Some Bourgain-Brezis type solutions via complex interpolation

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Abstract

In 2002 Bourgain and Brezis proved that given a vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{W}^{1,d}(\mathbb{R}^d)$ there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{W}^{1,d}(\mathbb{R}^d)$ such that $\operatorname{div} u = \operatorname{div} v$. We prove several results of a similar nature in which we take into consideration the Fourier support of the solutions. For instance, in the case $d \geq 3$ we prove the following: for any vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{B}_q^{d/p,p}(\mathbb{R}^d)$ (where $p \in [2,\infty)$ and $q \in (1,2)$), with *supp* $\widehat{v} \subseteq \mathbb{R}^d \backslash (-\infty,0)^d$, there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{B}_2^{d/p,p}(\mathbb{R}^d)$, with *supp* $\widehat{v} \subseteq \mathbb{R}^d \backslash (-\infty,0)^d$, such that

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{B}_2^{d/p,p}} \lesssim \|v\|_{\dot{B}_q^{d/p,p}}.$$

Our arguments rely on a version of the complex interpolation method combined with some ideas of Bourgain and Brezis.

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1 Introduction

1.1 Overview

Suppose $d \geq 2$ is an integer and consider some compactly supported function $f \in L^d\left(\mathbb{R}^d\right)$. Standard Calderón-Zygmund theory shows that there exists a vector field $u \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{W}^{1,d}\left(\mathbb{R}^d\right)$ such that

$$\operatorname{div} u = f, \tag{1}$$

in the distributions sense on $\mathbb{R}^d$. Indeed, it suffices to set

$$u := \nabla|\nabla|^{-2} f, \tag{2}$$

and to use the fact that the components of ∇u are of the form $R_i R_j u$ where $R_1, ..., R_d$ are the Riesz transforms on $\mathbb{R}^d$ ($\widehat{R_j\varphi}(\xi) = (\xi_j/|\xi|)\widehat{\varphi}(\xi)$, for any Schwartz φ). Since each R_j is a Calderón-Zygmund operator, we easily get that each component of ∇u belongs to $L^d(\mathbb{R}^d)$.

Note that the space $\dot{W}^{1,d}$ does not embed in L^∞ and hence, the solution in $\dot{W}^{1,d}$ provided by the expression (2) may fall outside L^∞ (see for instance the example given by L. Nirenberg in [5, Remark 7, p. 400]). However, as it was shown[1] by Bourgain and Brezis (2002), the fact that (1) admits a (possibly another) solution $u \in L^\infty(\mathbb{R}^d)$ is a direct consequence of the Gagliardo embedding ($W^{1,1}(\mathbb{R}^d) \hookrightarrow L^{d'}(\mathbb{R}^d)$, where $d' := d/(d-1)$) (see [5, Proposition 1]).

Even more, Bourgain and Brezis have proved in [5, Theorem 1] the following striking fact: there exists a solution u to (1) that is *simultaneously* bounded and in the "right" Sobolev space $\dot{W}^{1,d}(\mathbb{R}^d)$. In other words, there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{W}^{1,d}(\mathbb{R}^d)$ which is a solution to (1). In the general case where $d \geq 2$, this result was proved by an involved approximation argument using the Littlewood-Paley square function. We mention that the complicated construction

[1] The results of Bourgain and Brezis were stated in the case of the torus $\mathbb{T}^d$, however, it is easy to transfer these results on $\mathbb{R}^d$.

used in [5] can also be used in more general situations. By similar constructive methods, Bourgain and Brezis proved an analogue existence result for more general under-determined Hodge systems. Following the ideas in [5] and [6], Bousquet, Mironescu and Russ ([7], 2014) and later Bousquet, Russ, Wang and Yung ([8], 2017) provided generalizations of the Bourgain-Brezis results in the scale of Triebel-Lizorkin spaces (see also [11] for a non-commutative version). For instance, adapted to the case of the divergence equation, Theorem 2 in [8] gives us:

Theorem 1. *Suppose that $1 < p, q < \infty$ and consider some vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{F}_q^{d/p,p}(\mathbb{R}^d)$. Then, there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{F}_q^{d/p,p}(\mathbb{R}^d)$ such that*

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{F}_q^{d/p,p}} \lesssim \|v\|_{\dot{F}_q^{d/p,p}}.$$

Remark 2. *Note that by Calderón-Zygmund theory any (compactly supported) $f \in \dot{F}_q^{d/p-1,p}(\mathbb{R}^d)$ can be written as the divergence of the vector field $v = \nabla|\nabla|^{-2} f \in \dot{F}_q^{d/p,p}(\mathbb{R}^d)$. Hence, since $\dot{F}_2^{1,d} = \dot{W}^{1,d}$, when $p = d$ and $q = 2$, from Theorem 1 above we recover the result of Bourgain and Brezis.*

A similar existence result holds for the scale of Besov spaces, the proof being technically the same as for Theorem 1. Since in this paper we are concerned more with the Besov version, we explicitly state it below:

Theorem 3. *Suppose that $1 < p, q < \infty$ and consider some vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{B}_q^{d/p,p}(\mathbb{R}^d)$. Then, there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{B}_q^{d/p,p}(\mathbb{R}^d)$ that satisfies*

$$\operatorname{div} u = \operatorname{div} v,$$

and such that

$$\|u\|_{L^\infty \cap \dot{B}_q^{d/p,p}} \lesssim \|v\|_{\dot{B}_q^{d/p,p}}.$$

Remark 4. *Throughout the paper we will call the given vector v the source and u will be called solution. A similar convention will also be applied to more general equations.*

It is worth noticing that in the special case where $d = 2$ (and hence, $p = 2$) Bourgain and Brezis have found a much simpler proof of their existence result (see [5, Section 4, p. 403]). In this case the proof is by duality and it is non-constructive. One of the keys in their proof is showing that a certain symbol has a bounded Fourier transform; this is achieved not by an explicit computation but by a "robust" analytical argument. Another proof was found by Mazya (2007) for the case $p = q = 2$ of Theorem 1 (or equivalently of Theorem 3) (see [22]). Again, the proof is by duality and strikingly simple. In contrast to the argument given by Bourgain and Brezis, Mazya's argument relies on some explicit algebraic manipulations[2]. (See also [23] for some related discussions.) However, both approaches, namely that of Bourgain and Brezis in the case $d = 2$ and that of Mazya, are based on L^2-Fourier analysis arguments that are unlikely to be extended to the case where $p \neq 2$.

There is yet another situation of a different nature:

[2]There are also more recent arguments related to this type of problem. For instance, essentially the same statement as in Mazya's theorem was proved for $d = 2$ by Da Lio and Rivière in [14, Section II.2] (2021). Here, the argument is in the spirit of Mazya's proof, although the motivation seems to be different–inspired by some compensation compactness phenomena. In [15] (2021), Da Lio, Rivière and Wettstein consider another related problem. In this case the arguments given are either in the spirit of Mazya's method either they rely on the Bourgain-Brezis technique. In all these arguments the fact that $p = 2$ is quite important.

Proposition 5. *Let $d \geq 2$ be an integer and consider some $r \in (1, \infty)$. Then, for any vector field $v \in \dot{B}_1^{d/r,r}(\mathbb{R}^d)$ there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{B}_1^{d/r,r}(\mathbb{R}^d)$ such that*

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{B}_1^{d/r,r}} \lesssim \|v\|_{\dot{B}_1^{d/r,r}}.$$

Indeed, since $\dot{B}_1^{d/r,r}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \cap \dot{B}_1^{d/r,r}(\mathbb{R}^d)$, it suffices to set $u := v$ (see Subsection 2.1 for the definition of $\dot{B}_1^{d/r,r}(\mathbb{R}^d)$ we use in this paper). Since Proposition 5 and Theorem 3 have much easier proofs than the constructive proof used in [5] and in [7], it would be interesting to find a way to "interpolate" between Proposition 5 and Theorem 3.

1.2 A naive interpolation strategy

One may try to interpolate in the following way. By duality (see for instance [5, Lemma 1, p. 395]) and the lifting property of the Besov spaces we can reformulate Proposition 5 and Theorem 3

$$\|g\|_{\dot{B}_\infty^{-d/r+1,r'}} \sim \|\nabla g\|_{\dot{B}_\infty^{-d/r,r'}} \lesssim \|\nabla g\|_{\mathcal{M} + \dot{B}_\infty^{-d/r,r'}}, \tag{3}$$

and respectively,

$$\|g\|_{\dot{H}^{-d/2+1}} \sim \|\nabla g\|_{\dot{H}^{-d/2}} \lesssim \|\nabla g\|_{\mathcal{M} + \dot{H}^{-d/2}}, \tag{4}$$

for any Schwartz function g with $\widehat{g}$ vanishing in a neighborhood of 0[3].

For each non-trivial Banach function space Y on $\mathbb{R}^d$ denote by $\mathcal{G}(Y)$ the space of all vector fields in Y that are gradients, i.e.,

$$\mathcal{G}(Y) := \{g \in Y \mid g = \nabla g_1, \text{ for some tempered distribution } g_1\},$$

endowed with the norm of Y.

With this notation one can view (3) and (4) as the embeddings

$$\mathcal{G}(\mathcal{M} + \dot{B}_\infty^{-d/r,r'}) \hookrightarrow \dot{B}_\infty^{-d/r+1,r'},$$

and respectively,

$$\mathcal{G}(\mathcal{M} + \dot{H}^{-d/2}) \hookrightarrow \dot{H}^{-d/2+1}.$$

By complex interpolation (we may consider the real interpolation as well), we conclude that, for any $\theta \in (0, 1)$,

$$(\mathcal{G}(\mathcal{M} + \dot{B}_\infty^{-d/r,r'}), \mathcal{G}(\mathcal{M} + \dot{H}^{-d/2}))_\theta \hookrightarrow (\dot{B}_\infty^{-d/r+1,r'}, \dot{H}^{-d/2+1})_\theta. \tag{5}$$

The right hand side of (5) can be easily computed explicitly (see for instance [2, Chapter 6]):

$$(\dot{B}_\infty^{-d/r+1,r'}, \dot{H}^{-d/2+1})_\theta = \dot{B}_{q'}^{-d/p+1,p'},$$

where $1/p = (1-\theta)/r + \theta/2$ and $1/q = (1-\theta)/1 + \theta/2$. Now we would like to have

$$\mathcal{G}(\mathcal{M} + \dot{B}_{q'}^{-d/p,p'}) \hookrightarrow (\mathcal{G}(\mathcal{M} + \dot{B}_\infty^{-d/r,r'}), \mathcal{G}(\mathcal{M} + \dot{H}^{-d/2}))_\theta, \tag{6}$$

[3] Although, the inequalities of this type are expressed in [5] mostly by using the space L^1, here, for the sake of consistency with the other parts of the paper we use everywhere $\mathcal{M}$ instead (the two versions being equivalent).

and combining this with (5), we would get via the closed range theorem the fact that for any vector field $v \in \dot{B}^{d/p,p}_q(\mathbb{R}^d)$ there exists another vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{B}^{d/p,p}_q(\mathbb{R}^d)$ of the same divergence as v. However, computing explicitly the left hand side of (5) or proving (6) only by interpolation theory is a quite difficult task. Naively we may have the following strategy for proving (6). We can observe that, since we have the embeddings $\mathcal{M}, \dot{B}^{-d/r,r'}_\infty \hookrightarrow \mathcal{M} + \dot{B}^{-d/r,r'}_\infty$ and $\mathcal{M}, \dot{H}^{-d/2} \hookrightarrow \mathcal{M} + \dot{H}^{-d/2}$, we can conclude by interpolation that

$$\mathcal{M}, \dot{B}^{-d/p,p'}_{q'} \hookrightarrow (\mathcal{M} + \dot{B}^{-d/r,r'}_\infty, \mathcal{M} + \dot{H}^{-d/2})_\theta$$

or equivalently

$$\mathcal{M} + \dot{B}^{-d/p,p'}_{q'} \hookrightarrow (\mathcal{M} + \dot{B}^{-d/r,r'}_\infty, \mathcal{M} + \dot{H}^{-d/2})_\theta.$$

Consequently,

$$\mathcal{G}(\mathcal{M} + \dot{B}^{-d/p,p'}_{q'}) \hookrightarrow \mathcal{G}(\mathcal{M} + \dot{B}^{-d/r,r'}_\infty, \mathcal{M} + \dot{H}^{-d/2}_\theta).$$

Hence, in order to obtain (6) it would be sufficient to have

$$\mathcal{G}((\mathcal{M} + \dot{B}^{-d/r,r'}_\infty, \mathcal{M} + \dot{H}^{-d/2})_\theta) \hookrightarrow (\mathcal{G}(\mathcal{M} + \dot{B}^{-d/r,r'}_\infty), \mathcal{G}(\mathcal{M} + \dot{H}^{-d/2}))_\theta,$$

or, equivalently, since the other embedding is trivial,

$$\mathcal{G}((\mathcal{M} + \dot{B}^{-d/r,r'}_\infty, \mathcal{M} + \dot{H}^{-d/2})_\theta) = (\mathcal{G}(\mathcal{M} + \dot{B}^{-d/r,r'}_\infty), \mathcal{G}(\mathcal{M} + \dot{H}^{-d/2}))_\theta. \tag{7}$$

We can further reformulate this fact as

$$\mathcal{N} \cap (Y_0, Y_1)_\theta = (\mathcal{N} \cap Y_0, \mathcal{N} \cap Y_1)_\theta\,, \tag{8}$$

where $Y_0 = \mathcal{M} + \dot{B}^{-d/r,r'}_\infty$, $Y_1 = \mathcal{M} + \dot{H}^{-d/2}$ and $\mathcal{N}$ is the spaces of the fields in $Y_0 + Y_1$ that are gradients, i.e, $\mathcal{N} := \mathcal{G}(Y_0 + Y_1)$. The difficulty of proving (7) consists in the fact that, for the general situation when Y_0, Y_1, $\mathcal{N}$ are Banach spaces the question whether or not (8) holds does not have yet a satisfactory answer. If we replace the complex interpolation method in (8) with the real K-method of interpolation, then, (8) may be false for some particular choice of the spaces Y_0, Y_1, $\mathcal{N}$ (see for instance [21]). In the case of the complex interpolation method it seems that even less is known about the validity of (8).

This naive interpolation strategy seems inappropriate to prove (6) or even a weaker statement like

$$\mathcal{G}(\mathcal{M} + \dot{B}^{-d/p,p'}_1) \hookrightarrow (\mathcal{G}(\mathcal{M} + \dot{B}^{-d/r,r'}_\infty), \mathcal{G}(\mathcal{M} + \dot{H}^{-d/2}))_\theta,$$

which corresponds to the following existence result:

Proposition 6. *Let $d \geq 2$ be an integer and consider some parameters $p \in (2, \infty)$ and $q \in (1, 2)$. Then, for any vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{B}^{d/p,p}_q(\mathbb{R}^d)$ there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{B}^{d/p,p}_\infty(\mathbb{R}^d)$ such that*

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{B}^{d/p,p}_\infty} \lesssim \|v\|_{\dot{B}^{d/p,p}_q}.$$

1.3 The main results

In this paper we take into consideration the spectrum of solutions. In what follows, the spectrum of a tempered distribution v is its Fourier support, i.e., $spec(v) := supp\ \widehat{v}$ (see Remark 14 and Subsection 2.3 for another version of the spectrum). Adapted to the case of the divergence equation, our main result reads:

Theorem 7. *Let $d \geq 3$ be an integer and consider the set $\Delta := \mathbb{R}^d \setminus (-\infty, 0)^d$. Consider some parameters $p \in [2, \infty)$ and $q \in (1, 2)$. Then, for any vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{B}_q^{d/p,p}(\mathbb{R}^d)$, with $spec(v) \subseteq \Delta$ there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{B}_2^{d/p,p}(\mathbb{R}^d)$, with $spec(u) \subseteq \Delta$ such that*

$$\operatorname{div} u = \operatorname{div} v, \tag{9}$$

and

$$\|u\|_{L^\infty \cap \dot{B}_2^{d/p,p}} \lesssim \|v\|_{\dot{B}_q^{d/p,p}}.$$

In the case where $d = 2$ our method does not provide solutions with the spectrum in Δ. Nevertheless, one can obtain solutions with the spectrum in a different type of sets. For each $\delta \in (0, \pi/4)$ let C_δ be the symmetric cone

$$C_\delta := \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid |\xi_1| \leq (\tan \delta)|\xi_2|\}.$$

With this notation we have:

Theorem 8. *Consider the numbers $\delta \in (0, \pi/8)$, $\varepsilon \in (0, 1]$ and some parameters $p \in [2, \infty)$ and $q \in (1, 2)$. Then, for any vector field $v \in \mathcal{S}'(\mathbb{R}^2) \cap \dot{B}_q^{2/p,p}(\mathbb{R}^2)$, with $spec(v) \subseteq C_\delta$, there exists a vector field $u \in L^\infty(\mathbb{R}^2) \cap \dot{B}_2^{2/p,p}(\mathbb{R}^2)$, with $spec(u) \subseteq C_{(1+\varepsilon)\delta}$, such that*

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{B}_2^{2/p,p}} \lesssim \|v\|_{\dot{B}_q^{2/p,p}}.$$

When compared with Theorem 3 one can observe that Theorem 7 (or Theorem 8) has two major drawbacks. First, we are not allowed to take $p < 2$ or $q \geq 2$ as parameters for the space $\dot{B}_q^{d/p,p}$ on the source side. Secondly, for the space in which we obtain the solution we lose some control of the "third parameter". In other words, we would prefer to obtain $L^\infty \cap \dot{B}_q^{d/p,p}$ for the solution space instead of $L^\infty \cap \dot{B}_2^{d/p,p}$ which is a slightly larger space. (See however, Lemma 37 for a "perfect" version of our results in the case $p = q = 2$.) On the other hand it is unlikely that one can deduce Theorem 7 directly from Theorem 3. Indeed, given a vector field $v \in \dot{B}_q^{d/p,p}$, with $spec(v) \subseteq \Delta$, by Theorem 3 one can find some vector field $u \in L^\infty \cap \dot{B}_q^{d/p,p}$ such that $\operatorname{div} u = \operatorname{div} v$, however, not necessarily with $spec(u) \subseteq \Delta$. It is not obvious that one can obtain a solution u with $spec(u) \subseteq \Delta$ by direct methods: suppose P_Δ is the Fourier projection on Δ, i.e., $P_\Delta = I - P_+$, where P_+ is the Riesz projection and I is the identity operator. We have $P_\Delta v = v$ and we can write

$$\operatorname{div} P_\Delta u = \operatorname{div} v.$$

However, since P_D is not bounded on L^∞, we may *not* have $P_\Delta u \in L^\infty$, i.e., $P_\Delta u$ is *not* in general a candidate for a solution. The same observation applies to Theorem 8. To our knowledge, except for the method we give in this paper, there is no other method in the literature able to prove results like Theorem 7 or Theorem 8.

In fact, when $d \geq 3$, we will prove a more general result than Theorem 7. Our methods allow us to work with more general Fourier multipliers than the usual derivatives. In order to formulate our result we first need some preparations.

Let $\sigma \in C^{d+2}(\mathbb{R}^d \backslash \{0\}, \mathbb{R})$ be a function. We consider the following properties (that may or not be satisfied by σ):

(P1) The function σ satisfies the estimate

$$|\nabla^\alpha \sigma(\xi)| \lesssim |\xi|^{1-|\alpha|},$$

on $\mathbb{R}^d \backslash \{0\}$, for any multiindex $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq d+2$;

(P2) The function σ is odd in the variable ξ_1 and even in any other variable, i.e.,

$$\sigma(\epsilon_1\xi_1, \epsilon_2\xi_2, ..., \epsilon_d\xi_d) = \epsilon_1 \sigma(\xi_1, \xi_2, ..., \xi_d),$$

on $\mathbb{R}^d$, for any signs $\epsilon_1, ..., \epsilon_d \in \{-1, 1\}$.

Introduce the new functions $\sigma_1, ..., \sigma_d$ defined by

$$\sigma_j(\xi_1, \xi_2, ..., \xi_d) := \sigma(\xi_j, \xi_2, ...\xi_{j-1}, \xi_1, \xi_{j+1}..., \xi_d), \tag{10}$$

on $\mathbb{R}^d$, for any index $j \in \{1, 2, ..., d\}$.

Consider some half-spaces[4] $D_1, ..., D_d \subset \mathbb{R}^d$ and a family of functions $G_1, ..., G_d : \mathbb{R}^d \to \mathbb{R}^{d-1}$. We say that the function G_j is *adapted* to the half-space D_j if there exists a rotation R on $\mathbb{R}^d$ (depending on j) and a function $\sigma : \mathbb{R}^d \to \mathbb{R}$ (depending on j) satisfying (P1), (P2) such that $D_j = R(U)$, where $U := \mathbb{R}^{d-1} \times (0, \infty)$, and

$$G_j = (\sigma_1 \circ R, ..., \sigma_{d-1} \circ R). \tag{11}$$

We say that the family of functions $G_1, ..., G_d : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is *adapted* to the family of half-spaces $D_1, ..., D_d \subset \mathbb{R}^d$, if for each $j \in \{1, ..., d\}$ the function G_j is adapted to D_j and

$$\sum_{j=1}^{d} |G_j(\xi)| \mathbf{1}_{D_j}(\xi) \sim |\xi| \mathbf{1}_D(\xi), \tag{12}$$

on $\mathbb{R}^d$, where $D := \cup_{j=1}^d D_j$.

Let us recall now some standard notation concerning the Fourier multipliers. To a scalar valued function $m \in L^1_{loc}(\mathbb{R}^d \backslash \{0\}, \mathbb{R})$ of polynomial growth ($|m(\xi)| \lesssim 1 + |\xi|^N$, for some positive number N and for any ξ with $|\xi| > 1$) we associate the Fourier multiplier $m(\nabla)$ defined by the relation

$$\widehat{m(\nabla) f}(\xi) := m(\xi) \widehat{f}(\xi),$$

[4]In this paper by a *half-space* we mean any rotation of the set $U = \mathbb{R}^{d-1} \times (0, \infty)$. We will not consider translations of these sets.

on $\mathbb{R}^d$, for any Schwartz function f whose Fourier transform $\widehat{f}$ is compactly supported and vanishing in a neighborhood of 0. In most of the cases one can extend the meaning of $m(\nabla)$ as follows. Let us denote by $\mathcal{S}_{\sharp,c}$ the space of all Schwartz function f whose Fourier transform $\widehat{f}$ is compactly supported and vanishing in a neighborhood of 0. Suppose E and F are some Banach function spaces on $\mathbb{R}^d$ such that $\mathcal{S}_{\sharp,c}$ is dense in E and

$$\|m(\nabla)f\|_F \lesssim \|f\|_E,$$

for any $f \in \mathcal{S}_{\sharp,c}$. Then, by linearity and density $m(\nabla)$ can be uniquely extended to a bounded operator $m(\nabla) : E \to F$ (see also Remark 13). We will often say that m is the symbol of the Fourier multiplier $m(\nabla)$.

To a vector valued function $G : \mathbb{R}^d \to \mathbb{R}^{d-1}$, with $G = (G^1, ..., G^{d-1})$, where $G^1, ..., G^{d-1} : \mathbb{R}^d \to \mathbb{R}$ are scalar functions of polynomial growth, we associate the vector-valued Fourier multiplier

$$G(\nabla) := (G^1(\nabla), ..., G^{d-1}(\nabla)).$$

In other words, if $f \in \mathcal{S}_{\sharp,c}$, by $G(\nabla)f$ we mean

$$G(\nabla)f := (G^1(\nabla)f, ..., G^{d-1}(\nabla)f).$$

Suppose $u^1, ..., u^{d-1} \in \mathcal{S}_{\sharp,c}$ and let u be the $(d-1)$-vector field[5] $u := (u^1, ..., u^{d-1})$. By $G(\nabla) \cdot u$ we mean

$$G(\nabla) \cdot u := G^1(\nabla)u^1 + ... + G^{d-1}(\nabla)u^{d-1}.$$

Now we can formulate our generalisation of Theorem 7.

Theorem 9. *Let $d \geq 3$ be an integer and consider some parameters $p \in [2, \infty)$ and $q \in (1, 2)$. Suppose that the family of functions $G_1, ..., G_d : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is adapted to the family of half-spaces $D_1, ..., D_d \subset \mathbb{R}^d$.*

Then, for any system of $(d-1)$-vector fields $(v_j)_{j=1,..,d}$ with $v_j \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{B}_q^{d/p,p}(\mathbb{R}^d)$ and $spec(v_j) \subseteq D_j$, there exists a system of $(d-1)$-vector fields $(u_j)_{j=1,..,d}$, with $u_j \in L^\infty(\mathbb{R}^d) \cap \dot{B}_2^{d/p,p}(\mathbb{R}^d)$ and $spec(u_j) \subseteq D_j$, such that

$$\sum_{j=1}^{d} G_j(\nabla) \cdot u_j = \sum_{j=1}^{d} G_j(\nabla) \cdot v_j, \tag{13}$$

and

$$\sum_{j=1}^{d} \|u_j\|_{L^\infty \cap \dot{B}_2^{d/p,p}} \lesssim \sum_{j=1}^{d} \|v_j\|_{\dot{B}_q^{d/p,p}}.$$

In this paper equations such as (9) or (13) will be called *divergence-like* equations.

In Subsection 5.1 we will see that Theorem 9 easily implies Theorem 7.

1.4 About the proofs

Our proofs of Theorem 9, Theorem 7 and Theorem 8 are based on two ingredients:

1. The $\mathcal{W}$-method of interpolation. This is the key method that we are using throughout the paper. Let (A_0, A_1) and (B_0, B_1) be compatible couples of Banach spaces and $T : A_0 + A_1 \to$

[5]In this paper the distributions with $d-1$ components will be called $(d-1)$-vector fields.

$B_0 + B_1$ be a linear operator such that $T(B_j) \hookrightarrow T(A_j)$, for any $j = 0, 1$ (see (60) for a precise meaning for the embedding of the images). Suppose we want to see under which conditions on the spaces involved and the operator T we have

$$T\left(F_\theta^1\left(B_0, B_1\right)\right) \hookrightarrow T\left(F_\theta^2\left(A_0, A_1\right)\right), \tag{14}$$

for some θ-interpolation functors F_θ^1, F_θ^2. One can say that, in some sense, we "interpolate" linear equations or that we preserve some form of surjectivity of the operator T. In order to give reasonable sufficient conditions for (14) to hold for some convenient interpolation functors, we introduce a variant of the complex interpolation method which will be called the $\mathcal{W}$-method[6]. In our case F_θ^1 will be given by the usual complex method of interpolation and F_θ^2 will be given by our $\mathcal{W}$-method. Roughly speaking this method consists in the following. Suppose (A_0, A_1) is a compatible couple of Banach spaces. In order to define the interpolation space of the couple (A_0, A_1) via the $\mathcal{W}$-method we use the three lines lemma on the strip as in the standard complex method of interpolation. However, instead of quantifying the endpoint regularity of the analytic functions involved via the norms $L^\infty(\mathbb{R}, A_j)$ $(j = 0, 1)$ we use slightly more complicated quantities that depend on some prescribed pair of non-trivial Banach spaces (X_0, X_1). In this way, for each $\theta \in (0, 1)$, we obtain an interpolation space that will be denoted by $(A_0, X_0 \mid A_1, X_1)_\theta$. The efficiency of the $\mathcal{W}$-method relies (between other facts) on properly choosing the spaces X_0, X_1.

When the spaces X_j have the UMD property, under some additional embedding assumptions concerning the Banach spaces involved, the $\mathcal{W}$-method of interpolation preserves the "surjectivity" of operators (i.e., (14) holds). The main requirements for applying the $\mathcal{W}$-method are twofold:

(i) On the one hand one needs to verify some embedding conditions for the domains and the co-domains of the operator. We give some simple necessary conditions that are easy to formulate, however not sharp. We also mention that, in the absence of any such conditions, surjectivity may not be preserved (see the examples in the second part of Subsection 3.3).

(ii) On the other hand explicitly computing the space $(A_0, X_0 \mid A_1, X_1)_\theta$ seems to be difficult in practice. However, there are particular situations in which we can embed $(A_0, X_0 \mid A_1, X_1)_\theta$ in some convenient space. More precisely, when A_j are of the form $A_j = A \cap X_j$, for some non-trivial Banach space A, we have

$$(A_0, X_0 \mid A_1, X_1)_\theta = (A \cap X_0, X_0 \mid A \cap X_1, X_1)_\theta \hookrightarrow A \cap (X_0, X_1)_\theta .$$

For instance, when we have $X_0 = \dot{B}_2^{d/r,r}$, $X_1 = \dot{B}_2^{d/2,2}$ and $A = L^\infty$, the above embedding becomes

$$(L^\infty \cap \dot{B}_2^{d/r,r}, \dot{B}_2^{d/p,p} \mid L^\infty \cap \dot{B}_2^{d/2,2}, \dot{B}_2^{d/2,2})_\theta \hookrightarrow L^\infty \cap (\dot{B}_2^{d/r,r}, \dot{B}_2^{d/2,2})_\theta.$$

By using only the result of Mazya (Theorem 3 in the case $p = q = 2$) and the $\mathcal{W}$-method together with the embedding $\dot{B}_1^{d/p,p}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ we easily obtain the following:

Theorem 10. *Let $d \geq 2$ be an integer and consider some parameters $p \in [2, \infty)$ and $q \in (1, 2)$. Then, for any vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{B}_q^{d/p,p}(\mathbb{R}^d)$ there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{B}_2^{d/p,p}(\mathbb{R}^d)$ such that*

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{B}_2^{d/p,p}} \lesssim \|v\|_{\dot{B}_q^{d/p,p}}.$$

[6]Here, "$\mathcal{W}$" stands for "weak".

One can even obtain an analogue of Theorem 10 for a class of Lorentz-Sobolev spaces (for definitions see Subsection 2.2). Namely, by using only the result of Mazya and the $\mathcal{W}$-method, together with some standard facts in the theory of Lorentz spaces, we easily obtain the following:

Theorem 11. *Let $d \geq 2$ be an integer and consider some parameters $p \in [2, \infty)$ and $q \in (1, 2)$. Then, for any vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{H}^{d/p}L^{p,q}(\mathbb{R}^d)$ there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{H}^{d/p}L^{p,2}(\mathbb{R}^d)$ such that*

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{H}^{d/p}L^{p,2}} \lesssim \|v\|_{\dot{H}^{d/p}L^{p,q}}.$$

The conditions $p \geq 2$ and $q < 2$ in Theorem 10, Theorem 11, as well as in Theorem 7 and Theorem 9, are induced by some technical limitations of the $\mathcal{W}$-method (see Subsection 5.2).

2. The Bourgain-Brezis technique. In [5, Section 4, p. 403] Bourgain and Brezis proved the torus analogue of Theorem 3 in the case where $p = q = d = 2$. They concluded the existence of solutions for the divergence equation by duality. Namely, they proved that (see [5, Lemma 2, p. 403])

$$\|u\|_{L^2(\mathbb{T}^2)} \lesssim \|\nabla u\|_{\mathcal{M}(\mathbb{T}^2)+H^{-1}(\mathbb{T}^2)}, \tag{15}$$

for any $u \in \mathcal{M}(\mathbb{T}^2)$, with $\widehat{u}(0) = 0$. In order to obtain this, they used the following estimate for the sin function (see [5, (4.20), p. 405]):

$$\left| \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \frac{n_1 n_2}{|n|^4} \sin n_1\theta_1 \sin n_2\theta_2 \right| \leq C,$$

uniformly in $\theta_1, \theta_2 \in \mathbb{T}$, for some numerical constant $C > 0$. By convexity this allows us to write

$$\left| \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \frac{n_1 n_2}{|n|^4} \widehat{F}_1(n)\widehat{F}_2(n) \right| \leq C\|F_1\|_{\mathcal{M}(\mathbb{T}^2)}\|F_2\|_{\mathcal{M}(\mathbb{T}^2)},$$

for any $F_1, F_2 \in \mathcal{M}(\mathbb{T}^2)$. This bilinear estimate turns out to be a key ingredient thanks to which we can deal with the space $\mathcal{M}(\mathbb{T}^2)$ in (15).

We use the technique introduced by Bourgain and Brezis in [5, Section 4, p. 403] and we prove a version of Theorem 9 (and Theorem 8) in the case where the source space is $\dot{H}^{d/2}$. As we will see, thanks to this technique we are able to work with more general Fourier multipliers than derivatives. Also, it is this technique that allows us to gain some control on the Fourier spectrum of solutions. The results obtained by this method are "perfect" in the sense that that the source space is $\dot{H}^{d/2}$ and the solution space is $L^\infty \cap \dot{H}^{d/2}$; there is no loss of regularity in the third parameter. The drawback of this technique is the fact that it does not apply to the case where $p \neq 2$.

As in the case of Theorem 10, we can easily obtain Theorem 9 using the $\mathcal{W}$-method. This time however, instead of using Mazya's result we use the more general results that we obtain via the Bourgain-Brezis technique. Using the properties of Lorentz spaces we can give a Lorentz-Sobolev version of Theorem 9. In fact, our methods will provide more general results. On the one hand the function spaces we work with can be more general than those in the statements of our final results (see for instance Theorem 39). One the other hand, the conditions imposed on the Fourier multipliers and the Fourier spectrum of the solutions can be more general. Also, by using the technique of Mazya, one can easily obtain a version of Theorem 3 in the case $p = q = 2$ that concerns general Hodge systems. Combining this result with the $\mathcal{W}$-method one can obtain an analogue of Theorem 10 for Hodge systems. We will not consider however, such issues here. In this paper, we limit ourselves to some model situations that are easier to describe.

Notation and conventions. Throughout the paper we use mainly standard notation. For instance, we often use the symbols $\lesssim$ and $\sim$. For two non-negative variable quantities a and b we write $a \lesssim b$ if there exists a constant $C > 0$ such that $a \leq Cb$. If $a \lesssim b$ and $b \lesssim a$, then we write $a \sim b$. If the implicit constant in $a \lesssim b$ depends on some parameters, say θ, ε,..., we write $a \lesssim_{\theta,\varepsilon,...} b$. Everywhere in this paper $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions on $\mathbb{R}^d$. We denote by $\mathcal{S}_{\sharp,c} = \mathcal{S}_{\sharp,c}(\mathbb{R}^d)$ the space of all Schwartz function f on $\mathbb{R}^d$ whose Fourier transform $\widehat{f}$ is compactly supported and vanishing in a neighborhood of 0. By $\mathcal{M}$ we mean the space of all complex Radon measures on $\mathbb{R}^d$. When X is a function space on $\mathbb{R}^d$ and $u = (u_1, ..., u_d)$ is a vector filed on $\mathbb{R}^d$ where each u_j belongs to X, we write $u \in X$ instead of $u \in X \times ... \times X = X^d$. A similar convention will be made for the $(d-1)$-vector fields. Other notation will be introduced when needed.

All the Banach spaces explicitly mentioned in this paper are complex (over $\mathbb{C}$).

2 Function spaces

In this Section we quickly recall the definition and some properties of some standard function spaces.

2.1 Sobolev and Besov spaces

When $1 < p < \infty$ and $\alpha \in \mathbb{R}$ the homogeneous space $\dot{H}^{\alpha,p}(\mathbb{R}^d)$ is obtained by completion of $\mathcal{S}_{\sharp,c}$ under the norm

$$\|f\|_{\dot{H}^{\alpha,p}} := \||\nabla|^{\alpha} f\|_{L^p}.$$

When $\alpha \in \mathbb{N}^*$ and $1 < p < \infty$ we have $\dot{H}^{\alpha,p}(\mathbb{R}^d) = \dot{W}^{\alpha,p}(\mathbb{R}^d)$. Also, when $p = 2$, the space $\dot{H}^{\alpha,2}(\mathbb{R}^d)$ is denoted by $\dot{H}^{\alpha}(\mathbb{R}^d)$.

We can see that we can also define the above homogeneous spaces $\dot{H}^{\alpha,p}$ by completion of the the normed function spaces $\dot{H}^{\alpha,p}_c(\mathbb{R}^d)$. Here, $\dot{H}^{\alpha,p}_c(\mathbb{R}^d)$ is the space of all the compactly supported functions whose $\dot{H}^{\alpha,p}$-norm is finite. The spaces $\dot{H}^{\alpha,p}$ as defined here are complete.

We continue by briefly recalling the definition of the Besov spaces (we do not define here the Triebel-Lizorkin spaces; see [31] for details). Consider a radial function $\Phi \in C_c^{\infty}(\mathbb{R}^d)$ such that *supp* $\Phi \subset B(0, 2)$ and $\Phi \equiv 1$ on $B(0, 1)$. For $k \in \mathbb{Z}$ we define the operators P_k, acting on the space of tempered distributions on $\mathbb{R}^d$, by the relation

$$\widehat{P_k f}\,(\xi) := \left(\Phi\left(\frac{\xi}{2^k}\right) - \Phi\left(\frac{\xi}{2^{k-1}}\right)\right)\widehat{f}\,(\xi)\,, \tag{16}$$

for any Schwartz function f on $\mathbb{R}^d$. The operators P_k will be called *Littlewood-Paley "projections"* adapted to $\mathbb{R}^d$. For any Schwartz function f we have that

$$f = \sum_{k\in\mathbb{Z}} P_k f,$$

in the sense of tempered distributions. The homogeneous Besov space $\dot{B}^{\alpha,p}_q(\mathbb{R}^d)$ (with $1 \le p, q \le \infty$ and α a real number) is obtained by completion of $\mathcal{S}_{\sharp,c}$ under the norm

$$\|f\|_{\dot{B}^{\alpha,p}_q} := \left(\sum_{j\in\mathbb{Z}} 2^{\alpha k q} \|P_k f\|^q_{L^p} \right)^{1/q}.$$

We have $\dot{B}^{\alpha,2}_2(\mathbb{R}^d) = \dot{H}^\alpha(\mathbb{R}^d)$ with equivalent norms.

The main advantage of our definition of the homogeneous Besov spaces is the fact that, whenever $\alpha_0 - d/p_0 = \alpha_1 - d/p_1$ and $\alpha_1 > \alpha_0$ we have the embedding

$$\dot{B}^{\alpha_1,p_1}_{q_1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{\alpha_0,p_0}_{q_0}(\mathbb{R}^d), \tag{17}$$

for any $q_0, q_1 \in [1,\infty)$ with $q_0 \le q_1$.

Note that we have the following dilation properties:

$$\|f(\lambda\cdot)\|_{\dot{B}^{\alpha,p}_q} \sim \lambda^{\alpha-d/p}\|f\|_{\dot{B}^{\alpha,p}_q}, \tag{18}$$

for any $f \in \dot{B}^{\alpha,p}_q(\mathbb{R}^d)$ respectively and any $\lambda > 0$. In particular, when $\alpha = d/p$ the spaces $\dot{B}^{\alpha,p}_q$ have the same scaling property as L^∞. In what follows the spaces of the form $\dot{B}^{d/p,p}_q$ will be called critical. It is worth recalling here, that, by a direct application of the Bernstein-Nikolskiĭ inequalities we get the embedding $\dot{B}^{d/p,p}_1(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. When $q > 1$ the critical spaces $\dot{B}^{d/p,p}_q$ do not embed in L^∞.

Any space of the form $\dot{B}^{\alpha,p}_q$ with $p, q \in (1,\infty)$ is embedded in $l^\alpha_q(L^p)$ and hence it has the UMD property (see for instance [26]).

Another aspect that we consider is the action of the multipliers $G_j(\nabla)$ (defined in Subsection 1.3) on the Besov spaces. For any real number α, any $p \in (1,\infty)$, $q \in [1,\infty]$ and any rotation $R : \mathbb{R}^d \to \mathbb{R}^d$ we have

$$\|(\sigma \circ R)(\nabla)f\|_{\dot{B}^{\alpha-1,p}_q} \sim \|(\sigma \circ R)(\nabla)|\nabla|^{-1}f\|_{\dot{B}^{\alpha,p}_q} \lesssim \|f\|_{\dot{B}^{\alpha,p}_q}, \tag{19}$$

for any $f \in \mathcal{S}_{\sharp,c}$. This is a direct consequence of Theorem 8.2 in [24], since by (P1) and the Leibniz rule, the function $\xi \to (\sigma \circ R)(\xi)|\xi|^{-1}$, defined on $\mathbb{R}^d\backslash\{0\}$, is a Mikhlin symbol. By (19) we have the boundedness of $G_j(\nabla) : \mathcal{S}_{\sharp,c} \cap \dot{B}^{\alpha,p}_q \to \dot{B}^{\alpha-1,p}_q$ which can be uniquely extended by linearity to a bounded operator

$$G_j(\nabla) : \dot{B}^{\alpha,p}_q \to \dot{B}^{\alpha-1,p}_q, \tag{20}$$

for each $j \in \{1, ...d\}$.

Remark 12. *Note that the spaces $\dot{B}^{d/p,p}_q$ (with $q > 1$) as defined here contain elements that are not tempered distributions. However, when $\alpha < d/p$ the elements of the space $\dot{B}^{\alpha,p}_q$ are all tempered distributions (see for instance [1, Remark 2.26, p.68] or [3]).*

Remark 13. *Since the operator* $\mathrm{div} : \mathcal{S}_{\sharp,c} \cap \dot{B}^{d/p,p}_q \to \dot{B}^{d/p-1,p}_q$ *is linear and bounded (here, $\mathcal{S}_{\sharp,c} \cap \dot{B}^{d/p,p}_q$ is endowed with the norm induced by $\dot{B}^{d/p,p}_q$), by density of $\mathcal{S}_{\sharp,c} \cap \dot{B}^{d/p,p}_q$ in $\dot{B}^{d/p,p}_q$ it extends uniquely to an operator* $\mathrm{div} : \dot{B}^{d/p,p}_q \to \dot{B}^{d/p-1,p}_q$. *Similar facts hold for other spaces and other operators as those obtained as linear combinations of the operators $G_j(\nabla)$ in (20). In this way we can remove from the hypotheses of Theorem 7, Theorem 8 and Theorem 9 the fact that the source v belongs to $\mathcal{S}'$. For instance (9) will be understood as an equality of two elements in $\dot{B}^{d/p-1,p}_2$. This formulation is the one that we prefer throughout the paper.*

Remark 14. *Note that the Laplacian* $\triangle : \mathcal{S}_{\sharp,c} \cap \dot{B}_q^{d/p,p} \to \dot{B}_q^{d/p-2,p}$ *is linear and bounded. Hence, by density of* $\mathcal{S}_{\sharp,c} \cap \dot{B}_q^{d/p,p}$ *in* $\dot{B}_q^{d/p,p}$*, it extends uniquely to an operator* $\triangle : \dot{B}_q^{d/p,p} \to \dot{B}_q^{d/p-2,p}$*. According to Remark 12, if* $v \in \dot{B}_q^{d/p,p}$*, then* $\triangle v \in \dot{B}_q^{d/p-2,p}$ *is a tempered distribution and we can define the spectrum of* v *as* $spec(v) = supp\ \widehat{\triangle v}$*. This notion will be applied for other function spaces as well.*

2.2 Lorentz-Sobolev spaces

Consider[7] some parameters $p \in (1,\infty)$, $q \in [1,\infty]$ and $\alpha \geq 0$. The homogeneous Lorentz-Sobolev spaces $\dot{H}^\alpha L^{p,q}(\mathbb{R}^d)$ is the completion of the normed space of Schwartz functions f on $\mathbb{R}^d$ under the norm

$$\|f\|_{\dot{H}^\alpha L^{p,q}} := \||\nabla|^\alpha f\|_{L^{p,q}},$$

where $L^{p,q}$ is the usual Lorentz space of parameters p and q.

Many of the embedding properties of the Besov and Triebel-Lizorkin spaces hold for the Lorentz-Sobolev spaces (see for instance [30] for detailes). We mention below only some properties of the Lorentz-Sobolev spaces that will be needed in the proof of Theorem 11. All of them are direct consequences of well-known facts from the theory of Lorentz spaces.

Lemma 15. *For any* $r \in [2,\infty)$*, we have that*

$$\dot{H}^{d/2}(\mathbb{R}^d) \hookrightarrow \dot{H}^{d/r} L^{r,2}(\mathbb{R}^d).$$

Proof. It is a consequence of the improved Sobolev embedding (see for instance [33, Theorem 2.10.2, p. 98]) that

$$\|f\|_{L^{r,2}} \lesssim \||\nabla|^{d(1/2-1/r)} f\|_{L^2},$$

for any Schwartz function f on $\mathbb{R}^d$. This can be rewritten as

$$\|f\|_{\dot{H}^{d/r}L^{r,2}} = \||\nabla|^{d/r} f\|_{L^{r,2}} \lesssim \||\nabla|^{d/2} f\|_{L^2} = \|f\|_{\dot{H}^{d/2}},$$

obtaining that $\dot{H}^{d/2} \hookrightarrow \dot{H}^{d/r}L^{r,2}$. □

Lemma 16. *For any* $r \in (1,\infty)$*, we have that*

$$\dot{H}^{d/r} L^{r,1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d).$$

Proof. For any Schwartz function f on $\mathbb{R}^d$ we have that

$$\|I_{d/r} * f\|_{L^\infty} \lesssim \|f\|_{L^{r,1}}, \tag{21}$$

where $I_{d/r}(x) = |x|^{d/r-d} = |x|^{-d/r'}$, for any $x \in \mathbb{R}^d$. Indeed, using [18, Theorem 1.4.17 (v), p. 52], we have

$$\int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d/r'}} dy \leq \|I_{d/r}\|_{(L^{r,1})^*} \|f\|_{L^{r,1}} = \|I_{d/r}\|_{L^{r',\infty}} \|f\|_{L^{r,1}},$$

and we can see that

$$\|I_{d/r}\|_{L^{r',\infty}} = \sup_{\lambda>0} \lambda \left| \left\{ x \in \mathbb{R}^d \mid |x| < (1/\lambda)^{r'/d} \right\} \right|^{1/r'} \sim 1.$$

Hence, (21) holds. We can reformulate (21) as

$$\|f\|_{L^\infty} \lesssim \||\nabla|^{d/r} f\|_{L^{r,1}} = \|f\|_{\dot{H}^{d/r}L^{r,1}},$$

This shows that $\dot{H}^{d/r}L^{r,1} \hookrightarrow L^\infty$. □

[7]The results of this Subsection will be used only in the proof of Theorem 11.

Lemma 17. *Suppose* $p_0, p_1, q_0, q_1 \in (1,\infty)$ *and* $\alpha_0, \alpha_1 \geq 0$. *Then, for any* $\theta \in (0,1)$ *we have*

$$(\dot{H}^{\alpha_0}L^{p_0,q_0}(\mathbb{R}^d), \dot{H}^{\alpha_1}L^{p_1,q_1}(\mathbb{R}^d))_\theta = \dot{H}^{\alpha}L^{p,q}(\mathbb{R}^d), \tag{22}$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $1/q = (1-\theta)/q_0 + \theta/q_1$.

Proof. This can be proved by a variant of Stein's method of interpolation given by Cwikel and Janson (see [13]) as follows. First, note that the function $\xi \to |\xi|^{it}$ defined on $\mathbb{R}^d \backslash \{0\}$ satisfies

$$|\nabla^k |\xi|^{it}| \lesssim (1+|t|)^{d+2} |\xi|^{-k},$$

a.e. in $\xi \in \mathbb{R}^d \backslash \{0\}$, for any $t \in \mathbb{R}$ and any non-negative integer $k \leq d+2$. It follows that (see [24, Theorem 8.2 , p. 197]) for any $a \in (1,\infty)$, the norm of the operator $|\nabla|^{it} : L^a \to L^a$ satisfies

$$\||\nabla|^{it}\|_{L^a \to L^a} \lesssim_a (1+|t|)^{d+2}.$$

This implies, via the real method of interpolation (see for instance [2, Theorem 5.2.1 (2), p. 109]), that for any $a \in (1,\infty)$ and any $b \in [1,\infty]$, we have

$$\||\nabla|^{it}\|_{L^{a,b} \to L^{a,b}} \lesssim_{a,b} (1+|t|)^{d+2}. \tag{23}$$

Let us consider the analytic family of operators $(T_z)_{z\in\overline{S}}$ (here, $\overline{S} := \{z \in \mathbb{C} \mid 0 \leq \Re z \leq 1\}$ is the closed standard strip; see also Subsection 3.1) with

$$T_z := exp(z^2 - \theta^2)|\nabla|^{(1-z)\alpha_0 + z\alpha_1},$$

for all $z \in \overline{S}$. One can check that for each $z \in \overline{S}$, $T_z : \dot{H}^{\alpha_0}L^{p_0,q_0} \cap \dot{H}^{\alpha_1}L^{p_1,q_1} \to L^{p_0,q_0} + L^{p_1,q_1}$ and that for each $\lambda \in (L^{p_0,q_0} + L^{p_1,q_1})^*$ and $f \in \dot{H}^{\alpha_0}L^{p_0,q_0} \cap \dot{H}^{\alpha_1}L^{p_1,q_1}$ the function $\lambda(T_z f)$ is analytic in the open strip S and continuous and bounded on $\overline{S}$. Also, by (23) we get that

$$\|T_{j+it}f\|_{L^{p_j,q_j}} \lesssim exp(-t^2)(1+|t|)^{d+2}\|f\|_{\dot{H}^{\alpha_j}L^{p_j,q_j}} \lesssim \|f\|_{\dot{H}^{\alpha_j}L^{p_j,q_j}},$$

for any $j = 0,1$ and any $f \in \dot{H}^{\alpha_0}L^{p_0,q_0} \cap \dot{H}^{\alpha_1}L^{p_1,q_1}$. Now one can check that the analytic family $(T_z)_{z\in\overline{S}}$ satisfies the hypotheses of [13, Theorem 1] and we get

$$T_\theta(\dot{H}^{\alpha_0}L^{p_0,q_0}, \dot{H}^{\alpha_1}L^{p_1,q_1})_\theta \hookrightarrow (L^{p_0,q_0}, L^{p_1,q_1})_\theta = L^{p,q}.$$

(Note that, since the spaces L^{p_j,q_j}, $j = 0,1$, are separable, with the notation used in [13, Theorem 1], we have $(L^{p_0,q_0}, L^{p_1,q_1})_\theta = (L^{p_0,q_0}, L^{p_1,q_1})_\theta^\alpha$.)

In a similar way (applying the same method to the family $(T_{-z})_{z\in\overline{S}}$),

$$T_{-\theta}(L^{p,q}) = T_{-\theta}(L^{p_0,q_0}, L^{p_1,q_1})_\theta \hookrightarrow (\dot{H}^{\alpha_0}L^{p_0,q_0}, \dot{H}^{\alpha_1}L^{p_1,q_1})_\theta.$$

Hence,

$$T_\theta(\dot{H}^{\alpha_0}L^{p_0,q_0}, \dot{H}^{\alpha_1}L^{p_1,q_1})_\theta = L^{p,q},$$

and (22) is proven. □

The spaces $\dot{H}^\alpha L^{p,q}$ have scaling properties that are similar to those of the Triebel-Lizorkin spaces (see (18)). In particular, the spaces $\dot{H}^{d/p}L^{p,q}$ have the same scaling as L^∞. As we have seen in Lemma 16 we have $\dot{H}^{d/p}L^{p,1} \hookrightarrow L^\infty$. However, when $q > 1$ the critical spaces $\dot{H}^{d/p}L^{p,q}$ do not embed in L^∞.

Let us see that the spaces $\dot{H}^{\alpha}L^{p,q}$ have the UMD property when $p,q \in (1,\infty)$. For this is sufficient to see that $L^{p,q}$ has the UMD property. Consider some $p_0, p_1 \in (1,\infty)$ such that $p_0 < p < p_1$. Since L^{p_0} and L^{p_1} are UMD spaces, by Burkholder's theorem (see [9]) the Hilbert transform is bounded on $L^2(\mathbb{T}, L^{p_0})$ and $L^2(\mathbb{T}, L^{p_1})$ respectively. Hence, the Hilbert transform is bounded on the space

$$\begin{aligned}(L^2(\mathbb{T}, L^{p_0}), L^2(\mathbb{T}, L^{p_1}))_{\eta,q} &= L^2(\mathbb{T}, (L^{p_0}, L^{p_1})_{\eta,q}) \\ &= L^2(\mathbb{T}, L^{p,q}),\end{aligned}$$

where $\eta \in (0,1)$ is such that $1/p = (1-\eta)/p_0 + \eta/p_1$ (see for instance [2, Theorem 5.6.2, p. 123]). By Bourgain's theorem ([4]), we get that $L^{p,q}$ has the UMD property.

Remark 18. *Note that for any $r > 1$ and $q \in [1, r]$ we have $L^{r,q}(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d)$ and hence,*

$$\dot{H}^{d/r}L^{r,q}(\mathbb{R}^d) \hookrightarrow \dot{H}^{d/r}L^{r}(\mathbb{R}^d) = \dot{H}^{d/r,r}(\mathbb{R}^d) \hookrightarrow \dot{B}^{d/r,r}_{r}(\mathbb{R}^d).$$

Therefore, as in Remark 13 we may consider the divergence operator as div : $\dot{H}^{d/r}L^{r,q} \to \dot{B}^{d/r-1,r}_{r}$. *Namely, its co-domain is, in this case also, a space that consists of temepered distributions. Similarly, we have $\triangle : \dot{H}^{d/r}L^{r,q} \to \dot{B}^{d/r-2,r}_{r}$.*

2.3 Some quotient spaces

In this paper a Banach function space on $\mathbb{R}^d$ is any Banach space $Y \supseteq \mathcal{S}_{\sharp,c}$ obtained by the completion of $\mathcal{S}_{\sharp,c}$ under the Y- norm. Let Y be a Banach function space on $\mathbb{R}^d$ with the property that $\triangle : \mathcal{S}_{\sharp,c} \cap Y \to F$ is bounded for some Banach space F that consists of tempered distributions on $\mathbb{R}^d$. Note that $\mathcal{S}_{\sharp,c} \cap Y$ is dense in Y and by density, $\triangle$ extends uniquely to an operator $\triangle : Y \to F$. As in Remark 14, for any $v \in Y$, the element $\triangle v \in F$ is a tempered distribution. For any $v \in Y$, we define its spectrum by

$$spec(v) = supp\ \widehat{\triangle v}.$$

We note that it suffices to prove our main results with this definition of the spectrum.

Consider now some set $D \subseteq \mathbb{R}^d$. Relative to the set D we define the closed subspace Y_D of Y by

$$Y_D := \{f \in Y \mid spec(f) \subseteq \overline{D}\},$$

(where $\overline{D}$ is the closure of D in $\mathbb{R}^d$) the norm being the one induced by Y. For simplicity we will denote the quotient space Y/Y_{D^c} by Y/D. The space Y/D is a Banach space whose elements are equivalence classes of the form $f + Y_{D^c}$, where $f \in Y$. We prefer to work with representatives of these equivalence classes rather than with the equivalence classes themselves. For any $f \in Y$ we define its Y/D-semi-norm by

$$\|f\|_{Y/D} := \|f + Y_{D^c}\|_{Y/D} = \inf_{f^- \in Y_{D^c}} \|f + f^-\|_Y.$$

Hence, any $f \in Y$ can be decomposed as

$$f = \tilde{f} + f^-, \tag{24}$$

for some $f^- \in Y_{D^c}$ and $\tilde{f} := f - f^- \in Y$ such that

$$\|\tilde{f}\|_Y \le 2\|f\|_{Y/D}.$$

In what follows we work only with regular sets D for which there exists a sequence of open sets $(\Omega_n)_{n\geq 1}$ for which $\overline{\Omega}_n$ is compact and included in $\Omega_{n+1} \subset D^c$, for all $n \geq 1$, and the set

$$D^c \setminus (\cup_{n=1}^{\infty} \Omega_n) \subset \mathbb{R}^d,$$

is of null Lebesgue measure.

Also, we will work only with quotient spaces of the form $\dot{H}^\alpha/D$ and $(\mathcal{M}+\dot{H}^\alpha)/D$. In the case where Y is the Sobolev space $\dot{H}^\alpha$, for some $\alpha \in \mathbb{R}$, one can efficiently compute the semi-norm induced by Y/D. Namely, let us see that for any $u \in \mathcal{S}_{\sharp,c}$ and any closed set $D \subseteq \mathbb{R}^d$ we have

$$\|u\|_{\dot{H}^\alpha/D} \sim \left(\int_{\mathbb{R}^d} |\xi|^{2\alpha} \mathbf{1}_D(\xi) \, |\widehat{u}(\xi)|^2 d\xi \right)^{1/2}. \tag{25}$$

Indeed, we can choose some functions $\eta_n \in C_c^\infty(\Omega_{n+1})$ such that, $0 \leq \eta_n \leq 1$ with $\eta_n \equiv 1$ on Ω_n, and define $v_n := -(\widehat{u}\eta_n)^\vee$, for any $n \geq 1$. We have that $v_n \in \mathcal{S}_{\sharp,c}$, *supp* $\widehat{v}_n \subset D^c$ and by the dominated convergence theorem,

$$\int_{D^c} |\xi|^{2\alpha} |\widehat{u}(\xi) + \widehat{v}_n(\xi)|^2 d\xi = \int_{D^c} |\xi|^{2\alpha} |\widehat{u}(\xi)|^2 |1 - \eta_n(\xi)|^2 d\xi \to 0,$$

when $n \to \infty$, and consequently,

$$\inf_{v \in \dot{H}^\alpha_{D^c}} \int_{D^c} |\xi|^{2\alpha} |\widehat{u}(\xi) + \widehat{v}(\xi)|^2 d\xi = 0.$$

Using this we can justify (25):

$$\begin{aligned}
\|u\|_{\dot{H}^\alpha/D} &\sim \inf_{v \in \dot{H}^\alpha_{D^c}} \int_{\mathbb{R}^d} |\xi|^{2\alpha} |\widehat{u}(\xi) + \widehat{v}(\xi)|^2 d\xi \\
&= \int_D |\xi|^{2\alpha} |\widehat{u}(\xi)|^2 d\xi + \inf_{v \in \dot{H}^\alpha_{D^c}} \int_{D^c} |\xi|^{2\alpha} |\widehat{u}(\xi) + \widehat{v}(\xi)|^2 d\xi \\
&= \int_D |\xi|^{2\alpha} |\widehat{u}(\xi)|^2 d\xi.
\end{aligned}$$

When D is a set as above we denote by P_D the Fourier projection on the set D, i.e., we have

$$\widehat{P_D f}(\xi) = \mathbf{1}_D(\xi) \widehat{f}(\xi),$$

for any $\xi \in \mathbb{R}^d$ and any Schwartz function f. Note that, in the case where $D = (0,\infty)^d$ the operator $P_{(0,\infty)^d}$ is the Riesz projection. In this case we will write P_+ in the place of $P_{(0,\infty)^d}$. With this notation (when D is closed) one can rewrite (25) as

$$\|u\|_{\dot{H}^\alpha/D} \sim \|P_D u\|_{\dot{H}^\alpha}. \tag{26}$$

3 The $\mathcal{W}$-method of complex interpolation

In this Section we introduce a variant of the complex interpolation and we prove several of its properties. We call this new method of interpolation the $\mathcal{W}$-method and, as stated in the Introduction (see Subsection 1.4), this will be used in the proof of Theorem 7, Theorem 8, Theorem 9, Theorem 11. We mainly study here only the properties of the $\mathcal{W}$-method that are used in the proof of our main results. In Subsection 3.1 we show that the $\mathcal{W}$-method is indeed an interpolation method. However, we ignore some issues specific to the interpolation methods in general such as computing dual interpolation spaces or reiteration theorems. These problems do not concern us here.

An important aspect is the relation of the $\mathcal{W}$-method with the classical complex method. We do not know in general how to compute efficiently the interpolation spaces obtained via the $\mathcal{W}$-method. However, as we will see in Subsection 3.2 the space obtained via the $\mathcal{W}$-method is, in many "convenient" cases, the same as the space obtained via the classical complex method.

The main feature of the $\mathcal{W}$-method is that one can use it to "interpolate" linear equations. It is one of the main ingredients that enter in the proof of our main results and it is the final goal of this Section.

3.1 Construction of the interpolation space

We describe here the $\mathcal{W}$-method and prove some basic properties. The proofs we give are straightforward adaptations of those that correspond to the classical complex interpolation as found in [2, Chapter 4]. Following the general presentation in [2, Chapter 4] let us introduce now the $\mathcal{W}$-method.

For the beginning, fix two non-trivial Banach spaces X_0 and X_1 and suppose (A_0, A_1) is a compatible couple of Banach spaces. Let $\mathcal{F}^2 = \mathcal{F}^2(A_0, X_0 \mid A_1, X_1)$ be the linear space of all bounded continuous functions f with values in $A_0 + A_1$, defined on the closed strip

$$\overline{S} := \{z \in \mathbb{C} \mid 0 \leq \Re z \leq 1\},$$

that are analytic in the open strip

$$S := \{z \in \mathbb{C} \mid 0 < \Re z < 1\},$$

and moreover, such that $f(j + it) \in A_j$ for any $j = 0, 1$ and any $t \in \mathbb{R}$, and

$$\|f\|_{\mathcal{F}^2} := \max_{j=0,1} \sup_{\|\Lambda_j\| \leq 1} \left(\int_{\mathbb{R}} \|\Lambda_j f(j + it)\|_{X_j}^2 dt \right)^{1/2} < \infty, \tag{27}$$

where, for each $j = 0, 1$, $\Lambda_j : A_j \to X_j$ are linear bounded operators. One can verify that $\|\cdot\|_{\mathcal{F}^2}$ defines a norm on $\mathcal{F}^2$.

Fix $0 < \theta < 1$. Consider the linear space $\mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$ defined by

$$\mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1) := \left\{a \in A_0 + A_1 \mid a = f(\theta), \text{ for some } f \in \mathcal{F}^2(A_0, X_0 \mid A_1, X_1)\right\}.$$

and define, for each $a \in \mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$, the quantity

$$\|a\|_\theta := \inf\left\{\|f\|_{\mathcal{F}^2} \mid a = f(\theta),\ f \in \mathcal{F}^2(A_0, X_0 \mid A_1, X_1)\right\}.$$

Lemma 19. *The mapping $a \to \|a\|_\theta$ is a norm on $\mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$.*

In order to prove Lemma 19 we rely on the following basic fact (and at least implicitly well-known):

Lemma 20. *Fix some $1 \le p < \infty$ and let Z be a Banach space. Suppose $F : \overline{S} \to Z$ is a bounded continuous function which is analytic in S such that the functions $t \to F(j+it)$ belong to the space $L^p(\mathbb{R}, Z)$.*

Then, for any $z \in S$, we have

$$F(z) = -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{F(it)}{it - z} dt + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{F(1+it)}{1+it-z} dt. \tag{28}$$

In particular, for any $\theta \in (0,1)$,

$$\|F(\theta)\|_Z \lesssim_{\theta,p} \max_{j=0,1} \left(\int_{\mathbb{R}} \|F(j+it)\|_Z^p dt \right)^{1/p}. \tag{29}$$

Proof of Lemma 20. Fix some $z \in S$. Consider some arbitrary $N \in \mathbb{N}^*$ and the curve γ_N given by the boundary of the rectangle $[0,1] \times [-N, N]$, oriented anti-clockwise. For N sufficiently large we have $z \in [0,1] \times [-N,N]$. By Cauchy's formula we get

$$F(z) = \frac{1}{2\pi i}\int_{\gamma_N} \frac{F(\zeta)}{\zeta - z} d\zeta,$$

and we can rewrite this as

$$\begin{aligned} F(z) \quad = \quad & -\frac{1}{2\pi i}\int_{-N}^{N} \frac{F(it)}{it-z} dt + \frac{1}{2\pi i}\int_{-N}^{N} \frac{F(1+it)}{1+it-z} dt \\ & + \frac{1}{2\pi i}\int_0^1 \frac{F(x+iN)}{x+iN-z} dx - \frac{1}{2\pi i}\int_0^1 \frac{F(x-iN)}{x-iN-z} dx. \end{aligned} \tag{30}$$

Note that, the functions $t \to F(j+it)/(j+it-\theta)$ belong to the space $L^1(\mathbb{R}, Z)$. Indeed, by Hölder's inequality (since we always have $p' > 1$) we can write

$$\begin{aligned} \int_{\mathbb{R}} \left\| \frac{F(j+it)}{j+it-z} \right\|_Z dt \quad & \le \quad \left(\int_{\mathbb{R}} \frac{1}{|j+it-z|^{p'}} dt \right)^{1/p'} \left(\int_{\mathbb{R}} \|F(j+it)\|_Z^p dt \right)^{1/p} \\ & \lesssim_{\theta,z} \left(\int_{\mathbb{R}} \|F(j+it)\|_Z^p dt \right)^{1/p} < \infty, \end{aligned} \tag{31}$$

with the natural modification in the case where $p' = \infty$.

Also, for any sign $\nu \in \{-1, 1\}$,

$$\left\| \frac{1}{2\pi i}\int_0^1 \frac{F(x+i\nu N)}{x+i\nu N - z} dx \right\|_Z \le \frac{1}{|x+i\nu N - z|} \|F\|_{L^\infty(S,Z)} \to 0, \tag{32}$$

when $N \to \infty$.

Using (30), (32), letting $N \to \infty$ and using the dominated convergence theorem ([19, Proposition 1.2.5]), we get the representation formula (28). Using (28) and (31), for $z = \theta$, we obtain (29). $\square$

Proof of Lemma 19. Clearly, $\|\cdot\|_\theta$ is a semi-norm on $\mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$. It remains to see that, if $\|a\|_\theta = 0$, for some $a \in \mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$, then $a = 0$. We prove this by showing that

$$\|a\|_{A_0+A_1} \lesssim_\theta \|a\|_\theta, \tag{33}$$

for all $a \in \mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$. For this purpose fix $a \in \mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$ and consider a functional $\lambda \in (A_0 + A_1)^*$, with $\|\lambda\| = 1$, such that $\|a\|_{A_0+A_1} \le 2\lambda(a)$. Consider also a function $f \in \mathcal{F}^2(A_0, X_0 \mid A_1, X_1)$, such that $f(\theta) = a$ and $\|f\|_{\mathcal{F}^2} \le 2\|a\|_\theta$.

Let us define, for each $j = 0, 1$, the linear operators $\Lambda_j : A_j \to X_j$ by

$$\Lambda_j(a_j) = \lambda(a_j)e_j, \quad \text{for any } a_j \in A_j,$$

where $e_j \in X_j$ are some fixed vectors with $\|e_j\|_{X_j} = 1$ (recall that X_j are non-trivial). Clearly, for any $j = 0, 1$,

$$\|\Lambda_j(a_j)\|_{X_j} = |\lambda(a_j)| \leq \|a_j\|_{A_0+A_1} \leq \|a_j\|_{A_j},$$

for any $a_j \in A_j$, and we get

$$\|\Lambda_j\| \leq 1.$$

Using this observation and introducing the function $F : \overline{S} \to \mathbb{C}$ defined by $F(z) := \lambda(f(z))$, one can write,

$$\begin{aligned} \max_{j=0,1} \left(\int_{\mathbb{R}} |F(j+it)|^2 dt \right)^{1/2} &= \max_{j=0,1} \left(\int_{\mathbb{R}} |\lambda(f(j+it))|^2 dt \right)^{1/2} \\ &= \max_{j=0,1} \left(\int_{\mathbb{R}} \|\Lambda_j(f(j+it))\|_{X_j}^2 dt \right)^{1/2} \\ &\leq \|f\|_{\mathcal{F}^2} \leq 2\|a\|_\theta < \infty. \end{aligned} \tag{34}$$

This shows, in particular, that the functions $t \to F(j+it)$ belong to the space $L^2(\mathbb{R}, \mathbb{C})$. We also see immediately that F is bounded, continuous on $\overline{S}$ and analytic in S. Hence, by applying Lemma 20 for $Z = \mathbb{C}$ and $p = 2$ (more precisely (29)), using (34), we get

$$\|a\|_{A_0+A_1} \leq 2\lambda(a) = 2F(\theta) \lesssim \max_{j=0,1} \left(\int_{\mathbb{R}} \|F(j+it)\|^2 dt \right)^{1/2} \lesssim \|a\|_\theta,$$

which proves (33). $\square$

Now, thanks to Lemma 19, we can define the interpolation space $(A_0, X_0 \mid A_1, X_1)_\theta$ as being the completion of the normed space $(\mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1), \|\cdot\|_\theta)$.

One can see that $(A_0, X_0 \mid A_1, X_1)_\theta$ is an intermediate space:

$$A_0 \cap A_1 \hookrightarrow (A_0, X_0 \mid A_1, X_1)_\theta \hookrightarrow A_0 + A_1. \tag{35}$$

The second embedding in (35), follows directly from the inequality (33). In order to see the first embedding, pick $a \in A_0 \cap A_1$ and consider the function $f(z) := \exp(z^2 - \theta^2)\, a$. One has that $f(\theta) = a$, $f \in \mathcal{F}^2$ and

$$\begin{aligned} \|a\|_\theta &\leq \max_{j=0,1} \sup_{\|\Lambda_j\| \leq 1} \left(\int_{\mathbb{R}} \|\Lambda_j f(j+it)\|_{X_j}^2 dt \right)^{1/2} \\ &\leq \max_{j=0,1} \left(\int_{\mathbb{R}} \|f(j+it)\|_{A_j}^2 dt \right)^{1/2} \sim_\theta \|a\|_{A_0 \cap A_1}. \end{aligned}$$

This gives us the embedding

$$A_0 \cap A_1 \hookrightarrow \mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1) \hookrightarrow (A_0, X_0 \mid A_1, X_1)_\theta.$$

Let us see now that the $\mathcal{W}$-method provides an exact interpolation functor:

Proposition 21. *Fix a parameter $\theta \in (0,1)$ and consider some non-trivial Banach spaces X_0, X_1. Let (A_0, A_1), (B_0, B_1) be two compatible couples of Banach spaces and $T : A_0 + A_1 \to B_0 + B_1$ be a linear operator such that $T : A_j \to B_j$ is bounded, of norm $\|T\|_{j\to j}$, for any $j = 0, 1$.*

Then, the operator

$$T : (A_0, X_0 \mid A_1, X_1)_\theta \to (B_0, X_0 \mid B_1, X_1)_\theta,$$

is bounded and of norm $\|T\|_{\theta\to\theta}$ satisfying

$$\|T\|_{\theta\to\theta} \le \|T\|_{0\to 0}^{1-\theta} \|T\|_{1\to 1}^{\theta}.$$

Proof. First, we suppose that $\|T\|_{j\to j} > 0$, for any $j = 0, 1$. For the brevity of notation we denote by $\|\cdot\|_\theta$, $\|\cdot\|_{\mathcal{F}^2}$, $\|\cdot\|'_\theta$ and $\|\cdot\|'_{\mathcal{F}^2}$ the norms on the spaces $(A_0, X_0 \mid A_1, X_1)_\theta$, $\mathcal{F}^2(A_0, X_0 \mid A_1, X_1)$, $(B_0, X_0 \mid B_1, X_1)_\theta$ and $\mathcal{F}^2(B_0, X_0 \mid B_1, X_1)$ respectively. Pick some $a \in \mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$ and fix some $\varepsilon > 0$. Consider a function $f \in \mathcal{F}^2(A_0, X_0 \mid A_1, X_1)$ such that $f(\theta) = a$ and $\|f\|_{\mathcal{F}^2} \le (1+\varepsilon)\|a\|_\theta$. The function

$$F(z) := \|T\|_{0\to 0}^{z-1} \|T\|_{1\to 1}^{-z} T f(z)$$

belongs to $\mathcal{F}^2(B_0, X_0 \mid B_1, X_1)$. Indeed, F is bounded and continuous on $\overline{S}$ with values in $B_0 + B_1$, analytic in S and, for any linear operators $\Lambda'_j : B_j \to X_j$ of norm at most 1, we have

$$\begin{aligned}\left(\int_{\mathbb{R}} \|\Lambda'_j F(j+it)\|_{X_j}^2 dt\right)^{1/2} &\le \left(\int_{\mathbb{R}} \|\left(\|T\|_{j\to j}^{-1} \Lambda'_j \circ T\right) f(j+it)\|_{X_j}^2 dt\right)^{1/2} \\ &\le \sup_{\|\Lambda_j\|\le 1} \left(\int_{\mathbb{R}} \|\Lambda_j f(j+it)\|_{X_j}^2 dt\right)^{1/2}, \end{aligned} \tag{36}$$

for any $j = 0, 1$, where the supremum is taken over all linear bounded operators $\Lambda_j : A_j \to X_j$ with $\|\Lambda_j\| \le 1$. Here, we have used the fact that $\|T\|_{j\to j}^{-1} \Lambda'_j \circ T : A_j \to X_j$ is a linear operator of norm at most 1, for any $j = 0, 1$.

Now, by (36), we get

$$\|T\|_{0\to 0}^{\theta-1} \|T\|_{1\to 1}^{-\theta} \|Tf(a)\|'_\theta = \|F(\theta)\|'_\theta \le \|F\|'_{\mathcal{F}^2} \le \|f\|_{\mathcal{F}^2} \le (1+\varepsilon)\|a\|_\theta,$$

and letting $\varepsilon \to 0$ one obtains,

$$\|Ta\|'_{\theta\to\theta} \le \|T\|_{0\to 0}^{1-\theta} \|T\|_{1\to 1}^{\theta} \|a\|_\theta,$$

for any $a \in \mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$. Since, by definition, $\mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$ is dense in $(A_0, X_0 \mid A_1, X_1)_\theta$, we get the conclusion in the case $\|T\|_{0\to 0}, \|T\|_{1\to 1} > 0$.

Suppose now that $\|T\|_{j\to j} = 0$, for at least one index $j = 0, 1$, say for $j = 0$. Fix a and f as before, and define for any $\lambda \in (B_0 + B_1)^*$ the function

$$F_\lambda(z) := \lambda(Tf(z)),$$

on $\overline{S}$. Note that F_λ is analytic in S, continuous and bounded on $\overline{S}$ (by the definition of f). We lave that $F(it) = 0$, for all $t \in \mathbb{R}$. Hence, by the three lines theorem (see for instance [2, Lemma 1.1.2]) we have that F_λ is identically zero on $\overline{S}$, for any $\lambda \in (B_0 + B_1)^*$ In particular, $Ta = 0$. Since $T = 0$ on $\mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$ we get by a density argument (as above) that $T = 0$ on $(A_0, X_0 \mid A_1, X_1)_\theta$, i.e., $\|T\|_{\theta\to\theta} = 0$. The inequality in the statement of Proposition 21 is trivially satisfied in this case.

The proof of Proposition 21 is now completed.

□

3.2 A particular case

In general, computing the interpolation space $(A_0, X_0 \mid A_1, X_1)_\theta$ seems to be a non-trivial task. However, there are some particular cases where an explicit computation is easy.

Let us restrict ourselves to the case where $A_0 = X_0$ and $A_1 = X_1$ and let us denote, for simplicity, the space $(X_0, X_0 \mid X_1, X_1)_\theta$ by $(X_0|X_1)_\theta$. Also, instead of $\mathcal{F}^2(X_0, X_0 \mid X_1, X_1)$ we write $\mathcal{F}^2(X_0|X_1)$ and instead of $\mathcal{C}_\theta(X_0, X_0 \mid X_1, X_1)$ we write $\mathcal{C}_\theta(X_0|X_1)$. In this case, formula (27) becomes

$$\|f\|_{\mathcal{F}^2} = \max_{j=0,1} \left(\int_{\mathbb{R}} \|f(j+it)\|_{X_j}^2 dt \right)^{1/2}. \tag{37}$$

Indeed, for any $j = 0, 1$, we have

$$\sup_{\|\Lambda_j\| \le 1} \left(\int_{\mathbb{R}} \|\Lambda_j f(j+it)\|_{X_j}^2 dt \right)^{1/2} \le \left(\int_{\mathbb{R}} \|f(j+it)\|_{X_j}^2 dt \right)^{1/2},$$

and

$$\begin{aligned} \sup_{\|\Lambda_j\| \le 1} \left(\int_{\mathbb{R}} \|\Lambda_j f(j+it)\|_{X_j}^2 dt \right)^{1/2} &\ge \left(\int_{\mathbb{R}} \|id_{X_j} f(j+it)\|_{X_j}^2 dt \right)^{1/2} \\ &= \left(\int_{\mathbb{R}} \|f(j+it)\|_{X_j}^2 dt \right)^{1/2}, \end{aligned}$$

where $id_{X_j} : X_j \to X_j$ is the identity mapping on X_j.

It turns out that, the space $(X_0|X_1)_\theta$ coincides with the space $(X_0, X_1)_\theta$ obtained via the classical complex interpolation method. The proof of this fact is easy, however we give it below for the sake of completeness.

Proposition 22. *Suppose (X_0, X_1) is a compatible couple of non-trivial Banach spaces. Then, for any $\theta \in (0, 1)$,*

$$(X_0|X_1)_\theta = (X_0, X_1)_\theta,$$

with equivalence of norms.

Remark 23. *Implicitly, the embedding $(X_0|X_1)_\theta \hookrightarrow (X_0, X_1)_\theta$ was already proved and used by Peetre in a different context (see [29, Lemme 1.1]). Both proofs, the one that we give below and Peetre's, are easy consequences of the ideas of Calderón from [10, Section 9.4].*

Proof. Consider some $a \in \mathcal{C}_\theta(X_0|X_1)$ and let some $f \in \mathcal{F}^2(X_0|X_1)$ be such that $f(\theta) = a$ and

$$\|f\|_{\mathcal{F}^2} \le 2\|a\|_{(X_0|X_1)_\theta}.$$

By [2, Lemma 4.3.2 (ii), p. 93] (or [10, Section 9.4, (ii)]) we have

$$\|a\|_{(X_0,X_1)_\theta} \le \left(\frac{1}{1-\theta} \int_{\mathbb{R}} \|f(i\tau)\|_{X_0} P_0(\theta, \tau) d\tau \right)^{1-\theta} \left(\frac{1}{\theta} \int_{\mathbb{R}} \|f(1+i\tau)\|_{X_1} P_1(\theta, \tau) d\tau \right)^{\theta}, \tag{38}$$

where P_j $(j = 0, 1)$ are the real Poisson kernels defined by

$$P_j(s+it, \tau) := \frac{e^{-\pi(\tau - t)} \sin \pi s}{\sin^2 \pi s + (\cos \pi s - e^{ij\pi - \pi(\tau - t)})^2},$$

for $s \in (0,1)$, $t, \tau \in \mathbb{R}$. Note that $P_j(\theta, \cdot) \in L^2(\mathbb{R}, \mathbb{R})$ and by the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_{\mathbb{R}} \|f(i\tau)\|_{X_0} P_0(\theta,\tau)d\tau &\le \left(\int_{\mathbb{R}} \|f(i\tau)\|_{X_0}^2 d\tau\right)^{1/2} \left(\int_{\mathbb{R}} P_0^2(\theta,\tau)d\tau\right)^{1/2} \\ &\lesssim \|f\|_{\mathcal{F}^2} \lesssim \|a\|_{(X_0|X_1)_\theta}.\end{aligned}$$

In a similar way we get

$$\int_{\mathbb{R}} \|f(i\tau)\|_{X_0} P_0(\theta,\tau)d\tau \lesssim \|a\|_{(X_0|X_1)_\theta},$$

and combining this with (38) one obtains

$$\|a\|_{(X_0,X_1)_\theta} \lesssim \|a\|_{(X_0|X_1)_\theta}. \tag{39}$$

By taking the completion we get $(X_0|X_1)_\theta \hookrightarrow (X_0, X_1)_\theta$.

Conversely, if $a \in (X_0, X_1)_\theta$, then there exists $g \in \mathcal{F}(X_0, X_1)$ (see [2, Chapter 4] for the standard notation $\mathcal{F}(X_0, X_1)$) such that $g(\theta) = a$ and

$$\max_{j=0,1} \sup_{t\in\mathbb{R}} \|g(j+it)\|_{X_j} \le 2\|a\|_{(X_0,X_1)_\theta}. \tag{40}$$

Introduce the function $\widetilde{g} : \overline{S} \to X_0 + X_1$ defined by $\widetilde{g}(z) := \exp(z^2 - \theta^2)g(z)$, for $z \in \overline{S}$. We observe that, for any $j = 0, 1$,

$$\begin{aligned}\left(\int_{\mathbb{R}} \|\widetilde{g}(j+it)\|_{X_j}^2 dt\right)^{1/2} &\lesssim \left(\int_{\mathbb{R}} e^{-2t^2} \|g(j+it)\|_{X_j}^2 dt\right)^{1/2} \\ &\le \left(\int_{\mathbb{R}} e^{-2t^2} dt\right)^{1/2} \sup_{t\in\mathbb{R}} \|g(j+it)\|_{X_j} \\ &\sim \sup_{t\in\mathbb{R}} \|g(j+it)\|_{X_j}.\end{aligned} \tag{41}$$

Hence, $\widetilde{g} \in \mathcal{F}^2(X_0|X_1)$, $a = \widetilde{g}(\theta) \in \mathcal{C}_\theta(X_0|X_1)$ and by (40), (41),

$$\|a\|_{(X_0|X_1)_\theta} \lesssim \|a\|_{(X_0,X_1)_\theta}, \tag{42}$$

We have now $(X_0, X_1)_\theta \hookrightarrow (X_0|X_1)_\theta$ and Proposition 22 is proven. □

An immediate consequence of Proposition 22 is the following useful embedding result:

Corollary 24. *Suppose (X_0, X_1) is a compatible couple of Banach spaces. Then, for any Banach space A, we have the embedding*

$$(A \cap X_0, X_0 \mid A \cap X_1, X_1)_\theta \hookrightarrow A \cap (X_0, X_1)_\theta\,.$$

Proof. Consider the canonical inclusion $\iota : A \cap X_0 + A \cap X_1 \to X_0 + X_1$ as a linear bounded operator $\iota : A \cap X_j \to X_j$ and apply Proposition 21. We get

$$(A \cap X_0, X_0 \mid A \cap X_1, X_1)_\theta \hookrightarrow (X_0, X_0 \mid X_1, X_1)_\theta = (X_0|X_1)_\theta\,.$$

Since by Proposition 22 we have $(X_0|X_1)_\theta = (X_0, X_1)_\theta$, we now obtain the embedding

$$(A \cap X_0, X_0 \mid A \cap X_1, X_1)_\theta \hookrightarrow (X_0, X_1)_\theta\,. \tag{43}$$

Also, using the fact that $(A \cap X_0, X_0 \mid A \cap X_1, X_1)_\theta$ is an intermediate space (see (35)), we have

$$(A \cap X_0, X_0 \mid A \cap X_1, X_1)_\theta \hookrightarrow A \cap X_0 + A \cap X_1 \hookrightarrow A,$$

which together with (43) proves Corollary 24. □

3.3 Solutions of linear equations

In this Subsection we highlight the main strength of the $\mathcal{W}$-method. Namely, we show here how the $\mathcal{W}$-method can be used in order to “interpolate ” under-determined equations. Here we make an essential use of the fact that the Hilbert transform is bounded on spaces of the form $L^2(\mathbb{R}, Z)$, where Z is an UMD space. The UMD property plays here a key role.

Before stating the results in this Subsection let us make some (common) notational conventions. The space $C_b^l(\mathbb{R}, Z)$, where $l \in \mathbb{N}$, is the space of all the functions $f : \mathbb{R} \to Z$ for which the k-th derivative $\partial^k f$ is a continuous and bounded Z-valued function on $\mathbb{R}$ for all $k \in \mathbb{N}$ with $k \leq l$. (For basic facts concerning the vector-valued differentiable functions see for instance [17, Chapter VIII].) We endow $C_b^l(\mathbb{R}, Z)$ with the norm

$$\|f\|_{C_b^l(\mathbb{R},Z)} := \sum_{k=0}^{l} \|\partial^k f\|_{L^\infty(\mathbb{R},Z)}.$$

Given a measurable function $w : \mathbb{R} \to (0, \infty)$ we denote by $L^2(w, Z)$ the space of the strongly measurable (see for instance [19, Chapter 1 and 2]) functions $f : \mathbb{R} \to Z$ for which the norm

$$\|f\|_{L^2(w,Z)} := \left(\int_{\mathbb{R}} \|f(t)\|_Z^2 w(t) dt \right)^{1/2},$$

is finite. When $w(t) = \exp(\alpha t^2)$ for some non-zero $\alpha \in \mathbb{R}$ the space $L^2(w, Z)$ is denoted by $L^2(\exp(\alpha t^2), Z)$. However, when $w \equiv 1$ we prefer to write $L^2(\mathbb{R}, Z)$ instead of $L^2(1, Z)$. In what follows we will work only with bounded continuous functions f and hence, we will not be concerned with the issue of strong measurability.

3.3.1 Boundary values of functions on the strip

Let us recall now some (at least implicitly) well-known facts related to the Hilbert transforms of vector-valued functions. Let Z be a Banach space and consider a function $f \in C_b^1(\mathbb{R}, Z) \cap L^2(\mathbb{R}, Z)$. The Hilbert transform of f is defined by

$$Hf(t) := \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon < |t-s| < 1/\varepsilon} \frac{f(s)}{t-s} ds = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon < |s| < 1/\varepsilon} \frac{f(t-s)}{s} ds, \tag{44}$$

for $t \in \mathbb{R}$. One can see that Hf is a well-defined bounded continuous function:

Lemma 25. *Suppose Z is a Banach space and consider a function $f \in C_b^1(\mathbb{R}, Z) \cap L^2(\mathbb{R}, Z)$. Then, the limit in (44) exists for any $t \in \mathbb{R}$ and it defines a function $Hf \in C_b(\mathbb{R}, Z)$.*

Proof. Fix some $t \in \mathbb{R}$. For any $\varepsilon \in (0, 1)$ and any $N \in \mathbb{N}^*$ we can write

$$\begin{aligned} \int_{\varepsilon<|s|<1/\varepsilon} \frac{f(t-s)}{s} ds &= \int_{\varepsilon<|s|<N} \frac{f(t-s) - f(t)}{s} ds \\ &\quad + \int_{N<|s|<1/\varepsilon} \frac{f(t-s)}{s} ds \\ &= \int_{-N}^{N} \frac{f(t-s) - f(t)}{s} \mathbf{1}_{(-\varepsilon,\varepsilon)^c}(s) ds \\ &\quad + \int_{(-N,N)^c} \frac{f(t-s)}{s} \mathbf{1}_{(-1/\varepsilon,1/\varepsilon)}(s) ds. \end{aligned} \tag{45}$$

Let us see first that the limit in (44) exists. We fix $N = 1$. By using the mean value theorem ([17, (8.5.2)]) we observe that

$$\left\| \frac{f(t-s) - f(t)}{s} \mathbf{1}_{(-\varepsilon,\varepsilon)^c}(s) \right\|_Z \leq \|f\|_{C_b^1(\mathbb{R},Z)}, \tag{46}$$

and by the dominated convergence theorem ([19, Proposition 1.2.5]) we get

$$\int_{-N}^{N} \frac{f(t-s) - f(t)}{s} \mathbf{1}_{(-\varepsilon,\varepsilon)^c}(s) ds \to \int_{-N}^{N} \frac{f(t-s) - f(t)}{s} ds, \tag{47}$$

strongly in Z. Note also that the Cauchy-Schwarz inequality gives

$$\int_{(-1,1)^c} \left\| \frac{f(t-s)}{s} \right\|_Z ds \lesssim \|f\|_{L^2(\mathbb{R},Z)}, \tag{48}$$

and hence, the function $s \to f(t-s)/s$ is Bochner integrable on $(-1,1)^c$. By the dominated convergence theorem we have

$$\int_{(-1,1)^c} \frac{f(t-s)}{s} \mathbf{1}_{(-1/\varepsilon,1/\varepsilon)}(s) ds \to \int_{(-1,1)^c} \frac{f(t-s)}{s} ds,$$

strongly in Z, which together with (45) and (47) gives the existence of the limit in (44).

In a similar way we can verify the boundedness of Hf. Indeed, by (45) for $N = 1$, (46) and (48) we have

$$\|Hf(t)\|_Z \leq \limsup_{\varepsilon \to 0} \left\| \int_{\varepsilon < |s| < 1/\varepsilon} \frac{f(t-s)}{s} ds \right\|_Z \lesssim \|f\|_{C_b^1(\mathbb{R},Z)} + \|f\|_{L^2(\mathbb{R},Z)}.$$

In order to see that Hf is continuous we fix some $t \in \mathbb{R}$ and consider some sequence $(t_n)_{n\geq 1}$ such that $t_n \to t$. By (45) we have

$$\|Hf(t) - Hf(t_n)\|_Z \leq \int_{-N}^{N} g_n(s) ds + \int_{(-N,N)^c} \frac{\|f(t-s) - f(t_n - s)\|_Z}{s} ds, \tag{49}$$

where $g_n : [-N, N] \backslash \{0\} \to [0, \infty)$ is defined by

$$g_n(s) := \left\| \frac{f(t-s) - f(t)}{s} - \frac{f(t_n - s) - f(t_n)}{s} \right\|_Z.$$

Note that by (46) we have $g_n(s) \leq 2\|f\|_{C_b^1(\mathbb{R},Z)}$, uniformly, and, since $g_n(s) \to 0$ point-wise, by the dominated convergence theorem we get

$$\int_{-N}^{N} \left\| \frac{f(t-s) - f(t)}{s} - \frac{f(t_n - s) - f(t_n)}{s} \right\|_Z ds \to 0,$$

when $n \to \infty$.

As in (48) we obtain

$$\int_{(-N,N)^c} \frac{\|f(t-s) - f(t_n - s)\|_Z}{s} ds \lesssim \|f\|_{L^2(\mathbb{R},Z)}/N,$$

and hence, (49) gives now that

$$\limsup_{n \to \infty} \|Hf(t) - Hf(t_n)\|_Z \lesssim \|f\|_{L^2(\mathbb{R},Z)}/N,$$

for any $N \in \mathbb{N}^*$. This concludes the proof of Lemma 25. □

In what follows we need a vector-valued version of the Plemelj formula. The proof we give is completely similar to the one in the scalar valued case, however, we include it here for completeness (see for instance [25]).

Lemma 26. *Suppose Z is a Banach space and consider a function $f \in C_b^1(\mathbb{R}, Z) \cap L^2(\mathbb{R}, Z)$. Then, when $\varepsilon \searrow 0$ we have*

$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(s)}{s-(t+i\nu\varepsilon)} ds \to \frac{\nu f(t) + iHf(t)}{2},$$

in the norm of Z, uniformly in $t \in \mathbb{R}$, for any sign $\nu \in \{-1, 1\}$.

Proof. We only consider the case of the sign $\nu = 1$, the other one being similar. We first show that

$$\left\| \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(s)}{s-i\varepsilon} dy - \frac{f(0)}{2} - \frac{i}{2\pi}\int_{|s|>\varepsilon} \frac{f(s)}{-s} ds \right\|_Z \lesssim \|f\|_{C_b^1(\mathbb{R},Z)} \varepsilon^{1/2}, \tag{50}$$

for any $\varepsilon \in (0, 1)$, where the implicit constant does not depend on f or ε. Changing the variables ($s = \varepsilon\tau$), this is equivalent to

$$\left\| \int_{\mathbb{R}} f(\varepsilon\tau) \left(\frac{1}{\tau - i} - \mathbf{1}_{|\tau|>1}(\tau) \frac{1}{\tau} \right) d\tau - \pi i f(0) \right\|_Z \lesssim \|f\|_{C_b^1(\mathbb{R},Z)} \varepsilon^{1/2}. \tag{51}$$

Notice that

$$\int_{\mathbb{R}} \left| \frac{1}{\tau - i} - \mathbf{1}_{|\tau|>1}(\tau) \frac{1}{\tau} \right| d\tau < \infty,$$

and

$$\int_{\mathbb{R}} \left(\frac{1}{\tau - i} - \mathbf{1}_{|\tau|>1}(\tau) \frac{1}{\tau} \right) d\tau = \pi i.$$

Hence, it remains to show that

$$\left\| \int_{\mathbb{R}} (f(\varepsilon\tau) - f(0)) \left(\frac{1}{\tau - i} - \mathbf{1}_{|\tau|>1}(\tau) \frac{1}{\tau} \right) d\tau \right\|_Z \lesssim \|f\|_{C_b^1(\mathbb{R},Z)} \varepsilon^{1/2}.$$

One can see this by a direct computation. Indeed, the quantity

$$\left\| \int_{\mathbb{R}} (f(\varepsilon\tau) - f(0)) \left(\frac{1}{\tau - i} - \mathbf{1}_{|\tau|>1}(\tau) \frac{1}{\tau} \right) d\tau \right\|_Z$$

is bounded by

$$\begin{aligned} \int_{\mathbb{R}} \|f(\varepsilon\tau) - f(0)\|_Z \left| \frac{1}{\tau - i} - \mathbf{1}_{|\tau|>1}(\tau) \frac{1}{\tau} \right| d\tau \ &\lesssim \int_{[-N,N]} \|f(\varepsilon\tau) - f(0)\|_Z \frac{1}{1+|\tau|} d\tau \\ &+ \int_{[-N,N]^c} \|f(\varepsilon\tau) - f(0)\|_Z \frac{1}{\tau^2} d\tau, \end{aligned} \tag{52}$$

for any $N > 1$. Since

$$\|f(\varepsilon\tau) - f(0)\|_Z \leq \varepsilon \|f\|_{C_b^1(\mathbb{R},Z)} |\tau|,$$

we get

$$\int_{[-N,N]} \|f(\varepsilon\tau) - f(0)\|_Z \frac{1}{1+|\tau|} d\tau \lesssim \|f\|_{C_b^1(\mathbb{R},Z)} \varepsilon N.$$

Also, since

$$\|f(\varepsilon\tau) - f(0)\|_Z \leq 2\|f\|_{C_b^1(\mathbb{R},Z)},$$

we have

$$\int_{[-N,N]^c} \|f(\varepsilon\tau) - f(0)\|_Z \frac{1}{\tau^2} d\tau \lesssim \|f\|_{C_b^1(\mathbb{R},Z)} / N.$$

From (52) we get now,

$$\int_{\mathbb{R}} \|f(\varepsilon\tau) - f(0)\|_Z \left| \frac{1}{\tau - i} - \mathbf{1}_{|\tau|>1}(\tau)\frac{1}{\tau} \right| d\tau \lesssim \|f\|_{C_b^1(\mathbb{R},Z)}(\varepsilon N + 1/N),$$

where the implicit constant does not depend on ε or N. Setting $N = \varepsilon^{-1/2}$ we obtain (51) and hence (50).

Now consider some function $f \in C_b^1(\mathbb{R}, Z)$ and for each $t \in \mathbb{R}$, define $f_t : \mathbb{R} \to Z$ by $f_t(x) := f(x+t)$ for all $x \in \mathbb{R}$. We can see that $f_t \in C_b^1(\mathbb{R}, Z)$ and

$$\|f_t\|_{C_b^1(\mathbb{R},Z)} = \|f\|_{C_b^1(\mathbb{R},Z)},$$

for any $t \in \mathbb{R}$. By a simple change of variables we can also observe that

$$\left\| \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f_t(s)}{s - i\varepsilon} ds - \frac{f_t(0)}{2} - \frac{i}{2\pi}\int_{|s|>\varepsilon} \frac{f_t(s)}{-s} ds \right\|_Z$$

equals

$$\left\| \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(s)}{s - (t + i\varepsilon)} ds - \frac{f(t)}{2} - \frac{i}{2\pi}\int_{|t-s|>\varepsilon} \frac{f(s)}{t - s} ds \right\|_Z.$$

Hence, by applying (50) to f_t and letting $\varepsilon \to 0$ we obtain the conclusion. □

The operators H_j^S and R_j^S. Fix some real number $\alpha > 0$. Let us introduce some operators that quantify the boundary behaviour of analytic functions on the strip. For each $j = 0, 1$ let

$$H_j^S, R_j^S : C_b^1(\mathbb{R}, Z) \cap L^2(\exp\left(\alpha t^2\right), Z) \to C_b(\mathbb{R}, Z),$$

be defined by

$$H_j^S f(t) := \frac{-if(t) - (-1)^j Hf(t)}{2},$$

and

$$R_j^S f(t) := \rho_j * f(t),$$

respectively, where $\rho_j \in C_b(\mathbb{R}, \mathbb{C})$ are the functions

$$\rho_j(t) := \frac{(-1)^j}{2\pi i}\frac{1}{1 - 2j + it}.$$

The operator H_j^S is well-defined thanks to Lemma 25. Also, note that for $f \in L^2(\exp(\alpha t^2), Z)$ the quantity $R_j^S f$ is well-defined and by the dominated convergence theorem ([19, Proposition 1.2.5, p. 16]) we have $R_j^S f \in C_b(\mathbb{R}, Z)$.

A consequence of Lemma 25 and Lemma 26 is the following fact:

Lemma 27. *Fix some $\alpha > 0$. Suppose (B_0, B_1) is a compatible couple of Banach spaces. Consider some functions $u_j \in C_b^1(\mathbb{R}, B_j) \cap L^2(\exp(\alpha t^2), B_j)$, $j = 0, 1$, and define $u : \overline{S} \to B_0 + B_1$ by*

$$u(z) := -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{u_0(t)}{it - z} dt + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{u_1(t)}{1 + it - z} dt,$$

for all $z \in S$, and

$$u(j + it) := H_j^S u_j(t) + R_j^S u_{1-j}(t),$$

for all $t \in \mathbb{R}$. Then, u is analytic in S and $u \in C_b\left(\overline{S}, B_0 + B_1\right)$.

Proof. First we note that if $\delta < \Re z < 1-\delta$, for some $\delta \in (0,1/2)$, then

$$\int_{\mathbb{R}} \frac{\|u_j(t)\|_{B_0+B_1}}{|it-z|} dt \lesssim \delta^{-1} \int_{\mathbb{R}} \|u_j(t)\|_{B_j} dt \lesssim \delta^{-1} \|u_j\|_{L^2(\exp(\alpha t^2),B_j)}$$

for any $j=0,1$, where for the second "$\lesssim$" we have used the Cauchy-Schwarz inequality. In particular, u is well-defined on S. Also, we get that u is bounded on any strip of the form $\{\delta < \Re z < 1-\delta\}$:

$$\sup_{y\in\mathbb{R}} \|u(x+iy)\|_{B_0+B_1} \lesssim \delta^{-1}\|u_0\|_{L^2(\exp(\alpha t^2),B_0)} + \delta^{-1}\|u_1\|_{L^2(\exp(\alpha t^2),B_1)}, \tag{53}$$

for any $x \in (\delta, 1-\delta)$.

Now, we are concerned with behaviour of u on the boundary of $\overline{S}$. By Lemma 25 (for $Z = B_0+B_1$) we have $H_j^S u_j \in C_b(\mathbb{R}, B_0+B_1)$. Also, by Lemma 26 (for $Z = B_0+B_1$) we have

$$\frac{(-1)^{j+1}}{2\pi i} \int_{\mathbb{R}} \frac{u_j(t)}{j+it-(j+i\tau+(-1)^j\varepsilon)} dt \to H_j^S u_j(\tau), \tag{54}$$

in B_0+B_1 uniformly in $\tau\in\mathbb{R}$, when $\varepsilon \searrow 0$. On the other hand, we have $R_j^S u_{1-j} \in C_b(\mathbb{R}, B_0+B_1)$ and, as one can see,

$$\frac{(-1)^{j+1}}{2\pi i} \int_{\mathbb{R}} \frac{u_{1-j}(t)}{j+it-(1-j+i\tau+(-1)^j\varepsilon)} dt \to R_j^S u_{1-j}(\tau), \tag{55}$$

in B_0+B_1 uniformly in $\tau\in\mathbb{R}$, when $\varepsilon \searrow 0$. This can be seen as follows. The quantity

$$\left\| \frac{(-1)^{j+1}}{2\pi i} \int_{\mathbb{R}} \frac{u_{1-j}(t)}{-1+i(t-\tau)+(-1)^{j+1}\varepsilon} dt - R_j^S u_{1-j}(\tau) \right\|_{B_0+B_1}$$

is bounded by

$$\int_{\mathbb{R}} \|u_{1-j}(t)\|_{B_0+B_1} \left| \frac{1}{-1+i(t-\tau)+(-1)^{j+1}\varepsilon} - \frac{1}{-1+i(t-\tau)} \right| dt,$$

which in turn is bounded by

$$\varepsilon \|u_{1-j}\|_{L^\infty(B_{1-j})} \int_{\mathbb{R}} \frac{1}{1/4+(t-\tau)^2} dt \lesssim \varepsilon \|u_{1-j}\|_{L^\infty(B_{1-j})},$$

for any $\varepsilon \in (0,1/2)$, and (55) is proved.

Hence, by (54) and (55), u is approaching its boundary values uniformly. In other words, introducing the quantity

$$\omega_j(\varepsilon) := \sup_{t\in\mathbb{R}} \left\| u\left(j+(-1)^j\varepsilon+it\right) - u(j+it) \right\|_{B_0+B_1}, \tag{56}$$

for $j=0,1$ and $\varepsilon\in(0,1)$, we have $\omega_j(\varepsilon)\to 0$, when $\varepsilon \to 0$. Fix some $\varepsilon_0 \in (0,1/2)$ such that $\omega_j(\varepsilon) < 1$, for any $\varepsilon\in(0,\varepsilon_0)$ and $j=0,1$. Since u is analytic on S, we obtain the conclusion. By (56) we have

$$\begin{aligned} \sup_{t\in\mathbb{R}} \left\| u\left(j+(-1)^j\varepsilon+it\right) \right\|_{B_0+B_1} &\le 1+\sup_{t\in\mathbb{R}} \|u(j+it)\|_{B_0+B_1} \\ &\le 1+\|u_0\|_{L^\infty(B_0)}+\|u_1\|_{L^\infty(B_1)}, \end{aligned}$$

and combining this with (53) we get the uniform boundedness of $u(z)$ in the norm of B_0+B_1 on $\overline{S}$.

In order to see that u is continuous we fix some $z \in \partial S$ and consider some sequence $(z_n)_{n\geq 1}$ in $\overline{S}$ such that $z_n \to z$. Suppose L is the line representing the connected component of ∂S that contains the point z. For each $n \geq 1$ let z_n' be the orthogonal projection of z_n on L. We have that

$$\begin{aligned}\|u(z) - u(z_n)\|_{B_0+B_1} &\leq \|u(z) - u(z_n')\|_{B_0+B_1} + \|u(z_n') - u(z_n)\|_{B_0+B_1} \\ &\leq \|u(z) - u(z_n')\|_{B_0+B_1} + \max_{j=0,1} \mu_j(|z_n' - z_n|).\end{aligned}$$

Since $u_j \in C(\mathbb{R}, B_j)$ and $\omega_j(|z_n' - z_n|) \to 0$, for any $j = 0, 1$, we get that $u(z_n) \to u(z)$ strongly in the norm of $B_0 + B_1$. Hence, u is continuous in any point of ∂S. The function u is weakly analytic, and hence, analytic on S. In particular, u is continuous on S. Lemma 27 is proved. □

3.3.2 Interpolation of equations

Let us illustrate by some examples the fact that, in general, the surjectivity of operators is not preserved by interpolation:

Example 1. Consider the operator $T_s : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to L^2(\mathbb{T})$, defined by the formula $T_s(f, g) := f + g$, for any $(f, g) \in L^2(\mathbb{T})$. One can see that $T_s : L^2(\mathbb{T}) \times L^4(\mathbb{T}) \to L^2(\mathbb{T})$ and $T_s : L^4(\mathbb{T}) \times L^2(\mathbb{T}) \to L^2(\mathbb{T})$ are surjective operators. However, the operator

$$T_s : (L^2(\mathbb{T}) \times L^4(\mathbb{T}), L^4(\mathbb{T}) \times L^2(\mathbb{T}))_{1/2} \to (L^2(\mathbb{T}), L^2(\mathbb{T}))_{1/2}, \tag{57}$$

is not surjective. Indeed, if T_s in (57) is surjective, then

$$T_s : L^{8/3}(\mathbb{T}) \times L^{8/3}(\mathbb{T}) \to L^2(\mathbb{T})$$

is surjective, and we get the false embedding $L^2(\mathbb{T}) \hookrightarrow L^{8/3}(\mathbb{T})$.

Example 2. Let us consider now another example which is more closely related to the equations we treat in this paper. For any $p \geq 1$ let $W_\sharp^{-1,p}(\mathbb{T}^2)$ be the spaces of those distributions f that are divergences of L^p-vector fields on $\mathbb{T}^2$. (In general, if Z is a function space on $\mathbb{T}^2$, we denote by $Z_\sharp$ the space of those $f \in Z$ with $\widehat{f}(0) = 0$.) The norm on $W_\sharp^{-1,p}(\mathbb{T}^2)$ is given by

$$\|f\|_{W_\sharp^{-1,p}(\mathbb{T}^2)} = \inf\left\{\|f_1\|_{L^p(\mathbb{T}^2)} + \|f_2\|_{L^p(\mathbb{T}^2)} \mid f = \partial_1 f_1 + \partial_2 f_2\right\}.$$

We have that $\operatorname{div} : L^1(\mathbb{T}^2) \to W_\sharp^{-1,1}(\mathbb{T}^2)$ and $\operatorname{div} : L^3(\mathbb{T}^2) \to W_\sharp^{-1,3}(\mathbb{T}^2)$ are surjective operators. However, the operator

$$\operatorname{div} : L^{3/2}(\mathbb{T}^2) \to (W_\sharp^{-1,1}(\mathbb{T}^2), W_\sharp^{-1,3}(\mathbb{T}^2))_{1/2}, \tag{58}$$

cannot be surjective. Indeed, since

$$\operatorname{div} : L^{3/2}(\mathbb{T}^2) \to W_\sharp^{-1,3/2}(\mathbb{T}^2),$$

the surjectivity of the operator in (58) would imply that

$$(W_\sharp^{-1,1}(\mathbb{T}^2), W_\sharp^{-1,3}(\mathbb{T}^2))_{1/2} \hookrightarrow W_\sharp^{-1,3/2}(\mathbb{T}^2),$$

which, by duality is equivalent to

$$W_\sharp^{1,3}(\mathbb{T}^2) \hookrightarrow (W_\sharp^{1,\infty}(\mathbb{T}^2), W_\sharp^{1,3/2}(\mathbb{T}^2))_{1/2}. \tag{59}$$

However, (59) is false (see cite [12, Section 4]).

In what follows we will work in a slightly different setting. We ask for a solution a in some interpolation space A_θ to an equation of the form

$$Ta = Tb,$$

where b is given in some interpolation space B_θ. Lemma 29 below gives some sufficient conditions under which one can find such solutions. To state and prove Lemma 29 we need the technical Lemma 28. We introduce first some notation needed in the statement of Lemma 28.

Embeddings of images. Suppose A, B, F are Banach spaces and $T_1 : A \to F$, $T_2 : B \to F$ are two linear operators. In what follows we write

$$T_2(B) \hookrightarrow T_1(A), \tag{60}$$

if there exists some constant $C > 0$ such that for any $b \in B$ there exists some $a \in A$ such that $T_1 a = T_2 b$, in F, and $\|a\|_A \le C\|b\|_B$.

Regularisation by convolutions. Let $\varphi \in C_c^\infty([-1,1], \mathbb{R})$ be a function of integral 1 and consider some $\varepsilon > 0$. Define the function φ_ε by $\varphi_\varepsilon(t) := \varepsilon^{-1}\varphi(\varepsilon^{-1}t)$, for any $t \in \mathbb{R}$. For any $\varepsilon > 0$, and any (other) function $g \in L^1_{loc}(\mathbb{R}, Z)$ taking values in some Banach space Z, we define the function $g_\varepsilon := g * \varphi_\varepsilon$.

With this notation we state the following:

Lemma 28. *Let A, X_0, X_1, B_0, B_1, E, F be non-trivial Banach spaces such that $A, B_0, B_1 \hookrightarrow E$ and consider a bounded linear operator $T : E \to F$. Denote $A_0 := A \cap X_0$ and $A_1 := A \cap X_1$. Suppose moreover that the following conditions are satisfied:*

(i) $B_1 \hookrightarrow X_1 \hookrightarrow X_0$ and $B_0 \hookrightarrow A_0$;

(ii) A, X_1 have separable preduals and E is a separable reflexive space;

(iii) $T(B_1) \hookrightarrow T(A_1)$.

Then, for each $j = 0, 1$ we have the following:

For any $v_j \in L^2(\exp(t^2), B_j)$ and any $\varepsilon \in (0,1)$ there exists a function $u_j^\varepsilon \in C_b^1(\mathbb{R}, A_j) \cap L^2(\exp(t^2/2), A_j)$, such that

$$Tu_j^\varepsilon(t) = Tv_{j,\varepsilon}(t), \tag{61}$$

for any $t \in \mathbb{R}$, and satisfying the estimates:

$$\|u_j^\varepsilon\|_{L^2(A_j)} \lesssim \|v_j\|_{L^2(B_j)}, \tag{62}$$

and

$$\|R^S_{1-j} u_j^\varepsilon\|_{L^2(\exp(-2t^2), A_{1-j})} \lesssim \|v_j\|_{L^2(\exp(t^2), B_j)} + \delta_{j0}\|R^S_{1-j} v_j\|_{L^2(\exp(-t^2), B_{1-j})}. \tag{63}$$

where all the implicit constants do not depend on v_j and ε.

(Here, $v_{j,\varepsilon}(t) = v_j * \varphi_\varepsilon(t) = (v_j)_\varepsilon(t)$ and δ_{j0} is the Kronecker symbol, i.e., we have $\delta_{j0} = 1$ if $j = 0$ and $\delta_{j0} = 0$ if $j \neq 0$.)

In order to prove Lemma 28 we need some simple facts that are easy consequences of classical inequalities.

Fact 1. Consider some numbers $\alpha > \beta > 0$. *Suppose Z is a Banach space and consider some function $g \in L^2_{loc}(\mathbb{R}, Z)$. Then, we have:*

(i) $\|g_\varepsilon\|_{L^2(\exp(-\alpha t^2),Z)} \lesssim_{\alpha,\beta} \|g\|_{L^2(\exp(-\beta t^2),Z)}$, *uniformly in* $\varepsilon \in (0,1)$;

(ii) $\|g_\varepsilon\|_{L^2(\exp(\beta t^2),Z)} \lesssim_{\alpha,\beta} \|g\|_{L^2(\exp(\alpha t^2),Z)}$, *uniformly in* $\varepsilon \in (0,1)$.

Proof of Fact 1. We prove only item (i), item (ii) being similar. By Minkowski's inequality we have

$$\begin{aligned}
\|g_\varepsilon\|_{L^2(\exp(-\alpha t^2),Z)} &= \left(\int_{\mathbb{R}} e^{-\alpha t^2} \|\int_{-1}^{1} g(t-\varepsilon s)\varphi(s)ds\|_Z^2 dt\right)^{1/2} \\
&\leq \left(\int_{\mathbb{R}} \left(\int_{-1}^{1} e^{-\alpha t^2/2}\|g(t-\varepsilon s)\|_Z \varphi(s)ds\right)^2 dt\right)^{1/2} \\
&\leq \int_{-1}^{1} \left(\int_{\mathbb{R}} e^{-\alpha t^2}\|g(t-\varepsilon s)\|_Z^2 dt\right)^{1/2} \varphi(s)ds \\
&= \int_{-1}^{1} \left(\int_{\mathbb{R}} e^{-\alpha (t+\varepsilon s)^2}\|g(t)\|_Z^2 dt\right)^{1/2} \varphi(s)ds \\
&\lesssim \left(\int_{\mathbb{R}} e^{-\beta t^2}\|g(t)\|_Z^2 dt\right)^{1/2} \\
&\lesssim \|g\|_{L^2(\exp(-\beta t^2),Z)},
\end{aligned}$$

where we have used the fact that $e^{-\alpha(t+\varepsilon s)^2} \lesssim e^{-\beta t^2}$, when $s \in [-1,1]$ and $\varepsilon \in (0,1)$. □

Fact 2. Consider some numbers $\alpha, \beta > 0$. *Suppose Z is a Banach space and consider some function $g \in L^2(\exp(\beta t^2), Z)$. Then, we have*

$$\|R_j^S g\|_{L^2(\exp(-\alpha t^2),Z)} \lesssim_{\alpha,\beta} \|g\|_{L^2(\exp(\beta t^2),Z)},$$

for any $j = 0, 1$.

Proof of Fact 2. Fix $j \in \{0,1\}$. Note first that if $g \in L^2(\exp(\beta t^2), Z)$, then $g \in L^1(\mathbb{R}, Z)$. Indeed, by the Cauchy-Schwarz inequality we get

$$\int_{\mathbb{R}} \|g(s)\|_Z ds = \int_{\mathbb{R}} e^{-\beta t^2/2} e^{\beta t^2/2} \|g(t)\|_Z dt \lesssim \|g\|_{L^2(\exp(\beta t^2),Z)}. \tag{64}$$

Using the boundedness of the function ρ_j on $\mathbb{R}$, one can write

$$\begin{aligned}
\|R_j^S g\|_{L^2(\exp(-\alpha t^2),Z)} &= \left(\int_{\mathbb{R}} e^{-\alpha t^2} \left\|\int_{\mathbb{R}} \rho_j(t-s)g(s)ds\right\|_Z^2 dt\right)^{1/2} \\
&\lesssim \left(\int_{\mathbb{R}} e^{-\alpha t^2} \left(\int_{\mathbb{R}} \|g(s)\|_Z ds\right)^2 dt\right)^{1/2} \\
&\lesssim \int_{\mathbb{R}} \|g(s)\|_Z ds,
\end{aligned}$$

which together with (64) proves Fact 2. □

Conditional expectation. Let us introduce some more notation. Let Z be a Banach space and fix some $N \in \mathbb{N}^*$. For any function $g \in L^1_{loc}(\mathbb{R}, Z)$ we denote by $\mathbb{E}_N g$ the conditional expectation of g with the respect to the σ-algebra generated by the intervals $I_N^k := [k/N, (k+1)/N)$, were $k \in \mathbb{Z}$. In other words, if $(g)_I$ is the mean of g on one of these intervals I, i.e.,

$$(g)_I := \frac{1}{|I|} \int_I g(t)dt,$$

we define the corresponding conditional expectation of g by

$$\mathbb{E}_N g := \sum_{k \in \mathbb{Z}} (g)_{I_N^k} \mathbf{1}_{I_N^k}.$$

See [16, Chapter 5] or [19, Section 2.6] for some fundamental properties of more general conditional expectation operators. We use the following properties of the operator $\mathbb{E}_N$. Suppose that $g \in L^2(\mathbb{R}, Z)$, for some Banach space Z. Then, we have (see [19, Corollary 2.6.30])

$$\|\mathbb{E}_N g\|_{L^2(\mathbb{R},Z)} \le \|g\|_{L^2(\mathbb{R},Z)}. \tag{65}$$

Also, for any compact interval $J \subset \mathbb{R}$ we have

$$\|\mathbb{E}_N g - g\|_{L^2(J,Z)} \to 0, \tag{66}$$

when $N \to \infty$ (see [16, Corollary 2, p. 126]).

Now we can pass to the proof of Lemma 28.

Proof of Lemma 28. For each $j = 0, 1$ consider some functions $v_j \in L^2(\exp(t^2), B_j)$. We split the proof in two parts corresponding to the situations when $j = 0$ and $j = 1$.

The case $j = 0$

Construction of the solution u_0^ε and the corresponding estimates. In this case one can simply set $u_0^\varepsilon := v_{0,\varepsilon}$. Clearly, $u_0^\varepsilon \in L^2(\exp(t^2/2), A_0) \cap C_b^1(\mathbb{R}, A_0)$. It is also clear that the conditions (61), (62) are satisfied thanks to the fact that $B_0 \hookrightarrow A_0$. Let us verify (63). We have

$$\|R_1^S u_0^\varepsilon\|_{L^2(\exp(-2t^2),A_1)} \sim \|R_1^S v_{0,\varepsilon}\|_{L^2(\exp(-2t^2),A)} + \|R_1^S v_{0,\varepsilon}\|_{L^2(\exp(-2t^2),X_1)}, \tag{67}$$

and it remains to bound each term in the right hand side of (67). Since $B_0 \hookrightarrow A \cap X_0$, we have in particular that $B_0 \hookrightarrow A$ and hence,

$$\|R_1^S v_{0,\varepsilon}\|_{L^2(\exp(-2t^2),A)} \lesssim \|R_1^S v_{0,\varepsilon}\|_{L^2(\exp(-2t^2),B_0)} \lesssim \|v_{0,\varepsilon}\|_{L^2(\exp(t^2/2),B_0)} \lesssim \|v_0\|_{L^2(\exp(t^2),B_0)}, \tag{68}$$

where for the second "$\lesssim$" we have used Fact 2 and for the third "$\lesssim$" we have used Fact 1 (ii). Since $B_1 \hookrightarrow X_1$,

$$\begin{aligned}\|R_1^S v_{0,\varepsilon}\|_{L^2(\exp(-2t^2),X_1)} &\lesssim \|R_1^S v_{0,\varepsilon}\|_{L^2(\exp(-2t^2),B_1)} = \|(R_1^S v_0)_\varepsilon\|_{L^2(\exp(-2t^2),B_1)} \\ &\lesssim \|R_1^S v_0\|_{L^2(\exp(-t^2),B_1)},\end{aligned} \tag{69}$$

where for "=" we have used (159) and for the last "$\lesssim$" we have used Fact 1 (i). From (67), (68) and (69) we obtain (63) in the case $j = 0$.

The case $j = 1$

Construction of the solution u_1^ε. Since $T : E \to F$ is bounded and A_j, $B_j \hookrightarrow E$, we have that $T : A_j \to F$ and $T : B_j \to F$ are bounded for each $j = 0, 1$. Since $T(B_1) \hookrightarrow T(A_1)$ (see (60)), if $b \in B_1$, then there exists $a \in A_1$ such that $Ta = Tb$ and $\|a\|_{A_1} \leq C\|b\|_{B_1}$ for some constant $C > 0$. As a consequence, for each $k \in \mathbb{Z}$ we can find some elements $a_N^k \in A_1$ with

$$Ta_N^k = T\,(v_1)_{I_N^k}$$

and such that

$$\|a_N^k\|_{A_1} \leq C\|(v_1)_{I_N^k}\|_{B_1}.$$

Hence, defining $u_N : \mathbb{R} \to A_1$ by

$$u_{1,N} := \sum_{k\in\mathbb{Z}} a_N^k \mathbf{1}_{I_N^k},$$

we have

$$Tu_{1,N}(t) = T\mathbb{E}_N v_1(t), \tag{70}$$

for any $t \in \mathbb{R}$, and

$$\|u_{1,N}(t)\|_{A_1} \lesssim \|\mathbb{E}_N v_1(t)\|_{B_1}, \tag{71}$$

uniformly in $t \in \mathbb{R}$.

Define now the function $u_{1,N}^\varepsilon := u_{1,N} * \varphi_\varepsilon = (u_{1,N})_\varepsilon$, the convolution being in the t variable. Thanks to (70) we have

$$Tu_{1,N}^\varepsilon(t) = T(\mathbb{E}_N v_{1,})_\varepsilon(t), \tag{72}$$

for any $t \in \mathbb{R}$.

Let us observe that, when $N \to \infty$,

$$\|(\mathbb{E}_N v_1)_\varepsilon(t) - v_{1,\varepsilon}(t)\|_{B_1} \to 0, \tag{73}$$

uniformly on compact sets in $t \in \mathbb{R}$.

Indeed, by Jensen's inequality and (66) we have

$$\begin{aligned}
\|(\mathbb{E}_N v_1)_\varepsilon(t) - v_{1,\varepsilon}(t)\|_{B_1} &\leq \int_{B_\mathbb{R}(t,1/\varepsilon)} \|\mathbb{E}_N v_1(s) - v_1(s)\|_{B_1}\varphi_\varepsilon(t-s)ds \\
&\leq \left(\int_{B_\mathbb{R}(t,1/\varepsilon)} \|(\mathbb{E}_N v_1)(s) - v_1(s)\|_{B_1}^2\varphi_\varepsilon(t-s)ds\right)^{1/2} \\
&\lesssim_\varepsilon \|\mathbb{E}_N v_1 - v_1\|_{L^2(B_\mathbb{R}(t,1/\varepsilon),B_1)} \to 0,
\end{aligned}$$

when $N \to \infty$.

Also one observe that the sequence of functions $(u_{1,N}^\varepsilon)_{N\geq 1}$ is equi-continuous and uniformly bounded. In fact we have more. For any $m \in \mathbb{N}^*$ and any $t \in \mathbb{R}$ (see (160)),

$$\partial^m u_{1,N}^\varepsilon(t) = \frac{1}{\varepsilon^m}\int_\mathbb{R} u_{1,N}(s)(\partial^m\varphi)_\varepsilon(t-s)ds$$

and hence,

$$\begin{aligned}
\|\partial^m u_{1,N}^\varepsilon(t)\|_{A_1} &\lesssim_{\varepsilon,m} \int_\mathbb{R} \|u_{1,N}(s)\|_{A_1}\|(\partial^m\varphi)_\varepsilon(t-s)\|ds \\
&\lesssim_{\varepsilon,m} \|u_{1,N}\|_{L^2(\mathbb{R},A_1)}\|\partial^m\varphi\|_{L^2(\mathbb{R})} \\
&\lesssim_{\varepsilon,m} \|u_{1,N}\|_{L^2(\mathbb{R},A_1)}.
\end{aligned} \tag{74}$$

It remains to notice that, by (71) and (65), we have

$$\|u_{1,N}\|_{L^2(\mathbb{R},A_1)} \leq \|\mathbb{E}_N v_1\|_{L^2(\mathbb{R},B_1)} \lesssim \|v_1\|_{L^2(\mathbb{R},B_1)} < \infty,$$

and since $A_1 \hookrightarrow E$, we can write (using (74))

$$\|\partial^m u^\varepsilon_{1,N}(t)\|_E \lesssim \|\partial^m u^\varepsilon_{1,N}(t)\|_{A_1} \lesssim_{\varepsilon,m} \|v_1\|_{L^2(\mathbb{R},B_1)}, \tag{75}$$

uniformly in t and N.

Since A, X and E have separable preduals[8], by Lemma 44 (for $l = 2$) and a diagonalization argument, we obtain that there exists a subsequence of $(u^\varepsilon_{1,N})_{N\geq 1}$ (which for simplicity will be denoted also by $(u^\varepsilon_{1,N})_{N\geq 1}$) and an element $u^\varepsilon_1 \in C^1_b(\mathbb{R}, A_1)$ such that $u^\varepsilon_{1,N} \to u^\varepsilon_1$ in the w^*-topology on A, X and E respectively, uniformly on the compact subsets of $\mathbb{R}$. Note that, since E is a reflexive space we get that $u^\varepsilon_{1,N} \to u^\varepsilon_1$, uniformly on the compact subsets of $\mathbb{R}$, in the w-topology on E. This gives in particular that $Tu^\varepsilon_{1,N}(t) \to Tu^\varepsilon_1(t)$, in the strong w-topology on F, for any $t \in \mathbb{R}$. Also, by the embedding $B_1 \hookrightarrow E$ and (73) we get $T(\mathbb{E}_N v_1)_\varepsilon(t) \to Tv_{1,\varepsilon}(t)$ in the strong topology on F, for any $t \in \mathbb{R}$. Hence, thanks to (72) one can write

$$Tu^\varepsilon_1(t) = Tv_{1,\varepsilon}(t),$$

for all $t \in \mathbb{R}$, which proves (61).

The estimates corresponding to u^ε_1. In order to verify (62) one uses the Young inequality (see [19, Lemma 1.2.30]) and (71):

$$\begin{aligned}\|u^\varepsilon_1\|_{L^2(\mathbb{R},A_1)} &\leq \liminf_{N\to\infty}\|u^\varepsilon_{1,N}\|_{L^2(\mathbb{R},A_1)} \leq \liminf_{N\to\infty}\|u_{1,N}\|_{L^2(\mathbb{R},A_1)} \\ &\lesssim \liminf_{N\to\infty}\|\mathbb{E}_N v_1\|_{L^2(\mathbb{R},B_1)} \lesssim \|v_1\|_{L^2(\mathbb{R},B_1)}.\end{aligned} \tag{76}$$

Fix two numbers $1/2 < \beta < \alpha < 1$. Observe now that, as in (76), we get

$$\begin{aligned}\|u^\varepsilon_1\|_{L^2(\exp(t^2/2),A_1)} &\lesssim \liminf_{N\to\infty}\|u^\varepsilon_{1,N}\|_{L^2(\exp(t^2/2),A_1)} \\ &\lesssim \liminf_{N\to\infty}\|u_{1,N}\|_{L^2(\exp(\beta t^2),A_1)} \\ &\lesssim \liminf_{N\to\infty}\|\mathbb{E}_N v_1\|_{L^2(\exp(\beta t^2),B_1)}.\end{aligned} \tag{77}$$

where for the second "$\lesssim$" we have used Fact 1 (ii) and for the third "$\lesssim$" we have used (71). We also have

$$\begin{aligned}\|\mathbb{E}_N v_1\|^2_{L^2(\exp(\beta t^2),B_1)} &= \int_{\mathbb{R}} e^{\beta t^2}\|\mathbb{E}_N v_1(t)\|^2_{B_1}dt \lesssim \sum_{k\in\mathbb{Z}} e^{\alpha k^2}\int_k^{k+1}\|\mathbb{E}_N v_1(t)\|^2_{B_1}dt \\ &\leq \sum_{k\in\mathbb{Z}} e^{\alpha k^2}\int_k^{k+1}\|v_1(t)\|^2_{B_1}dt \lesssim \int_{\mathbb{R}} e^{t^2}\|v_1(t)\|^2_{B_1}dt \\ &= \|v_1\|^2_{L^2(\exp(t^2),B_1)}.\end{aligned}$$

Combining this with (77) we get the estimate

$$\|u^\varepsilon_1\|_{L^2(\exp(t^2/2),A_1)} \lesssim \|v_1\|_{L^2(\exp(t^2),B_1)}. \tag{78}$$

In particular, we have that $u^\varepsilon_1 \in L^2(\exp(t^2/2), A_1) \cap C^1_b(\mathbb{R}, A_1)$.

[8] In the case of the reflexive space E we use the fact that $E = (E^*)^*$ is separable, and then, E^* is separable (see for instance [20, Theorem 4.6-8, p. 245]).

Let us verify now that u_1^ε also satisfies (63). We can write

$$\|R_0^S u_1^\varepsilon\|_{L^2(\exp(-2t^2),A_0)} \lesssim \|u_1^\varepsilon\|_{L^2(\exp(t^2/2),A_0)} \lesssim \|u_1^\varepsilon\|_{L^2(\exp(t^2/2),A_1)}, \tag{79}$$

where for the first "$\lesssim$" we have used Fact 2 and for the second "$\lesssim$" we have used the embedding $X_1 \hookrightarrow X_0$ (that implies $A_1 \hookrightarrow A_0$). Combining (78) with (79) we get

$$\|R_0^S u_1^\varepsilon\|_{L^2(\exp(-2t^2),A_0)} \lesssim \|v_1\|_{L^2(\exp(t^2),B_1)},$$

and (63) is proved in the case $j = 1$. Lemma 28 is proved. □

We are now able to state and prove the main result of Subsection 3.3:

Lemma 29. *Fix some number $\theta \in (0,1)$. Let the Banach spaces A, X_0, X_1, A_0, A_1, B_0, B_1, E, F and the operator T be as in Lemma 28. Moreover, we assume that X_0, X_1 and B_1 are UMD spaces and that $(X_0, X_1)_\theta$ has a separable predual. Then, for any $b \in (B_0, B_1)_\theta$ there exists some $a \in A \cap (X_0, X_1)_\theta$ such that*

$$Ta = Tb,$$

and

$$\|a\|_{A\cap(X_0,X_1)_\theta} \lesssim \|b\|_{(B_0,B_1)_\theta}. \tag{80}$$

Proof. Fix some $b \in \mathcal{C}_\theta(B_0|B_1)$. Consider $v \in \mathcal{F}^2(B_0|B_1)$ such that $v(\theta) = b$ and

$$\|v\|_{\mathcal{F}^2(B_0|B_1)} \le 2\|b\|_{(B_0|B_1)_\theta}. \tag{81}$$

Since we can replace (if necessary) v by $\exp(z^2 - \theta^2)v$, we can assume without loss of generality that $v_j \in L^2(\exp(t^2), B_j)$, and (by (81))

$$\|v_j\|_{L^2(\mathbb{R},B_j)} \le \|v_j\|_{L^2(\exp(t^2),B_j)} \lesssim \|b\|_{(B_0|B_1)_\theta}, \tag{82}$$

where $v_j(t) := v(j + it)$, $j = 0, 1$, for all $t \in \mathbb{R}$.

Define, for each $\varepsilon \in (0,1)$, the function v_ε on $\overline{S}$ by

$$v_\varepsilon(z) := \int_{\mathbb{R}} v(z - is)\varphi_\varepsilon(s)ds,$$

with φ_ε as in the statement of Lemma 28. Note that, with the notation used in Lemma 28, we have $v_{j,\varepsilon}(t) = v_\varepsilon(j + it)$. Now, thanks to Lemma 20 and Lemma 27,

$$v_{j,\varepsilon}(t) := H_j^S v_{j,\varepsilon}(t) + R_j^S v_{1-j,\varepsilon}(t),$$

for all $t \in \mathbb{R}$. From this identity, since B_1 is an UMD space, we can write

$$\begin{aligned}\|R_1^S v_{0,\varepsilon}\|_{L^2(\mathbb{R},B_1)} &\le \|v_{1,\varepsilon}\|_{L^2(\mathbb{R},B_1)} + \|H_1^S v_{1,\varepsilon}\|_{L^2(\mathbb{R},B_1)} \\ &\lesssim \|v_{1,\varepsilon}\|_{L^2(\mathbb{R},B_1)} \lesssim \|v_1\|_{L^2(\mathbb{R},B_1)} \\ &\lesssim \|b\|_{(B_0|B_1)_\theta},\end{aligned}$$

where for the second "$\lesssim$" we have used Young's inequality ([19, Lemma 1.2.30]) and for the last "$\lesssim$" we have used (82). In particular, we get

$$\|R_1^S v_{0,\varepsilon}\|_{L^2(\exp(-t^2),B_1)} \lesssim \|b\|_{(B_0|B_1)_\theta}. \tag{83}$$

By Lemma 28 there exist some functions $u_j^\varepsilon \in L^2(\exp(t^2/2), A_j) \cap C_b^1(\mathbb{R}, A_j)$ satisfying (61), (62), (63). Define $\tilde{u}^\varepsilon : \overline{S} \to A_0 + A_1$ by

$$\tilde{u}^\varepsilon(z) := -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{u_0^\varepsilon(t)}{it - z} dt + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{u_1^\varepsilon(t)}{1 + it - z} dt, \tag{84}$$

for all $z \in S$, and

$$\tilde{u}_j^\varepsilon(t) = \tilde{u}^\varepsilon(j + it) := H_j^S u_j^\varepsilon(t) + R_j^S u_{1-j}^\varepsilon(t), \tag{85}$$

for all $t \in \mathbb{R}$. Notice that, since $u_j^\varepsilon \in L^2(\exp(t^2/2), A_j) \cap C_b^1(\mathbb{R}, A_j)$, thanks to Lemma 27, $\tilde{u}^\varepsilon$ is well-defined and $\tilde{u}^\varepsilon \in C_b\left(\overline{S}, A_0 + A_1\right)$.

Let us verify that $\exp(z^2 - \theta^2)\tilde{u}^\varepsilon \in \mathcal{F}^2(A_0, X_0 \mid A_1, X_1)$. We show that, for any $j = 0, 1$, we have the estimate:

$$\left(\int_{\mathbb{R}} e^{-2t^2} \|\Lambda_j \tilde{u}^\varepsilon(j + it)\|_{X_j}^2 dt\right)^{1/2} \lesssim \|b\|_{(B_0|B_1)_\theta}, \tag{86}$$

for any bounded linear operator $\Lambda_j : A_j \to X_j$ with $\|\Lambda_j\| \le 1$, the implicit constant not depending on Λ_j.

Using (85) and (158) we write:

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-2t^2} \|\Lambda_j \tilde{u}^\varepsilon(j + it)\|_{X_j}^2 dt\right)^{1/2} &\le \left(\int_{\mathbb{R}} e^{-2t^2} \|H_j^S \Lambda_j u_j^\varepsilon(t)\|_{X_j}^2 dt\right)^{1/2} \\ &\quad + \left(\int_{\mathbb{R}} e^{-2t^2} \|\Lambda_j R_j^S u_{1-j}^\varepsilon(t)\|_{X_j}^2 dt\right)^{1/2}.\end{aligned} \tag{87}$$

Since the spaces X_j have the UMD property, we get

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-2t^2} \|H_j^S \Lambda_j u_j^\varepsilon(t)\|_{X_j}^2 dt\right)^{1/2} &\le \|H_j^S \Lambda_j u_j^\varepsilon\|_{L^2(\mathbb{R}, X_j)} \lesssim \|\Lambda_j u_j^\varepsilon\|_{L^2(\mathbb{R}, X_j)} \\ &\le \|u_j^\varepsilon\|_{L^2(\mathbb{R}, A_j)} \le \|v_j\|_{L^2(\mathbb{R}, B_j)} \\ &\lesssim \|b\|_{(B_0|B_1)_\theta},\end{aligned} \tag{88}$$

where for the third "$\le$" we have used (62) and for the last "$\lesssim$" we have used (82). It remains to estimate the second term in the right hand side of (87):

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-2t^2} \|\Lambda_j R_j^S u_{1-j}^\varepsilon(t)\|_{X_j}^2 dt\right)^{1/2} &\le \|R_j^S u_{1-j}^\varepsilon\|_{L^2(\exp(-t^2) A_j)} \\ &\lesssim \|v_{1-j}\|_{L^2(\exp(t^2), B_{1-j})} + \delta_{j1} \|R_j^S v_{1-j}\|_{L^2(\exp(-t^2), B_j)} \\ &\lesssim \|b\|_{(B_0|B_1)_\theta},\end{aligned} \tag{89}$$

where for the first "$\lesssim$" we have used (63) and for the last "$\lesssim$" we have used (82) and (83). By (87), (88), (89) we have proved (86). Hence, we have obtained

$$\|\exp(z^2 - \theta^2)\tilde{u}^\varepsilon\|_{\mathcal{F}^2(A_0, X_0|A_1, X_1)} \lesssim \|b\|_{(B_0|B_1)_\theta}.$$

This implies that for $a^\varepsilon := \tilde{u}^\varepsilon(\theta)$ we have

$$\|a^\varepsilon\|_{(A_0, X_0|A_1, X_1)_\theta} \lesssim \|b\|_{(B_0|B_1)_\theta}. \tag{90}$$

Note that, by Proposition 22, $(B_0|B_1)_\theta = (B_0, B_1)_\theta$, and by Corollary 24, $(A_0, X_0 \mid A_1, X_1)_\theta \hookrightarrow A \cap (X_0, X_1)_\theta$. From this and (90) we get

$$\|a^\varepsilon\|_{A\cap(X_0,X_1)_\theta} \lesssim \|b\|_{(B_0,B_1)_\theta}. \tag{91}$$

We observe that for $b_\varepsilon := v_\varepsilon(\theta)$ we have

$$Ta^\varepsilon = Tb_\varepsilon. \tag{92}$$

Indeed, by applying Lemma 20, (84), the continuity of $T : E \to F$ and (61), one gets

$$T\tilde{u}^\varepsilon(\theta) - Tv_\varepsilon(\theta) = -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{Tu_0^\varepsilon(t) - Tv_{0,\varepsilon}(t)}{it-\theta}dt + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{Tu_1^\varepsilon(t) - Tv_{1,\varepsilon}(t)}{1+it-\theta}dt = 0.$$

We let $\varepsilon \to 0$. Since $v_{j,\varepsilon} \to v_j$ in $L^2(\mathbb{R}, B_j)$, for each $j = 0, 1$ (see [19, Proposition 1.2.32]) we get that $b_\varepsilon \to b$ in $(B_0|B_1)_\theta = (B_0, B_1)_\theta$. Also, thanks to (91), since A and $(X_0, X_1)_\theta$ have separable preduals, by the sequential Banach-Aloglu theorem, there exists some $a \in A \cap (X_0, X_1)_\theta$ such that $a^{1/n} \to a$ ($n \in \mathbb{N}^*$) in the w^*-topology on A and in the w^*-topology on $(X_0, X_1)_\theta$, up to a subsequence. Also, by (91) we get

$$\|a\|_{A\cap(X_0,X_1)_\theta} \lesssim \liminf_{n\to\infty} \|a^{1/n}\|_{A\cap(X_0,X_1)_\theta} \lesssim \|b\|_{(B_0,B_1)_\theta}.$$

It follows that $Tb_{1/n} \to Tb$ and $Ta^{1/n} \to Ta$ in the w^*-topology of F, up to a subsequence. Consequently, by (92) we have

$$Ta = Tb.$$

Since $\mathcal{C}_\theta(B_0|B_1)$ is dense in $(B_0|B_1)_\theta = (B_0, B_1)_\theta$ we can use the above compactness argument in order to obtain a solution for any $b \in (B_0|B_1)_\theta$. Lemma 29 is proved. □

4 Spectral analysis

In this Section we study the solutions of divergence-like equations via L^2-based Fourier analysis methods. This is done by a slight modification of the ideas of Bourgain and Brezis used in the proof of [5, Lemma 2]. While in this case the techniques we use are essentially those of Bourgain and Brezis, we give more general existence results that take into account the shape of the Fourier spectrum of solutions. The final results of this Section will represent the "1-endpoint" when we apply the $\mathcal{W}$-method (see Lemma 37 and Lemma 38 below).

4.1 Overview: existence results for divergence-like equations in $\dot{H}^{d/2}$

The proof of Lemma 37 (and Lemma 38) is rather technical. In this overview we present only a non-rigurous sketch of the steps that lead to Lemma 37. The technique used is based exclusively on the one in [5, Section 4]. In turn, [5, Section 4] can be seen as a simplified version of the argument presented here. A similar, but much simpler situation is described in [15, Subsection 3.1] which is also based on the Bourgain-Brezis argument. While the situation in [15, Subsection 3.1] is too particular to be an illustration for the technique we use here, it may represent however, a better introduction to this technique.

The proof of Lemma 38 is using the same ideas as the proof of Lemma 37. It is also technically simpler than Lemma 37 and we will give only a sketch for the proof of Lemma 38. In this overview we focus on the more complicated existence result provided by Lemma 37. Assume that $d \geq 3$. As in the Introduction (see (11)) we consider a family of functions $G_1, ..., G_d : \mathbb{R}^d \to \mathbb{R}^{d-1}$ which is adapted to the family of half-spaces $D_1, ..., D_d \subset \mathbb{R}^d$. We want to show that for any system of $(d-1)$-vector fields $(v_j)_{j=1,..,d}$ with $v_j \in \dot{H}^{d/2}(\mathbb{R}^d)$ and $spec(v_j) \subseteq D_j$, there exists a system of $(d-1)$-vector fields $(u_j)_{j=1,..,d}$, with $u_j \in L^\infty(\mathbb{R}^d) \cap \dot{H}^{d/2}(\mathbb{R}^d)$ and $spec(u_j) \subseteq D_j$, such that

$$\sum_{j=1}^{d} G_j(\nabla) \cdot u_j = \sum_{j=1}^{d} G_j(\nabla) \cdot v_j. \tag{93}$$

We divide the proof of this existence result in four steps:

Step 1. We show first that any symbol m, satisfying some properties, has a bounded Fourier transform (see Lemma 31 and Lemma 32). These properties of m are related to the growth of its derivatives (see (103), (104)) and some symmetry that m has to obey (namely to be odd in some coordinates; see (105)). A symbol with such properties will be called a BB-symbol (more precisely, ℓ-BB symbol, where ℓ is a parameter between 1 and d that depends on m).

The argument used in Lemma 31 is technical and we do not describe it in this overview. One can say however, that it is based on a cancellation phenomenon in which the fact that m is odd in some coordinates plays an important role (see (107)).

Some examples of symbols on $\mathbb{R}^d$ ($d \geq 2$) with bounded Fourier transform are:

$$m_1(\xi) := \frac{\xi_1 \xi_2}{|\xi|^{d+2}} \quad \text{and} \quad m_2(\xi) := \frac{\xi_1^2 - \xi_2^2}{|\xi|^{d+2}}. \tag{94}$$

One can observe that m_1 is a BB-symbol for $\ell = 2$ (it is a 2-BB symbol) and hence, by Lemma 32 below we have $\widehat{m}_1 \in L^\infty(\mathbb{R}^d)$. The symbol m_2 is not a BB-symbol (it is not odd). However, m_2 can be obtained from m_1 by rotating the coordinates. Hence, we have as well that $\widehat{m}_2 \in L^\infty(\mathbb{R}^d)$. Mironescu showed in [23] that the boundedness of $\widehat{m}_1, \widehat{m}_2$ can be justified by a direct explicit computation involving "the" fundamental solution of $\triangle^2$ (see Proposition 1 and Corollary 2 in [23]). By contrast, the method used in Lemmas 31, 32 below is dealing with a more general situation where an explicit computation may not be possible (see also Remark 34).

Step 2. Let $U := \mathbb{R}^{d-1} \times (0, \infty)$ be the upper half-plane and let Y be the Sobolev space $\dot{H}^{d/2}(\mathbb{R}^d)$. In Lemma 35 we prove the estimate

$$\left\| |\nabla|^{-1} |\nabla_2^\sigma|^2 u \right\|_{Y^*/U} \lesssim \|\nabla_2^\sigma u\|_{(\mathcal{M}+Y^*)/U} + \|\nabla_2^\sigma u\|_{Y^*/U}^{1/2} \|\nabla_2^\sigma u\|_{(\mathcal{M}+Y^*)/U}^{1/2}, \tag{95}$$

where

$$\nabla_2^\sigma u := (\partial_1^\sigma u, \partial_2^\sigma u),$$

and ∂_j^σ being the Fourier multiplier $\partial_j^\sigma := \sigma_j(\nabla)$ (see (10) for the definition of the functions σ_j).

By means of Parseval's theorem and composition with rotations, we reduce (95) to another estimate

$$\int_{\mathbb{R}^d} \frac{\sigma_1^2(\xi)\sigma_2^2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) |\widehat{u}(\xi)|^2 d\xi \lesssim \|\nabla_2^\sigma u\|_{(\mathcal{M}+Y^*)/U}^2 + \|\nabla_2^\sigma u\|_{Y^*/U} \|\nabla_2^\sigma u\|_{(\mathcal{M}+Y^*)/U}, \tag{96}$$

Next, we decompose $\nabla_2^\sigma u$ in $(\mathcal{M}+Y^*)/U$:

$$\nabla_2^\sigma u = (F_1, F_2) + (h_1, h_2) + (F_1^-, F_2^-),$$

with $F_1, F_2 \in \mathcal{M}$, $h_1, h_2 \in Y^*$ and $spec(F_1^-), spec(F_2^-) \subseteq U^c$. Using this decomposition we split the left hand side of (95) in a term only containing F_j and terms containing at least one function h_j (the terms containing F_j^- are cancelling out by the orthogonality of F_j^- and $\mathbf{1}_U$). The terms containing at least one function h_j can be estimated by using standard L^2 theory (see (124)). For the term containing both F_1, F_2 we need to employ the estimates that we obtained for the BB-symbols (see (122)).

Step 3. We would like to absorb terms or factors of the form $\|\nabla_2 u\|_{Y^*}$ (those not containing any reference to the spaces $\mathcal{M}$) in the right hand side of (95). By (95) we have

$$\left\||\nabla|^{-1}|\nabla_2^\sigma|^2 u\right\|_{Y^*/U} \lesssim \varepsilon \left\|\nabla_2^\sigma u\right\|_{Y^*/U} + \varepsilon^{-1} \left\|\nabla_2^\sigma u\right\|_{(\mathcal{M}+Y^*)/U}, \tag{97}$$

for any $\varepsilon \in (0,1)$ ((97) follows from (95) by the standard use of the inequality $2a^{1/2}b^{1/2} = 2(\varepsilon a)^{1/2}(\varepsilon^{-1}b)^{1/2} \leq \varepsilon a + \varepsilon^{-1}b$, for any $a, b > 0$).

One can get in the same way (or by composing the function u in (97) with appropriate rotations) that

$$\left\||\nabla|^{-1}|(\partial_1^\sigma, \partial_j^\sigma)|^2 u\right\|_{Y^*/U} \lesssim \varepsilon \left\|(\partial_1^\sigma, \partial_j^\sigma) u\right\|_{Y^*/U} + \varepsilon^{-1} \left\|(\partial_1^\sigma, \partial_j^\sigma) u\right\|_{(\mathcal{M}+Y^*)/U}, \tag{98}$$

for any $j \in \{1, ..., d-1\}$. By adding up all these inequalities for $j \in \{1, ..., d-1\}$ we get

$$\left\||\nabla|^{-1}|(\partial_1^\sigma, ..., \partial_{d-1}^\sigma)|^2 u\right\|_{Y^*/U} \lesssim \varepsilon \left\|(\partial_1^\sigma, ..., \partial_{d-1}^\sigma) u\right\|_{Y^*/U} + \varepsilon^{-1} \left\|(\partial_1^\sigma, ..., \partial_{d-1}^\sigma) u\right\|_{(\mathcal{M}+Y^*)/U}. \tag{99}$$

(This after recalling that whenever Z is the space Y^*/U or $(\mathcal{M}+Y^*)/U$, by our conventions we have

$$\|(v_1, ..., v_k)\|_Z \sim \|v_1\|_Z + ... + \|v_k\|_Z,$$

for any family $v_1, ..., v_k \in Z$.)

By composing the function u in (99) with appropriate rotations (that transform U in D_j) we get

$$\left\||\nabla|^{-1}|G_j(\nabla)|^2 u\right\|_{Y^*/D_j} \lesssim \varepsilon \left\|G_j(\nabla) u\right\|_{Y^*/D_j} + \varepsilon^{-1} \left\|G_j(\nabla) u\right\|_{(\mathcal{M}+Y^*)/D_j}, \tag{100}$$

At this point we observe that, for a fixed j, the term $\||\nabla|^{-1}|G_j(\nabla)|^2 u\|_{Y^*/D_j}$ still can be much smaller than $\|G_j(\nabla)u\|_{Y^*/D_j}$. However, by adding all these terms up we get equivalent quantities. Namely, using the Parseval estimate we have:

$$\sum_{j=1}^d \left\||\nabla|^{-1}|G_j(\nabla)|^2 u\right\|_{Y^*/D_j} \sim \sum_{j=1}^d \left\|G_j(\nabla) u\right\|_{Y^*/D_j} \sim \|u\|_{\dot{H}^{-d/2+1}/D},$$

(where $D = D_1 \cup ... \cup D_d$) which combined with (100) gives

$$\|u\|_{\dot{H}^{-d/2+1}/D} \lesssim \varepsilon \|u\|_{\dot{H}^{-d/2+1}/D} + \varepsilon^{-1} \sum_{j=1}^d \left\|G_j(\nabla) u\right\|_{(\mathcal{M}+Y^*)/D_j}, \tag{101}$$

Fixing a sufficiently small ε we can absorb $\|u\|_{\dot{H}^{-d/2+1}}$ in the right hand side of (101). We now have

$$\|u\|_{\dot{H}^{-d/2+1}/D} \lesssim \varepsilon^{-1} \sum_{j=1}^d \left\|G_j(\nabla) u\right\|_{(\mathcal{M}+Y^*)/D_j}, \tag{102}$$

which is the inequality proved in Lemma 36.

Step 4. Now we can apply the closed range theorem (Theorem 4.13 in [27]) to deduce from (102) the existence result in (93) (see the end proof of Lemma 37 for details).

4.2 Symbols with bounded Fourier transform

Let $1 \le \ell \le d$ be some integers and let $m : \mathbb{R}^d \to \mathbb{C}$ be a function. We say that m is an ℓ-*BB* symbol[9] if the following conditions are satisfied:

(i) there exists a constant $C > 0$ such that, in the case $\ell < d$,

$$\int_{\mathbb{R}^{d-\ell}} |\partial_1^{\alpha_1} ... \partial_\ell^{\alpha_\ell} m(\xi', \xi'')| d\xi'' \le \frac{C}{|\xi'|^{\ell+|\alpha|}}, \tag{103}$$

for all $\alpha = (\alpha_1, ..., \alpha_\ell) \in \{0,1\}^\ell$ and all $\xi' \in (0,\infty)^\ell$, and, in the case $\ell = d$,

$$|\partial_1^{\alpha_1} ... \partial_d^{\alpha_d} m(\xi)| \le \frac{C}{|\xi|^{\ell+|\alpha|}}, \tag{104}$$

for all $\alpha = (\alpha_1, ..., \alpha_d) \in \{0,1\}^d$ and all $\xi \in (0,\infty)^d$;

(ii) m is an odd function in each of the components $\xi_1, \xi_2, ..., \xi_\ell$, i.e.,

$$m(\xi_1, ..., \xi_{j-1}, -\xi_j, \xi_{j+1} ..., \xi_d) = -m(\xi_1, ..., \xi_{j-1}, \xi_j, \xi_{j+1} ..., \xi_d), \tag{105}$$

for all $1 \le j \le \ell$, and all $\xi_1, \xi_2, ..., \xi_d \in \mathbb{R}$.

Remark 30. *In the item (i) above we assume that the derivatives $\partial_1^{\alpha_1} ... \partial_l^{\alpha_l} m(\xi', \xi'')$ exist in the classical sense and are continuous for any $(\xi', \xi'') \in (0,\infty)^l \times \mathbb{R}^{d-l}$, when $l = \ell$ or $l = d$ respectively (with the convention that $(0,\infty)^l \times \mathbb{R}^{d-l} = (0,\infty)^d$, when $l = d$).*

For any $\nu \in \mathbb{Z}$ we denote by I_ν the interval $[2^{\nu-1}, 2^\nu]$. For every $k' = (k_1, ..., k_\ell) \in \mathbb{Z}^\ell$ we consider the positive dyadic box $I_{k'} := I_{k_1} \times ... \times I_{k_\ell}$ and we associate to it the ℓ-symmetric set:

$$s_\ell(I_{k'}) := \bigcup_{\alpha_1,...,\alpha_\ell \in \{0,1\}} ((-1)^{\alpha_1} I_{k_1}) \times ... \times ((-1)^{\alpha_\ell} I_{k_\ell}) \subset \mathbb{R}^\ell.$$

The next technical Lemma is the basis for all of our results in this Section. Its proof consists in slightly adapting some arguments of Bourgain and Brezis (see [5, Lemma 3, p. 404]):

Lemma 31. *Let $d \ge 1$ be an integer and let $m : \mathbb{R}^d \to \mathbb{C}$ be an ℓ-BB symbol for some $1 \le \ell \le d$ and some constant C.*

Then,

$$\sum_{k' \in \mathbb{Z}^\ell} \left| \int_{s_\ell(I_{k'}) \times M_{k'}} m(\xi) e^{i\langle \xi, x\rangle} d\xi \right| \lesssim C,$$

for any measurable subsets $M_{k'} \subseteq \mathbb{R}^{d-\ell}$ of finite measure, uniformly in $x \in \mathbb{R}^d$ and in $M_{k'}$. (By convention, if $\ell = d$, then $s_\ell(I_{k'}) \times M_{k'}$ is replaced by $s_d(I_{k'})$.)

Proof. We first prove Lemma 31 in the case where m is a d-*BB* symbol. In this case we have to prove that

$$\sum_{k \in \mathbb{Z}^d} \left| \int_{s_d(I_k)} m(\xi) e^{i\langle \xi, x\rangle} d\xi \right| \lesssim C, \tag{106}$$

uniformly in $x = (x_1, ..., x_d) \in \mathbb{R}^d$.

[9] Here, *BB* stands for "Bourgain-Brezis".

Since m is odd in the variables $\xi_1, ..., \xi_d$, for any $k \in \mathbb{Z}^d$ one can write,

$$\int_{s_d(I_k)} m(\xi) e^{i\langle \xi, x\rangle} d\xi = \int_{s_d(I_k)} m(\xi) \prod_{j=1}^{d} e^{i\xi_j x_j} d\xi = (2i)^d \int_{I_k} m(\xi) \prod_{j=1}^{d} \sin(\xi_j x_j)\, d\xi. \tag{107}$$

Now, let us notice that whenever $a : \mathbb{R}^d \to \mathbb{C}$, $b_1, ..., b_d : \mathbb{R} \to \mathbb{C}$ are sufficiently smooth functions and $J_j = [q_j, r_j]$, $(q_j < r_j)$, $j = 1, ..., d$ are d intervals, we have

$$\int_{J_1\times...\times J_d} a(\xi) \prod_{j=1}^{d} b_j(\xi_j)\, d\xi = \sum_{\alpha\in\{0,1\}^d} \int_{J_1\times...\times J_d} (-\nabla)^\alpha a(\xi) \prod_{j=1}^{d} [b_j]_{J_j}^{\alpha_j}(\xi_j)\, d\xi, \tag{108}$$

where, for each $1 \le j \le d$, the quantity $[b_j(\xi_j)]_{J_j}^{\alpha_j}$ is defined as follows

$$[b_j]_{J_j}^{0}(\xi_j) := \left(\int_{q_j}^{r_j} b_j(t)\, dt\right) \delta_{r_j}(\xi_j) \quad \text{and} \quad [b_j]_{J_j}^{1}(\xi_j) := \int_{q_j}^{\xi_j} b_j(t)\, dt,$$

where δ_{r_j} is the Dirac measure on $\mathbb{R}$ concentrated in r_j. The case $d = 1$ of (108) is directly obtained by integration by parts. The general case of (108) follows by a repeated application of the case $d = 1$ and Fubini's theorem.

Fix some integers $k_1, ..., k_d$ and consider the intervals $I_{k_j} = \left[2^{k_j-1}, 2^{k_j}\right]$, $j = 1, ..., d$. By applying (108) to the functions $a = m$ and $b_j = sin_{x_j}$ (where $sin_{x_j}(\xi_j) := \sin(\xi_j x_j)$), we obtain

$$\int_{I_{k_1}\times...\times I_{k_d}} m(\xi) \prod_{j=1}^{d} \sin(\xi_j x_j)\, d\xi = \sum_{\alpha\in\{0,1\}^d} \int_{I_{k_1}\times...\times I_{k_d}} (-\nabla)^\alpha m(\xi) \prod_{j=1}^{d} \left[\sin_{x_j}\right]_{I_{k_j}}^{\alpha_j}(\xi_j)\, d\xi. \tag{109}$$

By a direct computation,

$$\left|\int_J \sin(t\cdot x_j)\, dt\right| \lesssim \min\left(4^{k_j}|x_j|, \frac{1}{|x_j|}\right),$$

for any subinterval $J \subset I_{k_j}$, and consequently,

$$|\left[\sin_{x_j}\right]_{I_{k_j}}^{\alpha_j}(\xi_j)| \lesssim \min\left(4^{k_j}|x_j|, \frac{1}{|x_j|}\right), \tag{110}$$

for any $j = 1, ..., d$, any $\alpha_j \in \{0, 1\}$ and any $\xi_j \in I_{k_j}$.

Let us fix some integer $0 \le l \le d$ and consider the case of $\alpha = (1, ..., 1, 0, ..., 0) \in \{0, 1\}^d$ (l values equal to 1). By (110) we get that the quantity

$$\left|\int_{I_{k_1}\times...\times I_{k_d}} (-\nabla)^\alpha m(\xi) \prod_{j=1}^{d} \left[\sin_{x_j}\right]_{I_{k_j}}^{\alpha_j}(\xi_j)\, d\xi\right|$$

is bounded by

$$\left(\int_{I_{k_1}\times...\times I_{k_l}} |\partial_1 ... \partial_l m\left(\xi_1, ..., \xi_l, 2^{k_{l+1}}, ..., 2^{k_d}\right)| d\xi_1 ... d\xi_l\right) \prod_{j=1}^{d} \min\left(4^{k_j}|x_j|, \frac{1}{|x_j|}\right). \tag{111}$$

Fix some index $\mu \in \{1, ..., d\}$. Using the fact that m is a *d-BB* symbol one can write (see (104))

$$\begin{aligned}\int_{I_{k_1}\times...\times I_{k_l}} |\partial_1 ... \partial_l m\left(\xi_1, ..., \xi_l, 2^{k_{l+1}}, ..., 2^{k_d}\right)| d\xi_1 ... d\xi_l &\le C \int_{I_{k_1}\times...\times I_{k_l}} \frac{1}{2^{k_\mu(d+l)}} d\xi_1 ... d\xi_l \\ &\sim C \frac{2^{k_1+...+k_l}}{2^{k_\mu(d+l)}}.\end{aligned}$$

Combining this with (111) we get that

$$\left|\int_{I_{k_1}\times\ldots\times I_{k_d}} (-\nabla)^{\alpha} m(\xi) \prod_{j=1}^{d} \left[\sin_{x_j}\right]_{I_{k_j}}^{\alpha_j} (\xi_j)\, d\xi\right| \lesssim C\frac{2^{k_1+\ldots+k_l}}{2^{k_\mu(d+l)}} \prod_{j=1}^{d} \min\left(4^{k_j}|x_j|, \frac{1}{|x_j|}\right),$$

and hence, for $\Gamma_\mu := \left\{k = (k_1, ..., k_d) \in \mathbb{Z}^d \mid k_\mu = \max_{1\le j\le d} k_j\right\}$,

$$\sum_{k\in\Gamma_\mu}\left|\int_{I_{k_1}\times\ldots\times I_{k_d}} (-\nabla)^{\alpha} m(\xi) \prod_{j=1}^{d} \left[\sin_{x_j}\right]_{I_{k_j}}^{\alpha_j} (\xi_j)\, d\xi\right|$$

is bounded by

$$C\sum_{k\in\Gamma_\mu}\frac{2^{k_1+\ldots+k_l}}{2^{k_\mu(d+l)}} \prod_{j=1}^{d} \min\left(4^{k_j}|x_j|, \frac{1}{|x_j|}\right) = C\sum_{k\in\Gamma_\mu}\frac{2^{k_1+\ldots+k_l}}{2^{k_\mu(d+l)}}2^{k_1+\ldots+k_d} \prod_{j=1}^{d} \min\left(2^{k_j}|x_j|, \frac{1}{2^{k_j}|x_j|}\right). \tag{112}$$

For any $k \in \Gamma_\mu$ we have $k_\mu = \max_{1\le j\le d} k_j$. Hence, we can write

$$\frac{2^{k_1+\ldots+k_l}}{2^{k_\mu(d+l)}}2^{k_1+\ldots+k_d} = \prod_{j=1}^{l} 2^{k_j-k_\mu} \prod_{j=1}^{d} 2^{k_j-k_\mu} \le 1,$$

and therefore, the right hand side of (112) is at most

$$\begin{aligned} C\sum_{k\in\Gamma_\mu}\prod_{j=1}^{d} \min\left(2^{k_j}|x_j|, \frac{1}{2^{k_j}|x_j|}\right) &\le C\sum_{k_1,k_2,\ldots,k_d\in\mathbb{Z}}\prod_{j=1}^{d} \min\left(2^{k_j}|x_j|, \frac{1}{2^{k_j}|x_j|}\right) \\ &= C\prod_{j=1}^{d}\sum_{k_j\in\mathbb{Z}} \min\left(2^{k_j}|x_j|, \frac{1}{2^{k_j}|x_j|}\right) \lesssim C. \end{aligned}$$

In other words, we have seen that for any $\mu \in \{1, ..., d\}$ and any multiindex $\alpha \in \{0, 1\}^d$ of the form $\alpha = (1, ...1, 0, ..., 0)$,

$$\sum_{k\in\Gamma_\mu}\left|\int_{I_{k_1}\times\ldots\times I_{k_d}} (-\nabla)^{\alpha} m(\xi) \prod_{j=1}^{d} \left[\sin(\xi_j x_j)\right]_{I_{k_j}}^{\alpha_j} d\xi\right| \lesssim C. \tag{113}$$

Clearly, (113) remains true for any $\alpha \in \{0, 1\}^d$. Now, observing that $\mathbb{Z}^d$ is covered by the union of the sets $\Gamma_1, ...\Gamma_d$, one can write

$$\sum_{k\in\mathbb{Z}^d}\left|\int_{I_{k_1}\times\ldots\times I_{k_d}} (-\nabla)^{\alpha} m(\xi) \prod_{j=1}^{d} \left[\sin(\xi_j x_j)\right]_{I_{k_j}}^{\alpha_j} d\xi\right| \le \sum_{\mu=1}^{d}\sum_{k\in\Gamma_\mu} ... \lesssim C.$$

This, together with (107) and (109) proves Lemma 31 in the case $\ell = d$ (i.e., (106)). The case where $1 \le \ell < d$, can be obtained from the case $\ell = d$ as follows.

Suppose that $1 \le \ell < d$ and m is an ℓ-BB symbol. Let $M_{k'}$ be measurable subsets of $\mathbb{R}^{d-\ell}$ of finite measure, for each $k' \in \mathbb{Z}^\ell$. For each $x'' \in \mathbb{R}^{d-\ell}$ and each $k' \in \mathbb{Z}^\ell$ define the function $\widetilde{m}_{x''}^{k'} : \mathbb{R}^\ell \to \mathbb{C}$ by

$$\widetilde{m}_{x''}^{k'}(\xi') := \int_{M_{k'}} m(\xi', \xi'')e^{i\langle\xi'',x''\rangle}d\xi'',$$

for all $\xi' \in \mathbb{R}^\ell$. Thanks to the fact that m satisfies the condition (103), $\widetilde{m}_{x''}^{k'}$ is well-defined and it satisfies uniformly in x'' and k' the condition (104) as a symbol on $\mathbb{R}^\ell$. Indeed,

$$\begin{aligned} |\partial_1^{\alpha_1}...\partial_\ell^{\alpha_\ell}\widetilde{m}_{x''}^{k'}(\xi')| &= \left|\int_{M_{k'}} \partial_1^{\alpha_1}...\partial_\ell^{\alpha_\ell} m(\xi',\xi'')e^{i\langle\xi'',x''\rangle}d\xi''\right| \\ &\leq \int_{M_{k'}} |\partial_1^{\alpha_1}...\partial_\ell^{\alpha_\ell} m(\xi',\xi'')|d\xi'' \\ &\leq \int_{\mathbb{R}^{d-\ell}} |\partial_1^{\alpha_1}...\partial_\ell^{\alpha_\ell} m(\xi',\xi'')|d\xi'' \\ &\leq \frac{C}{|\xi'|^{\ell+|\alpha|}}, \end{aligned} \tag{114}$$

for all $\alpha = (\alpha_1, ..., \alpha_\ell) \in \{0,1\}^\ell$ and all $\xi' \in (0,\infty)^\ell$. (Note that the final estimate in (114) does not depend on the sets $M_{k'}$.)

Also, $\widetilde{m}_{x''}^{k'}$ is odd in each variable and hence, we have (105). Now, using (106) for $\widetilde{m}_{x''}^{k'}$, we have

$$\begin{aligned} \sum_{k'\in\mathbb{Z}^\ell}\left|\int_{s_\ell(I_{k'})\times M_{k'}} m(\xi)e^{i\langle\xi,x\rangle}d\xi\right| &= \sum_{k'\in\mathbb{Z}^\ell}\left|\int_{s_\ell(I_{k'})}\left(\int_{M_{k'}} m(\xi',\xi'')e^{i\langle\xi'',x''\rangle}d\xi''\right)e^{i\langle\xi',x'\rangle}d\xi'\right| \\ &= \sum_{k'\in\mathbb{Z}^\ell}\left|\int_{s_\ell(I_{k'})} \widetilde{m}_{x''}^{k'}(\xi')\, e^{i\langle\xi',x'\rangle}d\xi'\right| \lesssim C, \end{aligned}$$

uniformly in $x \in \mathbb{R}^d$, which proves Lemma 31. □

By applying Lemma 31 we can deduce the following useful fact:

Lemma 32. *Let $d \geq 1$ and $1 \leq \ell \leq d$ be some integers, and let $m : \mathbb{R}^d \to \mathbb{C}$ be an ℓ-BB symbol satisfying condition (103) or (104) (when $\ell = d$) for some constant C. Then, there exists some kernel $K \in L^\infty(\mathbb{R}^d)$ such that $\widehat{K}(\xi) = m(\xi)$ on $\mathbb{R}^d \setminus \{0\}$ and $\|K\|_{L^\infty} \lesssim C$, the implicit constant not depending on C.*

Remark 33. *The meaning of Lemma 32 is that there exists a unique function $K \in L^\infty$, such that $\|K\|_{L^\infty} \lesssim C$ and*

$$\langle K, \psi\rangle = \langle m, \widehat{\psi}\rangle,$$

for any function $\psi \in \mathcal{S}_{\sharp,c}$.

Proof. For each $n \in \mathbb{N}$, we consider the functions K_n defined by

$$\widehat{K_n}(\xi) = m(\xi)\mathbf{1}_{J_n^d}(\xi),$$

for all $\xi \in \mathbb{R}^d$, where $J_n := [-2^n, -2^{-n}] \cup [2^{-n}, 2^n]$ and $J_n^d = J_n \times ... \times J_n$ (d times). It is easy to see that K_n are well-defined continuous functions. One can also see that K_n are uniformly bounded. Indeed, applying Lemma 31 for $M_{k'} = J_n^{d-\ell}$, for any $k' \in \mathbb{Z}^\ell$ (we suppose that $\ell < d$; the case $\ell = d$ is similar) we can write (using the triangle inequality),

$$
\begin{aligned}
|K_n(x)| &\sim \left|\int_{J_n^\ell\times J_n^{d-\ell}} m(\xi)e^{i\langle\xi,x\rangle}d\xi\right| \\
&= \left|\sum_{\substack{k'\in\mathbb{Z}^\ell\\ s_\ell(I_{k'})\subseteq J_n^\ell}} \int_{s_\ell(I_{k'})\times J_n^{d-\ell}} m(\xi)e^{i\langle\xi,x\rangle}d\xi\right| \\
&\leq \sum_{k'\in\mathbb{Z}^\ell}\left|\int_{s_\ell(I_{k'})\times J_n^{d-\ell}} m(\xi)e^{i\langle\xi,x\rangle}d\xi\right| \lesssim C,
\end{aligned}
$$

uniformly in $x \in \mathbb{R}^d$, and in $n \in \mathbb{N}$. (Here, for "=", we have used the fact that the sets $s_\ell(I_{k'})$ with $s_\ell(I_{k'}) \subseteq J_n^\ell$ are pairwise almost disjoint and they cover J_n^ℓ.) Hence, $\|K_n\|_{L^\infty} \lesssim C$ uniformly in $n \in \mathbb{N}$. Using the sequential Banach-Alaoglu theorem, we can find some $K \in L^\infty$, with $\|K\|_{L^\infty} \lesssim C$ and such that $K_n \to K$ in the w^*-topology on L^∞, up to a subsequence.

Consider now some $\psi \in \mathcal{S}_{\sharp,c}$. Clearly, for any n,

$$\langle K_n, \psi\rangle = \langle m\mathbf{1}_{J_n^d}, \widehat{\psi}\rangle. \tag{115}$$

Since $\psi \in L^1$, we have $\langle K_n, \psi\rangle \to \langle K, \psi\rangle$ up to a subsequence when $n \to \infty$. Also, we have

$$\langle m\mathbf{1}_{J_n^d}, \widehat{\psi}\rangle \to \langle m, \widehat{\psi}\rangle,$$

and by (115) we get

$$\langle K, \psi\rangle = \langle m, \widehat{\psi}\rangle. \tag{116}$$

The uniqueness of K immediately follows from (116). Lemma 32 is proved. □

Remark 34. *One can "extract" the corresponding symbols m that are used by the arguments given in [5, Section 4], [15, Subsection 3.1] and [15, Subsection 5.3]. In the case of [5, Section 4] we would have $m^1(\xi) = \xi_1\xi_2|\xi|^{-4}$ (in fact $m^1(n) = n_1n_2|n|^{-4}$, $n \in \mathbb{Z}^2 \setminus \{0\}$, for the version on $\mathbb{T}^2$), with $\ell = d = 2$. In the case of [15, Subsection 3.1] we would have $m^2(\xi) = sign(\xi)|\xi|^{-1}$ (in fact $m^2(n) = sign(n)|n|^{-1} = 1/\xi$, $n \in \mathbb{Z} \setminus \{0\}$, for the version on $\mathbb{T}$), with $\ell = d = 1$. One can also find a similar symbol in [15, Subsection 5.3]. Namely, there we have the 1-BB symbol $m^3(\xi) = \xi_1|\xi|^{-d-1}$ (in fact $m^3(n) = n_1|n|^{-d-1}$, $n \in \mathbb{Z}^d \setminus \{0\}$, for the version on $\mathbb{T}^d$).*[10]

If we (formally) denote by K^j the corresponding inverse Fourier transforms of m^j, $j = 1,2,3$, we can see that Lemma 32 gives the boundedness of K^j. This is essentially the approach in [5, Section 4], [15, Subsection 3.1] and [15, Subsection 5.3] based on the Bourgain-Brezis technique. However, we remark that this technique is unnecessary in all the above cases $j = 1,2,3$. In these cases one can compute the kernels K^j explicitly:

(i) up to a multiplicative constant, m^1 is the kernel of a double Riesz transform and its (inverse) Fourier transform is a constant multiple of $(x_1|x|^{-1})(x_2|x|^{-1})$ which is bounded;

(ii) up to a multiplicative constant, m^2 is the kernel of the Hilbert transform and its (inverse) Fourier transform is a constant multiple of $sign(x)$ which is bounded;

(iii) up to a multiplicative constant, m^3 is the kernel of a Riesz transform and its (inverse) Fourier transform is a constant multiple of $x_1|x|^{-1}$ which is bounded.

[10] The main result of [15, Section 5] concerns the existence of regular bounded solutions for a divergence type equation that has an additional term (the "free" term f_0 in [15, (119)]). This is weaker than the result of Mazya in [22] which does not need a "free" term.

As we mentioned in Subsection 4.1 the Bourgain-Brezis technique has a great advantage when explicit computations are inconvenient. On the other hand, as we can see from Lemma 31, the Bourgain-Brezis technique give slightly more than the boundedness of Fourier transforms of the BB-symbols, even in the case of m^j above.

4.3 Divergence-like equations in $\dot{H}^{d/2}$

We can now prove our existence result for divergence-like equations in a particular type of critical Sobolev spaces. We start by an analogue of [5, Lemma 2]. The proof we give below rests on some elaborations of the main ideas used by Bourgain and Brezis in the proof of [5, Lemma 2]. Lemma 32 from the previous Subsection will play here an important role.

Recall (see (95)) that we denoted by $\nabla_2^\sigma u$ the first two components of the "σ-gradient" of u, namely,

$$\nabla_2^\sigma u := (\partial_1^\sigma u, \partial_2^\sigma u),$$

where ∂_j^σ is the Fourier multiplier $\partial_j^\sigma := \sigma_j(\nabla)$. (See (10) for the definition of the functions σ_j.)

Lemma 35. *Let $d \geq 3$ be an integer and consider the set $U := \mathbb{R}^{d-1} \times (0, \infty)$. If $\sigma : \mathbb{R}^d \to \mathbb{R}$ satisfies (P1),(P2) then,*

$$\begin{aligned} \| |\nabla|^{-1} |\nabla_2^\sigma|^2 u \|_{Y^*/U} &\lesssim \|\nabla_2^\sigma u\|_{(\mathcal{M}+Y^*)/U} \\ &\quad + \|\nabla_2^\sigma u\|_{Y^*/U}^{1/2} \|\nabla_2^\sigma u\|_{(\mathcal{M}+Y^*)/U}^{1/2}, \end{aligned} \tag{117}$$

for any $u \in \mathcal{S}_{\sharp,c}$, where $Y := \dot{H}^{d/2}$.

(For the meaning of the notation Y^*/U and $(\mathcal{M} + Y^*)/U$ see Subsection 2.3.)

Proof. Clearly, $Y^* = \dot{H}^{-d/2}$. First we show that

$$\int_{\mathbb{R}^d} \frac{\sigma_1^2(\xi)\sigma_2^2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) |\widehat{u}(\xi)|^2 d\xi \lesssim \|\nabla_2^\sigma u\|_{(\mathcal{M}+Y^*)/U}^2 + \|\nabla_2^\sigma u\|_{Y^*/U} \|\nabla_2^\sigma u\|_{(\mathcal{M}+Y^*)/U}, \tag{118}$$

for any function $u \in \mathcal{S}_{\sharp,c}$.

By (24) we can find $\tilde{F}_j \in \mathcal{M} + Y^*$ and $F_j^- \in (\mathcal{M} + Y^*)_{U^c}$, for $j = 0, 1$, such that

$$\nabla_2^\sigma u = (\tilde{F}_1, \tilde{F}_2) + (F_1^-, F_2^-),$$

and

$$\|\tilde{F}_1\|_{\mathcal{M}+Y^*} + \|\tilde{F}_2\|_{\mathcal{M}+Y^*} \leq 2 \|\nabla_2^\sigma u\|_{(\mathcal{M}+Y^*)/U}.$$

We split each $\tilde{F}_j \in \mathcal{M} + Y^*$ as $\tilde{F}_j = F_j + h_j$, with

$$\|F_j\|_{\mathcal{M}} + \|h_j\|_{Y^*} \leq 2 \|\tilde{F}_j\|_{\mathcal{M}+Y^*},$$

for $j = 0, 1$. Hence, there exists $F_1, F_2 \in \mathcal{M}$, $h_1, h_2 \in Y^*$, $F_1^-, F_2^- \in \mathcal{M}+Y^*$ with $spec(F_1^-), spec(F_2^-) \subseteq U^c$ such that

$$\nabla_2^\sigma u = (F_1, F_2) + (h_1, h_2) + (F_1^-, F_2^-), \tag{119}$$

and

$$\|F_1\|_{\mathcal{M}} + \|F_2\|_{\mathcal{M}} + \|h_1\|_{Y^*} + \|h_2\|_{Y^*} \lesssim \|\nabla_2^\sigma u\|_{(\mathcal{M}+Y^*)/U}. \tag{120}$$

We may also assume without loss of generality that F_j, h_j, F_j^- are smooth functions with $spec(F_j), spec(h_j), spec(F_j^-) \subset B(0, r_2) \backslash B(0, r_1)$, for some real numbers $0 < r_1 < r_2$. Indeed, since $u \in \mathcal{S}_{\sharp,c}$, we have that $spec(\nabla_2^\sigma u) \subset B(0,r_2)\backslash B(0,r_1)$, if $r_1 > 0$ is sufficiently small and $r_2 > 0$ is sufficiently large. Let P_k be the Littlewood-Paley "projections" (see (16)) and

$$P_{k_1 \le \cdot \le k_2} := \sum_{k=k_1}^{k_2} P_k,$$

where, if $k_1 \in \mathbb{Z}$ is sufficiently small and $k_2 \in \mathbb{Z}$ is sufficiently large. We have $P_{k_1 \le \cdot \le k_2} \nabla_2^\sigma u = \nabla_2^\sigma u$ and by (119),

$$\nabla_2^\sigma u = (P_{k_1 \le \cdot \le k_2} F_1, P_{k_1 \le \cdot \le k_2} F_2) + (P_{k_1 \le \cdot \le k_2} h_1, P_{k_1 \le \cdot \le k_2} h_2) + (P_{k_1 \le \cdot \le k_2} F_1^-, P_{k_1 \le \cdot \le k_2} F_2^-).$$

The claim about F_j, h_j, F_j^- follows by noticing that $P_{k_1 \le \cdot \le k_2}$ is bounded on $\mathcal{M}$ and Y^*, with the norm $\lesssim 1$, uniformly in k_1, k_2.

Now we go back to the proof of Lemma 35. We have

$$\int_{\mathbb{R}^d} \frac{\sigma_1^2(\xi)\sigma_2^2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) |\widehat{u}(\xi)|^2 d\xi = c \int_{\mathbb{R}^d} \frac{\sigma_1(\xi)\sigma_2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) \widehat{\partial_1^\sigma u}(\xi) \overline{\widehat{\partial_2^\sigma u}(\xi)} d\xi.$$

Using this, (119) and the fact that $spec(F_j^-) \subseteq U^c$ (and hence $\mathbf{1}_U(\xi)\widehat{F_j^-}(\xi) = 0$, for $j = 1, 2$ and for all $\xi \in \mathbb{R}^d$), we can write

$$\int_{\mathbb{R}^d} \frac{\sigma_1^2(\xi)\sigma_2^2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) |\widehat{u}(\xi)|^2 d\xi = I + II,$$

where

$$I := c \int_{\mathbb{R}^d} \frac{\sigma_1(\xi)\sigma_2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) \widehat{F_1}(\xi) \overline{\widehat{F_2}(\xi)} d\xi,$$

and II is the sum of a finite number of terms of the form

$$c \int_{\mathbb{R}^d} \frac{\sigma_1(\xi)\sigma_2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) g_1(\xi) g_2(\xi) d\xi, \tag{121}$$

where each $g_\nu : \mathbb{R}^d \to \mathbb{C}$ is one of the functions

$$\widehat{h}_1,\ \widehat{h}_2,\ \overline{\widehat{h}}_1,\ \overline{\widehat{h}}_2,\ \widehat{F}_1,\ \widehat{F}_2,\ \overline{\widehat{F}}_1,\ \overline{\widehat{F}}_2$$

and at least one g_ν is $\widehat{h}_j$ or $\overline{\widehat{h}}_j$ for some $j \in \{1, 2\}$.

Using (P1) and the Leibniz rule, one can verify that the symbol m defined by

$$m(\xi) := \frac{\sigma_1(\xi)\sigma_2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi),$$

for any $\xi \in \mathbb{R}^d \backslash \{0\}$, satisfies the conditions in Lemma 32. Hence, by a applying Lemma 32 and (120),

$$\begin{aligned} |I| &\sim |\langle K * F_1, F_2 \rangle| \le \|K * F_1\|_{L^\infty} \|F_2\|_{\mathcal{M}} \\ &\le \|K\|_{L^\infty} \|F_1\|_{\mathcal{M}} \|F_2\|_{\mathcal{M}} \lesssim \|\nabla_2^\sigma u\|^2_{(\mathcal{M}+Y^*)/U}. \end{aligned} \tag{122}$$

(Here, we have used the notation from Lemma 32: $\widehat{K} = m$.)

In order to estimate II we estimate each of its terms of the form (121). By the Cauchy-Schwarz inequality, we get

$$\left| \int_{\mathbb{R}^d} \frac{\sigma_1(\xi)\sigma_2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) g_1(\xi) g_2(\xi) d\xi \right| \leq \prod_{\nu=1}^{2} \left(\int_{\mathbb{R}^d} \mathbf{1}_U(\xi) \frac{|g_\nu(\xi)|^2}{|\xi|^d} d\xi \right)^{1/2}, \tag{123}$$

where we have used the inequality

$$\left| \frac{\sigma_1(\xi)\sigma_2(\xi)}{|\xi|^2} \right| \lesssim 1,$$

which follows directly from (P1).

Note that, if $|g_\nu| = |\widehat{h}_j|$, for some $j \in \{1,2\}$, then, by (120)

$$\left(\int_{\mathbb{R}^d} \mathbf{1}_U(\xi) \frac{|g_\nu(\xi)|^2}{|\xi|^d} d\xi \right)^{1/2} \leq \|h_j\|_{Y^*} \lesssim \|\nabla_2^\sigma u\|_{(\mathcal{M}+Y^*)/U}. \tag{124}$$

If $|g_\nu| = |\widehat{F}_j|$, for some $j \in \{1,2\}$, then, since $\widehat{F}_j \mathbf{1}_U = (\widehat{\partial_j^\sigma u} - \widehat{h}_j)\mathbf{1}_U$, the triangle inequality together with (120) gives

$$\begin{aligned} \left(\int_{\mathbb{R}^d} \mathbf{1}_U(\xi) \frac{|g_\nu(\xi)|^2}{|\xi|^d} d\xi \right)^{1/2} &= \left(\int_{\mathbb{R}^d} \mathbf{1}_U(\xi) |\xi|^{-d} |\widehat{\partial_j^\sigma u}(\xi) - \widehat{h}_j(\xi)|^2 d\xi \right)^{1/2} \\ &\leq \|\partial_j^\sigma u\|_{Y^*/U} + \|h_j\|_{Y^*} \\ &\lesssim \|\nabla_2^\sigma u\|_{Y^*/U} + \|\nabla_2^\sigma u\|_{(\mathcal{M}+Y^*)/U} \\ &\lesssim \|\nabla_2^\sigma u\|_{Y^*/U}. \end{aligned} \tag{125}$$

Since $|g_\nu| = |\widehat{h}_j|$ (for some j) for at least one ν, we get from (123), (124) and (125) that

$$\left| \int_{\mathbb{R}^d} \frac{\sigma_1(\xi)\sigma_2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) g_1(\xi) g_2(\xi) d\xi \right| \lesssim \|\nabla_2^\sigma u\|_{Y^*/U} \|\nabla_2^\sigma u\|_{(\mathcal{M}+Y^*)/U}.$$

Hence,

$$|II| \lesssim \|\nabla_2^\sigma u\|_{Y^*/U} \|\nabla_2^\sigma u\|_{(\mathcal{M}+Y^*)/U}. \tag{126}$$

By (125) and (126) we get (118).

Consider the rotation

$$R(\xi) = (\xi_1 - \xi_2, \xi_1 + \xi_2, \xi_3, ..., \xi_d),$$

for any $\xi \in \mathbb{R}^d$. Consider now the functions

$$\sigma_1' := \sigma_1 \circ R - \sigma_2 \circ R, \quad \text{and} \quad \sigma_2' := \sigma_1 \circ R + \sigma_2 \circ R.$$

By a direct computation (using (P2) and (10)) one can check that the function

$$\sigma_1' \sigma_2' := (\sigma_1 \circ R)^2 - (\sigma_2 \circ R)^2$$

is odd in each of the variables ξ_1, ξ_2. Using this, we observe that the symbol m' defined by

$$m'(\xi) := \frac{\sigma_1'(\xi)\sigma_2'(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi),$$

for every $\xi \in \mathbb{R}^d \setminus \{0\}$, satisfies the conditions in Lemma 32. Hence, as in (118), we obtain that

$$\int_{\mathbb{R}^d} \frac{\sigma_1'(\xi)^2 \sigma_2'(\xi)^2}{|\xi|^{d+2}} \mathbf{1}_U(\xi) |\widehat{v}(\xi)|^2 d\xi \lesssim \|\nabla_2^{\sigma'} v\|_{(\mathcal{M}+Y^*)/U}^2 + \|\nabla_2^{\sigma'} v\|_{Y^*/U} \|\nabla_2^{\sigma'} v\|_{(\mathcal{M}+Y^*)/U}, \tag{127}$$

for any function $v \in \mathcal{S}_{\sharp,c}$, where

$$\nabla_2^{\sigma'} v := (\partial_1^{\sigma'} v, \partial_2^{\sigma'} v),$$

and $\partial_j^{\sigma'}$ is the Fourier multiplier of symbol σ_j', for any $j = 1, 2$.

Since the spaces Y^*/U and $(\mathcal{M}+Y^*)/U$ are invariant under the rotation R, by applying (127) to the function $v = u \circ R^t$ we obtain (by changing the variables) that

$$\int_{\mathbb{R}^d} \frac{(\sigma_1(\xi) - \sigma_2(\xi))^2 (\sigma_1(\xi) + \sigma_2(\xi))^2}{|\xi|^{d+2}} \mathbf{1}_U(\xi) |\widehat{u}(\xi)|^2 d\xi$$

is bounded by

$$\|\nabla_2^{\sigma} u\|^2_{(\mathcal{M}+Y^*)/U} + \|\nabla_2^{\sigma} u\|_{Y^*/U} \|\nabla_2^{\sigma} u\|_{(\mathcal{M}+Y^*)/U}, \tag{128}$$

for any function $u \in \mathcal{S}_{\sharp}$. By adding up, we get from (118) and (128) that

$$\int_{\mathbb{R}^d} \frac{\widetilde{\sigma}(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) |\widehat{u}(\xi)|^2 d\xi \lesssim \|\nabla_2^{\sigma} u\|^2_{(\mathcal{M}+Y^*)/U} + \|\nabla_2^{\sigma} u\|_{Y^*/U} \|\nabla_2^{\sigma} u\|_{(\mathcal{M}+Y^*)/U}, \tag{129}$$

where

$$\widetilde{\sigma}(\xi) := \sigma_1^2(\xi)\sigma_2^2(\xi) + (\sigma_1(\xi) - \sigma_2(\xi))^2 (\sigma_1(\xi) + \sigma_2(\xi))^2,$$

for any $\xi \in \mathbb{R}^d$. Since for any real numbers a, b we have

$$a^2 b^2 + (a-b)^2 (a+b)^2 \sim a^4 + b^4,$$

we obtain

$$\widetilde{\sigma}(\xi) \sim \sigma_1^4(\xi) + \sigma_2^4(\xi),$$

for all $\xi \in \mathbb{R}^d$, and now (129) gives us

$$\int_{\mathbb{R}^d} \frac{\sigma_1^4(\xi) + \sigma_2^4(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) |\widehat{u}(\xi)|^2 d\xi \lesssim \|\nabla_2^{\sigma} u\|^2_{(\mathcal{M}+Y^*)/U} + \|\nabla_2^{\sigma} u\|_{Y^*/U} \|\nabla_2^{\sigma} u\|_{(\mathcal{M}+Y^*)/U}. \tag{130}$$

Note that by (25) (with $D = U$), we have

$$\||\nabla|^{-1} |\nabla_2^{\sigma}|^2 u\|^2_{Y^*/U} \sim \int_{\mathbb{R}^d} \frac{\sigma_1^4(\xi) + \sigma_2^4(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) |\widehat{u}(\xi)|^2 d\xi,$$

and together with (130) this concludes the proof of Lemma 35. □

By standard arguments involving Parseval's theorem, Lemma 35 implies the following:

Lemma 36. *Let $d \geq 3$ be an integer. Suppose that the family of functions $G_1, ..., G_d : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is adapted to the family of half-spaces $D_1, ..., D_d \subset \mathbb{R}^d$. For any $\varphi \in \mathcal{S}_{\sharp,c}$ we have*

$$\|\varphi\|_{\dot{H}^{-d/2+1}/D} \lesssim \sum_{j=1}^{d} \|G_j(\nabla)\varphi\|_{(\mathcal{M}+\dot{H}^{-d/2})/D_j}, \tag{131}$$

where $D = D_1 \cup ... \cup D_d$.

(For the properties of the functions $G_1, ..., G_d$ and their relation with the half-spaces $D_1, ..., D_d$ see the Subsection 1.3 in the Introduction of this paper.)

Proof. Let Y be the space $Y := \dot{H}^{d/2}$. According to Lemma 35, for any $\varphi \in \mathcal{S}_{\sharp,c}$, we have

$$\begin{aligned} \||\nabla|^{-1} |\nabla_2^{\sigma}|^2 \varphi\|_{Y^*/U} &\lesssim \|\nabla_2^{\sigma} \varphi\|_{(\mathcal{M}+Y^*)/U} \\ &\quad + \|\nabla_2^{\sigma} \varphi\|^{1/2}_{Y^*/U} \|\nabla_2^{\sigma} \varphi\|^{1/2}_{(\mathcal{M}+Y^*)/U} \\ &\lesssim \varepsilon \|\nabla_2^{\sigma} \varphi\|_{Y^*/U} + \varepsilon^{-1} \|\nabla_2^{\sigma} \varphi\|_{(\mathcal{M}+Y^*)/U}, \end{aligned}$$

for any $\varepsilon \in (0,1)$ and any $j \in \{1,...,d\}$. Similar inequalities hold for the operators $(\partial_1^\sigma, \partial_j^\sigma)$, for $j = 1,...,d-1$, instead of $\nabla_2^\sigma = (\partial_1^\sigma, \partial_2^\sigma)$. By adding up the corresponding inequalities we get

$$\||\nabla|^{-1}|\nabla_{d-1}^\sigma|^2\varphi\|_{Y^*/U} \lesssim +\varepsilon\|\nabla_2^\sigma\varphi\|_{Y^*/U} + \varepsilon^{-1}\|\nabla_{d-1}^\sigma\varphi\|_{(\mathcal{M}+Y^*)/U}, \tag{132}$$

where

$$\nabla_{d-1}^\sigma := (\partial_1^\sigma, ..., \partial_{d-1}^\sigma).$$

By composition with the appropriate rotations (recall (11)), we get from (132) the inequality

$$\begin{aligned}\||\nabla|^{-1}|G_j(\nabla)|^2\varphi\|_{Y^*/D_j} &\lesssim \varepsilon\|G_j(\nabla)\varphi\|_{Y^*/D_j} \\ &\quad +\varepsilon^{-1}\|G_j(\nabla)\varphi\|_{(\mathcal{M}+Y^*)/D_j},\end{aligned}$$

for each $j \in \{1,...,d\}$. By adding up these inequalities we get

$$\begin{aligned}\sum_{j=1}^{d}\||\nabla|^{-1}|G_j(\nabla)|^2\varphi\|_{Y^*/D_j} &\lesssim \varepsilon\sum_{j=1}^{d}\|G_j(\nabla)\varphi\|_{Y^*/D_j} \\ &\quad +\varepsilon^{-1}\sum_{j=1}^{d}\|G_j(\nabla)\varphi\|_{(\mathcal{M}+Y^*)/D_j}.\end{aligned} \tag{133}$$

Since the family G_1,...,G_d is adapted to D_1,...,D_d, (see (12)) we get

$$\sum_{j=1}^{d}|G_j(\xi)|^\beta\mathbf{1}_{D_j}(\xi) \sim_\beta |\xi|^\beta\mathbf{1}_D(\xi), \tag{134}$$

on $\mathbb{R}^d$, for any $\beta > 0$. Using now (134), with $\beta = 2$, we can write

$$\begin{aligned}\sum_{j=1}^{d}\||\nabla|^{-1}|G_j(\nabla)|^2\varphi\|_{Y^*/D_j} &\sim \sum_{j=1}^{d}\||\xi|^{-1}|G_j(\xi)|^2\mathbf{1}_{D_j}(\xi)|\widehat{\varphi}(\xi)||\xi|^{-d/2}\|_{L^2_\xi} \\ &\sim \||\xi|^{-1}\left(\sum_{j=1}^{d}|G_j(\xi)|^2\mathbf{1}_{D_j}(\xi)\right)|\widehat{\varphi}(\xi)||\xi|^{-d/2}\|_{L^2_\xi} \\ &\sim \||\xi|^{-1}|\xi|^2\mathbf{1}_D(\xi)|\widehat{\varphi}(\xi)||\xi|^{-d/2}\|_{L^2_\xi} \\ &= \||\xi|\mathbf{1}_D(\xi)|\widehat{\varphi}(\xi)||\xi|^{-d/2}\|_{L^2_\xi}.\end{aligned} \tag{135}$$

In a similar way (by (12)) we can write

$$\begin{aligned}\sum_{j=1}^{d}\|G_j(\nabla)\varphi\|_{Y^*/D_j} &\sim \sum_{j=1}^{d}\||G_j(\xi)|\mathbf{1}_{D_j}(\xi)\widehat{\varphi}(\xi)||\xi|^{-d/2}\|_{L^2_\xi} \\ &\sim \|\left(\sum_{j=1}^{d}|G_j(\xi)|\mathbf{1}_{D_j}(\xi)\right)\widehat{\varphi}(\xi)||\xi|^{-d/2}\|_{L^2_\xi} \\ &\sim \||\xi|\mathbf{1}_D(\xi)|\widehat{\varphi}(\xi)||\xi|^{-d/2}\|_{L^2_\xi}.\end{aligned} \tag{136}$$

Note that, by (25),

$$\||\xi|\mathbf{1}_D(\xi)|\widehat{\varphi}(\xi)||\xi|^{-d/2}\|_{L^2_\xi} \sim \|\varphi\|_{\dot{H}^{-d/2+1}/D}.$$

Using this together with (135) and (136) we obtain

$$\sum_{j=1}^{d}\||\nabla|^{-1}|G_j(\nabla)|^2\varphi\|_{Y^*/D_j}\sim\sum_{j=1}^{d}\|G_j(\nabla)\varphi\|_{Y^*/D_j}\sim\|\varphi\|_{\dot{H}^{-d/2+1}/D},$$

and together with (133) this yields

$$\|\varphi\|_{\dot{H}^{-d/2+1}/D}\lesssim\varepsilon\|\varphi\|_{\dot{H}^{-d/2+1}/D}+\varepsilon^{-1}\sum_{j=1}^{d}\|G_j(\nabla)\varphi\|_{(\mathcal{M}+Y^*)/D_j}.$$

Choosing ε sufficiently small one can write

$$\|\varphi\|_{\dot{H}^{-d/2+1}/D}\lesssim\varepsilon^{-1}\sum_{j=1}^{d}\|G_j(\nabla)\varphi\|_{(\mathcal{M}+Y^*)/D_j},$$

which is the desired inequality. □

We can now state and prove the existence results of this Subsection:

Lemma 37. *Let $d\geq 3$ be an integer. Suppose that the family of functions $G_1,...,G_d:\mathbb{R}^d\to\mathbb{R}^{d-1}$ is adapted to the family of half-spaces $D_1,...,D_d\subset\mathbb{R}^d$. Then, for any system of $(d-1)$-vector fields $(v_j)_{j=1,..,d}$ with $v_j\in\dot{H}^{d/2}(\mathbb{R}^d)$ and spec$(v_j)\subseteq D_j$, there exists a system of $(d-1)$-vector fields $(u_j)_{j=1,..,d}$, with $u_j\in L^\infty(\mathbb{R}^d)\cap\dot{H}^{d/2}(\mathbb{R}^d)$ and spec$(u_j)\subseteq D_j$, such that*

$$\sum_{j=1}^{d}G_j(\nabla)\cdot u_j=\sum_{j=1}^{d}G_j(\nabla)\cdot v_j,$$

and

$$\sum_{j=1}^{d}\|u_j\|_{L^\infty\cap\dot{H}^{d/2}}\lesssim\sum_{j=1}^{d}\|v_j\|_{\dot{H}^{d/2}}.$$

Proof. We show that (131) can be written as

$$\|\varphi\|_{B^*}\lesssim\|T^*\varphi\|_{A^*},\tag{137}$$

for some Banach spaces A, B and some appropriate operator $T:A\to B$.

Let

$$A:=(C_0\cap\dot{H}^{d/2}_{D_1})\times....\times(C_0\cap\dot{H}^{d/2}_{D_d}),$$

and

$$B:=\dot{H}^{d/2-1}_{D}.$$

By [27, Section 4.8] we have

$$B^*=\dot{H}^{-d/2+1}/D.\tag{138}$$

Similarly, one can write

$$\begin{aligned}A^*&=((C_0\cap\dot{H}^{d/2})_{D_1}\times....\times(C_0\cap\dot{H}^{d/2})_{D_d})^*\\&=((M+\dot{H}^{-d/2})/D_1)\times....\times((M+\dot{H}^{-d/2})/D_d).\end{aligned}$$

We now define the operator T. First, observe that for any $j \in \{1, ..., d\}$, the multiplier $G_j(\nabla) : \dot{H}^{d/2} \to \dot{H}^{d/2-1}$ is well-defined and bounded. This follows by (P1) since for any $f \in \mathcal{S}_{\sharp,c}$ and any rotation $R : \mathbb{R}^d \to \mathbb{R}^d$ we have[11]

$$\int_{\mathbb{R}^d} |\xi|^{d-2} |(\sigma \circ R)(\xi)|^2 |\widehat{f}(\xi)| d\xi \lesssim \int_{\mathbb{R}^d} |\xi|^d |\widehat{f}(\xi)| d\xi. \tag{139}$$

Thanks to this property of the operators $G_j(\nabla)$, we can define the bounded linear operator $T : \dot{H}^{d/2}_{D_1} \times ... \times \dot{H}^{d/2}_{D_d} \to \dot{H}^{d/2-1}_D = B$ by

$$Tv = G_1(\nabla) \cdot v_1 + ... + G_d(\nabla) \cdot v_d$$

for any system of $(d-1)$-vector fields $v = (v_1, ..., v_d)$ belonging to $\dot{H}^{d/2}_{D_1} \times ... \times \dot{H}^{d/2}_{D_d}$. Since $A \hookrightarrow \dot{H}^{d/2}_{D_1} \times ... \times \dot{H}^{d/2}_{D_d}$, we also have that $T : A \to B$ is well-defined and bounded. Computing T^* we obtain

$$T^* \varphi = (G_1(\nabla) \varphi, ..., G_d(\nabla) \varphi),$$

for any $\varphi \in B^*$. (Note also that one can define in the same way $T^*\varphi$, for any $\varphi \in \mathcal{S}_{\sharp,c}$.)

This, (138) and (131) justifies (137) for any $\varphi \in \mathcal{S}_{\sharp,c}$. By density arguments (137) holds for any $\varphi \in B^*$.

Using now the closed range theorem (see Theorem 4.13 in [27]) we get that any $\tilde{v} \in B$ can be written as

$$\tilde{v} = \sum_{j=1}^{d} G_j(\nabla) \cdot u_j, \tag{140}$$

for some $(d-1)$-vector fields $(u_j)_{j=1,..,d}$, with $u_j \in C_0 \cap \dot{H}^{d/2}$ and $spec(u_j) \subseteq D_j$. Given some $(d-1)$-vector fields $(v_j)_{j=1,..,d}$, with $v_j \in \dot{H}^{d/2}$ and $spec(v_j) \subseteq D_j$, it suffices now to apply (140) to the element

$$\tilde{v} := \sum_{j=1}^{d} G_j(\nabla) \cdot v_j,$$

noticing that by (139) we have indeed $\tilde{v} \in B$. Since $C_0 \hookrightarrow L^\infty$, we conclude the proof of Lemma 37. ☐

With the same methods one can prove an analogue of Theorem 8 when $d = 2$ and the source space is $\dot{H}^1(\mathbb{R}^2)$:

Lemma 38. *Consider the numbers $\delta \in (0, \pi/8)$ and $\varepsilon \in (0, 2]$. Then, for any vector field $v \in \dot{H}^1(\mathbb{R}^2)$, with $spec(v) \subseteq C_\delta$, there exists a vector field $u \in L^\infty(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)$, with $spec(u_j) \subseteq C_{(1+\varepsilon)\delta}$, such that*

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{H}^1} \lesssim \|v\|_{\dot{H}^1}.$$

(For the meaning of C_δ see Subsection 1.3.)

Sketch of the proof. First we establish the result for $\varepsilon = 2$. Let us denote by $\mathcal{D}(C_\delta)$ the set defined by

$$\mathcal{D}(C_\delta) := \bigcup_{I^\epsilon_\nu \subset C_\delta} I^\epsilon_\nu,$$

[11] This also follows from (20), since $\dot{H}^{d/2} = \dot{B}^{d/2,2}_2$. However, the argument given here is more direct, showing that for this particular Lemma it suffices to have the inequality in (P1) valid for any α with $|\alpha| \le 2$ (instead $|\alpha| \le d+2$).

the union being taken after all the dyadic boxes of the form $I_\nu^\epsilon = (\epsilon_1[2^{\nu_1-1}, 2^{\nu_1}]) \times (\epsilon_1[2^{\nu_2-1}, 2^{\nu_2}])$ (where $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$ and $\epsilon = (\epsilon_1, \epsilon_2) \in \{0,1\}^2$) that are included in C_δ. One can find a finite number of rotations $R_1, ..., R_n : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$C_\delta \backslash B_{\mathbb{R}^2}(0,r) \subset \widetilde{C}_\delta := \bigcup_{j=1}^n R_j(\mathcal{D}(C_\delta)) \subset C_{2\delta}, \tag{141}$$

for some sufficiently large $r > 0$. As in the proof of (118) we get

$$\int_{\mathbb{R}^d} \frac{(\xi_1\xi_2)^2}{|\xi|^{d+2}} \mathbf{1}_{\mathcal{D}(C_\delta)}(\xi)|\widehat{u}(\xi)|^2 d\xi \lesssim \|\nabla u\|^2_{(\mathcal{M}+Y^*)/\mathcal{D}(C_\delta)} + \|\nabla u\|_{Y^*/\mathcal{D}(C_\delta)} \|\nabla u\|_{(\mathcal{M}+Y^*)/\mathcal{D}(C_\delta)}, \tag{142}$$

for any $u \in \mathcal{S}_{\sharp,c}$, where $Y := \dot{H}^{d/2}$. As in the proof of Lemma 32, by Lemma 31 (applied in the case $\ell = d = 2$) one can see that there exists $K \in L^\infty$ such that

$$\widehat{K}(\xi) = \frac{\xi_1\xi_2}{|\xi|^{d+2}} \mathbf{1}_{\mathcal{D}(C_\delta)}(\xi),$$

with the same meaning as in Lemma 32. We use then the same method as in (122). The rest of the argument remains essentially the same as the one used in the proof of (118).

Note that, by (142) we get (by composition with rotations) that

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{(R_j^1(\xi)R_j^2(\xi))^2}{|\xi|^{d+2}} \mathbf{1}_{R_j(\mathcal{D}(C_\delta))}(\xi)|\widehat{u}(\xi)|^2 d\xi \quad &\lesssim \quad \|\nabla u\|^2_{(\mathcal{M}+Y^*)/R_j(\mathcal{D}(C_\delta))} \\ &+ \|\nabla u\|_{Y^*/R_j(\mathcal{D}(C_\delta))} \|\nabla u\|_{(\mathcal{M}+Y^*)/R_j(\mathcal{D}(C_\delta))},\end{aligned}$$

for all $j \in \{1, ..., n\}$, where $R_j^l(\xi)$ is the l-th coordinate of the vector $R_j(\xi)$. Using (141) we get

$$\|\nabla u\|_{Y^*/R_j(\mathcal{D}(C_\delta))} \le \|\nabla u\|_{Y^*/\widetilde{C}_\delta} \quad \text{and} \quad \|\nabla u\|_{(\mathcal{M}+Y^*)/R_j(\mathcal{D}(C_\delta))} \le \|\nabla u\|_{(\mathcal{M}+Y^*)/\widetilde{C}_\delta},$$

for all $j \in \{1, ..., n\}$, and one can write

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{(R_j^1(\xi)R_j^2(\xi))^2}{|\xi|^{d+2}} \mathbf{1}_{R_j(\mathcal{D}(C_\delta))}(\xi)|\widehat{u}(\xi)|^2 d\xi \quad &\lesssim \quad \|\nabla u\|^2_{(\mathcal{M}+Y^*)/\widetilde{C}_\delta} \\ &+ \|\nabla u\|_{Y^*/\widetilde{C}_\delta} \|\nabla u\|_{(\mathcal{M}+Y^*)/\widetilde{C}_\delta},\end{aligned} \tag{143}$$

for all $j \in \{1, ..., n\}$. One can check that

$$\sum_{j=1}^n (R_j^1(\xi)R_j^2(\xi))^2 \mathbf{1}_{R_j(\mathcal{D}(C_\delta))}(\xi) \sim |\xi|^4 \mathbf{1}_{\widetilde{C}_\delta}(\xi),$$

for all $\xi \in \mathbb{R}^2$. Hence, by adding up the inequalities (143) we get

$$\int_{\mathbb{R}^d} \frac{|\xi|^2}{|\xi|^d} \mathbf{1}_{\widetilde{C}_\delta}(\xi)|\widehat{u}(\xi)|^2 d\xi \lesssim \|\nabla u\|^2_{(\mathcal{M}+Y^*)/\widetilde{C}_\delta} + \|\nabla u\|_{Y^*/\widetilde{C}_\delta} \|\nabla u\|_{(\mathcal{M}+Y^*)/\widetilde{C}_\delta},$$

which can be rewritten as

$$\|\nabla u\|_{Y^*/\widetilde{C}_\delta} \lesssim \|\nabla u\|_{(\mathcal{M}+Y^*)/\widetilde{C}_\delta} + \|\nabla u\|^{1/2}_{Y^*/\widetilde{C}_\delta} \|\nabla u\|^{1/2}_{(\mathcal{M}+Y^*)/\widetilde{C}_\delta}.$$

This implies that for any $\varepsilon \in (0,1)$ we have

$$\|\nabla u\|_{Y^*/\widetilde{C}_\delta} \lesssim \varepsilon \|\nabla u\|_{Y^*/\widetilde{C}_\delta} + \varepsilon^{-1} \|\nabla u\|_{(\mathcal{M}+Y^*)/\widetilde{C}_\delta},$$

and choosing a sufficiently small ε we get (as in the proof of Lemma 36)

$$\|\nabla u\|_{Y^*/\widetilde{C}_\delta} \lesssim \|\nabla u\|_{(\mathcal{M}+Y^*)/\widetilde{C}_\delta}.$$

By duality (as in the proof of Lemma 37) we obtain that for any vector field $v \in \dot{H}^1(\mathbb{R}^2)$, with $spec(v) \subseteq \widetilde{C}_\delta$, there exist a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{H}^1(\mathbb{R}^2)$, with $spec(u_j) \subseteq \widetilde{C}_\delta$, such that

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{H}^1} \lesssim \|v\|_{\dot{H}^1}.$$

Thanks to (141), this implies Lemma 38 in the case $\varepsilon = 2$. To obtain the result for any $\varepsilon \in (0,1)$ simply cover the symmetric cone C_δ with a small union of rotated copies of the symmetric cones $C_{\delta/n}$ for some large integer $n > 0$. It suffices now to apply the result corresponding to the case $\varepsilon = 2$ to each rotated copy of $C_{\delta/n}$ and then add the obtained solutions. □

5 Solutions in interpolation spaces

5.1 Proof of the main results

We now discuss some immediate applications of the $\mathcal{W}$-method to the divergence-like equations. First we formulate a general result:

Theorem 39. *Let X, $\tilde{X}$, Y, $\tilde{Y}$, F be Banach function spaces on $\mathbb{R}^d$ satisfying the embeddings $X \hookrightarrow \tilde{X}$, $Y \hookrightarrow \tilde{Y} \hookrightarrow \tilde{X}$ and consider a bounded linear operator $T : \tilde{X} \to F$. Suppose moreover that the following conditions are satisfied:*

(i) $\tilde{X}$, Y, $\tilde{Y}$ are UMD separable spaces;

(ii) T is bounded from X to F and from Y to F, and $T(Y) \hookrightarrow T(L^\infty \cap \tilde{Y})$.

Fix some $\theta \in (0,1)$. Then, for any vector field $v \in (L^\infty \cap X, Y)_\theta$, there exists a vector field $u \in L^\infty \cap (\tilde{X}, \tilde{Y})_\theta$ such that

$$Tu = Tv,$$

and

$$\|u\|_{L^\infty \cap (\tilde{X},\tilde{Y})_\theta} \lesssim \|v\|_{(L^\infty \cap X, Y)_\theta}.$$

Proof. We apply Lemma 29 for the Banach spaces $X_0 = E = \tilde{X}$, $X_1 = \tilde{Y}$, $A = L^\infty$, $A_0 = L^\infty \cap \tilde{X}$, $B_0 = L^\infty \cap X$, $A_1 = L^\infty \cap \tilde{Y}$, $B_1 = Y$ and the operator T. One can easily observe that in this setting the conditions of Lemma 29 (part of them are explicitly stated in Lemma 28) are satisfied. Indeed, in order to verify the condition (i) in Lemma 28 it suffices to see that $X, Y \hookrightarrow \tilde{X}$ and hence,

$$B_1 \hookrightarrow X_1 = \tilde{Y} \hookrightarrow X_0 = \tilde{X},$$

and $B_0 \hookrightarrow A_0$.

The space $\tilde{Y}$ is reflexive (since it has the UMD property) and hence, it has a separable predual[12]. Also, $A = (L^1)^*$ has a separable predual. Thus, the condition (ii) in Lemma 28 is verified. Condition (iii) in Lemma 28 is ensured by condition (ii) in Theorem 39.

[12]Here we use the fact that if the dual X^* of a Banach space X is separable, then, X is separable (see for instance [20, Theorem 4.6-8, p. 245]).

Notice that $\tilde{X} \cap \tilde{Y}$ is a separable space that is dense in $(\tilde{X}, \tilde{Y})_\theta$ (see [2, Theorem 4.2.2 (a), p. 91]). It follows that $(\tilde{X}, \tilde{Y})_\theta$ is a reflexive and separable space and consequently it has a separable predual. We also have by (i) that the spaces $\tilde{X}$, Y, $\tilde{Y}$ have the UMD property. We can apply now Lemma 29 and we get Theorem 39. □

Proof of Theorem 10. Let us consider some parameter $r \in [2, \infty)$ such that $1/p = (1-\theta)/r + \theta/2$. We apply Theorem 39 for the Banach spaces $X = \dot{B}_1^{d/r,r}$, $\tilde{X} = \dot{B}_2^{d/r,r}$ and $Y = \tilde{Y} = \dot{B}_2^{d/2,2}$. Since $r \geq 2$, we have $\dot{B}_2^{d/2,2}, B_1^{d/r,r} \hookrightarrow \dot{B}_2^{d/r,r}$. We set $F = \dot{B}_2^{d/r-1,r}$. We have the embeddings $X \hookrightarrow \tilde{X}$ and $Y \hookrightarrow \tilde{Y} \hookrightarrow \tilde{X}$. Also, by Mazya's theorem (Theorem 3 in the case $p = q = 2$) we have $T(Y) \hookrightarrow T(L^\infty \cap \tilde{Y})$, for the operator $T = \text{div} : \dot{B}_2^{d/r,r} \to \dot{B}_2^{d/r-1,r}$. Now the hypotheses of Theorem 39 are satisfied.

Observe that, since $X = \dot{B}_1^{d/r,r} \hookrightarrow L^\infty$, we have $L^\infty \cap X = \dot{B}_1^{d/r,r}$ and we can write

$$(L^\infty \cap X, \dot{B}_2^{d/2,2})_\theta = (B_1^{d/r,r}, \dot{B}_2^{d/2,2})_\theta = B_q^{d/p,p}.$$

Since we also have

$$(\dot{B}_2^{d/r,r}, \dot{B}_2^{d/2,2})_\theta = \dot{B}_2^{d/p,p},$$

it remains to apply Theorem 39. Theorem 10 is proved. □

Proof of Theorem 11. As in the proof of Theorem 10 let us consider $r \in [2, \infty)$ such that $1/p = (1-\theta)/r + \theta/2$. We apply Theorem 39 for the Banach spaces $X = \dot{H}^{d/r} L^{r,1}$, $\tilde{X} = \dot{H}^{d/r} L^{r,2}$, $Y = \tilde{Y} = \dot{H}^{d/2}$, $F = \dot{B}_r^{d/r-1,r}$ and the operator $T = \text{div} : \dot{H}^{d/r} L^{r,2} \to \dot{B}_r^{d/r-1,r}$ (see Remark 18). It remains to verify that the hypotheses of Theorem 39 are satisfied. Indeed, by the monotonicity properties of the Lorentz spaces we also have $\dot{H}^{d/r} L^{r,1} \hookrightarrow \dot{H}^{d/r} L^{r,2}$, i.e., $X \hookrightarrow \tilde{X}$. By Lemma 15 we get $\dot{H}^{d/2} \hookrightarrow \tilde{X}$, i.e., $Y = \tilde{Y} \hookrightarrow \tilde{X}$. Also, by Mazya's theorem (Theorem 3 in the case $p = q = 2$) we have $T(Y) \hookrightarrow T(L^\infty \cap \tilde{Y})$ and now the hypotheses of Theorem 39 are satisfied.

By Lemma 16 we have $\dot{H}^{d/2} \hookrightarrow \tilde{X}$ and $X = \dot{H}^{d/r} L^{r,1} \hookrightarrow L^\infty$ and hence $L^\infty \cap X = \dot{H}^{d/r} L^{r,1}$. From this and Lemma 17 we get

$$(L^\infty \cap X, \dot{H}^{d/2})_\theta = (\dot{H}^{d/r} L^{r,1}, \dot{H}^{d/2})_\theta = \dot{H}^{d/p} L^{p,q}. \tag{144}$$

Lemma 17 also gives

$$(\dot{H}^{d/r} L^{r,2}, \dot{H}^{d/2})_\theta = \dot{H}^{d/p} L^{p,2}, \tag{145}$$

and now one can conclude the proof of Theorem 11 by a direct application of Theorem 39. □

Proof of Theorem 9. The proof is very similar to the one of Theorem 10. Suppose p, r, θ are as in the proof of Theorem 10. We put

$$X = \prod_{j=1}^{d} (\dot{B}_1^{d/r,r})_{D_j}, \quad \widetilde{X} = \prod_{j=1}^{d} (\dot{B}_2^{d/r,r})_{D_j} \text{ and } Y = \tilde{Y} = \prod_{j=1}^{d} (\dot{B}_2^{d/2,2})_{D_j}.$$

Now, the operator $T : \widetilde{X} \to \dot{B}_2^{d/r-1,r}$ is defined by the formula

$$T(v_1, ..., v_d) = \sum_{j=1}^{d} G_j(\nabla) \cdot v_j,$$

where $v_1, ..., v_d$ are $(d-1)$-vector fields with $(v_1, ..., v_d) \in \widetilde{X}$ (recall (20) in order to see that T is well-defined and bounded). In order to verify the item (ii) in Theorem 39 we use Lemma

37 instead of Mazya's theorem. It remains to apply Theorem 39 and to observe that, by the retraction method,

$$((\dot{B}_1^{d/r,r})_{D_j}, (\dot{B}_2^{d/2,2})_{D_j})_\theta = (\dot{B}_q^{d/p,p})_{D_j}$$

and

$$((\dot{B}_2^{d/r,r})_{D_j}, (\dot{B}_2^{d/2,2})_{D_j})_\theta = (\dot{B}_2^{d/p,p})_{D_j},$$

for any $j \in \{1, ..., d\}$. □

Proof of Theorem 8. Again, suppose p, r, θ are as in the proof of Theorem 10. We put $X = (\dot{B}_1^{d/r,r})_{C_\delta}$, $\widetilde{X} = (\dot{B}_2^{d/r,r})_{C_{(1+\varepsilon)\delta}}$ and $Y = (\dot{B}_2^{d/2,2})_{C_\delta}$, $\widetilde{Y} = (\dot{B}_2^{d/2,2})_{C_{(1+\varepsilon)\delta}}$, $F = \dot{B}_2^{d/r-1,r}$. The operator T is the usual divergence operator $T = \text{div} : \widetilde{X} \to \dot{B}_2^{d/r-1,r}$. It remains to apply Theorem 39 and to observe that, by the retraction method,

$$((\dot{B}_1^{d/r,r})_{C_\delta}, (\dot{B}_2^{d/2,2})_{C_\delta})_\theta = (\dot{B}_q^{d/p,p})_{C_\delta}$$

and

$$((\dot{B}_2^{d/r,r})_{C_{(1+\varepsilon)\delta}}, (\dot{B}_2^{d/2,2})_{C_{(1+\varepsilon)\delta}})_\theta = (\dot{B}_2^{d/p,p})_{C_{(1+\varepsilon)\delta}}.$$

Both of these equalities rest on the fact that the Fourier projections P_{C_δ} on the sets C_δ are a sum of two rotated and dilated Riesz projections. Hence, P_{C_δ} is bounded on each of the spaces $\dot{B}_1^{d/r,r}$, $\dot{B}_2^{d/r,r}$ and $\dot{B}_2^{d/2,2}$. □

Let us see now that Theorem 9 implies Theorem 7. For this we need only some elementary geometry. Suppose $d \geq 3$ and consider the unit vectors $\nu_j := (1, ..., 1, 2, 1, ..., 1)/\sqrt{d+1}$ in $\mathbb{R}^d$ (with value 2 on the j-th position), $j \in \{1, ..., d\}$. For each $j \in \{1, ..., d\}$ define the half-spaces

$$D_j := \left\{\xi \in \mathbb{R}^d \mid \langle \xi, \nu_j \rangle > 0\right\},$$

and let D be the set $D := D_1 \cup ... \cup D_d$. By $p_{D_j}(\xi)$ we denote the orthogonal projection of the point ξ on the support hyperplane Π_j of D_j:

$$\Pi_j := \left\{\xi \in \mathbb{R}^d \mid \langle \xi, \nu_j \rangle = 0\right\}.$$

Note that

$$|p_{D_j}(\xi)| = \left(|\xi|^2 - |\langle \xi, \nu_j \rangle|^2\right)^{1/2},$$

for any $j \in \{1, ..., d\}$.

Consider the function $\sigma : \mathbb{R}^d \to \mathbb{R}$ defined by $\sigma(\xi) = \xi_1$. We can immediately see that this σ satisfies the conditions (P1), (P2). We have now $\sigma_j(\xi) = \xi_j$, for all $j \in \{1, ..., d\}$. Consider $G_0 := (\sigma_1, ..., \sigma_{d-1})$ and let G_j be obtained by composing G_0 with a rotation that transforms $\mathbb{R}^{d-1} \times (0, \infty)$ in D_j. One can write that $|G_j(\xi)| \sim |p_{D_j}(\xi)|$, for all $j \in \{1, ..., d\}$. In order to see that the family of functions $G_1, ..., G_d$ is adapted to the family $D_1, ..., D_d$ of half-spaces it remains to prove the following equivalence that corresponds to (12):

Lemma 40. *With the above notation we have*

$$\sum_{j=1}^{d} |p_{D_j}(\xi)| \mathbf{1}_{D_j}(\xi) \sim |\xi| \mathbf{1}_D(\xi),$$

for any $\xi \in \mathbb{R}^d$.

Proof. Observe that $|\nu_i - \nu_j| < 1$ and $| \langle \nu_i, \nu_j \rangle | < 1$ for any $i, j \in \{1, ..., d\}$, $i \neq j$. It follows from this that we can find some sufficiently small number $\alpha \in (0, 1)$ such that

$$|\nu_i - \nu_j| < \sqrt{1-\alpha}, \tag{146}$$

for any $i, j \in \{1, ..., d\}$, and

$$\sqrt{\frac{\alpha}{1-\alpha}} + | \langle \nu_1, \nu_2 \rangle | < 1. \tag{147}$$

Let $c \in (0, 1)$ such that $\sqrt{1-c}$ equals the left hand side of (147). In order to prove Lemma 40 it suffices to see that, for any $\xi \in \mathbb{R}^d$,

$$\sum_{j=1}^{d} |p_{D_j}(\xi)| \mathbf{1}_{D_j}(\xi) \geq c_1 |\xi| \mathbf{1}_D(\xi), \tag{148}$$

with $c_1 = \min(\sqrt{\alpha}, \sqrt{c})$.

Pick some $\xi \in D_1 \cap ... \cap D_d$. If $| \langle \xi, \nu_1 \rangle | \leq |\xi| \sqrt{1-\alpha}$, then the left hand side of (148) is at least

$$|p_{D_1}(\xi)| = \left(|\xi|^2 - |\langle \xi, \nu_1 \rangle|^2\right)^{1/2} \geq |\xi| \sqrt{\alpha},$$

and we are done. Else, we have $| \langle \xi, \nu_1 \rangle | > |\xi| \sqrt{1-\alpha}$ and decomposing ξ as $\xi = \beta \nu_1 + w$, for some $\beta \in \mathbb{R}$ and $w \in \mathbb{R}^d$ with $w \perp \nu_1$, we can rewrite this inequality as

$$|\beta| > \left(|\beta|^2 + |w|^2\right)^{1/2} \sqrt{1-\alpha},$$

or, equivalently,

$$|w| < |\beta| \sqrt{\frac{\alpha}{1-\alpha}}. \tag{149}$$

Now, note that, using (147)

$$\begin{aligned} |\langle \xi, \nu_2 \rangle| &\leq |\beta| | \langle \nu_1, \nu_2 \rangle | + | \langle w, \nu_2 \rangle | \\ &\leq |\beta| | \langle \nu_1, \nu_2 \rangle | + |w| \\ &< |\beta| \left(| \langle \nu_1, \nu_2 \rangle | + \sqrt{\frac{\alpha}{1-\alpha}} \right) \\ &\leq |\xi| \sqrt{1-c}. \end{aligned}$$

As above we get $|p_{D_2}(\xi)| = (|\xi|^2 - | \langle \xi, \nu_1 \rangle |^2)^{1/2} \geq \sqrt{c} |\xi|$ and we are done. It remains to treat the case $\xi \in D \backslash (D_1 \cap ... \cap D_d)$.

Suppose $\xi \in D_k \backslash (D_1 \cap ... \cap D_d)$ for some $k \in \{1, ..., d\}$. Then, we cannot have $\langle \xi, \nu_k \rangle > |\xi| \sqrt{1-\alpha}$. Otherwise, using (146),

$$\begin{aligned} \langle \xi, \nu_j \rangle &= \langle \xi, \nu_k \rangle - \langle \xi, \nu_k - \nu_j \rangle \\ &> |\xi| \sqrt{1-\alpha} - |\xi| |\nu_i - \nu_j| > 0, \end{aligned}$$

for any $j \in \{1, ..., d\}$, and we obtain that $\xi \in D_1 \cap ... \cap D_d$ contradicting the choice of ξ. Hence, we must have $\langle \xi, \nu_k \rangle \leq |\xi| \sqrt{1-\alpha}$ and since $\langle \xi, \nu_k \rangle > 0$ (thanks to the fact that $\xi \in D_k$) we get $| \langle \xi, \nu_k \rangle | \leq |\xi| \sqrt{1-\alpha}$. Now we can conclude as above. □

By applying Theorem 9 we get a version of Theorem 7 in which the Fourier support of the solutions lie in the set D instead of $\Delta = \mathbb{R}^d \backslash (-\infty, 0)^d$. One can deduce from this the original version of Theorem 7 by composing the involved functions (the source term and the solution) by rotations and dilations.

Remark 41. *It would be interesting if one could replace the set $\Delta = \mathbb{R}^d \backslash (-\infty, 0)^d$ in Theorem 7 with the set $[-\infty, 0]^d$. This will give a stronger version of Theorem 7. It is not known whether this stronger version is true or not. The methods used in this paper seem to not apply in the case of the set $[-\infty, 0]^d$.*

5.2 Remark concerning the "third" parameter

Let us consider here the problem related to the non-optimality of the third parameter. For the sake of simplicity we are concerned here only with the divergence equation. Similar observations can be made for the case of the divergence-like equations.

Recall that, in Theorem 10 in contrast to Theorem 3, we lose some control of the parameter q of the Besov spaces involved: we start with a source term in $\dot{B}_q^{d/p,p}$ and we end up with a solution in $\dot{B}_2^{d/p,p}$ which, despite the fact that it has the "right" differential regularity (the exponents p and $s = d/p$ are the right ones), it is a space strictly larger than $\dot{B}_q^{d/p,p}$. This is due to the fact that in order to easily compute the source space we have chosen X such that $X \hookrightarrow L^\infty$. Consequently, we have to take $\tilde{X}$ strictly larger than X. Indeed, choosing $\tilde{X} = X$ the hypotheses of Theorem 39 imply that $\dot{H}^{d/2} \hookrightarrow X \hookrightarrow L^\infty$, however, $\dot{H}^{d/2}$ is not embedded in L^∞. By the method we used to prove Theorem 10 it is unlikely to improve the solution space to $L^\infty \cap \dot{B}_q^{d/p,p}$. A similar remark can be made for Theorem 11.

When we use Theorem 39, in order to not lose any regularity, we would like to have that $\tilde{X} = X$ and

$$(L^\infty \cap X, \dot{H}^{d/2})_\theta = (X, \dot{H}^{d/2})_\theta. \tag{150}$$

Since $\dot{H}^{d/2} \hookrightarrow X$ we cannot impose the condition $X \hookrightarrow L^\infty$. Apart from this situation, there are other natural candidates for the space X that one may expect to satisfy (150). However, this condition (150) is too restrictive. For instance we cannot pick $X = \dot{B}_r^{d/r,r}$ for some $r \in (2, \infty)$. Indeed, in this case, we have the following negative result:

Proposition 42. *Let $r \in (2, \infty)$ and $\theta \in (0, 1)$ be some fixed parameters. Then,*

$$(L^\infty(\mathbb{R}^d) \cap \dot{B}_r^{d/r,r}(\mathbb{R}^d), \dot{H}^{d/2}(\mathbb{R}^d))_\theta \neq (\dot{B}_r^{d/r,r}(\mathbb{R}^d), \dot{H}^{d/2}(\mathbb{R}^d))_\theta.$$

(The corresponding norms on the two interpolation spaces are not equivalent.)

Proof. Suppose by contradiction that we have

$$(L^\infty \cap \dot{B}_r^{d/r,r}, \dot{H}^{d/2})_\theta = (\dot{B}_r^{d/r,r}, \dot{H}^{d/2})_\theta.$$

This implies that

$$\begin{aligned} (L^\infty \cap \dot{B}_r^{d/r,r}, \dot{H}^{d/2})_\theta &= (\dot{B}_r^{d/r,r}, \dot{B}_2^{d/2,2})_\theta \\ &= \dot{B}_p^{d/p,p}, \end{aligned} \tag{151}$$

where $1/p = (1-\theta)/r + \theta/2$. On the other hand, since $p > 2$, there exists some $\eta \in (0, 1)$ such that

$$(\dot{B}_p^{d/p,p}, \dot{B}_1^{d,1})_\theta = \dot{B}_2^{d/2,2}.$$

This, together with (151) and T. Wolff's interpolation theorem (see [32, Theorem 2]) implies that, there exists some $\theta_1 \in (0, 1)$ such that

$$(L^\infty \cap \dot{B}_r^{d/r,r}, \dot{B}_1^{d,1})_{\theta_1} = \dot{B}_2^{d/2,2}. \tag{152}$$

Consider some function $\psi \in C_c^\infty(B(0,1))$ such that $\psi \equiv 1$ on $B(0,1/2)$ and define the operator T_ψ by

$$T_\psi f = f - f * \widehat{\psi},$$

for any Schwartz function f. We extend T_ψ by continuity to the spaces $L^\infty \cap \dot{B}_r^{d/r,r}$, $\dot{B}_1^{d,1}$, $\dot{B}_2^{d/2,2}$ and we observe that we have the embeddings $T_\psi(L^\infty \cap \dot{B}_r^{d/r,r}) \hookrightarrow L^\infty$ and $T_\psi(\dot{B}_1^{d,1}) \hookrightarrow B_1^{d,1} \hookrightarrow L^\infty$.

Thanks to this and (152), $T_\psi(\dot{B}_2^{d/2,2})$ must be embedded in L^∞. In other words,

$$\|f - f * \widehat{\psi}\|_{L^\infty} \lesssim \|f\|_{\dot{B}_2^{d/2,2}} = \|f\|_{\dot{H}^{d/2}} \lesssim \|f\|_{H^{d/2}}, \tag{153}$$

(where $H^{d/2} = L^2 \cap \dot{H}^{d/2}$) for any Schwartz function f. Young's inequality ([19, Lemma 1.2.30]) and the fact that $\widehat{\psi}$ is Schwartz gives us that

$$||f * \widehat{\psi}||_{L^\infty} \lesssim \|\widehat{\psi}\|_{L^2}\|f\|_{L^2} \lesssim \|f\|_{H^{d/2}},$$

which together with (152) yields

$$\|f\|_{L^\infty} \lesssim \|f - f * \widehat{\psi}\|_{L^\infty} + \|f * \widehat{\psi}\|_{L^\infty} \lesssim \|f\|_{H^{d/2}}.$$

In other words we have obtained the embedding $H^{d/2} \hookrightarrow L^\infty$, which is false. □

Open problem. *Suppose $D = \mathbb{R}^{d-1} \times (0,\infty)$ and X is a function space on $\mathbb{R}^d$ such that*

$$\dot{H}^{d/2} \hookrightarrow X \hookrightarrow BMO.$$

Is it true that

$$\operatorname{div}(L^\infty \cap X_D, \dot{H}_D^{d/2})_\theta = \operatorname{div}(X_D, \dot{H}_D^{d/2})_\theta$$

(in the sense that any divergence of a $(X_D, \dot{H}_D^{d/2})_\theta$-vector field is a divergence of a $(L^\infty \cap X_D, \dot{H}_D^{d/2})_\theta$-vector field)?

If the answer to this question is yes, then, by using Theorem 39 we would be able to provide a version of Theorem 7 with no loss of regularity in the third parameter:

Conjecture. *Let $d \geq 3$ be an integer and consider the set $\Delta := \mathbb{R}^d \backslash (-\infty, 0)^d$. Consider some parameters $p \in [2,\infty)$ and $q \in (1,\infty)$. Then, for any vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{F}_q^{d/p,p}(\mathbb{R}^d)$, with spec$(v) \subseteq \Delta$ there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{F}_q^{d/p,p}(\mathbb{R}^d)$, with spec$(u) \subseteq \Delta$ such that*

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{F}_q^{d/p,p}} \lesssim \|v\|_{\dot{F}_q^{d/p,p}}.$$

(And a similar statement with $\dot{B}_q^{d/p,p}$ in place of $\dot{F}_q^{d/p,p}$.)

One can formulate similar conjectures corresponding to the statements of Theorem 9 and Theorem 8.

6 Appendix

All the facts proved in this Appendix are at least implicitly well-know. We present them here for the convenience of the reader.

6.1 A compactness result

The following lemma is a version of standard compactness results (see for instance the Ascoli theorem in [17, (7.5.7)]).

Lemma 44. *Let X be a separable Banach space. Fix some $l \in \mathbb{N}^*$ and suppose $(u_n)_{n\geq 1}$ is a sequence in $C_b^l(\mathbb{R}, X^*)$ such that*

$$\sup_{t\in\mathbb{R}} \|\partial^m u_n(t)\|_{X^*} \leq 1,$$

for any $m \in \{0, 1, ..., l\}$ and any $n \geq 1$. Then, there exists some $u \in C_b^{l-1}(\mathbb{R}, X^*)$, with $\partial^{l-1}u$ being Lipschitz and a subsequence $(u_{n_k})_{k\geq 1}$ such that

$$\lim_{k\to\infty} \sup_{t\in I} |\langle \partial^m u_{n_k}(t) - \partial^m u(t), x\rangle| = 0, \tag{154}$$

for any $x \in X$, any $m \in \{0, 1, ..., l-1\}$ and any compact interval $I \subset \mathbb{R}$.

Proof. We first prove the version corresponding to the case $l = 1$. Let $(q_\nu)_{\nu\geq 1}$ a sequence of rational numbers such that $\mathbb{Q} = \{q_\nu \mid \nu \geq 1\}$. Since

$$\|u_n(q_\nu)\|_{X^*} \leq 1,$$

for all $n, \nu \geq 1$, by the Banach-Alaoglu theorem and a diagonalization argument there exists a subsequence $(u_{n_k})_{k\geq 1}$ such that each $u_{n_k}(q_\nu)$ is convergent to some element $u(q_\nu) \in X^*$, in the w^*-topology in X^*.

Note that, by the mean value theorem ([17, (8.5.2)] we have

$$\|u_n(t) - u_n(s)\|_{X^*} \leq |t - s|, \tag{155}$$

for any $t, s \in \mathbb{R}$ and any $n \geq 1$.

Fix some $t \in \mathbb{R}$ and let $(t_\nu)_{\nu\geq 1}$ be a sequence in $\mathbb{Q}$ such that $t_\nu \to t$. By (155), we have

$$\|u(t_\nu) - u(t_{\nu'})\|_{X^*} \leq \liminf_{n\to\infty} \|u_n(t_\nu) - u_n(t_{\nu'})\|_{X^*} \leq |t_\nu - t_{\nu'}|,$$

which shows in particular that $(u(t_\nu))_{\nu\geq 1}$ is a Cauchy sequence in X^*. By denoting its limit with $u(t)$, we have obtained a well-defined function $u : \mathbb{R} \to X^*$. By taking the inferior limit in (155) we also observe that u is Lipschitz of constant 1.

Let $I \subset \mathbb{R}$ be a compact interval and fix some $\varepsilon \in (0, 1)$. Consider some rational numbers $s_1 < .... < s_N$ in I such that $s_{j+1} - s_j < \varepsilon$, for any $j \in \{1, ..., N-1\}$. For any $t \in I$ we can find some s_j such that $|t - s_j| < \varepsilon/2$. For such s_j we have ,for any $x \in X$, that

$$\begin{aligned} |\langle u_{n_k}(t) - u(t), x\rangle - \langle u_{n_k}(s_j) - u(s_j), x\rangle| &\leq \|u_{n_k}(t) - u_{n_k}(s_j)\|_{X^*} + \|u(t) - u(s_j)\|_{X^*} \\ &\leq 2|t - s_j| < \varepsilon, \end{aligned}$$

which gives

$$\sup_{t\in I} |\langle u_{n_k}(t) - u(t), x\rangle| \leq \max_{j=1,...,N} |\langle u_{n_k}(s_j) - u(s_j), x\rangle| + \varepsilon.$$

Since for each j the sequence $u_{n_k}(s_j)$ converges to $u(s_j)$ in the w^*-topology on X^*, we get the conclusion of Lemma 44 in the case $l = 1$.

Now, suppose $l = 2$. By applying the case $l = 1$ to the sequences $(u_n)_{n\geq 1}$ and $(\partial u_n)_{n\geq 1}$, we obtain that there exists a subsequence $(u_{n_k})_{k\geq 1}$ and two functions $u \in C_b(\mathbb{R}, X^*)$ and $u^1 \in C_b(\mathbb{R}, X^*)$ such that (154) holds for $m = 0$ and

$$\lim_{k\to\infty} \sup_{t\in I} | \left\langle \partial u_n(t) - u^1(t), x \right\rangle | = 0, \tag{156}$$

for any $x \in X$, and any compact interval $I \subset \mathbb{R}$.

Fix some $t \in \mathbb{R}$. Thanks to (156), for any $h \in \mathbb{R}\backslash\{0\}$ we can write

$$\left\| \frac{u(t+h) - u(t)}{h} - u^1(t) \right\|_{X^*} \leq \liminf_{k\to\infty} \left\| \frac{u_{n_k}(t+h) - u_{n_k}(t)}{h} - \partial u_{n_k}(t) \right\|_{X^*}. \tag{157}$$

However, by the mean value theorem ([17, (8.6.2) and (8.5.2)]),

$$\left\| \frac{u_{n_k}(t+h) - u_{n_k}(t)}{h} - \partial u_{n_k}(t) \right\|_{X^*} \leq \sup_{\xi\in[t,t+h]} \|\partial u_{n_k}(\xi) - \partial u_{n_k}(t)\|_{X^*} \leq |h|,$$

which together with (157) implies that

$$\left\| \frac{u(t+h) - u(t)}{h} - u^1(t) \right\|_{X^*} \leq |h|,$$

and we must have that u is differentiable and $u^1 = \partial u$.

Now we have the conclusion of Lemma 44 in the case $l \leq 2$ and inductively we can obtain the general version. □

6.2 Commutativity of some operators

We list here several facts used in Subsection 3.3.2 that concern the commutativity of some operators acting on vector-valued functions.

The Hilbert transform and bounded linear operators. Let H be the Hilbert transform as defined in (44). Suppose A and X are some Banach spaces and $\Lambda : A \to X$ is a bounded linear operator. Then, for any $u \in C_b^1(\mathbb{R}, A) \cap L^2(\mathbb{R}, A)$ we have the equality

$$H\Lambda u(t) = \Lambda H u(t), \tag{158}$$

for any $t \in \mathbb{R}$. This can be seen to be a direct consequence of the definition (44) as follows. Fix some $t \in \mathbb{R}$. Since the limit in (44) is in the strong topology, using twice the boundedness of Λ we can write

$$\begin{aligned}
\Lambda H u(t) &= \Lambda \left(\lim_{\varepsilon\to 0} \int_{\varepsilon<|s|<1/\varepsilon} \frac{u(t-s)}{s} ds \right) \\
&= \lim_{\varepsilon\to 0} \Lambda \left(\int_{\varepsilon<|s|<1/\varepsilon} \frac{u(t-s)}{s} ds \right) \\
&= \lim_{\varepsilon\to 0} \int_{\varepsilon<|s|<1/\varepsilon} \frac{\Lambda u(t-s)}{s} ds \\
&= H\Lambda u(t).
\end{aligned}$$

Comutativity of convolution. Consider some functions $\varphi \in C_c^\infty(\mathbb{R}, \mathbb{C})$, $\rho \in C_b(\mathbb{R}, \mathbb{C})$ and suppose that $v \in L^2(\exp(\alpha t^2), Z)$ for some $\alpha > 0$ and some Banach space Z. Then,

$$\rho * (\varphi * v) = \varphi * (\rho * v). \tag{159}$$

First note that according to Fact 1 in Subsection 3.3.2 the function $\varphi * v$ belongs to $L^2(\exp(\alpha t^2/2), Z)$ and hence, $\rho * (\varphi * v)$ is a well-defined function. The dominated convergence theorem (see [19, Proposition 1.2.5]) shows that $\rho * (\varphi * v)$ is continuous. Also, $\rho * v \in C_b(\mathbb{R}, Z)$ and we get that $\varphi * (\rho * v)$ is well-defined and continuous. To see that (159) holds, we use the Fubini theorem (see [19, Proposition 1.2.7]) and write

$$\int_{\mathbb{R}} \rho(s_2) \left(\int_{\mathbb{R}} \varphi(s_1) v(t - s_2 - s_1) ds_1 \right) ds_2 = \int_{\mathbb{R}} \varphi(s_1) \left(\int_{\mathbb{R}} \rho(s_2) v(t - s_2 - s_1) ds_2 \right) ds_1.$$

Derivatives and convolution. Consider some function $\varphi \in C_c^\infty(\mathbb{R}, \mathbb{C})$ and suppose that $v \in L^2(\mathbb{R}, Z)$ for some Banach space Z. Then, $\varphi * v$ is differentiable and

$$\partial(\varphi * v) = (\partial \varphi) * v. \tag{160}$$

This follows from the dominated convergence theorem ([19, Proposition 1.2.5]) by writing

$$\frac{\varphi * v(t+h) - \varphi * v(t)}{h} = \int_{\mathbb{R}} \frac{\varphi(t+h-s) - \varphi(t-s)}{h} v(s) ds,$$

for any $t, h \in \mathbb{R}$, $h \neq 0$ and letting $h \to 0$.

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