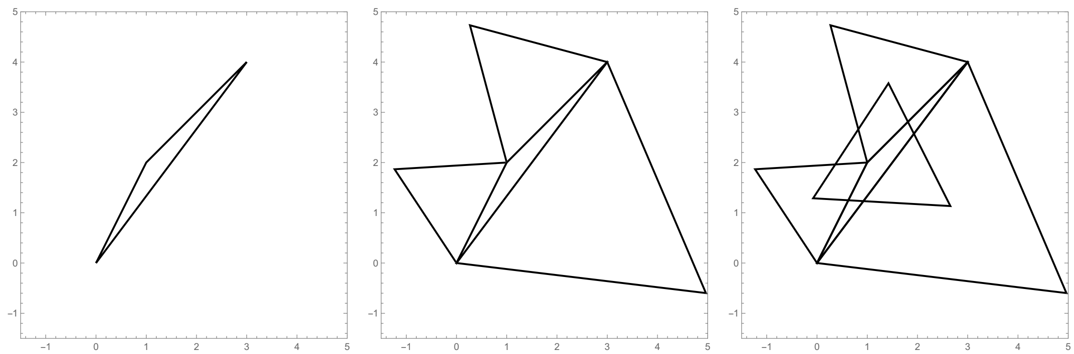

FIGURE 1.1. External Napoleonization of the Euclidean triangle with vertices in $(0,0)$, $(1,2)$ and $(3,4)$.

Napoleon's Theorem from planar triangles to polygons is known as the Petr-Douglas-Neumann problem, see [Pet08, Dou40, Neu41]. Napoleon's Theorem also finds practical applications in some optimization questions, such as the Fermat-Steiner-Torricelli problem, see [Nah21].

Among the many modern extensions of Napoleon's Theorem, a natural field of investigation is the discovery and classification of Napoleonic triangles in ambient surfaces different from the Euclidean plane. This happens to be a rather difficult problem about which little is known.

So far, the only ambient manifold for which all Napoleonic triangles have been classified is the *round sphere*. Specifically, in [DNV24] it is established that if a spherical triangle is Napoleonic then either it is equilateral or its congruency class lies in an explicit surfaces, which, in an appropriate coordinate system, can be written as a *two-dimensional rotational ellipsoid* (and, conversely, all congruency classes in this ellipsoid correspond to Napoleonic

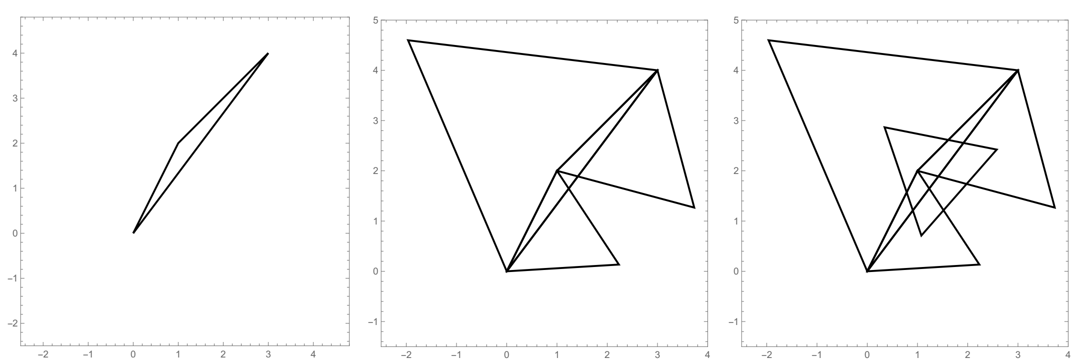

FIGURE 1.2. Internal Napoleonization of the Euclidean triangle with vertices in $(0,0)$, $(1,2)$ and $(3,4)$.

triangles). What is more, *all non-equilateral Napoleonic triangles on the sphere produce congruent Napoleonizations.*

The case of Napoleonic constructions in the *hyperbolic plane* appears then as a natural question. So far, to the best of our knowledge, the only available results in this setting go back to [McK01] and deal with an infinite sequence of recursively defined hyperbolic triangles (as explicitly mentioned in [Fis01] this construction is inspired by Napoleon's Theorem, but structurally quite different from it). In particular, no specific investigation of Napoleonic constructions in the hyperbolic plane has been carried out till now.

The goal of this paper is to fill this gap. For concreteness, as a model for the hyperbolic plane we consider here the upper sheet of the unit hyperboloid in the Minkowski space (see Section 2 for details). Our first result states that the only Napoleonic triangles are the trivial ones (i.e., the ones for which the initial triangle was equilateral):

Theorem 1.1. *If the Napoleonization of a hyperbolic triangle is equilateral, then the initial triangle is equilateral too.*

We stress that the hyperbolic case dealt with in Theorem 1.1 is surprisingly different both from the Euclidean case (in which all Napoleonizations are equilateral) and the spherical case (in which an ellipsoid of parameters produces non-trivial cases of equilateral Napoleonizations). We believe that the structural differences of Napoleon-like results depending on the geometry of the ambient surface is indeed a noteworthy phenomenon and a brand-new line of investigation which deserves a deeper understanding.

Another question of interest in this setting is what happens after repeated Napoleonizations, i.e. by taking a hyperbolic triangle to start with, and applying the Napoleonic construction over and over. For that target, given a hyperbolic triangle $P_0P_1P_2$, we denote its Napoleonization by $P_0^{(1)}P_1^{(1)}P_2^{(1)}$, and recursively we set $P_0^{(k)}P_1^{(k)}P_2^{(k)}$ to be the Napoleonization of $P_0^{(k-1)}P_1^{(k-1)}P_2^{(k-1)}$. In this framework, our result goes as follows:

Theorem 1.2. *As k increases, the triangles $P_0^{(k)}P_1^{(k)}P_2^{(k)}$ become smaller, more nearly equilateral and, as $k \to +\infty$, more nearly a single point.*

We point out that Theorem 1.2 is in sharp contrast with the Euclidean case (flat triangles remain equilateral and do not contract under repeated Napoleonizations, actually they just rotate by $60°$, up to relabeling vertices). The comparison with the Euclidean case also highlights an unavoidable difficulty intrinsically linked to the proof of Theorem 1.2: indeed, if repeated Napoleonizations tend to approach a point, the setting becomes "more and more Euclidean" during the iteration, thus making the convergence to a point problematic precisely when we approach the limit.

The rest of the paper is organized as follows. In Section 2 we gather some preliminary observations on the hyperboloid model and the hyperbolic triangles. In Section 3 we introduce a bespoke set of hyperbolic coordinates, which are different from the standard hyperbolic distance $\operatorname{arccosh}(-\langle \cdot, \cdot \rangle)$ and come in handy to simplify several otherwise cumbersome calculations. Sections 4 and 5 contain the proofs of Theorems 1.1 and 1.2, respectively.

2. Preliminaries on the Hyperboloid Model and the Hyperbolic Triangles

We recall that the Minkowski inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^3$ is given by

$$\langle (v_1, v_2, v_3)^{\mathbf{T}}, (w_1, w_2, w_3)^{\mathbf{T}} \rangle := -v_1 w_1 + v_2 w_2 + v_3 w_3.$$

Equivalently

$$\langle v, w \rangle = (Jv) \cdot w = v \cdot (Jw),$$

where $\cdot$ denotes the Euclidean inner product and J is the 3×3 diagonal matrix

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The (upper unit) hyperboloid is

$$\mathbb{H} := \left\{ P \in \mathbb{R}^3 \ \text{ s.t. } \ \langle P, P \rangle = -1 \ \text{ and } \ \langle P, E_1 \rangle \geqslant 1 \right\}$$

where E_1 is the timelike vector $(-1, 0, 0)^{\mathbf{T}}$. Moreover, we denote by $E_2 := (0, 1, 0)^{\mathbf{T}}$ and $E_3 := (0, 0, 1)^{\mathbf{T}}$.

As customary[1] we observe that if P, $Q \in \mathbb{H}$, then,

$$\langle P, Q \rangle \leqslant -1. \tag{2.1}$$

The hyperbolic cross-product $\tilde{\times}$ of vectors in Minkowski space $\mathbb{R}^3$ is given by

$$v \tilde{\times} w := J(v \times w)$$

where $\times$ is the Euclidean cross-product.

Since J is orthogonal with determinant -1, we deduce from the formula

$$(Jv) \times (Jw) = \det(J)(J^{-1})^T (v \times w) = -J(v \times w)$$

that, for any v, w, v', $w' \in \mathbb{R}^3$,

$$\langle v \tilde{\times} w, v' \tilde{\times} w' \rangle = J(v \times w) \cdot (v' \tilde{\times} w') = -((Jv) \times (Jw)) \cdot (v' \times w').$$

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[1]For an elementary proof of (2.1), one can write $P = (p, \tilde{P})$ and $Q = (q, \tilde{Q})$, with p, $q \in [1, +\infty)$ and $\tilde{P}$, $\tilde{Q} \in \mathbb{R}^2$, and use the standard Cauchy-Schwarz inequality. Indeed, we note that, for all a, $b \in \mathbb{R}$,

$$2pqab \leqslant p^2 a^2 + q^2 b^2.$$

Also, choosing $a := \sqrt{\tilde{Q} \cdot \tilde{Q}}$ and $b := \sqrt{\tilde{P} \cdot \tilde{P}}$, we see that $1 = -\langle \tilde{P}, \tilde{P} \rangle = p^2 - b^2$, and similarly $1 = q^2 - a^2$, yielding that $p \geqslant b$ and $q \geqslant a$ and that

$$1 = (p^2 - b^2)(q^2 - a^2) = p^2 q^2 + a^2 b^2 - p^2 a^2 - q^2 b^2 \leqslant p^2 q^2 + a^2 b^2 - 2pqab = (pq - ab)^2.$$

Consequently, $pq \geqslant ab$ and $pq - ab \geqslant 1$. Thus, since

$$-\langle P, Q \rangle = pq - \tilde{P} \cdot \tilde{Q} \geqslant pq - \sqrt{(\tilde{P} \cdot \tilde{P})(\tilde{Q} \cdot \tilde{Q})} = pq - ab,$$

the classical inequality in (2.1) plainly follows (and actually, tracing the equality cases in the above inequality, one also gets that equality in (2.1) holds if and only if $P = Q$).

Therefore, using the Binet-Cauchy Identity,

$$\begin{aligned} \langle v\tilde{\times}w, v'\tilde{\times}w'\rangle &= -((Jv)\cdot v')((Jw)\cdot w') + ((Jv)\cdot w')((Jw)\cdot v') \\ &= -\langle v, v'\rangle\langle w, w'\rangle + \langle v, w'\rangle\langle w, v'\rangle. \end{aligned} \tag{2.2}$$

The hyperbolic scalar triple product $\langle u, v\tilde{\times}w\rangle$ of u, v, $w \in \mathbb{R}^3$ is also the Euclidean scalar triple product $u\cdot(v\times w)$, so it has the same symmetries.

Given two points on the hyperboloid, a third point can be written as a combination of these two points and their cross-product. More explicitly, we have that:

Lemma 2.1. *Let $P_0 \neq P_1 \in \mathbb{H}$. Then, any $Q \in \mathbb{H}$ can be written in the form*

$$Q = a_0P_0 + a_1P_1 + bP_0\tilde{\times}P_1, \tag{2.3}$$

with

$$-a_0 = \frac{\langle Q, P_0\rangle + \langle P_0, P_1\rangle\langle Q, P_1\rangle}{1-\langle P_0, P_1\rangle^2} \quad \text{and} \quad -a_1 = \frac{\langle Q, P_1\rangle + \langle P_0, P_1\rangle\langle Q, P_0\rangle}{1-\langle P_0, P_1\rangle^2}. \tag{2.4}$$

Also,

$$-1 = \langle Q, Q\rangle = -a_0^2 - a_1^2 + 2a_0a_1\langle P_0, P_1\rangle - b^2(1-\langle P_0, P_1\rangle^2). \tag{2.5}$$

Proof. We point out that, in light of (2.1) and the fact that $P_0 \neq P_1$,

$$1-\langle P_0, P_1\rangle^2 \neq 0,$$

and therefore the coefficients in (2.4) are well defined.

Now, using the facts that $\langle P_0, P_0\rangle = -1 = \langle P_1, P_1\rangle$ and that $\langle P_0, P_0\tilde{\times}P_1\rangle = 0 = \langle P_1, P_0\tilde{\times}P_1\rangle$ and (2.2), we see that

$$\begin{aligned} \langle Q, Q\rangle &= \langle a_0P_0 + a_1P_1 + bP_0\tilde{\times}P_1, a_0P_0 + a_1P_1 + bP_0\tilde{\times}P_1\rangle \\ &= -a_0^2 - a_1^2 + 2a_0a_1\langle P_0, P_1\rangle + b^2\langle P_0\tilde{\times}P_1, P_0\tilde{\times}P_1\rangle \\ &= -a_0^2 - a_1^2 + 2a_0a_1\langle P_0, P_1\rangle + b^2(\langle P_0, P_1\rangle^2 - 1), \end{aligned}$$

which gives the claim in (2.5).

Thus, noticing that

$$\begin{aligned} & -a_0 + a_1\langle P_0, P_1\rangle = \langle Q, P_0\rangle \\ \text{and} \qquad & a_0\langle P_0, P_1\rangle - a_1 = \langle Q, P_1\rangle, \end{aligned}$$

we obtain (2.4), as desired. □

The case of isosceles and equilateral triangles on the hyperboloid are particular cases of Lemma 2.1 and go as follows:

Corollary 2.2. *Let P_0P_1Q be isosceles with $P_0 \neq P_1$ and $\langle P_0, Q\rangle = \langle P_1, Q\rangle$.*

Then, Q can be written as in (2.3), *with*

$$-a_0 = -a_1 = -a := \frac{\langle P_0, Q\rangle}{1-\langle P_0, P_1\rangle}. \tag{2.6}$$

Proof. When $\langle P_0, Q\rangle = \langle P_1, Q\rangle$, we have that (2.4) reduces to (2.6). □

Corollary 2.3. *Let $P_0P_1Q_2$ be equilateral with $P_0 \neq P_1 \neq Q_2$.*
Then,

$$Q_2 = \frac{-\langle P_0, P_1\rangle(P_0 + P_1) + \epsilon_2\sqrt{1 - 2\langle P_0, P_1\rangle}\,P_0 \tilde{\times} P_1}{1 - \langle P_0, P_1\rangle}$$

with $\epsilon_2 \in \{-1, 1\}$.

Proof. We recall that, being $P_0P_1Q_2$ equilateral, we have that

$$\langle P_0, P_1\rangle = \langle P_0, Q_2\rangle = \langle P_1, Q_2\rangle.$$

Therefore, by (2.5) and (2.6), used here with $Q := Q_2$, we see that

$$\begin{aligned} b^2(1 - \langle P_0, P_1\rangle^2) &= 1 - 2a^2(1 - \langle P_0, P_1\rangle) \\ &= 1 - 2\frac{\langle P_0, P_1\rangle^2}{1 - \langle P_0, P_1\rangle} = \frac{(1 - 2\langle P_0, P_1\rangle)(1 + \langle P_0, P_1\rangle)}{1 - \langle P_0, P_1\rangle}. \end{aligned}$$

As a consequence,

$$b = \epsilon_2 \frac{\sqrt{1 - 2\langle P_0, P_1\rangle}}{1 - \langle P_0, P_1\rangle}$$

and the desired result follows from (2.3). □

We now reconsider Corollary 2.3 with the aim of identifying the centroid of an equilateral triangle on the unit hyperboloid (where, by definition, the centroid of a triangle is the sum of the coordinates of its vertices projected over the hyperboloid):

Lemma 2.4. *Let $P_0P_1Q_2$ be equilateral with $P_0 \neq P_1 \neq Q_2$. Let $R_2 \in \mathbb{H}$ be its centroid.*
Then,

$$R_2 = \frac{\sqrt{1 - 2\langle P_0, P_1\rangle}(P_0 + P_1) + \epsilon_2 P_0 \tilde{\times} P_1}{\sqrt{3}(1 - \langle P_0, P_1\rangle)}$$

with $\epsilon_2 \in \{-1, 1\}$.

Proof. The centroid R_2 of the equilateral hyperbolic triangle $P_0P_1Q_2$ is $\hat{R}_2/\sqrt{-\langle \hat{R}_2, \hat{R}_2\rangle}$, where

$$\hat{R}_2 := (1 - 2\langle P_0, P_1\rangle)(P_0 + P_1) + \epsilon_2\sqrt{1 - 2\langle P_0, P_1\rangle}\,P_0 \tilde{\times} P_1$$

and, thanks to the equality in (2.2),

$$\begin{aligned} & -\langle \hat{R}_2, \hat{R}_2\rangle \\ = \;& 2(1 - 2\langle P_0, P_1\rangle)^2 - 2(1 - 2\langle P_0, P_1\rangle)^2\langle P_0, P_1\rangle + (1 - 2\langle P_0, P_1\rangle)(1 - \langle P_0, P_1\rangle^2) \\ = \;& (1 - 2\langle P_0, P_1\rangle)\big(2(1 - 2\langle P_0, P_1\rangle) - 2(1 - 2\langle P_0, P_1\rangle)\langle P_0, P_1\rangle + 1 - \langle P_0, P_1\rangle^2\big) \\ = \;& 3(1 - 2\langle P_0, P_1\rangle)(1 - \langle P_0, P_1\rangle)^2. \end{aligned}$$

The desired result now plainly follows. □

3. A Bespoke Set of Hyperbolic Coordinates

From now on we consider three distinct points P_0, P_1, $P_2 \in \mathbb{H}$ and define

$$\alpha := -1 + \langle P_0, P_1\rangle + \langle P_1, P_2\rangle + \langle P_2, P_0\rangle \tag{3.1}$$

and

$$\chi := \langle P_0 \tilde{\times} P_1, P_2\rangle. \tag{3.2}$$

We stress that α is symmetric with respect to permutations of P_0, P_1 and P_2, and that χ is symmetric with respect to cyclic permutations.

Since Theorems 1.1 and 1.2 are invariant under permutations of P_0, P_1 and P_2, we can list the vertices P_0, P_1 and P_2 so that

$$\chi \geqslant 0. \tag{3.3}$$

This reordering of vertices will be implicitly assumed in what follows.

From now on, we will also consider Q_0, $Q_2 \in \mathbb{H}$ such that $P_0P_1Q_2$ and $P_1P_2Q_0$ are equilateral. Let also R_0 be the centroid of $P_1P_2Q_0$ and R_2 be the centroid of $P_0P_1Q_2$.

With this notation, we can take a step further from Lemma 2.4 and obtain that:

Lemma 3.1. *We have that*

$$\begin{aligned}
&3(1-\langle P_0,P_1\rangle)(1-\langle P_1,P_2\rangle)\langle R_2,R_0\rangle \\
&\quad= \alpha\sqrt{1-2\langle P_0,P_1\rangle}\sqrt{1-2\langle P_1,P_2\rangle} + \chi\Big(\epsilon_0\sqrt{1-2\langle P_0,P_1\rangle} + \epsilon_2\sqrt{1-2\langle P_1,P_2\rangle}\Big) \\
&\qquad - \epsilon_2\epsilon_0(\langle P_0,P_1\rangle\langle P_1,P_2\rangle + \langle P_0,P_2\rangle),
\end{aligned}$$

with ϵ_0, $\epsilon_2 \in \{-1, 1\}$.

Proof. By swapping indexes in Lemma 2.4, we see that the equilateral hyperbolic triangle $P_1P_2Q_0$ has centroid

$$R_0 = \frac{\sqrt{1-2\langle P_1,P_2\rangle}(P_1+P_2) + \epsilon_0 P_1 \tilde{\times} P_2}{\sqrt{3}(1-\langle P_1,P_2\rangle)}.$$

Therefore, exploiting (2.2) and the cyclic symmetry of the hyperbolic scalar triple product,

$$\begin{aligned}
&3(1-\langle P_0,P_1\rangle)(1-\langle P_1,P_2\rangle)\langle R_2,R_0\rangle \\
&\quad= \langle\sqrt{1-2\langle P_0,P_1\rangle}(P_0+P_1) + \epsilon_2 P_0\tilde{\times}P_1, \sqrt{1-2\langle P_1,P_2\rangle}(P_1+P_2) + \epsilon_0 P_1\tilde{\times}P_2\rangle \\
&\quad= \sqrt{1-2\langle P_0,P_1\rangle}\sqrt{1-2\langle P_1,P_2\rangle}\langle P_0+P_1, P_1+P_2\rangle \\
&\qquad+\epsilon_0\sqrt{1-2\langle P_0,P_1\rangle}\langle P_0+P_1, P_1\tilde{\times}P_2\rangle + \epsilon_2\sqrt{1-2\langle P_1,P_2\rangle}\langle P_0\tilde{\times}P_1, P_1+P_2\rangle \\
&\qquad+\epsilon_2\epsilon_0\langle P_0\tilde{\times}P_1, P_1\tilde{\times}P_2\rangle \\
&\quad= \sqrt{1-2\langle P_0,P_1\rangle}\sqrt{1-2\langle P_1,P_2\rangle}(-1+\langle P_0,P_1\rangle+\langle P_1,P_2\rangle+\langle P_2,P_0\rangle) \\
&\qquad+(\epsilon_0\sqrt{1-2\langle P_0,P_1\rangle} + \epsilon_2\sqrt{1-2\langle P_1,P_2\rangle})\langle P_0\tilde{\times}P_1, P_2\rangle \\
&\qquad-\epsilon_2\epsilon_0(\langle P_0,P_1\rangle\langle P_1,P_2\rangle + \langle P_0,P_2\rangle).
\end{aligned}$$

From this the desired result follows. ∎

We now introduce a new set of "hyperbolic units of measurements", technically and conceptually different from the standard hyperbolic distance $\operatorname{arccosh}(-\langle\cdot,\cdot\rangle)$, which come in handy to simplify several otherwise cumbersome calculations. Namely, we define

$$\begin{aligned} & d_0 := \sqrt{1-2\langle P_1,P_2\rangle}, \\ & d_1 := \sqrt{1-2\langle P_2,P_0\rangle} \\ \text{and}\quad & d_2 := \sqrt{1-2\langle P_0,P_1\rangle}. \end{aligned} \tag{3.4}$$

We point out that (d_0, d_1, d_2) defines the congruency class of the hyperbolic triangle, and that these hyperbolic coordinates are well defined, thanks to (2.1).

The main features of these hyperbolic coordinates are the following:

Lemma 3.2. *For all $i \in \mathbb{Z}/3\mathbb{Z}$, we have that*

$$d_i \geqslant \sqrt{3} \tag{3.5}$$

and

$$d_i^2 - 1 \leqslant (d_{i+1}^2 - 1)(d_{i+2}^2 - 1). \tag{3.6}$$

Proof. The claim in (3.5) follows directly from (2.1).

Also, by the triangle inequality in $\mathbb{H}$ (see e.g. page 70 in [Ive92]), for $i \in \mathbb{Z}/3\mathbb{Z}$,

$$\operatorname{arccosh}\left(\frac{d_i^2-1}{2}\right) \leqslant \operatorname{arccosh}\left(\frac{d_{i+1}^2-1}{2}\right) + \operatorname{arccosh}\left(\frac{d_{i+2}^2-1}{2}\right).$$

We recall that

$$\cosh\big(\operatorname{arccosh} x + \operatorname{arccosh} y\big) = xy + \sqrt{(x^2-1)(y^2-1)}.$$

Therefore, using this formula with $x := \frac{d_{i+1}^2-1}{2}$ and $y := \frac{d_{i+2}^2-1}{2}$, we find that

$$\begin{aligned} 2d_i^2 - 2 &\leqslant (d_{i+1}^2-1)(d_{i+2}^2-1) + \sqrt{(d_{i+1}^4 - 2d_{i+1}^2 - 3)(d_{i+2}^4 - 2d_{i+2}^2 - 3)} \\ &\leqslant 2(d_{i+1}^2-1)(d_{i+2}^2-1), \end{aligned}$$

from which one obtains (3.6). □

Now we calculate α and χ:

Proposition 3.3. *We have that*

$$2\alpha = 1 - d_0^2 - d_1^2 - d_2^2 \tag{3.7}$$

and

$$2\chi = \sqrt{3(d_0^2+d_1^2+d_2^2) - (d_0^2d_1^2 + d_1^2d_2^2 + d_2^2d_0^2 + d_0^4 + d_1^4 + d_2^4) + d_0^2d_1^2d_2^2}. \tag{3.8}$$

Proof. The claim in (3.7) follows from (3.1) and (3.4).

To calculate χ note first that, by Lemma 2.1,

$$P_2 = -\frac{\langle P_2,P_0\rangle + \langle P_0,P_1\rangle\langle P_1,P_2\rangle}{1-\langle P_0,P_1\rangle^2} P_0 - \frac{\langle P_1,P_2\rangle + \langle P_0,P_1\rangle\langle P_2,P_0\rangle}{1-\langle P_0,P_1\rangle^2} P_1 + bP_0 \tilde{\times} P_1.$$

Hence, using also (2.2),

$$\chi = \langle P_0 \tilde{\times} P_1, P_2\rangle = b\langle P_0 \tilde{\times} P_1, P_0 \tilde{\times} P_1\rangle = -b(1-\langle P_0,P_1\rangle^2).$$

Moreover, by (2.5),

$$\begin{aligned}
& b^2(1-\langle P_0,P_1\rangle^2)^3 \\
= \;& (1-\langle P_0,P_1\rangle^2)^2(1-a_0^2-a_1^2+2a_0a_1\langle P_0,P_1\rangle) \\
= \;& (1-\langle P_0,P_1\rangle^2)^2-(\langle P_2,P_0\rangle+\langle P_0,P_1\rangle\langle P_1,P_2\rangle)^2-(\langle P_1,P_2\rangle+\langle P_0,P_1\rangle\langle P_2,P_0\rangle)^2 \\
& \quad +2(\langle P_2,P_0\rangle+\langle P_0,P_1\rangle\langle P_1,P_2\rangle)(\langle P_1,P_2\rangle+\langle P_0,P_1\rangle\langle P_2,P_0\rangle)\langle P_0,P_1\rangle \\
= \;& (1-\langle P_0,P_1\rangle^2)^2+2\langle P_0,P_1\rangle^3\langle P_0,P_2\rangle\langle P_1,P_2\rangle-2\langle P_0,P_1\rangle\langle P_0,P_2\rangle\langle P_1,P_2\rangle \\
& \quad +\langle P_0,P_1\rangle^2\langle P_0,P_2\rangle^2+\langle P_0,P_1\rangle^2\langle P_1,P_2\rangle^2-\langle P_0,P_2\rangle^2-\langle P_1,P_2\rangle^2 \\
= \;& (1-\langle P_0,P_1\rangle^2)(1-\langle P_0,P_1\rangle^2-\langle P_1,P_2\rangle^2-\langle P_2,P_0\rangle^2-2\langle P_0,P_1\rangle\langle P_1,P_2\rangle\langle P_2,P_0\rangle).
\end{aligned}$$

Thus, we find that

$$\begin{aligned}
\chi &= -b(1-\langle P_0,P_1\rangle^2) \\
&= \pm\sqrt{1-\langle P_0,P_1\rangle^2-\langle P_1,P_2\rangle^2-\langle P_2,P_0\rangle^2-2\langle P_0,P_1\rangle\langle P_1,P_2\rangle\langle P_2,P_0\rangle}
\end{aligned}$$

and then (3.8) follows from (3.3). □

Now we define

$$\gamma := 3(d_0^2+1)(d_1^2+1)(d_2^2+1) \tag{3.9}$$

and we have the following two estimates:

Lemma 3.4. *It holds that*

$$(2\chi)^2 \leqslant \frac{1}{3}\sum_{i=0}^{2} d_i^2(d_{i+1}^2-3)(d_{i+2}^2-3) \tag{3.10}$$

$$\text{and} \qquad -24\alpha\chi \leqslant \gamma. \tag{3.11}$$

Proof. By (3.8) and the standard Cauchy-Schwarz inequality applied to the 3-dimensional vectors (d_0^2, d_1^2, d_2^2) and (d_1^2, d_2^2, d_0^2), we see that

$$\begin{aligned}
(2\chi)^2 &= 3(d_0^2+d_1^2+d_2^2)-(d_0^2d_1^2+d_1^2d_2^2+d_2^2d_0^2+d_0^4+d_1^4+d_2^4)+d_0^2d_1^2d_2^2 \\
&\leqslant 3(d_0^2+d_1^2+d_2^2)-2(d_0^2d_1^2+d_1^2d_2^2+d_2^2d_0^2)+d_0^2d_1^2d_2^2 \\
&= \frac{1}{3}\sum_{i=0}^{2} d_i^2(d_{i+1}^2-3)(d_{i+2}^2-3).
\end{aligned}$$

This proves (3.10).

To prove (3.11), we recall (3.7) and (3.8) and we write

$$\begin{aligned} -24\alpha\chi &= 6(-2\alpha)(2\chi) \\ &\leqslant 3((2\alpha)^2+(2\chi)^2) \\ &= 3((d_0^2+d_1^2+d_2^2-1)^2+3(d_0^2+d_1^2+d_2^2) \\ &\qquad -(d_0^2d_1^2+d_1^2d_2^2+d_2^2d_0^2+d_0^4+d_1^4+d_2^4)+d_0^2d_1^2d_2^2) \\ &= 3(d_0^2d_1^2d_2^2+d_0^2d_1^2+d_1^2d_2^2+d_2^2d_0^2+d_0^2+d_1^2+d_2^2+1). \end{aligned}$$

From this and (3.9), we obtain (3.11), as desired. □

4. Napoleonic Triangles and Proof of Theorem 1.1

From now on, we suppose that P_0, P_1 and P_2 are not cogeodesic, namely $\chi \neq 0$. Hence, by (3.3),

$$\chi > 0.$$

Set also[2] $\epsilon_0 = \epsilon_1 = \epsilon_2 = \epsilon := \pm 1$.

Lemma 4.1. *We have that, for $i \in \mathbb{Z}/3\mathbb{Z}$,*

$$\begin{aligned} &\gamma\langle R_{i+1}, R_{i+2}\rangle \\ &\quad = (d_i^2+1)\Big(4\big(\alpha d_{i+1}d_{i+2}+\epsilon\chi(d_{i+1}+d_{i+2})\big)-(d_{i+1}^2-1)(d_{i+2}^2-1)+2(d_i^2-1)\Big) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\gamma\big(\langle R_{i+2}, R_i\rangle - \langle R_i, R_{i+1}\rangle\big) \\ &\quad = 4(d_{i+2}-d_{i+1})\Big(\alpha(d_0+d_1+d_2-d_0d_1d_2)+\epsilon\chi(1-d_0d_1-d_1d_2-d_2d_0)\Big). \end{aligned} \tag{4.2}$$

Proof. Using Lemma 2.4 we write that

$$\begin{aligned} &3(1-\langle P_{i+2},P_i\rangle)(1-\langle P_i,P_{i+1}\rangle)\langle R_{i+1},R_{i+2}\rangle \\ = &\left\langle \sqrt{1-2\langle P_{i+2},P_i\rangle}(P_{i+2}+P_i)+\epsilon P_{i+2}\tilde{\times}P_i, \sqrt{1-2\langle P_i,P_{i+1}\rangle}(P_i+P_{i+1})+\epsilon P_i\tilde{\times}P_{i+1}\right\rangle. \end{aligned}$$

Therefore, recalling also the definitions in (3.1), (3.2) and (3.4) and exploiting (2.2),

$$\begin{aligned} &3(1-\langle P_{i+2},P_i\rangle)(1-\langle P_i,P_{i+1}\rangle)\langle R_{i+1},R_{i+2}\rangle \\ &= d_{i+1}d_{i+2}\big(\langle P_0,P_1\rangle+\langle P_1,P_2\rangle+\langle P_0,P_2\rangle-1\big)+\epsilon\chi d_{i+1}+\epsilon\chi d_{i+2} \\ &\qquad -\langle P_i,P_{i+2}\rangle\langle P_i,P_{i+1}\rangle-\langle P_{i+1},P_{i+2}\rangle \\ &= \alpha d_{i+1}d_{i+2}+\epsilon\chi\big(d_{i+1}+d_{i+2}\big)-\frac{(d_{i+1}^2-1)(d_{i+2}^2-1)}{4}+\frac{d_i^2-1}{2}. \end{aligned} \tag{4.3}$$

[2]Because of these choices, if $\langle P_2, P_0\rangle = \langle P_0, P_1\rangle$ then $\langle R_2, R_0\rangle = \langle R_0, R_1\rangle$. Similarly, if $\langle P_0, P_1\rangle = \langle P_1, P_2\rangle$ then $\langle R_0, R_1\rangle = \langle R_1, R_2\rangle$, and if $\langle P_1, P_2\rangle = \langle P_2, P_0\rangle$ then $\langle R_1, R_2\rangle = \langle R_2, R_0\rangle$.

Our attention is not limited to the case where $P_0P_1P_2$ is isosceles.

Also,

$$\begin{aligned}(1-\langle P_{i+2},P_i\rangle)(1-\langle P_i,P_{i+1}\rangle) &= \left(1-\frac{1-d_{i+1}^2}{2}\right)\left(1-\frac{1-d_{i+2}^2}{2}\right)\\ &= \frac{1}{4}(1+d_{i+1}^2)(1+d_{i+2}^2).\end{aligned}$$

Plugging this information into (4.3), we conclude that

$$\begin{aligned}\langle R_{i+1},R_{i+2}\rangle = &\frac{1}{3(1+d_{i+1}^2)(1+d_{i+2}^2)}\\ &\times\Big(4\big(\alpha d_{i+1}d_{i+2}+\epsilon\chi\big(d_{i+1}+d_{i+2}\big)\big)-(d_{i+1}^2-1)(d_{i+2}^2-1)+2(d_i^2-1)\Big).\end{aligned}$$

Thus, the claim in (4.1) follows from this and the definition of γ in (3.9).

Now, to prove (4.2), we set

$$T_i := (d_{i+1}^2+1)\big(\alpha d_{i+2}d_i+\epsilon\chi(d_{i+2}+d_i)\big)-(d_{i+2}^2+1)\big(\alpha d_i d_{i+1}+\epsilon\chi(d_i+d_{i+1})\big)$$

and we see that T_i expands as

$$\begin{aligned}&\alpha d_i\big(d_{i+2}(d_{i+1}^2+1)-d_{i+1}(d_{i+2}^2+1)\big)\\ &\qquad+\epsilon\chi\big((d_{i+1}^2+1)(d_{i+2}+d_i)-(d_{i+2}^2+1)(d_i+d_{i+1})\big)\\ =\ &(d_{i+2}-d_{i+1})\big(\alpha d_i(1-d_{i+1}d_{i+2})+\epsilon\chi(1-d_0d_1-d_1d_2-d_2d_0)\big).\end{aligned}$$

Using this and (4.1), we find that

$$\begin{aligned}&\gamma\big(\langle R_{i+2},R_i\rangle-\langle R_i,R_{i+1}\rangle\big)-4T_i\\ &\quad= (d_{i+1}^2+1)\Big(4\big(\alpha d_i d_{i+2}+\epsilon\chi(d_i+d_{i+2})\big)-(d_i^2-1)(d_{i+2}^2-1)+2(d_{i+1}^2-1)\Big)\\ &\qquad-(d_{i+2}^2+1)\Big(4\big(\alpha d_{i+1}d_i+\epsilon\chi(d_{i+1}+d_i)\big)-(d_{i+1}^2-1)(d_i^2-1)+2(d_{i+2}^2-1)\Big)\\ &\qquad-4(d_{i+1}^2+1)\big(\alpha d_{i+2}d_i+\epsilon\chi(d_{i+2}+d_i)\big)+4(d_{i+2}^2+1)\big(\alpha d_i d_{i+1}+\epsilon\chi(d_i+d_{i+1})\big)\\ &\quad= -\left((d_{i+1}^2+1)(d_{i+2}^2-1)-(d_{i+2}^2+1)(d_{i+1}^2-1)\right)(d_i^2-1)+2(d_{i+1}^4-d_{i+2}^4)\\ &\quad= -2(d_{i+2}^2-d_{i+1}^2)(d_i^2-1)+2(d_{i+1}^4-d_{i+2}^4)\\ &\quad= -2(d_{i+2}^2-d_{i+1}^2)(d_0^2+d_1^2+d_2^2-1)\\ &\quad= 4\alpha(d_{i+2}^2-d_{i+1}^2).\end{aligned}$$

From these considerations, we obtain (4.2). □

Corollary 4.2. *Suppose that $P_0P_1P_2$ is not equilateral.*

Then, $R_0R_1R_2$ is equilateral if and only if

$$\alpha(d_0+d_1+d_2-d_0d_1d_2)+\epsilon\chi(1-d_0d_1-d_1d_2-d_2d_0)=0. \tag{4.4}$$

Proof. By Lemma 4.1, we have that if (4.4) is satisfied then $R_0R_1R_2$ is equilateral.

Conversely, if $R_0R_1R_2$ is equilateral, then $\langle R_{i+2},R_i\rangle=\langle R_i,R_{i+1}\rangle$ for $i\in\mathbb{Z}/3\mathbb{Z}$. In light of Lemma 4.1, this is true if and only if either $d_{i+1}=d_{i+2}$ or (4.4) holds true. Notice that it cannot be that $d_{i+1} = d_{i+2}$ for all $i \in \mathbb{Z}/3\mathbb{Z}$, since $P_0P_1P_2$ is not equilateral. Therefore, in this case, we see that (4.4) must be satisfied. □

With this preliminary work, we can now complete the proof of the non-existence of non-trivial Napoleonic triangles in the hyperbolic plane.

Proof of Theorem 1.1. Suppose that $P_0P_1P_2$ is Napoleonic, i.e. $R_0R_1R_2$ is equilateral. If $P_0P_1P_2$ is not equilateral then (4.4) implies that

$$\alpha^2(d_0 + d_1 + d_2 - d_0d_1d_2)^2 - \chi^2(1 - d_0d_1 - d_1d_2 - d_2d_0)^2 = 0, \tag{4.5}$$

with α and χ^2 given by (3.7) and (3.8) as symmetric polynomials in d_0, d_1 and d_2. After these substitutions, one sees that the left-hand side of (4.5) is

$$\frac{1}{4}\Big((1 - d_0^2 - d_1^2 - d_2^2)^2(d_0 + d_1 + d_2 - d_0d_1d_2)^2$$
$$-\Big(3(d_0^2 + d_1^2 + d_2^2) - (d_0^2d_1^2 + d_1^2d_2^2 + d_2^2d_0^2 + d_0^4 + d_1^4 + d_2^4) + d_0^2d_1^2d_2^2\Big)(1 - d_0d_1 - d_1d_2 - d_2d_0)^2\Big),$$

that in turn equals to

$$\frac{\gamma}{24}\Big((d_0 - d_1)^2 + (d_1 - d_2)^2 + (d_2 - d_0)^2\Big)\Big(d_0^2 + d_1^2 + d_2^2 + d_0d_1 + d_1d_2 + d_2d_0 - 2\Big),$$

which is strictly positive.

Accordingly, we have that $P_0P_1P_2$ must be equilateral. In conclusion, all Napoleonic triangles in $\mathbb{H}$ are equilateral, as claimed. □

5. Napoleonic Progressions and proof of Theorem 1.2

From now on, we deal with repeated Napoleonization. For this purpose, we use the notation introduced in Section 3 and, after a cyclic relabelling, we can suppose that

$$d_0 = \max\{d_0, d_1, d_2\}. \tag{5.1}$$

For $i \in \mathbb{Z}/3\mathbb{Z}$, we let

$$e_i := \sqrt{1 - 2\langle R_{i+1}, R_{i+2}\rangle}. \tag{5.2}$$

We remark that e_i are well defined, in light of (2.1) and the fact that R_{i+1}, $R_{i+2} \in \mathbb{H}$.

We also define

$$r_d := \frac{4}{\gamma}\Big(-2\alpha(d_0d_1d_2 - d_0 - d_1 - d_2) - 2\epsilon\chi(d_0d_1 + d_1d_2 + d_2d_0 - 1)\Big)$$

and

$$r_i := \frac{|r_d|}{d_{i+1} + d_{i+2}}. \tag{5.3}$$

We point out that, with this notation, equation (4.2) reads as

$$e_{i+2}^2 - e_{i+1}^2 = r_d(d_{i+2} - d_{i+1}). \tag{5.4}$$

Moreover, we observe that for $\epsilon = -1$, we have that $r_d > 0$. Therefore, in this case, e_0, e_1 and e_2 are in the same order as d_0, d_1 and d_2.

The main calculation needed to understand repeated Napoleonization is as follows:

Proposition 5.1. *We have that*

$$r_i \leqslant \rho := \frac{2}{3} + \frac{2}{27} + \frac{1}{3\sqrt{3}} \approx 0.93319. \tag{5.5}$$

Proof. Applying the Cauchy-Schwarz inequality to the vectors (d_0, d_1, d_2) and (d_1, d_2, d_0), we see that

$$d_0 d_1 + d_1 d_2 + d_2 d_0 \leqslant d_0^2 + d_1^2 + d_2^2.$$

This observation and (3.11) give that

$$\begin{aligned} |r_d| &\leqslant \frac{4}{\gamma}\Big(- 2\alpha(d_0 d_1 d_2 - d_0 - d_1 - d_2) + 2\chi(d_0 d_1 + d_1 d_2 + d_2 d_0 - 1)\Big) \\ &\leqslant \frac{4}{\gamma}\Big(- 2\alpha(d_0 d_1 d_2 - d_0 - d_1 - d_2) + 2\chi(d_0^2 + d_1^2 + d_2^2 - 1)\Big) \\ &\leqslant \frac{4}{\gamma}\Big(- 2\alpha(d_0 d_1 d_2 - d_0 - d_1 - d_2) - 4\alpha\chi\Big) \\ &\leqslant \frac{4}{\gamma}\Big(- 2\alpha(d_0 d_1 d_2 - d_0 - d_1 - d_2)\Big) + \frac{2}{3}. \end{aligned} \tag{5.6}$$

Furthermore,

$$\begin{aligned} \frac{4}{\gamma}\Big(- 2\alpha(d_0 d_1 d_2 - d_0 - d_1 - d_2)\Big) &= \frac{4}{3}\frac{(d_0^2 + d_1^2 + d_2^2 - 1)(d_0 d_1 d_2 - d_0 - d_1 - d_2)}{(d_0^2+1)(d_1^2+1)(d_2^2+1)} \\ &\leqslant \frac{4}{3}\left(\frac{d_0^2 + d_1^2 + d_2^2 - 1}{d_0 d_1 d_2}\right) \\ &\leqslant \frac{4}{3}\left(\frac{(d_1^2-1)(d_2^2-1) + d_1^2 + d_2^2}{d_0 d_1 d_2}\right) \\ &= \frac{4}{3}\left(\frac{d_1^2 d_2^2 + 1}{d_0 d_1 d_2}\right), \end{aligned}$$

where the last inequality uses (3.6).

Accordingly, using (3.5), we conclude that

$$\frac{4}{\gamma}\Big(- 2\alpha(d_0 d_1 d_2 - d_0 - d_1 - d_2)\Big) \leqslant \frac{4}{3}\left(\frac{d_1 d_2}{d_0} + \frac{1}{3\sqrt{3}}\right).$$

We now observe that

$$2 d_1 d_2 = d_1 d_2 + d_1 d_2 \leqslant d_0 \left(d_1 + d_2\right),$$

thanks to (5.1), and therefore

$$\frac{4}{\gamma}\Big(- 2\alpha(d_0 d_1 d_2 - d_0 - d_1 - d_2)\Big) \leqslant \frac{2}{3}\left(d_1 + d_2\right) + \frac{4}{9\sqrt{3}}.$$

Plugging this information into (5.6) we thereby find that

$$|r_d| \leqslant \frac{2}{3}\left(d_1 + d_2\right) + \frac{4}{9\sqrt{3}} + \frac{2}{3}.$$

Consequently, recalling (5.3) and using (3.5) and (5.1), we have that, for any $i \in \mathbb{Z}/3\mathbb{Z}$,

$$r_i \leqslant \frac{1}{d_{i+1}+d_{i+2}} \left(\frac{2}{3}(d_1+d_2) + \frac{4}{9\sqrt{3}} + \frac{2}{3} \right) \leqslant \frac{2}{3} + \frac{2}{27} + \frac{1}{3\sqrt{3}},$$

which gives (5.5), as desired. □

For our purposes, Proposition 5.1 is important, since it shows that repeated Napoleonization, with $\epsilon = \pm 1$, gives a sequence $\{P_0^{(k)} P_1^{(k)} P_2^{(k)} : k \geqslant 0\}$ of hyperbolic triangles, with geometrically decreasing differences of lengths of sides. This allows us to complete the proof of Theorem 1.2: in this respect, we distinguish the contractive iterations when $\epsilon = 1$ and when $\epsilon = -1$.

Proof of Theorem 1.2 when $\epsilon = 1$. For $\epsilon = 1$, we deduce from (3.9), (4.1) and (5.2) that

$$\begin{aligned} e_i^2 - 3 &= 1 - 2\langle R_{i+1}, R_{i+2}\rangle - 3 \\ &= -2(1 + \langle R_{i+1}, R_{i+2}\rangle) \\ &= -\frac{2}{\gamma}\Big(\gamma + 4(d_i^2+1)\big(\alpha d_{i+1}d_{i+2} + \chi(d_{i+1}+d_{i+2})\big) \\ &\qquad - (d_i^2+1)(d_{i+1}^2-1)(d_{i+2}^2-1) + 2(d_i^2-1)(d_i^2+1)\Big) \\ &= -\frac{2}{3(d_{i+1}^2+1)(d_{i+2}^2+1)}\Big(3(d_{i+1}^2+1)(d_{i+2}^2+1) + 4\alpha d_{i+1}d_{i+2} \\ &\qquad + 4\chi(d_{i+1}+d_{i+2}) - (d_{i+1}^2-1)(d_{i+2}^2-1) + 2(d_i^2-1)\Big) \\ &= \frac{2}{3(d_{i+1}^2+1)(d_{i+2}^2+1)} U_i, \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} U_i \quad := \quad & -4\chi(d_{i+1}+d_{i+2}) - 4\alpha d_{i+1}d_{i+2} + (d_{i+1}^2-1)(d_{i+2}^2-1) \\ & -2(d_i^2-1) - 3(d_{i+1}^2+1)(d_{i+2}^2+1). \end{aligned}$$

We also observe that, recalling (3.7),

$$
\begin{aligned}
U_i \leqslant & -4\alpha d_{i+1}d_{i+2} + (d_{i+1}^2-1)(d_{i+2}^2-1) - 2(d_i^2-1) - 3(d_{i+1}^2+1)(d_{i+2}^2+1) \\
= & -4\alpha(d_{i+1}d_{i+2}-1) - 2 + 2d_0^2 + 2d_1^2 + 2d_2^2 + (d_{i+1}^2-1)(d_{i+2}^2-1) \\
& \qquad - 2(d_i^2-1) - 3(d_{i+1}^2+1)(d_{i+2}^2+1) \\
= & -4\alpha(d_{i+1}d_{i+2}-1) - 2(d_{i+1}^2+1)(d_{i+2}^2+1) \\
= & -2\Big((1-d_0^2-d_1^2-d_2^2)(d_{i+1}d_{i+2}-1) + (d_{i+1}^2+1)(d_{i+2}^2+1)\Big) \\
= & \, 2\Big(-d_{i+1}d_{i+2} + d_i^2 d_{i+1}d_{i+2} + d_{i+1}^3 d_{i+2} + d_{i+1}d_{i+2}^3 - d_i^2 \\
& \qquad - 2d_{i+1}^2 - 2d_{i+2}^2 - d_{i+1}^2 d_{i+2}^2\Big) \\
= & \, 2\Big((d_{i+1}d_{i+2}-1)(d_i^2-3) + (d_{i+1}^2+d_{i+2}^2-3-d_{i+1}d_{i+2}) \\
& \qquad + (d_{i+1}^2 - d_{i+1}d_{i+2} + d_{i+2}^2)(d_{i+1}d_{i+2}-3)\Big).
\end{aligned} \tag{5.8}
$$

Now we set $\mu_d := \max\{d_j^2 - 3 : j \in \mathbb{Z}/3\mathbb{Z}\}$ and we claim that

$$
\begin{aligned}
&(d_{i+1}d_{i+2}-1)(d_i^2-3) + (d_{i+1}^2+d_{i+2}^2-3-d_{i+1}d_{i+2}) \\
&\qquad + (d_{i+1}^2 - d_{i+1}d_{i+2} + d_{i+2}^2)(d_{i+1}d_{i+2}-3) \leqslant (d_{i+1}^2+d_{i+2}^2+1)\mu_d.
\end{aligned} \tag{5.9}
$$

Indeed, if $\mu_d = 0$ then, in light of (3.5), we have that $d_i = \sqrt{3}$ for all $i \in \mathbb{Z}/3\mathbb{Z}$ and so also the left-hand side of (5.9) equals zero. Therefore, in this case we are done. Hence, from now on we suppose that $\mu_d \neq 0$.

In this case, we point out that

$$
\begin{aligned}
& d_{i+1}d_{i+2} - 1 + \frac{d_{i+1}^2 + d_{i+2}^2 - 3 - d_{i+1}d_{i+2}}{\mu_d} + d_{i+1}^2 - d_{i+1}d_{i+2} + d_{i+2}^2 \\
= \; & \frac{d_{i+1}^2 + d_{i+2}^2 - 3 - d_{i+1}d_{i+2} - \mu_d}{\mu_d} + d_{i+1}^2 + d_{i+2}^2 \\
\leqslant \; & \frac{d_{i+2}^2 - d_{i+2}d_{i+1}}{\mu_d} + d_{i+1}^2 + d_{i+2}^2 \\
\leqslant \; & \frac{d_{i+2}^2 - 3}{\mu_d} + d_{i+1}^2 + d_{i+2}^2 \\
\leqslant \; & 1 + d_{i+1}^2 + d_{i+2}^2,
\end{aligned}
$$

where (3.5) has also been used. This implies (5.9), as desired.

Using the information in (5.9) into (5.8), we obtain that

$$
U_i \leqslant 2(d_{i+1}^2 + d_{i+2}^2 + 1)\mu_d.
$$

Hence, by (5.7),

$$e_i^2 - 3 \leqslant \frac{4}{3}\left(\frac{d_{i+1}^2 + d_{i+2}^2 + 1}{d_{i+1}^2 d_{i+2}^2 + d_{i+1}^2 + d_{i+2}^2 + 1}\right)\mu_d. \tag{5.10}$$

Now we point out that

$$\frac{d_{i+1}^2 + d_{i+2}^2 + 1}{d_{i+1}^2 d_{i+2}^2} = \frac{1}{d_{i+2}^2} + \frac{1}{d_{i+1}^2} + \frac{1}{d_{i+1}^2 d_{i+2}^2} \leqslant \frac{1}{3} + \frac{1}{3} + \frac{1}{9} = \frac{7}{9},$$

thanks to (3.5), which gives that

$$\frac{d_{i+1}^2 + d_{i+2}^2 + 1}{d_{i+1}^2 d_{i+2}^2 + d_{i+1}^2 + d_{i+2}^2 + 1} \leqslant \frac{7}{16}.$$

From this and (5.10) we deduce that

$$e_i^2 - 3 \leqslant \frac{7}{12}\mu_d.$$

Consequently, setting $\mu_e := \max\{e_i^2 - 3 : i \in \mathbb{Z}/3\mathbb{Z}\}$, we find that

$$\mu_e \leqslant \frac{7}{12}\mu_d. \tag{5.11}$$

Thus, for $\epsilon = 1$ the hyperbolic triangles $P_0^{(k)} P_1^{(k)} P_2^{(k)}$ satisfy

$$0 \leqslant \max\left\{(d_i^{(k)})^2 - 3 : i \in \mathbb{Z}/3\mathbb{Z}\right\} \leqslant \left(\frac{7}{12}\right)^k \mu_d.$$

In particular, for $i = 0, 1, 2$,

$$\lim_{k \to +\infty} d_i^{(k)} = \sqrt{3},$$

establishing the desired result. □

Proof of Theorem 1.2 when $\epsilon = -1$. For $\epsilon = -1$, we have that $r_d > 0$. Accordingly, without loss of generality, we may suppose that $e_{i_1} \leqslant e_{i_2} \leqslant e_{i_0}$, where $d_{i_1} \leqslant d_{i_2} \leqslant d_{i_0}$, with $i_0 = 0$ and $\{i_1, i_2\} = \{1, 2\}$.

By (3.9), (4.1) and (5.2), we have that

$$
\begin{aligned}
e_0^2 - 3 &= 1 - 2\langle R_1, R_2\rangle - 3 \\
&= -2(1 + \langle R_1, R_2\rangle) \\
&= -\frac{2}{\gamma}\Big(\gamma + 4(d_0^2+1)\big(\alpha d_1 d_2 - \chi(d_1+d_2)\big) \\
&\qquad - (d_0^2+1)(d_1^2-1)(d_2^2-1) + 2(d_0^2+1)(d_0^2-1)\Big) \\
&= -\frac{2}{3(d_1^2+1)(d_2^2+1)}\Big(3(d_1^2+1)(d_2^2+1) + 4\big(\alpha d_1 d_2 - \chi(d_1+d_2)\big) \\
&\qquad - (d_1^2-1)(d_2^2-1) + 2(d_0^2-1)\Big) \\
&= \frac{2}{3(d_1^2+1)(d_2^2+1)} U_0,
\end{aligned} \tag{5.12}
$$

with

$$U_0 := 4\chi(d_1+d_2) - 4\alpha d_1 d_2 + (d_1^2-1)(d_2^2-1) - 2(d_0^2-1) - 3(d_1^2+1)(d_2^2+1).$$

We remark that, using (3.7),

$$
\begin{aligned}
U_0 &= 4\chi(d_1+d_2) + 4\alpha(1 - d_1 d_2) - 2 + 2d_0^2 + 2d_1^2 + 2d_2^2 \\
&\qquad + (d_1^2-1)(d_2^2-1) - 2(d_0^2-1) - 3(d_1^2+1)(d_2^2+1) \\
&= 4\chi(d_1+d_2) + 4\alpha(1 - d_1 d_2) - 2(d_1^2+1)(d_2^2+1) \\
&= 4\chi(d_1+d_2) + 2(1 - d_0^2 - d_1^2 - d_2^2)(1 - d_1 d_2) - 2(d_1^2+1)(d_2^2+1) \\
&= 2\Big(2\chi(d_1+d_2) - d_0^2 - 2d_1^2 - 2d_2^2 - d_1 d_2 + (d_0^2 + d_1^2 + d_2^2)d_1 d_2 - d_1^2 d_2^2\Big) \\
&= 2\Big(2\chi(d_1+d_2) + (d_1 d_2 - 1)(d_0^2 - 3) + (d_1^2 + d_2^2 - 3 - d_1 d_2) \\
&\qquad + (d_1^2 - d_1 d_2 + d_2^2)(d_1 d_2 - 3)\Big) \\
&\leqslant 2\Bigg((d_1+d_2)\sqrt{\frac{1}{3}\sum_{i=0}^{2} d_i^2(d_{i+1}^2-3)(d_{i+2}^2-3)} \\
&\qquad + (d_1 d_2 - 1)(d_0^2 - 3) + (d_1^2 + d_2^2 - 3 - d_1 d_2) \\
&\qquad + (d_1^2 - d_1 d_2 + d_2^2)(d_1 d_2 - 3)\Bigg),
\end{aligned} \tag{5.13}
$$

where (3.10) is used for the inequality.

Also, we claim that

$$
\begin{aligned}
&(d_1 d_2 - 1)(d_0^2 - 3) + (d_1^2 + d_2^2 - 3 - d_1 d_2) + (d_1^2 - d_1 d_2 + d_2^2)(d_1 d_2 - 3) \\
&\qquad \leqslant (d_1^2 + d_2^2)(d_0^2 - 3).
\end{aligned} \tag{5.14}
$$

Indeed, if $d_0 = \sqrt{3}$, then (3.5) and (5.1) give that also $d_1 = d_2 = \sqrt{3}$, and so the left-hand side of (5.14) vanishes as well, thus establishing the desidered inequality.

If instead $d_0 > \sqrt{3}$, we recall (5.1) and we see that

$$\begin{aligned} & d_1 d_2 - 1 + \frac{d_1^2 + d_2^2 - 3 - d_1 d_2}{d_0^2 - 3} + d_1^2 - d_1 d_2 + d_2^2 \\ = \; & \frac{d_1^2 + d_2^2 - 3 - d_1 d_2 - d_0^2 + 3}{d_0^2 - 3} + d_1^2 + d_2^2 \\ = \; & \frac{\min\{d_1^2, d_2^2\} + \max\{d_1^2, d_2^2\} - d_1 d_2 - d_0^2}{d_0^2 - 3} + d_1^2 + d_2^2 \\ \leqslant \; & \frac{\min\{d_1^2, d_2^2\} - d_1 d_2}{d_0^2 - 3} + d_1^2 + d_2^2 \\ = \; & \frac{\min\{d_1, d_2\} \left(\min\{d_1, d_2\} - \max\{d_1, d_2\}\right)}{d_0^2 - 3} + d_1^2 + d_2^2 \\ \leqslant \; & d_1^2 + d_2^2, \end{aligned}$$

which implies the desired inequality in (5.14) in this case as well.

From (5.13) and (5.14), and recalling also (5.1), we thereby conclude that

$$\begin{aligned} U_0 &\leqslant 2 \left((d_1 + d_2) \sqrt{\frac{1}{3} \sum_{i=0}^{2} d_i^2} + d_1^2 + d_2^2 \right) (d_0^2 - 3) \\ &\leqslant 2 \left(2 d_0 \sqrt{\frac{1}{3} \sum_{i=0}^{2} d_0^2} + 2 d_0^2 \right) (d_0^2 - 3) \leqslant 8 d_0^2 (d_0^2 - 3). \end{aligned}$$

This and (5.12) entail that

$$e_0^2 - 3 \leqslant \frac{16 d_0^2 (d_0^2 - 3)}{3(d_1^2 + 1)(d_2^2 + 1)}. \tag{5.15}$$

Now, writing

$$D_i^{(k)} := (d_i^{(k)})^2 := 1 - 2\langle P_{i+1}^{(k)}, P_{i+2}^{(k)} \rangle,$$

we have that

$$3 \leqslant D_1^{(k)}, D_2^{(k)} \leqslant D_0^{(k)}.$$

With this notation, we deduce from (5.4) and (5.5) that

$$D_0^{(k+1)} - D_j^{(k+1)} \leqslant \rho (D_0^{(k)} - D_j^{(k)}) \qquad \text{for} \quad j = 1, 2, \tag{5.16}$$

where $\rho \in (0,1)$ is independent of k, and also independent of $P_0^{(0)} P_1^{(0)} P_2^{(0)} := P_0 P_1 P_2$.

Moreover (5.15) becomes

$$D_0^{(k+1)} - 3 \leqslant \frac{16 D_0^{(k)} (D_0^{(k)} - 3)}{3(D_1^{(k)} + 1)(D_2^{(k)} + 1)}. \tag{5.17}$$

By (5.16), we can choose k_0 so large that

$$\delta_0 := \rho^{k_0} \max\left\{D_0^{(0)} - D_1^{(0)}, D_0^{(0)} - D_2^{(0)}\right\} < 1.$$

In this way, we see that, for all $k \geqslant k_0$,

$$D_0^{(k)} - D_j^{(k)} \leqslant \rho^{k+1}\big(D_0^{(0)} - D_j^{(0)}\big) \leqslant \rho^{k_0}\big(D_0^{(0)} - D_j^{(0)}\big) \leqslant \delta_0.$$

Hence, by this and (5.17), for $k \geqslant k_0$,

$$\begin{aligned} D_0^{(k+1)} &\leqslant 3 + \frac{16 D_0^{(k)}(D_0^{(k)} - 3)}{3(D_0^{(k)} + 1 - \delta_0)^2} \\ &= 3 + \frac{16}{3}\left(1 - \frac{1-\delta_0}{D_0^{(k)} + 1 - \delta_0}\right)\left(1 - \frac{4-\delta_0}{D_0^{(k)} + 1 - \delta_0}\right) < \frac{25}{3}, \end{aligned}$$

namely $\beta_0 := 25/3$ is an upper bound for the sequence $\{D_0^{(k)} : k > k_0\}$.

We will now iterate this argument as follows. For $p \in \mathbb{N}$, suppose that we have $\delta_p \geqslant 0$ and an upper bound β_p for the sequence $\{D_0^{(k)} : k > k_p\}$ for some $k_p \in \mathbb{N}$. Notice that, in light of (5.17), the upper bound $\beta_p \geqslant 3$. Suppose also that $\delta_p > 0$ if $\beta_p > 3$, and $\delta_p = 0$ otherwise.

If $\beta_p = 3$, set $\delta_{p+1} := 0$, $\beta_{p+1} := 3$ and $k_{p+1} := k_p + 1$.

If instead $\beta_p > 3$, by (5.16), we can choose $k_{p+1} > k_p$ so large that

$$\delta_{p+1} := \rho^{k_{p+1}} \max\left\{D_0^{(0)} - D_1^{(0)}, D_0^{(0)} - D_2^{(0)}\right\} \leqslant \min\left\{1, \frac{\delta_p}{2}, \beta_p + 1 - 4\sqrt{\frac{\beta_p}{3}}\right\},$$

and we stress that the right hand side is positive because $\beta_p > 3$.

Also, by (5.17), for $k \geqslant k_{p+1} > k_p$, we have that $D_0^{(k)} \leqslant \beta_p$, and therefore

$$\begin{aligned} D_0^{(k+1)} &\leqslant 3 + \frac{16}{3}\left(1 - \frac{1-\delta_{p+1}}{D_0^{(k)} + 1 - \delta_{p+1}}\right)\left(1 - \frac{4-\delta_{p+1}}{D_0^{(k)} + 1 - \delta_{p+1}}\right) \\ &\leqslant 3 + \frac{16}{3}\left(1 - \frac{1-\delta_{p+1}}{\beta_p + 1 - \delta_{p+1}}\right)\left(1 - \frac{4-\delta_{p+1}}{\beta_p + 1 - \delta_{p+1}}\right) =: \beta_{p+1}, \end{aligned}$$

namely β_{p+1} is an upper bound for the sequence $\{D_0^{(k)} : k > k_{p+1}\}$.

Also,

$$\begin{aligned} \beta_{p+1} - \beta_p &= \frac{16}{3} \cdot \frac{\beta_p(\beta_p - 3)}{(\beta_p + 1 - \delta_{p+1})^2} - (\beta_p - 3) \\ &= \frac{(\beta_p - 3)}{3} \cdot \frac{16\beta_p - 3(\beta_p + 1 - \delta_{p+1})^2}{(\beta_p + 1 - \delta_{p+1})^2}. \end{aligned} \tag{5.18}$$

We point out that, from the definition of δ_{p+1}, we have that $16\beta_p - 3(\beta_p + 1 - \delta_{p+1})^2 \leqslant 0$, and thus $\beta_{p+1} \leqslant \beta_p$. We conclude that the sequence of upper bounds β_p is nonincreasing and bounded below by 3, and accordingly the following limit exists:

$$\beta_\infty := \lim_{p\to+\infty} \beta_p$$

with $\beta_\infty \geqslant 3$.

Moreover, evidently,

$$\lim_{p\to+\infty} \delta_p = 0.$$

As a result, taking limits in (5.18),

$$0 = \frac{(\beta_\infty - 3)}{3} \cdot \frac{16\beta_\infty - 3(\beta_\infty + 1)^2}{(\beta_\infty + 1)^2} = -\frac{(\beta_\infty - 3)^2(3\beta_\infty - 1)}{3(\beta_\infty + 1)^2}.$$

This entails that $\beta_\infty = 3$.

Hence, for $j = 1, 2$, we have that

$$3 \leqslant \lim_{k\to+\infty} D_j^{(k)} \leqslant \lim_{k\to+\infty} D_0^{(k)} = 3,$$

namely

$$\lim_{k\to+\infty} d_i^{(k)} = \sqrt{3},$$

yielding the desired result. □

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