GENERIC REGULARITY OF CONSERVATIVE SOLUTIONS TO THE $N - abc$ FAMILY OF CAMASSA-HOLM TYPE EQUATION

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ABSTRACT. In this paper, we investigate the generic regularity of conservation solutions to the $N - abc$ family of Camassa-Holm type equation with $(N + 1)$-order nonlinearities. This quasi-linear equation is nonlocal with higher order nonlinearities, compared to the Camassa-Holm equation ($N = 1$) and Novikov eqution ($N = 2$). For an open dense set of C^3 initial data, we prove that the solution is piecewise smooth in the $t - x$ plane, while the gradient $u^{N-1}u_x$ can blow up along finitely many characteristic curves. Moreover, we provide a detailed asymptotic description of the solution in a neighborhood of each generic singular point. Our strategy mainly relies on the variable transformation introduced in [72], which reduces the equation to a new semi-linear system semilinear system on new characteristic coordinates and Thom's transversality technique introduced in [52]. This result improves earlier ones in the literatures, such as [52], [66] and [41].
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Keywords: the $N - abc$ family of Camassa-Holm type equation; generic regularity; conservative weak solution; singularity

1. Introduction

In this paper, we consider the Cauchy problem for the $N - abc$ family of Camassa-Holm type equation

$$\begin{cases} u_t - u_{txx} - cu^N u_{xxx} - bu^{N-1}u_x u_{xx} + au^N u_x = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x), x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $N \in Z^+, N \geq 1$, a, b, c are positive constants and $a = b + c$. More details are present in [71–73].

It is well known that (1.1) is an evolution equation with $(N + 1)$-order nonlinearities and contains three integrable dispersive equations: the $b-$family equation, the Camassa-Holm (CH) equation, and the Degasperis-Procesi (DP) equation as well as the Novikov equation. There are lots of literatures devoted to the equations with respect to (1.1) for the initial condition $u_0 \in H^s(\mathbb{R})$, such as the local well-posedness, global well-posedness, blow up criterion and wave breaking phenomenon as well as persistence properties and unique continuation properties. Next, let us recall some known contributions about (1.1).

(1.1) was first considered by Himonas and Holliman [39]. Using a Galerkin-type approximation technique, they demonstrated that (1.1) is well-posed in Sobolev spaces H^s with $s > \frac{3}{2}$ on both the circle and the line in the sense of Hadamard. In [74], Zhou an Mu established the local well-posedness of strong solutions in H^s with $s > \frac{3}{2}$ and the persistence properties of strong solutions as well as the existence of its global weak solutions and peakon solution of (1.1). Later on, Himonas and Mantzavinos [42] established well-posedness in H^s with $s > \frac{5}{2}$. They also showed a sharpness result on the data-to-solution map and proved that it is not uniformly continuous from any bounded subset of H^s into $C([0, T); H^s)$. In [7], Barostichi et al. studied (1.1) and obtained an abstract Cauchy-Kovalevsky type theorem by using a power series method in abstract Banach spaces with analytic initial data. Recently, in [35], Guo et al. investigated the large time behavior of compact support of the potential for (1.1), if the compactly supported initial potential keeps its sign. Moreover, using time-frequency analysis, they established the pointwise decay estimates and demonstrated the persistence property in weighted Sobolev spaces. In [65], using the Littlewood-Paley decomposition technique and transport equation theory, the author first obtained the local well-posedness result in Besov and Sobolev spaces for (1.1). Also, using the Littlewood-Paley decomposition technique and the commutator estimate as well as Morse-type inequality, the author established a crucial blow up criterion in Besov spaces, which is different from the usual results only in Sobolev spaces. Furthermore, using the characteristic method, the author established an invariant property of the solution and established the precise blow-up scenario for strong solutions to (1.1). On the other hand, the author established some global existence results for the strong solutions by deriving two useful conservation laws. Finally, the author demonstrated wave breaking phenomenon and established a new finite time blow-up solution to (1.1) with respect to the initial data u_0. It is worth mentioning that Zhou and Ji [70] investigated some properties for the solutions to (1.1), including the local well-posedness, wave breaking, blow-up rate, global existence and uniqueness. More precisely, firstly, using Kato's theorem, they established the local well-posedness. Then, using the lower order energy conservation law and the commutator estimate technique, we derive the wave breaking mechanism of solutions to (1.1). In particular, when the power of the higher order nonlinearities N is even, using the characteristic method and differential inequality technique, they demonstrated two sufficient conditions on the initial datum for the occurrence of wave breaking, and established the upper bound of the maximal existence time and the blow-up rate. Finally, they proved the global existence and uniqueness of solutions to (1.1).

• If $c = 1$ and $N = 1$, then (1.1) reduces the $b-$family equation

$$\begin{cases} u_t - u_{txx} - uu_{xxx} - bu_x u_{xx} + (b+1)uu_x, x \in \mathbb{R} = 0, t > 0, \\ u(x,0) = u_0(x), x \in \mathbb{R}, \end{cases}$$

with quadratic nonlinearity, which can be obtained as the family of asymptotically equivalent shallow water wave equations [21]. In [23], we know that it admits peakon solutions for any constant b. The local well-posedness and global well-posedness as well as blow up criterion for the $b-$family equation have been considered in [25–27]. By using Painlevé analysis [23,24], we can find that there are two integrable equations in the $b-$family equation, i.e, the Camassa-Holm equation (when $b = 2$) and the Degasperis-Procesi equation (when $b = 3$), which will be presented as follows.

• If $c = 1, N = 1, b = 2$ and $a = 3$, then (1.1) becomes to the well-known CH equation

$$\begin{cases} u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x), x \in \mathbb{R}. \end{cases} \tag{1.2}$$

(1.2) models one-dimensional unidirectional propagation of shallow water waves over a flat bottom under the influence of gravity, and $u(t, x)$ represents the fluid velocity at time t in the horizontal direction x [16]. It is well known that (1.2) enjoys two local Hamilton structures [55]

$$m_t = B_0 \frac{\delta H_2}{\delta m} = B_1 \frac{\delta H_1}{\delta m},$$

$$B_0 = -\partial_x + \partial_x^3 = -\mathcal{L}, B_1 = -(m\partial_x + \partial_x m),$$

$$H_1 = \frac{1}{2}\int (u^3 + uu_x^2)dx, H_2 = \frac{1}{2}\int (u^2 + u_x^2)dx,$$

where $m = u - u_{xx}$, whose compatibility condition was shown in [28]. In [16], Camassa and Holm demonstrated that (1.2) admits peakon solutions $u(t, x) = ce^{-|x-ct|}$, which have discontinuous first derivative at the wave peak in contrast to the smoothness of most previously known specious of solitary waves and thus are called peakons. CH equation (1.2) is integrable in the sense of an infinite-dimensional Hamiltonian system and arises as model for shallow water waves [16, 46]. As a matter of fact, in [29], Fokas and Fuchssteiner established the bi-Hamiltonian structure for CH equation (1.2). Moreover, it also related to describe small amplitude radial deformation waves in cylindrical compressible hyper-elastic rods in Ref. [19]. The local well-posedness and global well-posedness of the CH equation (1.2) has extensively been considered in Ref. [14]. It was demonstrated that there exist strong solutions to the CH equation (1.2) [14] and the finite time blow-up strong solutions to the CH equation (1.2) in Ref. [14, 15]. It is worth mentioning that, using vanishing viscosity method, Xin and Zhang [62, 63] established some new a priori one-sided supernorm and space-time higher-norm estimates on the first-order derivatives. Also, using viscous approximate solutions method and energy estimate, they obtained the existence and uniqueness of global weak solutions to the CH equation (1.2). Related works can be found in the literatures, such as [31, 32, 34].

• If $c = 1, N = 1, b = 3$ and $a = 4$, then (1.1) reads the DP equation

$$\begin{cases} u_t - u_{txx} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0, x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x), x \in \mathbb{R}. \end{cases} \tag{1.3}$$

DP equation (1.3) was obtained by the method of asymptotic integrability in Ref. [22, 24]. (1.3) arises from the asymptotic theory of shallow water waves [21] and has peakon and multi-peakon solutions, see for instance [22, 50, 51], as well as shock-type solutions in Ref. [49]. DP equation (1.3) is also an integrable system with quadratic nonlinearity, but with 3×3 Lax pairs [22] a bi-Hamiltonian structure, and an infinite hierarchy of symmetries and conservation laws. In [55], we know that DP equation (1.3) enjoys only one local Hamiltonian structure and second Hamiltonian structure

$$m_t = B_0\frac{\delta H_{-1}}{\delta m} = B_1\frac{\delta H_0}{\delta m},$$
$$B_0 = \mathcal{L}(4-\partial_x^2), B_1 = (\partial_x m + 3m\partial_x)\mathcal{L}^{-1}(2\partial_x m + 3m\partial_x),$$
$$H_1 = -\frac{1}{6}\int u^3 dx, H_2 = -\frac{1}{2}\int m dx,$$

whose compatibility condition was demonstrated in Ref. [44]. The local existence, global existence and blow-up phenomena of the DP equation was investigated, see for instance [25,26,37,40,64].

• If $c = 1, N = 2, b = 3$ and $a = 4$, then (1.1) becomes the Novikov equation

$$\begin{cases} u_t - u_{txx} + 4u^2u_x - 3u_xu_{xx} - uu_{xxx} = 0, x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x), x \in \mathbb{R}. \end{cases} \tag{1.4}$$

Novikov equation (1.4) was obtained by Novikov [54] in a symmetry classification of nonlocal partial differential equation with cubic nonlinearity. In [53], using the perturbative symmetry approach and formal recursion operator technique, Mikhailov and Novikov established the existence of infinite hierarchies of higher symmetries and local conservation laws. On the other hand, Hone and Wang [45] recently found a matrix Lax-pair representation of the Novikov equation (1.4). In particular, they demonstrated that Novikov equation (1.4) arises as a zero curvature equation

$$F_t - G_x + [F,G] = 0,$$

which is the compatibility condition for the linear system

$$\Psi_x = F(m,\lambda)\Psi, \quad \Psi_t = G(m,\lambda)\Psi,$$

where $m = u - u_{xx}$ and the matrices F and G are defined by

$$F = \begin{pmatrix} 0 & \lambda m & 1 \\ 0 & 0 & \lambda m \\ 1 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} \frac{1}{3\lambda^2} - uu_x & \frac{1}{\lambda}u_x - \lambda u^2 m & u_x^2 \\ \frac{1}{\lambda}u & -\frac{2}{3\lambda^2} & -\frac{1}{\lambda}u_x - \lambda u^2 m \\ -u^2 & \frac{1}{\lambda}u & \frac{1}{3\lambda^2} + uu_x \end{pmatrix}.$$

Also, using this matrix Lax-pair representation, Hone and Wang [45] demonstrated how the Novikov equation (1.4) is related by a reciprocal transformation to a negative flow in the Sawada Kotera hierarchy. Moreover, they proved that the Novikov equation (1.4) enjoys infinitely many conservation law, among which, the most crucial ones given by

$$E(u) \triangleq \int_{\mathbb{R}} (u^2(t,x) + u_x^2(t,x))dx \tag{1.5}$$

and

$$F(u) \triangleq \int_{\mathbb{R}} (u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4)dx. \tag{1.6}$$

It is well known that the Novikov equation (1.4) is also integrable peakon model with 3×3 Lax pairs and the peakon solution $u(x,t) = \sqrt{\omega}e^{-|x-\omega t|}$ with $\omega > 0$. The great difference between the Novikov equation (1.4) and the CH (1.2), DP (1.3) is that the former one has cubic nonlinearity and the latter ones have quadratic nonlinearity. Novikov equation (1.4) is among the class of integrable equations with the Lax pair given in Ref. [45] as

$$\begin{cases} \Psi_{xxx} = \Psi_x + \lambda m^2\Psi + \frac{2m_x}{m}\Psi_{xx} + \frac{mm_{xx}-2m_x^2}{m^2}\Psi_x, \\ \Psi_t = \frac{u}{\lambda_m}\Psi_{xx} - \frac{mu_x+um_x}{m^2}\Psi_x - u^2\Psi_x. \end{cases}$$

Moreover, they obtained a 3×3 matrix Lax pair representation to the Novikov equation (1.4). Also, they found that the Novikov equation (1.4) admits a bi-Hamiltonian structure

$$m_\tau = J_2\frac{\delta H_1}{\delta m} = J_1\frac{\delta H_2}{\delta m}$$

with the Hamiltonian operators

$$J_2 = -2(3m\partial_x + 2\partial_x m)(4\partial_x - \partial_x^3)^{-1}(3m\partial_x + \partial_x m),$$
$$J_1 = (1-\partial_x^2)\frac{1}{m}\partial_x\frac{1}{m}(1-\partial_x^2),$$

and the corresponding Hamiltonians

$$H_1 = \frac{1}{3}\int (m^{-\frac{8}{3}}\partial_x^2 m + 9m^{-\frac{2}{3}})dx,\ \ H_2 = \frac{1}{8}\int (u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4)dx.$$

Related works for the Novikov equation (1.4) can be found in the literatures, including the local existence, global existence and blow-up phenomena, such as [17, 38, 59–61, 67, 68]. It is worth mentioning that Ji and Zhou [47] investigated some properties for the solutions to NE (1.4) with a weakly dissipative effect, including the local well-posedness, wave breaking, blow-up rate, global existence and uniqueness. More precisely, firstly, using Kato's theorem, they established the local well-posedness. Then, using the time dependent conserved quantity and the commutator estimate technique, we established the wave breaking mechanism of solutions to NE (1.4). In particular, using the characteristic method and differential inequality technique, they showed two sufficient conditions on the initial datum for the occurrence of wave breaking, and established the upper bound of the maximal existence time and the blow-up rate. Finally, they proved the global existence and uniqueness of solutions to NE (1.4).

Due to the singularity of strong solutions in finite time, we are forced to study weak solutions. In particular, in order to go beyond the breaking wave (i.e, the wave profile remains bounded but its slope becomes unbounded in finite time) [11, 14, 15], if one considers global weak solution, it is natural to consider Hölder continuous solution, for instance, H^1 solution for CH (1.2). It is well known that there are two methods to deal with the global existence of weak solutions to CH (1.2), Novikov equation (1.4) and (1.1). One method is the vanishing viscosity technique, see Ref. [62, 63]. The other method is to introduce a new semi-linear system on new characteristic coordinates, see Ref. [3, 4, 12, 58, 72, 76].

Also, it is well known that wave breaking is a common phenomenon, in Ref. [10, 72]. To further understand the wave breaking phenomena, we found that some works with respect to wave breaking could be divided into three different categories:
•(1) before wave breaking,
•(2) during wave breaking,
•(3) after wave breaking.
Based on the above three categories, however, there are still many issues that need to be addressed, in particular for question (3). After wave-breaking, there exist peakons and the multi-peakon solutions to the CH equation (1.2), Novikov equation (1.4) and (1.1). Let us recall known results with these issues. Owing to the different wave speed for the CH equation (1.2), Novikov equation (1.4) and (1.1), the authors established the existence of a different peaked solutions [33, 36]. On one hand, it is well known that (1.2) admits the multi-peakon solutions [16]

$$u(t,x) = \sum_{i=1}^{n} p_i(t)e^{-|x-q_i(t)|},$$

where $p_i(t)$ and $q_i(t)$ satisfy the Hamiltonian system

$$\begin{cases} \frac{dp_i}{dt} = \sum\limits_{j\neq i} p_i p_j sgn(q_i - q_j)e^{-|q_i-q_j|} = -\frac{\partial H}{\partial q_i}, \\ \frac{dq_i}{dt} = \sum\limits_{j} p_j e^{-|q_i-q_j|-|q_i-q_k|} = \frac{\partial H}{\partial p_i}, \end{cases}$$

with the Hamiltonian

$$H = \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j e^{-|q_i-q_j|}.$$

In [43], Holden and Raynaud demonstrated the rigorous analysis for the systems of $p_i(t)$ and $q_i(t)$. In [33], they obtained the solutions, containing infinite many peaked solitary waves. From [33, 34], we know that peakons are a kind of weak solutions, and have a feature that is characteristic for the waves of great height, i.e, waves of the largest amplitude that are exact solutions of the governing equations for water waves. On the other hand, in [45], we know that, one of the most crucial features of the Novikov equation (1.4) is the existence of peakon and anti-peakon solutions, which are peaked traveling waves with a discontinuous derivative at the crest. Peakon and anti-peakon solutions are explicitly given by

$$\pm\phi_c(x-ct) = \pm\sqrt{c}\phi(x-ct) \triangleq \pm\sqrt{c}e^{-|x-ct|}, c > 0.$$

Furthermore, in [45], Hone and Wang demonstrated that Novikov equation (1.4) admits the multi-peakons solutions. That is, for any given number $n \in \mathbb{N}$, let $\vec{q} = (q_1, \cdots, q_n)$ and $\vec{p} = (p_1, \cdots, p_n)$ be the position and momenta vectors, respectively. Then, one has the n-peaked traveling wave solution

$$u(t,x)=\sum_{i=1}^{n}p_i(t)e^{-|x-q_i(t)|}$$

on the line, where p_i and q_i verify the following $2n$ dimensional differential equations

$$\begin{cases} \frac{dq_i}{dt}=u^2(q_i)=\sum\limits_{j,k=1}^{n}p_jp_ke^{-|q_i-q_j|-|q_i-q_k|}, \\ \frac{dp_i}{dt}=-p_iu(q_i)u_x(q_i)=p_i\sum\limits_{j,k=1}^{n}p_jp_k sgn(q_i-q_j)e^{-|q_i-q_j|-|q_i-q_k|}. \end{cases}$$

Clearly, after wave breaking, it is curial to comprehend this type of weak solution to the wave propagation in the shallow water. There are two aspects to the weak solution, that is, existence and uniqueness. After wave-breaking as either global conservative or global dissipative weak solution, the solution to the CH (1.2) and Novikov equation (1.4) can be uniquely continued, respectively, was established in [3,4,6,12,58,74,76].

In [1,65], they demonstrated (1.1) admits the single peakon and multi-peakon solutions if and only if $a\geq 0$ and $c\neq 0$. That is, the single peakon solutions for (1.1) enjoy the form as follows:

$$u=u(x-\upsilon t)=(\upsilon/c)^{1/N}e^{-|x-\upsilon t|},\quad a=b+c,\quad c\neq 0,\quad N\geq 1,$$

where $\upsilon=constant$ is the wave speed. Also, they obtained the N-peakon solutions ($N\geq 2$) for (1.1)

$$u(t,x)=\sum_{i=1}^{N}\alpha_i(t)e^{-|x-\beta_i(t)|},\quad N=2,\ldots,$$

where the amplitudes $\alpha_i(t)$ and positions $\beta_i(t)$ satisfy a Hamiltonian system of ordinary differential equations (ODEs)

$$\begin{cases} \alpha_i{}'=(b-ac)\alpha_i(\alpha_j+\sum\limits_{j=1,j\neq i}^{N}\alpha_je^{-|\beta_i-\beta_j|})^{a-1}\sum\limits_{k=1,k\neq i}^{N}sgn(\beta_i-\beta_k)\alpha_ke^{-|\beta_i-\beta_k|}, \\ \beta_i{}'=c(\alpha_j+\sum\limits_{j=1,j\neq i}^{N}\alpha_je^{-|\beta_i-\beta_j|})^{a}, \end{cases} \tag{1.7}$$

Indeed, the above ODEs (1.7) is equivalent to the following system

$$\begin{cases} \alpha_i{}'=\{\alpha_i,H\}, \\ \beta_i{}'=\{\beta_i,H\}, \end{cases}$$

where $\{\ \}$ is the Poisson bracket and H is the Hamiltonian function given by

$$H=\frac{1}{2}\sum_{j,k=1}^{N}\alpha_j\alpha_ke^{-|\beta_i-\beta_j|},\ \ i=1,\ldots,N.$$

It is worth mentioning that Zhang et al. [72] investigated the global energy conservation solution for (1.1). More precisely, using both the lower order and the higher order energy conservation laws, as well as the characteristic method, they established the global existence and uniqueness of the Hölder continuous energy weak solution to (1.1) in the energy space $H^1(\mathbb{R})\times W^{1,2N}(\mathbb{R})$. Also, they demonstrated that a very natural and interesting problem is to study how the regularity of solution changes with respect to N. Namely, they established Hölder continuous energy weak solutions with the exponent $1-\frac{1}{2N}$. This result precisely shows how the regularity of solution changes with respect to the power of nonlinear wave speed N, and it reveals an intrinsic relation between Camassa-Holm equation (linear wave speed u^1), Novikov equation (quadratic wave speed u^2) and the $N-abc$ family of Camassa-Holm type equation (wave speed u^N).

Remark 1.1. *We observe that the general the N-peakon solutions ODEs (1.7) yields to the well-known multi-peakon systems for the b-family equation, when $(p,a,b,c)=(1,b+1,b,1)$, which includes the CH equation (1.2) and the DP equation (1.3), when $b=2$ and $b=3$, respectively, as well as for the Novikov equation (1.4), when $(p,a,b,c)=(2,4,3,1)$.*

In this paper, we mainly focus on the generic regularity of conservative solutions to (1.1). The generic property is a property which is satisfied by almost all elements of the whole set. Considering our problem here, generic regularity is the regularity of the solutions solved from an open and dense subset in the space of initial data. Generic property is an interesting problem with respect to the hyperbolic conservation laws. It has been such a long time that many researchers are interested in this issues. The original result was investigated by Schaeffer in [56], which demonstrated that for $1-D$ conservation law, the generic solutions are piecewise smooth, with finitely shocks in a bounded domain in the (t, x) plane. The proof relies on the Hopf-lax representation formula. Later on, in [20], using the generalized characteristics method, Dafermos and Geng obtained the uniqueness and regularity of solutions of the Cauchy problem for a special system of hyperbolic conservation laws. Also, they obtained a similar result for the 2×2 temple class systems. However, for the general $n \times n$ system with $n \geq 3$, Caravenna and Spinolo [18] exhibited an explicit counter example and proved that the generic property does not hold true. The proof mainly relies on careful interaction estimates and uses fine properties of the wave front-tracking approximation. Recently, Bressan and his co-workers [5, 8, 9] investigated the generic property and singularity behavior for the nonlinear variational wave equation. In [5], they established that for an open dense set of C^3 initial data, the solution is piecewise smooth in the (t, x) plane, while the gradient u_x will blow up along finite many characteristic curves. For an open dense set of initial data, the authors in [8] provided a detailed asymptotic description of the solution in a neighborhood of each singular point, where $|u_x| \to \infty$ and established the different structure of conservative and dissipative solutions. Li and Zhang [52] obtained the generic property and the singular behavior of (1.2) and the two-component CH equation in a sense of Baire category. Later on, Yang [66] established the generic regularity of energy conservative solutions to (1.2) with rotation effect. However, because of the energy concentration when the finite time gradient blow-up occurs, the solution flow is in general not Lipschitz continuous with respect to the natural H^1 distance. In order to establish the Lipschitz property, one needs to introduce some new metric. Recently, Cai et al. [13] studied the generic property of conservative solutions to the Hunter-Saxton type equations and presented a new way to establish a Finsler type metric, which renders the solution flow uniformly Lipschitz continuous on bounded subsets of $H^1(\mathbb{R}^+)$. Also, the distance will be determined by the minimum cost to transport an energy measure from one solution to the other. Quite recently, benefited from some ideas of [52], He et al. [41] the generic properties of the Novikov equation (1.4).

Our approach is inspired by the work on the CH (1.2) in [52, 66] and on the integrable Novikov equation (1.4) in [41], however the difficulty is that (1.4) includes the nonlocal higher nonlinear terms P_1 and P_2 in the following equivalent form (2.4)(below in section 2). The situations should oblige us to needs the fine analysis and estimates to treat with the Lipschitz continuity for (2.4). More precisely, we will need to overcome some difficulties as follows:

• For CH (1.2) in [52, 66] (that is, $N = 1$ in (1.1) or an equivalent form (2.4)), they obtained the lower conservation law (1.5) and established the global existence of solution $u(\cdot, t) \in W^{1,2}$ (or $C^{0,\frac{1}{2}}$ by Sobolev embedding). Unfortunately, this issue is invalid if there are the nonlocal higher nonlinear term P_2 in (2.4). In fact, the energy (1.5) is not enough to control the cubic nonlinearity u_x^3 in P_2 in (2.4). For that purpose, we have to establish the higher order energy balance law (2.15)(see below). Using Sobolev inequality and Gagliado-Nirenberg interpolation inequality, we can control the cubic nonlinearity u_x^3 in P_2 in (2.4).

• For Novikov equation (1.4) in [41] (that is, $N = 2$ in (1.1) or an equivalent form (2.4)), we find that another energy conservation law (1.6) including u_x^4 is available. Using both the lower order (1.5) (on u_x^2) and the higher order (1.6) (on u_x^4) energy conservation laws, as well as the characteristic method framework first established in [3], Chen et al. [12] and Zhou et al. [74] proved the global existence of solution $u(\cdot, t) \in W^{1,4}$ (or $C^{0,\frac{3}{4}}$ by Sobolev embedding). Indeed, a very interesting observation on existing results for CH (1.2) and Novikov equation (1.4) shows global well-posedness in different spaces. A very natural and interesting problem is to investigate how the regularity of solution for (1.1) changes with respect to the parameter N (that is, the order of nonlocal higher nonlinear terms P_1 and P_2 in (2.4)). For the high-order case, the existing methods fail. For that purpose, we will construct energy conservative Hölder continuous solution with the exponent $1 - \frac{1}{2N}$ for (1.1) and make some efforts to establish a uniform estimate by using the higher order energy balance law (2.15). This result precisely shows how the regularity of solution changes with respect to the parameter N, so it reveals an intrinsic relation between CH (1.2)(linear wave speed u^1), Novikov equation (1.4) (quadratic wave speed u^2) and the general equation (1.1) (wave speed u^N).

To the best of our knowledge, the generic properties of (1.1) has not been reported yet. Because of the difference nonlinearity of wave speed, the singularity behaviour is various. In our contribution, we give an exactly proof to show that why the singularity is various between the CH (1.2) and the Novikov equation (1.4).

The paper is organized as follows: In section 2, we recall some basic definitions and the related results with respect to the conservative solutions of (1.1). In section 3, we show a perturbation lemma and construction families of perturbed solutions to the semilinear system. In section 4, we investigate the generic property of (1.1). In section 5 we demonstrate the asymptotic behavior for generic singularity.

We are now in the position to state the main results as follows.

Theorem 1.2. *(Generic property) For any $T>0$ fixed, there exists an open dense set of initial data*

$$\mathscr{D} \subset (C^3(\mathbb{R}) \cap H^1(\mathbb{R}) \cap W^{1,2N}(\mathbb{R}))$$

such that for $u_0 \in \mathscr{D}$, the energy conservative solutions $u = u(t,x)$ of (1.1) is differentiable in the complement of finitely many characteristic curves γ_i, within the domain $[0,T]\times\mathbb{R}$.

Based on the above generic regularity, we demonstrate the asymptotic description of the solution in a neighborhood of each singular point, where $|u_x| \to \infty$.

Theorem 1.3. *(Asymptotic behavior for generic singularity) Consider generic initial data $u_0 \in \mathscr{D}$ as in Theorem 1.2 with $u_0 \in C^\infty(\mathbb{R})$. Let $u = u(x,t)$ be the solution to the original equation (1.1) or an equivalent form (2.4) and (u, v, w, x, t) is the corresponding solution of the semilinear system (2.44) and (2.45). Consider a singular point $P = (t_0, Y_0)$ where$v = \pi$, and set $(x_0, t_0) = (x(t_0, Y_0), t(t_0, Y_0))$. Generically, at the singular point $P = (t_0, Y_0)$, u has parameteric expression as follows:*

(1) If P is a point of case 1, i.e. $v_Y = 0$ and $v_{YY} \neq 0$, $v = \pi$, $v_{YY} \neq 0$, then

$$u(t,x) = A(x-x_0)^{1-\frac{1}{2N}} + B(t-t_0) + O(1)(|t-t_0|^2 + |x-x_0|^{1-\frac{1}{2N+1}}) \tag{1.8}$$

for some constant A, B.

(2) If P is a point of case 2 , i.e. $v_Y \neq 0$ and $v_{YY} = 0$, $v = \pi$, $v_{YY} \neq 0$, then

$$u(t,x) = A(x-x_0)^{\frac{2N}{2N+1}} + B(t-t_0) + O(1)(|t-t_0|^2 + |x-x_0|) \tag{1.9}$$

for some constant A, B.

2. Preliminaries

In this section, to further our analysis, we present some notations, including transversality and genericity, the basic definitions and results for conservation solutions for (1.1).

Definition 2.1. *[2, 30, 48, 57] (Map transverse to a submanifold) Let $F : X \to Y$ be a smooth map from manifold X to manifold Y. W is a submanifold of Y. We say F is transverse to W at a point$x \in X$, denoted by $F \pitchfork_x W$, if*

- *either $F(x) \notin W$,*
- *or $F(x) \in W$ and $T_{F(x)}Y = (dF)_x(T_xX) + T_{F(x)}W$. Here T_xX means the tangent space of X at point x.*

If $F \pitchfork_x W$ for every $x \in X$, we say F is transverse to W, and denote as $F \pitchfork W$.

Definition 2.2. *[2, 30, 48, 57] Let $F : X \to Y$ be a smooth map from manifold X to Y. A point $y \in Y$ is a regular value if for every $x \in X$ one has $T_yY = (dF)_x(T_xX)$. In the special case where $W = \{y\}$ consists of a single point, $F \pitchfork W$ if and only if y is a regular value of F.*

Theorem 2.3. *[2, 30, 48, 57] (Thom's transversality theorem) Let X, Θ and Y be smooth manifolds and W a submanifolds of Y. Let $\theta \to \phi^\theta$ be a smooth map that to each $\theta \in \Theta$ associates a function $\phi^\theta \in C^\infty(X,Y)$, and define $\Phi : X \times \Theta \to Y$ by setting $\Phi(x,\theta) = \phi^\theta(x)$. If $\Phi \pitchfork W$ then the set $\theta \in \Theta; \phi^\theta \pitchfork W$ is dense in Θ.*

Lemma 2.4. *[52] Consider an ODE system*

$$\frac{d}{dt}u^\epsilon = f(u^\epsilon), \quad u^\epsilon(0) = u_0 + \epsilon_1 v_1 + \cdots + \epsilon_m v_m, \tag{2.1}$$

where $u^\epsilon(t) : \mathbb{R} \to \mathbb{R}$, f is a Lipschitz function. The system is well posed in $[0,T)$. Assume the matrix

$$D_\epsilon u_0^\epsilon = (v_1, v_2, ..., v_m) \in \mathbb{R}^{n\times m}, \tag{2.2}$$

and the rank of this matrix is

$$rank(D_\epsilon u_0^\epsilon) = k. \tag{2.3}$$

Then for any $t \in [0,T)$, $rank(D_\epsilon u^\epsilon) = k$.

In what follows, we will recall several fundamental results with respect to (1.1). First, we can rewrite (1.1) in the equivalent nonlocal form

$$\begin{cases} u_t + cu^N u_x + \partial_x P_1 + P_2 = 0, x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x), x \in \mathbb{R}, \end{cases} \tag{2.4}$$

in the Sobolev space $H^1(\mathbb{R}) \times W^{1,2N}(\mathbb{R})$. Here

$$P_1 = p * (\frac{b}{N+1}u^{N+1} + \frac{3cN-b}{2}u^{N-1}u_x^2), \quad P_2 = p * [\frac{(N-1)(b-cN)}{2}u^{N-2}u_x^3]$$

with $p(x) = \frac{1}{2}e^{-|x|}$, and we demonstrate the definition and the global existence and uniqueness of the Hölder continuous energy weak solution as follows.

Definition 2.5. *[72] Let $c = 1$, $b = N+1$ and $a = b + c$. The energy conservative solution $u(t,x)$ of the Cauchy problem (1.1) or (2.4) enjoys the following properties.*

Property 1. *The map $t \to u(t,\cdot)$ is Lipschitz continuous from $\mathbb{R}$ into $L^{2N}(\mathbb{R})$ with $u(t,\cdot) \in L^{2N}(\mathbb{R})$, $\forall t \geq 0$.*

Property 2. *The solution $u = u(t,x)$ satisfies the initial condition $u_0 \in H^1(\mathbb{R}) \times W^{1,2N}(\mathbb{R})$ and*

$$0 = \int\int_{\Sigma} \left\{ -u_x \cdot (\psi_t + u^N \psi_x) + [-u^{N+1} - \frac{2N-1}{2}u^{N-1}u_x^2 + (P_1 + \partial_x P_2)] \cdot \psi \right\} dxdt$$
$$+ \int_{\mathbb{R}} u_{0,x}\psi(0,x)dx,$$

for any test function $\psi(t,x) \in C_c^1(\Sigma)$, where $\Sigma \triangleq \{(t,x) \mid t \in [0,+\infty), \ \ x \in \mathbb{R}\}$.

Property 3. *The solution $u = u(t,x)$ is conservative, that is, the higher order energy balance law (2.15) is satisfied in the following sense.*

There exists a family of positive Radon measures $\mu_{(t)}$, depending continuously on time and with respect to the topology of weak convergence of measures. For every $t \in \mathbb{R}^+$, $\mu_{(t)}$ is the sum of an absolutely continuous part and a singular part, and the absolutely continuous part of has density $u_x^{2N}(t,\cdot)$ with respect to the Lebesgue measure, which provides a measure-valued solution to the balance law

$$\int_{\mathbb{R}+} \left\{ \int_{\mathbb{R}} (\psi_t + u^N\psi_x) d\mu(t) + \int_{\mathbb{R}} 2Nu_x^{2N-1}[u^{N+1} - (P_1 + \partial_x P_2)]\psi dx \right\} dt + \int_{\mathbb{R}} u_{0,x}^{2N}\psi(0,x)dx = 0,$$

for any test function $\psi(t,x) \in C_c^1(\Sigma)$.

Theorem 2.6. *[72] Let the initial data $u_0 \in H^1(\mathbb{R}) \times W^{1,2N}(\mathbb{R})$ be an absolutely continuous function on variable x. Then there exists a uniquely global energy conservative solution $u(t,x)$ for the Cauchy problem (1.1) or (2.4) in the sense of Definition 2.5. Moreover, the solution satisfies the following properties.*

Property 1. *$u = u(t,x)$ is uniformly Hölder continuous with exponent $1 - \frac{1}{2N}$ on both t and x.*

Property 2. *The energy density $u^2 + u_x^2$ is almost conserved, that is,*

$$E(t) = \|u(t)\|_{H^1(\mathbb{R})} = \|u_0\|_{H^1(\mathbb{R})}, \ \ for \ \ t \notin \mathbb{S}, \ \ E(t) < E(0), \ \ for \ \ t \in \mathbb{S}.$$

Property 3. *There exists a null set $\mathbb{S}$ with $meas(\mathbb{S}) = 0$, such that for every $t \notin \mathbb{S}$, the measure $\mu_{(t)}$ is absolutely continuous and has density $u_x^{2N}(t,\cdot)$ with respect to the Lebesgue measure.*

Property 4. *The solution is continuously depending on the initial data, that is, continuous dependence property holds. More precisely, for a sequence of initial data $u_{0,n}$, such that*

$$\|u_{0,n} - u_0\|_{H^1(\mathbb{R})\times W^{1,2N}(\mathbb{R})} \to 0, \ \ as \ \ n \to \infty.$$

Then the corresponding solutions $u_n(t,x)$ converge to $u(t,x)$ uniformly for (t,x) in any bounded sets.

For smooth solutions, differentiating the first equation in (2.4) about the variable x, and using the fact

$$\partial_x^2 p * f = p * f - f,$$

one has

$$u_{xt} + cu^N u_{xx} + P_1 + \partial_x P_2 + \frac{b-cN}{2}u^{N-1}u_x^2 - \frac{b}{N+1}u^{N+1} = 0. \tag{2.5}$$

Multiplying the first equation in (2.4) by u and (2.5) by u_x, we establish the two lower order energy conservation laws with source term P_1 and P_2

$$(\frac{u^2}{2})_t + (\frac{c}{N+2}u^{N+2} + uP_1)_x + uP_2 = u_xP_1, \tag{2.6}$$

and

$$\begin{aligned}(\frac{u_x^2}{2})_t + [\frac{cu^N u_x^2}{2} - \frac{b}{(N+1)(N+2)}u^{N+2} + u\partial_x P_2]_x - uP_2 \\ - \frac{N[b-c(N+1)]}{2}u^{N-1}u_x^3 = -u_xP_1,\end{aligned} \tag{2.7}$$

respectively. Next, our aim is to obtain some energy conservation laws. For that purpose, we must make technical assumption with respect to the parameters N, a, b and c as follows:

$$a = b + c, \quad b = c(N+1). \tag{2.8}$$

In fact, it follows from (2.7) and (2.8) that

$$(\frac{u_x^2}{2})_t + [\frac{cu^N u_x^2}{2} - \frac{b}{(N+1)(N+2)}u^{N+2} + u\partial_x P_2]_x - uP_2 = -u_xP_1. \tag{2.9}$$

After a scaling transformation $t \longrightarrow \frac{t}{c}$, we take $c = 1$. It follows from (2.6), (2.8) and (2.9) that

$$(\frac{u^2}{2})_t + (\frac{1}{N+2}u^{N+2} + uP_1)_x + uP_2 = u_xP_1, \tag{2.10}$$

and

$$(\frac{u_x^2}{2})_t + (\frac{u^N u_x^2}{2} - \frac{1}{N+2}u^{N+2} + u\partial_x P_2)_x - uP_2 = -u_xP_1. \tag{2.11}$$

Thus, combining (2.10) with (2.11), we establish the lower conservation law

$$(\frac{u^2+u_x^2}{2})_t + (\frac{u^N u_x^2}{2} + uP_1 + u\partial_x P_2)_x = 0. \tag{2.12}$$

Integrating both sides of (2.12) with respect to the variable x, we conclude that the total lower order energy

$$E(t) \triangleq \int_{\mathbb{R}} [u^2(t,x) + u_x^2(t,x)]dx$$

is conserved with respect to the time variable t, i.e,

$$E(t) = E(0) \triangleq E_0. \tag{2.13}$$

In what follows, our aim is to obtain the higher order energy balance law. For that purpose, multiplying the first equation in (2.4) by $2Nu^{2N-1}$, one has

$$(u^{2N})_t + (\frac{2}{3}u^{3N} + 2Nu^{2N-1}P_1)_x = -2Nu^{2N-1}P_2 + 2N(2N-1)u^{2N-2}u_xP_1. \tag{2.14}$$

Multiplying (2.5) by $2Nu_x^{2N-1}$, we have

$$(u_x^{2N})_t + (u^N u_x^{2N})_x = 2Nu^{N+1}u_x^{2N-1} - 2Nu_x^{2N-1}(P_1 + \partial_x P_2). \tag{2.15}$$

It follows from (2.13) and the Sobolev inequality that

$$\|u\|_{L^\infty}^2 \leq \frac{1}{2}\|u\|_{H^1}^2 = \frac{1}{2}E_0. \tag{2.16}$$

We now demonstrate the uniform bounds on P_1, $\partial_x P_1$, P_2 and $\partial_x P_2$ given by (2.4). First, we will establish the uniform bounds on P_1 and $\partial_x P_1$. It follows from (2.4), (2.13) and (2.16) that

$$\begin{aligned}\|P_1(t)\|_{L^\infty},\ \|\partial_x P_1(t)\|_{L^\infty} \leq C_1 E_0^{\frac{N+1}{2}},\\ \|P_1(t)\|_{L^\infty},\ \|\partial_x P_1(t)\|_{L^i} \leq C_2 E_0^{\frac{N+1}{2}},\end{aligned} \tag{2.17}$$

for some positive constants C_1, C_2 and any $i \geq 1$.

Using the higher order energy balance law (2.13) and Gagliado-Nirenberg interpolation inequality, we establish a uniform estimate on $\|u_x\|_{L^{2N}}$, in order to obtain a uniform bound on P_2 and $\partial_x P_2$, which both contain u_x^3 in (2.4). Actually, from (2.4), (2.15) and (2.16), we arrive at

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} u_x^{2N}dx &= \int_{\mathbb{R}}[2Nu^{N+1}u_x^{2N-1} - 2Nu_x^{2N-1}(P_1+\partial_x P_2)]dx\\
&\le 2N(\|u\|_{L^\infty}^{N+1} + \|P_1\|_{L^\infty})\int_{\mathbb{R}}|u_x|^{2N-1}dx\\
&\quad + 2N\|\frac{1}{2}e^{|-x|}\|_{L^\infty}\cdot\frac{N-1}{2}\|u\|_{L^\infty}^{N-2}\int_{\mathbb{R}}|u_x|^3dx\cdot\int_{\mathbb{R}}|u_x|^{2N-1}dx\\
&\le 2N(E_0^{\frac{N+1}{2}} + C_1E_0^{\frac{N+1}{2}})\int_{\mathbb{R}}|u_x|^{2N-1}dx\\
&\quad + 2N\cdot\frac{1}{2}\cdot\frac{N-1}{2}E_0^{\frac{N-2}{2}}\int_{\mathbb{R}}|u_x|^3dx\cdot\int_{\mathbb{R}}|u_x|^{2N-1}dx\\
&\triangleq C_3(E_0)\int_{\mathbb{R}}|u_x|^{2N-1}dx + C_4(E_0)\int_{\mathbb{R}}|u_x|^3dx\cdot\int_{\mathbb{R}}|u_x|^{2N-1}dx,
\end{aligned}
\tag{2.18}
$$

for some positive constants

$$C_3(E_0) = 2N(1+C_1)E_0^{\frac{N+1}{2}},\quad C_4(E_0) = \frac{N(N-1)}{2}E_0^{\frac{N-2}{2}},$$

depending only upon E_0.

In what follows, we will estimate the second term in (2.18). It follows from Gagliado-Nirenberg interpolation inequality that

$$\|u_x\|_{L^3} \le C_5\|u_x\|_{L^2}^{\frac{2N-3}{3(N-1)}}\|u_x\|_{L^{2N}}^{\frac{N}{3(N-1)}},\tag{2.19}$$

$$\|u_x\|_{L^{2N-1}} \le C_6\|u_x\|_{L^2}^{\frac{1}{(N-1)(2N-1)}}\|u_x\|_{L^{2N}}^{\frac{N(2N-3)}{(N-1)(2N-1)}},\tag{2.20}$$

for some positive constants C_5 and C_6. Using (2.19) and (2.20), we obtain

$$\int_{\mathbb{R}}|u_x|^3dx \le C_5(\int_{\mathbb{R}}|u_x|^2dx)^{\frac{2N-3}{2(N-1)}}(\int_{\mathbb{R}}|u_x|^{2N}dx)^{\frac{1}{2(N-1)}},\tag{2.21}$$

$$\int_{\mathbb{R}}|u_x|^{2N-1}dx \le C_6(\int_{\mathbb{R}}|u_x|^2dx)^{\frac{1}{2(N-1)}}(\int_{\mathbb{R}}|u_x|^{2N}dx)^{\frac{2N-3}{2(N-1)}}.\tag{2.22}$$

By (2.21) and (2.22), one has

$$\int_{\mathbb{R}}|u_x|^3dx\cdot\int_{\mathbb{R}}|u_x|^{2N-1}dx \le C_7\int_{\mathbb{R}}|u_x|^2dx\cdot\int_{\mathbb{R}}|u_x|^{2N}dx,\tag{2.23}$$

for some positive constant C_7. It follows from (2.18), (2.22) and (2.23) that

$$\frac{d}{dt}\int_{\mathbb{R}}u_x^{2N}dx \le C_3(E_0)\int_{\mathbb{R}}|u_x|^{2N-1}dx + C_4(E_0)\int_{\mathbb{R}}|u_x|^2dx\cdot\int_{\mathbb{R}}|u_x|^{2N}dx,$$

that is,

$$\frac{d}{dt}\int_{\mathbb{R}}u_x^{2N}dx \le C_8(E_0)(C_9(E_0) + \int_{\mathbb{R}}|u_x|^{2N}dx),\tag{2.24}$$

for some positive constants $C_8(E_0)$ and $C_9(E_0)$, which depend only upon E_0. Using Gronwall's inequality and (2.24), for any time $t \in [0,T]$, one has

$$\int_{\mathbb{R}}u_x^{2N}dx \le e^{C_8(E_0)}((C_9(E_0) + \int_{\mathbb{R}}|u_x|^{2N}dx(0))(e^{C_8(E_0)}-1) \triangleq \Xi(T).\tag{2.25}$$

Then, using (2.13), (2.23) and (2.25), we can establish the bound

$$\int_{\mathbb{R}}u_x^3dx \le C_{10}(E_0,T)\tag{2.26}$$

with a positive constant C_{10} depending upon E_0 and T. Thus, using (2.4), (2.13), (2.16) and (2.26), for any given $t \in [0,T]$, $T>0$, we demonstrate the time dependent bounds as follows

$$\begin{aligned} &\|P_2(t)\|_{L^\infty},\ \|\partial_x P_2(t)\|_{L^\infty} \le C_{11}(E_0,T),\\ &\|P_2(t)\|_{L^\infty},\ \|\partial_x P_2(t)\|_{L^i} \le C_{12}(E_0,T), \end{aligned} \tag{2.27}$$

for some positive constants C_{11}, C_{12} and any $i \ge 1$.

Next, we will focus on establishing a semi-linear system for smooth solutions. For that purpose, we introduce the characteristic equation [71–73]

$$\begin{cases} \frac{dx(t)}{dt} = u^N(t, x(t,Y)),\\ x(0,Y) = x_0(Y). \end{cases} \tag{2.28}$$

For any fixed point (t,x), the characteristic curve passing through the point (t,x) is defined by $\tau \mapsto x(\tau;t,x)$. Using the energy density $(1+u_x^2)^N$, we define the characteristic coordinate $Y = Y(t,x)$,

$$Y \equiv Y(t,x) = \int_0^{x(0;t,x)} (1+u_x^2(0,x))^N dx. \tag{2.29}$$

Thus, we have

$$Y_t + u^N Y_x = 0,\ \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{2.30}$$

Also, we define $T = t$ to get the new coordinate (T,Y). It follows from (2.30) and the chain rule for derivatives that

$$\begin{cases} h_T = h_T(T_t + u^N T_x) + h_Y(Y_t + u^N Y_x) = h_t + u^N h_x,\\ h_x = h_T T_x + h_Y Y_x = h_Y Y_x, \end{cases}$$

where $h(T,Y) = H(t, Y(t,x))$ is a smooth function. That is,

$$\begin{cases} h_T = h_t + u^N h_x,\\ h_x = h_Y Y_x. \end{cases} \tag{2.31}$$

We introduce the new variables v and w as follows

$$v \triangleq 2\arctan u_x \ \ \textit{and}\ \ w \triangleq \frac{(1+u_x^2)^N}{Y_x} = (1+u_x^2)^N \frac{\partial x}{\partial Y} \tag{2.32}$$

with $u_x = u_x(T, x(T,Y))$. Using (2.32), we have

$$u_x = \tan\frac{v}{2},\ \ \frac{1}{1+u_x^2} = \cos^2\frac{v}{2},\ \ \frac{u_x^2}{1+u_x^2} = \sin^2\frac{v}{2},\ \ \frac{u_x}{1+u_x^2} = \frac{1}{2}\sin v, \tag{2.33}$$

and

$$\frac{\partial x}{\partial Y} = \frac{w}{(1+u_x^2)^N} = w\cos^{2N}\frac{v}{2}. \tag{2.34}$$

By (2.34), for any time $t = T$, one has

$$x(T,\overline{Y}) - x(T,Y) = \int_Y^{\overline{Y}} w\cos^{2N}\frac{v}{2}(T,s)ds. \tag{2.35}$$

Let $y = x(T,\overline{Y})$ and $x = x(T,Y)$. It follows from (2.33)-(2.35) that

$$\begin{aligned} P_1(Y) \triangleq P_1(T,Y) &= p * (u^{N+1} + \tfrac{2N-1}{2}u^{N-1}u_x^2)\\ &= \tfrac12 \int_{-\infty}^{+\infty} e^{-|y-x|}(u^{N+1} + \tfrac{2N-1}{2}u^{N-1}u_x^2)\tfrac{w}{(1+u_x^2)^N}dy\\ &= \tfrac12 \int_{-\infty}^{+\infty} e^{-|\int_Y^{\overline{Y}} w(s)\cos^{2N}\frac{v(s)}{2}ds|}[u^{N+1}\cos^{2N}\tfrac{v(\overline{Y})}{2}\\ &\quad + \tfrac{2N-1}{2}u^{N-1}\sin^2\tfrac{v(\overline{Y})}{2}\cos^{2N-2}\tfrac{v(\overline{Y})}{2}]w(\overline{Y})d\overline{Y}, \end{aligned} \tag{2.36}$$

$$\begin{aligned} \partial_x P_1(Y) \triangleq \partial_x P_1(T,Y) &= \partial_x p * (u^{N+1} + \tfrac{2N-1}{2}u^{N-1}u_x^2)\\ &= \tfrac12 \Big(\int_{x(T,Y)}^{+\infty} - \int_{-\infty}^{x(T,Y)}\Big)e^{-|y-x|}(u^{N+1} + \tfrac{2N-1}{2}u^{N-1}u_x^2)\tfrac{w}{(1+u_x^2)^N}dy\\ &= \tfrac12\Big(\int_Y^{+\infty} - \int_{-\infty}^{Y}\Big)e^{-|\int_Y^{\overline{Y}} w(s)\cos^{2N}\frac{v(s)}{2}ds|}[u^{N+1}\cos^{2N}\tfrac{v(\overline{Y})}{2}\\ &\quad + \tfrac{2N-1}{2}u^{N-1}\sin^2\tfrac{v(\overline{Y})}{2}\cos^{2N-2}\tfrac{v(\overline{Y})}{2}]w(\overline{Y})d\overline{Y}, \end{aligned} \tag{2.37}$$

$$
\begin{aligned}
P_2(Y) \triangleq P_2(T,Y) &= p * [\tfrac{N-1}{2}u^{N-2}u_x^3] \\
&= \tfrac{N-1}{4}\int_{-\infty}^{+\infty} e^{-|\int_Y^{\overline{Y}} w(s)\cos^{2N}\frac{v(s)}{2}ds|}[u^{N-1}\sin^3\tfrac{v(\overline{Y})}{2}\cos^{2N-3}\tfrac{v(\overline{Y})}{2}]w(\overline{Y})d\overline{Y},
\end{aligned}
\tag{2.38}
$$

and

$$
\begin{aligned}
\partial_x P_2(Y) \triangleq \partial_x P_2(T,Y) &= \partial_x p * [\tfrac{N-1}{2}u^{N-2}u_x^3] \\
&= \tfrac{N-1}{4}(\int_Y^{+\infty} - \int_{-\infty}^{Y})e^{-|\int_Y^{\overline{Y}} w(s)\cos^{2N}\frac{v(s)}{2}ds|}[u^{N-1}\sin^3\tfrac{v(\overline{Y})}{2}\cos^{2N-3}\tfrac{v(\overline{Y})}{2}]w(\overline{Y})d\overline{Y}.
\end{aligned}
\tag{2.39}
$$

In what follows, we obtain the evolution equations with respect to the unknown variables u, v and w under the new coordinate (T, Y). It follows from (2.5) and (2.31) that

$$
u_T(T,Y) = u_t(T,Y) + u^N(T,Y)u_x = -\partial_x P_1 - P_2. \tag{2.40}
$$

Using (2.5) (2.31) (2.32) and (2.33), we establish the equation with respect to the variable v

$$
\begin{aligned}
v_T &= \frac{2}{1+u_x^2}(u_{xt} + u^N u_{xx}) \\
&= \frac{2}{1+u_x^2}(-u^N u_{xx} - \frac{1}{2}u^{N-1}u_x^2 + u^{N+1} - P_1 - \partial_x P_2 + u^N u_{xx}) \\
&= -\frac{u_x^2}{1+u_x^2}u^{N-1} + 2\frac{1}{1+u_x^2}u^{N+1} - \frac{2}{1+u_x^2}(P_1 + \partial_x P_2) \\
&= -u^{N-1}\sin^2\frac{v}{2} + 2u^{N+1}\cos^2\frac{v}{2} - 2\cos^2\frac{v}{2}(P_1 + \partial_x P_2).
\end{aligned}
\tag{2.41}
$$

Now, we establish the equation with respect to the variable w. To do this, by (2.30), we arrive at

$$
Y_{tx} + u^N Y_{xx} = -Nu^{N-1}u_x Y_x. \tag{2.42}
$$

Using (2.5) (2.31) (2.33) and (2.42), we conclude

$$
\begin{aligned}
w_T &= \frac{1}{Y_x}[((1+u_x^2)^N)_T + (1+u_x^2)^N) \cdot (-\frac{Y_{xT}}{Y_x^2})] \\
&= \frac{1}{Y_x}[2Nu_x(1+u_x^2)^{N-1}(u_{xt} + u^N u_{xx}) - (1+u_x^2)^N) \cdot \frac{Y_{tx} + u^N Y_{xx}}{Y_x^2} \\
&= \frac{N}{Y_x}(1+u_x^2)^{N-1}[(u_x^2)_t + u^N(u_x^2)_x + (1+u_x^2)u^{N-1}u_x] \\
&= \frac{(1+u_x^2)^N}{Y_x} \cdot \frac{Nu^{N-1}u_x}{1+u_x^2} + \frac{(1+u_x^2)^N}{Y_x} \cdot \frac{u_x}{1+u_x^2} \cdot \frac{1}{u_x^{2N-1}}[(u_x^{2N})_t + (u^N u_x^{2N-1})_x] \\
&= \frac{(1+u_x^2)^N}{Y_x} \cdot \frac{Nu^{N-1}u_x}{1+u_x^2} + \frac{(1+u_x^2)^N}{Y_x} \cdot \frac{u_x}{1+u_x^2} \cdot \frac{1}{u_x^{2N-1}}[u^{N+1} - (P_1 + \partial_x P_2)] \cdot 2Nu_x^{2N-1} \\
&= Nwu^{N-1} \cdot \frac{1}{2}\sin v + 2Nw \cdot \frac{1}{2}\sin v[u^{N+1} - (P_1 + \partial_x P_2)] \\
&= Nw\sin v[\frac{1}{2}u^{N-1} + u^{N+1} - (P_1 + \partial_x P_2)].
\end{aligned}
\tag{2.43}
$$

Based on the above arguments, we can transfer (1.1) to the semi-linear system with respect to the unknown variables u, v and w under the new coordinate (T, Y). In fact, using (2.41), (2.42) and (2.43), we arrive at

$$
\begin{cases}
u_T(T,Y) = -\partial_x P_1 - P_2, \\
v_T(T,Y) = -u^{N-1}\sin^2\frac{v}{2} + 2u^{N+1}\cos^2\frac{v}{2} - 2\cos^2\frac{v}{2}(P_1 + \partial_x P_2), \\
w_T(T,Y) = Nw\sin v[\frac{1}{2}u^{N-1} + u^{N+1} - (P_1 + \partial_x P_2)]
\end{cases}
\tag{2.44}
$$

with the initial data

$$
\begin{cases}
u(0,Y) = u_0(x(0,Y)), \\
v(0,Y) = 2\arctan(u_0'(x(0,Y))), \\
w(0,Y) = 1,
\end{cases}
\tag{2.45}
$$

where P_1, $\partial_x P_1$, P_2 and $\partial_x P_2$ are given by (2.36)-(2.39), respectively. Thus, the energy conservative solution is constructed by the following semilinear system

Lemma 2.7. *[72] Let (u, v, w, Y) be the solution to (2.44) and (2.45), with $w > 0$. Then the set of points*

$$\{(t, x(t, Y), u(t, Y)); (t, Y) \in \mathbb{R}^+ \times \mathbb{R}\} \tag{2.46}$$

is the graph of a conservative solution to (2.4)

To investigate the singularities of the solution u of (2.4), we are interested in the level sets

$$\{v(t, Y) = \pi\}.$$

We now construct families $(\bar{u}^\theta, \bar{v}^\theta, \bar{q}^\theta)$ of perturbations of the initial data along the charactristic

3. Families of perturbed solutions

In this section, benefited from the ideas of [41] and [52, 66, 69], we construction families of perturbed solutions to the semilinear system (2.44) and (2.45) with the initial data, depending smoothly on $\theta \in \mathbb{R}^3$, such that the Jacobian matrix has full rank at the point $(t_0, Y_0) \in \mathbb{R}^+ \times \mathbb{R}$. From Lemma 2.4, it suffices to prove the Lipschitz continuity of f. Thus, based on the ideas of [72] and due to the smoothness of (u, v, w), it is crucial for us to prove the Lipschitz continuity of P_i and $\partial_x P_i$ $(i = 1, 2)$ in (2.36)-(2.39) with respect to the new variables u, v and w in every bounded domain Ω. To do this, we need the following lemma.

Lemma 3.1. *Let (u, v, w) be a smooth solution of the semilinear system (2.44) and (2.45), and let a point $(t_0, Y_0) \in \mathbb{R}^+ \times \mathbb{R}$ be given. If $(v, v_Y, v_{YY})(t_0, Y_0) = (\pi, 0, 0)$, then there exists a three parameter family of smooth solutions $(u^\theta, v^\theta, w^\theta)$ depending smoothly on $\theta \in \mathbb{R}^3$ such that*

(i) *When $\theta = 0 \in \mathbb{R}^3$, one recovers the original solution namely $(u^0, v^0) = (u, v)$;*
(ii) *At the point (t_0, Y_0), when $\theta = 0$, we obtain*

$$\text{rank } D_\theta(v^\theta, v^\theta_Y, v^\theta_{YY}) = 3.$$

Proof. Let (u, v, w) be a smooth solution of the semilinear system (2.44) and (2.45). Taking derivative to the equation with respect to v, one has

$$\begin{aligned}\frac{\partial}{\partial_T} v_Y = &-(N-1)u_Y u^{N-2}\sin^2\frac{v}{2} - \frac{1}{2}u^{N-1}v_Y\sin v + 2(N+1)u^N u_Y\cos^2\frac{v}{2} - u^{N+1}v_Y\sin v \\ &+ \frac{1}{2}\sin v v_Y(P_1 + \partial_x P_2) - (\cos v + 1)(\partial_Y P_1 + \partial_Y\partial_x P_2) \triangleq F_1.\end{aligned} \tag{3.1}$$

Similarly, we have

$$\begin{aligned}\frac{\partial}{\partial_T} v_{YY} = &\, u_{YY}[-(N-1)u^{N-2}\sin^2\frac{v}{2} + 2(N+1)u^N\cos^2\frac{v}{2}] + u_Y^2[-(N-1)(N-2)u^{N-3}\sin^2\frac{v}{2} \\ &+ 2N(N+1)u^{N-1}\cos^2\frac{v}{2}] + v_{YY}[-\frac{1}{2}u^{N-1}\sin v - \frac{1}{2}u^{N+1}\sin v + \sin v(P_1 + \partial_x P_2)] \\ &+ v_Y^2(-\frac{1}{2}u^{N-1}\sin v - \frac{1}{2}u^{N+1}\sin v) + v_Y[-\frac{N-1}{2}u^{N-2}u_Y\sin v - \frac{3(N-1)}{2}u^N u_Y\sin v \\ &+ \cos v(P_1 + \partial_x P_2) + 2\sin v(\partial_Y P_1 + \partial_Y\partial_x P_2)] - (\cos v + 1)(\partial_Y\partial_x P_1 + \partial_Y^2\partial_x P_2) \triangleq F_2,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\frac{\partial}{\partial_T} w_Y = &\, w_Y[\frac{N}{2}u^{N-1} + Nu^{N+1} - N(P_1 + \partial_x P_2)]\sin v + w[\frac{N}{2}u^{N-1} + \frac{N}{2}u^{N+1} - N(P_1 + \partial_x P_2)]\cos v v_Y \\ &+ w[\frac{N(N-1)}{2}u^{N-2}u_Y + N(N+1)u^N u_Y - N(\partial_Y P_1 + \partial_Y\partial_x P_2)]\sin v,\end{aligned} \tag{3.3}$$

where P_1, $\partial_x P_1$, P_2 and $\partial_x P_2$ are given by (2.36)-(2.39), respectively. $\partial_Y P_i (i = 1, 2)$ and $\partial_Y\partial_x P_2$ as well as $\partial_Y^2\partial_x P_2$ in (3.1)-(3.3) are given by (3.4) and (3.9) (see below). In what follows, we need the expression of $\partial_Y P_i (i = 1, 2)$ and $\partial_Y\partial_x P_2$ as well as $\partial_Y^2\partial_x P_2$ in (3.1)-(3.3).

We first observe that

$$\begin{aligned}\partial_Y P_i &= w\cos^{2N}\tfrac{v}{2}\partial_x P_i, \quad (i = 1, 2) \\ \partial_Y\partial_x P_2 &= -\tfrac{N-1}{2}wu^{N-2}\sin^3\tfrac{v}{2}\cos^{2N-3}\tfrac{v}{2} + P_2 w\cos^{2N}\tfrac{v}{2},\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\partial_Y^2\partial_x P_2 = &-\tfrac{N-1}{2}[w_Y u^{N-2}\sin^3\tfrac{v}{2}\cos^{2N-3}\tfrac{v}{2} + (N-2)wu^{N-3}u_Y\sin^3\tfrac{v}{2}\cos^{2N-3} \\ &+\tfrac{3}{2}u^{N-2}wv_Y\sin^2\tfrac{v}{2}\cos^{2N-2}\tfrac{v}{2} + \tfrac{2N-3}{2}u^{N-2}wv_Y\sin^3\tfrac{v}{2}\cos^{2N-4}\tfrac{v}{2}] \\ &+\partial_Y P_2 w\cos^{2N}\tfrac{v}{2} + P_2 w_Y\cos^{2N}\tfrac{v}{2} - NP_2 wv_Y\sin\tfrac{v}{2}\cos^{2N-1}\tfrac{v}{2}.\end{aligned} \tag{3.5}$$

In order to estimate u_Y in (3.5), we first claim that

$$u_Y = \frac{u_x}{Y_x} = u_x \cdot \frac{w}{(1+u_x^2)^N} = \frac{u_x}{1+u_x^2}\frac{w}{(1+u_x^2)^{N-1}} = \frac{1}{2} w \sin v \cos^{2N-2}\frac{v}{2}. \tag{3.6}$$

as long as the local solution of the semilinear system (2.44) and (2.45) is defined. To obtain (3.1), we need to check

$$u_{YT} = u_{TY}.$$

It follows from (2.4) that

$$\begin{aligned}
u_{YT} &= (\frac{1}{2} w \sin v \cos^{2N-2}\frac{v}{2})_T \\
&= \frac{1}{2} w_T \sin v \cos^{2N-2}\frac{v}{2} + \frac{1}{2} w v_T \cos v \cos^{2N-2}\frac{v}{2} + \frac{1}{2} w \sin v [2(N-1)\cos^{2N-3}\frac{v}{2}\cdot(-\sin\frac{v}{2})\cdot\frac{1}{2}v_T] \\
&= \frac{1}{2} N w \sin v [\frac{1}{2} u^{N-1} + u^{N+1} - (P_1+\partial_x P_2)] \sin v \cos^{2N-2}\frac{v}{2} \\
&\quad + \frac{1}{2} w [-u^{N-1}\sin^2\frac{v}{2} + 2u^{N+1}\cos^2\frac{v}{2} - 2\cos^2\frac{v}{2}(P_1+\partial_x P_2)] \cos v \cos^{2N-2}\frac{v}{2} \\
&\quad - \frac{N-1}{2} w \sin v \cos^{2N-3}\frac{v}{2}\sin\frac{v}{2}\cdot[-u^{N-1}\sin^2\frac{v}{2} + 2u^{N+1}\cos^2\frac{v}{2} - 2\cos^2\frac{v}{2}(P_1+\partial_x P_2)] \\
&= w\cos^{2N-2}\frac{v}{2}[u^{N+1}\cos^2\frac{v}{2} + \frac{2N-1}{2}u^{N-1} - \cos^2\frac{v}{2}(P_1+\partial_x P_2)].
\end{aligned} \tag{3.7}$$

On the other hand, by (2.4), (2.38) and (2.39), we have

$$\begin{aligned}
u_{TY} &= (-\partial_x P_1 - P_2)_Y \\
&= [u^{N+1}\cos^{2N}\frac{v}{2} + \frac{2N-1}{2}u^{N-1}\sin^2\frac{v}{2}\cos^{2N-2}\frac{v}{2}] w - w\cos^{2N}\frac{v}{2} P_1 \\
&\quad + \frac{N-1}{4}[u^{N-1}\sin^3\frac{v}{2}\cos^{2N-3}\frac{v}{2}] w - w\cos^{2N}\frac{v}{2}\partial_x P_2 \\
&= w\cos^{2N-2}\frac{v}{2}[u^{N+1}\cos^2\frac{v}{2} + \frac{2N-1}{2}u^{N-1} - \cos^2\frac{v}{2}(P_1+\partial_x P_2)].
\end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that $u_{YT} = u_{TY}$, which implies (3.6) holds. It follows from (3.4),(3.5) and (3.6) that

$$\begin{aligned}
\partial_Y^2\partial_x P_2 = &-\tfrac{N-1}{2}[w_Y u^{N-2}\sin^3\tfrac{v}{2}\cos^{2N-3}\tfrac{v}{2} + (N-2) w u^{N-3} u_Y \sin^3\tfrac{v}{2}\cos^{2N-3} \\
&- w v_Y[\tfrac{N-1}{2}u^{N-2}(\tfrac{3}{2}\sin^2\tfrac{v}{2}\cos^{2N-2}\tfrac{v}{2} + \tfrac{2N-3}{2}\sin^3\tfrac{v}{2}\cos^{2N-4}\tfrac{v}{2}) + N P_2 \sin\tfrac{v}{2}\cos^{2N-1}\tfrac{v}{2}] \\
&+ w^2\cos^{4N}\tfrac{v}{2} + P_2 w_Y \cos^{2N}\tfrac{v}{2}.
\end{aligned} \tag{3.9}$$

We now construct families $(\bar{u}^\theta, \bar{v}^\theta, \bar{q}^\theta)$ of perturbations of the initial data along the charactristic as

$$\bar{u}^\theta(Y) = \bar{u}(Y) + \sum_{i=1}^{3}\theta_i U_i(Y), \tag{3.10}$$

$$\bar{v}^\theta(Y) = \bar{v}(Y) + \sum_{i=1}^{3}\theta_i V_i(Y), \tag{3.11}$$

$$\bar{w}^\theta(Y) = \bar{w}(Y) + \sum_{i=1}^{3}\theta_i W_i(Y). \tag{3.12}$$

Thus, by (2.44) and (3.1)-(3.3), a 5-D ODE system can be established as follows:

$$\frac{\partial}{\partial T}\begin{pmatrix} u \\ v \\ w \\ v_Y \\ v_{YY}\end{pmatrix} = \begin{pmatrix} -\partial_x P_1 - P_2 \\ -u^{N-1}\sin^2\frac{v}{2} + 2u^{N+1}\cos^2\frac{v}{2} - 2\cos^2\frac{v}{2}(P_1+\partial_x P_2) \\ N w \sin v[\frac{1}{2}u^{N-1} + u^{N+1} - (P_1+\partial_x P_2)] \\ F_1 \\ F_2 \end{pmatrix}. \tag{3.13}$$

We construct a family of solutions $(\bar{u}^\theta, \bar{v}^\theta, \bar{\xi}^\theta)$ to (3.13) of perturbations of the initial data as in (3.10)-(3.12). Taking derivative with respect to θ, we have

$$\frac{\partial}{\partial t}\begin{pmatrix} D_\theta u^\theta \\ D_\theta v^\theta \\ D_\theta q^\theta \\ D_\theta v_Y^\theta \\ D_\theta v_{YY}^\theta \end{pmatrix} = \begin{pmatrix} D_\theta f_1^\theta \\ D_\theta f_2^\theta \\ D_\theta f_3^\theta \\ D_\theta f_4^\theta \\ D_\theta f_5^\theta \end{pmatrix} \tag{3.14}$$

where f_i^θ(i=1,2,3,4,5) are the perturbation of the right-hand side of (3.14). Then we arrive at

$$\frac{\partial}{\partial T}\begin{pmatrix} D_\theta u^\theta \\ D_\theta v^\theta \\ D_\theta q^\theta \\ D_\theta v_Y^\theta \\ D_\theta v_{YY}^\theta \end{pmatrix} = \begin{pmatrix} D_u f_1^\theta & D_v f_1^\theta & D_q f_1^\theta & D_{v_Y} f_1^\theta & D_{v_{YY}} f_1^\theta \\ D_u f_2^\theta & D_v f_2^\theta & D_q f_2^\theta & D_{v_Y} f_2^\theta & D_{v_{YY}} f_2^\theta \\ D_u f_3^\theta & D_v f_3^\theta & D_q f_3^\theta & D_{v_Y} f_3^\theta & D_{v_{YY}} f_3^\theta \\ D_u f_4^\theta & D_v f_4^\theta & D_q f_4^\theta & D_{v_Y} f_4^\theta & D_{v_{YY}} f_4^\theta \\ D_u f_5^\theta & D_v f_5^\theta & D_q f_5^\theta & D_{v_Y} f_5^\theta & D_{v_{YY}} f_5^\theta \end{pmatrix}\begin{pmatrix} D_{\theta_1}\bar{u}^\theta & D_{\theta_2}\bar{u}^\theta & D_{\theta_3}\bar{u}^\theta \\ D_{\theta_1}\bar{v}^\theta & D_{\theta_2}\bar{v}^\theta & D_{\theta_3}\bar{v}^\theta \\ D_{\theta_1}\bar{q}^\theta & D_{\theta_2}\bar{q}^\theta & D_{\theta_3}\bar{q}^\theta \\ D_{\theta_1}\bar{v}_Y^\theta & D_{\theta_2}\bar{v}_Y^\theta & D_{\theta_3}\bar{v}_Y^\theta \\ D_{\theta_1}\bar{v}_{YY}^\theta & D_{\theta_2}\bar{v}_{YY}^\theta & D_{\theta_3}\bar{v}_{YY}^\theta \end{pmatrix}.$$

Utilizing Lemma 2.4 , we only need to prove the Lipschitz continuity of f. Based on the ideas of [72] and due to the smoothness of (u, v, w), we only need to consider the Lipschitz continuity of P_i and $\partial_x P_i$ $(i = 1, 2)$ in (2.36)-(2.39) with respect to the new variables u, v and w in every bounded domain $\Omega \subset X$ of the form

$$\Omega \triangleq \{(u, v, w) : \|u\|_{H^1} + \|u\|_{W^{1,2N}} = A,\ \ \|v\|_{L^2} \le B,\ \ \|v\|_{L^\infty} \le \frac{3\pi}{2},\ \ w^- \le w \le w^+,\ \ a.e.\ x\}, \tag{3.15}$$

for some positive constants A, B, w^- and w^+. Here

$$X \triangleq [H^1(\mathbb{R}) \times W^{1,2N}(\mathbb{R})] \cap [L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})] \times L^\infty(\mathbb{R}),$$

which endowed with the norm

$$\|(u, v, w)\|_X \triangleq \|u\|_{H^1(\mathbb{R})\times W^{1,2N}(\mathbb{R})} + \|v\|_{L^2(\mathbb{R})} + \|v\|_{L^\infty(\mathbb{R})} + \|w\|_{L^\infty(\mathbb{R})}.$$

For sake of simplicity, we shall only present the detailed estimates for $\frac{\partial P_1}{\partial u}$ and $\frac{\partial P_{1x}}{\partial u}$, because all other partial derivatives can be estimated in the same manners. For that purpose, from (2.36) and (2.37), we know that, for every test function $\phi \in H^1(\mathbb{R})$, the functions $\frac{\partial P_1}{\partial u}$ and $\frac{\partial P_{1x}}{\partial u}$, are defined by

$$\begin{aligned}(\tfrac{\partial P_1}{\partial u}\cdot\phi)(Y) = &\tfrac{1}{2}\int_{-\infty}^{+\infty} e^{-|\int_Y^{\bar{Y}} w(s)\cos^{2N}\frac{v(s)}{2}ds|} \\ &\times[(N+1)u^N\cos^{2N}\tfrac{v(\bar{Y})}{2} + \tfrac{(2N-1)(N-1)}{2}u^{N-2}\sin^2\tfrac{v(\bar{Y})}{2}\cos^{2N-2}\tfrac{v(\bar{Y})}{2}]w(\bar{Y})\phi(\bar{Y})d\bar{Y}\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}(\tfrac{\partial P_{1x}}{\partial u}\cdot\phi)(Y) = &\tfrac{1}{2}(\int_Y^{+\infty} - \int_{-\infty}^{Y})e^{-|\int_Y^{\bar{Y}} w(s)\cos^{2N}\frac{v(s)}{2}ds|} \\ &\times[(N+1)u^N\cos^{2N}\tfrac{v(\bar{Y})}{2} + \tfrac{(2N-1)(N-1)}{2}u^{N-2}\sin^2\tfrac{v(\bar{Y})}{2}\cos^{2N-2}\tfrac{v(\bar{Y})}{2}]w(\bar{Y})\phi(\bar{Y})d\bar{Y}\end{aligned} \tag{3.17}$$

First, we estimate (3.17). To do this, we now introduce the exponentially decaying function

$$\Gamma(\xi) \triangleq \min\{1, e^{\frac{w^-}{2N}(\frac{9}{2}B^2 - |\xi|)}\}.$$

Then, we deduce

$$\begin{aligned}\|\Gamma\|_{L^1} &= (\int_{|\xi|\le\frac{9}{2}B^2} + \int_{|\xi|\ge\frac{9}{2}B^2})\Gamma(\xi)d\xi \\ &= \int_{-\frac{9}{2}B^2}^{\frac{9}{2}B^2} d\xi + \int_{\frac{9}{2}B^2}^{+\infty} e^{\frac{w^-}{2N}(\frac{9}{2}B^2-\xi)}d\xi + \int_{-\infty}^{-\frac{9}{2}B^2} e^{\frac{w^-}{2N}(\frac{9}{2}B^2+\xi)}d\xi \\ &= 9B^2 + \tfrac{w^-}{2N}.\end{aligned} \tag{3.18}$$

Using standard properties of convolutions, Young's inequality, (3.17) and (3.18), we have

$$\begin{aligned}\|\tfrac{\partial P_{1x}}{\partial u}\cdot\phi\|_{L^{2N}} \le &\ \tfrac{w^+}{2}\|\Gamma * [(N+1)u^N\cos^{2N}\tfrac{v}{2} \\ &+ \tfrac{(2N-1)(N-1)}{2}u^{N-2}\sin^2\tfrac{v}{2}\cos^{2N-2}\tfrac{v}{2}]\|_{L^{2N}}\|\phi\|_{L^\infty} \\ \le &\ \tfrac{w^+}{2}\|\Gamma\|_{L^1}[(N+1)\|u^N\|_{L^{2N}} + \tfrac{(2N-1)(N-1)}{2}\|u^{N-2}\|_{L^{2N}}]\|\phi\|_{H^1} \\ \triangleq &\ \tfrac{w^+}{2}\|\Gamma\|_{L^1}\|\phi\|_{H^1} J < \infty,\end{aligned}$$

where

$$J = (N+1)\|u^N\|_{L^{2N}} + \frac{(2N-1)(N-1)}{2}\|u^{N-2}\|_{L^{2N}}.$$

Similarly, we can obtain the boundedness of $\frac{\partial P_1}{\partial u}$.

Choosing suitable perturbation V_i $(i = 1, 2, 3)$, we can make

$$rank\begin{pmatrix} D_\theta \bar{v}^\theta \\ D_\theta \bar{v}^\theta_Y \\ D_\theta \bar{v}^\theta_{YY} \end{pmatrix} = 3, \tag{3.19}$$

when $\theta = 0$. □

4. Generic property

In this section, first, using perturbed technique (i.e Lemma 3.1) and Thom's transversality theorem (i.e Theorem 2.3), we investigate the generic solutions to the semilinear system (2.44) and (2.45), determining the generic structure of the level sets $\{v = \pi\}$ (i.e Lemma 4.1). Then, based on the blow-up criteria in [65] and the singularity in [72] on the level set $\{u = 0\}$, as well as Lemma 4.1, we will prove the generic regularity for the energy conservative solutions $u = u(t, x)$ of (1.1).

4.1. Generic solutions of the semilinear system.

In this subsection, we first consider smooth solutions to the semilinear system (2.44) and (2.45), determining the generic structure of the level sets $\{v = \pi\}$. To do this, we need the following Lemma 4.1.

Lemma 4.1. *Consider a compact domain of the form*

$$\Theta \triangleq \{(t, Y);\ \ 0 \le t \le T, |Y| \le M\}.$$

Let $\mathbb{S}$ be the family of all C^2 solutions to the semilinear system (2.44) and (2.45), with $w{>}0$ for all $(t, Y) \in \mathbb{R}_+ \times \mathbb{R}$. Furthermore, call $\mathbb{S}' \subset \mathbb{S}$ the subfamily of all solutions (u, v, w), such that for $(t, Y) \in \Gamma$ the following value is never attained:

$$(v, v_Y, v_{YY}) = (\pi, 0, 0). \tag{4.1}$$

Then $\mathbb{S}'$ is a relatively open and dense subset of $\mathbb{S}$, in the topology induced by $C^2(\Theta)$.

Proof. By the perturbed technique (i.e Lemma 3.1) and Thom's transversality theorem (i.e Theorem 2.3), we divide our arguments into four steps.

Step 1. Let $\mathbb{S}_1$ be the subset of solutions for which $(v, v_Y, v_{YY}) = (\pi, 0, 0)$ is never attained on Θ. Since Θ is a compact domain, each $\mathbb{S}_1$ is a relatively open subset of $\mathbb{S}$, in the topology of $C^2(\Theta)$.

Step 2. Let (u, v, w) be any C^2 solution of the semilinear system (2.44) and (2.45), with $w{>}0$. For any $(t_0, Y_0) \in \Theta$, two cases can occur as follows:

Case 1. $(v, v_Y, v_{YY})(t_0, Y_0) \neq (\pi, 0, 0)$. In this case, by continuity, we know that there exists a neighborhood $\mathcal{N}$ of (t_0, Y_0) in the $t - Y$ plane where $(v, v_Y, v_{YY}) \neq (\pi, 0, 0)$.

Case 2. $(v, v_Y, v_{YY}) = (\pi, 0, 0)$. By Lemma 3.1, we can find a three-parameter family of solutions $(u^\theta, v^\theta, w^\theta)$, such that the $3{\times}3$ Jacobian matrix of the map

$$(\theta_1, \theta_2, \theta_3) \to (v^\theta(t, Y), v^\theta_Y(t, Y), v^\theta_{YY}(t, Y)) \tag{4.2}$$

has rank 3 at the point (t_0, Y_0), when $\theta = 0$. By continuity, this matrix still has rank 3 on a neighborhood $\mathcal{N}$ of (t_0, Y_0), for θ small enough.

Choosing finitely many points $(t_i, Y_i)(i = 1, ..., n)$, such that the corresponding open neighborhoods $\mathcal{N}_{(t_i, Y_i)}$ cover the compact set Θ. Let $n_{\mathcal{I}}$ be the cardinality of the set of indices

$$\mathcal{I} \triangleq \{i; (v, v_Y, v_{YY})(t_i, Y_i) = (\pi, 0, 0)\}$$

for which Case 2 applies. For each $i \in \mathcal{I}$, based on a 3-parameter family of perturbations, we know that all these perturbed solutions depend on $n = 3n_{\mathcal{I}}$ parameters.

Step 3. Let $\Theta' \supset \Theta$ be an open set contained in the union of the neighborhoods $\mathcal{N}_{(t_i, Y_i)}$ and $B_\epsilon \triangleq \{\theta \in \mathbb{R}^n; |\theta| \le \epsilon\}$ be the open ball of radius ϵ in $\mathbb{R}^n$. We will construct a family $(u^\theta, v^\theta, w^\theta)$ of smooth solutions to the semilinear system (2.44) and (2.45), such that the map with parameter θ

$$(t, Y, \theta) \to (v^\theta(t, Y), v^\theta_Y(t, Y), v^\theta_{YY}(t, Y))$$

from $\Theta' \times B_\epsilon$ into $\mathbb{R}^3$ has $(\pi, 0, 0)$ as a regular value. For that purpose, we need to combine perturbations based at possibly different points (t_i, Y_i) into a single N-parameter family of perturbed solutions. Let $(u, x, w)(t, Y)$ be a solution to the semilinear system (2.44) and (2.45). For each $k = 1, ..., n$, let a point (t_k, Y_k) be given, together with a number $U_k \in \mathbb{R}$ and functions $V_k, W_k \in C^\infty_c(\mathbb{R})$. Based on the previous arguments, if we construct a family of initial data $(\bar{v}, \bar{v}_Y, \bar{v}_{YY})$ with rank 3, we will establish a one-parameter family of perturbed solutions to the semilinear system with rank 3 in the neighborhood of 0 for the parameter. Thus, the family of the perturbed solution is determined as follows. For $|\epsilon|<\epsilon_k$ sufficiently small, we can determine the unique solution of the perturbed semilinear system as

$$(u^\epsilon, v^\epsilon, w^\epsilon) \triangleq \Phi_k^\epsilon(u, v, w). \tag{4.3}$$

Given $(\theta_1, ..., \theta_n)$, we can define a perturbation of the original solution (u, v, w) to the semilinear system (2.44) and (2.45) as the composition of n-parameter perturbations:

$$(u^\theta, v^\theta, w^\theta) = \Phi_n^{\theta_n} \circ ... \circ \Phi_1^{\theta_1}(u, v, w).$$

Step 4. At every point (t_i, Y_i), where $i \in \mathcal{I}$, using Lemma 3.1, we can establish a three-parameter families of perturbed solutions, so that Jacobian matrix of (4.2) is of full rank on the neighborhood $\mathcal{N}$ of a point (t_0, Y_0), for θ sufficiently small. Choosing finitely many points (t_i, Y_i), $(i = 1, ..., n)$, such that the corresponding open neighborhood $\mathbb{N}_{(t_i,Y_i)}$ covers the compact set Θ. Thus, we deduce a $n-$parameter family of solutions, such that the value $(\pi, 0, 0)$ is a regular value for the map (4.2) from $\Theta \times B_\epsilon$ into $\mathbb{R}^3$. Using Thom's transversality theorem (Theorem 2.3), for a.e. θ, we know that the map with parameter θ:

$$F : (t, Y) \to (v^\theta(t, Y), v_Y^\theta(t, Y), v_{YY}^\theta(t, Y)) \tag{4.4}$$

is transverse to $(\pi, 0, 0)$.

By the definition of transversality, either

$$F(t, Y) \neq (\pi, 0, 0)$$

or

$F(t, Y) = (\pi, 0, 0)$ and $T_{(\pi,0,0)}\mathbb{R}^3 = (dF)_{(t,Y)}(T_{(t,Y)}\Theta')$

Since Θ' is only two-dimensional, $T_{(\pi,0,0)}\mathbb{R}^3 = (dF)_{(t,Y)}(T_{(t,Y)}\Theta')$ cannot happen. Thus, the only choice is $F(t, Y) \neq (\pi, 0, 0)$, which means that, for a.e. θ suffiently small, the corresponding solution $(u^\theta, v^\theta, w^\theta)$ enjoy property that $(v^\theta, v_Y^\theta, v_{YY}^\theta) \neq (\pi, 0, 0)$ for all $(t, Y) \in \Theta$. This proves that the set $\mathbb{S}_1$ is dense in $\mathbb{S}$. This completes the proof of Lemma 4.1. □

We are now position to prove our main result.

4.2. Proof of Theorem 1.2.

Proof. Consider the space $\mathbb{S} = C^3(\mathbb{R}) \cap H^1(\mathbb{R}) \cap W^{1,2N}(\mathbb{R})$, equipped with norm

$$\|u\|_{\mathbb{S}} \triangleq \|u_0\|_{C^3} + \|u_0\|_{H^1} + \|u_0\|_{W^{1,2N}}.$$

Given the initial data $\tilde{u}_0 \in \mathbb{S}$ and set the open ball

$$B_\delta \triangleq \{u_0 \in \mathbb{S}; \|u_0 - \tilde{u_0}\|<\delta\}.$$

In what follows, our aim is to prove that, for any $\tilde{u}_0 \in \mathbb{S}$, there exists a radius $\delta>0$ and an open dense subset $\mathscr{D} \subset B_\delta$, such that for every initial data $u_0 \in \mathscr{D}$, the energy conservative solution $u = u(t, x)$ to (1.1) is of class C^1 in the complement of finitely many characteristic curves γ_i within the domain $[0, T] \times \mathbb{R}$. To achieve our aim, we divide our further arguments into four steps.

Step 1 (Construction of $\mathscr{D}$). Based on the blow-up criteria in [65], we know that, if $u_0^{N-1}u_{0,x} = O(\epsilon^{\frac{1}{N}})$, then the blow up time of $u^{N-1}u_x$ along the charactersitics is of order $\frac{1}{\epsilon^N}$. Since $u_0 \in \mathbb{S}$, we know $u_0^{N-1}u_{0,x} \to 0$ as $x \to \infty$. Consequently, taking $r>0$, such that when $|x|>r$, we will have $|u^{N-1}u_x|<\frac{1}{T_1}$, which implies the singularity of u in set $[0, T_1] \times \mathbb{R}$ only appears in the compact set

$$\mathcal{A} \triangleq [0, T_1] \times [-r - \|u^N\|_{L^\infty}T_1, r + \|u^N\|_{L^\infty}T_1], \tag{4.5}$$

where

$$\|u^N\|_{L^\infty} \triangleq \max\{u^N(t, x), (t, x) \in [0, T_1] \times \mathbb{R}\}.$$

In the (T, Y) plane, from Lemma 4.1, it is reasonable for us to take a domain Δ, such that $\mathcal{A} \subset \Lambda(\Delta)$, where the map $\Lambda : (t, Y) \to (t, x(t, Y))$.

The subset $\mathscr{D} \subset B_\delta$ is defined as follows. $u_0 \in \mathscr{D}$ if $u_0 \in B_\delta$ and for the corresponding solution (u, v, w) of (2.44) and (2.45), the value (4.1) are never attained for $(t, x) \in \mathcal{A}$. Later in our arguments, we will validate $\mathscr{D}$ is an open dense set.

Step 2 ($\mathscr{D}$ is open). In this step, our aim is to prove that $\mathscr{D}$ is open, in the topology of C^3. Take a sequence of initial data $\{u_0^\nu\}_{\nu\geq 1}$, such that converges to u_0 and $u_0^\nu \notin \mathscr{D}$. By definition of $\mathcal{D}$ in the Step 1, we know that there exists points (t^ν, Y^ν), such that the corresponding solutions (u^ν, v^ν, w^ν) fulfill

$$(v^\nu, v_Y^\nu, v_{YY}^\nu)(t^\nu, Y^\nu) = (\pi, 0, 0),\ (t^\nu, x^\nu(t^\nu, Y^\nu)) \in \mathcal{A},$$

for all $\nu \geq 1$. Since we have the domain $\mathcal{A}$ (in (4.5)) is compact, taking a subsequence, we can assume (t^ν, Y^ν) converges to some point (t, Y) and

$$(v, v_Y, v_{YY})(t, Y) = (\pi, 0, 0),\ (t, x(t, Y)) \in \mathcal{A},$$

which implies $u_0 \notin \mathscr{D}$. This means that $\mathscr{D}$ is an open set.

Step 3 ($\mathscr{D}$ is dense). Given $u_0 \in B_\delta$, by a small perturbation, we can assume $u_0 \in C^\infty$. By Lemma 4.1, we can construct a sequence of solutions (u^ν, v^ν, ξ^ν) of (2.44) and (2.45), such that

(i) for every bounded set $I \subset \mathbb{R}^2$ and $k \geq 1$,

$$\lim_{\nu\to+\infty} \|(u^\nu - u, v^\nu - v, \xi^\nu - \xi, x^\nu - x)\|_{C^k(I)} = 0;$$

(ii) for every $\nu \geq 1$ the values in (4.1) are never attained for any $(t, Y) \in \Theta$, where Θ is given in Lemma 4.1.

Consider the sequence of solution $u^\nu(t, x)$ with the graph

$$\left\{(u^\nu(t, Y), t^\nu(t, Y), x^\nu(t, Y)); (t, Y) \in \mathbb{R}^2\right\} \subset \mathbb{R}^3 \tag{4.6}$$

and the corresponding sequence of initial data satisfies

$$\|u^\nu(0, \cdot) - u_0\|_{C^l(I)} \to 0,\ as\ \nu \to \infty \tag{4.7}$$

for every bounded set I.

In order to obtain the convergence for the far field, we modify the sequence. To do this, we introduce a cutoff function $\phi \in C_c^\infty$, such that

$$\phi(x) = \begin{cases} 1, & if\ |x| \leq \rho, \\ 0, & if\ |x| \geq \rho + 1, \end{cases}$$

where $\rho \gg r + \|u^N\|_{L^\infty} T$ is large enough. For each $\nu \geq 1$, consider the initial data

$$\tilde{u}_0^\nu \triangleq \phi u_0^\nu + (1 - \phi) u_0.$$

Then, we have

$$\lim_{\nu\to+\infty} \|\tilde{u}^\nu - u_0\|_{\mathbb{S}} = 0,$$

where $\mathbb{S} = C^3(\mathbb{R}) \cap H^1(\mathbb{R}) \cap W^{1,2N}(\mathbb{R})$. Furthermore, if $\rho>0$ large enough, we have

$$\tilde{u}^\nu(t, x) = u^\nu(t, x),\ \forall (t, x) \in \mathcal{A},$$

while $\tilde{u}^\nu(t, x)$ is C^2 on the outer domain. Then we can find that $\tilde{u}^\nu(t, x) \in \mathscr{D}$, for all $\nu \geq 1$ sufficiently large, which implies that $\mathscr{D}$ is dense on B_δ.

Step 4 (u is piecewise C^2). In this step, we need to demonstrate that, for every initial data $u_0 \in \mathscr{D}$, the corresponding solution $u(t, x)$ is piecewise C^2 on the domain $[0, T] \times \mathbb{R}^+$. For that purpose, based on the previous analysis, we deduce that u is C^2 on the outer domain

$$\{(t, x) | 0 \leq\ t \leq T, |x| > \|u^N\|_{L^\infty} T|\}.$$

Thus, it suffices to investigate the singularity of u on the inner domain $\mathcal{A}$. Consider a point $(t_0, Y_0) \in \Theta$, where Θ is given in Lemma 4.1. Thus, there are two situations to consider as follows.

Situation 1 $v(t_0, Y_0) \neq \pi$. It follows from (2.34) that

$$\frac{\partial x}{\partial Y} = \frac{w}{(1 + u_x^2)^N} = w \cos^{2N} \frac{v}{2}.$$

Thus, we know that the map $(t, Y) \to (t, x)$ is locally invertible in a neighborhood of (t_0, Y_0). Then, we conclude that the function u is C^2 in a neighborhood of the point $(t_0, x(t_0, Y_0))$.

Situation 2 $v(t_0, Y_0) = \pi$. In [72], we know that the singularity happens on the level set $\{u = 0\}$. By continuity, there exists $\epsilon>0$, such that the values (4.1) are never attained in the open neighborhood

$$\Theta' \triangleq \{(t, Y); t \in [0, T], |Y| \leq M + \epsilon\}.$$

For (4.1), there are two situations that can occur.

Situation 2.1 $v_Y = 0, v_{YY} \neq 0$. It follows from the implicit function theorem and (4.1) that the set

$$\mathcal{S}^{v_Y} \triangleq \{(t, Y) \in \Theta'; \ \ v_Y(t, Y) = 0\}$$

is a one dimensional embedded manifold of class C^1.

We now claim that the number of connected components of $\mathcal{S}^{v_Y}$ that intersect the compact set Θ is finite. In what follows, we argue by contradiction. For that purpose, we shall assume the contrary that $\{P_i\}$ $(i = 1, 2, ...)$ is a sequence of points in $\mathcal{S}^{v_Y} \cap \Theta$ belonging to distinct components. Taking a subsequence we can assume the convergence

$$P_i \to \bar{P}, \ \ for \ \ some \ \ \bar{P} \in \mathcal{S}^{v_Y} \cap \Theta.$$

By assumption, $v_{YY} \neq 0$. Hence, by the implicit function theorem, we know that there is a neighborhood $\mathcal{N}$ of $\bar{P}$, such that $\gamma \triangleq \mathcal{S}^{v_Y} \cap \mathcal{N}$ is a connected C^1 curve. Thus, $P_i \in \Theta$ for all i large enough, which implies a contradiction.

Situation 2.2 $v_Y \neq 0$. By the implicit function theorem and (4.1), we deduce that the set

$$\mathcal{S}^{v} \triangleq \{(t, Y) \in \Theta'; v(t, Y) = \pi\}$$

is a one dimensional embedded manifold of class C^2. In the same way, we prove that the number of connected components of S^v is finite, thus it will be omitted. □

Remark 4.2. *We observe that the arguments in Step 4 are different from the CH (1.2) and Novikov equation (1.4). For the CH (1.2), if $v(t_0, Y_0) = 0$, for some point $(t_0, Y_0) \in \Gamma$, then one can observe that $v_t \neq 0, v_Y \neq 0$, see [52, 66]. For Novikov equation (1.4), the case $v_Y = 0$ will happens, see [41]. However, for (1.1), the case $v_T = -u^{N-1}$ will happens, which is maybe zero. We believe that these differences are caused by the energy concentration phenomenon. Due to this transversality property, the energy conservative solution for the CH (1.2) has no energy concentration for almost every time t. More details are presented in [3]. Indeed, in terms of the CH (1.2), v is defined in a way similar to what we investigated in our contributions, while when singularity happens, $v_T = -1$, which is never zero. For the CH (1.2), when the characteristic meet tangentially, they will separate immediately. In terms of Novikov equation (1.4), energy density $\mu_{(t)}$ might be concentrated on a set of time whose measure is not zero. If energy concentration of $\mu_{(t)}$ happens, some characteristics tangentially touch each other, then stay together for a period of time t. On this piece of characteristic, we have $\cos\frac{v}{2} = 0$ and $u = 0$. However, in terms of the (1.1), $v_T = -u^{N-1}$.*

On the other hand, in appearance of this kind of phenomenon, we believe that these differences are caused by the nonlinearity of the wave speed $c(u)$ and cusp singularity. In fact, the wave speed $c(u) = u$, so $c'(u) \equiv 1$ for CH (1.2), and $c(u) = u^2$, so $c'(u) = 2u$ for Novikov equation (1.4). However, $c(u) = u^N$, so $c'(u) = Nu^{N-1}$ for (1.1). In terms of the wave speed $c(u)$ and cusp singularity, we show an intrinsic relation between CH (1.2) (linear wave speed, i.e $N = 1$), Novikov equation (1.4) (quadratic wave speed, i.e $N = 2$) and (1.1) (wave speed $c(u) = u^N$). In this sense, our paper improve the contributions in the literature in [52, 66] and in [41].

Remark 4.3. *From [72], we know that the global energy conservative solution is uniformly Hölder continuous with exponent $1 - \frac{1}{2N}$. We observe that this result precisely demonstrates how the regularity of solution changes with respect to N (the order of higher order nonlinearity). If $N = 1$ and $N = 2$, then the global energy conservative solutions are uniformly Hölder continuous with exponent $\frac{1}{2}$ (see [3]) and $\frac{3}{4}$ (see [12]) respectively.*

Remark 4.4. *During the arguments in Step 4, benefited from the ideas of [72], we know that, for almost every $t \in \mathbb{R}$, the singular part of $\nu_{(t)}$ concentrates on $u = 0$. In fact, if the solution blows up, then $|u_x| \to \infty$, which implies $\cos\frac{v}{2} = 0$. Also, we know that $v_T = -u^{N-1}$, which is nonzero only when u is nonzero. Thus, we can demonstrate that, for almost every $t \in \mathbb{R}$, the singular part of $\nu_{(t)}$ concentrates on $u = 0$.*

5. Generic singularity behavior

Based on [69,72], we know that, for smooth the initial data $u_0 \in C^\infty(\mathbb{R})$, the solution $(t, Y) \to (x, t, u, v, w)(t, Y)$ of the semilinear system (2.44) and (2.45), remains smooth on the entire $t - Y$ plane. However, the smoothness of the solution u of (2.4) is still needed to investigate because the coordinate change:$(Y, t) \to (x, t)$ is not smoothly invertible. This leads to the asymptotic behavior for generic singularity. In fact, by definition, it follows from (2.34) that its Jacobian matrix is given by

$$\mathcal{M} = \begin{pmatrix} x_Y & x_t \\ t_Y & t_t \end{pmatrix} = \begin{pmatrix} w\cos^{2N}\frac{v}{2} & u^{N-1} \\ 0 & 1 \end{pmatrix}. \tag{5.1}$$

It is easy to see that the determinant of $\mathcal{M}$, $det(\mathcal{M}) = w\cos^{2N}\frac{v}{2}$, where the new variables u, v and w in every bounded domain Ω in (3.15). Thus, we know that the matrix $\mathcal{M}$ in (5.1) is invertible, having a strictly positive determinant, when $v \neq \pi$. To investigate the set of points in the $t - x$ plane, where u is singular, we thus need to look at points where $v = \pi$.

Theorem 1.2 provides us with a detailed description of the solution $u(t, x)$ in the neighborhood of each one of these singular points. For sake of simplicity, we shall assume that the initial data u_0 are smooth, thus we shall not need to count how many derivatives are actually used to derive the Talyor approximations.

We now are position to prove our main result.

Proof of Theorem 1.3

Proof. According to the type of singular point, there are two cases as follows:

Case 1. $v_Y = 0,\ v_t = 0,\ v_{YY} \neq 0$;

Case 2. $v = \pi,\ v_Y \neq 0,\ v_{YY} = 0$.

First, we consider **Case 1**. Let P be the point of **Case 1**. It is follows from (3.6) that

$$u_Y = \frac{u_x}{1+u_x^2}\frac{w}{(1+u_x^2)^{N-1}} = \frac{1}{2}w\sin v\cos^{2N-2}\frac{v}{2}.$$

In the same way, one has

$$\begin{aligned} u_{YY} &= \tfrac{1}{2}w_Y\cdot\sin v\cdot\cos^{2N-2}\tfrac{v}{2} + \tfrac{1}{2}w\cdot v_Y\cdot\cos v\cdot\cos^{2N-2}\tfrac{v}{2} - \tfrac{N-1}{4}v_Y\cdot w\sin^2 v\cos^{2N-4}\tfrac{v}{2}, \\ u_{Yt} &= \tfrac{1}{2}w_t\sin v\cos^{2N-2}\tfrac{v}{2} + \tfrac{1}{2}w\cos v\cdot v_t\cos^{2N-2}\tfrac{v}{2} + \tfrac{1}{2}w\sin v\cdot(2N-2)\cos^{2N-3}\tfrac{v}{2}\cdot(-sin\tfrac{v}{2})\cdot\tfrac{1}{2}v_t \\ &= w\cos^{2N-2}\tfrac{v}{2}[u^{N+1}\cos^2\tfrac{v}{2} + \tfrac{2N-1}{2}u^{N-1} - \cos^2\tfrac{v}{2}(P_1 + \partial_x P_2)]. \end{aligned}$$

We observe that $v_Y = 0,\ v_t = 0$ and $v_{YY} \neq 0$ at the singular point P with Case 1. We denote u_{Y^n} that u is differentiated with Y by n times. It is easy to see that

$$u_{Y^3} = 0,\ u_{Y^4} = 0,\ u_{Y^5} = 0, \cdots,\ u_{Y^{2(2^N-1)}} \neq 0.$$

Abiding by the same line, one has

$$u_{Y^2t} = 0,\ u_{Y^3t} = 0,\ u_{Y^4t} = 0, \cdots,\ u_{Y^{2^{N+1}-3}t} = 0.$$

Thus, using Taylor approximations of u at the singular point $P = (t_0, Y_0)$, we arrive at

$$u(t, Y) = B_1(t-t_0) + B_3(Y-Y_0)^{2(2^N-1)} + O(1)(|t-t_0|^2, |Y-Y_0|^{2^{N+1}-1}) \tag{5.2}$$

It follows from (2.33), (2.34) and (2.44) that

$$x_Y \triangleq \frac{\partial x}{\partial Y} = \frac{w}{(1+u_x^2)^N} = w\cos^{2N}\frac{v}{2}.$$

Similarly, one has

$$\begin{aligned} x_{YY} &= w_Y\cos^{2N}\tfrac{v}{2} - Nwv_Y\cos^{2N-1}\tfrac{v}{2}\sin\tfrac{v}{2}, \\ x_{Yt} &= w_t\cos^{2N}\tfrac{v}{2} - Nwv_t\cos^{2N-1}\tfrac{v}{2}\sin\tfrac{v}{2}. \end{aligned}$$

Thus, at the singular point $P = (t_0, Y_0)$, we have

$$x_{Y^j} = 0,\ (j = 1, 2, \cdots, 2^{N+1}-1)$$

and

$$x_{Y^{2^{N+1}}} \neq 0,\quad x_{Y^{2^{N+1}-1}t} \neq 0.$$

Thus, using Taylor approximations of x at the singular point $P = (t_0, Y_0)$, we have

$$x(t, Y) = x(t_0, Y) + A_2(Y-Y_0)^{2^{N+1}} + O(1)(|t-t_0|^2, |Y-Y_0|^{2^{N+1}+1}). \tag{5.3}$$

By (5.2) and (5.3), we obtain (1.8) in Theorem 1.3.

In what follows, we consider **Case 2**. In this case, we have

$$u_Y = 0,\ u_{YY} = 0,\ u_{Y^3} = 0, \cdots,\ u_{Y^{2^N}} \neq 0$$

and

$$u_{Yt} = 0,\ u_{Y^2t} = 0, \cdots,\ u_{Y^{2^N-1}t} = 0.$$

Then, using Taylor approximations of u at the singular point $P = (t_0, Y_0)$, we deduce

$$u(t, Y) = B_1(t - t_0) + B_2(Y - Y_0)^{2^N} + O(1)(|t - t_0|^2 + |Y - Y_0|^{2^N+1}). \tag{5.4}$$

Similarly, at the singular point $P = (t_0, Y_0)$, one has

$$x_Y = 0,\ x_{Y^2} = 0,\ x_{Y^3} = 0, \cdots,\ x_{Y^{2^N}} = 0,\ x_{Y^{2^N+1}} \neq 0,$$
$$x_{Yt} = 0,\ x_{Y^2t} = 0,\ x_{Y^3t} = 0 , \cdots,\ x_{Y^{2^N}t} = 0.$$

Thus, using Taylor approximations of x at the singular point $P = (t_0, Y_0)$, we have

$$x(t, Y) = x(t_0, Y_0) + A_2(Y - Y_0)^{2^N+1} + O(1)(|t - t_0|^2, |Y - Y_0|^{2^N+2}). \tag{5.5}$$

It follows from (5.4) and (5.5) that (1.9) holds. This completes the proof of Theorem 1.3. □

Remark 5.1. *During the arguments of Theorem 1.3, we observe that this result precisely demonstrates how the asymptotic behavior for generic singularity changes with respect to N. If $N = 1$, (1.8) becomes*

$$u(t, x) = A(x - x_0)^{\frac{1}{2}} + B(t - t_0) + O(1)(|t - t_0|^2 + |x - x_0|^{\frac{3}{4}})$$

for some constant A, B. Also,(1.9) becomes

$$u(t, x) = A(x - x_0)^{\frac{2}{3}} + B(t - t_0) + O(1)(|t - t_0|^2 + |x - x_0|)$$

for some constant A, B. More details are present in [52] and [66].

If $N = 2$, (1.8) becomes

$$u(t, x) = A(x - x_0)^{\frac{3}{4}} + B(t - t_0) + O(1)(|t - t_0|^2 + |x - x_0|^{\frac{7}{8}})$$

for some constant A, B. Also,(1.9) becomes

$$u(t, x) = A(x - x_0)^{\frac{4}{5}} + B(t - t_0) + O(1)(|t - t_0|^2 + |x - x_0|)$$

for some constant A, B. More details are present in [41]. Thus, in this sense, our result improves earlier ones in the literatures, such as [52], [66] and [41].

Remark 5.2. *In [69], Yang et al. considered the generalized Camassa-Holm equation with higher-order nonlinearity (gCH for short)*

$$\begin{cases} u_t - u_{txx} - u^m u_{xxx} + \frac{(m+2)(m+1)}{2} u^m u_x = (\frac{m}{2} u^{m-1} u_x^2 u^m u_x)_x, & x \in \mathbb{R}, t > 0, \\ u(x, 0) = u_0(x), x \in \mathbb{R}, \end{cases} \tag{5.6}$$

where m is a positive integer. Let $p(x) = \frac{1}{2} e^{-|x|}$. Then $(1 - \partial_x^2)^{-1} f = p * f$ for all $f \in L^2(\mathbb{R})$. Thus, the gCH equation (5.6) becomes

$$\begin{cases} u_t + u^m u_x = -P_x, x \in \mathbb{R}, t > 0, \\ u(x, 0) = u_0(x), x \in \mathbb{R}, \end{cases} \tag{5.7}$$

where

$$P = p * (\frac{m}{2} u^{m-1} u_x^2 + \frac{m(m + 3)}{2(m + 1)} u^{m+1}).$$

They investigated the continuation of solutions to the gCH equation (5.6) or (5.7) beyond wave breaking. First, by introducing new variables, they transformed the gCH equation (5.6) or (5.7) to a semi-linear system and establish the global solutions to this semi-linear system, and by returning to the original variables, they established the existence of global conservative solutions to the original equation. Second, they introduced a set of auxiliary variables tailored to a given conservative solution, which satisfy a suitable semi-linear system, and demonstrated that the solution for the semi-linear system is unique. Moreover, they obtained that the original equation has a unique global conservative solution. Finally, by Thom's transversality lemma, they proved that piecewise smooth solutions with only generic singularities are dense in the whole solution set, which means the generic regularity for the gCH equation (5.6) or (5.7)

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References

[1] S. C. Anco, P. L. da Silva and I. L. Freire, *A family of wave-breaking equations generalizing the Camassa-Holm and Novikov equations*, J. Math. Phys. **56** (2015), 091506.
[2] J. M. Bloom, "The local structure of smooth maps of manifolds ", PhD thesis, Harvard University, Cambrige, MA, 2004.
[3] A. Bressan and A. Constantin, *Global conservative solutions to the Camassa-Holm equation*, Arch. Ration. Mech. Anal. **183** (2007), 215-239.
[4] A. Bressan and A. Constantin, *Global dissipative solutions to the Camassa-Holm equation*, Anal. Appl. **5** (2007), 1-27.
[5] A. Bressan and G. Chen, *Generic regularity of conservative solutions to a nonlinear wave equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **34(2)** (2017), 335-354.
[6] A. Bressan, G. Chen and Q. Zhang, *Uniqueness of conservative solutions to the Camassa-Holm equation via characteristics*, Discrete Contin. Dyn. Syst. **35** (2015), 25-42.
[7] R. Barostichi, A. Himonas and G. Petronilho, *The power series method for nonlocal and nonlinear evolution equations*, J. Math. Anal. Appl. **443** (2016), 834-847.
[8] A. Bressan, T. Huang and F. Yu., *Structurally stable singularities for a nonlinear wave equation*, Bull. Inst. Math. Acad. Sin. (N.S.) **10(4)** (2015), 449-478.
[9] A. Bressan and Y. Zheng, *Conservative solutions to a nonlinear variational wave equation*, Comm. Math. Phys. **266(2)** (2006), 471-497.
[10] E. D. Cokelet, *Breaking waves*, Nature **267** (1977), 769-774.
[11] A. Constantin, *Existence of permanent and breaking waves for a shallow water equation: a geometric approach*, Ann. Inst. Fourier (Grenoble) **50** (2000), 321-362.
[12] G. Chen, M. Chen and Y. Liu, *Existence and uniqueness of the global conservative weak solutions for the integrable Novikov equation*, Indiana Univ. Math. J. **67** (2018), 2393-2433.
[13] H. Cai, G. Chen, Y. Shen and Z. Tan, *Generic regularity and Lipschitz metric for the Hunter-Saxton type equations*, J. Differential Equations, **262(2)** (2017), 1023-1063.
[14] A. Constantin and J. Escher, *Well-posedness, global existence, and blow up phenomena for a periodic quasi-linear hyperbolic equation*, Comm. Pure Appl. Math. **51** (1998), 475-504.
[15] A. Constantin and J. Escher, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Math. **181** (1998), 229-243.
[16] R. Camassa and D. D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. **71** (1993), 1661-1664.
[17] A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations*, Arch. Ration. Mech. Anal. **192** (2009), 165-186.
[18] L. Caravenna and L. V. Spinolo, *Schaeffer's regularity theorem for scalar conservation laws does not extend to systems*, Indiana Univ. Math. J. **66(1)** (2017), 101-160.
[19] H. Dai, *Model equations for nolinear dispersive waves in compressible Mooney-Rivlin rod*, Acta Mech. **127** (1998), 193-207.
[20] C. M. Dafermos and X. Geng, *Generalized characteristics uniqueness and regularity of solutions in a hyperbolic system of conservation laws*, Ann. Inst. H. Poincaré Anal. Non Linéaire **8(3-4)** (1991), 231-269.
[21] H. R. Dullin, G. A. Gottwald and D. D. Holm, *On asymptotically equivalent shallow water wave equations*, Phys. D **190** (2004), 1-14.
[22] A. Degasperis, D. D. Holm and A. N. W. Hone, *A new integral equation with peakon solutions*, Theor. Math. Phys. **133** (2002), 1463-1474.
[23] A. Degasperis, D. D. Holm and A. N. W. Hone, *Integral and non-integrable equations with peakons*, Nonlinear Phys.:Theory Exp. **II** (2003) 37-43.
[24] A. Degasperis and M. Procesi, *Asymptotic integrability, in: A. Degasperis, G. Gaeta (Eds.), Symmetry and Perturbation Theory*, World Scientific, 1999, 23-37.
[25] J. Escher, Y. Liu and Z. Yin, *Global weak solutions and blow-up structure for the Degasperis-Procesi equation*, J. Funct. Anal. **241** (2006), 257-485.
[26] J. Escher, Y. Liu and Z. Yin, *Shock waves and blow-up phenomena for the periodic Degasperis-Procesi equation*, Indiana Univ. Math. J. **56** (2007), 87-117.
[27] J. Escher and Z. Yin, *Well-posedness, blow-up phenomena and global solutions for the b- equation*, J. Reine Angew. Math. **624** (2008), 51-80.
[28] B. Fuchssteiner, *Some tricks from the symmetry-tool box for nonlinear equations: generalizations of the Camassa-Holm equation*, Phys. D **95** (1996), 229-243.
[29] A. Fokas and B. Fuchssteiner, *Symplectic structures, their Bäcklund transformation and hereditary symmetries*, Phys. D **4** (1981), 47-66.
[30] M. Golubitsky and V. Guillemin, "Stable mappings and their singularities ", Graduate Texts in Mathematics, Vol. 14. Springer-Verlag, New York-Heidelberg, 1973.
[31] G. Gui and Y. Liu, *On the Cauchy problem for the two-component Camassa-Holm system*, Math. Z. **268** (2011), 45-66.

[32] G. Gui and Y. Liu, *Global well-posedness and blow-up of solutions for the Camassa-Holm equations with fractional dissipation*, Math. Z. **281** (2015), 993-1020.
[33] G. Gui, Y. Liu, P. J. Olver and C. Qu , *Wave-breaking and peakons for a modified Camassa-Holm equation*, Comm. Math. Phys. **319** (2013), 731-759.
[34] G. Gui, Y. Liu and T. Luo, *Model equations and traveling wave solutions for shallow-water waves with the Coriolis effect*, J. Nonlinear Sci. **29** (2019), 993-1039.
[35] Z. Guo, X. Li and C. Yu, *Some properties of solutions to the Camassa-Holm-type equation with higher-order nonlinearities*, J. Nonlinear Sci. **28** (2018), 1901-1914.
[36] G. Gui, Y. Liu and M. Zhu, *On the wave-breaking phenomena and global existence for the generalized periodic Camassa-Holm equation*, Int. Math. Res. Not. IMRN **21** (2012), 4858-4903.
[37] A. Himonas and C. Holliman, *On well-posedness of the Degasperis-Procesi equation*, Discrete Contin. Dyn. Syst. **31** (2011), 469-488.
[38] A. Himonas and C. Holliman, *The Cauchy problem for the Novikov equation*, Nonlinearity **25** (2012), 449-479.
[39] A. Himonas and C. Holliman, *The Cauchy problem for a generalized Camassa-Holm equation*, Adv. Differential Equations **19** (2013), 161-200.
[40] A. Himonas, C. Holliman and K. Grayshan, *Norm inflation and illposedness for the Degasperis-Procesi equation*, Commun. Partial Diff. Eqs. **39** (2014), 2198-2215.
[41] Z. He, W. Luo and Z. Yin, *Generic regularity of conservative solutions to the Novikov equation*, preprint, 2023, arXiv: 2308.04107:2023.
[42] A. Himonas and D. Mantzavinos, *The Cauchy problem for a 4-paramter family of equations with peakon traveling waves*, Nonlinear Anal. **133** (2016), 161-199.
[43] H. Holden and X. Raynaud, *A convergent numerical scheme for the Camassa-Holm equation based on multipeakons*, Discrete Contin. Dyn. Syst. Ser. A **14** (2006), 505-523.
[44] A. N. W. Hone and J.P. Wang, *Prolongation algebras and Hamiltonian operators for peakon equations*, Inverse Problems **19** (2002), 129-145.
[45] A. N. W. Hone and J. P. Wang, *Integrable peakon equations with cubic nonlinearity*, J. Phys. A: Math. Theor. **41** (2008), 372002.
[46] R. S. Johnson, *Camassa Holm, Korteweg de Vries and related models for water waves*, J. Fluid Mech. **455** (2002), 63-82.
[47] S. G. Ji and Y. H. Zhou, *Wave-breaking phenomena and global existence for the weakly dissipative generalized Novikov equation*, Proc. R. Soc. A **477** (2021), 20210532.
[48] J. M. Lee, "Introduction to smooth manifolds ", volume 218 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2003.
[49] H. Lundmark, *Formation and dynamics of shock waves in the Degasperis Procesi equation*, J. Nonlinear Sci. **17** (2007), 169-198.
[50] H. Lundmark and J. Szmigeilski, *Multi-peakon solutions of the Degasperis-Procesi equation*, Inverse Problems **19** (2003), 1241-1245.
[51] H. Lundmark and J. Szmigeilski, *Degasperis-Procesi peakons and the discrete cubic string*, Int. Math. Res. Papers **2** (2005) 53-116.
[52] M. Li and Q. Zhang, *Generic regularity of conservative solutions to Camassa-Holm type equations*, SIAM J. Math. Anal. **49(4)** (2017), 2920-2949.
[53] A. V. Mikhailov and V. Novikov, *Perturbative symmetry approach*, J. Phys. A: Math. Theor. **35** (2002), 4775-4790.
[54] V. Novikov, *Generalizations of the Camassa-Holm equation*, J. Phys. A: Math. Theor. **42** (2009), 342002.
[55] Z. Popowicz, *A Camassa-Holm equation interacted with the Degasperis-Procesi equation*, Czech. J. Phys. **56** (2006), 1263-1268.
[56] D. G. Schaeffer, *A regularity theorem for conservation laws*, Adv. Math. **11** (1973), 368-386.
[57] R. Thom, "Structural stability and morphogenesis ", Advanced Book Classics, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989.
[58] X. Tu, Y. Liu and C. Mu, *Existence and uniqueness of the global conservative weak solutions to the rotation-Camassa-Holm equation*, J. Differential Equations **266** (2019), 4864-4900.
[59] X. Wu and Z. Yin, *Global weak solutions for the Novikov equation*, J. Phys. A: Math. Theor. **44** (2011), 055202.
[60] X. Wu and Z. Yin, *Well-posedness and global existence for the Novikov equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **11** (2012), 707-727.
[61] X. Wu and Z. Yin, *A note on the Cauchy problem of the Novikov equation*, Appl. Anal. **92** (2013), 1116-1137.
[62] Z. Xin and P. Zhang, *On the weak solutions to a shallow water equation*, Comm. Pure Appl. Math. **53** (2000), 1411-1433.
[63] Z. Xin and P. Zhang, *On the uniqueness and large time behavior of the weak solution to a shallow water equation*, Comm. Partial Differential Equations **27** (2002), 1815-1844.
[64] Z. Yin, *On the Cauchy problem for an integrable equation with peakon solutions*, Illinois J. Math. **47** (2003), 649-666.
[65] K. Yan, *Wave breaking and global existence for a family of peakon equations with high order nonlinearity*, Nonlinear Anal. RWA **45** (2019), 721-735.
[66] S. J. Yang, *Generic regularity of conservative solutions to the rotational Camassa-Holm equation*, J. Math. Fluid Mech. **22(4)** (2020) Paper No. 49, 11.
[67] W. Yan, Y. Li and Y. Zhang, *The Cauchy problem for the integrable Novikov equation*, J. Differential Equations **253** (2012), 298-318.
[68] W. Yan, Y. Li and Y. Zhang, *The Cauchy problem for the Novikov equation*, NoDEA Nonlinear Differential Equations Appl. **20** (2013), 1157-1169.
[69] L. Yang, C. Mu, S. Zhou and X. Tu, *The global conservative solutions for the generalized camassa-holm equation*, Electron. Res. Arch. **27** (2019), 37-67.
[70] Y. H. Zhou, and S. G. Ji, *Well-posedness and wave breaking for a class of integrable equations* , Math. Z. **302** (2022), 1527-1550.
[71] Z. Y. Zhang, Z. H. Liu, Y. J. Deng, et al., *Global well-posedness and infinite propagation speed for the $N-abc$ family of Camassa-Holm type equation with both dissipation and dispersion*, J. Math. Phys. **231** (2020), 071502.

[72] Z. Y. Zhang, Z. H. Liu and Y. J. Deng, *Global energy conservation solution for the $N-abc$ family of Camassa-Holm type equation*, Nonlinear Anal. RWA **78** (2024), 104093.
[73] Z. Y. Zhang, L. M. Li, C. H. Fang, et al., *A new blow-up criterion for the $N-abc$ family of Camassa-Holm type equation with both dissipation and dispersion*, Open Math. **18** (2020), 194-203.
[74] S. Zhou and C. Mu, *The properties of solutions for a generalized b family equation with peakons*, J. Nonlinear Sci. **23** (2013), 863-889.
[75] S. Zhou and C. Mu, *Global conservative and dissipative solutions of generalized Camassa-Holm equation*, Discrete Contin. Dyn. Syst. **16** (2018), 1615-1633.
[76] S. Zhou, L. Yang and C. Mu, *Global dissipative solutions of the Novikov equation*, Commun. Math. Sci. **16** (2018), 1615-1633.

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