CONTINUITY OF ENTROPY FOR ALL α-DEFORMATIONS OF AN INFINITE CLASS OF CONTINUED FRACTION TRANSFORMATIONS

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Abstract. We extend the results of our 2020 paper in the Annali della Scuola Normale Superiore di Pisa, Classe di Scienze. There, we associated to each of an infinite family of triangle Fuchsian groups a one-parameter family of continued fraction maps and showed that the matching (or, synchronization) intervals are of full measure. Here, we find planar extensions of each of the maps, and prove the continuity of the entropy function associated to each one-parameter family.

We also introduce a notion of "first pointwise expansive power" of an eventually expansive interval map. We prove that for every map in one of our one-parameter families its first pointwise expansive power map has its natural extension given by the first return of the geodesic flow to a cross section in the unit tangent bundle of the hyperbolic orbifold uniformized by the corresponding group. We conjecture that this holds for all of our maps. We give numerical evidence for the conjecture.

1. Introduction

Shortly after the introduction at the end of the 1950s of the Kolmogorov–Sinai measure theoretic entropy, hereafter simply *entropy*, Rohlin [30] introduced the notion of the natural extension of a dynamical system. In briefest terms, the natural extension is the minimal invertible dynamical system of which the original system is a factor under a surjective map. Rohlin showed that the original system and its natural extension share entropy values.

At least since the 1977 [29], natural extensions have been exploited to determine various properties of continued fractions and related interval maps. In particular, the continuity of the entropy function on the parameter interval for Nakada's α-continued fractions, of the 1981 [27], was proven using natural extensions. Initiated in [27], this approach was pursued in [20], with a further significant step provided in 2008 by [24], who conjectured that continuity holds on the full parameter interval. The conjecture was proven in 2012 independently by [12] and [23].

A common technique in those 2012 proofs is the use of 'matching' (also known in the literature as: synchronization, cycle property, etc.). On a matching parameter interval, for each map the orbits of endpoints of the interval of definition meet one another in finitely many steps and do so in a common fashion; planar models of the natural extensions vary continuously over the interval. This causes the entropy to change continuously along these parameter intervals.

Katok-Ugarcovici [18] introduced the family of (a, b)-continued fraction maps and determined the full subset of its two-dimensional parameter set for which matching occurs; see also [11, 19]. These continued fractions, as also the Nakada family, are associated to the modular group $\mathrm{PSL}_2(\mathbb{Z})$. To each of the triangle Fuchsian groups known as the Hecke groups, [14] associated a one-parameter family of continued fraction maps and began the study of their entropy functions; see also [21]. In [8], to each of another infinite family of triangle Fuchsian groups we associated

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a one-parameter family of continued fraction maps and determined the set of matching intervals (there called "synchronization intervals").

Series [31] showed that the regular continued fraction interval map is a factor of the first return of the geodesic flow to a cross section in the unit tangent bundle of the hyperbolic orbifold uniformized by the modular group. Luzzi and Marmi [24] conjectured that this is also true for all of the Nakada α-continued fraction maps. Using the 2012 results on the entropy function, this was proven in [4]. Related results for (a, b)-continued fractions maps, using coding of geodesics, can be found in [19].

See § 1.2 for more on our motivation for this project.

1.1. **Main results.** The bulk of this paper is devoted to determining explicit planar natural extensions for the infinite family of continuous deformations of interval maps defined and studied in [8]. We give detailed descriptions of candidate domains for the planar natural extensions for maps parametrized along matching intervals; these candidates are bijectivity domains for the planar two-dimensional map associated to the given interval map. This is fairly straightforward for small values of α, see § 4, 5; but rather intricate for larger values, see § 7– 10. With these domains in hand, we can prove both that they do indeed form the planar natural extensions and furthermore that their appropriate limits give natural extensions for the remaining maps. The explicit natural extensions $\Omega_{n,\alpha}$ are key to proving the following dynamical properties of the systems of the interval maps. (The basic terminology and notation of [8] are recalled in §§ 1.3 and 7.1.)

Theorem 1. *Fix $n \geq 3$. For $\alpha \in (0, 1)$, let μ_α be the normalization of μ to a probability measure on $\Omega_\alpha = \Omega_{n,\alpha}$, and $\mathscr{B}'_\alpha$ denote the Borel sigma algebra on Ω_α. Then the system $(\mathcal{T}_\alpha, \Omega_\alpha, \mathscr{B}'_\alpha, \mu_\alpha)$ is the natural extension of $(T_\alpha, \mathbb{I}_\alpha, \mathscr{B}_\alpha, \nu_\alpha)$, where ν_α is the marginal measure of μ_α and $\mathscr{B}_\alpha$ the Borel sigma algebra on $\mathbb{I}_\alpha$. Furthermore, both systems are ergodic.*

For the proof of the theorem, as well as the following step in its proof, we rely upon the theoretical tools established in [9].

Proposition 2. *Every $T_{n,\alpha}$ with $\alpha \in (0, 1)$ is eventually expansive.*

The detailed knowledge of the domains Ω_α allows us to prove the following.

Theorem 3. *For each $n \geq 3$, the function $\alpha \mapsto \mu(\Omega_\alpha)$ is continuous on $(0, 1)$.*

Theorem 4. *For each $n \geq 3$, the function $\alpha \mapsto h(T_\alpha)\mu(\Omega_\alpha)$ is constant on $(0, 1)$.*

The previous two results immediately imply our main result.

Theorem 5. *The function $\alpha \mapsto h(T_\alpha)$ is continuous on $(0, 1)$.*

We introduce the notion of "first pointwise expansive power" of an eventually expansive interval map, see Definition 153. Numerical evidence leads us to the following conjecture. See, say, [25] for related background.

Conjecture 6. *For all $n \geq 3$ and for all $\alpha \in (0, 1)$ we conjecture that the first pointwise expansive power of $T_{n,\alpha}$ has its natural extension given by the first return of the geodesic flow to a cross section in the unit tangent bundle of the hyperbolic orbifold uniformized by G_n.*

We prove the following, where the "set of Möbius transformations" of a piecewise Möbius interval map T denotes the set of Möbius transformations required to define T.

Theorem 7. *Suppose that the set of Möbius transformations of a piecewise Möbius interval map T and its first pointwise expansive power U generate the same Fuchsian group G. Suppose*

further that the unit tangent bundle of $G\backslash\mathbb{H}$ has finite volume equal to the product $h(T)\,\mu(\Omega)$. Then there is a cross section in the unit tangent bundle of $G\backslash\mathbb{H}$ to the geodesic flow, for which the first return map of the geodesic flow gives the natural extension of the system of U.

From this last, the conjecture is equivalent to the following.

Conjecture 8. *For all $n \geq 3$ and for all $\alpha \in (0,1)$ one has that $h(T_\alpha)\,\mu(\Omega_\alpha)$ equals the volume of the unit tangent bundle of the hyperbolic orbifold uniformized by the triangle group G_n.*

We report on numerical corroboration of this latter conjecture in § 15.4, and also prove the case when $n = 3$.

Theorem 9. *Conjecture 8, and therefore also Conjecture 6, holds when $n = 3$.*

Conventions Throughout, we will allow ourselves the minor abuse of using adjectives such as injective, surjective and bijective to mean in each case *up to measure zero*, and thus similarly where we speak of disjointness and the like we again will assume the meaning being taken to include the proviso "up to measure zero" whenever reasonable. For legibility, we suppress various subscripts, trusting that this causes no confusion for the reader. We interchangeably use our old term "synchronization" and the now more standard "matching". Finally, it is convenient to refer to the complement of the closure of the synchronization intervals as the "non-synchronization" points.

1.2. **Overarching questions.** We undertook the project of which this is the third paper motivated by a desire to gather more information to hopefully contribute to eventual answers to the following questions.

Our core motivation springs from the connection between Diophantine approximation and metric number theory. The central result is that of Lévy [22], which can be seen as relating the entropy value of the regular continued fractions map to a value determined by the convergents to x of almost every x. See say [3, 28] for explicit formulations of this. Thus, for any continued fraction-like map, we desire to know its entropy. For any family of such maps, we desire to know properties such as continuity of the function that gives the entropy values of the maps.

The question presents itself: *Are their properties of a family of interval maps which ensure that the associated entropy map is continuous?* To date, it seems that only direct argumentation, as we give here, has resulted in proofs of such continuity. We are most familiar with the proofs of continuity of this function in the setting of the Nakada [27] α-continued fractions which are directly related to the modular group, and in the setting of their generalizations to similar maps related to each of the Hecke groups, [14, 21]. The proofs there center on planar natural extensions for the interval maps; with the following Steps: (1) the natural extensions vary nicely as geometric objects on 'certain' parameter intervals, (2) the natural extensions Ω in fact vary continuously on the full parameter space, (3) the normalizing constant $\mu(\Omega)$ for the mass of each of these natural extensions then varies continuously, (4) the product of the entropy $h(T)$ of each map with the normalizing constant for the measure of the natural extension defines a constant function on the parameter space: $h(T)\mu(\Omega)$ is constant.

The 'certain' parameter intervals mentioned above have to date been intervals on which 'matching' (or, synchronization, cycle property, etc.) occurs. On such an interval, for each map the orbits of endpoints of the interval of definition meet one another in finitely many steps. A second question arises: *Are their properties of a family of interval maps, parametrized by an interval, which ensure that matching intervals exist? That they are of full measure?* We ourselves tend towards the use of the technique of 'quilting' [21], which directly shows Step (1) at least locally. In general, Step (2) requires refined analysis (in the case of 'small α' our very

detailed analysis is contained in three sections, beginning with § 4). Upon showing finiteness of the normalizing constants, Step (3) follows. Step (4) results from successful use of Abramov's formula for entropy, for example see § 14.2.

Alternately, if the family of maps is associated to a Fuchsian group uniformizing a hyperbolic surface (or orbifold) of finite volume, one can hope to show that each of the interval maps has an extension given by the return to a cross section by the geodesic flow on the unit tangent bundle of the surface. It then follows from the ergodicity of this flow with respect to the standard measure that the integral which appears in Rohlin's formula is equal to the quotient of the volume of the unit tangent bundle by the normalizing constant. We first saw this in Arnoux's [3] (to be precise, in that setting of the regular continued fractions, due to the presence of Möbius transformations of negative determinant, an extra factor of two belongs in the quotient that we just mentioned). A third question presents itself: *Which interval maps have extensions given by the return of the geodesic flow to a cross section in the unit tangent bundle of a hyperbolic surface/orbifold?*

Our favorite technique for showing that the first return system to a cross section to the geodesic flow gives an extension of a given interval map uses "Arnoux's transversal", see ([9], Section 2.5) for a brief overview. For this, the interval map itself must be expansive. However, for each of our infinitely many groups there is a nonempty parameter subinterval along which our maps are *non-expansive*. For each of these, there is a region in the cross section of the corresponding tangent bundle where our image unit tangent vectors point so that their geodesics flow backwards; see p. 64 for details. Still, experimentation led us to conjecture that all of our maps are such that the product of their entropy with the normalizing constant of Step (3) above is equal to the volume of the unit tangent bundle of the orbifold uniformized by the group generated by their set of Möbius transformations. Thus our Conjecture 8, see above. In particular, we realized that there is a way to pointwise "accelerate" any eventually expansive map so as to obtain a candidate for having an extension given by the *first* return map of the geodesic flow to a cross section. This we defined as the first pointwise expansive power of the map, and Theorem 7 presented itself to us. Having proved it, we have Conjecture 6. Of course, we also ask for a proof of the conjectures.

In [8] we also defined a one parameter family of interval maps related to each of the groups denoted there as $G_{m,n}$ with $n \geq m > 3$. We fully expect that for each of these that matching occurs on a full measure subset of the parameter interval, as for the case of $m = 3$ proven in [8], and even that the tree of words $\mathcal{V}$ defined in that paper can play an analogous central role in the description of the matching intervals. As m increases, the direct analog of the set of small α occupies an ever decreasing portion of the parameter interval; for large m and large α, computer experimentation indicates that the dynamics of $T_{m,\alpha}$ is quite complicated. We mention in passing that one can also define families of interval maps for each of these groups where the interval maps also involve Möbius transformations which are orientation-reversing. Vaguely, we ask: *Can one obtain the analog of the results here for these other families?*

1.3. **Terse review of notation and terminology.** We give a quick review of some of the notation and terminology from [8]. In fact, we simplify some of that notation as throughout this paper the index m there is set here to equal 3.

1.3.1. *Groups, maps, digits, cylinders, parameter interval partition, "small" α.* For integer $n \geq 3$ we let $\nu = \nu_n = 2\cos\pi/n$ and

$$t := t_n = 1 + 2\cos\pi/n.$$

We use the projective group G_n generated by

$$A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \; C = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1}$$

(We will have no direct use of the matrix $B = A^{-1}C$ of [8].)

Fix $n \geq 3$. For each $\alpha \in [0,1]$, let $\ell_0(\alpha) = (\alpha - 1)t, r_0(\alpha) = \alpha t$ and $\mathbb{I}_\alpha = \mathbb{I}_{n,\alpha} = [\ell_0(\alpha), r_0(\alpha))$. Our interval maps are piecewise Möbius, of the form

$$T_\alpha = T_{n,\alpha} : [\ell_0(\alpha), r_0(\alpha)] \to \mathbb{I}_\alpha, \quad x \mapsto A^k C^l \cdot x \tag{2}$$

where $\ell \in \{1,2\}$ is minimal such that $C^l \cdot x \notin \mathbb{I}_\alpha$ and k is the unique integer such that then $A^k C^l \cdot x \in \mathbb{I}_\alpha$. We call $b_\alpha(x) = (k,l)$ the α-digit of such an x, and say that x lies in the cylinder $\Delta_\alpha(k,l)$. To each element of $\mathbb{I}_\alpha$ we then associate the sequence of its α-digits. The T_α-orbits of the endpoints $\ell_0(\alpha), r_0(\alpha)$ are of extreme importance in our discussions, one writes

$$\underline{b}^\alpha_{[1,\infty)} = (k_1,l_1)(k_2,l_2)\dots$$

for the α-digits of $\ell_0(\alpha)$ and replaces the lower bar by an upper bar, $\overline{b}^\alpha_{[1,\infty)}$ for $r_0(\alpha)$. We also label entries in these orbits by $\ell_0, \ell_1, \dots$ and $r_0, r_1, \dots$. The collection of all finite words in the (k,l) which arise in the expansions of any $x \in \mathbb{I}_\alpha$ form the language $\mathcal{L}_\alpha$ for T_α; any word in $\mathcal{L}_\alpha$ is called α-admissible. There is an order, $\prec$ on the words in the (k,l) such that given α, for any $x_1, x_2 \in \mathbb{I}_\alpha$ one has $x_1 \leq x_2$ if and only if their α-digits sequences are similarly related under $\prec$. From this, a word is in $\mathcal{L}_\alpha$ if and only if it lies between $\underline{b}^\alpha_{[1,\infty)}$ and $\overline{b}^\alpha_{[1,\infty)}$. See ([8], Subsection 1.6) for more about this order.

The parameter interval is naturally partitioned, $(0,1] = (0,\gamma_n) \cup [\gamma_n, \epsilon_n) \cup [\epsilon_n, 1]$, where $\alpha < \gamma_n$ if $\forall x \in [\ell_0(\alpha), r_0(\alpha)]$ the α-digit (k,l) of x has $l = 1$ and $\alpha \geq \epsilon_n$ if and only if both $\alpha > \gamma_n$ and the α-digit of $\ell_0(\alpha)$ equals $(k,1)$ with $k \geq 2$, see ([8], Figure 4.1) for plots indicating dynamical behavior related to this partition. We informally refer to the set of $\alpha < \gamma_n$ as the *small* α, and all others as *large* α. For small α, we use *simplified digits:* since $l = 1$ we only report the exponent of A; in this setting we use $\underline{d}^\alpha_{[1,\infty)}, \overline{d}^\alpha_{[1,\infty)}$ in place of $\underline{b}^\alpha_{[1,\infty)}, \overline{b}^\alpha_{[1,\infty)}$, respectively.

1.3.2. *Expansions given words, synchronization intervals, tree of words; for small α.* For small α, we define parameter intervals on which an initial portion of the expansion of the $r_0(\alpha)$ are fixed. That is, there is a common prefix of the $\overline{d}^\alpha_{[1,\infty)}$. Given a word $v = c_1 d_1 \cdots d_{s-1} c_s$ of positive integers, let $\overline{S}(v) = \sum_{i=1}^s c_i + \sum_{j=1}^{s-1} d_j$ and for each $k \in \mathbb{N}$ let

$$\overline{d}(k,v) = k^{c_1}, (k+1)^{d_1}, \cdots, (k+1)^{d_{s-1}}, k^{c_s}, \quad \text{when } \; v = c_1 d_1 \cdots d_{s-1} c_s. \tag{3}$$

The corresponding subinterval of parameters is $I_{k,v} = \{\alpha \mid \overline{d}^\alpha_{[1,\overline{S}(v)]} = \overline{d}(k,v)\}$; in words, this is the set of α for which the α-expansion (in simplified digits) of $r_0(\alpha)$ has $\overline{d}(k,v)$ as a prefix. Thus, setting

$$R_{k,v} = (A^k C)^{c_s} \; (A^{k+1}C)^{d_{s-1}} (A^k C)^{c_{s-1}} \cdots (A^{k+1}C)^{d_1} (A^k C)^{c_1}, \tag{4}$$

for $\alpha \in I_{k,v}$ one has $T_\alpha^{\overline{S}(v)}(\, r_0(\alpha)\,) = R_{k,v} \cdot r_0(\alpha)$. The left endpoint of $I_{k,v}$ is denoted $\zeta_{k,v}$, one finds that $R_{k,v} \cdot r_0(\zeta_{k,v}) = \ell_0(\zeta_{k,v})$. We define $J_{k,v} = [\zeta_{k,v}, \eta_{k,v})$ where with $L_{k,v} = C^{-1}ACR_{k,v}$ we have $L_{k,v} \cdot r_0(\eta_{k,v}) = r_0(\eta_{k,v})$. Confer ([8], Figure 4.2).

A main result of [8] is that each $J_{k,v}$ is a *synchronization interval*: for all α in the interior of $J_{k,v}$, the T_α-orbits of $\ell_0(\alpha)$ and of $r_0(\alpha)$ meet and do so in a common fashion. We furthermore showed that the complement in $(0,\gamma_n)$ of the union of the $J_{k,v}$ is of measure zero. Key to this was determining a maximal common prefix of the $\underline{d}^\alpha_{[1,\infty)}$ for $\alpha \in J_{k,v}$. For that, we used $w = w_n = (-1)^{n-2}, -2, (-1)^{n-3}, -2$ and for k fixed, we let $\mathcal{C} = \mathcal{C}_k = (-1)^{n-3}, -2, w^{k-1}$ and

$\mathcal{D} = C_{k+1}$, and defined $\underline{d}(k,v) = w^k, \mathcal{C}^{c_1-1}\mathcal{D}^{d_1}\cdots\mathcal{D}^{d_{s-1}}\mathcal{C}^{c_s}, (-1)^{n-2}$. This is the common prefix, and also

$$\underline{d}^{\eta_{k,v}}_{[1,\infty)} = \overline{w^k, \mathcal{C}^{c_1-1}\mathcal{D}^{d_1}\cdots\mathcal{D}^{d_{s-1}}\mathcal{C}^{c_s}, (-1)^{n-3}, -2}\,, \tag{5}$$

where the overline indicates a period. We denote the length as a word in $\{-1,-2\}$ of $\underline{d}(k,v)$ by $\underline{S}(k,v)$. There is an expression for $L_{k,v}$ related to $\underline{d}(k,v)$ in a manner similar to how (4) relates $R_{k,v}$ to $\overline{d}(k,v)$. The aforementioned synchronization is of the form

$$T_\alpha^{\overline{S}(v)+1}(\, r_0(\alpha)\,) = T_\alpha^{\underline{S}(v)+1}(\, \ell_0(\alpha)\,). \tag{6}$$

The words v that we use form a tree, $\mathcal{V}$. For general $v \in \mathcal{V}$ and nonnegative integers q, we define a new word $\Theta_q(v) = v(v')^q v''$ where v' is defined — see (10) below — so that any $\overline{d}^{\eta_{k,v}}_{[1,\infty)}$ equals $\overline{d}(k, v(v')^\infty)$ using the obvious extension of our notation, and v'' is an appropriate suffix of v. The tree is rooted at $v = 1$, and for words of particularly short length there are special details of the definition of the Θ_q, see ([8], §4.2). There is also a type of self-similarity of $\mathcal{V}$ which allows the explicit definition of the *derived words* operator $\mathscr{D}$ such that (again for general v) $\mathscr{D} \circ \Theta_q(v) = \Theta_q \circ \mathscr{D}(v)$. In general $\mathscr{D}$ decreases the length of words while preserving various properties, and hence assists in induction proofs. For example, we applied it to prove that every $v \in \mathcal{V}$ is a palindrome.

Very similar structures were used in [8] for the large values of α. We will remind the reader of any of those used in this work, nearer to where they are employed; see § 7.1.

1.3.3. *Two-dimensional maps.* The standard number theoretic planar map associated to a Möbius transformation M is

$$\mathcal{T}_M(x,y) := \begin{pmatrix} M\cdot x, N\cdot y \end{pmatrix} := \begin{pmatrix} M\cdot x, RMR^{-1}\cdot y \end{pmatrix} \quad \text{for } \ x \in \mathbb{I}_M,\ y \in \mathbb{R}\setminus\{(RMR^{-1})^{-1}\cdot\infty\}\,,$$

where $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus, $\mathcal{T}_M(x,y) = (\, M\cdot x, -1/(M\cdot(-1/y))\,)$. An elementary Jacobian matrix calculation verifies that the measure μ on $\mathbb{R}^2$ given by

$$d\mu = \frac{dx\,dy}{(1+xy)^2} \tag{7}$$

is (locally) $\mathcal{T}_M$-invariant.

Fixing n, each T_α is piecewise Möbius, so that there is some partition of its domain into subintervals, $\mathbb{I}_\alpha = \cup_\beta K_\beta$, such that $T_\alpha(x) = M_\beta \cdot x$ for all $x \in K_\beta$. We thus set

$$\mathcal{T}_\alpha(x,y) = \begin{pmatrix} M_\beta\cdot x, RM_\beta R^{-1}\cdot y \end{pmatrix} \quad \text{for } \ x \in K_\beta,\ y \in \mathbb{R}\setminus\{N^{-1}\cdot\infty\}\,,$$

We thus consider $\mathcal{T}_\alpha$ as being defined on the infinite cylinder fibering over I_α and seek a subset on which this map is bijective.

1.4. **Outline of remainder of paper.** The bulk of this paper is devoted to explicitly describing the domains $\Omega = \Omega_\alpha = \Omega_{n,\alpha}$, and then proving that each is a bijectivity domain, up to measure zero, for the corresponding two dimensional map. We give the theorem announcing this in § 2. We collect some main background results in § 3.

In § 4 and 5, we verify that the proposed domains are bijectivity domains, in the case of small α in synchronization intervals. In § 6 we treat the remaining small α; the section ends with the proof of the continuity of the function sending small α to the μ-measure of the associated bijectivity domain.

§ 7 tersely reviews notation and terminology for the setting of large α. § 8 and 9 are the analogs of § 4 and 5, but now for the more challenging setting of large α in the left portion of synchronization intervals (see § 7.1 for the notion of the left- and right-portions). § 10 treats the case of large α in the right portion of synchronization intervals. § 11 and § 12 together form the analog of § 6 for large α.

In § 13 we rely on results of [9] to prove: our maps are eventually expansive; the systems on the bijectivity domains give natural extensions of the interval maps; and, these are all ergodic. In § 14 we use Abramov's formula for the entropy value of an induced system to prove that the function assigning to α the product of the entropy and μ-measure is constant on $(0, 1)$.

Finally, in § 15 we treat the setting of the conjectures, by way of a connection between Rohlin's integral formula for entropy of an interval map and the volume of the unit tangent bundle of the hyperbolic surface/orbifold associated to the map.

1.5. **Thanks.** The last named authors thankfully acknowledge the kind hospitality extended by the Mathematics and Statistics Department of Vassar College during a visit where this work was furthered.

2. Bijectivity domains

The bulk of this paper is involved in providing explicit bijectivity domains for our maps. For parameter α in the closure of synchronization intervals the behavior of the associated two-dimensional map leads to a partitioning of the domain into an upper Ω^+, and a lower portion Ω^-.

2.1. **Two dimensional bijectivity domain associated to every map.** The overall shape of these upper and lower portions of these domains depends upon whether α is in the interior, or is an endpoint of its synchronization interval; in the case of large α, synchronization intervals naturally divide into two portions, as we recall in § 7.1. In each case, we give the domain in detail. In order to announce this fundamental result, here we simply point forward to where these corresponding portions are defined.

	α in Interior of $J_{k,v}$	$\alpha = \zeta_{k,v}$	$\alpha = \eta_{k,v}$
Ω^+	Definition 11	Definition 11	Proposition 60
Ω^-	Definition 12	Proposition 59	Definition 12

Table 1. Location of definitions of $\Omega^{\pm}$ for small α in the closure of a synchronization interval.

	$\eta_{-k,v} < \alpha < \delta_{-k,v}$	$\delta_{-k,v} < \alpha < \zeta_{-k,v}$	$\alpha = \eta_{-k,v}$	$\alpha = \zeta_{-k,v}$	$\alpha = \delta_{-k,v}$
Ω^+	Definition 80	Definition 80	Definition 80	Proposition 109	Definition 80
Ω^-	Definition 81	Definition 106	Definition 81	Definition 106	Definition 81

Table 2. Location of definitions of $\Omega^{\pm}$ for large α in the closure of a synchronization interval.

Compare the following with Figures 1, 7 and 8 for small α values; and with 10, 15 for large α values.

Theorem 10. *Fix $n \geq 3$ and $\alpha \in (0,1)$. If α is in the closure of a synchronization interval, let $\Omega_{n,\alpha} := \Omega^+ \cup \Omega^-$ where $\Omega^\pm$ are given according to the locations pointed to in Tables 1 or 2. For all other α, let $\Omega_{n,\alpha}$ be given as in Propositions 65 or 116 according to whether $\alpha < \gamma_n$ or not.*

Then $\mathcal{T}_{n,\alpha}$ is bijective on $\Omega_{n,\alpha}$, up to μ-measure zero.

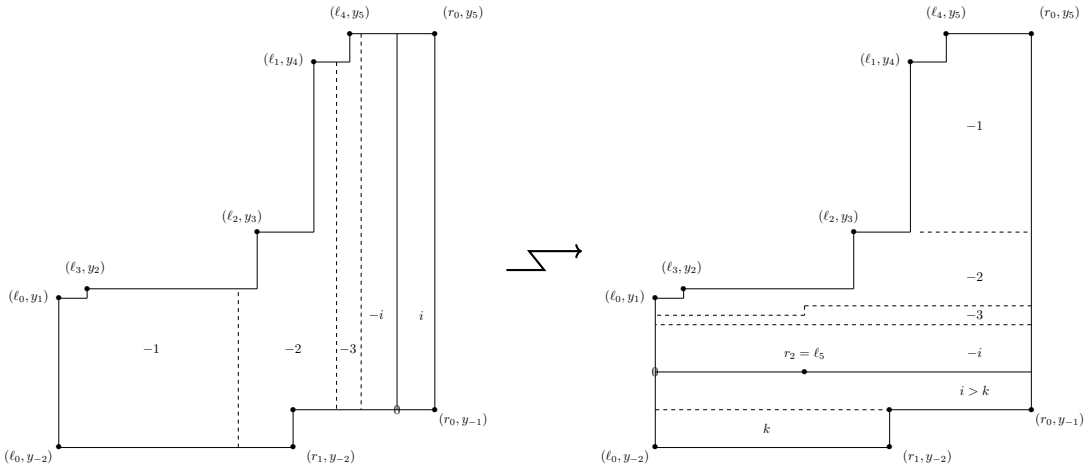

FIGURE 1. The domain $\Omega_{3,0.14}$, with blocks $\mathcal{B}_i$ (see Subsection 5.1), and their images, both denoted by i. Here $R_{k,v} = AC$ and $L_{k,v} = A^{-1}CA^{-2}CA^{-2}CA^{-1}CA^{-1}$, and α is an interior point of $J_{1,1}$.

For α in a synchronization interval, the surjectivity of $\mathcal{T}_\alpha$ requires that the upper boundary of Ω^+ surjects onto itself, and the lower boundary of Ω^- does as well; this imposes certain relations on the heights of their constituent rectangles. Using these relations, we solved for the heights, and thus simply define $\Omega^\pm$ appropriately. Surjectivity also requires that the top and bottom heights satisfy what we call lamination relations, see Subsection 3.2. Section 4 is devoted to verifying that these relations do indeed hold for small α lying in any synchronization interval $J_{k,v}$. We treat the remaining cases of α in subsequent sections.

2.2. **Definition of Ω_α for small α in synchronization interval.** We define our domains Ω_α as the union of an upper and lower region. Compare each of the following two definitions with both Figures 1, 2.

Definition 11. Fix $n \geq 3, v \in \mathcal{V}, k \in \mathbb{N}$ and $\alpha \in (\zeta_{k,v}, \eta_{k,v})$ and let $\underline{S} = \underline{S}(k,v)$.

i.) The upper region is the union of rectangles. We let

$$\Omega^+ = \bigcup_{a=1}^{\underline{S}+1} K_a \times [0, y_a], \tag{8}$$

where K_a and y_a are defined below.

ii.) for $1 \leq a \leq \underline{S}$ we let

$$K_a = [\ell_{i_a}, \ell_{i_{a+1}}),$$

after having labelled as $\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_{\underline{S}+1}}$ the first $\underline{S}+1$ elements of the T_α-orbit of $\ell_0(\alpha)$ in increasing real order.

iii.) Ordering and labelling the elements of the $T_{\zeta_{k,v}}$-orbit of $\ell_0(\zeta_{k,v})$ in exactly the same way, also let $\tau : \{0, \dots, \underline{S}\} \to \{1, \dots, \underline{S}+1\}$, be defined by $\tau(j) = a$ exactly when $i_a = j$. And, finally, for $i \in \{0, \dots, \underline{S}\}$, set

$$y_{\tau(i)} = -\ell_{\underline{S}-i}(\zeta_{k,v}).$$

The ordering of the $\ell_i(\alpha)$ is the same as that of the $\ell_i(\zeta_{k,v})$ because the initial $\underline{S}$ simplified digits of $\ell_0(\alpha)$ are constant on $J_{k,v}$, and by Lemma 32 below.

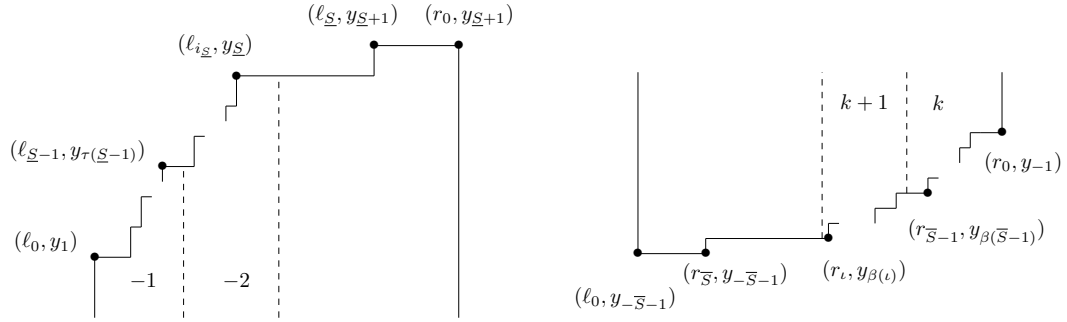

FIGURE 2. Schematic representations showing the most important vertices of the tops and bottoms of blocks, for general $\alpha \in (\zeta_{k,v}, \eta_{k,v})$. Labels in part justified by Lemmas 37 and 38. See Definitions 11, 12 and 28 for notation at vertices. Interior label i denotes partitioning block $\mathcal{B}_i$, see Subsection 5.1 for definitions.

The lower region is also a union of rectangles. (In the following we will use singly subscripted L to denote subintervals, we trust that there will be no confusion with our notation for certain group elements.)

Definition 12.

i.) Let

$$\Omega^- = \bigcup_{b=-1}^{-\overline{S}-1} L_b \times [y_b, 0], \tag{9}$$

where the L_b and y_b in the following items.

ii.) For $-\overline{S} \le b \le -1$ we let

$$L_b = [r_{j_{b-1}}, r_{j_b}]$$

and $L_{-\overline{S}-1} = [\ell_0, r_{\overline{S}}]$, after having labelled as $r_{j_{-1}}, r_{j_{-2}}, \dots, r_{j_{-\overline{S}-1}}$ the points $r_0, r_1, \dots, r_{\overline{S}}$ in *decreasing* order as real numbers.

(Lemma 37 below guarantees that there is no nontrivial intersection of the various L_b.)

iii.) Ordering and labelling the elements of the $T_{\eta_{k,v}}$-orbit of $r_0(\eta_{k,v})$ in exactly the same way, also let $\beta : \{0, \dots, \overline{S}\} \to \{-1, \dots, -\overline{S}-1\}$, be defined by $\beta(i) = b$ exactly when $j_b = i$. And, finally, for $j \in \{0, \dots, \overline{S}\}$, set

$$y_{\beta(j)} = -r_{\overline{S}-j}(\eta_{k,v}).$$

	$\zeta_{1,1}$	approx. value	$\eta_{1,1}$	approx. value
r_0	$(5-\sqrt{21})/2$	0.209	$(-1+\sqrt{21})/2$	0.358
r_1	$(1-\sqrt{21})/2$	-1.791	$(5-\sqrt{21})/2$	0.209
ℓ_0	$(1-\sqrt{21})/2$	-1.791	$(-21+\sqrt{21})/10$	-1.642
ℓ_1	$(-9+\sqrt{21})/10$	-0.442	$(-21+\sqrt{21})/42$	-0.391
ℓ_2	$(-9+\sqrt{21})/6$	-0.736	$(-9+\sqrt{21})/10$	-0.442
ℓ_3	$(-21+\sqrt{21})/10$	-1.642	$(-9+\sqrt{21})/6$	-0.736
ℓ_4	$(-21+\sqrt{21})/42$	-0.391	$\ell_0(\eta_{1,1})$	
ℓ_5	$\ell_1(\zeta_{1,1})$			

TABLE 3. Endpoint orbits for $\alpha = \zeta_{1,1}$ and $\alpha = \eta_{1,1}$ when $n = 3$.

	approx. value		approx. value
y_5	1.791	y_{-1}	-0.209
y_4	1.642	y_{-2}	-0.358
y_3	0.736		
y_2	0.442		
y_1	0.391		

TABLE 4. The horizontal boundary values, the "heights", of $\Omega_{3,\alpha}$ for $\alpha \in (\zeta_{1,1}, \eta_{1,1})$.

Example 13. Fix $n = 3$, $k = 1$, $v = 1$. As indicated in the caption to Figure 1, we have $R_{1,1} = AC$ and $L_{1,1} = A^{-1}CA^{-2}CA^{-2}CA^{-1}CA^{-1}$. In general, $AR_{k,v}$ fixes $r_0(\zeta_{k,v}) = \zeta_{k,v}t$; here, we find $\zeta_{1,1} = (5-\sqrt{21})/4 \sim 0.104$. Similarly, $L_{k,v}$ fixes $r_0(\eta_{k,v})$ and thus here $\eta_{1,1} = (-1+\sqrt{21})/20 \sim 0.179$. Table 3 gives the initial portions of the orbits of each of the endpoints of the interval of definition for both $\alpha = \zeta_{1,1}$ and $\alpha = \eta_{1,1}$.

Since $\underline{S}(1,1) = 4$ and $\overline{S}(1,1) = 1$, the various $y_a, a > 0$ and $y_b, b < 0$ for the Ω_α with $\alpha \in (\zeta_{1,1}, \eta_{1,1})$ can now be determined, see Table 4. In particular, this second table gives the values of the various "heights" of rectangles shown in Figure 1.

3. Elementary identities and another order on words

We collect a few basic tools for use in proving that the images of regions under the various two-dimensional maps $\mathcal{T}_M$ are as we claim. We imagine that the reader will initially skip this section and return to it as its results are applied.

3.1. Relating y-coordinate action of $\mathcal{T}_M$ to x-coordinate action.

In what follows, a left arrow over a word indicates the word taken in reverse order. Recall from Subsection 1.3.3 that our two-dimensional maps are defined using the matrix R.

Lemma 14. *If X is a word in $\{A^pC,\, p \in \mathbb{Z},\}$ and $x_1, x_2 \in \mathbb{R}$, then*

$$RXR^{-1} \cdot (-x_1) = -x_2 \text{ if and only if } x_1 = \overleftarrow{X} \cdot x_2\,.$$

In particular, for real x,

$$(RA^pCR^{-1})^{-1} \cdot (-x) = -\,(A^pC \cdot x).$$

Proof. We prove the statements in opposite order. First, observe that A^pC is of the form $\begin{pmatrix} a & b \\ -b & 0 \end{pmatrix}$. And therefore, using M^t to denote the transpose of a matrix M, $(A^pC)^t \cdot (-x) = -(A^pC \cdot x)$ for any real x. Since for any $M \in \mathrm{SL}_2(\mathbb{R})$, $RMR^{-1} = (M^t)^{-1}$ (projectively), we have that $(RA^pCR^{-1})^{-1} \cdot (-x) = -(A^pC \cdot x)$ for any real x.

Now, $RXR^{-1} \cdot (-x_1) = -x_2$ if and only if $-x_1 = (RXR^{-1})^{-1} \cdot (-x_2)$. Writing $X = X_1 \cdots X_{|X|}$, we have $(RXR^{-1})^{-1} = (RX_{|X|}R^{-1})^{-1} \cdots (RX_1R^{-1})^{-1}$, and induction gives $RXR^{-1} \cdot (-x_1) = -x_2$ if and only if $x_1 = \overleftarrow{X} \cdot x_2$. □

3.2. **Relations for lamination.** The simplest manner for regions to have images under $\mathcal{T}_{n,\alpha}$ that abut one another, thus to *laminate*, occurs when regions fibering above two distinct cylinders are mapped to lie directly one above the other. We give the simple formulas that imply such behavior in the special cases that arise in this work.

Lemma 15. *For any $n \geq 3$ the matrices A and CR commute. Furthermore, $RAR^{-1} = C^{-1}AC$.*

Proof. First, $CR = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and thus clearly commutes with A. The stated equality then easily follows. □

Lemma 16. *Fix $n \geq 3$ and $\alpha \in [0,1]$. Suppose that $a < b, c < d$ are real numbers. The rectangle $\Delta_\alpha(k,l) \times [a,b]$ is mapped by $\mathcal{T}_{n,\alpha}$ below the image of $\Delta_\alpha(k+1,l) \times [c,d]$ so as to share a common horizontal line segment if and only if*

$$b = R^{-1}C^{-l}AC^lR \cdot c\,.$$

In particular, if $l = 1$ then this holds if and only if $b = c + t$.

Proof. The second coordinate of $\mathcal{T}_{n,\alpha}(x,y)$ for $x \in \Delta(k,l)$ is given by $RA^kC^lR^{-1} \cdot y$, and similarly when $x \in \Delta(k+1,l)$. Since all of the matrices here are of positive determinant, the images of the first rectangle highest horizontal edge of y-value $RA^kC^lR^{-1} \cdot b$, while the second will have lowest horizontal edge $RA^{k+1}C^lR^{-1} \cdot c$, respectively. Setting these equal and solving gives the first result.

The second result follows since CR and A commute, and R^2 acts as the identity. □

Lemma 17. *The rectangle $\Delta(k,l) \times [a,b]$ is mapped by $\mathcal{T}_{n,\alpha}$ above the image of $\Delta(k,l+1) \times [e,f]$ so as to share exactly a common horizontal line if and only if*

$$a = RCR^{-1} \cdot f\,.$$

Proof. Here we simply solve for a in the equation $RA^kC^lR^{-1} \cdot a = RA^kC^{l+1}R^{-1} \cdot f$. □

The next lemma is also immediate.

Lemma 18. *The rectangle $\Delta(k,l) \times [a,b]$ is mapped by $\mathcal{T}_{n,\alpha}$ below the image of $\Delta(k+1,l+1) \times [g,h]$ so as to share exactly a common horizontal line if and only if*

$$b = RC^{-l}AC^{l+1}R^{-1} \cdot g\,.$$

3.3. **An order on words.** The order on words mentioned in Subsection 1.3.1 agrees with the standard ordering of real numbers, and since each T_α takes on only values less than $r_0(\alpha)$, this leads to a corresponding order on words of $\mathcal{V}$. This is what might be called an alternating dictionary order. In fact, we made use this order in ([8], Lemmas 4.16 and 6.7) without giving it a distinct notation, including stating, without proof, a variant of our Lemma 20 below.

Definition 19. Let $\prec\!\!\!\prec$ be the dictionary order on words $a = a_1a_2\cdots a_m$ in natural numbers induced by a dictionary ordering extending (for all words a, b and any $i, j \in \mathbb{N}$): $a_{2i-1} \prec\!\!\!\prec b_{2i-1}$ if $a_{2i-1} < b_{2i-1}$; $a_{2j} \prec\!\!\!\prec b_{2j}$ if $a_{2j} > b_{2j}$; $b_{2j-1} \prec\!\!\!\prec a_{2i}$.

The following result justifies our introduction of the second word order; recall that $\overline{d}(k, v)$ is given in (3).

Lemma 20. *Fix n, k. Suppose $a, b \in \mathcal{V}$, then $\overline{d}(k, a) \preceq \overline{d}(k, b)$ if and only if $a \preceq\!\!\!\prec b$.*

Proof. The word order on simplified digits is such that $k + 1 \prec k$ and hence $\overline{d}(k, a) \prec \overline{d}(k, b)$ if and only if there is some i such that $\overline{d}(k, a)_{[1,i-1]} = \overline{d}(k, b)_{[1,i-1]}$ and $\overline{d}(k, b)_{[i,i]} = k$ while $\overline{d}(k, a)_{[i,i]} = k + 1$. In particular, either (1) $\overline{d}(k, a)_{[i-1,i-1]} = \overline{d}(k, b)_{[i-1,i-1]} = k$ and thus there is some j such that $b_{2j-1} \geq a_{2j-1} + 1$; or, (2) $\overline{d}(k, a)_{[i-1,i-1]} = \overline{d}(k, b)_{[i-1,i-1]} = k + 1$ and thus there is some j such that $a_{2j} \geq b_{2j} + 1$. In either case, we find that $a \prec\!\!\!\prec b$. The case of equality is easily verified. □

A proof of the following can be given by use of the derived words map $\mathscr{D}$. We let σ be the left-shift on letters, thus $\sigma(c_1d_1\cdots d_{s-1}c_s) = d_1\cdots d_{s-1}c_s$.

Lemma 21. *Each $v \in \mathcal{V}$ is* self-dominant *in the sense that $\sigma^j(v) \preceq\!\!\!\prec v$ for all $1 \leq j \leq |v|$.*

Recall that ([8], Definition 4.8) defines

$$v' = \begin{cases} 1(c_1 - 1)1c_2\cdots 1c_s & \text{if } \ c_1 \neq 1\,, \\ (d_1 + 1)1\cdots d_{s-1}1 & \text{otherwise}\,, \end{cases} \tag{10}$$

and that we show in [8] that when $c_1 > 1$ then for all i, $c_i \in \{c_1, c_1 - 1\}$ as well as that when $c_1 = 1$, each $d_i \in \{d_1, d_1 + 1\}$. In ([8], Definition 4.10), we give $\Theta_q(v)$ for short words, and then recursively define values of the operators Θ_q: whenever $v = \Theta_p(u) = uv''$ for some $p \geq 0$ and some suffix v'', we let $\Theta_q(v) = v(v')^q v''$.

Lemma 22. *Suppose that $v \in \mathcal{V}$. Then $v' \prec\!\!\!\prec \sigma^j(v)$ for any $j < |v|$.*

Proof. Note that if v is such that no $c_i = c_1 - 1$ or no $d_i = d_1 + 1$ then it is immediate that every $v' \prec\!\!\!\prec \sigma^j(v)$. Furthermore, for v of length one that the result holds is easily verified. We therefore proceed by induction, assuming our result for a given v and proving that it holds for $\Theta_q(v)$.

Now suppose that $v' \prec\!\!\!\prec \sigma^j(v)$ holds for all $j < |v|$. Since, $\Theta_q(v) = v(v')^q v''$ we have $(\,\Theta_q(v)\,)' = (v')^{q+1}v''$. We aim to show that $(\,(\Theta_q(v)\,)'\,)^\infty \prec\!\!\!\prec \sigma^j(\,\Theta_q(v)\,)$ for all $j \leq |\Theta_q(v)|$. From our hypothesis, if $j < |v|$ then $(\,\Theta_q(v)\,)' \prec\!\!\!\prec \sigma^j(\,\Theta_q(v)\,)$. Also by this hypothesis we have $v' \prec\!\!\!\prec v''$, and hence $(\,\Theta_q(v)\,)' \prec\!\!\!\prec \sigma^j(\,\Theta_q(v)\,)$ when $j = |v| + p|v'|$, with $0 \leq p \leq q$. The inequality for the values $j = 1 + |v| + p|v'|$ holds due to the form of v'. For $j = r + |v| + p|v'|$, with $2 \leq r < |v|$, $\sigma^j(\,\Theta_q(v)\,)$ begins with a suffix of v, and thus our induction hypothesis shows that the inequality holds for all of these values. □

Combined with the property of the self-dominance of v, the following result aids in determining explicit orbit elements with extremal value. See for instance the proof of Lemma 32.

Lemma 23. *Suppose that $v \in \mathcal{V}$ is of length greater than one. The parent of v is the longest proper suffix of v that is also a prefix.*

Proof. We have that $v = u(u')^q u''$ with $q \geq 0$ and u the parent of v. Since every word in $\mathcal{V}$ is a palindrome, certainly u is a suffix of v. Any longer suffix/prefix must of course admit u as both a prefix and a suffix. In particular, it suffices to show that there are no internal appearances of

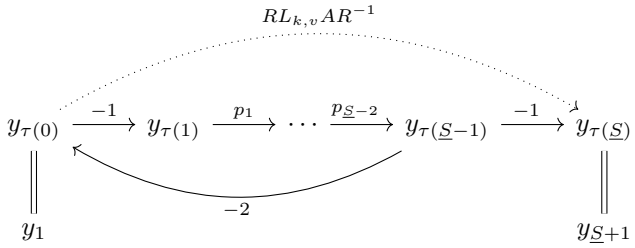

FIGURE 3. Relations on the top heights of rectangles comprising Ω^+ for $\alpha \in [\zeta_{k,v}, \eta_{k,v})$. Here -1 denotes $RA^{-1}CR^{-1}$, and -2 denotes $RA^{-2}CR^{-1}$, similarly each p_i denotes $RA^{p_i}CR^{-1}$ with notation as in Definition 24. The leftmost vertical equality sign is due to Definition 11, the rightmost is due to Lemma 32. Note that up to relabeling vertices and arrows, the top row is the reverse diagram of that for the orbit of $\ell_0(\zeta_{k,v})$; see Figure 4.

u within v. Since this sort of containment is a property preserved by the derived words operator $\mathscr{D}$, it suffices to check this in the setting of the shortest possible v. But for these short v, the result is clear. □

4. Relations on heights of rectangles, for small α

Throughout this section, unless otherwise stated, we assume that α is a fixed value in the interior of some synchronization interval $J_{k,v}$. Recall that a review of definitions and notation is given in Subsection 1.3.

We state and prove the main results on the heights of the rectangles of Definitions 11 and 12 whose union gives Ω_α. For larger values of α, see Subsection 8.3 and Section 10.

4.1. **Heights of rectangles, top half.**

Definition 24. Considering $\underline{d}(k,v)$ as a word in $\{-1,-2\}$, for $i \in \{0,\dots,\underline{S}\}$ let p_i be the first letter of $\sigma^i(\,\underline{d}(k,v)\,)$, where σ is the left-shift on letters. That is, p_i is the first simplified digit of $\ell_i = \ell_i(\alpha)$; equivalently, $\ell_i \in \Delta_\alpha(p_i, 1)$.

Compare the following with Figures 2, 3 and 4.

Lemma 25. *The following hold:*

i.) *For all* $i \in \{0,\dots,\underline{S}-1\}$, $y_{\tau(i+1)} = RA^{p_i}CR^{-1}\cdot y_{\tau(i)}$;

ii.) $y_{\tau(\underline{S})} = RL_{k,v}AR^{-1}\cdot(-\ell_{\underline{S}}(\zeta_{k,v})\,)$;

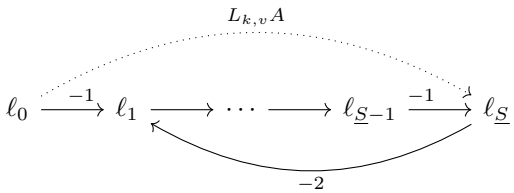

FIGURE 4. The orbit of $\ell_0 = \ell_0(\zeta_{k,v})$ under $T_{n,\zeta_{k,v}}$. Here -1 denotes $A^{-1}C$, and -2 denotes $A^{-2}C$. Compare with Table 3.

iii.) $RA^{-2}CR^{-1} \cdot y_{\tau(\underline{S}-1)} = y_{\tau(0)}$;
iv.) $y_1 = RA^{-1}R^{-1} \cdot (-\ell_0(\zeta_{k,v}))$.

Proof. Definition 11 gives $y_{\tau(i)} = -\ell_{\underline{S}-i}(\zeta_{k,v})$ and $y_{\tau(i+1)} = -\ell_{\underline{S}-i-1}(\zeta_{k,v})$. Of course, $A^{p_{\underline{S}-i-1}}C \cdot \ell_{\underline{S}-i-1}(\zeta_{k,v}) = \ell_{\underline{S}-i}(\zeta_{k,v})$. Since $v \in \mathcal{V}$ is a palindrome, it follows that $\underline{d}(k,v)$ is a palindrome as a word in $\{-1,-2\}$; in particular, $p_i = p_{\underline{S}-i-1}$. Hence, $-y_{\tau(i)} = -A^{p_i}C \cdot (-y_{\tau(i+1)})$. Lemma 14 now implies $y_{\tau(i+1)} = RA^{p_i}CR^{-1} \cdot y_{\tau(i)}$.

That $y_{\tau(\underline{S})} = RL_{k,v}AR^{-1} \cdot (-\ell_{\underline{S}}(\zeta_{k,v}))$ follows by elementary induction.

By ([8], Lemma 4.5), see also ([8], Lemma 4.33), $A^{-2}C \cdot \ell_{\underline{S}}(\zeta_{k,v}) = \ell_1(\zeta_{k,v})$. Lemma 14 yields $y_{\tau(0)} = RA^{-2}CR^{-1} \cdot y_{\tau(\underline{S}-1)}$.

Finally, since $\underline{d}(k,v)$ ends with a -1, we find $y_1 = y_{\tau(0)} = RA^{-2}C(A^{-1}C)^{-1}R^{-1} \cdot y_{\tau(\underline{S})}$, and thus $y_1 = RA^{-1}R^{-1} \cdot (-\ell_0(\zeta_{k,v}))$. □

Lemma 26. *We have $\mu(\Omega^+) < \infty$.*

Proof. To show that $\mu(\Omega^+) < \infty$, we show that for each negative $x \in \mathbb{I}_\alpha$, the fiber in Ω^+ above it, say $\{x\} \times [0, y_a]$, lies below the curve $y = -1/x$. Set $\zeta = \zeta_{k,v}$; by Lemma 25, $y_1 = RA^{-1}R \cdot (-\ell_0(\zeta)) = 1/(t - 1/\ell_0(\zeta))$ and thus $y_1 < -1/\ell_0(\zeta)$ holds if $\ell_0(\zeta)$ lies between the roots of $x^2 + tx - 1 = 0$. Since this interval has left endpoint less than $-t$ and has a positive right endpoint, this condition holds and hence $y_1 < -1/\ell_0(\zeta)$. Thus, all of the line segment $\{\ell_{i_1}(\zeta)\} \times [0, y_1]$ lies below $y = -1/x$.

Recall that $\ell_{i_1} = \ell_0$. Lemma 25 shows that for each a, there is an $M \in G_n$ sending $(\ell_{i_1}(\zeta), y_1)$ to $(\ell_{i_a}(\zeta), y_a)$; compare Figures 3 and 4. A simple check confirms that any M is such that $\mathcal{T}_M$ sends the locus $y = -1/x$ to itself (and since M^{-1} exists this is the only preimage of the locus). Since every $\mathcal{T}_M$ is both orientation preserving and μ-mass preserving, each $\{\ell_{i_a}(\zeta)\} \times [0, y_a]$ lies below $y = -1/x$. For each $\alpha \in J_{k,v}$, one has $\ell_{i_a}(\zeta) < \ell_{i_a}(\alpha)$ and it follows that each $\{\ell_{i_a}(\alpha)\} \times [0, y_a]$ lies below $y = -1/x$ (whenever $\ell_{i_a}(\alpha) < 0$), and thus all of $K_a \times [0, y_a]$ does also. Thus, the result holds. □

4.2. **Heights of rectangles, bottom half.**

Definition 27. Considering $\overline{d}(k,v)$ as a word in $\{k, k+1\}$, for each $j \in \{-1, \ldots, -\overline{S}\}$ let q_j be the first letter of $\sigma^j(\overline{d}(k,v))$. That is, q_j is the first simplified digit of r_j; equivalently, $r_j \in \Delta(q_j, 1)$.

To simplify typography, we define the following.

Definition 28. If $u \in \mathcal{V}$ is the parent of v, let

$$\iota = |\overline{d}(k,u)|,$$

where the length of $\overline{d}(k,u)$ is as a word in $\{k, k+1\}$. When $v = 1$, let $\iota = 0$.

Compare the following with Figures 2, 5 and 6. Recall that the details of the construction of Ω^- are given in Definition 12.

Lemma 29. *The following hold:*

i.) *For all $j \in \{-1, \ldots, -\overline{S} - 2\}$, $y_{\beta(j+1)} = RA^{q_j}CR^{-1} \cdot y_{\beta(j)}$;*
ii.) $y_{\beta(\overline{S})} = RR_{k,v}R^{-1} \cdot y_{\beta(0)}$;
iii.) $y_{\beta(0)} = RA^{k+1}CR^{-1} \cdot y_{\beta(\overline{S}-1)}$;
iv.) *if v has parent u, then $y_{\beta(\iota)} = RR_{k,u}R^{-1} \cdot y_{\beta(0)}$;*
v.) $y_{\beta(0)} = RAR^{-1} \cdot (-r_0(\eta_{k,v}))$.

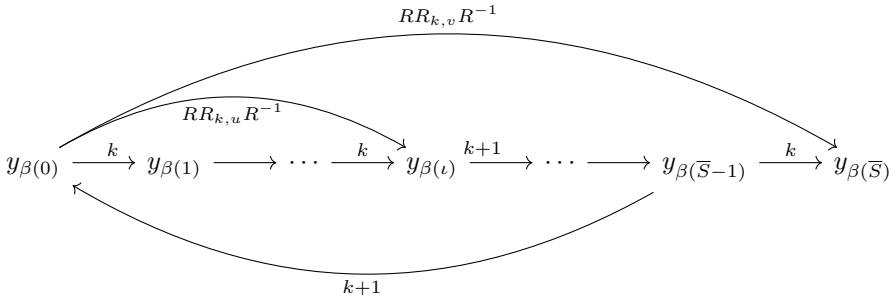

FIGURE 5. Relations on the bottom heights of rectangles comprising Ω^- for $\alpha \in (\zeta_{k,v}, \eta_{k,v})$, when v has parent u. Arrow label k denotes RA^kCR^{-1}, and $k+1$ denotes $RA^{k+1}CR^{-1}$.

Proof. As in the proof of Lemma 25, the first two statements follow from Lemma 14. By ([8], Lemma 4.5), $A^{k+1}C \cdot r_{\overline{S}}(\eta_{k,v}) = r_1(\eta_{k,v})$, and thus Lemma 14 yields the third statement. Since v is palindromic and has prefix u, it also has suffix u (itself a palindrome). From this, Lemma 14 yields the fourth statement. Finally, due to the palindromic nature of v, we know that the final digit of $\overline{d}(k,v)$ is k and hence from (i), $y_{\beta(\overline{S})} = RA^kCR^{-1} \cdot y_{\beta(\overline{S}-1)}$. Thus, (iii) gives that $y_{\beta(0)} = RAR^{-1} \cdot y_{\beta(\overline{S}-1)}$. The result holds, since by definition $y_{\beta(\overline{S})} = -r_0(\eta_{k,v})$. □

Lemma 30. *We have $\mu(\Omega^-) < \infty$.*

Proof. To show that $\mu(\Omega^-) < \infty$, we show that for each positive $x \in \mathbb{I}_\alpha$, its fiber $\{x\} \times [y_b, 0]$ lies above the curve $y = -1/x$. Set $r_0 = r_0(\eta_{k,v})$; simplifying in Lemma 29 (v) gives $y_{\beta(0)} = -r_0/(t\,r_0 + 1)$. Since the interval between the roots of $x^2 - tx - 1 = 0$ contains all positive numbers up to at least t, it easily follows that $y_{\beta(0)} > -1/r_0$. The proof of Lemma 26 can be further adjusted *mutatis mutandis*. □

4.3. **Ordering the ℓ_i; top heights increase.** To further simplify typography, we define the following.

Definition 31. Let $d_\alpha(x)$ denote the sequence of *simplified* α-digits of x. Similarly, set $\underline{S} = \underline{S}(k,v)$ and for all i let ℓ_i denote $\ell_i(\alpha)$.

Lemma 32. *Fix $n \geq 3$ and $k \in \mathbb{N}$ and $\alpha \in J_{k,v}$. We have that*

i.)
$$\ell_{\underline{S}} = \max_{0 \leq i \leq \underline{S}} \ell_i \,.$$
Equivalently, $\tau(\underline{S}) = \underline{S} + 1$ and $i_{\underline{S}+1} = \underline{S}$.

ii.) *Moreover,*
$$d_\alpha(\ell_{i_{\underline{S}}}) = \begin{cases} -2, (-1)^{n-3}, -2, (-1)^{n-2},\, d_\alpha(\ell_{\underline{S}}) & \text{if } v = 1; \\ -2, (-1)^{n-3}, -2,\, \sigma^{n-1}(\,\underline{d}(k,u)\,),\, d_\alpha(\ell_{\underline{S}}) & \text{if } u \text{ is the parent of } v. \end{cases}$$

Proof. Since $\ell_{\underline{S}}(\alpha) \geq \ell_{\underline{S}}(\zeta_{k,v})$ for any $\alpha \in J_{k,v}$, and our order of words agrees with the usual order on real numbers, it suffices to show that $\sigma^j(\,\underline{d}(k,v)\,)\, d_\alpha(\,\ell_{\underline{S}}(\alpha)\,) \prec d_\alpha(\,\ell_{\underline{S}}(\zeta_{k,v})\,)$ for all $0 \leq j < \underline{S}$.

Letting $\zeta = \zeta_{k,v}$, ([8], Lemma 4.33) gives

$$\begin{aligned} \underline{d}^{\zeta}_{[1,\infty)} &= -1, \overline{(-1)^{n-3}, -2, (-1)^{n-3}, -2, w^{k-1}, \mathcal{C}^{c_1-1}\mathcal{D}^{d_1}\cdots\mathcal{D}^{d_{s-1}}\mathcal{C}^{c_s}, (-1)^{n-2}, -2} \\ &= \underline{d}(k,v), \overline{-2, (-1)^{n-3}, -2, \mathcal{C}^{c_1}\mathcal{D}^{d_1}\cdots\mathcal{D}^{d_{s-1}}\mathcal{C}^{c_s}, (-1)^{n-2}}. \end{aligned} \tag{11}$$

Hence, for the first result, it is sufficient to prove that

$$\sigma^j(\,\underline{d}(k,v)\,) \prec -2, (-1)^{n-3}, -2, \mathcal{C}^{c_1}, \mathcal{D}^{d_1}\cdots\mathcal{C}^{c_s}, (-1)^{n-2} \tag{12}$$

for all $0 \le j < \underline{S}$.

Temporarily viewing $\underline{d}(k,v)$ as a word in $a := (-1)^{n-2}, b := (-1)^{n-3}, c := -2$, and letting $\star$ denote the expansion of $\ell_{\underline{S}}$,

$$\underline{d}^{\alpha}_{[1,\infty)} = ac[bc(acbc)^{k-1}]^{c_1}\,[bc(acbc)^{k}]^{d_1}\,\cdots\,[bc(acbc)^{k}]^{d_{s-1}}\,[bc(acbc)^{k-1}]^{c_s}\,a\star\,.$$

We may use $a \prec b \prec c \preceq \star$ and easily determine that the greatest subword of $\underline{d}(k,v)$ in two of these letters is cb, in three is cbc and so forth (of course, if $n = 3$ special considerations are necessary). Thus, when $v = 1$ we find that the greatest suffix of $\underline{d}^{\alpha}_{[1,\infty)}$ is $cbca\star$. For the remainder of the proof, we no longer explicitly refer to the letters $a, b, c, \star$, but the reader may still find them of use.

Due to the maximality of c_1 and the self-domination of v, we find that when c_1 appears only as the initial and final letter of v, $\sigma^j(\,\underline{d}(k,v)\,)$ has maximal value $-2, (-1)^{n-3}, -2, \mathcal{C}^{c_1}, (-1)^{n-2}$. Such appearances of c_1 implies that $v = \Theta_q(c_1)$ for $q \ge 1$. These cases are also easily confirmed.

For all other v, due to Lemma 21, the maximal value of $\sigma^j(\,\underline{d}(k,v)\,)$ occurs when $\sigma^{j+n-1}(\,\underline{d}(k,v)\,)$ equals some prefix of $\mathcal{C}^{c_1}, \mathcal{D}^{d_1}\cdots\mathcal{C}^{c_s}, (-1)^{n-2}$. Of course, by its very definition, $\sigma^{j+n-1}(\,\underline{d}(k,v)\,)$ is a suffix of $\underline{d}(k,v)$. From this simultaneous prefix/suffix property, there is an ancestor $u \in \mathcal{V}$ of v such that $\sigma^{j+n-1}(\,\underline{d}(k,v)\,) = \sigma^{n-1}(\,\underline{d}(k,u)\,)$. Maximality combined with Lemma 23 implies that u is in fact the direct parent of v. We have thus proved the second statement.

Of course, $\underline{d}(k,u)$ ends with $w, \mathcal{C}^{c_s}, (-1)^{n-2}$. But, v is strictly longer than u, and hence the prefix corresponding to u, in σ^{n-1} applied to the period of $\ell_{\underline{S}(\zeta_{k,v})}$, is followed by a $-1^{n-3}, -2$. Therefore, inequality (12) holds, and the result is proven. □

Lemma 33. *The T_α-image of the set of ℓ_i in $\Delta_\alpha(-1,1)$ with $i < \underline{S}$ lies to the right of the T_α-image of the set of $\ell_i, i < \underline{S}$ in $\Delta_\alpha(-2,1)$. In particular, $\tau(1) = 1 + \tau(1 + i_{\underline{S}})$.*

Proof. Certainly ℓ_0 is the smallest ℓ_i in $\Delta_\alpha(-1,1)$ with $i < \underline{S}$. By definition, $\ell_{i_{\underline{S}}}$ is the largest ℓ_i with $i < \underline{S}$; it certainly lies in $\Delta_\alpha(-2,1)$. Since each A^pC is an order preserving function, it suffices to show that $\ell_1 > \ell_{1+i_{\underline{S}}}$. We thus refer to Lemma 32; our inequality when $v = c_1$ can be directly verified. For general v, since the expansion of $\ell_{i_{\underline{S}}}$ begins with a -2, inequality (12) implies that $\ell_{i_{\underline{S}}+1}$ has expansion less than the expansion of ℓ_1. The result thus holds. □

Lemma 34. *For all $0 \le i < \underline{S}$,*

$$\tau(i) + \tau(\underline{S} - i) = \underline{S} + 2.$$

Proof. Since certainly $\tau(0) = 1$, Lemma 32 yields the result when $i = 0$.

Lemma 33 implies that $\tau(i+1) - \tau(i)$ is determined by the simplified digit of ℓ_i. Indeed, if $\ell_i \in \Delta_\alpha(-1,1)$, then $\tau(i+1) = \tau(i) + (\tau(1) - \tau(0))$. Similarly, if $\ell_i \in \Delta_\alpha(-2,1)$, then $\tau(i+1) = \tau(i) + (\tau(1 + i_{\underline{S}}) - \tau(i_{\underline{S}}))$.

Due to the palindromic nature of $\underline{d}(k,v)$, the expansions of $\ell_{\underline{S}-i-1}$ and ℓ_i share the same initial digit. Therefore, $\tau(\underline{S} - i) - \tau(\underline{S} - i - 1) = \tau(i+1) - \tau(i)$. The result thus follows from induction. □

Lemma 35. *For $1 \le a \le \underline{S}$,*

$$y_{a+1} > y_a \,.$$

Proof. By definition, there is an i such that $y_a = y_{\tau(i)} = -\ell_{\underline{S}-i}(\zeta_{k,v})$. Due to the negative sign, this is the $\underline{S} + 2 - \tau(\underline{S} - i)^{\text{th}}$ of the values in $\{-\ell_{i'}(\zeta_{k,v}) \,|\, i' < \underline{S}\}$ ordered as increasing real numbers. Lemma 34 yields that this is the $\tau(i)^{\text{th}}$ value. But, $a = \tau(i)$, and thus the position in this ordered set increases with a. □

Corollary 36. *We have*

$$y_{\underline{S}} = -\ell_{i_2}(\zeta_{k,v}).$$

Proof. The lemma yields that $y_{\underline{S}}$ is the second largest height. Since these heights are the various $-\ell_i(\zeta_{k,v})$, this second largest height is indeed negative one times the second smallest $\ell_i(\zeta_{k,v})$. □

4.4. **Ordering the r_j; bottom heights increase.**

Lemma 37. *We have that*

$$r_{\overline{S}} = \min_{0 \le j \le \overline{S}} r_j \,.$$

Equivalently, $\beta(\overline{S}) = -\overline{S} - 1$.

Proof. To show that $r_{\overline{S}}$ is the minimal value in the initial portion of the orbit of r_0 for each $\alpha \in J_{k,v} = [\zeta_{k,v}, \eta_{k,v})$, it suffices to show that $\sigma^j(\,\overline{d}(k,v)\,) \succ d_{[1,\infty)}(\,r_{\overline{S}}\,)$ for all $j < \overline{S}$, where $d_{[1,\infty)}(x)$ denotes the sequence of simplified digits of x. Since $r_{\overline{S}}(\eta_{k,v})$ is larger than $r_{\overline{S}}(\alpha)$ for all $\alpha \in J_{k,v}$, and $\overline{d}^{\eta_{k,v}}_{[1,\infty)} = \overline{d}(k, v(v')^\infty)$, it in fact suffices to show the $\sigma^j(\,\overline{d}(k,v)\,) \succ \overline{d}(k,(v')^\infty)$. But, Lemma 20 shows that this is equivalent to the result of Lemma 22. □

We can also identify the second smallest r_j. Recall that $\iota = \overline{S}(u)$, when $v = \Theta_q(v)$ and $\iota = 0$ when $v = 1$.

Lemma 38. *We have*

$$r_\iota = \min_{0 \le j < \overline{S}} r_j \,.$$

Equivalently, $\beta(\iota) = -\overline{S}$.

Proof. The statement for $v = 1$ is trivial.

Due to Lemma 37, we can repose the statement for $v \ne 1$ as $\sigma^\iota(\,\Theta_q(u)\epsilon\,) \prec \sigma^j(\,\Theta_q(u)\epsilon\,)$ for all $j < |v|$, where ϵ is any word such that $\epsilon \prec \sigma^j(\,\Theta_q(u)\epsilon\,)$ for all $j < |v|$. (That is, ϵ is simply any sufficiently small word, allowing comparisons to accord with the minimality of $r_{\overline{S}(v)}$.)

Now, this reformulated statement is easily verified for v and for any $\Theta_0^h \circ \Theta_p(v)$ when v is of length one. We can thus apply the morphism $\mathscr{D}$ and deduce that it is true for all $v \in \mathcal{V}$. □

Lemma 39. *We have that $\beta(1 + \iota) = 1 + \beta(1)$.*

Proof. The argument here is the exact analog of the proof of Lemma 33. □

Lemma 40. *For $-1 \le b \le -\overline{S} + 1$,*

$$y_{b-1} < y_b \,.$$

Proof. The proof here is the exact analog of that for Lemma 35. □

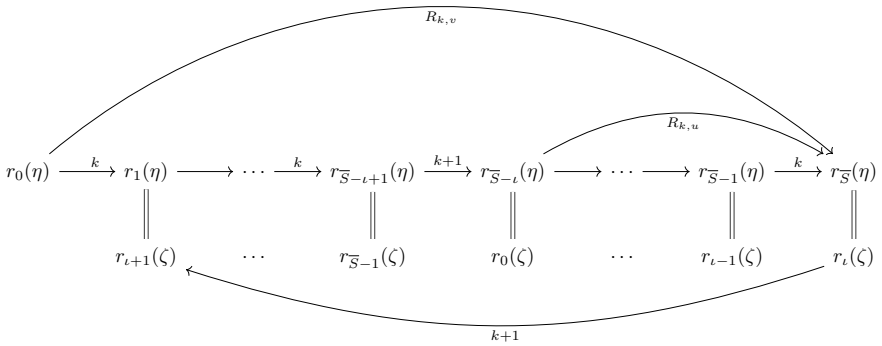

FIGURE 6. The $T_{\eta_{k,v}}$-orbit of $r_0(\eta_{k,v})$ contains the initial $\overline{S}$ elements of the $T_{\zeta_{k,v}}$-orbit of $r_0(\zeta_{k,v})$, see Proposition 41. Here η, ζ denote $\eta_{k,v}, \zeta_{k,v}$, respectively. Arrow labels $k, k+1$ denote $A^kC, A^{k+1}C$, respectively.

4.5. **Meeting of $T_{\zeta_{k,v}}$-orbits with $T_{\eta_{k,v}}$-orbits.** We now explore a phenomenon that already can be noticed in the setting of Example 13, see in particular Table 3.

In the following, note that although in general for different values of α, a fixed sequence of (simplified) digits determines distinct real values, (after having fixed n) purely periodic digit sequences correspond to fixed points of well defined elements of the group G_n, independent of choice of α.

Recall from Definition 28 that if u is the parent of v in $\mathcal{V}$ and $k \in \mathbb{N}$ are fixed, then ι is short hand notation for $|\overline{d}(k,u)|$. Compare the following with Figure 6.

Proposition 41. *Fix $n \geq 3, k \in \mathbb{N}, v \in \mathcal{V}$. Then $T_{\eta_{k,v}}$-orbit of $r_0(\eta_{k,v})$ contains the initial $\overline{S}$ elements of the $T_{\zeta_{k,v}}$-orbit of $r_0(\zeta_{k,v})$.*

In particular,

i.) *For any length one v, we have $r_1(\eta_{k,v}) = r_0(\zeta_{k,v})$;*

ii.) *For $|v| > 1$,*

$$r_{\overline{S}}(\eta_{k,v}) = r_\iota(\zeta_{k,v}).$$

Proof. We first prove the enumerated statements. For all v, ([8], Lemma 4.5) implies that $A^{k+1}CR_{k,v}(A^kC)^{-1}$ fixes $r_1(\eta_{k,v})$, while $AR_{k,v}$ fixes $r_0(\zeta_{k,v})$. When $v = c_1$, we have $R_{k,v} = (A^kC)^{c_1}$ and thus $r_1(\eta_{k,v}), r_0(\zeta_{k,v})$ are fixed by the same element of G_n. Both $r_1(\eta_{k,v}), r_0(\zeta_{k,v})$ and their respective orbits are certainly positive and less than $r_0(\gamma_n)$ which in turn is less than one; in this interval each A^jC defines an expansive map. Therefore, both $r_1(\eta_{k,v}), r_0(\zeta_{k,v})$ must be the repulsive fixed point of the hyperbolic $AR_{k,v}$, and hence equality does hold.

For longer length v, let u be the parent of v. Then $r_{\overline{S}}(\eta_{k,v})$ is fixed by $R_{k,v}(A^kC)^{-1}A^{k+1}C$ and $r_\iota(\zeta_{k,v})$ by $R_{k,u}AR_{k,v}(R_{k,u})^{-1}$. With mild abuse of notation, the first of these is $R_{k,v'}$, the second is $R_{k,u^{-1}\overleftarrow{v'}u}$. (Recall that v' is given in (10).) Now, ([8], Remark 4.14) gives $v = uaz$, with v, u, a, z and $z'a$ all being palindromes. Thus, $\overleftarrow{v'} = \overleftarrow{(zau)'} = \overleftarrow{z'au} = u\overleftarrow{(z'a)} = uz'a$. Hence, $u^{-1}\overleftarrow{v'}u = z'au = v'$, and therefore $r_{\overline{S}}(\eta_{k,v}) = r_\iota(\zeta_{k,v})$ does hold.

Again by the palindromic nature of v, one easily sees that $r_{\overline{S}-\iota+i}(\eta_{k,v}) = r_i(\zeta_{k,v})$ for $0 \leq i \leq \iota$ and $r_j(\eta_{k,v}) = r_{\iota+j}(\zeta_{k,v})$ for $1 \leq i \leq \overline{S} - \iota - 1$. □

Corollary 42. *Fix $n \geq 3, k \in \mathbb{N}, v \in \mathcal{V}$. Then $\ell_0(\eta_{k,v})$ is in the $T_{\zeta_{k,v}}$-orbit of $\ell_0(\zeta_{k,v})$.*

More precisely,

i.) $\ell_{|w|}(\zeta_{k,1}) = \ell_0(\eta_{k,1})$;

ii.) *For* $v \neq 1$,

$$\ell_{|\underline{d}(k,u)|}(\zeta_{k,v}) = \ell_0(\eta_{k,v}).$$

Proof. From (5) and (11) we have $\underline{d}^{\eta_{k,1}}_{[1,\infty)} = \overline{w^k, (-1)^{n-3}, -2}$ and $\underline{d}^{\zeta_{k,1}}_{[1,\infty)} = w^{k+1}, \overline{(-1)^{n-3}, -2, w^k}$. The case of $v = 1$ thus visibly holds.

We now suppose that u is the parent of v. In order to apply the results of the previous proposition, we use the relation $L_{k,v} = C^{-1}ACR_{k,v}$. Note that in general $\ell_{|\underline{d}(k,u)|}(\alpha) = A^{-1}L_{k,u}A{\cdot}\ell_0(\alpha)$, instead of what might be naively expected. Indeed, the definition of the $L_{k,v}$ in general, see ([8], Definition 4.32), is such that the prefix of $(A^{-1}C)^{n-2}$ of $L_{k,u}$ must be replaced by $A^{-2}C(A^{-1}C)^{n-3}$ in order to achieve an admissible suffix of $L_{k,v}$.

First suppose that $v = c_1$ with $c_1 > 1$, thus of parent $u = (c_1 - 1)$. Then $\ell_{|\underline{d}(k,u)|}(\zeta_{k,c_1}) = A^{-1}L_{k,c_1-1}A \cdot \ell_0(\zeta_{k,c_1}) = A^{-1}C^{-1}ACR_{k,c_1-1} \cdot r_0(\zeta_{k,c_1}) = A^{-1}C^{-1}ACR_{k,c_1-1} \cdot r_1(\eta_{k,c_1}) = A^{-1}C^{-1}ACR_{k,c_1} \cdot r_0(\eta_{k,c_1}) = A^{-1}L_{k,c_1}A \cdot \ell_0(\eta_{k,c_1}) = A^{-1} \cdot r_0(\eta_{k,c_1}) = \ell_0(\eta_{k,c_1})$.

From Proposition 41, $L_{k,u}A \cdot \ell_0(\zeta_{k,v}) = L_{k,u} \cdot r_0(\zeta_{k,v}) = C^{-1}ACR_{k,u} \cdot r_0(\zeta_{k,v}) = C^{-1}AC \cdot r_\iota(\zeta_{k,v}) = C^{-1}AC \cdot r_{\overline{S}}(\eta_{k,v}) = C^{-1}ACR_{k,v} \cdot r_0(\eta_{k,v}) = L_{k,v} \cdot r_0(\eta_{k,v}) = r_0(\eta_{k,v})$. Thus, $\ell_{|\underline{d}(k,u)|}(\zeta_{k,v}) = A^{-1} \cdot r_0(\eta_{k,v}) = \ell_0(\eta_{k,v})$. □

Proposition 43. *Fix* $n \geq 3, k \in \mathbb{N}, v \in \mathcal{V}$. *Then* $\ell_0(\eta_{k,v})$ *is the second smallest element of the* $T_{\zeta_{k,v}}$*-orbit of* $\ell_0(\zeta_{k,v})$. *That is,*

$$\ell_0(\eta_{k,v}) = \ell_{i_2}(\zeta_{k,v}).$$

Proof. Suppose that $|v| > 1$. The previous result implies that the $T_{\zeta_{k,v}}$-orbit of $\ell_0(\zeta_{k,v})$ not only reaches $\ell_0(\eta_{k,v})$ but also thereafter agrees with the $T_{\eta_{k,v}}$-orbit of $\ell_0(\eta_{k,v})$. Suppose that i is such that $\ell_i(\zeta_{k,v}) < \ell_0(\eta_{k,v})$. Since every element of the $T_{\eta_{k,v}}$-orbit of $\ell_0(\eta_{k,v})$ is greater than or equal to $\ell_0(\eta_{k,v})$, we must have $i < |\underline{d}(k,u)|$. Now, the first $|\underline{d}(k,v)| - 1$ simplified digits of $\ell_0(\zeta_{k,v})$ agree with those of $\ell_0(\eta_{k,v})$, and of course $\ell_0(\zeta_{k,v}) \leq \ell_i(\zeta_{k,v})$. Thus, the simplified digits of $\ell_i(\zeta_{k,v})$ begin $w^k, \mathcal{C}^{c_1-1}\mathcal{D}^{d_1} \cdots \mathcal{D}^{d_{s-1}}\mathcal{C}^{c_s}, (-1)^{n-3} \cdots$. Recall that $\mathcal{D} = \mathcal{C}w = (-1)^{n-3}, -2, w^k$.

If $i > 0$, then in light of (11), the digit sequence of $\ell_i(\zeta_{k,v})$ begins with the final w^k obtained by factoring some $\mathcal{D}^{d_j}$ in the prefix $\underline{d}(k,u)$ of $\underline{d}(k,v)$ as $\mathcal{D}^{-1+d_j}, (-1)^{n-3}, -2, w^k$. But then $\ell_{i-(n-2)}(\zeta_{k,v})$ corresponds to factoring instead as $\mathcal{D}^{d_j} = \mathcal{D}^{-1+d_j}, \mathcal{D}$. Indeed, in light of the digits of $\ell_i(\zeta_{k,v})$, the expansion of $\ell_{i-(n-2)}(\zeta_{k,v})$ is associated with a copy of v', refer to (10). But, this copy of v' begins within the prefix u of the word $v = u(u')^q u''$. This in turn implies that there is a copy of u' contained in v, beginning within the prefix u. However, Lemma 22 shows that this is impossible. Therefore, the case of $i > 0$ is void.

Finally, the case of $|v| = 1$ is argued similarly, but is more straightforward. □

Corollary 44. *We have*

$$y_{\underline{S}} = -\ell_0(\eta_{k,v}).$$

Proof. By definition, $y_{\underline{S}}$ is the second largest of the values $-\ell_i(\zeta_{k,v})$. Thus, $y_{\underline{S}} = \ell_{i_2}(\zeta_{k,v})$ and by the proposition this equals $-\ell_0(\eta_{k,v})$. □

4.6. **Differing by t: two paired top/bottom heights.**

Lemma 45. *We have* $y_{\underline{S}+1} = t + y_{-\overline{S}}$.

Proof. By Lemma 38, $-\overline{S} = \beta(\iota)$. By definition, $y_{\beta(\iota)} = -r_{\overline{S}-\iota}(\eta_{k,v})$. Proposition 41 now gives $y_{\beta(\iota)} = -r_0(\zeta_{k,v})$.

From Lemma 32, $\underline{S}{+}1 = \tau(\underline{S})$. By definition, $y_{\tau(\underline{S})} = -\ell_0(\zeta_{k,v})$, and thus the result holds. □

Lemma 46. *We have* $y_{\underline{S}} = t + y_{-\overline{S}-1}$.

Proof. By Corollary 44, $y_{\underline{S}} = -\ell_0(\eta_{k,v})$. On the other hand, from Lemma 37, $y_{-\overline{S}-1} = y_{\beta(\overline{S})}$, which by definition has the value $-r_0(\eta_{k,v})$. The result thus holds. □

5. Bijectivity of $\mathcal{T}_\alpha$ on Ω_α for synchronizing small α

In this section, we prove Theorem 10 for $\alpha \in (0, \gamma_n)$. This bijectivity of $\mathcal{T}_\alpha$ on Ω_α follows from: (*i*) the upper boundary of Ω^+ surjects onto itself; (*ii*) the lower boundary of Ω^- surjects onto itself; (*iii*) the images of the "blocks", defined directly below, laminate.

5.1. **Partitioning Ω_α by blocks $\mathcal{B}_i$.** Recall that the upper and lower parts of Ω_α are given in (8) and (9).

For each $i \in \{-1, -2, \cdots\} \cup \{k, k+1, \dots\}$, let the *block* $\mathcal{B}_i$ be the closure of the set $\{(x, y) \in \Omega \,|\, x \in \Delta_\alpha(i, 1)\}$. Thus the blocks partition Ω up to μ-measure zero, confer Figures 1 and 2.

Since $\mathcal{T}$ is invertible, it is clear that for the values of α under consideration, Theorem 10 follows from the following two results.

Proposition 47. *The union of the $\mathcal{T}(\mathcal{B}_i)$ taken over all negative i equals Ω^+ up to μ-measure zero.*

This holds because: the various blocks appropriately laminate (proven in Subsection 5.2); their image includes the upper boundary of Ω^+(proven in Subsection 5.3); and, it is directly verified that the limit as $i \to -\infty$ of $\mathcal{T}(\mathcal{B}_i)$ is $\mathbb{I}_\alpha \times \{0\}$.

Proposition 48. *The union of the $\mathcal{T}(\mathcal{B}_i)$ taken over all positive $i \geq k$ equals Ω^-, up to μ-measure zero.*

This follows similarly from the results of Subsections 5.2 and 5.4, along with the easily verified fact that the limit as $i \to \infty$ of $\mathcal{T}(\mathcal{B}_i)$ is $\mathbb{I}_\alpha \times \{0\}$.

5.2. **Blocks laminate one above the other.** As an initial step to proving these two results, we have the following results.

Lemma 49. *Let s denote the first simplified digit of $r_{\overline{S}}$. Then for all $i \notin \{k+1, k\}$, the lower boundary of the block $\mathcal{B}_i$ has height given by*

$$\begin{cases} y = y_{-\overline{S}-1} & \text{if } s \succ i\,; \\ (y = y_{-\overline{S}-1}) \cup (y = y_{-\overline{S}}) & \text{if } s = i\,; \\ y = y_{-\overline{S}} & \text{if } s \prec i\,. \end{cases}$$

Proof. Recall that $L_{-\overline{S}-1} = [\ell_0, r_{\overline{S}}]$. Hence, the leftmost piece in the partition defining Ω^- is $L_{-\overline{S}-1} \times [y_{-\overline{S}-1}, 0]$. Therefore, we certainly find that whenever $s \succ i$, the lower boundary of the block $\mathcal{B}_i$ is given by $y = y_{-\overline{S}-1}$. Since the next partition piece to the right is $J_{-\overline{S}} \times [y_{-\overline{S}}, 0]$, and for all $j < \overline{S}$ we have $r_j \in \Delta_\alpha(k+1, 1) \cup \Delta_\alpha(k, 1)$, the rest of the statement follows. □

Lemma 50. *Let u denote the first simplified digit of $\ell_{\underline{S}}$. Then for all $i \notin \{-1, -2\}$, the top boundary of the block $\mathcal{B}_i$ has height given by*

$$\begin{cases} y = y_{\underline{S}+1} & \text{if } u \succ i\,; \\ (y = y_{\underline{S}+1}) \cup (y = y_{\underline{S}}) & \text{if } u = i\,; \\ y = y_{\underline{S}} & \text{if } u \prec i\,. \end{cases}$$

Proof. Due to Lemma 32, $K_{\underline{S}+1} = [\ell_{\underline{S}}, r_0]$. Hence, the rightmost piece in the partition defining Ω^+ is $K_{\overline{S}+1} \times [0, y_{\underline{S}+1}]$. With this, one argues as for the bottoms of the blocks. □

Lemma 51. *Suppose that $i \notin \{-1, k+1, k\}$. Then $\mathcal{T}_\alpha(\mathcal{B}_i)$ laminates above $\mathcal{T}_\alpha(\mathcal{B}_{i-1})$.*

Proof. Since $L_{k,v} = C^{-1}AC\,R_{k,v}$, ([8], Lemma 4.2) and its proof show that $r_{\overline{S}} \in \Delta_\alpha(u+1, 1)$ if and only if $\ell_{\underline{S}} \in \Delta_\alpha(u, 1)$.

Case: $s \succ i$. Here, $\mathcal{B}_i$ has bottom boundary height $y_{-\underline{S}-1}$. Also, $u = s-1 \succ i-1$, and hence $\mathcal{B}_{i-1}$ has top boundary height $y_{\underline{S}}$. Lemma 46 states that $y_{\underline{S}} = t + y_{-\overline{S}-1}$, and thus Lemma 16 yields the result in this case.

Case: $s \prec i$. Here, $\mathcal{B}_i$ has bottom boundary height $y_{-\underline{S}}$. And, $u = s-1 \prec i-1$, gives that $\mathcal{B}_{i-1}$ has top boundary height $y_{\underline{S}+1}$. Lemma 45 and Lemma 16 yield the result in this case.

Case: $s = i$. Since $T_\alpha(r_{\overline{S}}) = T_\alpha(\ell_{\underline{S}})$ and the corresponding two pieces of each of the bottom boundary of $\mathcal{B}_i$ and top boundary of $\mathcal{B}_{i-1}$ are sent to the same height (by the arguments for the previous two cases), the result holds in this final case as well. □

Lemma 52. *The $\mathcal{T}_\alpha$-image of the block $\mathcal{B}_{-1}$ laminates above a portion of $\mathcal{T}_\alpha(\mathcal{B}_{-2})$.*

Proof. We have that $\mathcal{B}_{-1}$ has bottom height $y_{-\overline{S}-1}$ if $r_{\overline{S}} \notin \Delta_\alpha(-1, 1)$, and both heights $y_{-\overline{S}-1}, y_{-\overline{S}}$ otherwise. The block $\mathcal{B}_{-2}$ has multiple heights along its upper boundary, with its rightmost height being either $y_{\underline{S}}$ if $\ell_{\underline{S}} \notin \Delta_\alpha(-2, 1)$ and otherwise its two rightmost heights being $y_{\underline{S}}, y_{\underline{S}+1}$. In either case, the corresponding portion of the upper boundary fibers above an interval whose left endpoint is $\ell_{i_{\underline{S}}}$.

Arguments as in the proof of the previous lemma, combined with Lemma 33, show that the lower boundary of $\mathcal{B}_{-1}$ has image equalling the image of a rightmost portion of the upper boundary of $\mathcal{B}_{-2}$. □

Lemma 53. *A portion of $\mathcal{T}_\alpha(\mathcal{B}_{k+1})$ laminates above $\mathcal{T}_\alpha(\mathcal{B}_k)$.*

Proof. Here Lemma 39 combines with the above arguments to imply the result. □

5.3. **Upper boundary is in image.**

Lemma 54. *There is a partition*

$$K_{\tau(\underline{S}-1)} = K'_{\tau(\underline{S}-1)} \cup K''_{\tau(\underline{S}-1)}$$

such that

$$A^{-1}C \cdot K'_{\tau(\underline{S}-1)} = K_{\underline{S}+1}, \quad A^{-2}C \cdot K''_{\tau(\underline{S}-1)} = K_1.$$

Furthermore, $\mathcal{T}$ sends $K_{\tau(\underline{S}-1)} \times \{y_{\tau(\underline{S}-1)}\}$ to $(K_1 \times \{y_1\}) \cup (K_{\underline{S}+1} \times \{y_{\underline{S}+1}\})$.

Proof. Since $\alpha \in J_{k,v}$ certainly $\ell_{\underline{S}} < r_0$, and since $\underline{d}(k, v)$ ends with -1, it follows that $\ell_{\underline{S}-1}$ lies in the interior of $\Delta_\alpha(-1, 1)$. Furthermore, Lemma 32 yields that $\ell_{\underline{S}-1}$ is the largest ℓ_i lying in $\Delta_\alpha(-1, 1)$. By definition, $K_{\tau(\underline{S}-1)} = [\ell_{\tau(\underline{S}-1)}, \ell_{1+\tau(\underline{S}-1)}]$, and it follows that $\ell_{1+\tau(\underline{S}-1)}$ is the smallest $\ell_i \in \Delta_\alpha(-2, 1)$. Thus, $K_{\tau(\underline{S}-1)}$ is partitioned into two pieces, which we define as $K'_{\tau(\underline{S}-1)} \cup K''_{\tau(\underline{S}-1)}$, where the first of these is mapped by $A^{-1}C$ to $[\ell_{\underline{S}}, r_0] = K_{\underline{S}+1}$ and the second is mapped by $A^{-2}C$ to $[\ell_0(\alpha), \ell_i]$ for some i. Since there is certainly at least one sequence $-2, (-1)^{n-2}$ in $\underline{d}(k, v)$, surely the smallest $\ell_i \in \Delta_\alpha(-2, 1)$ has image under $A^{-2}C$ that is larger than any $A^{-1}C \cdot \ell_{i'}$ for $i' \in \Delta_\alpha(-1, 1)$, as these latter images have digits starting at least (in the sense of our ordering) with $(-1)^{n-3}$. Therefore, $A^{-2}C$ sends $K''_{\tau(\underline{S}-1)}$ to $[\ell_0, \ell_{i_2}] = K_1$.

Finally, Lemma 25 implies the result on images of the top height. □

Lemma 55. *Let $K'_{\underline{S}}$ be the closure of $K_{\underline{S}} \cap \Delta_\alpha(-2,1)$. Then*

$$A^{-2}C \cdot K'_{\underline{S}} = \begin{cases} K_{\tau(1+i_{\underline{S}})} & \text{if } \ell_{\underline{S}} \in \Delta_\alpha(-2,1); \\ K_{\tau(1+i_{\underline{S}})} \cup [\ell_{1+i_{\underline{S}}}, r_0] & \text{otherwise.} \end{cases}$$

Furthermore, $\mathcal{T}$ correspondingly sends $K'_{\underline{S}} \times \{y_{\underline{S}}\}$ to $K_{\tau(1+i_{\underline{S}})} \times \{y_{\tau(1+i_{\underline{S}})}\}$, respectively $(K_{\tau(1+i_{\underline{S}})} \times \{y_{\tau(1+i_{\underline{S}})}\}) \cup ([\ell_{1+i_{\underline{S}}}, r_0] \times \{y_{\tau(1+i_{\underline{S}})}\})$.

Proof. The first simplified digit of $\ell_{\underline{S}}$ is at least -2 (in our usual ordering). On the other hand, all ℓ_i with $i < \underline{S}$ are contained in $\Delta_\alpha(-1,1) \cup \Delta_\alpha(-2,1)$, and there do exist i with $\ell_i \in \Delta_\alpha(-2,1)$. Thus certainly, $K_{\underline{S}} \cap \Delta_\alpha(-2,1) \neq \emptyset$. This intersection is all of $K_{\underline{S}}$ if $\ell_{\underline{S}} \in \Delta_\alpha(-2,1)$; otherwise, this intersection includes the right endpoint of $\Delta_\alpha(-2,1)$. The first statement thus holds. The second statement is an immediate consequence of the first. □

Recall that the various p_i are defined in Definition 11.

Lemma 56. *For each $1 \leq a < \underline{S}$ with $a \neq \tau(\underline{S}-1)$,*

$$\mathcal{T}(\, K_a \times \{y_a\}\,) = K_{\tau(i_a+1)} \times \{y_{\tau(i_a+1)}\}.$$

In particular,

$$\bigcup_{a=1}^{\underline{S}} \mathcal{T}(\, K_a \times \{y_a\}) \supset \bigcup_{a=1}^{\underline{S}+1} K_a \times \{y_a\}.$$

Proof. If $a < \underline{S}+1$ and $a \neq \tau(\underline{S}+1)$, then $K_a \subset \Delta_\alpha(p_{i_a}, 1)$. Thus, $\mathcal{T}(\, K_a \times \{y_a\}) = K_{\tau(i_a+1)} \times \{y_{\tau(i_a+1)}\}$.

Lemmas 54 and 55 show that $a \in \{\tau(\underline{S}+1), \underline{S}\}$ account for images indexed by $a \in \{1, \tau(1+i_{\underline{S}}), \underline{S}+1\}$. An elementary counting argument shows that the remaining values of a account for images indexed by the remaining $\underline{S}-2$ values. □

5.4. **Lower boundary is in image.**

Lemma 57. *Let*

$$L'_{\beta(\overline{S}-1)} = L_{\beta(\overline{S}-1)} \cap \Delta_\alpha(k+1,1)$$

and

$$L''_{\beta(\overline{S}-1)} = L_{\beta(\overline{S}-1)} \cap \Delta_\alpha(k,1).$$

Then

$$A^{k+1}C \cdot L'_{\beta(\overline{S}-1)} \supset L_{-1}, \quad A^kC \cdot L''_{\beta(\overline{S}-1)} = L_{-\overline{S}-1}.$$

Furthermore, $\mathcal{T}$ applied to $L_{\beta(\overline{S}-1)} \times \{y_{\beta(\overline{S}-1)}\}$ contains $(L_{-1} \times \{y_{-1}\}) \cup (L_{-\overline{S}-1)} \times \{y_{-\overline{S}-1)}\})$.

Proof. Since $\overline{d}(k,v)$ ends with a k, Lemma 37 implies that $r_{\overline{S}-1}$ is the least r_j in $\Delta_\alpha(k,1)$. If $v \neq c_1$, there exist $r_j \in \Delta_\alpha(k+1,1)$. In this case, $L_{\beta(\overline{S}-1)}$ is partitioned as $L'_{\beta(\overline{S}-1)} \cup L''_{\beta(\overline{S}-1)}$. Since $L'_{\beta(\overline{S}-1)}$ contains the right endpoint of $\Delta_\alpha(k+1,1)$, it easily follows that $A^{k+1}C \cdot L'_{\beta(\overline{S}-1)} = L_{-1}$. Similar considerations show $A^kC \cdot L''_{\beta(\overline{S}-1)} = L_{-\overline{S}-1}$. The rest of the proof follows easily in this case.

If $v = c_1$, then $L_{\beta(\overline{S}-1)} = [r_{\overline{S}}, r_{\overline{S}-1}]$ and here $A^{k+1}C \cdot L'_{\beta(\overline{S}-1)} \supset L_{-1}$ (with equality in general not holding). Once again, the rest of the proof follows easily. □

The following now easily follows.

Lemma 58. *We have*

$$\bigcup_{b=-1}^{-\overline{S}} \mathcal{T}(\, L_b \times \{y_b\}) \supset \bigcup_{b=-1}^{-\overline{S}-1} L_b \times \{y_b\}.$$

5.5. **Left endpoint values: $\alpha = \zeta_{k,v}$.** Recall that for any v and with $k \in \mathbb{N}$, when $\alpha = \zeta_{k,v}$ one has $r_{\overline{S}} = \ell_0$ and $A^{-2}C \cdot \ell_{\underline{S}} = \ell_1$. The first of these equations shows that the interval $L_{-\overline{S}-1}$ of Definition 12 degenerates to a point in this case; correspondingly, Ω_- will now have one fewer lower height. As well, Lemma 45 gives $y_{\underline{S}+1} = -\ell_0(\zeta_{k,v})$ and $y_{\beta(\iota)} = -r_0(\zeta_{k,v})$. Note that Lemma 39 implies that $A^{k+1}C$ sends r_ι to the right of r_1. Thus the various preceding sections now easily imply the following, see Figure 7.

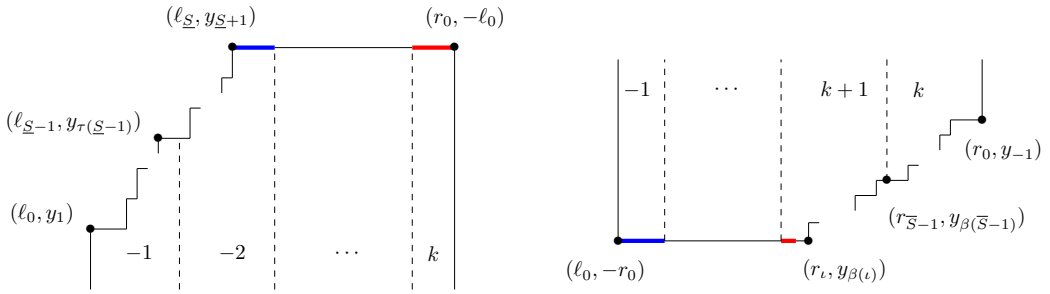

FIGURE 7. Schematic representations (not to scale) showing the most important vertices of the tops and bottoms of blocks, for $\alpha = \zeta_{k,v}$ (and $|v| > 1$). See Proposition 59. Interior label i denotes partitioning block $\mathcal{B}_i$, see Subsection 5.1 for definitions. Thick line segments of same color have equal images under $\mathcal{T}_{n,\zeta_{k,v}}$, where the bottommost thick red segment has right endpoint at $(A^{k+1}C)^{-1} \cdot r_1$.

Proposition 59. *Fix $n \geq 3, k \in \mathbb{N}, v \in \mathcal{V}$ and $\alpha = \zeta_{k,v}$. Let Ω^+ be as in Definition 11. Let Ω^- be as in Definition 12, except that we redefine $L_{-\overline{S}}$ to be $[\ell_0, r_\iota]$, and set* $\Omega^- = \bigcup_{b=-1}^{-\overline{S}} L_b \times [y_b, 0]$.

Then $\mathcal{T}_{n,\alpha}$ is bijective on $\Omega_{n,\alpha} := \Omega^+ \cup \Omega^-$, up to μ-measure zero. Furthermore, $\mu(\Omega)_{n,\alpha}) < \infty$.

5.6. **Right endpoint values: $\alpha = \eta_{k,v}$.** Fix $n \geq 3$, and $v \in \mathcal{V}$. Although when $\alpha = \eta_{k,v}$, the T_α-obits of $\ell_0(\alpha), r_0(\alpha)$ do not synchronize, we still can precisely describe the domain on which the associated two-dimensional maps is bijective.

Recall that $\ell_{\underline{S}} = \ell_0(\eta_{k,v})$ and $r_{\overline{S}+1}(\eta_{k,v}) = A^{k+1}C \cdot r_{\overline{S}}(\eta_{k,v}) = r_1(\eta_{k,v})$. Furthermore, from Lemmas 46 and 37, $y_{\underline{S}} = -\ell_0(\eta_{k,v})$ and $y_{-\overline{S}-1} = y_{\beta(\overline{S})} = -r_0(\eta_{k,v})$. In particular, here Ω^+ has one less height than the general case.

Thus the various preceding sections now easily imply the following results, see Figure 8.

Proposition 60. *Fix $n \geq 3, k \in \mathbb{N}, v \in \mathcal{V}$ and $\alpha = \eta_{k,v}$. Let Ω^- be as in Definition 12. Let Ω^+ be as in Definition 11 except that we redefine $K_{\underline{S}}$ to be $[\ell_{i_{\underline{S}}}, r_0]$, and $\Omega^+ = \bigcup_{a=1}^{\underline{S}} K_a \times [0, y_a]$.*

Then $\mathcal{T}_{n,\alpha}$ is bijective on $\Omega_{n,\alpha} := \Omega^+ \cup \Omega^-$, up to μ-measure zero. Furthermore, $\mu(\Omega_{n,\alpha}) < \infty$.

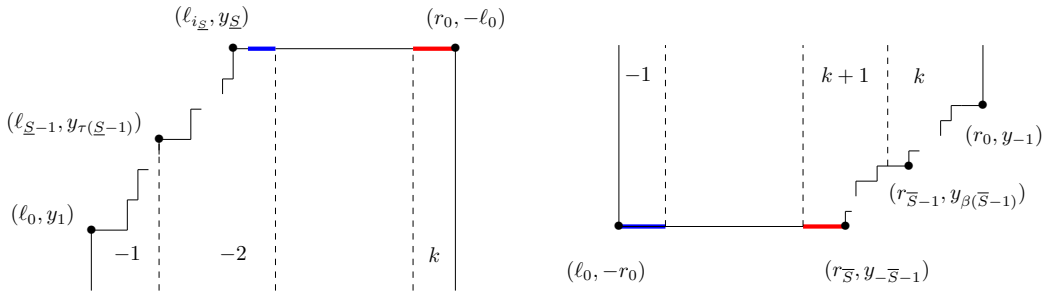

FIGURE 8. Schematic representations (not to scale) showing the most important vertices of the tops and bottoms of blocks, for $\alpha = \eta_{k,v}$ (and $|v| > 1$). See Proposition 60. Interior label i denotes partitioning "block" $\mathcal{B}_i$, see Subsection 5.1 for definitions. Thick line segments of same color have equal images under $\mathcal{T}_{n,\eta_{k,v}}$; the topmost thick blue segment in the left figure has left endpoint at $(\,(A^{-2}C)^{-1} \cdot \ell_1, -\ell_0)$.

Corollary 61. *Fix $n \geq 3, k \in \mathbb{N}, v \in \mathcal{V}$. Then the union of $\Omega_{\eta_{k,v}}$ with the line segment $r_0(\eta_{k,v}) \times [-\ell_0(\eta_{k,v}), -\ell_0(\zeta_{k,v})]$ is the limit, with respect to the Hausdorff metric topology on compact subsets of $\mathbb{R}^2$, of Ω_α as α tends to $\eta_{k,v}$ from the left.*

Proof. When evaluating the limit from the left as $\alpha \to \eta_{k,v}$, we can assume that $\alpha \in J_{k,v}$. Recall that we define both $\Omega_\alpha, \Omega_{\eta_{k,v}}$ as the union of a top piece and a bottom piece: in simple notation $\Omega^+ \cup \Omega^-$. The bottom piece of each is defined as in Definition 12. In particular, each is the union of the same number of rectangles; the vertices of these rectangles are such that their first coordinates clearly vary continuously in α, while their second coordinates are fixed throughout the closure of $J_{k,v}$. The top pieces are similarly defined, however for $\alpha \in J_{k,v}$ the number of rectangles is greater by one, with the "extra" rectangle being $[\ell_{i_{\underline{S}+1}}(\alpha), r_0(\alpha)] \times [0, -\ell_0(\zeta_{k,v})]$. Since $\ell_{i_{\underline{S}+1}}(\alpha)$ and $r_0(\alpha)$ both converge to $r_0(\eta_{k,v})$, the limit of this rectangle is simply a vertical line segment. The remaining rectangles comport themselves continuously in the manner discussed above for the rectangles comprising the bottom portion. The result thus holds. □

6. Continuity of mass, for small α

We aim to show that the bijectivity domains vary continuously at points of non-synchronization. Since these are limit points of sequences of the types of α which we have already considered, we bootstrap on the above work.

We prove Proposition 63, giving in particular the ergodicity for the systems indexed by endpoints of synchronization intervals, the $\zeta_{k,v}$ and $\eta_{k,v}$, by applying a main result of [9]. After establishing prerequisite results, we use this to show Lemma 72: For these values of α, reversing words is a bijection on the sublanguage formed by the admissible words whose letters are only $\{-1, -2\}$ and similarly for the sublanguage formed by restricting to $\{k, k+1\}$. A limiting argument then shows that this is also true for the systems of the non-synchronization α-values. This bijection is key in the proof of Proposition 65, which states that the domain we define there for each non-synchronization α is a domain of bijectivity for its associated 2-dimensional map. The reversal of words is naturally required in that argument as it is related to the piecewise action on y-values, see Lemma 68.

This section closes with its main result, Theorem 79, the continuity of the function taking α to the mass of its associated planar domain.

6.1. **Ergodicity at endpoints of synchronization intervals.** Here and later, we rely on ([9], Main Theorem 1), one of whose hypotheses is that the interval map in question satisfy the "bounded non-full cylinders" condition.

Remark 62. The bounded non-full cylinders condition of [9] is that the orbits of the endpoints of all non-full cylinders avoid the interior of some full cylinder. All of our maps have infinitely many full cylinders. Hence, the condition is certainly satisfied by any of our maps whose non-full cylinders have endpoints with expansions involving only finitely many digits. Recall, see ([8], § 1.6.2), that for any of our maps, the only possible non-full cylinders are those containing one of: $r_0(\alpha)$; $\ell_0(\alpha)$ or (for large α) $\mathfrak{b}_\alpha$. Furthermore, the remaining endpoint of each of these possible non-full cylinders always meets the orbit of one of the endpoints.

Recall that our underlying two-dimensional measure is μ, see (7).

Proposition 63. *Fix $n \geq 3, k \in \mathbb{N}, v \in \mathcal{V}$ and $\alpha \in \{\zeta_{k,v}, \eta_{k,v}\}$. Let μ_α be the normalization of μ to a probability measure on $\Omega_{n,\alpha}$, and $\mathscr{B}'_\alpha$ denote the Borel sigma algebra on Ω_α. Then the system $(\mathcal{T}_\alpha, \Omega_\alpha, \mathscr{B}'_\alpha, \mu_\alpha)$ is ergodic. Furthermore, this two dimensional system is the natural extension of $(T_\alpha, \mathbb{I}_\alpha, \mathscr{B}_\alpha, \nu_\alpha)$, where ν_α is the marginal measure of μ_α and $\mathscr{B}_\alpha$ the Borel sigma algebra on $\mathbb{I}_\alpha$. In particular, the one dimensional system is ergodic.*

Sketch. For the given values of α, the digits of $\overline{d}^{\alpha}_{[1,\infty)}$ and $\underline{d}^{\alpha}_{[1,\infty)}$ are contained in $\{-1, -2, k, k+1\}$. In particular the non-full cylinders of T_α are bounded in range. Due to this and the finiteness of $\mu(\Omega_\alpha)$ and of its vertical fibers, the hypotheses of ([9], Main Theorem 1) are met, thus giving the result here. □

We will use the ergodicity to argue that a subsystem gives rise to another description of the Ω_α when $\alpha = \zeta_{k,v}$ or $\eta_{k,v}$. This will then be key to showing continuity of $\alpha \mapsto \Omega_\alpha$, as we will define the two-dimensional domain for non-synchronizing α analogously.

6.2. **Bijectivity domain for non-synchronizing α.** In this subsection, we announce the bijectivity domain for non-synchronizing α.

We introduce some more notation and introduce basic sets for our construction. For legibility, we suppress various subscripts, trusting that this causes no confusion for the reader.

Definition 64. Fix $n \geq 3, k \in \mathbb{N}$ and $\alpha < \gamma_n$. Let $\mathcal{L} = \mathcal{L}_\alpha$ denote the language of admissible (simplified) digits. For finite length $u \in \mathcal{L}$, as usual let $\Delta_\alpha(u)$ be the corresponding T_α-cylinder. Let λ_u, ρ_u denote the left- and right endpoints, respectively of $\Delta_\alpha(u)$.

We then define

$$\Phi = [-r_0, -\ell_0],\ \Phi^- = [-r_0, -\lambda_{-3}],\ \text{and}\ \Phi^+ = [-\rho_{k+2}, -\ell_0]\,,$$
$$\mathcal{H} = \mathbb{I}_\alpha \times (\Phi^- \cap \Phi^+)$$

and

$$\mathcal{Z} = [\ell_0, \lambda_{-3}) \times \Phi^- \ \cup\ [\lambda_{-3}, \rho_{k+2}) \times \Phi \ \cup\ [\rho_{k+2}, r_0) \times \Phi^+,$$

see Figure 9. Note that $\mathcal{Z} \supset \mathcal{H}$.

For each non-synchronizing α, we determine a bijectivity domain for its two dimensional map. Recall that we allow ourselves the freedom to write set equalities (strict inclusion, disjoint unions, etc) when in fact these only hold up to μ-measure zero (and usually, the measure zero exceptional set is easily found). As usual, for any finite set S of digits, we let S^* denote the set of finite words in S.

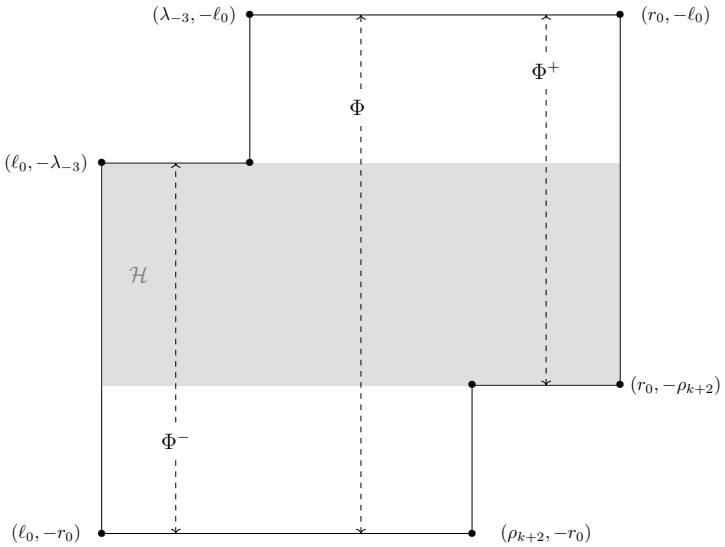

FIGURE 9. Schematic representation of $\mathcal{Z} = \mathcal{Z}_\alpha$, see Definition 64. For $\alpha < \gamma_n$ non-synchronizing or the endpoint of a synchronization interval, the central "vertical" region $[\lambda_{-3}, \rho_{k+2}) \times \Phi$ is sent by $\mathcal{T}_\alpha$ to the (gray) "horizontal" region $\mathcal{H} = \mathcal{H}_\alpha = \mathbb{I}_\alpha \times (\Phi^- \cap \Phi^+)$ while the complementary pieces are sent by powers of $\mathcal{T}_\alpha$ to fill in a region that strictly includes the complement of $\mathcal{H}_\alpha$ in $\mathcal{Z}_\alpha$.

Proposition 65. *Fix a non-synchronizing α with $\alpha < \gamma_n$. With notation as above, let $\mathcal{T} = \mathcal{T}_\alpha$ and define*

$$\Omega = \overline{\cup_{j=1}^{\infty} \mathcal{T}^j(\,\mathcal{Z}\,)}\,.$$

Then $\Omega \supset \mathcal{Z}$ and $\mathcal{T}$ is bijective up to μ-measure zero on Ω. Furthermore,

$$\Omega = \mathcal{H} \bigsqcup \overline{\sqcup_{u \in \{-1,-2\}^* \cap \mathcal{L}_\alpha} \mathcal{T}^{|u|}(\,\Delta_\alpha(u) \times \Phi^-\,)} \bigsqcup \overline{\sqcup_{u \in \{k+1,k\}^* \cap \mathcal{L}_\alpha} \mathcal{T}^{|u|}(\,\Delta_\alpha(u) \times \Phi^+\,)}. \tag{13}$$

The surjectivity of $\mathcal{T}$ on Ω follows from the definition of Ω. Key to proving the remaining statements of the proposition is to show that each $x \in \mathbb{I}_\alpha$ of y-fiber Φ in $\mathcal{Z}$ have exactly that same fiber in Ω. Our choice of the basic shapes of $\mathcal{Z}$ and $\mathcal{H}$ is motivated by Lemma 68 below, which also begins the proof of (13). Together, Lemmas 74 and 75 establish the key step. From this, both injectivity and (13) then follow.

We first introduce notation.

Definition 66. Given any simplified digit d, we let $M_d = A^d C$. For any finite word $u = u_1\, u_2\, \ldots u_{|u|}$ with the $u_i \in \mathbb{Z}$, we let $M_u = M_{u_{|u|}} \cdots M_{u_1}$. We also define $\overleftarrow{u}$ to be the reversed word of u, thus $\overleftarrow{u} = u_{|u|} \ldots u_2\, u_1$.

We have the following immediate implication of Lemma 14. (Temporarily, we allow intervals to include ∞.)

Lemma 67. *Let u be a finite length word in nonzero integers and $x \in \mathbb{R}$. Then $RM_uR^{-1} \cdot (-x) = -(M_{\overleftarrow{u}})^{-1} \cdot x$.*

Lemma 68. *Fix $n \geq 3, k \in \mathbb{N}$ and $\alpha < \gamma_n$. Suppose that finite length u is in $\mathcal{L}_\alpha$ and $a, b \in \overline{\mathbb{I}}_\alpha$ with $a < b$. Then*

$$\mathcal{T}^{|u|}(\,\Delta_\alpha(u) \times [-b, -a]\,) = [\,M_u \cdot \lambda_u,\, M_u \cdot \rho_u\,) \times [-(M_{\overleftarrow{u}})^{-1} \cdot b, -(M_{\overleftarrow{u}})^{-1} \cdot a].$$

Proof. For $x \in \Delta_\alpha(u)$, we have that $T^{|u|}(x) = M_u \cdot x$. Since each M_{u_j} defines an increasing function, the image of $\Delta_\alpha(u)$ under $T^{|u|}$ is the interval $[\, M_u \cdot \lambda_u,\, M_u \cdot \rho_u\,)$. Similarly, the image of the y-coordinates is given by $y \mapsto RM_uR^{-1} \cdot y$. Lemma 67 shows that our result holds. □

Lemma 69. *For each full cylinder $\Delta_\alpha(j)$, we have that*

$$\mathcal{T}_\alpha(\, \Delta_\alpha(j) \times \Phi\,) = \mathbb{I}_\alpha \times -\Delta_\alpha(j).$$

Furthermore,

$$\mathcal{T}_\alpha(\, [\lambda_{-3}, \rho_{k+2}) \times \Phi\,) = \mathcal{H}.$$

Proof. The interval $[\lambda_{-3}, \rho_{k+2})$ is the union of the full cylinders $\Delta_\alpha(j), k+2 \succeq j \succeq -3$. Since $\lambda_j = (M_j)^{-1} \cdot \ell_0$ and $\rho_j = (M_j)^{-1} \cdot r_0$, we find that $\mathcal{T}_\alpha$ sends $\Delta_\alpha(j) \times \Phi$ to $\mathbb{I}_\alpha \times -\Delta_\alpha(j)$. Taking the union over the various j gives $\mathcal{T}_\alpha(\, [\lambda_{-3}, \rho_{k+2}) \times \Phi\,) = \mathbb{I}_\alpha \times -[\lambda_{-3}, \rho_{k+2})$. The result now follows. □

When applying Lemma 68, the following is a natural aid.

Lemma 70. *Fix $n \geq 3$, and either a non-synchronizing $\alpha < \gamma_n$ or $\alpha \in \{\zeta_{k,v}, \eta_{k,v}\}$ with $k \in \mathbb{N}$ and $v \in \mathcal{V}$. Then*

$$[\ell_0, \lambda_{-3}] = \overline{\sqcup_{u \in \{-1,-2\}^* \cap \mathcal{L}_\alpha} [\lambda_{u,-3},\, \rho_u)}$$

and

$$(\rho_{k+2}, r_0] = \overline{\sqcup_{u \in \{k+1,k\}^* \cap \mathcal{L}_\alpha} [\lambda_u,\, \rho_{u,k+2})}\,.$$

Proof. We consider the first equality. Note first that $[\ell_0, \lambda_{-3}) = \Delta_\alpha(-1) \sqcup \Delta_\alpha(-2)$. Fix any $u \in \{-1,-2\}^* \cap \mathcal{L}_\alpha$. Then $u, -2$ is also an admissible word, and of course, $\rho_{u,-2} = \lambda_{u,-3}$. It follows that

$$\Delta_\alpha(u) = \Delta_\alpha(u,-1) \sqcup \Delta_\alpha(u,-2) \sqcup [\lambda_{u,-3},\, \rho_u),$$

where by definition $\Delta_\alpha(u,-1)$ is empty whenever $u, -1 \notin \mathcal{L}_\alpha$.

Thus, we can iterate the partitioning of $\Delta_\alpha(u)$ beginning with each of $u = -1, u = -2$. By the (eventual) expansiveness of the accelerated T_0 map, any infinite admissible word of simplified digits in $\{-1,-2\}$ corresponds to a unique point. Therefore, $\overline{\Delta_\alpha(-1) \sqcup \Delta_\alpha(-2)} = \overline{\sqcup_{u \in \{-1,-2\}^* \cap \mathcal{L}_\alpha} [\lambda_{u,-3},\, \rho_u)}$ and our equality holds.

The veracity of the second equation is similarly argued. □

6.3. Sweeping out Ω_α for α an endpoint of a synchronization interval.

Lemma 71. *Fix $n \geq 3, k \in \mathbb{N}, v \in \mathcal{V}$ and $\alpha \in \{\zeta_{k,v}, \eta_{k,v}\}$. Then $\Omega_\alpha \supset \mathcal{Z}_\alpha$ and*

$$\Omega_\alpha = \overline{\cup_{j=1}^\infty \mathcal{T}_\alpha^j(\, \mathcal{Z}_\alpha)}$$

up to μ-measure zero.

Proof. The containment follows from Propositions 59 and 60. Since $\overline{\cup_{j=1}^\infty \mathcal{T}_\alpha^j(\, \mathcal{Z})}$ is $\mathcal{T}_\alpha$-invariant, the ergodicity, shown in Proposition 63, implies the equality. □

Lemma 72. *Fix $n \geq 3, k \in \mathbb{N}, v \in \mathcal{V}$ and $\alpha \in \{\zeta_{k,v}, \eta_{k,v}\}$. The map $u \mapsto \overleftarrow{u}$ defines a self-bijection on each of $\{-1,-2\}^* \cap \mathcal{L}_\alpha$ and $\{k+1,k\}^* \cap \mathcal{L}_\alpha$.*

Proof. **Reversed admissible words are admissible.** Certainly taking the reversed word has no effect on the set of digits appearing in a word, and also defines a bijective function on the set of words. Since whenever u is of length one we trivially have $\overleftarrow{u}$ also in $\mathcal{L}_\alpha$, and also the admissibility of any word implies the admissibility of any of its subwords (that is, of any of its factors), we prove that the reverses of admissible words are admissible by induction on the

lengths of words. Thus, we suppose that for u given every admissible word of length less than that of u (satisfying our digit restrictions) has its reversed word also admissible.

We first consider $u \in \{-1,-2\}^* \cap \mathcal{L}_\alpha$. For $u = u_1 \cdots u_s$ with $s > 1$, we let $u' = u_2 \cdots u_s$.

Step 1: Reduction to special case. Were $\overleftarrow{u}$ inadmissible, then there is some suffix of this word, of say length j, which is less than $\underline{d}^\alpha_{[1,j]}$ (see [8], Lemma 1.5). Invoking our induction hypothesis, we find that it must be the word itself which has this property; that is, $\overleftarrow{u} \prec \underline{d}^\alpha_{[1,s]}$. On the other hand, $\overleftarrow{u'} \succeq \underline{d}^\alpha_{[1,s-1]}$, but since we are using a 'dictionary order' $\overleftarrow{u'} = \underline{d}^\alpha_{[1,s-1]}$ and (since only digits $-1,-2$ are involved) $u_1 = -1$ while $\underline{d}^\alpha_{[s,s]} = -2$.

Thus, we can and do assume that $\overleftarrow{u'} = \underline{d}^\alpha_{[1,s-1]}$ and $u_1 = -1$.

We remind the reader that the expansion for $\eta_{k,v}$ is given above in (5) and that of $\zeta = \zeta_{k,v}$ in (11).

Step 2: Special case is void. Since appearances of -2 in $\underline{d}^\alpha_{[1,\infty)}$ are separated by either $(-1)^{n-2}$ or $(-1)^{n-3}$, but the admissible u cannot begin with $(-1)^{n-1}$, it must be the case that $\underline{d}^\alpha_{[1,s]}$ has suffix $-2,(-1)^{n-3},-2$ and u thus has prefix $(-1)^{n-2},-2$. (It is easily seen that shorter u do have admissible $\overleftarrow{u}$; as we proceed other "short" possibilities will similarly arise, but these also are easily seen to give admissible $\overleftarrow{u}$ and thus we will not mention them.)

Now, the admissible u cannot begin $(-1)^{n-2},-2,(-1)^{n-2},-2$ and hence $\underline{d}^\alpha_{[1,s]}$ has suffix $-2,(-1)^{n-3},-2,(-1)^{n-3},-2$ and u has prefix w. We now note that for any $j \in \mathbb{N}$

$$\overleftarrow{-2,(-1)^{n-3},-2,w^j,(-1)^{n-2}} = w^{j+1}. \tag{14}$$

This then gives $\overleftarrow{-2,\mathcal{C},(-1)^{n-2}} = w^k$ and thus $\overleftarrow{-2,\mathcal{D},(-1)^{n-2}} = w^{k+1}$. Since w^{k+1} is not admissible, we see that $\underline{d}^\alpha_{[1,s]}$ cannot be of suffix $-2,\mathcal{D},(-1)^{n-3},-2$.

Appearances of $-2,(-1)^{n-3},-2,(-1)^{n-3},-2$ in $\underline{d}^\alpha_{[1,\infty)}$ occur only when the suffix $(-1)^{n-3},-2$ of some final w of a $\mathcal{C}$ or $\mathcal{D}$ is followed by the prefix $(-1)^{n-3},-2$ of again some $\mathcal{C}$ or $\mathcal{D}$. (When $k=1$ there is also the possibility of consecutive $\mathcal{C}$ giving this.) Hence, we must have that $\underline{d}^\alpha_{[1,s]}$ has suffix $-2,\mathcal{C},(-1)^{n-3},-2$, and u has prefix w^k. (The identity $\mathcal{D} = [\mathcal{C},(-1)^{n-2}],-2,(-1)^{n-3},-2$ shows that we have yet to find an inadmissible $\overleftarrow{u}$.) Equation (14) gives that $\overleftarrow{-2,\mathcal{C}^i,(-1)^{n-2}} = w^k\mathcal{C}^{i-1}$ for any positive i. Since $c_s = c_1 = \max c_i$ and $\mathcal{D} \prec \mathcal{C}$, admissibility of u entails that $\underline{d}^\alpha_{[1,s]}$ has suffix $-2,\mathcal{C}^{c_1},(-1)^{n-3},-2 = -2,\mathcal{C}^{c_s},(-1)^{n-3},-2$.

Recall from [8] that either all c_i equal one and each $d_i \in \{d_1, d_1+1\}$, or all d_i equal one and each $c_i \in \{c_1, c_1-1\}$. Since (14) gives that the reverse of any $-2,\mathcal{D}^{j_1},\mathcal{C}^{i_1},\cdots,\mathcal{D}^{j_r},\mathcal{C}^{i_r},(-1)^{n-2}$ is $w^k\mathcal{C}^{i_r-1},\mathcal{D}^{j_r},\ldots,\mathcal{C}^{i_1},\mathcal{D}^{j_1}$, we next see that $\underline{d}^\alpha_{[1,s]}$ has suffix $-2,\mathcal{D}^{d_1},\mathcal{C}^{c_1},(-1)^{n-3},-2$. We now do mention a short possibility: If the c_i are all one and $d_2 = d_1+1$ then u of the form $w^k,\mathcal{C}^{c_1-1}\mathcal{D}^{d_1},\mathcal{C}^{c_2},\mathcal{D}^{d_2-1}$ is admissible, however as usual $\overleftarrow{u}$ is admissible. With appropriate generalization of this, we can continue our arguments and find that (under our restrictions on u) no finite length u gives an inadmissible $\overleftarrow{u}$. That is, this case is empty and thus for all $u \in \{-1,-2\}^* \cap \mathcal{L}_\alpha$ we have also $\overleftarrow{u} \in \{-1,-2\}^* \cap \mathcal{L}_\alpha$.

Involution shows onto. Since any admissible u is the reverse of its own reverse, we now have that $u \mapsto \overleftarrow{u}$ defines an involution on $\{-1,-2\}^* \cap \mathcal{L}_\alpha$.

The proof in the case of $u \in \{k+1,k\}^* \cap \mathcal{L}_\alpha$ is given *mutatis mutandis*. □

6.4. **Finishing the proof of bijectivity, and finite mass for non-synchronizing α.** Throughout this subsection, we again fix $n \geq 3, k \in \mathbb{N}$ and a non-synchronizing $\alpha < \gamma_n$. Together, Lemma 69 above, and Lemmas 74 and 75 show the statement of Proposition 65 that $\Omega \supset \mathcal{Z}$. The latter two lemmas rely on the following corollary to Lemma 72.

Corollary 73. *The map $u \mapsto \overleftarrow{u}$ defines a self-bijection on each of $\{-1,-2\}^* \cap \mathcal{L}_\alpha$ and $\{k+1,k\}^* \cap \mathcal{L}_\alpha$.*

Proof. Lemma 72 holds directly for the various $\eta_{k,v}$. Any other non-synchronizing α is such that there is a sequence of $v_i \in \mathcal{V}$ and $q_i \in \mathbb{Z}_{\geq 0}$ such that $v_{i+1} = \Theta_i(v_i)$ with $\underline{d}^\alpha_{[1,\infty)} = \lim_{i\to\infty} \underline{d}(k, v_i)$ and $\overline{d}^\alpha_{[1,\infty)} = \lim_{i\to\infty} \overline{d}(k, v_i)$ (with the usual metric on sequences). Furthermore, $\alpha = \lim_{i\to\infty} \eta_{k,v_i}$ and hence for each $L \in \mathbb{N}$, there is some i such that for any word u with $|u| = L$, we have $u \in \mathcal{L}_\alpha$ if and only if $u \in \mathcal{L}_{\eta_{k,v_i}}$. Therefore the symmetry for $\{-1,-2\}^* \cap \mathcal{L}_\alpha$ and $\{k+1,k\}^* \cap \mathcal{L}_\alpha$ follows from the symmetry for each of the corresponding sets of words for every η_{k,v_i}. □

The following two results provide the heart of the justification of injectivity.

Let $\mathscr{L} = \sup_j \{\ell_j(\alpha)\}$. Note that $\mathscr{L} < \lambda_{-3}$.

Lemma 74. *We have that*

$$\overline{\sqcup_{u\in\{-1,-2\}^*\cap\mathcal{L}_\alpha} \mathcal{T}_\alpha^{|u|}(\,\Delta_\alpha(u)\times\Phi^-\,)} \cap \{(x,y) \mid x \in [\mathscr{L}, r_0)\,\} = [\mathscr{L}, r_0)\times\Phi\setminus\Phi^-\,.$$

Proof. Since $\underline{d}^\alpha_{[1,\infty)} \in \{-1,-2\}^*$ whereas $\overline{d}^\alpha_{[1,\infty)} \in \{k+1,-k\}^*$, for all $u \in \{-1,-2\}^* \cap \mathcal{L}_\alpha$ $T_\alpha^{|u|}(\Delta_\alpha)$ has some some $\ell_i(\alpha), i \geq 0$ as its left endpoint and has right endpoint equal to r_0. In particular, $T_\alpha^{|u|}(\Delta_\alpha) \supset [\mathscr{L}, r_0)$. Furthermore, $RM_uR^{-1}\cdot\Phi^- = -[\lambda_{\overleftarrow{u},-3}, \rho_{\overleftarrow{u}})$.

Since $\Phi\setminus\Phi^- = -[\ell_0, \lambda_{-3})$, it remains to show that (ℓ_0, λ_{-3}) is the union over all $u \in \{-1,-2\}^* \cap \mathcal{L}_\alpha$ of $[\lambda_{\overleftarrow{u},-3}, \rho_{\overleftarrow{u}})$. By Lemma 70, $(\ell_0, \lambda_{-3}) = \sqcup_{u\in\{-1,-2\}^*\cap\mathcal{L}_\alpha}\ [\lambda_{u,-3}, \rho_u)$ and hence Corollary 73 now implies $(\ell_0, \lambda_{-3}) = \sqcup_{u\in\{-1,-2\}^*\cap\mathcal{L}_\alpha}\ [\lambda_{\overleftarrow{u},-3}, \rho_{\overleftarrow{u}})$. □

The next result is proven analogously. Let $\mathscr{R} = \inf_j \{r_j(\alpha)\}$ and note that $\mathscr{R} > \rho_{k+2}$.

Lemma 75. *We have that*

$$\overline{\sqcup_{u\in\{k+1,k\}^*\cap\mathcal{L}_\alpha} \mathcal{T}_\alpha^{|u|}(\,\Delta_\alpha(u)\times\Phi^+\,)} \cap \{(x,y) \mid x \in [\ell_0, \mathscr{R})\,\} = [\ell_0, \mathscr{R})\times\Phi\setminus\Phi^+\,.$$

Although not necessary for the proof of bijectivity, more can be shown about the vertical fibers.

Lemma 76. *For each $x \in \mathbb{I}_\alpha$, let $\mathcal{F}_x$ be the vertical fiber of Ω above x. Then for all $x \in \mathbb{I}_\alpha$, the fiber $\mathcal{F}_x$ is an interval. In particular, $\mathcal{F}_{\ell_0} = [-r_0, -L]$ where $L = \sup_j \{\ell_j(\alpha)\}$ and $\mathcal{F}_{\ell_0} = [-\mathscr{R}, -\ell_0]$ where $\mathscr{R} = \inf_j \{r_j(\alpha)\}$. Furthermore, (i) if $x < x' < \rho_{k+2}$ then $\mathcal{F}_x \subseteq F_{x'}$, and (ii) if $\rho_{k+2} < x < x'$ then $\mathcal{F}_x \supseteq F_{x'}$.*

Proof. The arguments for "left" versus "right" sides of our setting are completely analogous. We thus only give the arguments for the latter case.

We first show containment of fibers. For $L < x < \rho_{k+2}$, we have that $\mathcal{F}_x = \Phi$. For $x \leq L$, we have $\mathcal{F}_x = \Phi^- \bigcup \overline{\cup_{u\in\{-1,-2\}^*\cap\mathcal{L}_\alpha} RM_uR^{-1}\cdot\Phi^-}$, thus certainly if $x < x' \leq L$ then $\mathcal{F}_x \subseteq F_{x'}$.

Now, $\mathcal{F}_{\ell_0} = \Phi^- \bigcup \overline{\cup^{\substack{x\geq T_\alpha^{|u|}(\lambda_u)\\ \Delta_\alpha(u)\text{ full}}}_{u\in\{-1,-2\}^*\cap\mathcal{L}_\alpha} - [\lambda_{\overleftarrow{u},-3}, \rho_{\overleftarrow{u}})}$. It is easily confirmed that Lemma 71, with Propositions 59 and 60, implies that when α is replaced by any α' equal to some $\zeta_{k,v}$ or $\eta_{k,v}$ then the leftmost fiber equals $[-r_0(\alpha'), -L(\alpha')]$ where $L(\alpha')$ is the largest element of the $T_{\alpha'}$-orbit of $\ell_0(\alpha')$.

We thus now aim to argue that $\underline{d}^\alpha_{[1,\infty)} = \lim_{i\to\infty} \underline{d}(k, v_i)$ leads to the result holding here. It is easily established that Φ^- is the limit of the corresponding interval for each η_{k,v_i} and also L

is the limit from below of the $L(\eta_{k,v_i})$. For any $u \in \{-1,-2\}^* \cap \mathcal{L}_\alpha$ there is some i_0 (depending only on $|u|$) such that for all $i \geq i_0$ we have $u \in \mathcal{L}_{\eta_{k,v_i}}$. The non-full cylinders are given by those u of suffix $\underline{d}^\alpha_{[1,j]}$ for some $j \in \mathbb{N}$. For each such j, there is some i_1 such that for all $i \geq i_1$ we have $\underline{d}^\alpha_{[1,j]} = \underline{d}^{\eta_{k,v_i}}_{[1,j]}$. Thus, each "full" u is such that $\Delta_{\eta_{k,v_i}}(u)$ is full for all i sufficiently large.

On the other hand, again since $\underline{d}^\alpha_{[1,\infty)} = \lim_{i\to\infty} \underline{d}(k,v_i)$, for each i every $u \in \mathcal{L}_{\eta_{k,v_i}}$ of length less than $\underline{d}(k,v_i)$ is α-admissible and if $\Delta_{\eta_{k,v_i}}(u)$ is full then also $\Delta_\alpha(u)$ is full. Thus, $\mathcal{F}_{\ell_0}$ is the limit of ever increasing subsets of the $\mathcal{F}_{\ell_0(\eta_{k,v_i})} = [-r_0(\alpha'), -L(\alpha')]$ from which the result now follows.

Finally, since $\mathcal{F}_{\ell_0}$ is an interval so also is each $\mathcal{F}_{\ell_j}$. Since any left non-full cylinder $\Delta_\alpha(u)$ has its left endpoint equal to some ℓ_j by $T_\alpha^{|u|}$, the fibers are constant (and hence intervals) between the various ℓ_j.

We repeat that the analogous arguments succeed for the remaining cases. □

6.5. **Ergodic natural extension for non-synchronizing α.** The proof of Theorem 1 for small, non-synchronizing α relies on ([9], Main Theorem 1). The following verifies one of the hypotheses of that theorem for such α.

Proposition 77. *With Ω as in Proposition 65, $\mu(\Omega) < \infty$.*

Proof. Let $\Omega^+ = \{(x,y) \in \Omega \mid y \geq 0\}$ and $\Omega^- = \{(x,y) \in \Omega \mid y \leq 0\}$.

Since $\underline{d}^\alpha_{[1,\infty)} \in \{-1,-2\}^{\mathbb{N}}$ the T_α-orbit of $\ell_0(\alpha)$ agrees with the T_0-orbit of this same value. The vertical fiber in Ω^+ above ℓ_0 is $-[\mathscr{L}, 0]$, where $\mathscr{L} = \sup_j \{\ell_j(\alpha)\}$. Now, $-\mathscr{L} = \lim_{i\to\infty} -\ell_{\underline{S}(\zeta_{k,v_i})} = \lim_{i\to\infty} RA^{-1}R^{-1} \cdot -\ell_0(\zeta_{k,v_i}) = RA^{-1}R^{-1} \cdot -\ell_0$. Since $-t < \ell_0$, we have $RA^{-1}R^{-1} \cdot -\ell_0 < RA^{-1}R^{-1} \cdot t$. Direct computation shows $RA^{-1}R^{-1} \cdot t = 1/(t+1/t)$. This value is certainly less than $1/t$, and hence it follows that the fiber $\mathcal{F}_{\ell_0}$ lies inside Ω_0, see [10]. That is, the $\mathcal{T}_0$-orbit of the vertical fiber lies strictly inside Ω_0. But, Ω_0 touches the locus $y = -1/x$, the locus of non-finiteness for $d\mu = (1+xy)^{-2}\,dx\,dy$, only at its vertices. Since the vertical fibers of Ω^+ are constant between the various $\ell_j(\alpha)$ we find that all of $\Omega^+ \cap \{x \leq 0\}$ lies within a set of finite μ-measure. Elementary considerations show that all of Ω^+ has finite μ-mass.

With $\mathscr{R} = \inf_j \{r_j(\alpha)\}$, Lemma 29, Proposition 60, and an elementary calculation give that $-\mathscr{R} = \lim_{i\to\infty} RAR^{-1} \cdot -r_0(\eta_{k,v_i}) = RAR^{-1} \cdot -r_0 = -r_0/(1+tr_0)$. Clearly, $(r_0, -\mathscr{R})$, and hence all of $\mathcal{F}_{r_0}$, lies above the locus $y = -1/x$. Since the locus is invariant under $\mathcal{T}_M$ for any $M \in \mathrm{SL}_2(\mathbb{R})$, we find every fiber $\mathcal{F}_{r_j}$ also lies above this locus. Since the fibers are increasing with respect to $x \geq 0$ we find that $\Omega^- \cap \{x \geq 0\}$ lies within a set of finite μ-measure. Elementary considerations show that all of Ω^- has finite μ-mass. □

Proposition 78. *With α as in the notation of Proposition 65, the system $(\mathcal{T}_\alpha, \Omega_\alpha, \mathscr{B}'_\alpha, \mu_\alpha)$ is the natural extension of $(T_\alpha, \mathbb{I}_\alpha, \mathscr{B}_\alpha, \nu_\alpha)$, where ν_α is the marginal measure of μ_α and $\mathscr{B}_\alpha$ the Borel sigma algebra on $\mathbb{I}_\alpha$. Finally, both systems are ergodic.*

Proof. From Proposition 65, $\Omega_\alpha = \Omega$ is a bijectivity domain for $\mathcal{T}_\alpha$. From Proposition 77 this domain has finite measure. From its description in Proposition 65 the domain has fibers of bounded Lebesgue measure. Since α is a non-synchronizing value, it corresponds to a limit under a sequence Θ_{q_i} applied to a word in $\mathcal{V}$; the digits which enter the expansion of the endpoints of such sequences form a finite set. In light of Remark 62, T_α satisfies the bounded non-full cylinders hypothesis for ([9], Main Theorem 1) and indeed all other hypotheses are then also easily verified. That is, the proof of the proposition is then complete. □

6.6. **Continuity of $\alpha \mapsto \mu(\Omega_\alpha)$.** We can now prove the main aim of this section.

Theorem 79. *The function $\alpha \mapsto \mu(\Omega_\alpha)$ is continuous on $(0, \gamma_n)$.*

Proof. Since the both top and bottom heights are constant along any synchronization interval $J_{k,v}$, with fibers constant between the various $r_i(\alpha), \ell_j(\alpha)$, continuity and indeed smoothness, of $\alpha \mapsto \mu(\Omega_\alpha)$ in the interior of the interval is clear. Propositions 59 and 60 easily imply that the continuity holds on the closed interval.

Now fix k. To show continuity throughout $I_k := \cup_{v\in\mathcal{V}} I_{k,v}$, it only remains to show that $\mu(\Omega_{\alpha'}) \to \mu(\Omega_\alpha)$ when α is non-synchronizing or (for the remaining one sided limits to) some $\eta_{k,v}$ or $\zeta_{k,v}$ with all values in I_k. Fix one such value $\alpha \in I_k$, it suffices to show that $\mu(\Omega_{\alpha'}) \to \mu(\Omega_\alpha)$ when the α' are non-synchronizing or various $\eta_{k,v}$ or $\zeta_{k,v}$ also in I_k. Such α' have

$$\Omega_{\alpha'} = \mathcal{H}_{\alpha'} \bigsqcup \overline{\sqcup_{u\in\{-1,-2\}^*\cap\mathcal{L}_{\alpha'}} \mathcal{T}_{\alpha'}^{|u|}(\,\Delta_{\alpha'}(u)\times\Phi^-_{\alpha'}\,)} \bigsqcup \overline{\sqcup_{u\in\{k+1,k\}^*\cap\mathcal{L}_{\alpha'}} \mathcal{T}_{\alpha'}^{|u|}(\,\Delta_{\alpha'}(u)\times\Phi^+_{\alpha'}\,)}.$$

Now, as $\alpha' \to \alpha$ we have $\mu(\mathcal{Z}_{\alpha'} \setminus \mathcal{Z}_\alpha) \to 0, \mu(\mathcal{H}_{\alpha'} \setminus \mathcal{H}_\alpha) \to 0$. Furthermore both $\underline{d}^{\alpha'}_{[1,\infty)} \to \underline{d}^{\alpha}_{[1,\infty)}$ and $\overline{d}^{\alpha'}_{[1,\infty)} \to \overline{d}^{\alpha}_{[1,\infty)}$. In particular, there are $b_{\alpha'}, c_{\alpha'} \in \mathbb{N}$ each going to infinity as $\alpha' \to \alpha$ such that for all $u \in \{-1,-2\}^*$ with $|u| < b_{\alpha'}$, one has $u \in \mathcal{L}_{\alpha'}$ if and only if $u \in \mathcal{L}_\alpha$, and similarly for all $u \in \{k+1,k\}^*$ with $|u| < c_{\alpha'}$ we have $u \in \mathcal{L}_{\alpha'}$ if and only if $u \in \mathcal{L}_\alpha$. For these u, we have also $\Delta_{\alpha'}(u) \to \Delta_\alpha(u)$. Since Ω_α is of finite μ-measure, as $|u| \to \infty$ we have $\mu(\Delta_\alpha(u)\times\Phi^-_\alpha) \to 0$ and similarly $\mu(\Delta_\alpha(u)\times\Phi^+_\alpha) \to 0$. Taken all together, we find that $\mu(\Omega_{\alpha'}) \to \mu(\Omega_\alpha)$ as $\alpha' \to \alpha$.

It only remains to show continuity at the boundary of each of the various I_k. Fix k, we must show continuity from the left at $\alpha = \zeta_{k,1}$, we do so by considering the limit given by $\zeta_{k+1,h} \to \zeta_{k,1}$ as $h \to \infty$. As in (11), with $\mathcal{D} = (-1)^{n-3}, -2, w^{k+1}$,

$$\underline{d}^{\zeta_{k,1}}_{[1,\infty)} = (-1)^{n-2}, -2, \overline{\mathcal{D}} \quad \text{and} \quad \underline{d}^{\zeta_{k+1,h}}_{[1,\infty)} = (-1)^{n-2}, -2, \overline{\mathcal{D}^h, w}$$

Therefore, in terms of the argument of the previous paragraph, here also there is agreement among ever greater length words of $\{-1,-2\}^* \cap \mathcal{L}_{\alpha'}$ with those of $\{-1,-2\}^* \cap \mathcal{L}_\alpha$. That is, the arguments above apply and give $\mu(\Omega^+_{\zeta_{k+1,h}}) \to \mu(\Omega^+_{\zeta_{k,1}})$.

Note that that the only words in $\mathcal{L}_{\zeta_{k,1}}$ which contain the letter k are the prefixes of $\overline{d}^{\zeta_{k,1}}_{[1,\infty)}$; in each, k occurs only as the initial letter. Complementing this, words consisting of $k+1$ repeated any number of times are admissible. On the other hand, in each $\mathcal{L}_{\zeta_{k+1,h}}$ there is no word with the letter k, and the words consisting of $k+1$ repeated up to h of times are admissible. Thus, the arguments from above apply to show that $\mu(\Omega^-_{\zeta_{k+1,h}}) \to \mu(\Omega^-_{\zeta_{k,1}})$. Therefore, continuity of $\alpha \mapsto \mu(\Omega_\alpha)$ holds on all of $(0, \gamma_n)$. □

7. Intermezzo: Preparation for the treatment for larger values of α

7.1. **Terse review of further notation and terminology.** We complement Subsection 1.3 with reminders to the reader of further terminology and notation from [8]. For the following, confer ([8], Figures 4.1 and 1.3).

Fix $n \geq 3$. For each α, let

$$\mathfrak{b}_\alpha = C^{-1} \cdot \ell_0(\alpha). \tag{15}$$

Then the right endpoint γ_n of the set of small α is the value of α such that $\mathfrak{b}_\alpha = r_0(\alpha)$. The left endpoint of the set of large α is ϵ_n, the value of α such that $A^{-1}C \cdot \ell_0(\alpha) = r_0(\alpha)$. Thus, the large α are exactly the values such that the first α-digit of $\ell_0(\alpha)$ is $(-k,1)$ for some integer $k \geq 2$.

For large α, we use exactly the same tree of words $\mathcal{V}$ as for small α. For the following, confer ([8], Figure 6.1). For each $k \in \mathbb{N}, k \geq 2$ we let

$$\underline{d}(-k,v) = (-k)^{c_1}, (-k-1)^{d_1}, \cdots, (-k-1)^{d_{s-1}}, (-k)^{c_s}, \tag{16}$$

and so that the corresponding subinterval of parameters is $I_{-k,v} = \{\alpha \mid \underline{d}^{\alpha}_{[1,|v|]} = \underline{d}(-k,v)\}$. Thus, (using redundant notation) set $\underline{S}(-k,v)$ to be the sum of the c_i and d_j of v, and

$$L_{-k,v} = (A^{-k}C)^{c_s}\ (A^{-k-1}C)^{d_{s-1}}(A^{-k}C)^{c_{s-1}}\cdots(A^{-k-1}C)^{d_1}(A^{-k}C)^{c_1}A^{-1}, \tag{17}$$

then for $\alpha \in I_{-k,v}$ one has $\ell_{\underline{S}(-k,v)}(\alpha) = L_{-k,v}A \cdot \ell_0(\alpha)$. The right endpoint of $I_{k,v}$ is denoted $\zeta_{-k,v}$; one finds $L_{-k,v}A \cdot \ell_0(\zeta_{-k,v}) = r_0(\zeta_{-k,v})$. Set $R_{-k,v} = CA^{-1}CL_{-k,v}$ and define $J_{-k,v} = [\eta_{-k,v}, \zeta_{-k,v})$ where $R_{-k,v} \cdot r_0(\eta_{-k,v}) = \mathfrak{b}_{\eta_{-k,v}}$.

Each $J_{-k,v}$ is partitioned into two left-closed subintervals, whose common endpoint, $\delta_{-k,v}$, is characterized by $\ell_{\underline{S}}(\alpha) = \mathfrak{b}_\alpha$ when $\alpha = \delta_{-k,v}$. We will refer to these as the *left portion* and *right portion* of the interval. Recall that synchronization occurs after one step further in the T_α-orbit of $r_0(\alpha)$ for α in the right portion as opposed to in the left portion. To be precise, the synchronication is given by

$$\begin{cases} \ell_{1+\underline{S}}(\alpha) = r_{1+\overline{S}}(\alpha) & \text{if } \alpha < \delta_{-k,v}\,, \\ \\ \ell_{1+\underline{S}}(\alpha) = r_{2+\overline{S}}(\alpha) & \text{if } \alpha > \delta_{-k,v}\,. \end{cases} \tag{18}$$

Key to proving the synchronization results for large α was the determination of a maximal common prefix of the $\overline{b}^{\alpha}_{[1,\infty)}$ for $\alpha \in J_{-k,v}$. For that, we set $u = u_n = (1,2)^{n-2},(1,1)$ and for $k \geq 2$, $\mathcal{E} = \mathcal{E}_k = (1,1)u^{k-2}(1,2)^{n-3}$ and $\mathcal{F} = \mathcal{E}_{k+1}$. The common prefix is

$$\overline{b}(-k,v) = (1,2)^{n-2}\mathcal{E}^{c_1}\mathcal{F}^{d_1}\,\mathcal{E}^{c_2}\,\mathcal{F}^{d_2}\cdots\mathcal{E}^{c_{s-1}}\,\mathcal{F}^{d_{s-1}}\,\mathcal{E}^{c_s}, \tag{19}$$

denote its length as a word in the α-digits by $\overline{S}(-k,v)$. There is an expression for $R_{-k,v}$ related to $\overline{b}(-k,v)$ similar to how (17) relates $L_{-k,v}$ to $\underline{b}(-k,v)$; see (20).

Finally, in the case of intermediate sized parameters, that is for $\alpha \in [\gamma_n, \epsilon_n)$, one has as for small α that the first α-digit of $\ell_0(\alpha)$ is $(-1,1)$ and thus $k = 1$; however, the dynamics are mainly of the type of the large α. To see this, first note that $\gamma_n = \eta_{-1,n-2}$ and furthermore, there exists $\alpha \in J_{-1,n-2}$ such that $(A^{-1}C)^{n-2} \cdot \ell_0(\alpha) = C^{-1} \cdot \ell_0(\alpha)$. For this value of α, $AC^2 \cdot r_0(\alpha) = \ell_0(\alpha)$. Arguing as in ([8], Section 6), it follows that $(1,2)$ is the initial digit of r_0 for all parameter values at least as large as this α. Thus for α to the right of $J_{-1,n-2}$, the dynamics are indeed of the type of the large α. On the other hand, see Subsection 8.2, synchronization holds on $J_{-1,n-2}$ with $\overline{S}(-1,n-2) = 0$ and hence the fact that there are $\alpha \in J_{-1,n-2}$ whose digit of r_0 is less than $(1,2)$ is insignificant for our discussions.

Therefore to describe the dynamics for the intermediate sized parameters, we restrict to words in the subtree $\check{\mathcal{V}} \subset \mathcal{V}$ defined by (i) $v = c_1d_1\cdots c_s \in \check{\mathcal{V}}_n$, is such that $c_i \leq n-2$, and (ii) the only word in $\check{\mathcal{V}}_n$ with prefix $n-2$ is $v = n-2$ itself. As discussed in ([8], Section 7) we can still use (19) since the implied appearances of $u^{k-2} = u^{-1}$ turn out to always have neighboring appearances of u to positive powers. Thus, we treat the two-dimensional maps for the cases of $\alpha \geq \gamma_n$ all at the same time. We give various examples with intermediate sized α, see Subsection 8.2 and Examples 103, 108, 110, 112.

8. The case of large α in left portion of synchronization interval

Devoted to $\alpha \geq \gamma_n$, this section and the next parallel Sections 4 and 5. The relationships between the various rectangle heights in this current are more intricate, see Figure 13 for an indication of this.

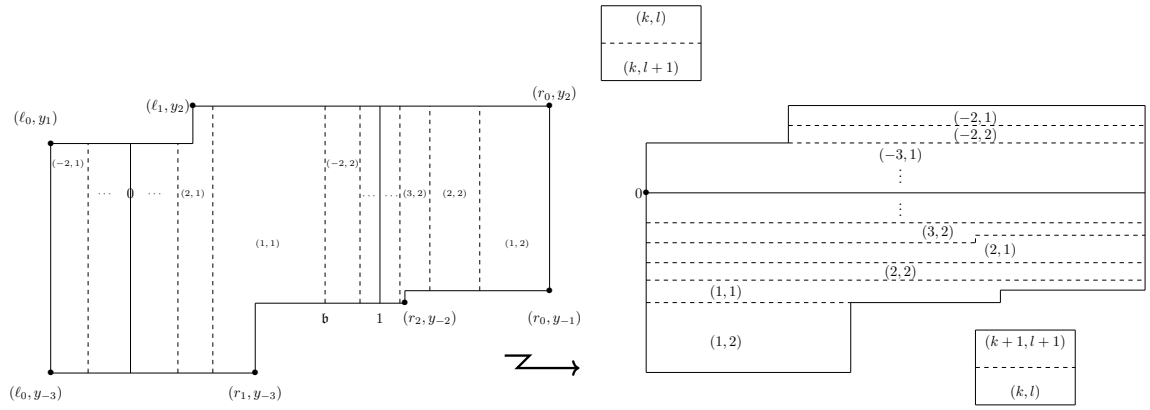

FIGURE 10. The domain $\Omega_{3,0.86}$. Blocks $\mathcal{B}_{i,j}$ and their images, both denoted by (i, j). Here $L_{-k,v} = A^{-2}CA^{-1}$ and $R_{-k,v} = AC\,AC^2$, and α is an interior point of $J_{-2,1}$ lying to the left of $\delta_{-2,1}$. Also, hints as to the lamination ordering given in small boxes.

8.1. **Definition of Ω_α for large α in left portion of synchronization interval.** Fix $n \geq 3, k \in \mathbb{N}$. If $k \geq 2$, fix $v \in \mathcal{V}$, and otherwise fix $v \in \check{\mathcal{V}}$. Finally, fix α in the interior of $J_{-k,v}$.

Compare the following definition with Figures 10 and 11.

Definition 80. Let $\underline{S} = \underline{S}(-k, v)$ and define Ω^+ as in Definition 11 except that

$$\text{For all } i \in \{0, \dots, \underline{S}\}, \text{ we set } y_{\tau(i)} = -\ell_{\underline{S}-i}(\eta_{-k,v}).$$

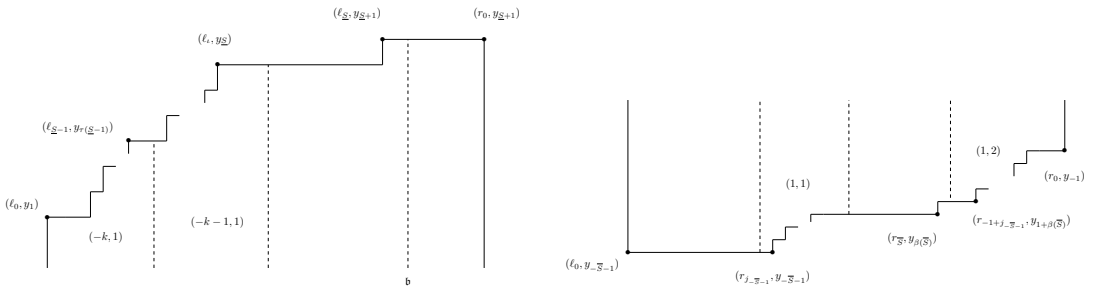

FIGURE 11. Schematic representations showing the most important vertices of the tops and bottoms of blocks (not to scale), for $\alpha \in (\eta_{-k,v}, \delta_{-k,v})$. Blocks $\mathcal{B}_{(i,j)}$ denoted by (i, j).

Definition 81. Let $\overline{S} = \overline{S}(-k, v)$ and define Ω^- as in Definition 12 except that we let

$$y_b = \begin{cases} -\hat{r}_{j_{-(b+e+1)}} & \text{if } -1 \geq b \geq -e; \\ \\ -\hat{r}_{j_{-(b+e+\overline{S}+2)}} & \text{otherwise,} \end{cases}$$

where for $0 \le i \le \overline{S}$,

$$\hat{r}_i = \begin{cases} C \cdot r_i(\eta_{-k,v}), & \text{if } r_i(\eta_{-k,v}) \in \Delta_{\eta_{-k,v}}(1,2); \\ r_i(\eta_{-k,v}), & \text{otherwise,} \end{cases}$$

and $e = e(-k, v)$ is the number of occurrences of $(1, 2)$ in $\overline{b}(-k, v)$. See Figure 12 for an indication of these values.

Note that since $\overline{b}(-k, v)$ only has digits $\{(1, 2), (1, 1)\}$, each $\hat{r}_i \le \mathfrak{b}_{\eta_{-k,v}}$ with equality exactly when $i = \overline{S}$. Furthermore, $e(-k, v) = \#\{i \mid \hat{r}_i = C \cdot \hat{r}_i\}$.

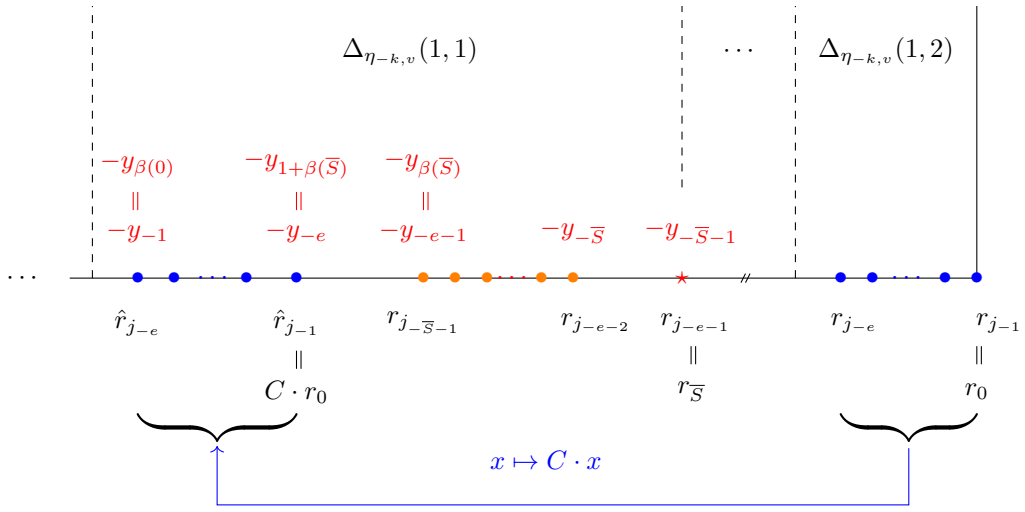

FIGURE 12. Bottom heights are of the form $-\hat{r}_j$, see Definition 81. For $r_j(\eta_{k,v}) \in \Delta_{\eta_{k,v}}(1, 2)$, we have $\hat{r}_j = C \cdot r_j(\eta_{k,v})$. That $\hat{r}_{j-1} < r_{j_{-\overline{S}-1}}(\eta_{k,v})$ is shown in Lemma 90.

The following result gives a hint of the use of the $\hat{r}_j$.

Lemma 82. *Considering $\overline{b}(-k, v)$ as a word in $\{(1, 2), (1, 1)\}$, let q_j be the first letter of $\sigma^j(\overline{b}(k, v))$, and set $M_j = AC^{l_j}$ where $q_j = (1, l_j)$ for $0 \le j < \overline{S}$. Then*

$$RM_jR^{-1} \cdot -\hat{r}_j = \begin{cases} -\hat{r}_{j-1} & \text{if } \ 1 \le j < \overline{S}; \\ -\hat{r}_{\overline{S}} & \text{if } \ j = 0 \end{cases}$$

and $RACR^{-1} \cdot -\hat{r}_{\overline{S}} = -\hat{r}_{\overline{S}-1}$.

Proof. Since the initial digit of $\overline{b}(-k, v)$ is $(1, 2)$, using Lemma 14 we evaluate

$$\begin{aligned} RAC^2R^{-1} \cdot -\hat{r}_0 &= -(CAC)^{-1} \cdot (C \cdot r_0(\eta_{-k,v})) = -(AC)^{-1} \cdot r_0(\eta_{-k,v}) \\ &= -C^{-1} \cdot \ell_0(\eta_{-k,v}) = -\mathfrak{b}_{\eta_{-k,v}} = -r_{\overline{S}}(\eta_{-k,v}) = -\hat{r}_{\overline{S}}. \end{aligned}$$

Similarly, we treat the four possible cases when $1 \le j < \overline{S}$. If $M_j = AC$ and $r_{j-1}(\eta_{-k,v}) \in \Delta_{\eta_{-k,v}}(1, 1)$, then

$$RACR^{-1} \cdot -\hat{r}_j = -(AC)^{-1} \cdot r_j(\eta_{-k,v}) = -r_{j-2}(\eta_{-k,v}) = -\hat{r}_{j-1}.$$

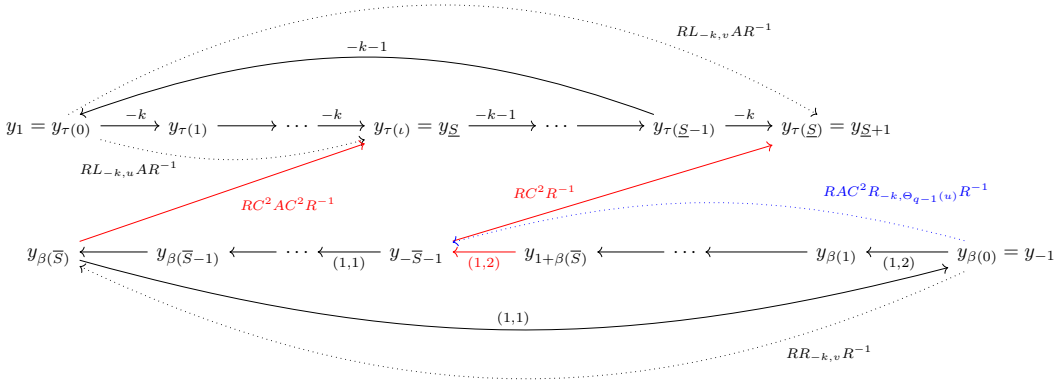

FIGURE 13. **Relations on the heights of rectangles for general k, v and $\alpha \in (\eta_{-k,v}, \delta_{-k,v})$.** The equality $\tau(\iota) = \underline{S}$ is Lemma 93. The red paths are used to prove that lamination occurs; the red arrows are due to Proposition 94 and Corollary 84. The blue dotted arrow is due to Lemma 85. The highest solid arrow is due to Lemma 91, the lowest solid arrow is due to Corollary 84. The lowest dotted arrow is due to Corollary 89. The top horizontal arrows are as for the small α setting, the lower horizontal arrows follow mainly from the definition of the y_b combined with Lemma 82. The top two dotted arrows are implied by definitions, in direct analogy with the case of small α.

Whereas if instead $r_{j-1}(\eta_{-k,v}) \in \Delta_{\eta_{-k,v}}(1,2)$, then

$$RACR^{-1} \cdot -\hat{r}_j = -(AC)^{-1} \cdot (AC^2 \cdot r_{j-1}(\eta_{-k,v})) = -C \cdot r_{j-1}(\eta_{-k,v}) = -\hat{r}_{j-1}.$$

If $M_j = AC^2$ and $r_{j-1}(\eta_{-k,v}) \in \Delta_{\eta_{-k,v}}(1,1)$, then

$$RAC^2R^{-1} \cdot -\hat{r}_j = -(CAC)^{-1} \cdot C \cdot r_j(\eta_{-k,v}) = -r_{j-1}(\eta_{-k,v}) = -\hat{r}_{j-1}.$$

And, if instead $r_{j-1}(\eta_{-k,v}) \in \Delta_{\eta_{-k,v}}(1,2)$, then

$$RAC^2R^{-1} \cdot -\hat{r}_j = -(CAC)^{-1} \cdot C \cdot r_j(\eta_{-k,v}) = -C \cdot r_{j-1}(\eta_{-k,v}) = -\hat{r}_{j-1}.$$

Finally,

$$RACR^{-1} \cdot -\hat{r}_{\overline{S}} = -(AC)^{-1} \cdot r_{\overline{S}}(\eta_{-k,v}) = -\hat{r}_{\overline{S}-1},$$

by considering the two possibilities for $l_{\overline{S}-1}$. □

8.2. **Examples with 'intermediate' values.** Recall from ([8], Example 7.4) that $R_{-1,n-2} = CA^{-1}C\,(A^{-1}C)^{n-2}A^{-1} = \mathrm{Id}$ holds for all $n \geq 3$; thus $\overline{S}(-1, n-2) = 0$. Recall that $\eta_{-1,n-2} = \gamma_n$, from whose definition it easily follows that the initial digit of $r_0(\gamma_n)$ is $(-1, 2)$. Thus, on $J_{-1,n-2}$ we have a single bottom height of $y_{-1} = -r_0(\gamma_n)$.

We now fix $n = 3$. Here $\eta_{-1,1} = \gamma_3 = g^2/2$ where $g = G - 1$ and $G = (1 + \sqrt{5})/2$. Thus $r_0(\eta_{-1,1}) = g^2$; one easily verifies that the $T_{g^2/2}$-orbit of $\ell_0(g^2/2)$ is periodic of length one with preperiod $-G \mapsto -g^2$. Therefore, for all $\alpha \in (\gamma_3, \delta_{-1,1})$ we have $y_1 = g^2, y_2 = G$ and $y_{-1} = -g^2$. That is, we find a "backwards L"-shape:

$$\Omega_{3,\alpha} = ([\ell_0(\alpha), \ell_1(\alpha)] \times [-g^2, g^2]) \cup ([\ell_1(\alpha), r_0(\alpha)] \times [-g^2, G]) \;\; \forall \alpha \in (\gamma_3, \delta_{-1,1}).$$

8.3. **Relations on heights of rectangles, for large α in left portion of synchronization interval.** The main aim of this subsection is to establish the relations needed to show that $\mathcal{T}_{n,\alpha}$ is bijective up to measure zero on $\Omega_{n,\alpha}$. Other than the result that the vertices have the correct images (see Lemma 88, below), these relations are all indicated in Figure 13. We also show that $\Omega_{n,\alpha}$ has the expected shape — the vertices increase in y-value as their x-coordinates increase. See Lemma 90 for the bottom vertices, the similar behavior of top vertices follows directly from arguments of the small α setting.

Note that since Subsection 8.2 directly establishes that the lower boundary of Ω_α is of constant height when $\alpha \in J_{-1,n-2}$, in what follows we will occasionally tacitly assume that $\alpha > \zeta_{-1,n-2}$.

Lemma 83. *For $\alpha \in [\eta_{-k,v}, \delta_{-k,v})$, we have*

i.) $r_{\overline{S}}(\alpha) \geq \mathfrak{b}_\alpha$ *with equality only for* $\alpha = \eta_{-k,v}$,

ii.) $r_{\overline{S}}(\alpha)$ *lies to the left of* $\Delta_\alpha(1,2)$,

iii.) $\beta(\overline{S}) = -1 - e(-k,v)$,

iv.) $\beta(-1 + j_{-\overline{S}-1}) = -e(-k,v)$.

Proof. For all $k \in \mathbb{N}$ and $v \in \mathcal{V}$, the definition of $\eta_{-k,v}$ results in $r_{\overline{S}}(\eta_{-k,v}) = \mathfrak{b}_{\eta_{-k,v}}$. The commonality of the initial digits along the synchronization interval then leads to the inequality.

For $\alpha \in (\eta_{-k,v}, \delta_{-k,v})$ we have $\ell_{\underline{S}}(\alpha) < \mathfrak{b}_\alpha$, and hence $C^{-1}AC^{-1} \cdot r_{\overline{S}}(\alpha) < C^{-1} \cdot \ell_0(\alpha)$, whence $AC^{-1} \cdot r_{\overline{S}}(\alpha) < \ell_0(\alpha)$. Since C^{-1} acts as C^2, each $r_{\overline{S}}(\alpha)$ lies to the left of $\Delta_\alpha(1,2)$.

We thus have that there are $e(-k,v)$ orbit entries larger than $r_{\overline{S}}(\alpha)$, and hence $\beta(\overline{S}) = -1 - e(-k,v)$.

We certainly have that $r_{j_{-\overline{S}-1}}$ is the leftmost r_j $(j < \overline{S})$ in the cylinder $\Delta_\alpha(1,1)$. Since all r_j with $j < \overline{S}$ are in $\Delta_\alpha(1,1) \cup \Delta_\alpha(1,2)$ while $r_{\overline{S}}$ lies between these cylinders, it follows that $r_{-1+j_{-\overline{S}-1}}$ is the image of the leftmost r_j in $\Delta_\alpha(1,2)$. That is, $\beta(-1 + j_{-\overline{S}-1}) = -e(-k,v)$. □

Compare the following with Figures 13 and 11.

Corollary 84. *The following hold:*

i.) $RC^2R^{-1} \cdot y_{-\overline{S}-1} = y_{\underline{S}+1}$,

ii.) $y_{-\overline{S}-1} = RAC^2R^{-1} \cdot y_{1+\beta(\overline{S})}$,

iii.) $y_{\underline{S}+1} = RC^{-1}AC^2R^{-1} \cdot y_{1+\beta(\overline{S})}$,

iv.) $y_{-1} = RACR^{-1} \cdot y_{\beta(\overline{S})}$.

Proof. By definition, $y_{-\overline{S}-1} = -\hat{r}_{j_{-e(-k,v)-1}}$. The lemma then implies that $y_{-\overline{S}-1} = -r_{\overline{S}}(\eta_{-k,v})$. By definition, this equals $-\mathfrak{b}_{\eta_{-k,v}}$. Lemma 14 gives $RC^2R^{-1} \cdot (-\mathfrak{b}_{\eta_{-k,v}}) = -C \cdot \mathfrak{b}_{\eta_{-k,v}} = -\ell_0(\eta_{-k,v})$. This final value equals $y_{\underline{S}+1}$.

From the lemma, $\beta(\overline{S}) + 1 = -e(-k,v)$. Hence $y_{1+\beta(\overline{S})} = -\hat{r}_{j_{-1}}$. Since $j_{-1} = 0$ we have that $y_{1+\beta(\overline{S})} = -\hat{r}_0$ and Lemma 82 gives $RAC^2R^{-1} \cdot (-\hat{r}_0) = -\hat{r}_{\overline{S}}$. From the above, this value is indeed $y_{\underline{S}+1}$.

The third equality follows directly from the first two.

Finally, from Definition 81, Lemma 82, Lemma 83 and again Definition 81, we have

$$RACR^{-1} \cdot y_{\beta(\overline{S})} = RACR^{-1} \cdot (-\hat{r}_{j_{-\overline{S}-1}}) = -\hat{r}_{-1+j_{-\overline{S}-1}} = -\hat{r}_{j_{-e}} = y_{-1}.$$

□

Recall from [8] that $U = AC(AC^2)^{n-2}$, $\tilde{\mathcal{E}} = (AC^2)^{n-3}U^{k-2}AC$, and $\tilde{\mathcal{F}} = (AC^2)^{n-3}U^{k-1}AC$. And,

$$R_{-k,v} = \tilde{\mathcal{E}}^{c_s}\tilde{\mathcal{F}}^{d_{s-1}}\,\tilde{\mathcal{E}}^{c_{s-1}}\tilde{\mathcal{F}}^{d_{s-2}}\cdots\tilde{\mathcal{E}}^{c_2}\tilde{\mathcal{F}}^{d_1}\tilde{\mathcal{E}}^{c_1}(AC^2)^{n-2}. \tag{20}$$

Recall also that $r_{j_{-\overline{S}-1}}$ is the least element in the initial T_α-orbit of $r_0 = r_0(\alpha)$; from § 8.2, when $(k, v) = (1, n-2)$ that initial orbit consists only of r_0 itself. In the following, we temporarily use s to also denote a word in $\mathcal{V}$.

Lemma 85. *Suppose that $(k, v) \neq (1, n-2)$. Then*

$$r_{j_{-\overline{S}-1}} = \begin{cases} (AC^2)^{n-3-c}\cdot r_0 & \text{if } v = c, k = 1\,; \\ (AC^2)^{n-2}U^{k-2}\cdot r_0 & \text{if } v = c, k > 1\,; \\ AC^2 R_{-k,\Theta_{q-1}(s)}\cdot r_0 & \text{if } v = \Theta_0^h\circ\Theta_q(s), \text{ with } q \geq 1, h \geq 0. \end{cases}$$

Proof. We first treat the case of $k \geq 2$. We can uniquely express $\overline{b}(-k, v)$ as a word in the letters that we temporarily use: $a := (1,1), b := (1,2)^{n-3}, c := (1,2)^{n-2}$, where we view these letters as having increasing value (of course, when $n = 3$ we suppress b). We introduce a fourth symbol, $\star$ to represent the expansion of $r_{\overline{S}}$, so that $\overline{b}^{\alpha}_{[1,\infty)}$ can be viewed as a word of length $\overline{S}(-k, v) + 1$ in these four letters, with the final symbol appearing only as a suffix. Of course, $r_{j_{-\overline{S}-1}}$ has expansion given by the least suffix of this word.

In terms of these letters, $u = ca, \mathcal{E} = a(ca)^{k-2}b, \mathcal{F} = a(ca)^{k-1}b$. By Lemma 83, $a \prec \star \prec b \prec c$, and thus $\mathcal{E} \prec \mathcal{F} \prec u$.

Since (20) gives the expression for cancelling the initial digits of $\overline{b}^{\alpha}_{[1,\infty)}$,

$$\overline{b}^{\alpha}_{[1,\infty)} = c[a(ca)^{k-2}b]^{c_1}[a(ca)^{k-1}b]^{d_1}\cdots[a(ca)^{k-2}b]^{c_{s-1}}\,[a(ca)^{k-1}b]^{d_{s-1}}\,[a(ca)^{k-2}b]^{c_s}\,\star\,. \tag{21}$$

If $v = 1$, then $\overline{b}^{\alpha}_{[1,\infty)} = c(ac)^{k-2}ab\star$. The least suffix is visibly $ab\star$ and we find

$$r_{j_{-\overline{S}-1}} = [(AC^2)^{n-2}AC]^{k-2}(AC^2)^{n-2}\cdot r_0 = (AC^2)^{n-2}U^{k-2}\cdot r_0\,.$$

Similarly, if $v = c_1, c_1 > 1$ or $|v| > 1$ then the least three letter word (when $n > 3$) is aba (when $n = 3$ then this is the least two letter word). Thus, when $v = c_1$ we find that the least suffix begins with the letters ab found at the end of the first copy of $\mathcal{E}$. (Indeed, all other suffixes beginning with an ab reach $\star$ at a length at which this suffix reaches a copy of a.) That is, here also $r_{j_{-\overline{S}-1}} = (AC^2)^{n-2}U^{k-2}\cdot r_0$.

When $|v| > 1$, recall that $c_s = c_1 = \max\{c_j\}$ and $d_1 = \min\{d_j\}$. There are no internal occurrences of $c_j = c_1$ in v exactly when v is of the form $v = \Theta_q(c_1)$. In this case, the minimal suffix is $ab\,[a(ca)^{k-2}b]^{c_s}\,\star$. Note that $AC^2\tilde{\mathcal{E}}\cdot\mathcal{F} = (1,1)(1,2)^{n-3}$. Hence, depending on whether $c_1 = 1$ or not, we find $M = AC^2\tilde{\mathcal{E}}\tilde{\mathcal{F}}^{q-1}\tilde{\mathcal{E}}(AC^2)^{n-2}$ or $M = AC^2\tilde{\mathcal{E}}\,[\tilde{\mathcal{E}}^{c_1-1}\tilde{\mathcal{F}}]^q\tilde{\mathcal{E}}^{c_1}(AC)^{n-2}$ sends r_0 to $r_{j_{-\overline{S}-1}}$. Therefore,

$$r_{j_{-\overline{S}-1}} = AC^2\,R_{-k,\Theta_{q-1}(c_1)}\cdot r_0\,.$$

If there are internal occurrences of $c_j = c_1$, then arguing as in in the proof to Lemma 32 , we find that the minimal word, up to its suffix of $\star$ and prefix ab, corresponds to the parent of v.

Thus, when $v = \Theta_q(s)$, we find $r_{j_{-\overline{S}-1}} = (1,1),(1,2)^{n-3},\sigma^{n-2}(\,\overline{b}(-k,s)\,),\star$. Following again the proof of Lemma 32, we find $AC^2R_{-k,\Theta_{q-1}(s)}\cdot r_0 = AC^2\tilde{\mathcal{E}}\cdot\mathcal{F}\,\sigma^{n-2}(\,\overline{b}(-k,s)\,),\star$. Thus, this case holds as well.

We turn to the case $k=1$. Recall that the only $v\in\check{\mathcal{V}}$ containing the letter $n-2$ is the one letter word $v=n-2$ itself, thus the case $n=3$ here is excluded by hypothesis. For $n>3$, the brief discussion at the end of § 7 can be shown to imply that in fact

$$R_{-1,v} = (AC^2)^{n-3-c_s}AC\,\tilde{\mathcal{F}}^{-1+d_{s-1}}\,(AC^2)^{n-3-c_{s-1}}AC\,\tilde{\mathcal{F}}^{-1+d_{s-2}}\cdots$$

$$\cdots(AC^2)^{n-3-c_2}AC\tilde{\mathcal{F}}^{-1+d_1}(AC^2)^{n-2-c_1}.$$

Equivalently, using $\star$ as above,

$$\overline{b}^{\alpha}_{[1,\infty)} = (1,2)^{n-2-c_1}(1,1)[(1,2)^{n-3}(1,1)]^{-1+d_1}(1,2)^{n-3-c_2}(1,1)[(1,2)^{n-3}(1,1)]^{-1+d_2}\cdots$$
$$\ldots(1,2)^{n-3-c_{s-1}}(1,1)[(1,2)^{n-3}(1,1)]^{-1+d_{s-1}}(1,2)^{n-3-c_s}\star\,.$$

If $v=c$ with $1\le c<n-2$, then $\overline{b}^{\alpha}_{[1,\infty)} = (1,2)^{n-2-c}\star$ and we find that our extremal suffix is given by $(AC^2)^{n-3-c}\cdot r_0$. For all other words, arguing as above we find that the extremal suffix begins with $(1,1)$ and continues with the digits corresponding to the longest suffix/prefix. That is, the argument for $k>1$ succeeds in this case as well. □

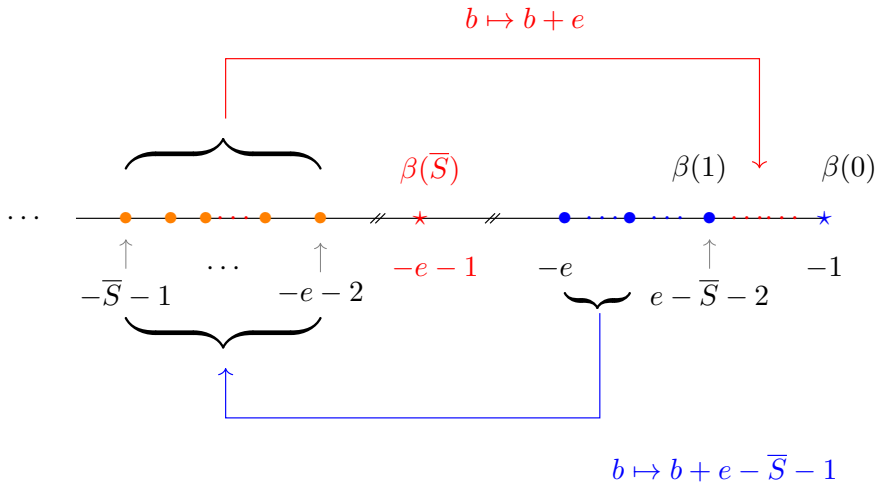

FIGURE 14. **Dynamics of the initial portion of the T_α-orbit of $r_0(\alpha)$.** Recall that $\beta(j)=j_b$ means that r_j is the j_b-th real value of this orbit, from the right. Indicated below the x-axis are the values of b, above it a few corresponding $\beta(j)$. See Corollaries 86 and 87.

Compare the following with Figures 11 , 13, 12 and 14.

Corollary 86. *We have* $1+\beta(1) = \beta(1+j_{-\overline{S}-1})$.

Sketch. We treat only the main case of $v = \Theta_q(s)$. The proof of the lemma implies that $r_{1+j_{-\overline{S}-1}} = (1,2)^{n-3}, \sigma^{n-2}(\,\overline{b}(-k,s)\,), \star$. We easily find that $r_1 = (1,2)^{n-3}, \sigma^{n-2}(\,\overline{b}(-k,v)\,), \star$. Now, $(1,2)^{n-3}, \sigma^{n-2}(\,\overline{b}(-k,s)\,)$ is a prefix of $(1,2)^{n-3}, \sigma^{n-2}(\,\overline{b}(-k,s)\,)(1,2)^{n-3}, \sigma^{n-2}(\,\overline{b}(-k,v)\,)$ which is followed by a digit of $(1,1)$. Since (mixing notation and temporary notation), $(1,1) \prec \star$ we have $r_{1+j_{-\overline{S}-1}} > r_1$. As r_1 is the rightmost of the $AC^2 \cdot r_j$ while $r_{1+j_{-\overline{S}-1}}$ is the leftmost of the $AC \cdot r_j$, the statement holds. □

A second direct result of the previous lemma gives the basic dynamics of the T_α-orbit of $r_0(\alpha)$ in terms of the indices for the real ordering. See Figure 14.

Corollary 87. *The position of the successor of r_{j_b} is given by*

$$\beta(1+j_b) = \begin{cases} b+e & \text{if } b \le -e-2; \\ \\ b+e-\overline{S}-1 & \text{if } b \ge -e. \end{cases}$$

Proof. The $r_j, j < \overline{S}$ in the cylinder $\Delta_\alpha(\,(1,2)^{n-2}\,)$ are $r_0 = r_{j_{-1}}$ and the images of the larger $r_j, j < \overline{S}$ in $\Delta_\alpha(1,1)$. In particular, the largest of these latter values, $r_{j_{-e-2}}$, is sent to $r_{j_{-2}}$. On the other hand, we have that the smallest of the $r_j, j < \overline{S}$ in $\Delta_\alpha(1,1)$ has image greater than r_1. Hence, the image of $\Delta_\alpha(\,(1,2)^{n-2}\,)$ lies to the left of the image of $\Delta_\alpha(1,1)$. It follows that $\beta(1+j_b) = b+e$ for all $r_{j_b} \in \Delta_\alpha(1,1)$. This is the set of r_{j_b} with $b \le -e-2$.

Since there are e values r_{j_b} in $\Delta_\alpha(1,2)$ and the one value, $r_{\overline{S}}$ outside of $\Delta_\alpha(1,1) \cup \Delta_\alpha(1,2)$, the set $\{b \mid r_{j_b} \in \Delta_\alpha(1,1)\}$ has cardinality $\overline{S} - e$. Hence,

$$\beta(1) = -\overline{S} + e - 2. \tag{22}$$

Therefore for all $r_b \in \Delta_\alpha(1,2)$, we have $\beta(1+j_b) = b+e-\overline{S}-1$. □

Lemma 88. *For $0 \le j < \overline{S}$, let M_j be as in Lemma 82. For each b we have*

$$\mathcal{T}_{M_{j_b}}(r_{j_b}, y_b) = (r_{1+j_b}, y_{\beta(1+j_b)}).$$

Proof. We treat the setting where $n > 3$ and hence $(1,1)$ appears only as an isolated letter in $\overline{b}(-k,v)\,)$. (When $n = 3$, it is $(1,2)$ that appears as an isolated letter, allowing analogous arguments.)

Case 1. $-1 \ge b \ge -e$**:** From Corollary 87, we have $\beta(1+j_b) = b+e-\overline{S}-1$. Thus, Definition 81 gives

$$y_{\beta(1+j_b)} = \begin{cases} -\hat{r}_{j_{-(b+2e+1)}} & \text{if} \quad -e > \beta(1+j_b) \\ \\ -\hat{r}_{j_{-(b+2e-\overline{S})}} & \text{otherwise.} \end{cases}$$

Definition 81 also gives $y_b = -\hat{r}_{j_{-(b+e+1)}}$. Note that $M_{j_b} = M_{j_{-(b+e+1)}} = AC^2$ here. If $b = -e$, then $y_b = -\hat{r}_0$ and Lemma 82 gives in this case $RM_{j_b}R^{-1} \cdot y_b = -\hat{r}_{\overline{S}}$. Here $y_{\beta(1+j_b)} = -\hat{r}_{j_{-e-1}} = -\hat{r}_{\overline{S}}$; the result holds for $b = -e$.

For $-1 \ge b > -e$, Lemma 82 combines with Corollary 87 to give

$$RM_{j_b}R^{-1} \cdot y_b = \begin{cases} -\hat{r}_{j_{-(b+2e+1)}} & \text{if} \quad \beta(-1+j_{-(b+e+1)}) \le -e-2, \\ \\ -\hat{r}_{j_{-(b+2e-\overline{S})}} & \text{if} \quad -1 \ge \beta(-1+j_{-(b+e+1)}) \ge -e. \end{cases}$$

We see that the result holds whenever $\beta(-1+j_{-(b+e+1)}), \beta(1+j_b)$ are both at least, or are both less than, $-e$.

In this first case, it remains to show that the two remaining possible combinations of inequalities on the values of $\beta(-1+j_{-(b+e+1)})$ and $\beta(1+j_b)$ can never be fulfilled. For this, note that the cardinality of the set $\{r_j \mid r_j \in \Delta_\alpha(\,(1,2)(1,1)\,), j \leq \overline{S}\}$ equals $\overline{S}-e$, as the set has image consisting of all of the r_j in $\Delta_\alpha(1,1)$. Therefore, the corresponding indices j are such that the real ordering gives $-e \leq \beta(j) \leq \overline{S}-2e-1$.

Thus, if $\beta(1+j_b) < -e-1$, then $-1 \geq -(b+e+1) \geq -\overline{S}+e$. From (22), $-(b+e+1) > \beta(1)$. Therefore, $\beta(-1+j_{-(b+e+1)}) < -e-1$. Now, if $-1 \geq \beta(-1+j_{-(b+e+1)}) \geq -e$, then (as also $-(b+e+1) \geq -e$), we have $-(b+e+1) \leq \beta(1)$; but then $\beta(1+j_b) \geq -e-1$. The result thus holds in this first case.

Case 2. $-e > b$**:** In this case, Definition 81 gives $y_b = -\hat{r}_{j_{-(b+e+\overline{S}+2)}}$. Of course $-\overline{S}-1 \leq b$ and thus we find $-(b+e+\overline{S}+2) < -e$. Therefore, $M_{j_b} = M_{j_{-(b+e+\overline{S}+2)}} = AC$ here; as well $-1 > \beta(-1+j_{-(b+e+\overline{S}+2)}) \geq -e$.

Hence, Lemma 82 combines with Corollary 87 to give

$$RM_{j_b}R^{-1} \cdot y_b = -\hat{r}_{j_{-(b+2e+1)}}.$$

From Corollary 87, we have $\beta(1+j_b) = b+e$, which must be at least $-e$ (assuming $n > 3$) Thus,

$$y_{\beta(1+j_b)} = -\hat{r}_{j_{-(b+2e+1)}}.$$

□

Corollary 89. *We have* $y_{\beta(\overline{S})} = RR_{-k,v}R^{-1} \cdot y_{\beta(0)}$.

Proof. Since $R_{-k,v} = M_{\overline{S}} \cdots M_0$, while for each $j \leq \overline{S}$ we have $M_j \cdots M_0 \cdot r_0 = r_j$, this does follow from the lemma. □

Lemma 90. *The values of the* $y_b, b < 0$ *increase with* b.

Proof. By Definition 81, $y_b = -C \cdot r_{j_{-e-1-b}}(\eta_{-k,v})$ when $b \in \{-e, \cdots, -1\}$. By definition, $r_{j_{-e}}(\eta_{-k,v}) \leq \cdots \leq r_{j_{-1}}(\eta_{-k,v})$. The map $b \mapsto -e-1-b$ is order reversing, the action of C is order preserving, and finally multiplication by -1 is order reversing. Thus, $b \mapsto y_b$ increases with b for these values of b.

Since $r_{1+j_{-\overline{S}-1}} > r_1$, we deduce that the e number of values $C \cdot r_{j_{-e-1-b}}(\eta_{-k,v})$ all lie to the left of the values of any of the $r_j(\eta_{-k,v}), 0 \leq j \leq \overline{S}$. In particular, those y_b with $-1 \geq b \geq -e$ are larger than the remaining values.

For these remaining values, $y_b = -\hat{r}_{j_{-(b+e+\overline{S}+2)}} = -r_{j_{-(e+\overline{S}+2)-b}}(\eta_{-k,v})$ increases with b because the multiplication by -1 reverses the decreasing nature of the $r_{j_{-(e+\overline{S}+2)-b}}(\eta_{-k,v})$. The result thus holds, we have: $y_{-\overline{S}-1} \leq \cdots \leq y_{-1}$. □

We give a result that could have been stated directly after the definition of the set of the various $y_{\tau(i)}$.

Lemma 91. *We have*

$$RA^{-k-1}CR^{-1} \cdot y_{\tau(\underline{S}-1)} = y_1.$$

Proof. By definition, $y_{\tau(\underline{S}-1)} = -\ell_1(\eta_{-k,v})$. By Lemma 14, $RA^{-k-1}CR^{-1}$ sends this to $-x$ where x satisfies $\ell_1 = A^{-k-1}C \cdot x$. Now, by ([8], Lemma 6.3), $x = \ell_{\underline{S}}(\eta_{-k,v})$. Since $y_1 = y_{\tau(0)}$, by definition this has the value $-\ell_{\underline{S}}(\eta_{-k,v})$, and the result holds. □

Since $\underline{d}(-k,v)$ differs only slightly from $\overline{d}(k,v)$, we use notation as in Section 4. We also revert to using u to denote a word in $\mathcal{V}$. (We trust that the reader can easily distinguish this from our other standard usage of u.)

Definition 92. Just as for $\alpha < \gamma_n$, we let

$$\iota = \begin{cases} 0 & \text{if } v = 1, \\ \underline{S}(-k,u) & \text{if } v = \Theta_q(u). \end{cases}$$

Recall from ([8], Lemma 6.7) that $v \prec w$ if and only if $\underline{d}(-k,v) \succ \underline{d}(-k,w)$. Lemma 38 thus implies the following.

Lemma 93. *We have*

$$\ell_\iota = \max_{0 \le i < \underline{S}} \ell_i \,.$$

Equivalently, $\tau(\iota) = \underline{S}$.

Compare the following with Figure 13.

Proposition 94. *We have*

$$y_{\underline{S}} = RC^2AC^2R^{-1} \cdot y_{\beta(\overline{S})}.$$

Sketch. We treat the main case of $v = \Theta_q(u)$, $q \ge 1$. From Corollary 84, combining Lemma 88 with Lemma 85, and Corollary 89,

$$y_{\beta(\overline{S})} = R\, R_{-k,v}\, [AC^2 R_{-k,\Theta_{q-1}(u)}]^{-1} C R^{-1} \cdot y_{\underline{S}+1}.$$

Using $R'_{-k,w}$ to denote $R_{-k,w}[(AC^2)^{n-2}]^{-1}$ (for general words w) as well as the notation of the proof of ([8], Proposition 4.13), we have

$$\begin{aligned} R_{-k,v}\, [AC^2 R_{-k,\Theta_{q-1}(u)}]^{-1} C &= R_{-k,u'u''} R_{u''}^{-1} (AC^2)^{-1} C \\ &= R'_{-k,\mathfrak{z}'\mathfrak{a}\mathfrak{z}\mathfrak{a}\mathfrak{z}} R'^{-1}_{\mathfrak{a}\mathfrak{z}} (AC^2)^{-1} C \\ &= R'_{-k,\mathfrak{a}\underset{\mathfrak{z}'}{\leftarrow}\mathfrak{z}\mathfrak{a}\mathfrak{z}} R'^{-1}_{\mathfrak{a}\mathfrak{z}} (AC^2)^{-1} C \\ & R'_{-k,\underset{\mathfrak{z}'}{\leftarrow} u} R'^{-1}_{\mathfrak{z}} (AC^2)^{-1} C \\ &= R'_{-k,u} \tilde{\mathcal{F}} \tilde{\mathcal{E}}^{-1} (AC^2)^{-1} C \\ &= R'_{-k,u} \tilde{\mathcal{F}} \tilde{\mathcal{E}}^{-1} (AC^2)^{-1} C \\ &= R'_{-k,u} (AC^2)^{n-3} U [(AC^2)^{n-3}]^{-1} (AC^2)^{-1} C \\ &= R'_{-k,u} (AC^2)^{n-2} \\ &= R_{-k,u} \,. \end{aligned}$$

Therefore, $y_{\beta(\overline{S})} = R\, R_{-k,u} R^{-1} \cdot y_{\underline{S}+1}$. Hence, $RC^2AC^2R^{-1} \cdot y_{\beta(\overline{S})} = RL_{k,u}R^{-1} \cdot y_{\underline{S}+1}$.

On the other hand, by Lemma 93, $y_{\underline{S}} = y_{\tau(\iota)}$. Therefore, from the definition of the y_a, we have that $y_{\underline{S}} = RL_{k,u}AR^{-1} \cdot y_{\tau(0)}$. Arguing as for Lemma 25, one has $RA^{-k-1}CR^{-1} \cdot y_{\tau(\underline{S}-1)} = y_{\tau(0)}$ and $RA^{-k}CR^{-1} \cdot y_{\tau(\underline{S}-1)} = y_{\tau(\underline{S})}$. Since $y_{\tau(\underline{S})} = y_{\underline{S}}$, one finds that $y_{\underline{S}}$ also is equal to $RL_{k,u}R^{-1} \cdot y_{\underline{S}+1}$. □

9. Bijectivity of $\mathcal{T}_\alpha$ on Ω_α for large α in left portion of synchronization interval

9.1. **Partitioning Ω by blocks $\mathcal{B}_i$.** For each $i \in \{-k, -k+1, \dots\} \cup \mathbb{N}$ and $j \in \{1, 2\}$, let the *block* $\mathcal{B}_{i,j}$ be the closure of the set $\{(x, y) \in \Omega \,|\, x \in \Delta_\alpha(i, j)\}$. Thus the blocks partition Ω up to μ-measure zero.

Since $\mathcal{T}$ is invertible, it is clear that Theorem 10 when $\alpha \in (\eta_{-k,v}, \delta_{-k,v})$ follows from the following two results, the proofs of which are given by combining the results of the ensuing three subsections.

Proposition 95. *The union of $\mathcal{T}(\mathcal{B}_{i,j})$ taken over all $i \in \{-k, -k+1, \dots\}$ and $j \in \{1, 2\}$ equals Ω^+ up to μ-measure zero.*

Proposition 96. *The union of $\mathcal{T}(\mathcal{B}_{i,j})$ taken over all $i \in \mathbb{N}$ and $j \in \{1, 2\}$ equals Ω^- up to μ-measure zero.*

9.2. **Blocks laminate one above the other.** The arguments for the case of $\alpha < \gamma_n$ give also the following.

Lemma 97. *Let (c, d) denote the first digit of $\ell_{\underline{S}}$. Then for all $(i, j) \notin \{(-k, 1), (-k-1, 1)\}$, the top boundary of the block $\mathcal{B}_{i,j}$ is given by*

$$\begin{cases} y = y_{\underline{S}+1} & \text{if } (c, d) \prec (i, j)\,; \\ (y = y_{\underline{S}+1}) \cup (y = y_{\underline{S}}) & \text{if } (c, d) = (i, j)\,; \\ y = y_{\underline{S}} & \text{if } (c, d) \succ (i, j)\,. \end{cases}$$

Lemma 98. *Let $(a, 2)$ denote the first digit of $r_{\overline{S}}$. Then for all $(i, j) \notin \{(1, 1), (1, 2)\}$, the lower boundary of the block $\mathcal{B}_{i,j}$ is given by*

$$\begin{cases} y = y_{-\overline{S}-1} & \text{if } j = 1\,; \\ y = y_{\beta(\overline{S})} & \text{if } j = 2 \text{ and } (a, 2) \succ (i, j)\,; \\ (y = y_{\beta(\overline{S})}) \cup (y = y_{1+\beta(\overline{S})}) & \text{if } (a, 2) = (i, j)\,; \\ y = y_{1+\beta(\overline{S})} & \text{if } (a, 2) \prec (i, j)\,. \end{cases}$$

Proof. Since all of the r_j, with $j \le \overline{S}$, lie in or to the right of $\Delta_\alpha(1, 1)$, the result for $j = 1$ holds. All of the r_j with $j < \overline{S}$ lie in $\Delta_\alpha(1, 1) \cup \Delta_\alpha(1, 2)$, and thus the bottom heights of $\mathcal{B}_{i,2}$ for $i \neq 1$ are one or both $y_{\beta(\overline{S})}, y_{1+\beta(\overline{S})}$ as determined by the location of $r_{\overline{S}}$ as per the remaining statements. □

Lemma 99. *Suppose that $(i, j) \notin \{(1, 1), (1, 2), (-k, 2)\}$. Then $\mathcal{T}_\alpha(\mathcal{B}_{i,j})$ laminates above $\mathcal{T}_\alpha(\mathcal{B}_{i',j'})$, where*

$$(i', j') = \begin{cases} (i, 2) & \text{if } j = 1, \\ \\ (i-1, 1) & \text{otherwise.} \end{cases}$$

Proof. We first prove the result in the case of $j = 1$. By Lemma 98, $\mathcal{B}_{i,1}$ has lower boundary height $y_{-\overline{S}-1}$. Similarly, since $\ell_{\underline{S}}(\alpha) < \mathfrak{b}_\alpha$, Lemma 97 gives that for any i, $\mathcal{B}_{i,2}$ has upper boundary height $y_{\underline{S}+1}$. From Lemma 84, $y_{-\overline{S}-1} = RCR^{-1} \cdot y_{\underline{S}+1}$. All cylinders are full with the possible exception of those of index $(-k, 1), (-k, 2), (1, 2)$, and the T_α-images of the cylinders of index $(-k, 1), (-k, 2)$ agree. Therefore, the T_α-images of the respective $\Delta_\alpha(i, j)$ and $\Delta_\alpha(i', j')$ agree. Lemma 17 thus gives the result in this case.

We now treat the case of $j = 2$. The proof of ([8], Lemma 6.2) shows that $r_{\overline{S}} \in \Delta_\alpha(a, 2)$ (with $a \geq 2$ because $\alpha \in (\eta_{-k,v}, \delta_{-k,v})$) if and only if $\ell_{\underline{S}} \in \Delta_\alpha(a-1, 1)$. Thus, if $j = 2$ and $(a, 2) \succ (i, j)$ then also $(c, d) \succ (i-1, 1)$ (with the notation of Lemma 97); in this case, the top of $\mathcal{B}_{i-1,1}$ is $y_{\underline{S}}$ and the bottom of $\mathcal{B}_{i,2}$ is $y_{\beta(\overline{S})}$. Both $\Delta_\alpha(i-1, 1)$ and $\Delta_\alpha(i, 2)$ are full cylinders, thus as Proposition 94 shows that the hypothesis for Lemma 18 is fulfilled, the result holds in this subcase.

When $j = 2$ and $(a, 2) \prec (i, j)$ we have also $(c, d) \prec (i-1, 1)$; Lemmas 97, 98 and Corollary 84 show that the hypothesis for Lemma 18 is fulfilled, and the result also holds in this subcase.

Finally, in the case where $\mathcal{B}_{i,2}$ has two heights at its bottom, we find that $\mathcal{B}_{i-1,1}$ has two heights on its top, with the pairs of heights fulfilling the hypothesis of Lemma 18. Because of synchronization, the images of the pairs of boundary pieces indeed match perfectly. □

Lemma 100. *The $\mathcal{T}_\alpha$-image of the block $\mathcal{B}_{-k,2}$ laminates above a portion of $\mathcal{T}_\alpha(\mathcal{B}_{-k-1,1})$.*

Proof. The cylinder $\Delta_\alpha(-k, 2)$ is right full, with image having left endpoint $\ell_1(\alpha)$. The cylinder $\Delta_\alpha(-k-1, 1)$ is full. Otherwise, the lamination is as above. □

Lemma 101. *The $\mathcal{T}_\alpha$-image of the block $\mathcal{B}_{1,1}$ laminates above a portion of $\mathcal{T}_\alpha(\mathcal{B}_{1,2})$.*

Proof. The cylinder $\Delta_\alpha(1, 2)$ is left full, with image having right endpoint $r_1(\alpha)$. The cylinder $\Delta_\alpha(1, 1)$ is full. Otherwise, the lamination is as above. □

9.3. **Upper boundary is in image.** The statements and proofs in the case of $\alpha < \gamma_n$, see Subsection 5.3, go through here with only minor adjustments.

9.4. **Lower boundary is in image.** Using the definition of the y_b and Corollary 84, the arguments for the case of $\alpha < \gamma_n$, see Subsection 5.4, succeed here.

9.5. **Bijectivity and ergodicity at left endpoints.** Whereas the left endpoint of a synchronization interval for small α is determined by the orbit of $r_{\overline{S}}$ being at the left end of $\mathbb{I}_\alpha$ (and thus there is one less rectangle in Ω^- than for the values in the interior of its synchronization interval), that of $J_{-k,v}$ is announced by $r_{\overline{S}} = \mathfrak{b}$. Thus, see (15), we have $C \cdot r_{\overline{S}}(\eta_{-k,v}) = \ell_0(\eta_{-k,v})$. Hence, $r_{\overline{S}+1}(\eta_{-k,v}) = A^{-k}C^2 \cdot r_{\overline{S}}(\eta_{-k,v}) = \ell_1(\eta_{-k,v})$. That is, here the synchronization is of the same form as for the rest of $(\eta_{-k,v}, \delta_{k,v})$. Thus, there is no change necessary to the definitions of $\Omega^\pm$.

Proposition 102. *Fix $n \geq 3, k \in \mathbb{N}, v \in \mathcal{V}$ (with $v \in \check{\mathcal{V}}$ if $k = 1$) and $\alpha = \eta_{-k,v}$. Let Ω^+ be as in Definition 80 and Ω^- be as in Definition 81.*

Then $\mathcal{T}_{n,\alpha}$ is bijective on $\Omega_{n,\alpha} := \Omega^+ \cup \Omega^-$, up to μ-measure zero.

Example 103. Recall the setting of Subsection 8.2: $n = 3$, $k = 1$, $v = 1$. Since $g^2/2 = \gamma_3 = \eta_{-1,1}$, we find

$$\Omega_{3,g^2/2} = ([-G, -g^2] \times [-g^2, g^2]) \cup ([-g^2, g^2] \times [-g^2, G]).$$

Proposition 104. *With $\alpha = \eta_{-k,v}$ as in the previous proposition, The system $(\mathcal{T}_\alpha, \Omega_\alpha, \mathscr{B}'_\alpha, \mu_\alpha)$ is ergodic.*

Furthermore, this two dimensional system is the natural extension of $(T_\alpha, \mathbb{I}_\alpha, \mathscr{B}_\alpha, \nu_\alpha)$, where ν_α is the marginal measure of μ_α and $\mathscr{B}_\alpha$ the Borel sigma algebra on $\mathbb{I}_\alpha$. In particular, the one dimensional system is ergodic.

Sketch. From ([8], Lemma 6.3), $\ell_0(\alpha)$ has a periodic α-expansion. Thus, $r_0(\alpha)$ and $\mathfrak{b}_\alpha$ are also all of finitely many α-digits. In view of Remark 62, it follows that the non-full cylinders of T_α are bounded in range. Due to this and the finiteness of $\mu(\Omega_\alpha)$ and of its vertical fibers, the hypotheses of ([9], Main Theorem 1) are met, thus giving the result here. □

10. Bijectivity for large α in right portion of synchronization interval

This section treats the right portion of synchronization intervals for large α in a completely analogous manner to that of § 8 and § 9 for the left portions.

10.1. **Bijectivity on the interior.** Suppose now that $\alpha \in (\delta_{-k,v}, \zeta_{-k,v})$. We define Ω_α^- with "one more" rectangle than for the left portion, and sketch the proof that bijectivity of $\mathcal{T}_\alpha$ on Ω_α then holds here as well.

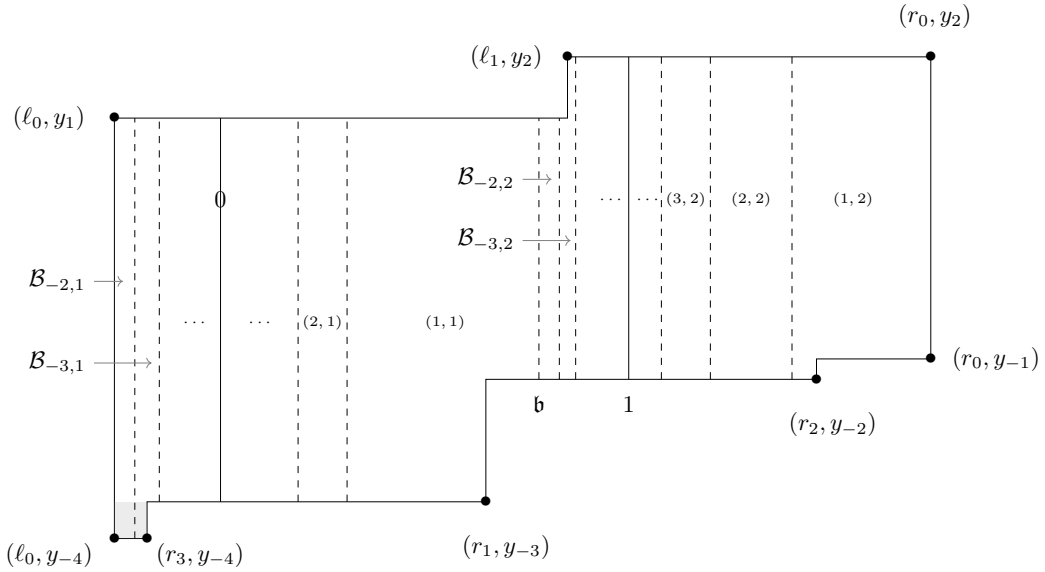

FIGURE 15. The domain $\Omega_{3,0.87}$. Blocks $\mathcal{B}_{i,j}$ also denoted by (i,j). Here $L_{-k,v} = A^{-2}CA^{-1}$ and $R_{-k,v} = AC\,AC^2$, and α is an interior point of $J_{-2,1}$ lying to the right of $\delta_{-2,1}$. Compare with $\Omega_{3,0.86}$, of Figure 10. Since both $0.86, 0.87$ give α values in $J_{-2,1}$, the heights $y_1, y_2, y_{-1}, y_{-2}, y_{-3}$ are the same. Highlighted in gray: rectangle of lower height not seen on left portion of $J_{-k,v}$; here, $y_{-4} = y_{-\overline{S}-2} = RAC^2R^{-1} \cdot y_{\beta(\overline{S})}$, as per Definition 106.

As a first step, we show that $r_{1+\overline{S}}$ lies to the left of the initial portion of the orbit of r_0.

Lemma 105. *We have* $r_{1+\overline{S}}(\alpha) < r_{j_{-\overline{S}-1}}(\alpha)$.

Proof. Here, $\ell_{\underline{S}}(\alpha) \geq \mathfrak{b}_\alpha$. The proof of Lemma 83 shows that here also $r_{\overline{S}}(\alpha) \in \Delta_\alpha(1,2)$, and that now synchronization occurs with $\ell_{1+\underline{S}}(\alpha) = r_{2+\overline{S}}(\alpha)$. Furthermore, ([8], Lemma 6.2) shows that in this setting we have $\ell_{\underline{S}}(\alpha) \in \Delta_\alpha(s,2)$ if and only if $r_{2+\overline{S}}(\alpha) = A^sCAC^2 \cdot r_{\overline{S}}(\alpha)$. In particular, $r_{\overline{S}}(\alpha)$ lies to the right of $\Delta_\alpha(1,1)$ and to the left of all values in $\Delta_\alpha(1,2)$ whose images are also in $\Delta_\alpha(1,2)$; thus, the proof of Lemma 85 succeeds also in this setting. When $r_{2+\overline{S}}(\alpha) = A^sCAC^2 \cdot r_{\overline{S}}(\alpha)$ with $(s,1) \preceq (2,1)$, we certainly have that $r_{1+\overline{S}}(\alpha) < r_{j_{-\overline{S}-1}}(\alpha)$. We must consider the extreme case when $s = 1$. Since $r_{\overline{S}}(\alpha) = CA^{-1}C \cdot \ell_{\underline{S}}(\alpha)$ and $\ell_{\underline{S}}(\alpha) < r_0(\alpha)$, we have that $r_{1+\overline{S}}(\alpha) < AC^2 \cdot (CA^{-1}C \cdot r_0(\alpha))$. That is, $r_{1+\overline{S}}(\alpha) < C \cdot r_0(\alpha)$ and therefore, $AC \cdot r_{1+\overline{S}}(\alpha) < r_1(\alpha)$. In terms of expansions, using the notation of the proof of Lemma 85, $r_{1+\overline{S}}(\alpha) < (1,1),(1,2)^{n-3}, \sigma^{n-2}(\,\overline{b}(-k,v)\,), \star$ which one easily finds is less than $(1,1),(1,2)^{n-3}, \sigma^{n-2}(\,\overline{b}(-k,u)\,), \star$. That is, $r_{1+\overline{S}}(\alpha) < r_{j_{-\overline{S}-1}}(\alpha)$. □

We must "update" Definition 81, confer Figure 15.

Definition 106. Define $L_{-\overline{S}-1} = [r_{1+\overline{S}}, r_{\overline{S}}]$, and let $L_{-\overline{S}-2} = [\ell_0, r_{1+\overline{S}})$. Set

$$y_{-\overline{S}-2} = RAC^2R^{-1} \cdot y_{\beta(\overline{S})}.$$

Let

$$\Omega^- = \bigcup_{b=-1}^{-\overline{S}-2} L_b \times [y_b, 0].$$

The following is necessary to show that the leftmost blocks $\mathcal{B}_{i,1}$ continue to laminate above $\mathcal{B}_{i,2}$.

Lemma 107. *We have $y_{-\overline{S}-2} = RCR^{-1} \cdot y_{\underline{S}}$.*

Proof. This follows directly from the definition of $y_{-\overline{S}-2}$ and Proposition 94. □

We must be sure that lamination, thus Lemma 99, still holds. The only change is that now there are blocks $\mathcal{B}_{i,1}$ whose bottom height is $y_{-\overline{S}-2}$. For each such $\mathcal{B}_{i,1}$, due to the relation between $\ell_{\underline{S}}$ and $r_{1+\overline{S}}$, the block $\mathcal{B}_{i,2}$ has top height $y_{\underline{S}}$ (or this and $y_{\underline{S}+1}$). Therefore, Lemma 107 shows that Lemma 17 continues to apply (in the two bottom heights case, synchronization yields that the x-coordinates are correct for lamination). The remainder of the proof goes through directly.

Example 108. As in Subsection 8.2 and Example 103, let $n = 3$, $k = 1$, $v = 1$; thus, $y_1 = g^2, y_2 = G$, $y_{-1} = -g^2$. We now consider $\alpha \in (\delta_{-1,1}, \zeta_{-1,1}) = (\delta_{-1,1}, \epsilon_3)$. From its definition, we find $y_{-2} = RAC^2R^{-1} \cdot (-g^2) = -(5+\sqrt{5})/10$. From the proof of Lemma 107, and the fact that $\Delta_\alpha(s,2)$ lies to the right of $\Delta_\alpha(s,1)$, we certainly have that $r_{\overline{S}+1} < \ell_{\underline{S}}$. Thus here we find that for all $\alpha \in (\delta_{-1,1}, \epsilon_3)$, one has

$$\Omega_{3,\alpha} = (\,[\ell_0, r_1] \times [-(5+\sqrt{5})/10, g^2]\,) \cup ([r_1, \ell_1] \times [-g^2, g^2]\,) \cup (\,[\ell_1, r_0] \times [-g^2, G]\,).$$

10.2. **Bijectivity and ergodicity at right endpoints.** Although for $k \in \mathbb{N}, v \in \mathcal{V}$, the $\zeta_{-k,v}$ are values of α for which the T_α-obits of $\ell_0(\alpha), r_0(\alpha)$ do not synchronize, we still can precisely describe domains on which their associated two-dimensional maps are bijective. Indeed, one has $\ell_{\underline{S}} = \ell_0$ and thus the various preceding sections now easily imply the following results.

Proposition 109. *Fix $n \geq 3, k \in \mathbb{N}, v \in \mathcal{V}$ (with $v \in \check{\mathcal{V}}$ if $k = 1$) and $\alpha = \zeta_{-k,v}$. Let Ω^- be as in Definition 106. Let Ω^+ be as in Definition 80 except that we redefine $K_{\underline{S}}$ to be $[\ell_{i_{\underline{S}}}, r_0]$, and $\Omega^+ = \bigcup_{a=1}^{\underline{S}} K_a \times [0, y_a]$.*

Then $\mathcal{T}_{n,\alpha}$ is bijective on $\Omega_{n,\alpha} := \Omega^+ \cup \Omega^-$, up to μ-measure zero.

Example 110. We continue the examples with $n = 3$, $k = 1$, $v = 1$; since $G/2 = \epsilon_3 = \zeta_{-1,1}$, we find

$$\Omega_{3,G/2} = (\,[-g^2, g^2] \times [-(5+\sqrt{5})/10, g^2]\,) \;\cup\; (\,[g^2, G] \times [-g^2, g^2]\,).$$

Proposition 111. *With $\alpha = \zeta_{-k,v}$ as in the previous proposition, the system $(\mathcal{T}_\alpha, \Omega_\alpha, \mathscr{B}'_\alpha, \mu_\alpha)$ is ergodic.*

Furthermore, this two dimensional system is the natural extension of $(T_\alpha, \mathbb{I}_\alpha, \mathscr{B}_\alpha, \nu_\alpha)$, where ν_α is the marginal measure of μ_α and $\mathscr{B}_\alpha$ the Borel sigma algebra on $\mathbb{I}_\alpha$. In particular, the one dimensional system is ergodic.

Proof. From ([8], Lemma 6.3), both $\ell_0(\alpha)$ and $r_0(\alpha)$ have periodic α-expansions. From the first of these, $\mathfrak{b}_\alpha$ also has of finitely many α-digits. In view of Remark 62, it follows that the non-full cylinders of T_α are bounded in range. Due to this and the finiteness of $\mu(\Omega_\alpha)$ and of its vertical fibers, the hypotheses of ([9], Main Theorem 1) are met, thus giving the result here. □

10.3. **Bijectivity and ergodicity at division point between portions: $\alpha = \delta_{-k,v}$.** At $\delta = \delta_{-k,v}$, we will find that the "extra" rectangle in Ω_α for $\alpha > \delta$ (as opposed for $\alpha < \delta$ in the same synchronization interval) has width zero and thus can be ignored. From the proof of Lemma 105, one has that for the value of s such that $\ell_1(\delta) = A^s C \cdot \ell_0(\delta)$ we also have $A^s C^2 \cdot \ell_{\overline{S}}(\delta) = \ell_1(\delta)$ and $A^s C \cdot r_{\overline{S}+1}(\delta) = \ell_1(\delta)$. (Of course, here $s = -k$.) That is, $r_{\overline{S}+1}(\delta) = \ell_0(\delta)$. Thus, Definition 106 reverts back to Definition 81 and the bijectivity of $\mathcal{T}_\delta$ on the resulting Ω_δ holds as above.

Example 112. We continue the examples with $n = 3$, $k = 1$, $v = 1$, as in Subsection 8.2 and Examples 103, 108. We have $\delta_{-1,1} = (5 - \sqrt{5})/4$ and thus $\ell_0(\delta_{-1,1}) = -g$. We find

$$\Omega_{3,\delta_{-1,1}} = ([-g, g] \times [-g^2, g^2]) \cup ([g, G] \times [-g^2, G]).$$

Proposition 113. *With $\alpha = \delta_{-k,v}$ as in the previous proposition, let $\mathscr{B}'_\alpha$ denote the Borel sigma algebra on Ω_α. The system $(\mathcal{T}_\alpha, \Omega_\alpha, \mathscr{B}'_\alpha, \mu_\alpha)$ is ergodic. Furthermore, this two dimensional system is the natural extension of $(T_\alpha, \mathbb{I}_\alpha, \mathscr{B}_\alpha, \nu_\alpha)$, where ν_α is the marginal measure of μ_α and $\mathscr{B}_\alpha$ the Borel sigma algebra on $\mathbb{I}_\alpha$. In particular, the one dimensional system is ergodic.*

Proof. Since $\ell_0(\alpha)$ has a periodic α-expansion and synchronization holds, in view of Remark 62, it follows that the non-full cylinders of T_α are bounded in range. Due to this and the finiteness of $\mu(\Omega_\alpha)$ and of its vertical fibers, the hypotheses of ([9], Main Theorem 1) are met, thus giving the result here. □

11. Limit values for larger α

The main result of this section are Propositions 116 and 132 giving a bijectivity domain for $\mathcal{T}_\alpha$ and then showing this to be an ergodic natural extension of T_α, for each large non-synchronizing α. As in § 6, due to the piecewise 'twisted' action on y-values, the proof of surjectivity requires that certain reversed words are admissible. Here also, we prove this admissibility by showing the admissibility of such words for the systems of endpoints of synchronization intervals and then arguing by taking limits.

Convention Just as in Subsection 6 where we used u as a general word in the setting of small α values, here we use w as a general word.

11.1. **Domain for non-synchronizing values.** We introduce some more notation and define basic sets for our construction. For legibility, we suppress various indications of dependence on α and the like, trusting that this causes no confusion for the reader.

Definition 114. Fix $n \geq 3, k \in \mathbb{N}$ and $\alpha > \gamma_n$. Let $\mathcal{L} = \mathcal{L}_\alpha$ denote the language of admissible digits. For finite length $u \in \mathcal{L}$, as usual let $\Delta_\alpha(u)$ be the corresponding T_α-cylinder. Let λ_u, ρ_u denote the left- and right endpoints, respectively of $\Delta_\alpha(u)$; note that $\rho_{1,1} = \lambda_{-k,2} = \mathfrak{b}$. For any $d \in \mathbb{N}$, let $\mu_{-d,1} = (A^{-d}C)^{-1} \cdot \mathfrak{b}$.

We then define vertical fibers, see Figure 16,

$$\Phi_1 = [-\mathfrak{b}, -\mu_{-k-1,1}],\ \Phi_2 = [-\mathfrak{b}, -\mu_{-k,1}],$$
$$\Phi_3 = [-\mathfrak{b}, -\ell_0],\ \Phi_4 = [-C \cdot r_0, -\ell_0],\ \text{and } \Phi_5 = [-\rho_{2,1}, -\ell_0]\,,$$

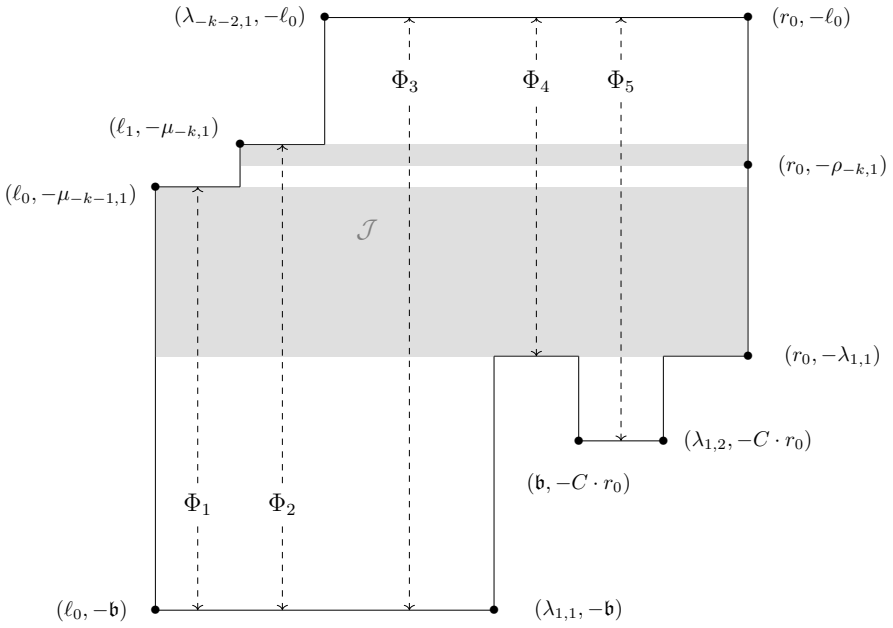

FIGURE 16. Schematic representation of $\mathcal{W}_\alpha$, see Definition 114. For large $\alpha = \eta_{-k,v}$ or for non-synchronizing α, Lemma 115 shows that the image of the regions in $\mathcal{W}_\alpha$ fibering over all cylinders other than those of the digits $(-k,1), (-k-1,1), (1,1), (1,2)$ gives the (gray) region $\mathcal{J} = \mathcal{J}_\alpha$ (of two connected components). Proposition 116 shows that the complementary pieces are sent by powers of $\mathcal{T}_\alpha$ to fill in a region that strictly includes the complement of $\mathcal{J}_\alpha$ in $\mathcal{W}_\alpha$.

and regions

$$\mathcal{J} = \mathbb{I}_\alpha \times [-\rho_{2,1}, -\mu_{-k-1,1}] \ \cup \ [\ell_1, r_0] \times [-\rho_{-k,1}, -\mu_{-k,1}]$$

and

$$\mathcal{W} = [\ell_0, \ell_1] \times \Phi_1 \ \cup \ [\ell_1, \lambda_{-k-2,1}) \times \Phi_2 \ \cup \ [\lambda_{-k-2,1}, \rho_{2,1}] \times \Phi_3 \\ \cup \ [\rho_{2,1}, \rho_{2,2}) \times \Phi_4 \ \cup \ [\rho_{2,2}, r_0) \times \Phi_5 \, .$$

Note that $\mathcal{W} \supset \mathcal{J}$.

Lemma 115. *Fix a non-synchronizing α with $\alpha > \gamma_n$. Then*

$$J_\alpha = \cup_{\substack{(i,1) \succ (-k-1,1) \\ i \neq 1}} \mathcal{T}_\alpha(\Delta_\alpha(i,1) \times \Phi_3) \bigcup \cup_{(i,2) \prec (1,2)} \mathcal{T}_\alpha(\Delta_\alpha(i,2) \times \Phi_5).$$

Proof. From Lemma 14, for each $i \in \mathbb{Z}$,

$$RA^iCR^{-1} \cdot -\ell_0 = -\lambda_{i,1}, \ RA^iCR^{-1} \cdot -\mathfrak{b} = -\mu_{i,1} = RA^iC^2R^{-1} \cdot -\ell_0, \ RA^iC^2R^{-1} \cdot (-C \cdot r_0) = \rho_{i,1}.$$

Since adjacent cylinders share a common endpoint, and $(-k,2)$ is the only non-full cylinders considered in the right hand side of the expression, the result follows. □

Proposition 116. *Fix a non-synchronizing α with $\alpha > \zeta_{-1,n-2}$. With notation as above, let $\mathcal{T} = \mathcal{T}_\alpha$ and define*

$$\Omega = \overline{\cup_{j=1}^\infty \mathcal{T}^j(\mathcal{W})} \, .$$

Then $\Omega \supset \mathcal{W}$ and $\mathcal{T}$ is bijective up to μ-measure zero on Ω. Furthermore, letting

$$\mathscr{A} = \{(-k,1),(-k-1,1)\}^* \cap \mathcal{L}_\alpha, \quad \mathscr{B} = \{(1,1),(1,2)\}^* \cap \mathcal{L}_\alpha$$

and

$$\Phi_2' = \Phi_2 \setminus (-\mu_{-k-1,1}, -\rho_{-k,1})$$

we have that Ω is the closure of

$$\mathcal{J} \bigsqcup \sqcup_{\substack{w\in\mathscr{A}\\ \rho_w<\ell_1}} \mathcal{T}^{|w|}(\,\Delta_\alpha(w)\times\Phi_1\,) \bigsqcup \sqcup_{\substack{w\in\mathscr{A}\\ \lambda_w>\ell_1}} \mathcal{T}^{|w|}(\,\Delta_\alpha(w)\times\Phi_2'\,)$$

$$\bigsqcup \sqcup_{s=2}^{\infty} \mathcal{T}^{|w|}(\,\Delta_\alpha(\underline{b}^\alpha_{[2,s]})\times\Phi_1\,) \sqcup [\ell_s, r_0)\times RM_{\underline{b}^\alpha_{[2,s]}}R^{-1}(\,\Phi_2'\setminus\Phi_1\,) \tag{23}$$

$$\bigsqcup \sqcup_{w\in\mathscr{B}\cap\mathcal{L}_\alpha} \mathcal{T}^{|w|}(\,\Delta_\alpha(w)\times\Phi_4\,).$$

Furthermore, the system $(\mathcal{T}_\alpha, \Omega_\alpha, \mathscr{B}'_\alpha, \mu_\alpha)$ is the natural extension of $(T_\alpha, \mathbb{I}_\alpha, \mathscr{B}_\alpha, \nu_\alpha)$, where ν_α is the marginal measure of μ_α and $\mathscr{B}_\alpha$ the Borel sigma algebra on $\mathbb{I}_\alpha$. Finally, both systems are ergodic.

The surjectivity of $\mathcal{T}$ on Ω follows from the definition of Ω. Key to proving the remaining statements of the proposition is to show that each $x \in \mathbb{I}_\alpha$ of first digit other than $\{(-k,1),(-k-1,1),(1,1),(1,2)\}$ has the same y-fiber in Ω as in $\mathcal{W}$.

11.2. **Agreement of fibers on lower portions of $\mathcal{W}, \Omega$.**

Lemma 117. *For*

$$w = (1,1)^{a_1}(1,2)^{b_1}\cdots(1,1)^{a_s}(1,2)^{b_s} \in \mathcal{L}_\alpha\,,$$

$\mathcal{T}^{|w|}(\,\Delta_\alpha(w)\times\Phi_4\,)$ *has y-fiber equal to*

$$\begin{cases} [-(M_{\overleftarrow{w}}AC)^{-1}\cdot\ell_0,\ -((AC)^{-1}M_{\overleftarrow{w}}AC)^{-1}\cdot\ell_0] & \text{if } a_1>0 \\ [-(M_{\overleftarrow{w}}AC)^{-1}\cdot\ell_0,\ -(AC(AC^2)^{-1}M_{\overleftarrow{w}}AC)^{-1}\cdot r_0] & \text{if } a_1=0 \end{cases}$$

Proof. We have $M_w = (AC^2)^{b_s}(AC)^{a_s}\cdots(AC^2)^{b_1}(AC)^{a_1}$, and for any real x Lemma 14 gives

$$RM_wR^{-1}\cdot -x = -(\,(AC)^{a_1}(CAC)^{b_1}\cdots(AC)^{a_s}(CAC)^{b_s}\,)^{-1}\cdot x.$$

Independent of the vanishing of a_1, one finds that $RM_wR^{-1}\cdot -x$ equals $-((AC)^{-1}M_{\overleftarrow{w}}AC)^{-1}\cdot x$. When $x = \lambda_{1,1} = (AC)^{-1}\cdot\ell_0$, this value equals $-(M_{\overleftarrow{w}}AC)^{-1}\cdot\ell_0$. When $a_1 = 0$, the expression $(AC)^{-1}M_{\overleftarrow{w}}$ factors as a C followed by a product of powers of AC^2 and AC; replacing ℓ_0 by $r_0 = A\cdot\ell_0$ results in an expression which is a word in our matrices, while giving the formula indicated for the largest value in the image of the fiber. □

The previous result shows that corresponding to w are fibers that, upon using a negative exponent to indicate cancelling a letter in a word, one might express as $[-\lambda_{(1,1)\overleftarrow{w}}, -\lambda_{(1,1)\overleftarrow{w}(1,1)^{-1}}]$ and $[-\lambda_{(1,1)\overleftarrow{w}}, -\rho_{(1,1)\overleftarrow{w}(1,2)^{-1}(1,1)}]$. For these expressions to be valid, we must then also prove that each of the subscripts is given by an admissible word. The next two lemmas give us this result.

Lemma 118. *Fix $n \geq 3$, and either a non-synchronizing $\alpha > \gamma_n$ or $\alpha = \eta_{-k,v}$ with $k \in \mathbb{N}$ and $v \in \mathcal{V}$. Then*

$$\Delta_\alpha(1,1) = \overline{\sqcup_{w\in(1,1)\{(1,1),(1,2)\}^*\cap\mathcal{L}_\alpha}\,[\lambda_w, \lambda_{w,(1,1)}) \cup [\rho_{w,(1,1)},\ \lambda_{w,(1,2)})}.$$

Proof. By definition in the union, we include $w = (1,1)$. Note that any $w \in \{(1,1),(1,2)\}^*$ is in $\mathcal{L}_\alpha$ if and only if $w \preceq \overline{d}^{\alpha}_{[1,\infty)}$; in particular, whenever $w \in \mathcal{L}_\alpha$ so is also $w,(1,1)$.

For any $w \in (1,1)\{(1,1),(1,2)\}^* \cap \mathcal{L}_\alpha$, we have

$$\Delta_\alpha(w) = [\lambda_w, \lambda_{w,(1,1)}) \cup \Delta_\alpha(w,(1,1)\,) \cup [\rho_{w,(1,1)},\, \lambda_{w,(1,2)}) \cup \Delta_\alpha(w,(1,2)\,),$$

where by definition $\Delta_\alpha(w,(1,2)\,) = \emptyset$ if $w,(1,2) \notin \mathcal{L}_\alpha$. From this, by beginning with $w = (1,1)$, it follows that $\Delta_\alpha(1,1)$ does equal the closure of the union over all $w \in (1,1)\{(1,1),(1,2)\}^* \cap \mathcal{L}_\alpha$ of $[\lambda_w, \lambda_{w,(1,1)}) \cup [\rho_{w,(1,1)},\, \lambda_{w,(1,2)})$.

We have tacitly used the fact that each admissible infinite word in $\{(1,1),(1,2)\}^*$ represents a unique point, this holds due to the dynamics of the $T_{n,1}$; see ([8], § 3) and ([9], § 3.6). □

Lemma 119. *Fix $n \geq 3$, and either a non-synchronizing $\alpha > \gamma_n$ or $\alpha = \eta_{-k,v}$ with $k \in \mathbb{N}$ and $v \in \mathcal{V}$. Let $w \in (1,1)\{(1,1),(1,2)\}^* \cap \mathcal{L}_\alpha$. Then $(1,1)\overleftarrow{w} \in \mathcal{L}_\alpha$ and so are $(1,1)\overleftarrow{w}(1,1)^{-1}$ if w has prefix $(1,1)$ and $(1,1)\overleftarrow{w}(1,2)^{-1}(1,1)$ otherwise.*

Proof. For w of digits in $\{(1,1),(1,2)\}$, we have w is admissible if and only if it and all its suffixes are smaller or equal to the expansion of $r_0(\alpha)$, thus than $\overline{b}^{\alpha}_{[1,\infty)}$. In particular, w is admissible if and only if $w,(1,1)$ is. Furthermore, $\overleftarrow{w} \in \mathcal{L}_\alpha$ if and only if the various $(1,1)\overleftarrow{w}, (1,1)\overleftarrow{w}(1,1)^{-1}, (1,1)\overleftarrow{w}(1,2)^{-1}(1,1)$ are (with cases depending again on whether w has prefix $(1,1)$ or not).

We thus aim to show that $w \in \mathcal{L}_\alpha$ implies $\overleftarrow{w} \in \mathcal{L}_\alpha$. For this, we argue by contradiction and can assume that w is of minimal length such that $\overleftarrow{w} \notin \mathcal{L}_\alpha$. We can now argue as in the proof of Lemma 72 and Corollary 73. □

Lemma 120. *With α as above, each $x \in \mathbb{I}_\alpha$ of first digit other than $\{(1,1),(1,2)\}$ has the same y-fiber in $\Omega \cap \{y \leq 0\}$ as in $\mathcal{W} \cap \{y \leq 0\}$.*

Proof. Lemmas 117, 118, and 119 combine to show that each x less than the infimum of the elements of the T_α-orbit of $r_0(\alpha)$ has y fiber in Ω_α meeting $\{y \leq 0\}$ in exactly $[\mathfrak{b}, -\lambda_{1,1}]$. Thus equality of fibers holds for all such x.

It remains to show the equality when $x \in [\mathfrak{b}, \lambda_{1,2})$. For this, it suffices to show that $[-\mathfrak{b}, -C\cdot r_0]$ equals the closure of the union of the images of Φ_4 corresponding to the cylinders $\Delta_\alpha(w)$ such that $T^{|w|}(\rho_w) = r_j$ for some j and such that $r_j \in \Delta_\alpha(1,1)$. Let us say that such a word w is *of $(1,1)$ right value* (with respect to α).

We first note that for any $\eta_{-k,v}$ this equality holds due to Proposition 102. Let α be a non-synchronizing value, then there there is a fixed k and a branch of $\mathcal{V}$ such that taking v along the branch gives values of the form $\eta_{-k,v}$ converging to α from the right, and such that the $\overline{b}^{\eta_{-k,v}}_{[1,\infty)}$ agree to ever greater length with $\overline{b}^{\alpha}_{[1,\infty)}$. Given $w \in (1,1)\{(1,1),(1,2)\}^* \cap \mathcal{L}_\alpha$, since in particular $w \preceq \overline{b}^{\alpha}_{[1,\infty)}$ the finiteness of w implies that for all v sufficiently far along the branch of $\mathcal{V}$ that also $w \in \mathcal{L}_{\eta_{-k,v}}$. Furthermore, if w is of $(1,1)$ right value with respect to α then it is also for all for all v sufficiently far along the branch.

The convergence of the various $\eta_{-k,v}$ to α gives that $RM_wR^{-1} \cdot -\ell_0(\eta_{-k,v}) \to RM_wR^{-1} \cdot -\ell_0(\alpha)$ and also $-C\cdot r_0(\eta_{-k,v}) \to -C \cdot r_0(\alpha)$. Since for each w of $(1,1)$ right value $RM_wR^{-1} \cdot -\ell_0(\eta_{-k,v})$ is below $-C\cdot r_0(\eta_{-k,v})$, we conclude that also $RM_wR^{-1} \cdot -\ell_0(\alpha) \leq -C \cdot r_0(\alpha)$. For any other $w \in (1,1)\{(1,1),(1,2)\}^* \cap \mathcal{L}_\alpha$, we argue similarly to find that $RM_wR^{-1} \cdot -\ell_0(\alpha) \geq -C \cdot r_0(\alpha)$. We conclude that $[-\mathfrak{b}, -C\cdot r_0]$ equals the closure of the union of the images of Φ_4 corresponding to the cylinders indexed by words of $(1,1)$ right value. The result thus holds. □

11.3. **Agreement of fibers on upper portions of $\mathcal{W}, \Omega$.**

11.3.1. *Statement and beginning results.* We announce the main result of this subsection.

Proposition 121. *With α as above, each $x \in \mathbb{I}_\alpha$ of first digit other than $\{(-k,1),(-k-1,1)\}$ has the same y-fiber in $\Omega \cap \{y \geq 0\}$ as in $\mathcal{W} \cap \{y \geq 0\}$.*

Convention. The arguments in this subsection mainly involve the digits $(-k,1)$ and $(-k-1,1)$, thus we will use simplified digits. (In particular, we consider $\underline{b}^\alpha_{[1,\infty)}$ to be equal to $\underline{d}^\alpha_{[1,\infty)}$.)

For further typographic ease, for $w \in \mathcal{L}_\alpha$, let ν_w be the point of simplified digit expansion $w,(-k-1)^\infty$.

The following is proved just as in the small α setting, see Lemma 72 and its corollary.

Lemma 122. *With α as above, if $w \in \{-k,-k-1\}^* \cap \mathcal{L}_\alpha$ then also $\overleftarrow{w} \in \mathcal{L}_\alpha$.*

11.3.2. *Images of fibers.* We begin by collecting images of Φ_1.

Lemma 123. *With α as above, if $w \in \{-k,-k-1\}^* \cap \mathcal{L}_\alpha$ then*

$$\cup_{j=0}^\infty RM_{w,(-k-1)^j} R^{-1} \cdot \Phi_1 = -(\nu_{\overleftarrow{w}}, \mu_{\overleftarrow{w}}].$$

Proof. Given $w \in \{-k,-k-1\}^* \cap \mathcal{L}_\alpha$ from the previous result we have that $\overleftarrow{w} \in \mathcal{L}_\alpha$. Furthermore, for words of digits $-k,-k-1$ admissibility is determined by being not less than $\underline{b}^\alpha_{[1,\infty)}$ and thus (since $-k \prec -k-1$) appending (or prepending) any power of $-k-1$ to such an admissible word results in an admissible word. Thus, the right hand side of the displayed equation is sensical.

When w is a word in simplified digits, Lemma 14 gives that $RM_w R^{-1} \cdot -x = (M_{\overleftarrow{w}})^{-1} \cdot x$. Thus, for $w \in \{-k,-k-1\}^* \cap \mathcal{L}_\alpha$ we find that $RM_w R^{-1} \cdot \Phi_1 = -[\mu_{\overleftarrow{w},-k-1}, \mu_{\overleftarrow{w}}]$. The result easily follows. □

Definition 124. For $w \in \{-k,-k-1\}^* \cap \mathcal{L}_\alpha$ of length at least two and with suffix $-k$, let

$$\widetilde{w} = \overleftarrow{w_{[1,|w|-1]}}.$$

Lemma 125. *With α as above, the function $w \mapsto \widetilde{w}$ gives a one-to-one correspondence between the set of $w \in \{-k,-k-1\}^* \cap \mathcal{L}_\alpha$ of length at least two and with suffix $-k$ and the set of $w \in \{-k,-k-1\}^* \cap \mathcal{L}_\alpha$ with $\widetilde{w} \succ \underline{b}^\alpha_{[2,\infty)}$.*

Proof. We first note that the function $w \mapsto \widetilde{w}$ is clearly injective into the set $\{-k,-k-1\}^*$.

If $w \in \{-k,-k-1\}^* \cap \mathcal{L}_\alpha$ of length at least two and with suffix $-k$, then we know that also $\overleftarrow{w} \in \mathcal{L}_\alpha$ and in particular $\overleftarrow{w} \succeq \underline{b}^\alpha_{[1,\infty)}$. Hence, $\widetilde{w} \succeq \underline{b}^\alpha_{[2,\infty)}$, since w is of finite length we have that $\widetilde{w} \succ \underline{b}^\alpha_{[2,\infty)}$.

On the other hand, if $w \in \{-k,-k-1\}^* \cap \mathcal{L}_\alpha$ is such that $w \succ \underline{b}^\alpha_{[2,\infty)}$, then $-k, w \succ \underline{b}^\alpha_{[1,\infty)}$ and thus $-k, w \in \mathcal{L}_\alpha$. From this, $\overleftarrow{-k, w} \in \mathcal{L}_\alpha$ is of length at least two and has $-k$ as a suffix. □

Lemma 126. *With α as above,*

$$-[\mu_w, \rho_w) = RM_{\widetilde{w}} R^{-1}(\,\Phi_2' \setminus \Phi_1\,).$$

Proof. Since $\Phi_2' \setminus \Phi_1 = -[\mu_{-k}, \rho_{-k})$, using Lemma 14 we find that $RM_{\widetilde{w}} R^{-1}(\,\Phi_2' \setminus \Phi_1\,) = -[\mu_w, \rho_w)$. □

11.3.3. *Partitioning an x-interval related to the upper portion of fibers Φ_3, Φ_4, Φ_5.*

Definition 127. Suppose that $w \in \{-k, -k-1\}^* \cap \mathcal{L}_\alpha$. Since every $w, (-k-1)^d \in \mathcal{L}_\alpha$, there is a minimal $d \geq 0$ such that $w, (-k-1)^d, -k$ is admissible. The *children* of w are the words $w, (-k-1)^i, -k$ with $i \geq d$.

Lemma 128. *With α as above and $w \in \{-k, -k-1\}^* \cap \mathcal{L}_\alpha$, all of the children of w are admissible. Furthermore, for each child w' of w, we have $\rho_{w'} \leq \nu_w$. Finally, if $x \in [\lambda_w, \nu_w)$ then there is a largest child w' of w such that $x \in \Delta_\alpha(w')$.*

Proof. Recall that for any $w \in \{-k, -k-1\}^* \cap \mathcal{L}_\alpha$ we have that every $w, (-k-1)^i$ with $i \in \mathbb{N}$ is admissible. Admissibility for words in $\{-k, -k-1\}^*$ is a matter of being at least as large as $\underline{b}^\alpha_{[1,\infty)}$, thus the requirement that j be at least as large as the minimal d with $w, (-k-1)^d, -k$ admissible does imply that all of the children are admissible.

Let w' be a child of w. For any $x \in \Delta_\alpha(w')$, there some i such that the simplified digits x has prefix w' since ν_w is the point of simplified digits $w, (-k-1)^\infty$ we find that $x < \nu_w$. By continuity we find that $\rho_{w'} \leq \nu_w$ and by induction on i one finds that strict inequality holds.

For any $j \geq 0$, we have $w, (-k-1)^j, -k \prec w, (-k-1)^{j+1}, -k$. Given $x \in [\lambda_w, \nu_w)$, there is a largest $i \geq 0$ such that the simplified digits of x begin $w, (-k-1)^{i+1}, -k$. Hence with $w' = w, (-k-1)^{i+1}, -k$ we have both that $x \in \Delta_\alpha(w')$ and that w' is the largest child of w with this property. □

Recall that the final letter of a word w is denoted $w_{[-1]}$.

Lemma 129. *With α as above,*

$$[\ell_0, \mu_{-k}] \cup [\lambda_{-k-1}, \mu_{-k-1}] = \overline{(\nu_{-k}, \mu_{-k}] \cup (\nu_{-k-1}, \mu_{-k-1}] \cup \bigcup_{\substack{w \in \{-k,-k-1\}^* \cap \mathcal{L}_\alpha \\ |w| \geq 2 \\ w_{[-1]} = -k}} (\nu_w, \rho_w)}\,.$$

Proof. Since $\ell_0 = \lambda_{-k}$, we have that each $x \in [\ell_0, \mu_{-k}] \cup [\lambda_{-k-1}, \mu_{-k-1}]$ is in $[\lambda_w, \nu_w) \cup [\nu_w, \mu_w]$ for one $w \in \{-k, -k-1\}$. If $x \notin [\nu_w, \mu_w]$ then there is a largest child w' such that $x \in [\lambda_{w'}, \nu_{w'}) \cup [\nu_{w'}, \mu_{w'}]$. Since every child w' is in the set $\{-k, -k-1\}^* \cap \mathcal{L}_\alpha$ with $|w| \geq 2$ and $w_{[-1]} = -k$, the result follows by induction. □

11.3.4. *Proof that fibers agree: reaching top.* Recycling notation, let $\mathscr{L} = \sup_j \{\ell_j\}$. Note that $\mathscr{L} < \lambda_{-k-2,1}$.

Lemma 130. *The intersection of $[\mathscr{L}, r_0) \times [-\mu_{-k-1}, \infty)$ with the closure of*

$$\Big\lfloor\Big\rfloor \sqcup_{\substack{w \in \mathscr{A} \\ \rho_w < \ell_1}} \mathcal{T}^{|w|}(\,\Delta_\alpha(w) \times \Phi_1\,) \Big\lfloor\Big\rfloor \sqcup_{\substack{w \in \mathscr{A} \\ \lambda_w > \ell_1}} \mathcal{T}^{|w|}(\,\Delta_\alpha(w) \times \Phi_2'\,)$$

$$\Big\lfloor\Big\rfloor \sqcup_{s=2}^\infty \mathcal{T}^{|w|}(\,\Delta_\alpha(\underline{b}^\alpha_{[2,s]}) \times \Phi_1\,) \sqcup [\ell_s, r_0) \times RM_{\underline{b}^\alpha_{[2,s]}} R^{-1}(\,\Phi_2' \setminus \Phi_1\,)$$

equals $[\mathscr{L}, r_0) \times -([\ell_0, \mu_{-k}] \cup [\lambda_{-k-1}, \mu_{-k-1}])$.

Proof. Since $\ell_1 \in [\lambda_w, \rho_w)$ exactly for w of the form $\underline{b}^\alpha_{[2,s]}$, and $\Phi_2' \supset \Phi_1$, we can rewrite the displayed quantity as

$$\Big\lfloor\Big\rfloor \sqcup_{w \in \mathscr{A}} \mathcal{T}^{|w|}(\,\Delta_\alpha(w) \times \Phi_1\,)$$

$$\Big\lfloor\Big\rfloor \sqcup_{\substack{w \in \mathscr{A} \\ \lambda_w > \ell_1}} \mathcal{T}^{|w|}(\,\Delta_\alpha(w) \times (\Phi_2' \setminus \Phi_1)\,) \Big\lfloor\Big\rfloor \sqcup_{s=2}^\infty [\ell_s, r_0) \times RM_{\underline{b}^\alpha_{[2,s]}} R^{-1}(\,\Phi_2' \setminus \Phi_1\,).$$

By Lemma 125, this equals

$$\bigsqcup \sqcup_{w\in\mathscr{A}} \mathcal{T}^{|w|}(\,\Delta_\alpha(w)\times\Phi_1\,)\bigsqcup \sqcup_{\substack{w\in\mathscr{A}\\ w_{[-1]}=-k\\ \text{or } w=-k-1}} \mathcal{T}^{|\widetilde{w}|}(\,\Delta_\alpha(\widetilde{w})\times(\Phi_2'\setminus\Phi_1)\,),$$

which we can of course rewrite as

$$\bigsqcup \sqcup_{w\in\mathscr{A}} T^{|w|}(\,\Delta_\alpha(w)\,)\times RM_wR^{-1}(\Phi_1)\,)\bigsqcup \sqcup_{\substack{w\in\mathscr{A}\\ w_{[-1]}=-k\\ \text{or } w=-k-1}} T^{|\widetilde{w}|}(\,\Delta_\alpha(\widetilde{w})\,)\times RM_{\widetilde{w}}R^{-1}(\,\Phi_2'\setminus\Phi_1\,),$$

Since the image of the left endpoint of every cylinder lies to the left of $\mathscr{L}$, we find that the y-fiber in Ω above any $x\in[\mathscr{L},r_0)$ includes

$$\mathcal{F}_x^+ = \sqcup_{\substack{w\in\mathscr{A}\\ w_{[-1]}=-k\\ \text{or } w=-k-1}} RM_{\widetilde{w}}R^{-1}(\,\Phi_2'\setminus\Phi_1\,),$$

For all w and all i, we have $\nu_{w,(-k-1)^i}=\nu_w$, and thus applying Lemmas 123 and 126 and then Lemma 129 shows that $\mathcal{F}_x^+ = -([\ell_0,\mu_{-k}]\cup[\lambda_{-k-1},\mu_{-k-1}])$. □

11.3.5. *Proof that fibers agree: filling in gap.* It remains to show that for $x\in[\ell_1,\mathscr{L})$ that $\Phi_2\setminus\Phi_2'$ is contained in the y-fiber of Ω over x. Since $RM_{-k-1}R^{-1}(\Phi_3)=[-\mu_{-k-1},-\lambda_{-k-1}]$ and of course $\lambda_{-k-1}=\rho_{-k}$ we must simply show that $(A^{-k-1}C)^{-1}\cdot\ell_1\geq\mathscr{L}$.

Just as Lemma 93 follows from Lemma 38, so also for any $\alpha=\eta_{-k,v}$ we have that $\ell_{\underline{S}}$ is maximal by arguing analogously to Lemma 37. For such α, we also have $A^{-k-1}C\cdot\ell_{\underline{S}}=\ell_1$, and thus our results holds for these values of α. By taking limits, we find that it holds for all of the values of α which we are considering. □

11.4. **Ergodic natural extension for non-synchronizing large α.**

Proposition 131. *Fix n and a non-synchronizing α with $\alpha>\gamma_n$. With Ω as in Proposition 116, $\mu(\Omega)<\infty$.*

Proof. As in the proof of Proposition 77, we can show the finiteness of the measure of the upper portion of Ω by comparing with $\Omega_{n,0}$. Similarly, here one can show the finiteness of the measure of the lower portion of Ω by comparing with $\Omega_{n,1}$, see ([9], Proposition 12). See also ([9], Figure 3). □

The following is proven as is Proposition 78, *mutatis mutandis*.

Proposition 132. *With α as in the notation of Proposition 116, the system $(\mathcal{T}_\alpha,\Omega_\alpha,\mathscr{B}'_\alpha,\mu_\alpha)$ is the natural extension of $(T_\alpha,\mathbb{I}_\alpha,\mathscr{B}_\alpha,\nu_\alpha)$, where ν_α is the marginal measure of μ_α and $\mathscr{B}_\alpha$ the Borel sigma algebra on $\mathbb{I}_\alpha$. Finally, both systems are ergodic.*

12. Continuity of mass for larger α; Completing proof of Theorem 3

In view of Theorem 79, the results of this section complete the proof of Theorem 3.

Theorem 133. *The function $\alpha\mapsto\mu(\Omega_\alpha)$ is continuous on $(\gamma_n,1)$.*

Proof. Since the both top and bottom heights are constant along any synchronization interval $J_{-k,v}$, with fibers constant between the various $\ell_i(\alpha),r_j(\alpha)$, continuity and indeed smoothness, of $\alpha\mapsto\mu(\Omega_\alpha)$ in each of $(\eta_{-k,v},\delta_{-k,v})$ and $(\delta_{-k,v},\eta_{-k,v})$ is clear. Propositions 102 and 109 along with Subsection 10.3 then imply that continuity holds on the closed interval.

Continuity throughout $I_{-k}:=\cup_{v\in\mathcal{V}}I_{k,v}$ for fixed k is shown as in the proof of Theorem 79, using that we can express every $\Omega_{\eta_{-k,v}}$ and $\Omega_{\zeta_{-k,v}}$ as the union of regions of the type of $\mathcal{W}$ similar to the result of Proposition 116.

Continuity at the boundaries of the various I_{-k} follows from arguing analogously to the proof of Theorem 79. Indeed, the right endpoint of I_{-k} is $\zeta_{-k,1}$, and one argues that $\Omega_{\zeta_{-k-1,h}} \to \Omega_{\zeta_{-k,1}}$ as $h \to \infty$. For this, ([8], Lemma 6.6) gives $\underline{d}_{[1,\infty)}^{\zeta_{-k,1}} = \overline{-k-1}$ while $\underline{d}_{[1,\infty)}^{\zeta_{-k-1,h}} = (-k)^{h-1}, \overline{-k-1}$ (in simplified digits) from which also the $\overline{d}_{[1,\infty)}^{\zeta_{-k,1}}$ and $\overline{d}_{[1,\infty)}^{\zeta_{-k-1,h}}$ sufficiently agree to deduce both the convergence of the $\Omega^+_{\zeta_{-k-1,h}}$ to $\Omega^+_{\zeta_{-k,1}}$ and similarly for the lower portions. □

There is only one remaining α value at which we must prove continuity.

Lemma 134. *Fix n. The function $\alpha \mapsto \mu(\,\Omega_\alpha\,)$ is continuous at $\alpha = \gamma_n$.*

Proof. Recall that $\gamma_n = \eta_{-1,n-2}$, thus we have already shown continuity from the right.

Since $\eta_{1,h}$ converges from the left to γ_n, by Theorem 79 it suffices to show that $\mu(\Omega_{\eta_{1,h}}) \to \mu(\Omega_{\eta_{-1,n-2}})$ as $h \to \infty$. By Lemma 71, for each h we have that $\Omega_{\eta_{1,h}}$ is swept out by the $\mathcal{T}_{\eta_{1,h}}$-orbit of $\mathcal{Z}_{\eta_{1,h}}$. Furthermore, a version of (13) holds for these values as well. That is, $\Omega_{\eta_{1,h}}$ is the union of the image of $\mathcal{Z}_{\eta_{1,h}}$ with pieces of the form $\mathcal{T}^{|u|}(\,\Delta_\alpha(u) \times \Phi^{\pm}\,)$, with signs depending upon whether $u \in \{-1,-2\}^*$ or $u \in \{2,1\}^*$ (in simplified digits).

Now, the only admissible words in $\mathcal{L}_{\eta_{-1,n-2}}$ of digit not of the form $(a,1)$ are prefixes of $\overline{b}_{[1,\infty)}^{\eta_{-1,n-2}}$. Thus we can temporarily delete these from the language and use simplified digits. One can then extend the definition of $\mathcal{Z}_\alpha$ in Definition 64 to include $\alpha = \eta_{-1,n-2}$, and easily find that (upon taking closures) here also a partition of the type described in (13) holds for $\alpha = \eta_{-1,n-2}$. Therefore, we can once again argue as in Theorem 79 and conclude that continuity does hold here. □

13. Ergodicity and natural extensions; Proof of Theorem 1

In [9], we gave theoretical tools which can be applied to show main dynamical properties of systems of the type studied here. (Indeed, we placed several theoretical tools in that paper so as to retain the brevity of this section.) There, the "bounded non-full range" condition is a key hypothesis to prove ergodicity of a planar system associated to interval maps such as our own, as well as to show that the planar system is the natural extension. See Remark 62 for a reminder of this condition. As recalled in the proof of Theorem 1, see page 58 below, we have already proven the theorem in various cases. This was done invoking ([9], Main Theorem 1). In § 13.3, we show the denseness of the set of parameter values indexing maps which satisfy the bounded non-full range condition. This dense set hence parametrizes maps whose planar natural extensions are ergodic.

A second result in [9] is that the properties of ergodicity and being a natural extension of a planar system are induced onto sufficiently nearby systems by way of our quilting. In § 13.2 establish that every α for which the bounded non-full range condition fails is contained in an open neighborhood of parameter values all of which are sufficiently nearby so that ergodicity and being a natural extension of a planar system could be shared with its system by way of quilting. The aforementioned denseness result than allows the section to end with a completion of the proof of Theorem 1.

We begin the section by proving that all of our maps are eventually expansive.

13.1. **Eventual expansivity; Proof of Proposition 2.** We show eventual expansiveness for all of our maps follows from results of ([9].

Proof of Proposition 2. Fix n, α. We must show that the hypotheses (a)–(c) of ([9], Proposition 9), listed in the statement of ([9], Main Theorem 1), are fulfilled. We have already established that for any n, α the domain $\Omega_{n,\alpha}$ has all vertical fibers being intervals of finite

length. In the case of $\alpha \notin \{0, 1\}$ we have also established that these fibers are a bounded distance from the locus of $y = -1/x$. Finally, we have also showed that every block of $\Omega_{n,\alpha}$ is such that its vertical fibers are intervals such that the ratio of the Lebesgue measure of the image by $\mathcal{T}_{n,\alpha}$-image to that of the receiving vertical fiber is bounded away from zero and one. That is, the proposition holds for all of our maps. □

13.2. **Close neighbors: shared properties and related entropy.** We aim to invoke a result from ([9], which depends on the use of "matching exponents" (there denoted m, n — here we will use e_ℓ, e_r). In our setting, if $\alpha < \gamma_n$ is in some synchronization interval $J_{k,v}$ then we let $e_\ell = \underline{S} + 1$ and $e_r = \overline{S} + 1$; if $\alpha > \gamma_n$ is in the left hand portion of some synchronization interval $J_{-k,v}$ then let $e_\ell = \underline{S} + 1$ and $e_r = \overline{S} + 1$; finally, if $\alpha > \gamma_n$ is in the left hand portion of some synchronization interval $J_{-k,v}$ then let $e_\ell = \underline{S} + 1$ and $e_r = \overline{S} + 2$. Then equations (6), (18) show that for all such α,

$$T_\alpha^{e_\ell}(\, \ell_0(\alpha)\,) = T_\alpha^{e_r}(\, r_0(\alpha)\,).$$

Furthermore, in [8], we show that for all but finitely many α in any synchronization interval that these exponents are appropriately minimal for such an equality to hold.

Now consider $n \geq 3$ fixed and suppose that $\alpha' \neq \alpha$. Denote the orbits of the respective endpoints of the intervals of definition by ℓ_i', r_i' for each $\ell_i(\alpha'), r_i(\alpha')$, and similarly simply ℓ_i, r_i for each $\ell_i(\alpha), r_i(\alpha)$, respectively. For the ease of the reader we include the following direct translation of ([9], Definition 4).

Definition 135. Suppose that J is a matching interval with corresponding matching exponents e_ℓ, e_r. We say that $\alpha, \alpha' \in J$ are *close neighbors* if $\ell_i', \ell_i, r_j', r_j \in \mathbb{I}_{\alpha'} \cap \mathbb{I}_\alpha$ for all $1 \leq i \leq e_\ell$ and $1 \leq j \leq e_r$.

Recall from § 1.3 that $b_\alpha(x)$ gives the α-digit of x. We let

$$\mathcal{C} = \{\, (x, y) \in \Omega_\alpha \mid x \in \mathbb{I}_\alpha \cap \mathbb{I}_{\alpha'}\,,\, b_\alpha(x) \neq b_{\alpha'}(x)\,\}\,.$$

Compare the following with Figure 17.

Proposition 136. *Fix n. Suppose that α, α', with $\alpha' < \alpha < \gamma_n$, are close neighbors. Then both of $\Omega_{\alpha'}, \Omega_\alpha$ can be finitely quilted from each other. In particular,*

$$\Omega_{\alpha'} = \left(\Omega_\alpha \setminus \coprod_{i=1}^{\overline{S}+1} \mathcal{T}_\alpha^i(\mathcal{C}\,) \right) \amalg \coprod_{j=1}^{\underline{S}+1} \mathcal{T}_{\alpha'}^j(\mathcal{C}\,). \tag{24}$$

Furthermore, if $\mathcal{T}_\alpha$ is the natural extension of T_α, then the entropy of $T_{\alpha'}$ is

$$h(T_{\alpha'}) = \left(1 + (\underline{S} - \overline{S})\, \nu_\alpha(\,[r_0', r_0]\,)\right)^{-1} h(T_\alpha),$$

where ν_α is T_α-invariant probability measure induced from μ on Ω_α. Finally, since we always have $\underline{S} > \overline{S}$, the entropy is a monotonically increasing continuous function in α' (for fixed α).

Proof. This follows from combining ([9], Proposition 20) with ([9], Main Theorem 2 and its associated propositions). □

Given large α, α', their respective digits of an $x \in \mathbb{I}_{\alpha'} \cap \mathbb{I}_\alpha$ can differ in two basic ways.

Definition 137. Fix n and suppose $\alpha > \alpha' > \gamma_n$ are as above. We define the two sets

$$\mathcal{C}_1 = \{\, (x, y) \in \Omega_\alpha \mid x \in \mathbb{I}_\alpha \cap \mathbb{I}_{\alpha'}\,,\, b_\alpha(x) = (i, j),\, b_{\alpha'}(x) = (i', j) \text{ with } i \neq i'\,\}\,,$$
$$\mathcal{C}_2 = \{\, (x, y) \in \Omega_\alpha \mid x \in \mathbb{I}_\alpha \cap \mathbb{I}_{\alpha'}\,,\, b_\alpha(x) = (i, j),\, b_{\alpha'}(x) = (i', j') \text{ with } j \neq j'\,\}.$$

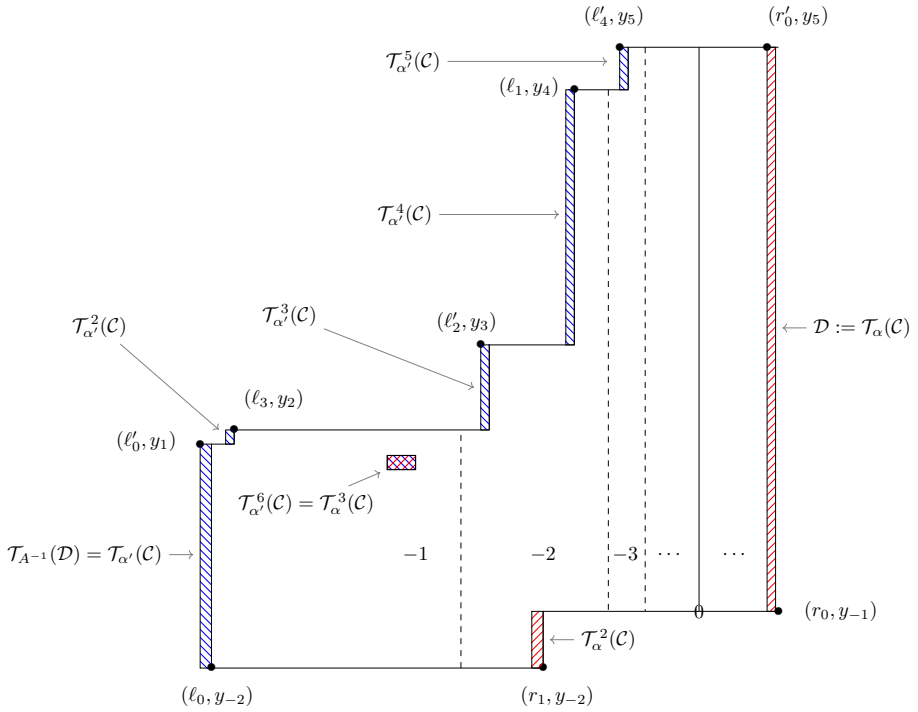

FIGURE 17. **Quilting for close neighbors.** Quilting from the $\Omega_{3,0.14}$ to $\Omega_{3,0.135}$, with blocks $\mathcal{B}_i, i = -1, -2, -3$ indicated (not fully to scale); compare with Figure 1. The forward $\mathcal{T}_\alpha$-orbit of $\mathcal{C}$ is deleted, while the forward $\mathcal{T}_{\alpha'}$-orbit of $\mathcal{C}$ is added, until the "hole" created by excising $\mathcal{T}^3_\alpha(\mathcal{C})$ is "patched" in by $\mathcal{T}^6_{\alpha'}(\mathcal{C})$. See Proposition 136.

Compare the following with Figure 18.

Proposition 138. *Suppose that α, α' are close neighbors in the synchronization interval $J_{-k,v}$ for some k, v, with $\alpha' < \alpha < \delta_{-k,v}$. Then both of $\Omega_{\alpha'}, \Omega_\alpha$ can be finitely quilted from each other. In particular,*

$$\Omega_{\alpha'} = \left(\Omega_\alpha \setminus \coprod_{i=1}^{\overline{S}+1} \mathcal{T}^i_\alpha(\mathcal{C}) \right) \amalg \coprod_{j=1}^{\underline{S}+1} \mathcal{T}^j_{\alpha'}(\mathcal{C}_1) \amalg \coprod_{j=1}^{\underline{S}} \mathcal{T}^j_{\alpha'}(\mathcal{C}_2). \tag{25}$$

Furthermore, if $\mathcal{T}_\alpha$ is the natural extension of T_α, then entropy of $T_{\alpha'}$ is

$$h(T_{\alpha'}) = \left(1 + (\underline{S} - \overline{S})\, \nu_\alpha([r'_0, r_0]) - \nu_\alpha([\mathfrak{b}_{\alpha'}, \mathfrak{b}_\alpha]) \right)^{-1} h(T_\alpha),$$

where ν_α is T_α-invariant probability measure induced from μ on Ω_α. Finally, the entropy is a monotonically decreasing continuous function in α' (for fixed α).

Proof. This follows from combining ([9], Proposition 21) with ([9], Main Theorem 2 and its associated propositions). □

We now treat the case of α, α' close neighbors in the right portion of some $J_{-k,v}$. We have $\mathcal{C}_1, \mathcal{C}_2$ as above.

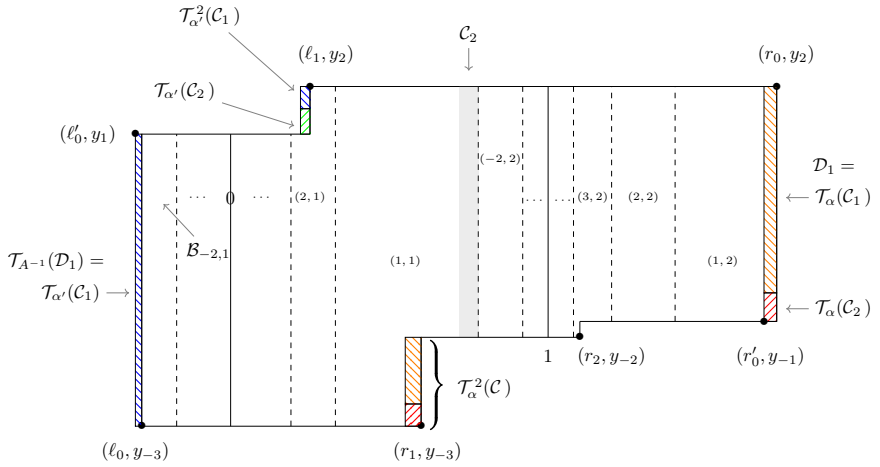

FIGURE 18. **Quilting for close neighbors, large α.** Quilting from $\Omega_{3,0.86}$ to $\Omega_{3,0.855}$ (not fully to scale); compare with Figure 15. Blocks $\mathcal{B}_{i,j}$, also denoted by (i,j). The forward $\mathcal{T}_\alpha$-orbit of $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is deleted, while the forward $\mathcal{T}_{\alpha'}$-orbit of $\mathcal{C}$ is added, until synchronization causes a "hole" excised due to the first of these, but to be "patched" due to the second (not shown here, but compare with Figure 17). See Proposition 138.

Proposition 139. *Suppose that α, α' are close neighbors in the synchronization interval $J_{-k,v}$ for some k, v, with $\delta_{-k,v} < \alpha' < \alpha$. Then both of $\Omega_{\alpha'}, \Omega_\alpha$ can be finitely quilted from each other. In particular,*

$$\Omega_{\alpha'} = \left(\Omega_\alpha \setminus \coprod_{i=1}^{\overline{S}+2} \mathcal{T}_\alpha^i(\mathcal{C}) \right) \amalg \coprod_{j=1}^{\underline{S}+1} \mathcal{T}_{\alpha'}^j(\mathcal{C}_1) \amalg \coprod_{j=1}^{\underline{S}} \mathcal{T}_{\alpha'}^j(\mathcal{C}_2).$$

Furthermore, if $\mathcal{T}_\alpha$ is the natural extension of T_α, then entropy of $T_{\alpha'}$ is

$$h(T_{\alpha'}) = \left(1 + (1 + \underline{S} - \overline{S})\, \nu_\alpha([r_0', r_0]) - \nu_\alpha([\mathfrak{b}_{\alpha'}, \mathfrak{b}_\alpha]) \right)^{-1} h(T_\alpha),$$

where ν_α is T_α-invariant probability measure induced from μ on Ω_α. Finally, the entropy is a monotonically decreasing continuous function in α' (for fixed α).

Proof. This again follows from combining ([9], Proposition 21) with ([9], Main Theorem 2 and its associated propositions). □

13.3. **Bounded non-full range values are dense.**

Proposition 140. *If T_α is not of bounded non-full range, then there is an open neighborhood of α within which every α' is a close neighbor of α.*

Proof. All endpoints of synchronization intervals, all $\delta_{-k,v}$, and all non-synchronizing α are such that T_α is of bounded non-full range. We thus fix α_0 in the interior of some interval J of the form $J_{k,v}$, $(\eta_{-k,v}, \delta_{-k,v})$ or $(\delta_{-k,v}, \zeta_{-k,v},)$.

On J, each $\alpha \mapsto \ell_i(\alpha)$ with $0 \le i \le \underline{S}$ and $\alpha \mapsto r_j(\alpha)$ with $0 \le j \le \overline{S}$ is continuous. Similarly, letting $\Delta(a,b) = \cup_{\alpha \in [0,1]} \Delta_\alpha(a,b)$, we have that the left and right boundaries $\lambda_{a,b}(\alpha)$ and $\rho_{a,b}(\alpha)$ also give continuous functions (at least for those (a,b) which are admissible for all of J), compare

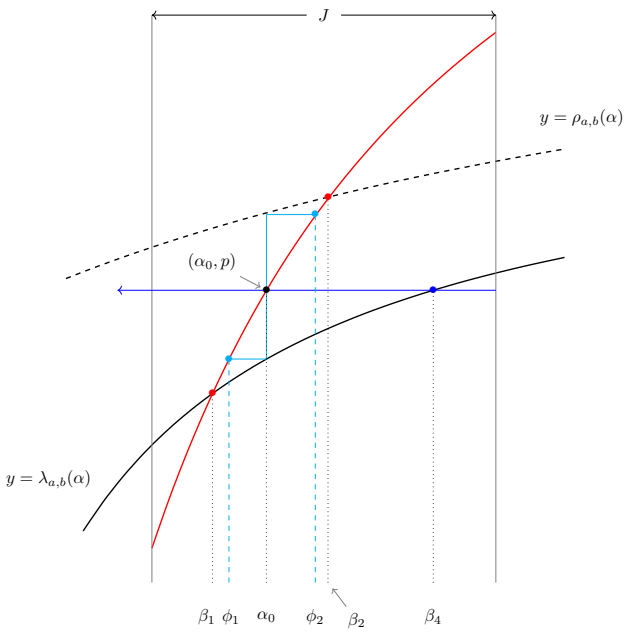

FIGURE 19. Determining the open set $N(p)$ of α with agreement of α- and α_0-digits for both p and corresponding T_α-image of endpoint, where p is in the T_{α_0}-orbit of an endpoint of $\mathbb{I}_{\alpha_0}$ (one possible configuration), see the proof of Proposition 140.

this with ([8], Figure 1.4), where such boundaries are plotted over all of $[0,1]$ in the setting of $n = 3$.

Fix $p \in \{\ell_i(\alpha_0) \mid 1 \le i \le \underline{S}\} \cup \{r_j(\alpha_0) \mid 1 \le j \le \overline{S}\}$, and thus also fix the value of i or j corresponding to p. We now solve for those α for which the T_α-digit of p agrees with its T_{α_0}-digit, confer Figure 19. Denote the T_{α_0}-digit of p by (a,b). Let β_1, β_2 be the endpoints of the subinterval of J where all $\ell_i(\alpha)$ (respectively, $r_j(\alpha)$) have α-digit equal to (a,b). Thus, $\beta_1 < \alpha_0 < \beta_2$. Let β_3, β_4 be defined by $\rho_{(a,b)}(\beta_3) = p$ and $\lambda_{(a,b)}(\beta_4) = p$. Then $\beta_3 < \alpha_0 < \beta_4$. Now let $M(p)$ be the open interval whose left endpoint is the greatest of β_1, β_3 and whose right endpoint is the least of β_2, β_4. Then for all $\alpha \in M(p)$, the α-digit of p is also (a,b). Thus, for all $\alpha \in M(p)$, both $p \in \mathbb{I}_{\alpha_0} \cap \mathbb{I}_\alpha$ and $T_{\alpha_0}(p) = A^a C^b \cdot p = T_\alpha(p)$.

Now let ϕ_1 be defined by $\lambda_{a,b}(\alpha_0) = \ell_i(\phi_1)$ and ϕ_2 by $\rho_{a,b}(\alpha_0) = \ell_i(\phi_2)$ (respectively, $r_j(\phi_1), r_j(\phi_2)$). Let $N(p)$ be the open interval whose left endpoint is the greatest of β_1, β_3, ϕ_1 and whose right endpoint is the least of β_2, β_4, ϕ_2. Then for all $\alpha \in N(p)$, $T_{\alpha_0}(\ell_i(\alpha)) = A^a C^b \cdot \ell_i(\alpha)) = T_\alpha(\ell_i(\alpha))$ (respectively, with r_j appropriately replacing ℓ_i).

We now let $\mathcal{N}(\alpha_0) = \cap_p N(p)$. Then every α in the open set $\mathcal{N}(\alpha_0)$ is a close neighbor of α_0. □

For our purposes, we only need that in every open set $\mathcal{N}(\alpha)$ of close neighbors there is some α' with $T_{\alpha'}$ of bounded non-full range. We show a stronger result.

Proposition 141. *The set of α with T_α of bounded non-full range is dense in $[0,1]$.*

Proof. The bounded non-full range condition is fulfilled by the maps indexed by: the endpoints of synchronization intervals, see the proofs of Propositions 63, 104, 111; and the division points of

synchronization intervals for large α, see the proof of Proposition 113; and, in light of Remark 62, also at all non-synchronization points: for each such α, both endpoints of the interval of definition have only finitely many distinct digits in their α-expansions.

Fix any α with T_α not of bounded non-full range. From the above, α lies in the interior of some synchronization interval; the T_α-orbits of $\ell_0(\alpha)$ and $r_0(\alpha)$ meet. Since T_α is not of bounded non-full range, the T_α-orbit of $\ell_0(\alpha)$ must visit every full cylinder at least once. Choose an increasing subsequence of $i_j, j \geq 1$ with $i_1 > \underline{S}$ such that $\ell_{i_j-1}(\alpha)$ is in a full cylinder. Zero is clearly included in the $T_\alpha^{i_j}$-image of the corresponding rank i_j cylinder, for each j . Set $z_j = \underline{b}^\alpha_{[1,i_j]}$ and let $a_j = M_{z_j}^{-1} \cdot 0$. For each sufficiently large j, there is an α_j such that $\ell_0(\alpha_j) = a_j$ and such that α_j is in the same synchronization interval as α_0. If M_{z_j} is α_j-admissible, then the T_{α_j}-orbit of $\ell_0(\alpha_j)$, and hence that of $r_0(\alpha_j)$, reaches zero in finitely many steps. In particular, these T_{α_j} are all of bounded non-full range.

Since the α_j certainly converge to α, we are done unless in fact we cannot find an infinite subsequence along which each M_{z_j} is α_j-admissible. If this obstruction exists, then it must be that for all i_j sufficiently large there is some $i < i_j$ such that $M_i \cdot a_j \notin [a_j, t + a_j)$, where M_i corresponds to $\underline{b}^\alpha_{[1,i]}$. For each such pair (i, j), either $M_i \cdot a_j < a_j$ or $M_i \cdot a_j \geq t + a_j$. In the first case, $M_i \cdot a_j < a_j < \ell_0(\alpha) < M_i \cdot \ell_0(\alpha)$ and hence M_i has a fixed point in $(a_j, \ell_0(\alpha))$. Recall that $A \cdot x = x + t$ for any x; in the second case, $A^{-1}M_i \cdot a_j < a_j < \ell_0(\alpha) < A^{-1}M_i \cdot \ell_0(\alpha)$ and it is $A^{-1}M_i$ that has a fixed point in $(a_j, \ell_0(\alpha))$.

We now define a new subsequence of indices, $(i_k)_{k>0}$ such that the distance of a fixed point of M_{i_k} or of $A^{-1}M_{i_k}$ to $\ell_0(\alpha)$ is less than all such distances for all $0 < i < i_k$. Let b_k be the fixed point defining i_k, and let α_k be such that $\ell_0(\alpha_k) = b_k$. By construction, each $M_i, i < i_k$ is α_k-admissible, and $T_{\alpha_k}^{i_k+1}$ fixes $\ell_0(\alpha_k)$. In particular, the T_{α_k}-orbit of $\ell_0(\alpha_k)$ meets only finitely many cylinders. We can and so assume that $i_k > \underline{S}$ holds for all k, and hence each T_{α_k} is of bounded non-full range. The result holds. □

Proof of Theorem 1. We have already shown that the result holds for: any endpoint of a synchronization interval, by Propositions 63 , 104, and 111; all of the $\delta_{-k,v}$ by Proposition 113; and, also all non-synchronizing values of α by Propositions 78 and 132. It thus remains to consider the case of α in an interval J of the form $J_{k,v}$, or $(\eta_{-k,v}, \delta_{-k,v})$, or $(\delta_{-k,v}, \zeta_{-k,v})$. By Proposition 140 , either α already satisfies the hypotheses of ([9], Main Theorem 1) and the result holds, or else the set of close neighbors of α is an open neighborhood. Thus, the denseness of the set of α' with $T_{\alpha'}$ of bounded non-full range implies that we can apply the results of Subsection 13.2 to conclude that our statements hold for this value of α as well. □

14. Entropy changes continuously

As in ([23], proof of Theorem 2), we employ one of the key tools for proving statements about change of entropy: Abramov's formula relates entropy values of systems which have a common subsystem to which the respective first return maps agree. We show such common first returns to subsystems of the $\alpha = 0$ system for a leftmost parameter subinterval, and similarly use $\alpha = 1$ for a rightmost parameter subinterval. With this in hand, continuity of entropy on each of these subintervals does hold; fortunately enough, the subintervals overlap and the continuity holds on the entire parameter interval.

14.1. **Common first returns.** For each $\alpha < 1$, we seek a subset of Ω_α contained in Ω_0 for which $\mathcal{T}_\alpha$ and $\mathcal{T}_0$ have common first returns. Due to the shape of Ω_0, such a subset must surely have negative x-values and positive y-values; this leads to the following set N, confer Figures 1 and 10.

Lemma 142. *Suppose that* $1/t_n \geq \alpha > 0$ *and set* $N = \bigcup_{i\in\mathbb{N}} \Delta_\alpha(-i,1) \cup \Delta_\alpha(-i,2)$. *Suppose* $x \in \mathbb{I}_\alpha$ *is negative and that there exists* $m \geq 0$ *such that both* $T_\alpha^m(x) \in N$ *and* $T_\alpha^{m+1}(x) < 0$. *Then there is some* $k \in \mathbb{N}$ *such that* $T_0^k(x) = T_\alpha^{m+1}(x)$. *Furthermore, for any* y *one has* $\mathcal{T}_0^k(x,y) = \mathcal{T}_\alpha^{m+1}(x,y)$.

Proof. Note that some of the cylinders in the expression defining N may be empty for any given α.

We can and do assume that m is minimal with respect to the hypotheses. From the definition of our maps, we can write

$$T_\alpha^{m+1}(x) = A^{a_{m+1}}C^{c_{m+1}}A^{a_m}C^{c_m}\cdots A^{a_1}C^{c_1}\cdot x,$$

with $c_1 = 1$ and both a_{m+1}, a_1 negative. If $m = 0$, then $A^{a_1}C^{c_1}\cdot x = T_0(x)$ and $k = 1$. We now assume that $m > 0$ and perform word processing on $A^{a_{m+1}}C^{c_{m+1}}A^{a_m}C^{c_m}\cdots A^{a_1}C^{c_1}$ with the goal to achieve an expression for this element of the group G_n that is in an admissible form for T_0.

We process from right to left, using the following substitution rules. Here ";" is placed so that the processed portion lies to its right.

Substitution rules:

(i)	$a < 0,\ A^aC;$	$\mapsto$	$A; A^{a-1}C$
(ii)	$a > 0,\ A^aCA;$	$\mapsto$	$CA; W^a$
(iii)	$A^aC^2A;$	$\mapsto$	$A^{a-1}C; (A^{-1}C)^{n-2}$
(iv)	$a > 0, b < 0,\ A^aC; A^bC$	$\mapsto$	$CA; W^aA^{b-1}C$
(v)	$b < 0,\ A^aC^2; A^bC$	$\mapsto$	$A^{a-1}C; (A^{-1}C)^{n-2}A^{b-1}C$

Recall from ([8], Equation 2.1) that

$$W = A^{-2}C(A^{-1}C)^{n-3}A^{-2}C(A^{-1}C)^{n-2}. \tag{26}$$

After a rule is applied, we either treat the $A^{a-1}C$; which has been defined, or we multiply on the left by the next unprocessed $A^{a_i}C^{c_i}$, and continue until only the empty word is to the left of the demarcating ";".

We claim that this process ends with a successful conversion of the element of G_n factored into a T_0-admissible form. Note first that each rule, up to ignoring the separation marker ";", is the result of applying a group identity. The first of these is obvious; the second and fourth result from induction and the simple identity $ACA = CAW$ see ([8], Equation 2.2) and compare with ([8], Lemma 5.1); finally, the third and fifth result from $C^2A = C^{-1}A = (A^{-1}C)^{n-1}$. We now argue that there is exactly one way to apply the rules. First note that no two rules are possible for any situation. It remains to show that there is always is a rule to apply at any (non-final) step.

First assume that $\alpha \leq \gamma_n$. Hence, there is no occurrence of $c_i = 2$. Since $x < 0$, we have $a_1 < 0$. Since $m > 0$, there are no consecutive a_i, a_{i+1} which are both negative. Thus, we apply the first rule immediately, and the second rule is always applied only after the first rule is. Both the second and the fourth result in a step where the third rule must be applied. The third and fifth result in a step where the first or fourth rule must be applied, unless in either case $a = 1$ when both result in the fifth rule being applied. It follows that there is always an applicable rule at each step. Since $a_{m+1} < 0$ we have $a_m > 0$ and thus the final step occurs with an application of the third or fifth rule; at this point, we can disregard the semi-colon, and have rewritten the original element of the group. The main requirements for T_0-admissibility of a word in letters of the form A^aC with each $a < 0$ come from the fact that $-t = \ell_0(0)$ is sent to itself by the

admissible word W. These requirements are that $A^{-1}C$ cannot appear more than $n-2$ times in a row, and appearances of $A^{-2}C\,(A^{-1}C)^{n-2}$ are isolated. It is easily checked that our processed word meets these requirements. The process fully succeeds in this case.

Now suppose $1/t_n \geq \alpha > \gamma_n$. Note that the complement of N in $\mathbb{I}_\alpha$ is $[0, \mathfrak{b}_\alpha)$. We of course can again assume $m > 0$, and also observe that at most one rule can be applied at any step of the word processing. Since $x < 0$ and we cannot have consecutive occurrences of $(a_i, 1)$ with a_i negative, the first rule is again applied immediately. It then is followed by an application of the second or third rule. Both the second and the fourth result in a step where the third rule must be applied, unless the next unprocessed portion is of the form $A^{a_i}C^2$. In this case, a simplification gives $A^{a_i}C^2CA; W^a = A^{a_i-1}C; (A^{-1}C)^{n-3}A^{-2}C(A^{-1}C)^{n-2}W^{a-1}$ and (assuming processing is not yet complete) either rule i or rule iv is applied, unless $a_i = 1$ in which case similar considerations are applied with $A^{a_{i+1}}C^{c_{i+1}}C; \cdots$. As well, there are further, easily determined, possibilities after either the third of fifth rule is applied and $a_i = 1$. That is, here also the process fully succeeds.

Finally, with $M = A^{a_{m+1}}C^{c_{m+1}}A^{a_m}C^{c_m}\cdots A^{a_1}C^{c_1}$, both $\mathcal{T}_0^k(x,y) = \mathcal{T}_M(x,y)$ and $\mathcal{T}_\alpha^{m+1}(x,y) = \mathcal{T}_M(x,y)$. □

We now wish to find a region for which the *first* return under $\mathcal{T}_0$ agrees with that of $\mathcal{T}_\alpha$. To this end, we restrict N and the region to which we return, so that $\{-1,-2\}$ digits introduced in the word processing could never index an 'early return' of a $\mathcal{T}_0$-orbit.

Lemma 143. *Suppose that* $1/t_n \geq \alpha > 0$. *Let* $d = \min\{-4, -2+d_\alpha(\ell_0(\alpha))\}$, $N_d = \bigcup_{i\leq d}\Delta_\alpha(i,1)\cup\Delta_\alpha(i,2)$ *and* $\Omega_d^+ = \Omega_\alpha \cap \{(x,y) \mid y>0, x<0, x\in N_d$ *and* $x'\in N_d$, *where* $\mathcal{T}_\alpha^{-1}(x,y) = (x',y')\}$. *The first return maps of* $\mathcal{T}_0$ *and of* $\mathcal{T}_\alpha$ *to* Ω_d^+ *agree.*

Proof. Since Ω_α^+ is of finite μ-mass, it follows that $\Omega_d^+ \subset \Omega_0$ and hence both return maps are defined. Suppose that $(x,y) \in \Omega_d^+$ and $\mathcal{T}_\alpha^{m+1}(x,y)$ is its first $\mathcal{T}_\alpha$-return. From the previous lemma, we can deduce that this first return map is also *some* return under $\mathcal{T}_0$.

If $m = 0$ then certainly $T_\alpha(x) = T_0(x)$, and the first return of the two-dimensional maps also agree. Now assume that $m > 0$. Suppose

$$T_\alpha^{m+1}(x) = A^{a_{m+1}}C^{c_{m+1}}A^{a_m}C^{c_m}\cdots A^{a_1}C^{c_1}\cdot x.$$

Of course, if for all i both $c_i = 1$ and a_i are negative, then the orbits are the same and the result clearly holds. In general, there may be intermediate consecutive appearances of this type. We thus add a special case to the first rule:

$$(i') \qquad A^{a'}CA^aC; \mapsto A^{a'}C; A^aC \ \text{ if } a<0, a'<0,$$

and process so as to achieve an admissible T_0 expression. Whenever any of our substitution rules inserts T_0-steps, these insertions are always of corresponding simplified 0-digits in $\{-1,-2\}$. Of course, any x-value with either of these as its simplified 0-digit cannot have simplified α-digit less than or equal to d.

On the other hand, our rules also make changes in the exponents of appearances of A. However, any such exponent is changed at most once throughout the process, and if it is changed then it is decreased exactly by one. Since each substitution realizes a group identity, it follows that any element in N_d that is in the T_0-orbit segment of x but not already present in the T_α-orbit segment is either (1) isolated (with respect to this property), or (2) is sent by T_0 to $T_\alpha^m(x)$ while its T_0-orbit predecessor is not in N_d. But, by hypothesis, $\mathcal{T}_\alpha^m(x,y) \notin \Omega_d^+$. Therefore the result holds. □

We now treat the case of large α. Note that $\alpha = 1$ gives a system defined on positive x, but importantly each digit is then of the form $(i, 1)$ or $(i, 2)$ with i *positive*.

Lemma 144. *Let $\gamma_n < \alpha < 1$ and set $P = \bigcup_{i\in\mathbb{N}} \Delta_\alpha(i, 1) \cup \Delta_\alpha(i, 2)$. Suppose $x \in P$ is such that there exists $m \geq 0$ such that both $T_\alpha^m(x)$ and $T_\alpha^{m+1}(x)$ are in P. Then there is some $k \in \mathbb{N}$ such that $T_1^k(x) = T_\alpha^{m+1}(x)$. Furthermore, for any y one has $\mathcal{T}_1^k(x, y) = \mathcal{T}_\alpha^{m+1}(x, y)$.*

Proof. We argue as in the case of small α. Here also the case of $m = 0$ is trivial, and we now suppose that $m > 0$ and minimal.

Recall that $U = AC(AC^2)^{n-2}$. The relation given in ([8], Lemma 8.1) directly implies that for any $i \in \mathbb{N}$ we have

$$A^{-i}CA^{-1} = C^2(AC^2)^{n-2}U^{i-1}.$$

Suppose

$$T_\alpha^{m+1}(x) = A^{a_{m+1}}C^{c_{m+1}}A^{a_m}C^{c_m} \cdots A^{a_1}C^{c_1} \cdot x,$$

with the right hand side being the α-admissible factorization.

We process $M = A^{a_{m+1}}C^{c_{m+1}}A^{a_m}C^{c_m} \cdots A^{a_1}C^{c_1}$ by the following rules. Note that if $c_2 = 2$ then $a_2 < 0$ and $T_\alpha(x) \in [\mathfrak{b}_\alpha, 1)$. It is easily verified that $[\mathfrak{b}_\alpha, 1) \subset \Delta_1(1, 1)$ and hence the 1-word retains $A^{a_1}C^{c_1}$ and continues by the result of a processing of $\cdots A^{a_2}C^2(AC)^{-1}AC$. Similar considerations apply whenever some intermediate $c_i = 2$.

$$A^{a_2}C^{c_2}A^{a_1}C^{c_1}; \mapsto \begin{cases} C^2(AC^2)^{n-2}; U^{-a_2-1}A^{1+a_1}C^{c_1} & \text{if } c_2 = 1 \\ C^2(AC^2)^{n-2}; U^{-a_2-1}ACA^{a_1}C^{c_1} & \text{otherwise.} \end{cases}$$

$$A^{a_i}C^{c_i}\, C^2(AC^2)^{n-2}; \mapsto \begin{cases} C^2(AC^2)^{n-2}; U^{-a_i-1}AC(AC^2)^{n-3} & \text{if } c_i = 1 \\ C^2(AC^2)^{n-2}; U^{-a_i-1}A^2C^2(AC^2)^{n-3} & \text{otherwise.} \end{cases}$$

$$A^{a_{m+1}}C^{c_{m+1}}\, C^2(AC^2)^{n-2}; \mapsto \begin{cases} ; A^{a_{m+1}}C(AC^2)^{n-2} & \text{if } c_{m+1} = 2 \\ ; A^{1+a_{m+1}}C^2(AC^2)^{n-3} & \text{otherwise.} \end{cases}$$

Once again, upon suppressing the semi-colon of demarcation, each rule is an identity. The processed word is 1-admissible, and the result holds. □

Lemma 145. *Suppose that $\gamma_n < \alpha < 1$, set $Q = \bigcup_{i\geq 4} \Delta_\alpha(i, 1) \cup \Delta_\alpha(i, 2)$, and let $\Omega^Q = \Omega_\alpha \cap \{(x, y) \mid y < 0; x, x' \in Q, \text{ where } \mathcal{T}_\alpha^{-1}(x, y) = (x', y')\}$. The first return maps of $\mathcal{T}_1$ and of $\mathcal{T}_\alpha$ to Ω^Q agree.*

Proof. We have that $\Omega^Q \subset \Omega_1$ and hence the return maps are defined. Suppose that $(x, y) \in \Omega^Q$ and $\mathcal{T}_\alpha^{m+1}(x, y)$ is its first $\mathcal{T}_\alpha$-return. From the previous lemma, we can deduce that this first return map is also *some* return under $\mathcal{T}_1$.

If $m = 0$ then certainly $T_\alpha(x) = T_1(x)$, and the first return of the two-dimensional maps also agree. Now assume that $m > 0$. Suppose

$$T_\alpha^{m+1}(x) = A^{a_{m+1}}C^{c_{m+1}}A^{a_m}C^{c_m} \cdots A^{a_1}C^{c_1} \cdot x.$$

Of course, if this x-orbit stays within Q, then it certainly stays within the positive reals and we find that this gives also the T_1-orbit and the result holds. In general, we process as in the previous lemma each subword corresponding to a return to a consecutive pair in Q (including the possibility of the use of the "$m = 0$ case"). Note that our rules insert only steps corresponding to one of AC, A^2C, AC^2, A^2C^2 and cause one-time changes of the form (a_i, c_i) replaced by $(1 + a_i, 1), (1 + a_i, 2)$ for certain positive a_i. These one time changes can only occur at the end of a return to a consecutive pair in Q, or at the beginning of an excursion from such a pair. In the latter case, the value of the point at the beginning of the excursion is unchanged. When

such an increase to $1 + a_i$ is made at the end of an excursion, the T_1-predecessor of the return lies in $\Delta_1(1 + a_i, 2)$; in all cases its own T_1-predecessor lies in $\Delta_1(1, 1) \cup \Delta_1(1, 2)$ (with only the case of $n = 3$ and $a_{i-1} = -1$ at all complicated).

Since the left endpoint of $\Delta_1(2, 2)$ is sent by A^2C^2 to 0, we find that $\Delta_1(2, 2) \cup \Delta_1(1, 2) \subset \Delta_\alpha(2, 2) \cup \Delta_\alpha(1, 2) \cup [r_0(\alpha), t)$. One similarly finds that $\Delta_1(2, 1) \cup \Delta_1(1, 1) \subset \Delta_\alpha(2, 1) \cup \Delta_\alpha(1, 1) \cup [\mathfrak{b}_\alpha, 1)$. Thus, $\bigcup_{i=1}^{2} \Delta_1(i, 1) \cup \Delta_1(i, 2)$ does not intersect Q. Thus, our 1-admissible word is such that it gives the *first* return under $\mathcal{T}_1$. □

14.2. **Entropy times mass is constant; Proofs of Theorems 4, 5.** We now argue by way of Abramov's formula, as in [9], that the following holds.

Proposition 146. *The function $\alpha \mapsto h(T_\alpha)\mu(\Omega_\alpha)$ is constant on each of the intervals $(0, 1/t_n]$ and $(\gamma_n, 1)$.*

Proof. Let α, α' both belong to one of the parameter intervals being considered. In the respective cases, the subset Ω_d^+ or Ω^Q of Ω_α is of positive μ-mass, and similarly for when α' replaces α. Let $\Omega_{\alpha,\alpha'}$ be the intersection of these respective subsets. From Lemmas 143 and 145 we deduce that the first return map of $\mathcal{T}_\alpha$ and $\mathcal{T}_{\alpha'}$ agree. These hence define the same dynamical system, and Abramov's formula yields that their entropy can be expressed as

$$\frac{h(\mathcal{T}_\alpha)\,\mu(\Omega_\alpha)}{\mu(\Omega_{\alpha,\alpha'})} = \frac{h(\mathcal{T}_{\alpha'})\,\mu(\Omega_{\alpha'})}{\mu(\Omega_{\alpha,\alpha'})}.$$

Since the entropy of a dynamical system equals the entropy of the natural extension system, the result holds. □

Since the two parameter subintervals of the proposition overlap, that is $1/t_n > \gamma_n$, Theorem 4 is proven. Recall that from this, and Theorem 3, Theorem 5 follows. □

15. Conjecture: Entropy times μ-mass equals volume of unit tangent bundle

The volume of the unit tangent bundle of the hyperbolic orbifold uniformized by the triangle group G_n is

$$\mathrm{vol}_n = \frac{2(2n-3)\pi^2}{3n}. \tag{27}$$

This is easily checked, as the hyperbolic area of the base orbifold is twice the hyperbolic area of any hyperbolic triangle of angles $\pi/3, \pi/n, 0$, and vol_n is this times π, see say [3] for this calculation when G_n is replaced by the standard modular group. From this, Conjecture 8 can be restated as the following.

Conjecture 147. *For all $n \geq 3$ and for all $\alpha \in (0, 1)$ we conjecture that*

$$h(T_\alpha)\,\mu(\Omega_\alpha) = \mathrm{vol}_n\,.$$

In § 15.1 we relate Rohlin's formula for entropy to volumes of unit tangent bundles. This reduces the above conjecture to the computation of an integral. § 15.2 shows that the conjecture holds when $n = 3$. § 15.3 gives in particular the proof of Theorem 7 (which implies the equivalence of Conjectures 6 and 8). Finally, in § 15.4 we report on computational confirmation for $4 \leq n \leq 12$.

15.1. **Rohlin's formula and the volume of unit tangent bundles.** We show that for each n, the conjecture can be reduced to an integral computation, involving either Ω_0 or Ω_1.

Definition 148. Fix $n \geq 3$. For $\alpha \in [0,1]$ and for each $x \in \mathbb{I}_\alpha$, let $\tau_\alpha(x) = -2\log|cx+d|$ where $T_\alpha(x) = (ax+b)/(cx+d)$. Of course, this is simply $\tau_\alpha(x) = \log|T'_\alpha(x)|$.

We now show that the integral of $\tau_\alpha(x)$ over Ω_α with respect to $d\mu$ gives the product of $h(T_\alpha)$ with $\mu(\Omega_\alpha)$. Furthermore, these integrals are constant with respect to α, including at the endpoints, this although both $\mu(\Omega_0)$ and $\mu(\Omega_1)$ are infinite.

Lemma 149. *Fix $n \geq 3$. For $\alpha \in [0,1]$ and for each $x \in \mathbb{I}_\alpha$, let $\tau_\alpha(x) = -2\log|cx+d|$ where $T_\alpha(x) = (ax+b)/(cx+d)$. Then for $0 < \alpha < 1$,*

$$h(T_\alpha)\,\mu(\Omega_\alpha) = \int_{\Omega_\alpha} \tau_\alpha(x)\,d\mu = \int_{\Omega_0} \tau_0(x)\,d\mu = \int_{\Omega_1} \tau_1(x)\,d\mu.$$

Proof. For $0 < \alpha < 1$, Rohlin's formula (see [15]) gives

$$h(T_\alpha) = \int_{\mathbb{I}_\alpha} \log|T'_\alpha(x)|\,d\nu_\alpha = \int_{\mathbb{I}_\alpha} \tau_\alpha(x) d\nu_\alpha = \frac{\int_{\Omega_\alpha} \tau_\alpha(x) d\mu}{\mu(\Omega_\alpha)},$$

where the last equality holds because ν_α is the marginal measure for the probability measure on Ω_α induced by μ. Since $\mu(\Omega_\alpha) < \infty$, our first equality holds. We now show the remaining two equalities hold by showing that each holds for α in a certain subinterval; these subintervals overlap, from this the result itself holds.

Assume first that $\alpha \leq 1/t$. We employ a tower construction, with base Ω_d^+. For this, note that Poincaré recurrence for each of $\mathcal{T}_\alpha^{\pm 1}$ to this positive measure subspace of Ω_α implies bijectivity of the first return of $\mathcal{T}_\alpha$ to Ω_d^+. For each $k \in \mathbb{N}$ we define $R_k \subset \Omega_d^+$ as the set of points whose first return to Ω_d^+ is given by applying $\mathcal{T}_\alpha^k$, thus the R_k partition Ω_d^+. We have that $\mathcal{T}_\alpha$ bijectively maps the tower formed by the disjoint union of the sets of the form $\mathcal{T}_\alpha^i(R_k)$ for all k and $0 \leq i < k$ to itself. By the ergodicity of $\mathcal{T}_\alpha$, we deduce that this union gives Ω_α up to measure zero.

Let ρ denote projection onto the x-coordinate of the first return map on Ω_d^+. Due to the logarithm turning the multiplication appearing in the Chain Rule to addition — in the sense that: For any reasonable map T, we have $\log|(T^k)'(x)| = \sum_{i=0}^{k-1} \log|T'(\,T^i(x)\,)|$ — and the additivity of integration, one deduces that

$$\int_{\Omega_\alpha} \tau_\alpha(x)\,d\mu = \int_{\Omega_d^+} \log|\rho'(x)|\,d\mu.$$

Due to the infinite measure of Ω_0, we cannot simply repeat this argument to directly achieve the equality with Ω_0 replacing Ω_α. However, the proof of Lemma 143 can easily be adapted to show that the first return map itself also agrees with the first return map for the accelerated system that [10] associates to $\mathcal{T}_0$. This system is again a finite μ-measure ergodic dynamical system and thus one finds that $\int_{\Omega_\alpha} \tau_\alpha(x)\,d\mu$ equals the integral of the logarithm of the absolute value of the first derivative of the accelerated one-dimensional map. Next, one can show that by taking the union of that accelerated two-dimensional domain with the forward $\mathcal{T}_0$ orbits of its points fibering over the subinterval where acceleration occurs (denoted $[-\tau, \epsilon_0)$ in [10], note that their τ is our t) gives Ω_0 up to zero measure. Putting this together, we do find that $\int_{\Omega_\alpha} \tau_\alpha(x)\,d\mu = \int_{\Omega_0} \tau_0(x)\,d\mu$.

We can now argue completely analogously, beginning with $\alpha > \gamma_n$, using Lemma 144, and the acceleration for the $\alpha = 1$ setting given in ([9], Lemma 13), to find that $\int_{\Omega_\alpha} \tau_\alpha(x)\,d\mu = \int_{\Omega_1} \tau_1(x)\,d\mu$. □

15.2. **Proof of Conjecture 8 when $n = 3$.** In light of the previous subsection, the following result proves Conjecture 8 when $n = 3$.

Lemma 150. *Let $n = 3$. We have*

$$\int_{\Omega_1} \tau_1(x)\, d\mu = \mathrm{vol}_3\,.$$

Proof. Restricting ([9], Proposition 12) to our case of $n = 3$, one has

$$\Omega_1 = (\,[0,1] \times [-1,0]\,) \;\cup\; (\,[1,2] \times [-1/2,0]\,).$$

For $x < 1, T_1(x) = A^{-k}C \cdot x$ for some $k \in \mathbb{Z}$ and in particular, $\tau_1(x) = -2\log x$. Similarly, for $2 \geq x > 1$ we have $\tau_1(x) = -2\log|x-1|$. Recall that for any $M \in \mathrm{SL}_2(\mathbb{R})$ that $\mathcal{T}_M$ locally preserves $d\mu$. One easily verifies that $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is such that $\mathcal{T}_M$ sends $[0,1] \times [-1,0]$ to $[1,2] \times [-1/2,0]$, and hence a change-of-variables calculation gives

$$\int_{[0,1]\times[-1,0]} -2\log x \; d\mu = \int_{[1,2]\times[-1/2,0]} -2\log|x-1|\, d\mu\,.$$

That is, $\int_{\Omega_1} \tau_1(x)\, d\mu = 2\int_{[0,1]\times[-1,0]} -2\log x \; d\mu$. The following lemma thus implies our result. □

Note that (27) specializes to $\mathrm{vol}_3 = 2\pi^2/3$.

Lemma 151. *We have*

$$\int_{[0,1]\times[-1,0]} -2\log x \; d\mu = \pi^2/3.$$

Proof. It is an exercise to show that the by-excess map on $[-1,0)$ to itself defined by $x \mapsto -1/x - \lfloor 1 - 1/x \rfloor$ has as planar natural extension domain exactly $[-1,0] \times [0,1]$. (Recall that this is the $\alpha = 0$ Nakada-continued fraction, see say [23].) Arguing as in [4] shows that $\int_{[-1,0]\times[0,1]} -2\log|x| \; d\mu = \pi^2/3$. (Again, see say [3] for related discussion.) Since $d\mu = (1+xy)^{-2}\, dx\, dy$, the measure is preserved under the map $(x,y) \mapsto (-x,-y)$ and hence a change-of-variables calculation gives our result. □

15.3. **First expansive power maps, Proofs of Theorems 7, 9.** We show that under Conjecture 8 every one of our maps can be appropriately accelerated so as to define an interval map which is naturally given by a geometric system.

15.3.1. *Cross sections for geodesic flow: $n = 3, \alpha > 1/2$.* We now show that the results of the previous subsection imply that each of infinitely many of our maps has its natural extension given by the geodesic flow returning to a cross section in the unit tangent bundle of the hyperbolic orbifold uniformized by G_3. See, say, [25] for related background.

Corollary 152. *For $n = 3$ and each $\alpha > 1/2$ there is a cross section in the unit tangent bundle of $G_3\backslash\mathbb{H}$ which when equipped with the first return map under geodesic flow gives the natural extension of the interval map $T_{3,\alpha}$.*

Sketch. For any $n \geq 3$ and $\alpha \in [0,1]$, Arnoux's transversal as in [4, 5], see also ([9], Section 2.5), allows for the map $v : \Omega_\alpha \to T^1(\,G_n\backslash\mathbb{H}\,)$ given by $(x,y) \mapsto [M \cdot \hat{i}]_{G_n}$ where $M = \begin{pmatrix} x & -1/(1+xy) \\ 1 & y/(1+xy) \end{pmatrix}$, $\hat{i}$ is the upwards directed unit tangent vector based at $i \in \mathbb{H}$, $M \cdot \hat{i}$ is given by the usual action, and $[M \cdot \hat{i}]_{G_n}$ denotes the projection to the quotient orbifold of the unit

tangent vector $M \cdot \hat{i}$. Thus, the cross section here is $v(\Omega_\alpha)$. From the cited works, the map v is injective up to measure zero. Furthermore, the map $v \circ \mathcal{T}_{n,\alpha} \circ v^{-1}$ has μ as invariant measure. (Note that the cited works have Lebesgue measure as invariant measure, but $(x, y) \mapsto (x, y/(1 + xy))$ pushes forward μ to Lebesgue measure. See ([9], Section 2.2). Since $\mathcal{T}_{n,\alpha}$ on Ω_α gives the natural extension for the interval map, so does the conjugate map on the cross section.

For any $n \geq 3$ and $\alpha \in [0, 1]$, the derivative of $T_{n,\alpha}(x)$ equals x^{-2} if $x < \mathfrak{b}_\alpha$ and $(x - 1)^{-2}$ otherwise. Thus, $T_{n,\alpha}(x)$ is expansive exactly when both (i) $\ell_0(\alpha) > -1$ and (ii) $r_0(\alpha) < 2$. Since $r_0(\alpha) = \ell_0(\alpha) + t = \alpha t$, these conditions are (i) $\alpha > 1 - 1/t$ and (ii) $\alpha < 2/t$.

We now fix $n = 3$, thus $t = 2$, and restrict to $1 > \alpha > 1/2$. Here each interval map $T_\alpha = T_{3,\alpha}$ is *expansive*. It follows, again from the cited works, that at each $v(x, y)$ the map $v \circ \mathcal{T}_\alpha \circ v^{-1}$ is given by *some* return of the positively oriented geodesic flow to the cross section. Key to this is that there is a geodesic arc connecting the basepoint of $v(x, y)$ to that of $v \circ \mathcal{T}_{n,\alpha}(x, y)$, whose length is in fact $\tau_\alpha(x)$; see ([5], Theorem 1 and its proof). By definition, the geodesic segments determined by each $v(x, y)$ and the first return under the flow to the cross section form a flow invariant set. The ergodicity of the flow implies that this set is all of the unit tangent bundle; the integral over the cross section of the lengths of the geodesic segments defined by the first return is hence equal to the volume of this bundle. (See, [25, 4] for details of the measures involved in this.) Since $\int_{\Omega_\alpha} \tau_\alpha(x)\, d\mu = \mathrm{vol}_3$ holds here, one finds that $v \circ \mathcal{T}_\alpha \circ v^{-1}$ agrees almost everywhere with the *first* return of the geodesic flow. □

15.3.2. *First expansive power maps and geodesic flow, under the conjecture.*

Definition 153. Let T be a function on an interval I, with an invariant probability measure ν which is equivalent to Lebesque measure. Suppose that T is eventually expansive, thus there is some $r \in \mathbb{N}$ such that T^r is expansive. By the Well Ordering Principle of the integers, for each $x \in I$ there is a least $\ell(x) \in \mathbb{N}$ such that $T^{\ell(x)}(x)$ is expansive in the sense that the derivative here is greater than 1 in absolute value. We define the *first pointwise expansive power of* T as the map $U : I \to I$ sending each x to $T^{\ell(x)}(x)$.

Of course, $\ell(x) \leq r$ for all $x \in I$ and also $U(x) = T(x)$ for all $x \in E$, where E is the maximal subset of I on which T itself is expansive. For each $k \leq r$, let $E_k \subset I$ be the set on which $\ell(x) = k$. Of course, the E_k give a finite partition of I. We have that $|(T^k)'(x)| > 1$ for all $x \in E_k$; by the Chain Rule, $|(T^k)'(x)| = \prod_{i=0}^{k-1} |T'(T^i(x))|$. The *minimality* of $\ell(x)$ hence implies both $T(E_k) \subseteq \cup_{i=1}^{k-1} E_i$ for $k \geq 2$ and $T^{k-1}(E_k) \subset E_1$ for all k.

Note that in the setting where I is an interval and T is given piecewise by Möbius transformations, then also U is so given.

We now identify the natural extension of the first pointwise expansive power of T as an induced system within the natural extension of T.

Proposition 154. *Suppose T as above is eventually expansive. Let now $\mathcal{T}, \Omega, \mu$ be the usual aspects of the natural extension of the system of T, and for each k let $\mathcal{E}_k \subset \Omega$ be the portion which projects to $E_k \subset I$. Set*

$$\mathcal{F} = \Omega \setminus \bigcup_{k=2}^{r} \bigcup_{j=1}^{k-1} \mathcal{T}^j(\mathcal{E}_k).$$

Suppose that there is a full T-cylinder contained in E_1. Then the system on $\mathcal{F} \subset \Omega$ induced by $\mathcal{T}$, in other words the first return system of $\mathcal{T}$ on $\mathcal{F}$, gives a natural extension for U.

Proof. We can and do assume that $r > 1$.

Since there is a full cylinder contained in E_1, there is positive measure subset of Ω consisting of points in $\mathcal{E}_1$ whose $\mathcal{T}^{-1}$-orbit segments of length r remain in $\mathcal{E}_1$. Since this set is contained in

$\mathcal{F}$, we find that $\mathcal{F}$ has positive measure and furthermore $\mathcal{F} \subset \Omega$ projects surjectively to I. Let $\mathcal{R} : \mathcal{F} \to \mathcal{F}$ be the first return map of $\mathcal{T}$. Since the $\mathcal{E}_k$ give a partition of Ω and $\mathcal{T}$ is bijective on Ω, the fact that one deletes at least $\mathcal{T}^{k-1}(\mathcal{E}_k)$ for $k \geq 2$ shows that $\mathcal{F} \subset \mathcal{T}(\mathcal{E}_1)$.

We now show that $\mathcal{R}$ on each $\mathcal{F} \cap \mathcal{E}_k, k \geq 1$ is given by $\mathcal{T}^k$. Given the definition of the set $\mathcal{F}$, certainly within this subset the first return map $\mathcal{R}$ is given by $\mathcal{T}^j$ with $j \geq k$. It suffices to show that $\mathcal{T}^k(\mathcal{F} \cap \mathcal{E}_k) \subset \mathcal{F}$.

Consider first $k \geq 2$. We thus show that for each $2 \leq i \leq r$ and $1 \leq j < i$ we have $\mathcal{T}^k(\mathcal{F} \cap \mathcal{E}_k) \cap \mathcal{T}^j(\mathcal{E}_i) = \emptyset$. We discern three cases. If $k < j$ then $\mathcal{F} \cap \mathcal{T}^{j-k}(\mathcal{E}_i) = \emptyset$ implies the result. Suppose $j < k$; if $x \in T^{k-j}(E_k)$ then we must have that $|(T^j)'(x)| > 1$ but since $j < i$ this implies that $x \notin E_i$. Should $j = k$ then since the $\mathcal{T}^j(\mathcal{E}_i), 1 \leq i \leq r$ partition Ω; we find that $i = k$ but $j < i = k$ allows the previous argument to apply. Therefore, $\mathcal{R}$ on $\mathcal{F} \cap \mathcal{E}_k, k \geq 2$ is given by $\mathcal{T}^k$.

Finally, if $k = 1$ there are two cases. When $j = 1$, since $\mathcal{E}_1 \cap \mathcal{E}_i = \emptyset$ the result holds. Otherwise, $1 < j < i$ and $\mathcal{F} \cap \mathcal{T}^{j-1}(\mathcal{E}_i) = \emptyset$ implies the result. Thus, we have shown that $\mathcal{R}$ on each $\mathcal{F} \cap \mathcal{E}_k, k \geq 1$ is given by $\mathcal{T}^k$.

It is now clear that the system of $\mathcal{R}$ on $\mathcal{F}$ is an extension of the system of U on I. (Note that μ and ν remain the respective invariant measures, up to normalization.) Since $\mathcal{T}$ on Ω gives the natural extension of T on I, the induced system on $\mathcal{F}$ inherits in particular the property of being a minimal extension. □

We can now show that under certain conditions, an interval map can be appropriately accelerated so as to define an interval map which is naturally given by a geometric system.

Proof of Theorem 7. Arguing as in the proof of Corollary 152, we need only show that $h(U)\,\mu(\mathcal{F}) = h(T)\,\mu(\Omega)$. But, this is immediate from Abramov's formula. □

Combining the above results and arguments yields Theorem 9.

15.4. **Computational confirmation for $4 \leq n \leq 12$.** In view of Subsection 15.1, we discuss $\int_{\Omega_0} \tau_0(x)\, d\mu$ for fixed n.

Let $s : (x, y) \mapsto (-y, -x)$ and note that for any reasonable region $\mathscr{R}$ in the plane one has

$$\int_{\mathscr{R}} -2 \log |xy| d\mu = \int_{\mathscr{R}} -2 \log |x| d\mu + \int_{s(\mathscr{R})} -2 \log |x| d\mu.$$

Now, for each n, the domain Ω_0 is symmetric under s, see [10]. We could thus integrate $-2 \log |xy|$ with respect to μ over half of the domain; indeed, it seems easier for Mathematica to evaluate integrals of this type. A further improvement is reached by noting that each Ω_0 contains the square $\mathcal{S} = [-1, 0] \times [0, 1]$; using Lemma 151, the integral of $-2 \log |x| d\mu$ over $\mathcal{S}$ equals $\pi^2/3$. We thus let $\mathcal{I}(n)$ denote the integration of $-2 \log |xy| d\mu$ over the portion of Ω_0 projecting to x-values less than -1.

Using Mathematica, we numerically approximated the values $Z(n) = \mathrm{vol}_n - (\mathcal{I}(n) + \pi^2/3)$ for $4 \leq n \leq 12$. Clearly, each $Z(n)$ is a real number, and our conjecture states that each $Z(n)$ equals zero. The approximated values of $Z(n), 4 \leq n \leq 6$ were real numbers of absolute value less than 10^{-8}. The approximated values of $Z(n), 7 \leq n \leq 11$ were complex (!) numbers whose real parts were of absolute value less than 10^{-8} and imaginary parts less than 10^{-12}. The same code run for $n = 12$ gave a complex number of large norm; however, upon using virtually the same code to separately recalculate one of constituent integrals, the reported value of $Z(12)$ was a complex number of real part less than 10^{-4} and imaginary part less than 10^{-6}.

We have the impression that as n increases the computational difficulty of evaluating $\mathcal{I}(n)$ increases due to both the increasing number of constituent integrals and also the increasing algebraic degree of the numbers defining the limits of integration.

15.5. **Remarks on the conjecture.** In the setting of *expansive* piecewise Möbius interval maps appropriately given by elements of some Fuchsian group that uniformizes a surface (or orbifold) of finite hyperbolic area, one can reasonably expect a result as conjectured above. Indeed, there is a long history of these matters, going back at least to E. Artin's [6] coding by way of regular continued fractions of the geodesic flow on the unit tangent bundle of the hyperbolic orbifold uniformized by the modular group; some 60 years later Series [31] determined a cross section to this flow for which the first return map has as a factor the interval map defining these continued fractions. For related work see [2, 18] and in particular the overview [17]. We ourselves learned of these matters from Arnoux, see [3] and also the discussion of 'Arnoux's transversal' in [4].

Some caution is necessary. First, an example of an expansive piecewise Möbius interval map, given by elements of the modular group $\mathrm{PSL}(2,\mathbb{Z})$, with a nice planar natural extension but which is not of "first return type" is given in [4]. To be of first return type, see [5], a map must be such that the induced function on Arnoux's cross section in the unit tangent bundle of the surface/orbifold agrees with the first return map of the geodesic flow. Under this property, again see [5], one has an equality of the form of our conjectures, but in the setting of expansive maps.

Various approaches to proving the conjectures naturally present themselves. First, by direct integration. The integrals which appear are naturally related to dilogarithmic functions and hence are notoriously difficult to evaluate exactly. See [16] for discussion of the appearance of the dilogarithm in a related setting. Secondly, one could attempt to prove that each of our continued fractions gives a coding for the geodesic flow on the corresponding orbifold and then argue as in [18]. Closely related to this, one could follow [26] to determine a natural extension for the interval maps as a cross section to the geodesic flow consisting of unit tangent vectors (mainly) along the boundary of a particular fundamental domain for the Fuchsian group in consideration. With this in hand, one could hope to use Bonahon's [7] geodesic current formula as in [1] to obtain the integral value. Thirdly, given any of our explicit planar natural extensions, Arnoux's transversal provides an easily determined cross section. When the cross section is lifted to the upper half-plane, each (x, y) in the planar natural extension is associated to an inward pointing unit tangent vector on a horocycle tangent to x. One could hope to determine if the interval map is of first return type by consideration of the flow in this model. We would very much like to see a method to pass from this cross section to one consisting of unit tangent vectors along the boundary of some fundamental domain for the Fuchsian group at hand.

References

[1] A. Abrams, S. Katok and I. Ugarcovici, *Rigidity and flexibility of entropies of boundary maps associated to Fuchsian groups*, Math. Res. Rep. 3 (2022), 1–19.

[2] R. Adler and L. Flatto, *Geodesic flows, interval maps, and symbolic dynamics*, Bull. Amer. Math. Soc. (N.S.) 25 (1991), no. 2, 229–334.

[3] P. Arnoux, *Le codage du flot géodésique sur la surface modulaire*, Enseign. Math. (2) 40 (1994), no. 1-2, 29–48.

[4] P. Arnoux and T. A. Schmidt, *Cross sections for geodesic flows and α-continued fractions*, Nonlinearity 26 (2013), 711–726.

[5] ———, *Commensurable continued fractions*, Discrete Contin. Dyn. Syst. 34 (2014), no. 11, 4389–4418.

[6] E. Artin, *Ein mechanisches System mit quasi-ergodischen Bahnen*, Abh. Math. Sem. Hamburg 3 (1924) 170–175 (and *Collected Papers*, Springer-Verlag, New York,1982, 499–505).

[7] F. Bonahon, *The geometry of Teichmüller space via geodesic currents*, Invent. Math. 92 (1988), no. 1, 139–162.

[8] K. Calta, C. Kraaikamp and T. A. Schmidt, *Synchronization is full measure for all α-deformations of an infinite class of continued fraction transformations*, Ann. Sc. Norm. Super. Pisa Cl. Sci.(5) Vol. XX (2020), 951–1008.
[9] ———, *Proofs of ergodicity of piecewise Möbius interval maps using planar extensions*, Expo. Math. 42 (2024), no. 4, Paper No. 125575, 39 pp.
[10] K. Calta and T. A. Schmidt, *Continued fractions for a class of triangle groups*, J. Austral. Math. Soc., 93 (2012) 21–42.
[11] C. Carminati, S. Isola and G. Tiozzo, *Continued fractions with $SL(2,\mathbb{Z})$-branches: combinatorics and entropy*, Trans. Amer. Math. Soc. 370 (2018), no. 7, 4927–4973.
[12] C. Carminati and G. Tiozzo, *A canonical thickening of $\mathbb{Q}$ and the entropy of α-continued fraction transformations*, Ergodic Theory Dynam. Systems 32 (2012), no. 4, 1249–1269.
[13] ———, *Tuning and plateaux for the entropy of α-continued fractions*, Nonlinearity 26 (2013), no. 4, 1049–1070.
[14] K. Dajani, C. Kraaikamp, W. Steiner, *Metrical theory for α-Rosen fractions*, J. Eur. Math. Soc. (JEMS) 11 (2009), no. 6, 1259–1283.
[15] M. Denker, G. Keller and M. Urbański, *On the uniqueness of equilibrium states for piecewise monotone mappings*, Studia Math. 97 (1990), no. 1, 27–36.
[16] D. Fried, *Symbolic dynamics for triangle groups*, Invent. Math. 125 (1996), no. 3, 487–521.
[17] S. Katok, I. Ugarcovici, *Symbolic dynamics for the modular surface and beyond*, Bull. Amer. Math. Soc. (N.S.) 44 (2007), no. 1, 87–132.
[18] ———, *Structure of attractors for (a,b)-continued fraction transformations*, J. Mod. Dyn. 4 (2010), no. 4, 637–691.
[19] ——— *Applications of (a,b)-continued fraction transformations* Ergodic Theory Dynam. Systems 32 (2012), no. 2, 755–777.
[20] C. Kraaikamp, *A new class of continued fraction expansions*, Acta Arith. 57 (1991), no. 1, 1–39.
[21] C. Kraaikamp, T. A. Schmidt and I. Smeets, *Natural extensions for λ-Rosen continued fractions*, J. Math. Soc. Japan 62, No. 2 (2010), 649–671
[22] P. Lévy, *Sur le développement en fraction continue d'un nombre choisi au hasard*, Compositio Math. 3 (1936), 286–303.
[23] C. Kraaikamp, T. A. Schmidt and W. Steiner, *Natural extensions and entropy of α-continued fractions*, Nonlinearity 25 (2012), no. 8, 2207–2243.
[24] L. Luzzi and S. Marmi, *On the entropy of Japanese continued fractions*, Discrete Contin. Dyn. Syst. **20** (2008), no. 3, 673–711.
[25] A. Manning, *Dynamics of geodesic and horocycle flows on surfaces of constant negative curvature*, In: "Ergodic theory, symbolic dynamics, and hyperbolic spaces (Trieste, 1989)", T. Bedford, M. Keane and C. Series (eds.), Oxford Sci. Publ., Oxford Univ. Press, New York, (1991), 71–91.
[26] D. Mayer and F. Strömberg, *Symbolic dynamics for the geodesic flow on Hecke surfaces*, J. Mod. Dyn. 2 (2008), no. 4, 581–627.
[27] H. Nakada, *Metrical theory for a class of continued fraction transformations and their natural extensions*, Tokyo J. Math. 4 (1981), 399-426.
[28] ———, *On the Lenstra constant associated to the Rosen continued fractions*, J. Eur. Math. Soc. (JEMS) 12 (2010), no. 1, 55–70.
[29] H. Nakada, S. Ito, and S. Tanaka, *On the invariant measure for the transformations associated with some real continued-fractions*, Keio Engrg. Rep. 30 (1977), no. 13, 159–175.
[30] V. A. Rohlin, *Exact endomorphisms of a Lebesgue space*, Izv. Akad. Nauk SSSR Ser. Mat. 25 (1961), 499–530; Am. Math. Soc. Transl. Ser. 2, 39 (1964)1–36 (Engl. transl.).
[31] C. Series, *The modular surface and continued fractions*, J. London Math. Soc. (2) **31** (1985), 69–80.

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