Unital C_∞-algebras and the real homotopy type of $(r-1)$-connected compact manifolds of dimension $\leq \ell(r-1)+2$

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Abstract. We encode the real homotopy type of an n-dimensional $(r-1)$-connected compact manifold M, $r \geq 2$ into a minimal unital C_∞-structure on $H^*(M, \mathbb{R})$, obtained via a Hodge homotopy transfer of the unital DGCA structure of the small quotient algebra associated with a Hodge decomposition of the de Rham algebra $\mathcal{A}^*(M)$, which has been proposed by Fiorenza-Kawai-Lê-Schwachhöfer in [11]. We prove that if $n \leq \ell(r-1)+2$, with $\ell \geq 4$, the multiplication μ_k on the minimal unital C_∞-algebra $H^*(M, \mathbb{R})$ vanishes for all $k \geq \ell - 1$. This extends the results from [11], extending the bound on the dimension from $5r-3$ the general bound $\ell(r-1)+2$. We also prove a variant of this result, conjectured by Zhou, stating that if $n \leq \ell(r-1)+4$ and $b_r(M) = 1$ then the multiplication μ_k for all $k \geq \ell - 1$ vanishes. This implies two formality results by Cavalcanti [3]. We show that in any dimension n the Harrison cohomology class $[\mu_3] \in \mathrm{HHarr}^{3,-1}(H^*(M,\mathbb{R}), H^*(M,\mathbb{R}))$ is a homotopy invariant of M and the first obstruction to formality, and provide a detailed proof that if $n \leq 4r-1$ this is the only obstruction. Furthermore, we show that in any dimension n the class $[\mu_3]$ and the Bianchi-Massey tensor invented by Crowley-Nordström in [5] define each other uniquely.

Contents

Date: March 29, 2025.

2010 *Mathematics Subject Classification.* Primary: 55P62, Secondary: 57R19, 13D03, 58A10.

Key words and phrases. real homotopy type, Hodge homotopy, unital C_∞-structure, Harrison cohomology, Bianchi-Massey tensor.

1. Introduction

In his seminal papers [33,34], Sullivan provided algebraic models of simply connected compact manifolds M of dimension at least 5, using the smooth differential forms on M.[1] Sullivan's theorem [33, Theorem 13.1] states that the diffeomorphism type of a simply connected manifold M of dimension at least 5 is defined by its real homotopy type, real Pontryagin classes, and by the torsion of M, up to finite ambiguity [33, Theorem 10.4], see also the extension of Sullivan's theorem by Kreck and Triantafillou [23, Theorem 1.1]. Furthermore, Sullivan proved that rational homotopy type of M is captured by the DGCA weak equivalence class of piecewise polynomial differential forms with rational coefficients on M and the real homotopy type by that of de Rham forms [34, Theorems A, B, C, D].

In 1979 Miller showed that any $(r-1)$-connected compact manifold of dimension less than or equal to $4r-2$ is formal [28]. In particular, the real homotopy type of any compact simply connected manifold M of dimension less than or equal 6 is defined by its cohomology ring $H^*(M, \mathbb{R})$. In 2020 Crowley-Nordström, using Sullivan minimal models, showed that the rational homotopy type of $(r-1)$-connected compact manifolds M of dimension less than or equal $(5r-3)$, $r \geq 2$, is determined by its cohomology algebra $H^*(M, \mathbb{Q})$ and by the Bianchi-Massey tensor they introduced in [5]. In the recent paper [29] Nagy and Nordström developed the techniques from [5] further and showed that the formality of a $(r-1)$-connected compact manifold M of dimension up to $(5r-2)$ is equivalent to the vanishing of the Bianchi-Massey tensor and the pentagon Massey tensor, a new homotopy invariant introduced by them.

In 2021 Fiorenza-Kawai-Lê-Schwachhöfer proposed a new method to study the real homotopy type of a compact simply connected smooth manifold by introducing the concept of a Poincaré DGCA admitting a Hodge homotopy [11], see also Definitions 2.1, 2.2 below. They made the crucial observation that a simply connected Poincaré DGCA $\mathcal{A}^*$ of degree n admitting a Hodge homotopy is weak equivalent to a finite dimensional non-degenerate Poincaré DGCA $\mathcal{Q}^*(\mathcal{A}^*)$ of degree n endowed with a Hodge homotopy [22, Theorem 3.10], recalled in Proposition 2.10 below. Using a Hodge homotopy transfer of the DCGA structure of $\mathcal{Q}^*(\mathcal{A}^*)$ to the cohomology algebra $H^*(\mathcal{Q}^*(\mathcal{A}^*)) = H^*(\mathcal{A}^*)$, they proved that if $\mathcal{A}^*$ is a n-dimensional $(r-1)$-connected Poicaré DGCA admitting a Hodge homotopy, $n \leq 5r-3$, $r \geq 2$, then $\mathcal{A}^*$ is C_∞-quasi-isomorphic to a minimal C_∞-algebra with vanishing higher multiplications m_k, for $k \geq 4$ [11, Theorem 4.1]. By Kadeishivili results [17,18,20], see also [21, Theorems 14, 15], the weak equivalence class of the DGCA of piecewise polynomial differential forms on M with rational

[1] The dimension 5 is motivated by Smale's solution of the Poincaré conjecture in dimension at least 5 [31], which opened the way to classification of simply connected differentiable manifolds in dimension 5 [32].

coefficients on M and that of the de Rham forms is captured by the homotopy type of the associated C_∞-algebra obtained by a homotopy transfer. Thus Fiorenza-Kawai-Lê-Schwachhöfer's results provide new information of real homotopy type of n-dimensional $(r-1)$-connected compact manifold M, $n \le 5r-3$, $r \ge 2$, in terms of a minimal C_∞-algebra structure on $H^*(M,\mathbb{R})$. Next, Zhou improved the result from [11] showing that generally, for any $\ell \ge 4$, if $n \le \ell(r-1)+2$, then $\mathcal{A}^*$ is A_∞-quasi-isomorphic to a minimal A_∞-algebra with vanishing higher multiplications m_k, for $k \ge \ell - 1$.

In this paper we show the effectiveness of the techniques from [11] to give a very quick proof of Zhou's result, as well as extending it from A_∞ to C_∞-algebras (Theorem 2.12), and develop the techniques in [11] further to study the real homotopy type of an n-dimensional $(r-1)$-connected compact manifold M, $r \ge 2$, in terms of a minimal unital C_∞-algebra structure on $H^*(M,\mathbb{R})$, whose multiplication μ_2 coincides with the usual multiplication $\cdot$ on $H^*(M,\mathbb{R})$. As a result, we proved a conjecture by Zhou stating that if $n \le \ell(r-1)+4$ and $b_r(M)=1$ then m_k vanishes for $k \ge \ell-1$ [38]. This is a generalization of two formality results by Cavalcanti [3]. We show that the Harrison cohomology class $[\mu_3] \in \mathrm{HHarr}^{3,-1}(H^*(M,\mathbb{R}), H^*(M,\mathbb{R}))$, where μ_3 is the ternary operation on the unital C_∞-algebra $(H^*(M,\mathbb{R}),\cdot,\mu_3,\ldots)$, is a homotopy invariant of M and the first obstruction to its formality. Moreover we show that $[\mu_3]$ and the Bianchi-Massey tensor define each other uniquely. For $\ell = 6$ one finds that if $n \le 6r-4$ then the multiplication μ_k on the unital C_∞-algebra $(H^*(M,\mathbb{R}),\cdot,\mu_3,\ldots)$ vanishes for $k > 4$, so that the unital C_∞-algebra structure is completely encoded into the multiplications μ_3 and μ_4. This complements to Nagy-Nordström's result on the equivalence of the formality of $(r-1)$-connected compact manifolds of dimension $n \le 5r-2$ and the vanishing of the Bianchi-Massey and pentagon Massey tensor [29, Theorem 1.7].

Our paper is organized as follows. In Section 2, we recall the concept of an $(r-1)$-connected Poincaré DGCA $\mathcal{A}^*$ of degree n admitting a Hodge homotopy (Definitions 2.1, 2.2 and 2.9) and the related concept of the small quotient algebra $\mathcal{Q}^*(\mathcal{A}^*)$ (Proposition 2.10). Using a Hodge homotopy transfer of the unital DGCA structure on $\mathcal{Q}^*(\mathcal{A}^*)$ to a minimal unital C_∞-structure on a space $\mathcal{H}^*$ of harmonic elements of $\mathcal{A}^*$, we show that if $n \le \ell(r-1)+2$ with $\ell \ge 4$, then all the higher operations m_k, $k > \ell$, on the minimal unital C_∞-algebra $(\mathcal{H}^*, m_2, m_3, \ldots,)$ vanish; in particular, m_4 vanishes if $n \le 5r-3$ and m_3 vanishes if $n \le 4r-2$ (Theorem 2.12). We then prove the Zhou's conjecture (Theorem 2.19) and derive from it and from Theorem 2.12 various previously known formality results. In Section 3, we transfer the operations m_k on $\mathcal{H}^*$ to operations μ_k on $H^*(\mathcal{A}^*)$ via the canonical linear isomorphism $\mathcal{H}^* \xrightarrow{\sim} H^*(\mathcal{A}^*)$ and prove that the Harrison cohomology class $[\mu_3] \in \mathrm{HHarr}^{3,-1}(H^*(\mathcal{A}^*), H^*(\mathcal{A}^*))$ is a homotopy invariant of $\mathcal{A}^*$ (Lemma 3.1) and the first obstruction to its formality (Lemma 3.2). Furthermore, $H^*(\mathcal{A}^*)$ and $[\mu_3]$ are complete invariants of the DGCA homotopy class of an $(r-1)$ connected Poincaré DGCA of degree $n \le 4r-1$ admitting a

Hodge homotopy (Theorem 3.9). In Section 4, using the spectral sequence symmetric-to-Harrison, we establish the equivalence between the Harrison cohomology class $[\mu_3]$ and the Bianchi-Massey tensor (Theorem 4.4). In the last Section 5, we summarize our results and suggest some problems for future investigations.

2. The small quotient algebra and the unital C_∞-algebra of a $(r-1)$ homologically connected $(r>1)$ compact manifold of dimension up to $(\ell(r-1)+2)$

In this section, we briefly recall the concept of a Poincaré DCGA admitting a Hodge homotopy, introduced and investigated by Fiorenza-Kawai-Lê-Schwachhöfer in [11], which shall be utilized to study unital C_∞-algebras associated with $(r-1)$-connected compact manifolds of dimension up to $\ell(r-1)+2$, with $\ell \geq 4$, in our paper. Then under the assumption the algebra admits a Hodge homotopy[2] we give a very quick proof of a theorem by Zhou on the homotopy type of $(r-1)$ connected Poincaré DGCAs of degree less than or equal to $\ell(r-1)+2$ (Theorem 2.12). This extend the analogous result in [11], the real (or rational) homotopy type of a smooth that was derived under the assumption of degree less than or equal to $5r-3$, corresponding to $\ell = 5$. We also give a proof to a Zhou's conjecture (Theorem 2.19) and derive from it Cavalcanti's formality results (Corollary 2.20).

2.1. Poincaré-DGCA of Hodge type.

Given a vector space V over a field $\mathbb{K}$ we denote by $V^\vee$ the dual space of V.

Definition 2.1. cf. [24, Def. 4.1], cf. [5, Def. 2.7], cf. [11, Def. 2.1]. A *Poincaré DGCA* $(\mathcal{A}^*, d)$ *of degree* n over a field $\mathbb{K}$ is a DGCA $\mathcal{A}^* = \oplus_{k=0}^n \mathcal{A}^k$ whose cohomology ring $H^*(\mathcal{A})$ is finite dimensional over $\mathbb{K}$ and has a *fundamental class* $\int \in (H^n(\mathcal{A}))^\vee$ such that the pairing, defined by using the cup-product,

$$\langle \alpha^k, \beta^l \rangle = \begin{cases} \int \alpha^k \cdot \beta^l & \text{if } k+l=n, \\ 0 & \text{else,} \end{cases} \tag{2.1}$$

is non-degenerate, i.e. $\langle \alpha, H^*(\mathcal{A}) \rangle = 0$ if and only if $\alpha = 0$.

The pairing on $H^*(\mathcal{A})$ induces a pairing on $\mathcal{A}^*$ as follows

$$\langle \alpha^k, \beta^l \rangle = \begin{cases} \int [\alpha^k \cdot \beta^l] & \text{if } k+l=n, \\ 0 & \text{else,} \end{cases} \tag{2.2}$$

where $[\cdot]$ stands for the projection $\mathcal{A}^n \supseteq \ker d \to H^n(\mathcal{A}^*)$. The Poincaré DCGA $(\mathcal{A}^*, d)$ is called *non-degenerate*, if the pairing $\langle \cdot, \cdot \rangle$ on $\mathcal{A}^*$ is non-degenerate.

[2]This assumption is always verified by the de Rham algebra of a compact oriented manifold.

Definition 2.2. Let $(\mathcal{A}^*, d, \cdot, \langle -, - \rangle)$ be a Poincaré DGCA. A *Hodge homotopy* for $(\mathcal{A}^*, d, \cdot, \langle -, - \rangle)$ is a degree -1 operator $d^-\colon \mathcal{A}^* \to \mathcal{A}^{*-1}$ such that:

$$d^- d^- = 0; \quad d^- d d^- = d^-; \quad d d^- d = d; \tag{2.3}$$

$$\langle \mathrm{Im}(d^-), \mathrm{Im}(d^-) \rangle = 0; \quad \langle \mathrm{Im}(\pi_{\mathcal{H}^*}), \mathrm{Im}(d^-) \rangle = 0, \tag{2.4}$$

where

$$\pi_{\mathcal{H}^*} := \mathrm{id}_{\mathcal{A}^*} - [d, d^{-1}]. \tag{2.5}$$

Remark 2.3. It follows from (2.3) that the two endomorphisms $dd^-, d^-d\colon \mathcal{A}^* \to \mathcal{A}^*$ are orthogonal idempotents. So they both, and also $\pi_{\mathcal{H}}$, are projection operators.

Definition 2.4. The graded subspace $\mathcal{H}^* := \mathrm{Im}(\pi_{\mathcal{H}^*})$ is called the *harmonic subspace* for the Hodge homotopy d^-.

Remark 2.5. It follows from the definition that $d\pi_{\mathcal{H}^*} = \pi_{\mathcal{H}^*} d = d^- \pi_{\mathcal{H}^*} = \pi_{\mathcal{H}^*} d^- = 0$, and that we have a direct sum decomposition

$$\mathcal{A}^* = \mathcal{H}^* \oplus \underbrace{dd^- \mathcal{A}^* \oplus d^- d \mathcal{A}^*}_{\mathcal{L}^*_{\mathcal{A}}}$$

with $(\mathcal{L}^*_{\mathcal{A}}, d)$ an acyclic complex and $d^-(\mathcal{L}^*_{\mathcal{A}}) \subseteq \mathcal{L}^*_{\mathcal{A}}$. Also, the composition

$$\mathcal{H}^* \to \ker(d) \to H^*(\mathcal{A})$$

is an isomorphism of graded vector spaces.

Remark 2.6. Since $\int d\omega = 0$, the pairing on $\mathcal{A}^*$ satisfies

$$\langle d\alpha, \beta \rangle = -(-1)^{\deg(\alpha)} \langle \alpha, d\beta \rangle.$$

This implies that the direct sum decomposition $\mathcal{A}^* = \mathcal{H}^* \oplus \mathcal{L}^*_{\mathcal{A}}$ is a orthogonal direct sum decomposition: $\mathcal{A}^* = \mathcal{H}^* \oplus^{\perp} \mathcal{L}^*_{\mathcal{A}}$.

Example 2.7. [11, Remark 2.7 (1)] Let (M, g) be an n-dimensional compact oriented Riemannian manifold, let $\mathcal{A}^* = \mathcal{A}^*(M)$ the de Rham algebra of M, let $\mathcal{H}^*$ be the space of harmonic forms and let $G = \Delta_g^{-1}$ be the Green operator of (M, g). Then integration over M makes $\mathcal{A}^*$ a Poincaré DGCA of degree n, and the operator d^- defined by

$$\begin{cases} d^-\big|_{\mathcal{H}^*} = 0 \\ \\ d^-\big|_{d\mathcal{A}^* \oplus d^* \mathcal{A}} = G d^* \end{cases}$$

is a Hodge homotopy.

Remark 2.8. A richer structure is obtained if instead of a single Hodge homotopy one has a pair of compatible Hodge homotopies. This is what happens, for instance, for Kähler manifolds, and the essential ingredient in Deligne-Griffiths-Morgan-Sullivan proof of formality of compact Kähler manifolds [6].

Definition 2.9. A DGCA $\mathcal{A}^*$ is said to be $(r-1)$-connected, with $r \geq 2$ if $H^0(\mathcal{A}) = \mathbb{K}$ and $H^k(\mathcal{A}) = 0$ for $1 \leq k \leq r-1$.

A crucial result from [11] is the following.

Proposition 2.10. *(cf.* [11, Theorem 3.10, (4.1)]*). Let $r \geq 2$ and let $\mathcal{A}^*$ be a $(r-1)$-connected Poincaré DGCA of degree n. If $\mathcal{A}^*$ admits a Hodge homotopy, then there exists a nondegenerate finite dimensional Poincaré DGCA $(\mathcal{Q}^*, d_{\mathcal{Q}})$ of degree n endowed with a Hodge homotopy $d^-_{\mathcal{Q}}$, homotopy equivalent to $\mathcal{A}^*$ via an explicit zig-zag of quasi-isomorphisms and such that*

$$\begin{cases} \mathcal{Q}^0 = \mathbb{K} & \\ \mathcal{Q}^k = 0, & 1 \leq k \leq r-1 \\ \mathcal{Q}^k = \mathcal{H}^k, & r \leq k \leq 2r-2 \\ \mathcal{Q}^k = \mathcal{H}^k \oplus \mathcal{L}^k_{\mathcal{Q}} & 2r-1 \leq k \leq n-2r+1 \\ \mathcal{Q}^k = \mathcal{H}^k, & n-2r+2 \leq k \leq n-r \\ \mathcal{Q}^k = 0, & n-r+1 \leq k \leq n-1 \\ \mathcal{Q}^n = \mathcal{H}^n \cong \mathbb{K}, & \end{cases} \tag{2.6}$$

with $\mathcal{L}^*_{\mathcal{Q}} = d_{\mathcal{Q}} d^-_{\mathcal{Q}} \mathcal{Q}^* \oplus d^-_{\mathcal{Q}} d_{\mathcal{Q}} \mathcal{Q}^*$ an acyclic subcomplex of $(\mathcal{Q}^*, d_{\mathcal{Q}})$.

Note that the Poincaré DGCA $(\mathcal{Q}^*, d_{\mathcal{Q}})$ in Proposition 2.10 is defined uniquely by a Hodge homotopy d^- for $(\mathcal{A}^*, d)$. It is called the *small quotient algebra of* $(\mathcal{A}^*, d, d^-)$ in[11, Defintion 3.2].

Remark 2.11. Even when $(\mathcal{A}^*, d)$ is formal, one should not expect $\mathcal{L}^*_{\mathcal{Q}}$ to be zero. That is, also for formal DGCAs admitting Hodge homotopies the small quotient algebra does not generally consist only of harmonic forms.

2.2. **The unital C_∞-algebra associated with $\mathcal{Q}^*$.** Let $j\colon \mathcal{H}^* \to \mathcal{Q}^*$ be the inclusion of the graded subspace of harmonic elements into $\mathcal{A}^*$. Then $j\colon (\mathcal{H}^*, d_{\mathcal{H}} = 0) \to (\mathcal{Q}^*, d_{\mathcal{Q}})$ is a quasi-isomorphism of cochain complexes. By definition of Hodge homotopy, $(d^-_{\mathcal{Q}}, \pi_{\mathcal{H}^*}, j)$ are homotopy data of cochain complexes, i.e. we have the following commutative diagram

$$d^-_{\mathcal{Q}} \circlearrowright (\mathcal{Q}^*, d_{\mathcal{Q}}) \underset{j}{\overset{\pi_{\mathcal{H}^*}}{\rightleftarrows}} (\mathcal{H}^*, d_{\mathcal{H}} = 0).$$

The DGCA structure on $\mathcal{Q}^*$ is a particular instance of a *unital C_∞-algebra* with unit $1 \in \mathbb{K} = \mathcal{Q}^0$, and the homotopy $d^-_{\mathcal{Q}}$ satisfies side conditions [4, Theorems 10, 12]

$$\pi_{\mathcal{H}^*} \circ d^-_{\mathcal{Q}} = 0,\ (d^-_{\mathcal{Q}})^2 = 0,\ d^-_{\mathcal{Q}}(1) = 0. \tag{2.7}$$

Thus we can transfer the unital DGCA structure of $(\mathcal{Q}^*, d_{\mathcal{Q}})$ to a minimal unital C_∞-structure on its harmonic subspace $\mathcal{H}^*$ [4, Theorems 10, 12]. Since this homotopy transfer is obtained by means of a Hodge homotopy, we will refer to it as a *Hodge homotopy transfer.* The minimal unital C^∞-structure on $\mathcal{H}^*$ is defined by a sequence of operations $m_k : \otimes^k \mathcal{H}^* \to \mathcal{H}^*[2-k]$, $k \geq 2$, whose unitality means that

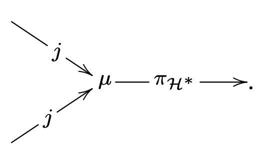
(2.8)
$$m_2(1,a) = m_2(a,1) = a, \qquad \text{and} \qquad m_n(a_1,\dots,1,\dots,a_n) = 0 \quad n \geq 3,$$

and which can be conveniently determined by the Kontsevich-Soilbeman formula for homotopy transfer [22] by [4, Theorem 12], cf. [7, Lemma 48], [36, Theorem 5.2]. Using these formulas we obtain the following short proof of the main theorem from [38], extending our previous result with Kawai and Schwachhöfer [11, Theorem 4.1], whose assertion is the next to last statement of Theorem 2.12.

Theorem 2.12. *Let $r \geq 2$ and let $\mathcal{A}^*$ be a $(r-1)$-connected Poincaré DGCA of degree n admitting a Hodge homotopy. If $n \leq \ell(r-1)+2$, with $\ell \geq 4$, then $\mathcal{H}^*$ carries a minimal unital C_∞-algebra structure making it C_∞-quasi-isomorphic to $\mathcal{A}^*$, whose multiplication m_2 is $m_2(\alpha,\beta) = \pi_{\mathcal{H}^*}(\alpha\cdot\beta)$, and whose multiplications m_k vanish for $k \geq \ell-1$. In particular, all the multiplications m_k with $k \geq 5$ vanish if $n \leq 6r-4$ all the multiplications m_k with $k \geq 4$ vanish if $n \leq 5r-3$. Also, all the multiplications m_k with $k \geq 3$ vanish if $n \leq 4r-2$, and so $\mathcal{A}^*$ is formal in this case.*

Proof. The DGCA $\mathcal{Q}^*$ is equivalent to $\mathcal{A}^*$. Hence, to prove Theorem 2.12 we need only to show that $\mathcal{H}^*$ carries a minimal unital C_∞-algebra structure whose multiplication m_2 is $\mu_2(\alpha,\beta) = \pi_{\mathcal{H}^*}(\alpha\cdot\beta)$ and whose multiplications m_k vanish for $k \geq \ell-1$, making it quasi-isomorphic to $\mathcal{Q}^*$. We can show this by means of the homotopy transfer theorem, using [4, Theorem 10]. This endows $\mathcal{H}^*$ with an explicit unital minimal C_∞-algebra structure making it C_∞-quasi-isomorphic to $\mathcal{Q}^*$, and so to prove the statement in the theorem we only need to show that this unital minimal C_∞-algebra structure actually has $\mu_2(\alpha,\beta) = \pi_{\mathcal{H}^*}(\alpha\cdot\beta)$ and m_k vanishing for $k \geq \ell-1$.

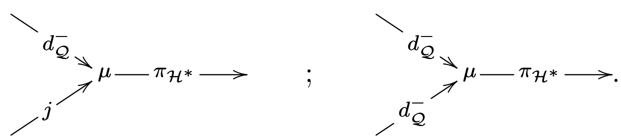
One has a convenient tree summation formula to express the higher multiplications m_k obtained by homotopy transfer, see [4, Theorem 12], [22]. Namely, m_k can be expressed as a sum over rooted trivalent trees with k leaves. Each tail edge of such a tree is decorated by the inclusion $j\colon \mathcal{H}^* \hookrightarrow \mathcal{Q}^*$, each internal edge is decorated by the operator $d^-_{\mathcal{Q}}\colon \mathcal{Q}^* \to \mathcal{Q}^{*-1}$ and the root edge is decorated by the operator $\pi_{\mathcal{H}^*}\colon \mathcal{Q}^* \to \mathcal{H}^*$; every internal vertex is decorated by the multiplication $\cdot$ in $\mathcal{Q}^*$. The only graph with two leaves appearing in the tree summation formula is



This gives the announced result for m_2, so to conclude the proof we only need to show that m_k vanishes for $k \geq \ell-1$. Since the multiplications m_k define a unital C_∞-algebra structure, the element $m_k(\alpha_1,\dots,\alpha_k)$ vanishes if any of the homogeneous entries α_i has degree zero. We can therefore assume $\deg(\alpha_i) \geq r$ for any $i = 1, \ldots, k$. Then $m_k(\alpha_1, \ldots, \alpha_k)$ will have degree h with $h \geq kr + (2-k) = k(r-1) + 2$. If $k \geq \ell - 1$ we have $h \geq (\ell-1)(r-1)+2 = \ell(r-1)+2-(r-1) \geq n-r+1$. The only possibly nonzero result is then obtained with $h = n$. Let us compute

$$\int m_k(\alpha_1, \ldots, \alpha_k).$$

Since $k \geq \ell - 1 \geq 3$, the root of a tree T contributing to m_k will have a neighbourhood of one of the following forms:



In degree n all elements in $\mathcal{Q}^*$ are harmonic, so the projection $\pi_{\mathcal{H}^*}$ is the identity in degree n. But then the orthogonality relations (2.4) tell us that, if we denote by m_T the contribution from the tree T to m_k, then we have

$$\int m_T(\alpha_1, \ldots, \alpha_k) = 0.$$

Since $\int : \mathcal{Q}^n \to \mathbb{K}$ is an isomorphism, this gives $m_T(\alpha_1, \ldots, \alpha_k) = 0$ for any tree T contributing to m_k and so $m_k(\alpha_1, \ldots, \alpha_k) = 0$.

□

Corollary 2.13 (Miller)**.** *Let $r \geq 2$ and let M be an $(r-1)$-connected compact manifold of dimension less than or equal to $4r-2$. Then M is formal.*

Remark 2.14. (1) The original proof by Miller [28] using the Quillen theory and the other two proofs of Miller's results using Sullivan minimal model by Fernandez-Munos [10] and Felix-Oprea-Tanré [9, Proposition 3.10, p. 110], as well as Zhou's proof of [38, Theorem 3.2] are considerably longer and less elementary than our proof of Theorem 2.12. This makes the proof of Theorem 2.12 a good example of the effectiveness of the use of Hodge homotopies in real homotopy theory.

(2) Recall that we have a linear isomorphism $\mathcal{H}^* \to \ker(d_{\mathcal{Q}}) \to H^*(\mathcal{Q})$ mapping each element in $\mathcal{H}^*$ to its cohomology class. One can use it to transfer the unital C_∞-structure on $\mathcal{H}^*$ from Theorem 2.12 to a unital C_∞-structure on $H^*(\mathcal{Q})$. This transferred structure is a *unital C_∞-enrichment* of the DGCA-structure on $H^*(\mathcal{Q})$ induced by the multiplication $\cdot$ on $\mathcal{Q}^*$. Indeed, let $\iota\colon H^*(\mathcal{Q}) \to \mathcal{H}^*$ be the inverse of $[-]\colon \mathcal{H}^* \to H^*(\mathcal{Q})$, and let a, b be two cohomology classes in $H^*(\mathcal{Q})$. Let $\hat{a}, \hat{b} \in \mathcal{Q}^*$ be such that $[\hat{a}] = a$ and $[\hat{b}] = b$, and let $\alpha = \pi_{\mathcal{H}^*}(\hat{a})$ and $\beta = \pi_{\mathcal{H}^*}(\hat{b})$, respectively. Then

$$\alpha = \hat{a} - d_{\mathcal{Q}}^- d_{\mathcal{Q}} \hat{a} - d_{\mathcal{Q}} d_{\mathcal{Q}}^- \hat{a} = \hat{a} - d_{\mathcal{Q}} d_{\mathcal{Q}}^- \hat{a},$$

and so $[\alpha] = a$. Therefore $\iota(a) = \alpha$. Similarly, $\iota(b) = \beta$. Therefore, one finds

$$\begin{aligned}[m_2(\iota(a), \iota(b))] &= [m_2(\alpha, \beta)] = [\pi_{\mathcal{H}^*}(\alpha \cdot \beta)] \\ &= [\alpha \cdot \beta - d_{\mathcal{Q}}^- d_{\mathcal{Q}}(\alpha \cdot \beta) - d_{\mathcal{Q}} d_{\mathcal{Q}}^-(\alpha \cdot \beta)] \\ &= [\alpha \cdot \beta] = [\alpha] \cdot [\beta] = a \cdot b.\end{aligned}$$

2.3. **Zhou's conjecture and two Cavalcanti's formality results.** In this subsection we prove Theorem 2.19, which was conjectured by Zhou in [38]. This theorem is a generalization of two Cavalcanti's formality results (Corollary 2.20) and a variation of Theorem 2.12 in the previous subsection.

We begin with the following remarks on operations m_k in our minimal unital C_∞-algebra obtained from a Hodge homotopy and the associated small quotient algebra $(\mathcal{Q}^*, \cdot, d_{\mathcal{Q}}, d_{\mathcal{Q}}^-)$.

Remark 2.15. Let $r \geq 2$ and let $\mathcal{A}^*$ be a $(r-1)$-connected Poincaré DGCA of degree n admitting a Hodge homotopy. The explicit expression for m_k are derived by using [27, Theorem 3.4], [4, Theorem 12]. One considers the recursive formula

$$\begin{aligned}&\widehat{m}_2(\alpha_1, \alpha_2) = \alpha_1 \cdot \alpha_2, \\ &\widehat{m}_k(\alpha_1, \ldots, \alpha_k) = (-1)^{k-1} d_{\mathcal{Q}}^- \widehat{m}_{k-1}(\alpha_1, \ldots, \alpha_{k-1}) \cdot \alpha_k \\ &\qquad - (-1)^{k \deg(\alpha_1)} \alpha_1 \cdot d_{\mathcal{Q}}^- \widehat{m}_{k-1}(\alpha_2, \cdots, \alpha_k) \\ &\qquad - \sum_{i=2}^{k-2} (-1)^{\nu} d_{\mathcal{Q}}^- \widehat{m}_i(\alpha_1, \ldots, \alpha_i) \cdot d_{\mathcal{Q}}^- \widehat{m}_{k-i}(\alpha_{i+1}, \ldots, \alpha_k),\end{aligned} \tag{2.9}$$

where the α_i's are element in $\mathcal{H}^* \subseteq \mathcal{Q}^*$ and $\cdot$ is the multiplication in the DGCA $(\mathcal{Q}^*, d_{\mathcal{Q}})$, and where $\nu = i + (k-i-1)(\deg(\alpha_1) + \cdots + \deg(\alpha_i))$. Then the multiplication m_k is defined by

$$m_k(\alpha_1, \ldots, \alpha_k) = \pi_{\mathcal{H}} \left(\widehat{m}_k(\alpha_1, \ldots, \alpha_k) \right). \tag{2.10}$$

One inductively sees that for any $k \geq 3$ and for any $\alpha \in \mathcal{H}^r$, one has

$$\widehat{m}_k(\alpha, \alpha, \ldots, \alpha) = 0. \tag{2.11}$$

The base of the induction are given by $k = 3$ and $k = 4$. For $k = 3$, (2.9) gives

$$\widehat{m}_3(\alpha, \beta, \gamma) = d_{\mathcal{Q}}^-(\alpha \cdot \beta) \cdot \gamma - (-1)^{\deg \alpha} \alpha \cdot d_{\mathcal{Q}}^-(\beta \cdot \gamma) \tag{2.12}$$

and one immediately sees that $m_3(\alpha, \alpha, \alpha) = 0$ both when r is even and when r is odd. For $k = 4$, (2.10) together with the vanishing we just proved for $\widehat{m}_3(\alpha, \alpha, \alpha)$, gives

$$\widehat{m}_4(\alpha, \alpha, \alpha, \alpha) = -d_{\mathcal{Q}}^-(\alpha^2) \cdot d_{\mathcal{Q}}^-(\alpha^2)$$

and this vanishes both when $\deg(\alpha)$ is odd, since $\alpha^2 = 0$ in this case, and when $\deg(\alpha)$ is even, since $\deg(d_{\mathcal{Q}}^-(\alpha^2))$ is odd in this case. For an $k > 4$ one has that both $k-1$ and at least one of the two indices (i, j) subject to the constrain $i + j = k$ have to be greater or equal to 3, so the inductive assumption applies. As an immediate consequence, one has

$$m_k(\alpha, \alpha, \ldots, \alpha) = 0 \tag{2.13}$$

for every $k \geq 3$.

Lemma 2.16. *Let $r \geq 2$ and let $\mathcal{A}^*$ be a $(r-1)$-connected Poincaré DGCA of degree n admitting a Hodge homotopy. If $n \leq \ell(r-1)+3$, then the multiplication $m_{\ell-1}(\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1})$ on $\mathcal{H}^*$ is zero unless $\deg(\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}) = (r, r, \ldots, r)$. Moreover, for any $\alpha \in \mathcal{H}^r$ we have $m_{\ell-1}(\alpha, \alpha, \ldots, \alpha) = 0$.*

Proof. We know from Theorem 2.12 that $m_{\ell-1}$ is zero if $n \leq \ell(r-1) + 2$, so we may assume $n = \ell(r-1) + 3$. By unitality, $m_{\ell-1}(\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}) = 0$ if one among the $\deg(\alpha_i)$'s is zero, so we may assume $\deg(\alpha_i) \geq r$ for $i = 1, \ldots, \ell-1$. If $\deg(\alpha_{i_0}) > r$ for some i_0, then $m_{\ell-1}(\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1})$ is an element in $\mathcal{H}^k$ with $k = \sum_i \deg(\alpha_i) - \ell + 3 > (\ell-1)r - \ell + 3 = n - r$, and so the only possibility for $m_{\ell-1}(\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1})$ to be nonzero is that $k = n$. But in this case $m_{\ell-1}(\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1})$ will vanish by the orthogonality relations. So we conclude that the only possibility for a nonzero $m_{\ell-1}(\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1})$ is $\deg(\alpha_i) = r$ for every $i = 1, \ldots, \ell-1$. Finally, $m_{\ell-1}(\alpha, \alpha, \ldots, \alpha) = 0$ is equation (2.13). □

Lemma 2.17. *Let $r \geq 2$ and let $\mathcal{A}^*$ be a $(r-1)$-connected Poincaré DGCA of degree n admitting a Hodge homotopy. If $n \leq \ell(r-1) + 4$, then the multiplication $m_\ell(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ on $\mathcal{H}^*$ is zero unless $\deg(\alpha_1, \alpha_2, \ldots, \alpha_\ell) = (r, r, \ldots, r)$. Moreover, for any $\alpha \in \mathcal{H}^r$ we have $m_\ell(\alpha, \alpha, \ldots, \alpha) = 0$.*

Proof. We know from Theorem 2.12 that m_ℓ is zero if $n \leq (\ell+1)(r-1)+2$, and since $r \geq 2$ we have $\ell(r-1)+3 \leq (\ell+1)(r-1)+2$. So we may assume $n = \ell(r-1)+4$. By unitality, $m_\ell(\alpha_1, \alpha_2, \ldots, \alpha_\ell) = 0$ if any of the $\deg(\alpha_i)$'s is zero, so we may assume $\deg(\alpha_i) \geq r$ for any $i = 1, \ldots, \ell$. If $\deg(\alpha_{i_0}) > r$ for some i_0, then $m_\ell(\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \mathcal{H}^k$ with $k > \ell r + 2 - \ell \geq n - r$ and so the only possibility for $m_\ell(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ to be nonzero is that $k = n$. But in this case $m_\ell(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ will vanish by the orthogonality relations. Finally, $m_\ell(\alpha, \alpha, \ldots, \alpha) = 0$ is equation (2.13).

□

Lemma 2.18. *Let $r \geq 2$ and let $\mathcal{A}^*$ be a $(r-1)$-connected Poincaré DGCA of degree $\ell(r-1)+4$ admitting a Hodge homotopy. The multiplication $m_{\ell-1}(\alpha_1, \ldots, \alpha_{\ell-1})$ is zero unless $\deg(\alpha_i) = r$ for all $i = 1, \ldots, \ell-1$ or $\deg(\alpha_i) = r$ for all $i = 1, \ldots, \ell-1$ except a single index i_0 for which one has $\deg(\alpha_{i_0}) = r+1$. Moreover, for any $\alpha \in \mathcal{H}^r$ we have $m_{\ell-1}(\alpha, \alpha, \ldots, \alpha) = 0$.*

Proof. By unitality, $m_{\ell-1}(\alpha_1, \ldots, \alpha_{\ell-1}) = 0$ if one among the $\deg(\alpha_i)$'s is zero, so we may assume $\deg(\alpha_i) \geq r$ for every $i = 1, \ldots, \ell-1$. If there is one index i_0 with $\deg(\alpha_{i_0}) \geq r+2$ or there are two indices i_0, i_1 with $\deg(\alpha_{i_j}) \geq r+1$, then $m_{\ell-1}(\alpha_1, \ldots, \alpha_{\ell-1})$ is an element in $\mathcal{H}^k$ with $k = \sum_i \deg(\alpha_i) - \ell + 1 \geq (\ell-1)r + 4 - (\ell-1) > \ell(r-1) + 4 - r = n - r$, and so the only possibility for $m_{\ell-1}(\alpha_1,\dots,\alpha_{\ell-1})=0$ to be nonzero is that $k=n$. In this case, $m_{\ell-1}(\alpha_1,\dots,\alpha_{\ell-1})=0$ will vanish by the orthogonality relations. Finally, $m_{\ell-1}(\alpha,\alpha,\dots,\alpha)=0$ is equation (2.13). This completes the proof of Lemma 2.18. □

Theorem 2.19 (Zhou's conjecture)**.** *Let $r\geq 2$ and let $\mathcal{A}^*$ be a $(r-1)$-connected Poincaré DGCA of degree n admitting a Hodge homotopy. If $n\leq \ell(r-1)+4$, with $\ell\geq 4$ and $b_r=1$, then $\mathcal{H}^*$ carries a minimal unital C_∞-algebra structure making it C_∞-quasi-isomorphic to $\mathcal{A}^*$, whose multiplication m_2 is $m_2(\alpha,\beta)=\pi_{\mathcal{H}^*}(\alpha\cdot\beta)$, and whose multiplications m_k vanish for $k\geq \ell-1$. In particular, all the multiplications m_k with $k\geq 5$ vanish if $n\leq 6r-2$ all the multiplications m_k with $k\geq 4$ vanish if $n\leq 5r-1$. Also, all the multiplications m_k with $k\geq 3$ vanish if $n\leq 4r$, and so $\mathcal{A}^*$ is formal in this case.*

Proof. If $n\leq \ell(r-1)+2$ this is Theorem 2.12. So there are only two cases to be considered: $n=\ell(r-1)+3$ and $\ell(r-1)+4$.

Case 1. If $n=\ell(r-1)+3$, we proceed as follows. Let x be a linear generator of $\mathcal{H}^r$. By Lemma 2.16, the only possible nonzero $m_{\ell-1}(\alpha_1,\dots,\alpha_{\ell-1})$'s are those with $\deg(\alpha_i)=r$ for $i=1,\dots,\ell-1$. So we are reduced to computing $m_{\ell-1}(x,x,\dots,x)$, and this is zero by Lemma 2.16 again. Finally, all the higher multiplications m_k with $k\geq\ell$ vanish due to Theorem 2.12, since $\ell(r-1)+3\leq(\ell+1)(r-1)+2$ for $r\geq 2$.

Case 2. If $n=\ell(r-1)+4$, we proceed as follows. Let x be a linear generator of $\mathcal{H}^r$. By Lemma 2.18, the multiplication $m_{\ell-1}(\alpha_1,\dots,\alpha_{\ell-1})$ is zero unless
(i) $\deg(\alpha_i)=r$ for all $i=1,\dots,\ell-1$,
(ii) or $\deg(\alpha_i)=r$ for all $i=1,\dots,\ell-1$ except a single index i_0 for which one has $\deg(\alpha_{i_0})=r+1$.

Case 2, step 1. We start by showing that $m_{\ell-1}$ vanishes.

In the first case (i) we are reduced to computing $m_{\ell-1}(x,x,\dots,x)$, and this vanishes by equation (2.13).

In the second case (ii) we are reduced to computing $m_{\ell-1}(x,\dots,x,v,x,\dots,x)$ with $v\in\mathcal{H}^{r+1}$, and we want to show that $m_{\ell-1}(x,\dots,x,v,x,\dots,x)=0$. Since $\deg(\widehat{m}_{\ell-1}(x,\dots,x,v,x,\dots,x))=\ell(r-1)+4-r=n-r$, from equation (2.10) we have $m_{\ell-1}(x,\dots,x,v,x,\dots,x)=\widehat{m}_{\ell-1}(x,\dots,x,v,x,\dots,x)$, so we want to show that $\widehat{m}_{\ell-1}(x,\dots,x,v,x,\dots,x)=0$. By nondegeneracy of the pairing, this is equivalent to

$$\int x\cdot\widehat{m}_{\ell-1}(x,\dots,x,v,x,\dots,x)=0.$$

Let us introduce the notation

$$\{x\}_a=(\underbrace{x,x,\dots,x}_{a});\qquad \{x,v,x\}_{a,b}=(\underbrace{x,\dots,x}_{a},v,\underbrace{x,\dots,x}_{b}),$$

where a and b are possibly zero. With this notation, what we want to prove is

$$\int x \cdot \widehat{m}_{\ell-1}(\{x, v, x\}_{a,\ell-2-a}) = 0$$

for any $a = 0, \ldots, \ell - 2$.

• We provide a detailed proof in the generic case $2 \le a \le \ell - 4$. Notice that we have $\ell \ge 6$ in this case. The remaining special cases with $a = 0, 1, \ell-3, \ell-2$ and/or with $\ell = 4, 5$ are handled similarly, so we will be more sketchy.

By (2.9) and by the vanishing (2.11), we have

$$\begin{aligned}
\widehat{m}_{\ell-1}(\{x, v, x\}_{a,\ell-2-a}) &= (-1)^{\ell-2} d^-_{\mathcal{Q}}\widehat{m}_{\ell-2}(\{x, v, x\}_{a,\ell-3-a}) \cdot x \\
&\quad - (-1)^{(\ell-1)r} x \cdot d^-_{\mathcal{Q}}\widehat{m}_{\ell-2}(\{x, v, x\}_{a-1,\ell-2-a}) \\
&\quad - \sum_{i=2}^{a} (-1)^{\nu} d^-_{\mathcal{Q}}\widehat{m}_i(\{x\}_i) \cdot d^-_{\mathcal{Q}}\widehat{m}_{\ell-1-i}(\{x, v, x\}_{a-i,\ell-2-a}) \\
&\quad - \sum_{i=a+1}^{\ell-3} (-1)^{\nu} d^-_{\mathcal{Q}}\widehat{m}_i(\{x, v, x\}_{a,i-a-1}) \cdot d^-_{\mathcal{Q}}\widehat{m}_{\ell-1-i}(\{x\}_{\ell-1-i}) \\
&= (-1)^{\ell} d^-_{\mathcal{Q}}\widehat{m}_{\ell-2}(\{x, v, x\}_{a,\ell-3-a}) \cdot x \\
&\quad - (-1)^{(\ell-1)r} x \cdot d^-_{\mathcal{Q}}\widehat{m}_{\ell-2}(\{x, v, x\}_{a-1,\ell-2-a}) \\
&\quad - d^-_{\mathcal{Q}}(x^2) \cdot d^-_{\mathcal{Q}}\widehat{m}_{\ell-3}(\{x, v, x\}_{a-2,\ell-2-a}) \\
&\quad - (-1)^{(\ell-1)r+\ell} d^-_{\mathcal{Q}}\widehat{m}_{\ell-3}(\{x, v, x\}_{a,\ell-4-a}) \cdot d^-_{\mathcal{Q}}(x^2).
\end{aligned}$$

so we want to prove that

$$\begin{aligned}
&(-1)^{\ell} \int x \cdot d^-_{\mathcal{Q}}\widehat{m}_{\ell-2}(\{x, v, x\}_{a,\ell-3-a}) \cdot x \\
&- (-1)^{(\ell-1)r} \int x^2 \cdot d^-_{\mathcal{Q}}\widehat{m}_{\ell-2}(\{x, v, x\}_{a-1,\ell-2-a}) \\
&- \int x \cdot d^-_{\mathcal{Q}}(x^2) \cdot d^-_{\mathcal{Q}}\widehat{m}_{\ell-3}(\{x, v, x\}_{a-2,\ell-2-a}) \\
&- (-1)^{(\ell-1)r+\ell} \int x \cdot d^-_{\mathcal{Q}}\widehat{m}_{\ell-3}(\{x, v, x\}_{a,\ell-4-a}) \cdot d^-_{\mathcal{Q}}(x^2) = 0.
\end{aligned}$$

This is trivially satisfied if r is odd, since in this case $x^2 = 0$.

For an even r, taking into account that in this case $\deg(d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-3}(\{x,v,x\}_{a,\ell-4-a})) \equiv \ell+1 \mod 2$, it reduces to

$$\begin{aligned}
&(-1)^{\ell}\int x^2\cdot d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-2}(\{x,v,x\}_{a,\ell-3-a})\\
&-\int x^2\cdot d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-2}(\{x,v,x\}_{a-1,\ell-2-a})\\
&-\int x\cdot d_{\mathcal{Q}}^{-}(x^2)\cdot d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-3}(\{x,v,x\}_{a-2,\ell-2-a})\\
&+\int x\cdot d_{\mathcal{Q}}^{-}(x^2)\cdot d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-3}(\{x,v,x\}_{a,\ell-4-a})=0.
\end{aligned}\tag{2.14}$$

We have

$$x^2=\pi_{\mathcal{H}}(x^2)+d_{\mathcal{Q}}d_{\mathcal{Q}}^{-}(x^2)$$

and

$$\widehat{m}_{\ell-2}(\{x,v,x\}_{a,b})=\pi_{\mathcal{H}}\widehat{m}_{\ell-2}(\{x,v,x\}_{a,b})+d_{\mathcal{Q}}^{-}d_{\mathcal{Q}}\widehat{m}_{\ell-2}(\{x,v,x\}_{a,b})+d_{\mathcal{Q}}d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-2}(\{x,v,x\}_{a,b}).$$

The orthogonality relations (2.4), the recursive formula (2.9) and the vanishing (2.11), together with $d_{\mathcal{Q}}^{-}(x^2)\cdot d_{\mathcal{Q}}^{-}(x^2)=0$ since $\deg(d_{\mathcal{Q}}^{-}(x^2))$ is odd, then give

$$\begin{aligned}
(-1)^{\ell}\int x^2\cdot d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-2}(\{x,v,x\}_{a,\ell-3-a}) &= (-1)^{\ell}\int d_{\mathcal{Q}}d_{\mathcal{Q}}^{-}(x^2)\, d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-2}(\{x,v,x\}_{a,\ell-3-a})\\
&= (-1)^{\ell}\int d_{\mathcal{Q}}^{-}(x^2)\cdot d_{\mathcal{Q}}d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-2}(\{x,v,x\}_{a,\ell-3-a})\\
&= (-1)^{\ell}\int d_{\mathcal{Q}}^{-}(x^2)\cdot \widehat{m}_{\ell-2}(\{x,v,x\}_{a,\ell-3-a})\\
&= -\int x\cdot d_{\mathcal{Q}}^{-}(x^2)\cdot d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-3}(\{x,v,x\}_{a,\ell-4-a})\\
&\quad -(-1)^{\ell}\int x\cdot d_{\mathcal{Q}}^{-}(x^2)\cdot d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-3}(\{x,v,x\}_{a-1,\ell-3-a})
\end{aligned}$$

and, similarly,

$$\begin{aligned}
-\int x^2\cdot d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-2}(\{x,v,x\}_{a-1,\ell-2-a}) &= (-1)^{\ell}\int x\cdot d_{\mathcal{Q}}^{-}(x^2)\cdot d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-3}(\{x,v,x\}_{a-1,\ell-3-a})\\
&\quad +\int x\cdot d_{\mathcal{Q}}^{-}(x^2)\cdot d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-3}(\{x,v,x\}_{a-2,\ell-2-a}).
\end{aligned}$$

Substituting these into the left hand side of (2.14) we see that (2.14) is indeed satisfied.

- For $a=0$ and $\ell\geq 6$, we have to prove that

$$\begin{aligned}
&(-1)^{\ell}\int x\cdot d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-2}(\{x,v,x\}_{0,\ell-3})\cdot x\\
&-(-1)^{(\ell-1)r+\ell}\int x\cdot d_{\mathcal{Q}}^{-}\widehat{m}_{\ell-3}(\{x,v,x\}_{0,\ell-4})\cdot d_{\mathcal{Q}}^{-}(x^2)=0.
\end{aligned}$$

Again, this is trivially satisfied if r is odd, while for an even r it is proved by the same argument used in the proof of (2.14).

The case $a = \ell - 2$ and $\ell \geq 6$ is perfectly analogous.

• For $a = 1$ and $\ell \geq 6$, we have to prove that

$$\begin{aligned}
&(-1)^{\ell} \int x \cdot d_Q^- \widehat{m}_{\ell-2}(\{x, v, x\}_{1,\ell-4}) \cdot x \\
&- (-1)^{(\ell-1)r} \int x^2 \cdot d_Q^- \widehat{m}_{\ell-2}(\{x, v, x\}_{0,\ell-3}) \\
&- (-1)^{(\ell-1)r+\ell} \int x \cdot d_Q^- \widehat{m}_{\ell-3}(\{x, v, x\}_{1,\ell-5}) \cdot d_Q^-(x^2) = 0,
\end{aligned}$$

and again this is trivial for an odd r, while for an even r it is proved by the same argument used in the proof of (2.14). The case $a = 1$ and $\ell \geq 6$ is perfectly analogous.

• For $\ell = 5$ we have four cases to consider; namely, (v, x, x, x), (x, v, x, x), (x, x, v, x) and (x, x, x, v). The fourth case is analogous to the first one, and the third one is analogous to the second one, so we need only investigate two cases. For the first one we have

$$\widehat{m}_4(v, x, x, x) = -d_Q^- \widehat{m}_3(v, x, x) \cdot x + d_Q^-(vx) \cdot d_Q^-(x^2)$$

so we want to prove

$$-\int x \cdot d_Q^- \widehat{m}_3(v, x, x) \cdot x + \int x \cdot d_Q^-(vx) \cdot d_Q^-(x^2) = 0.$$

Again this is trivial for an odd r, while for an even r we have

$$\begin{aligned}
-\int x \cdot d_Q^- \widehat{m}_3(v, x, x) \cdot x &= -\int x^2 \cdot d_Q^- \widehat{m}_3(v, x, x) \\
&= -\int d_Q^-(x^2) \cdot \widehat{m}_3(v, x, x) \\
&= -\int d_Q^-(x^2) \cdot \big(d_Q^-(vx) \cdot x + v \cdot d_Q^-(x^2)\big) \\
&= -\int d_Q^-(x^2) \cdot d_Q^-(vx) \cdot x \\
&= -\int x \cdot d_Q^-(vx) \cdot d_Q^-(x^2).
\end{aligned}$$

For the second case we have

$$\widehat{m}_4(x, v, x, x) = -d_Q^- \widehat{m}_3(x, v, x) \cdot x - x \cdot d_Q^- \widehat{m}_3(v, x, x) + d_Q^-(xv) \cdot d_Q^-(x^2)$$

so we want to prove

$$-\int x \cdot d_Q^- \widehat{m}_3(x, v, x) \cdot x - \int x^2 \cdot d_Q^- \widehat{m}_3(v, x, x) + \int x \cdot d_Q^-(xv) \cdot d_Q^-(x^2) = 0.$$

Again this is trivial for an odd r, while for an even r we have

$$\begin{aligned}-\int x\cdot d_{\mathcal{Q}}^{-}\widehat{m}_3(x,v,x)\cdot x &= -\int x^2\cdot d_{\mathcal{Q}}^{-}\widehat{m}_3(x,v,x)\\ &= -\int d_{\mathcal{Q}}^{-}(x^2)\cdot \widehat{m}_3(x,v,x)\\ &= -\int d_{\mathcal{Q}}^{-}(x^2)\cdot \big(d_{\mathcal{Q}}^{-}(xv)\cdot x - x\cdot d_{\mathcal{Q}}^{-}(vx)\big)\\ &= -0\end{aligned}$$

and

$$\begin{aligned}-\int x^2\cdot d_{\mathcal{Q}}^{-}\widehat{m}_3(v,x,x) &= -\int x\cdot d_{\mathcal{Q}}^{-}(vx)\cdot d_{\mathcal{Q}}^{-}(x^2)\\ &= -\int x\cdot d_{\mathcal{Q}}^{-}(xv)\cdot d_{\mathcal{Q}}^{-}(x^2)\end{aligned}$$

from the previous computation.

• For $\ell=4$ we have three cases to consider; namely, (v,x,x), (x,v,x), and (x,x,v). The third case is analogous to the first one, so we need only investigate two cases. For the first one we have

$$\begin{aligned}\widehat{m}_3(v,x,x) &= d_{\mathcal{Q}}^{-}(vx)\cdot x-(-1)^{r+1}v\cdot d_{\mathcal{Q}}^{-}(x^2)\\ &= d_{\mathcal{Q}}^{-}(vx)\cdot x - d_{\mathcal{Q}}^{-}(x^2)\cdot v,\end{aligned}$$

so we want to prove

$$\int x\cdot d_{\mathcal{Q}}^{-}(vx)\cdot x - \int x\cdot d_{\mathcal{Q}}^{-}(x^2)\cdot v = 0.$$

Once more this is trivial for an odd r, while for an even r we have

$$\begin{aligned}\int x\cdot d_{\mathcal{Q}}^{-}(vx)\cdot x &= \int x^2\cdot d_{\mathcal{Q}}^{-}(vx)\\ &= \int d_{\mathcal{Q}}^{-}(x^2)\cdot d_{\mathcal{Q}}d_{\mathcal{Q}}^{-}(vx)\\ &= \int d_{\mathcal{Q}}^{-}(x^2)\cdot vx\\ &= \int x\cdot d_{\mathcal{Q}}^{-}(x^2)\cdot v.\end{aligned}$$

In the second case we have

$$\widehat{m}_3(x,v,x) = d_{\mathcal{Q}}^{-}(xv)\cdot x-(-1)^r x\cdot d_{\mathcal{Q}}^{-}(vx).$$

So this time the odd r case is trivial by the usual reason, while also the even r case is trivial since for an even r we directly have $\widehat{m}_3(x,v,x)=0$.

Case 2, final step. Now we turn to the ℓ-ary multiplication m_ℓ. By Lemma 2.17, since $\dim\mathcal{H}^r=1$ we are reduced to computing $m_\ell(x,x,\dots,x)$

and this is zero, again by Lemma 2.17. Finally, all the higher multiplications m_k with $k \geq \ell + 1$ vanish due to Theorem 2.12, since $\ell(r-1)+4 \leq (\ell+2)(r-1)+2$ for $r \geq 2$.

□

The particular case $\ell = 4$ considered in the last statement in Theorem 2.19 recovers two results from [3], that we can jointly state as follows.

Corollary 2.20 (Cavalcanti)**.** *Let $r \geq 2$ and let $\mathcal{A}^*$ be a $(r-1)$-connected Poincaré DGCA of degree n with $n \leq 4r$ admitting a Hodge homotopy. If $\dim H^r(\mathcal{A}) = 1$ then $\mathcal{A}^*$ is formal. In particular an $(r-1)$-connected compact manifold M of dimension $n \leq 4r$ with $b_r = 1$ is formal.*

3. Harrison cohomology and formality

Let $\mathcal{A}^*$ be an arbitrary simply connected Poincaré DGCA endowed with a Hodge homotopy. In this section we prove several results relating the homotopy type, i.e., the weak equivalence class, of $\mathcal{A}^*$ to the Harrison cohomology class $[\mu_3] \in \mathrm{HHarr}^{3,-1}(H^*(\mathcal{A}))$ associated with the minimal unital C_∞-algebra structure on $H^*(\mathcal{A})$ defined via the homotopy transfer we described at the beginning of Subsection 2.2. As shown in Remark 2.14, such a minimal unital C_∞-structure on $H^*(\mathcal{A}^*)$ is an example of a *minimal unital C_∞-enrichment of* $(H^*(\mathcal{A}), \cdot)$. In particular, we show that $[\mu_3]$ is the first obstruction to the formality of $\mathcal{A}^*$ (Lemma 3.2), a homotopy invariant of $\mathcal{A}^*$ (Lemma 3.3), and that it defines completely the homotopy type of a $(r-1)$ connected Poincaré DGCA of degree $n \leq 4r-1$ admitting a Hodge homotopy (Theorem 3.9). Notice several results in this section are valid under slightly weaker conditions than, e.g., the Hodge type requirements, which are evident from their proofs.

If $(N, m_1 = 0, m_2, m_3, \ldots)$ is a minimal A_∞-algebra, then (N, m_2) is a graded associative algebra with multiplication $a \cdot b = m_2(a \otimes b)$. To avoid confusion and at the same time cumbersome notation, in what follows we will write $N_{[\infty]}$ for the minimal A_∞-algbera $(N, m_1 = 0, m_2, m_3, \ldots)$ and simply N for the graded associative algebra (N, m_2). The relation between the homotopy type of the minimal A_∞-algebra $N_{[\infty]}$ and the Hochschild cochain complex $\mathrm{CHoch}^{*,*}(N,N)$ of the graded associative algebra N, and in particular with the Hochschild cohomology groups $\mathrm{HHoch}^{n,2-n}(N,N)$, has first been considered by Kadeishvili [17], [19], see also [21]. Moving from A_∞-algebras to C_∞-algebras, any result concerning the relation between minimal A_∞-algebras and their Hochschild cochain complex $\mathrm{CHoch}^{*,*}(N,N)$ is also valid for minimal C_∞-algebras, replacing the Hochschild cochain complex by the Harrison cochain complex, i.e., by the subcomplex of the Hochschild complex $\mathrm{CHoch}^{*,*}(N,N)$ that consists of all cochains vanishing on shuffles. In particular, Kadeishvili proved that the homotopy type of a minimal C_∞-algebra $N_{[\infty]} = (N, 0, m_2, m_3, \ldots)$ is encoded in a suitable equivalence class of the Harrison cochain $m = m_3 + m_4 + \ldots \in \mathrm{CHarr}^{*,2-*}(N,N)$ for the DGCA

$N=(N,m_2)$. As a result, he showed that if $\mathrm{HHarr}^{n,2-n}(N,N)=0$ for $n\geq 3$ then the C_∞-algebra structure on N is formal, i.e., it is C_∞-isomorphic to one with $m_i=0$ for all $i\geq 3$ [17, Theorem 2]. His theorem implies Tanre's result on the formality of a simply connected space M if the Harrison cohomology $\mathrm{HHarr}^{*,2-*}(H^*(M),H^*(M))$ vanishes [35], cf. [21, Theorem 17]. We refer the reader to Kadeishvili works [17], [19],[21] for a detailed description of Harrison cohomology $\mathrm{HHarr}^{*,2-*}(N,N)$ and the aforementioned results due to Kadeishvili.

The real homotopy type of a simply connected compact manifold M is completely encoded in the homotopy class of its de Rham algebra of smooth forms [34, Theorem D], see also [14, Corollary 12.4, p. 117]. Since in characteristic zero the inclusion of DGCAs into C_∞-algebras induces an equivalence of the homotopy categories [26, Theorem 11.4.8, p.422], we see that the real homotopy type of M is completely encoded in the structure of the minimal C_∞-enhancement of $(H^*_{\mathrm{dR}}(M),\wedge)$ induced on the de Rham cohomology $H^*_{\mathrm{dR}}(M)$ of M by homotopy transfer via a Hodge homotopy as defined in Example 2.7.

Theorem 2.12 and Remark 2.14 then tells us that if $r\geq 2$, then the real homotopy type of an $(r-1)$-connected compact n-dimensional manifold M with $n\leq 6r-4$ is completely encoded in the unital C_∞-enrichment $(H^*_{\mathrm{dR}}(M),\wedge,\mu_3,\mu_4)$ of the de Rham cohomology algebra $(H^*_{\mathrm{dR}}(M),\wedge)$ of M by the ternary multiplication

$$\mu_3\colon H^{k_1}_{\mathrm{dR}}(M)\otimes H^{k_2}_{\mathrm{dR}}(M)\otimes H^{k_3}_{\mathrm{dR}}(M)\to H^{k_1+k_2+k_3-1}_{\mathrm{dR}}(M)$$

and the quaternary multiplication

$$\mu_4\colon H^{k_1}_{\mathrm{dR}}(M)\otimes H^{k_2}_{\mathrm{dR}}(M)\otimes H^{k_3}_{\mathrm{dR}}(M)\otimes H^{k_4}_{\mathrm{dR}}(M)\to H^{k_1+k_2+k_3+k_4-2}_{\mathrm{dR}}(M)$$

induced by m_3 and m_4 via the canonical linear isomorphism $\mathcal{H}^*(M)\xrightarrow{\sim} H^*_{\mathrm{dR}}(M)$. Moreover, if $n\leq 5r-3$ then μ_4 vanishes and so the real homotopy type of M is completely encoded in the unital C_∞-enrichment $(\mathcal{H}^*_{\mathrm{dR}}(M),\wedge,\mu_3)$ of $(\mathcal{H}^*_{\mathrm{dR}}(M),\wedge)$. These facts can be seen as a unital C_∞-algebra counterpart of results in rational homotopy theory by Crowley-Nordström [5, Theorems 1,2] for $n=5r-3$, and they are closely related to some result and conjecture by Nagy-Nordström [29, Theorem 1.7 and Conjecture 1.9] for $n=5r-2$.

Let now $\mathcal{A}^*$ be an arbitrary Poincaré DGCA of degree n endowed with a Hodge homotopy. Recall that the homotopy transfer works generally, and the result is a unital C_∞-algebra $(\mathcal{H}^*,m_2,m_3,m_4,m_5,\dots)$, where the multiplications m_2, m_3 and m_4 are the same as in Theorem 2.12: none of the trees contributing to m_2, m_3 and m_4 is discarded there due to the bound on the dimension. Via the canonical linear isomorphism $\mathcal{H}^*\to H^*(\mathcal{A})$ we get a unital C_∞-enrichment $(H^*(\mathcal{A}),\cdot,\mu_3,\mu_4,\mu_5,\dots)$ of the cohomology algebra $(H^*(\mathcal{A}),\cdot)$ of $\mathcal{A}^*$. We focus on the ternary multiplication μ_3. Since it is the ternary multiplication of a minimal C_∞-algebra which is a unital C_∞-enrichment of $(H^*(\mathcal{A}),\cdot)$, it is a cocycle of degree $(3,-1)$ in the Harrison complex of $(H^*(\mathcal{A}),\cdot)$. The first relevant fact on this cocycle is the following.

Lemma 3.1. *The Harrison cohomology class of μ_3 is independent of the Hodge homotopy used to define m_3. Therefore if $\mathcal{A}^*$ admits a Hodge homotopy we have a well defined cohomology class $[\mu_3] \in \mathrm{HHarr}^{3,-1}(H^*(\mathcal{A}), H^*(\mathcal{A}))$ only depending on the Poincaré DGCA $\mathcal{A}^*$ and not on the choice of a Hodge homotopy.*

Proof. Let $(H^*(\mathcal{A}),\cdot,\mu_3,\mu_4,\mu_5,\dots)$ and $(H^*(\mathcal{A}),\cdot,\mu_3',\mu_4',\mu_5',\dots)$ two unital C_∞-enrichments of $(H^*(\mathcal{A}),\cdot)$ induced by two Hodge homotopies. Then, by construction, the two unital C_∞-algebras $(H^*(\mathcal{A}),\cdot,\mu_3,\mu_4,\mu_5,\dots)$ and $(H^*(\mathcal{A}),\cdot,\mu_3',\mu_4',\mu_5',\dots)$ are isomorphic by a C_∞-isomorphism whose linear part is the identity of $H^*(\mathcal{A})$. In this case, the quadratic componenst ϕ_2 of this C_∞-isomorphism is a Harrison cochain with $d_{\mathrm{Harr}}\phi_2 = \mu_3 = \mu_3'$, and so $[\mu_3] = [\mu_3']$. □

Lemma 3.2. *The Harrison cohomology class $[\mu_3]$ is the first obstruction to the formality of a simply connected Poincaré DGCA $\mathcal{A}^*$.*

Proof. If $\mathcal{A}^*$ is formal, there exists an isomorphism of C_∞-algebras between $(H^*(\mathcal{A}),\cdot,\mu_3,\mu_4,\mu_5,\dots)$ and $(H^*(\mathcal{A}),\cdot)$. The component ϕ_2 of bidegree $(2,-1)$ of this C_∞-isomorphism is a Harrison cochain with $d_{\mathrm{Harr}}\phi_2 = \mu_3$. Hence if $\mathcal{A}^*$ is formal then $[\mu_3] = 0$. □

More generally one has the following.

Lemma 3.3. *The graded commutative algebra $(H^*(\mathcal{A}),\cdot)$ and the Harrison cohomology class $[\mu_3]$ are invariants of the DGCA homotopy class of a simply connected Poincaré DGCA $\mathcal{A}^*$ admitting a Hodge homotopy, i.e., if $\mathcal{A}^*$ and $\mathcal{A}'^*$ are in the same homotopy class, then there exists an isomorphism of graded commutative algebras $\phi\colon (H^*(\mathcal{A}),\cdot_{\mathcal{A}}) \xrightarrow{\sim} (H^*(\mathcal{A}'),\cdot_{\mathcal{A}'})$ mapping $[\mu_3]$ to $[\mu_3']$ via the induced isomorphism of Harrison cohomologies.*

Proof. If $\mathcal{A}^*$ and $\mathcal{A}'^*$ are in the same homotopy class then there exists a zig-zag of quasi-isomorphisms of DGCAs connecting $\mathcal{A}^*$ and $\mathcal{A}'^*$. So we are reduced to showing that if $\varphi\colon \mathcal{A}^* \to \mathcal{A}'^*$ is a quasi-isomorphism of DGCAs then there exists an isomorphism of graded commutative $\phi\colon (H^*(\mathcal{A}),\cdot_{\mathcal{A}}) \xrightarrow{\sim} (H^*(\mathcal{A}'),\cdot_{\mathcal{A}'})$ mapping $[\mu_3]$ to $[\mu_3']$ via the induced isomorphism of Harrison cohomologies. The same proof as for Lemma 3.2 shows that $\phi = H^*(\varphi)$ is such an isomorphism. □

Remark 3.4. The vanishing of $[\mu_3]$ is the first obstruction to formality of $\mathcal{A}^*$, but it is generally not the only obstruction. Indeed, if $[\mu_3] = 0$, then there exists a Harrison cochain ϕ_2 of bidegree $(2,-1)$ with $d_{\mathrm{Harr}}\phi_2 = \mu_3$ and one can use it to define a C_∞-isomorphism between $(H^*(\mathcal{A}),\cdot,\mu_3,\mu_4,\mu_5,\dots)$ and a C_∞-algebra of the form $(H^*(\mathcal{A}),\cdot,0,\tilde{\mu}_4,\tilde{\mu}_5,\dots)$, were now the ternary multiplication vanishes. We then have the Harrison cohomology class of the new quaternary multiplication $\tilde{\mu}_4$, that is an element in $\mathrm{HHarr}^{4,-2}(H^*(\mathcal{A}), H^*(\mathcal{A}))$ that is the secondary obstruction to formality. Notice however that, as usual in obstruction theory, while the first obstruction $[\mu_3]$ is canonical, the higher obstructions are not. Indeed the Harrison cohomology class $[\tilde{\mu}_4]$ depends on the choice of the cobounding cochain ϕ_2 for μ_3. This fact can be seen as a Harrison cohomology counterpart that the Bianchi-Massey tensor defined in [5] is canonical, while the pentagonal tensor of [29] is not. Similarly, the isomorphism class of the graded commutative algebra $(H^*(\mathcal{A}), \cdot)$ and the Harrison cohomology class $[\mu_3]$ are invariants of the homotopy class of $\mathcal{A}^*$, but generally they do not form a complete invariant.

Remark 3.5. The Harrison cohomology class $[\mu_3]$, its canonicity, and its role as the first obstruction to the formality, as well as the corresponding Hochschild cohomology class for noncommutative DGAs are widely discussed in the literature. See [2, 15] and the references therein for additional information.

Despite we have observed in Remark 3.4 that the vanishing of $[\mu_3]$ is generally not a sufficient condition for the formality of $\mathcal{A}^*$ and that the pair $((H^*(\mathcal{A}), \cdot), [\mu_3])$ is not a complete invariant, there is at least an interesting situation where this invariant is complete so that in particular the condition $[\mu_3] = 0$ is indeed sufficient for formality. This is the case when $\mathcal{A}^*$ is an $(r-1)$-connected Poincaré algebra of degree $4r-1$. In order to prove this result we need a couple of preliminary Lemmas.

Recall that the normalized Hochschild complex $\mathrm{CHoch}_\nu(A, A)$ of a graded associative $\mathbb{K}$-algebra A with coefficients in itself is the bigraded subcomplex of the Hochschild complex $(\mathrm{CHoch}(A, A), d_{\mathrm{Hoch}})$ consisting on those cochains that vanish on constants, i.e., in each bidegree $(n, \bullet)$ of those $\varphi\colon A^{\otimes n} \to A$ such that $\varphi(a_1, \ldots, a_n) = 0$ if one of the a_i is the unit of the algebra A (or, equivalently, by $\mathbb{K}$-linearity, it is an element of $\mathbb{K}$). Note that operations μ_3 and μ_4 in a unital minimal C_∞-enrichment of $(H^*(\mathcal{A}^*), \cdot)$ are elements in $(\mathrm{CHoch}_\nu(H^*(\mathcal{A}^*), H^*(\mathcal{A}^*)), d_{\mathrm{Hoch}})$. The most important feature of the normalized Hochschild complex is the fact that the inclusion $\mathrm{CHoch}_\nu(A, A) \hookrightarrow \mathrm{CHoch}(A, A)$ is a quasiisomorphism, i.e., induces an isomorphism in cohomology (see, e.g., [25, §1.5.7, p. 40]). However, in the next lemma we will not need this fact, but merely the fact that $\mathrm{CHoch}_\nu(A, A)$ is a subcomplex of $\mathrm{CHoch}(A, A)$.

Lemma 3.6. *Let $r \geq 2$ and let $\mathcal{A}^*$ be an $(r-1)$-connected Poincaré algebra of degree n with $n \leq 4r-1$. Let $(H^*(\mathcal{A}), \cdot, \mu_3)$ and $(H^*(\mathcal{A}), \cdot, \mu_3')$ be two minimal C_∞-enhancements of $(H^*(\mathcal{A}), \cdot)$ such that $\mu_3(a, b, c)$ and $\mu_3'(a, b, c)$ are zero unless $\deg(a, b, c) = (r, r, r)$. Then $[\mu_3] = [\mu_3']$ in $\mathrm{HHarr}^{3,-1}(H^*(\mathcal{A}), H^*(\mathcal{A}))$ if and only if there exists $\phi_2 \in \mathrm{CHarr}^{2,-1}(H^*(\mathcal{A}), H^*(\mathcal{A}))$ such that*

1. *$d_{\mathrm{Hoch}}\phi_2 = \mu_3 - \mu_3'$;*
2. *$\phi_2(a, b)$ vanishes unless $\deg(a)$ and $\deg(b)$ are strictly positive integer multiples of r.*

Proof. The "if" part is obvious, so let us prove the 'only if' part. By assumption, there exists $\tilde{\phi}_2 \in \mathrm{CHarr}^{2,-1}_{\nu}(H^*(\mathcal{A}), H^*(\mathcal{A}))$ such that $d_{\mathrm{Harr}}\tilde{\phi}_2 = \mu_3 - \mu_3'$. Let us set

$$\phi_2(a,b) = \begin{cases} \tilde{\phi}_2(a,b) & \text{if } \deg(a) \text{ and } \deg(b) \text{ are strictly positive integer multiples of } r \\ 0 & \text{otherwise} \end{cases}$$

We want to show that $d_{\mathrm{Harr}}\phi_2 = \mu_3 - \mu_3'$. If $\min(\deg(a), \deg(b), \deg(c)) = 0$, then by assumption, $\mu_3(a,b,c) = \mu_3'(a,b,c) = 0$. So we need to show that $d_{\mathrm{Harr}}\phi_2(a,b,c)$ is zero in this case. Since the Harrison complex is a subcomsplex of the Hochschild complex and $\phi_2(a,b)$ vanishes when either a or b has degree zero, the Harrison cochain ϕ_2 is an element in the normalized Hochschild complex of $H^*(\mathcal{A})$. Since the normalized Hochschild complex is a subcomplex of the Hochschild complex we have

$$d_{\mathrm{Harr}}\phi_2 = d_{\mathrm{Hoch}}\phi_2 \in \mathrm{CHoch}_{\nu}(H^*(\mathcal{A}), H^*(\mathcal{A})).$$

Since $H^0(\mathcal{A}) = \mathbb{K}$, this tells us that also $d_{Harr}\phi_2$ vanishes when $\min(\deg(a), \deg(b), \deg(c)) = 0$. Now let us assume $\deg(a), \deg(b), \deg(c) > 0$. If $\deg(a), \deg(b), \deg(c)$ are all multiples of r, then

$$\begin{aligned} d_{\mathrm{Harr}}\phi_2(a,b,c) &= a\phi_2(b,c) - \phi_2(ab,c) + \phi_2(a,bc) - \phi_2(a,b)c \\ &= a\tilde{\phi}_2(b,c) - \tilde{\phi}_2(ab,c) + \tilde{\phi}_2(a,bc) - \tilde{\phi}_2(a,b)c \\ &= d_{\mathrm{Harr}}\tilde{\phi}_2(a,b,c) \\ &= \mu_3(a,b,c) - \mu_3'(a,b,c). \end{aligned}$$

If $\deg(a), \deg(b), \deg(c)$ are not all multiples of r, then by assumption we have that $\mu_3(a,b,c) = \mu_3'(a,b,c) = 0$ so we have to show that also $d_{\mathrm{Harr}}\phi_2(a,b,c)$ vanishes in this case. Since we are assuming a, b, c of strictly positive degree and $\mathcal{A}^*$ is $(r-1)$-connected this means we can assume $\deg(a), \deg(b), \deg(c) \geq r$. Let us write $\deg(a) = r + k_a$, and similarly for b and c. Then the triple (k_a, k_b, k_c) is a triple of nonnegative integers with $k_a + k_b + k_c \geq 1$. The element $d_{\mathrm{Harr}}\phi_2(a,b,c)$ is an element in $H^k(\mathcal{A}^*)$ with $k = 3r - 1 + (k_a + k_b + k_c) \geq 3r \geq n - r + 1$. Since $\mathcal{A}^*$ is an $(r-1)$-connected, this element will automatically be zero unless $k = n$, i.e., unless $k_a + k_b + k_c = r$.

Since k_a, k_b, k_c are nonnegative integers, if they are all nonzero, then the condition $k_a + k_b + k_c = r$ implies $0 < k_a, k_b, k_c < r$, and so

$$r < \deg(a), \deg(b), \deg(c) < 2r < \deg(ab), \deg(bc).$$

This implies $d_{\mathrm{Harr}}\phi_2(a,b,c) = 0$. To analyze the remaining cases, let us begin by assuming $k_a = 0$. If $0 < k_b < r$ (or, equivalently, $0 < k_c < r$) then we have

$$r < \deg(b), \deg(c) < 2r < \deg(ab), \deg(bc),$$

so again $d_{\mathrm{Harr}}\phi_2(a,b,c) = 0$. The cases $k_b = 0$ and $0 < k_a < r$ (or, equivalently, $0 < k_c < r$), or $k - c = 0$ and $0 < k_a < r$ (or, equivalently, $0 < k_b < r$)

are analogous. We are therefore left with only three cases to consider:

$$(k_a, k_b, k_c) \in \{(0,0,r), (0,r,0), (r,0,0)\}.$$

But in all these three cases we have that $\deg(a), \deg(b), \deg(c)$ are all multiples of r, against our assumption. □

Remark 3.7. In the proof of Lemma 3.6 one may be tempted to avoid going through the normalized Hochschild complex and work directly with the normalized Harrison complex. Unfortunately the definition of the latter is not straightforward in general, see [13, Section 3.2], so we preferred to go through the simpler Hochschild framework.

Corollary 3.8. *Let $r \geq 2$ and let $\mathcal{A}^*$ be an $(r-1)$-connected Poincaré algebra of degree n with $n \leq 4r-1$. Let $(H^*(\mathcal{A}), \cdot, \mu_3)$ and $(H^*(\mathcal{A}), \cdot, \mu_3')$ be two minimal unital C_∞-enrichments of $(H^*(\mathcal{A}), \cdot)$ such that $\mu_3(a,b,c)$ and $\mu_3'(a,b,c)$ are zero unless $\deg(a,b,c) = (r,r,r)$. Then $(H^*(\mathcal{A}), \cdot, \mu_3)$ and $(H^*(\mathcal{A}), \cdot, \mu_3')$ are isomorphic C_∞-algebras via a C_∞-isomorphisms whose linear component is the identity if and only if $[\mu_3] = [\mu_3']$ in $\mathrm{HHarr}^{3,-1}(H^*(\mathcal{A}), H^*(\mathcal{A}))$.*

Proof. The "only if" part is obvious, so we only prove the "if" part. Assume $[\mu_3] = [\mu_3']$ in $\mathrm{HHarr}^{3,-1}(H^*(\mathcal{A}), H^*(\mathcal{A}))$. Then by Lemma 3.6 there exists ϕ_2 in $\mathrm{CHarr}^{2,-1}(H^*(\mathcal{A}), H^*(\mathcal{A}))$ such that $d_{\mathrm{Harr}}\phi_2 = \mu_3 - \mu_3'$ and with $\phi_2(a,b) = 0$ unless $\deg(a)$ and $\deg(b)$ are strictly positive integer multiples of r. We claim that $(\mathrm{id}_{H^*(\mathcal{A})}, \phi_2)$ is an isomorphism of C_∞-algebras between $(H^*(\mathcal{A}), \cdot, \mu_3)$ and $(H^*(\mathcal{A}), \cdot, \mu_3')$. Indeed, since we only have multiplications $\mu_2 = \mu_2' = \cdot, \mu_3, \mu_3'$ and components $\mathrm{id}_{H^*(\mathcal{A})}$ and ϕ_2 for the morphism, the C_∞-morphism equations

$$\sum_{k=1}^{p} \sum_{r_1+\ldots r_k = n} (-1)^\eta \mu_k'(\phi_{r_1}(a_1, \ldots, a_{r_1}), \ldots, \phi_{r_k}(a_{n-r_k+1}, \ldots, a_p))$$

(3.1)

$$= \sum_{k=1}^{p} \sum_{\lambda=0}^{p-k} (-1)^\xi \phi_{p-k+1}(a_1, \ldots, a_\lambda, m_k(a_{\lambda+1}, \ldots, a_{\lambda+k}), a_{\lambda+k+1}, \ldots, a_p)$$

where

$$\eta = \sum_{1 \leq \alpha < \beta \leq k} (|a_{r_1+\ldots+r_{\alpha-1}+1}| + \ldots + |a_{r_1+\ldots+r_\alpha}| + r_\alpha)(1 + r_\beta),$$

$$\xi = k + k\lambda + k(|a_1| + \ldots + |a_\lambda|) + p.$$

are trivial for $p \geq 7$. indeed, since $\phi_n = 0$ for $n > 2$ and $\mu_n, \mu_n' = 0$ for $n > 3$, there is no way of getting a nontrivial term with 7 or more arguments of the form $\mu_n'(\phi_{i_1}, \phi_{i_2}, \ldots, \phi_{i_n})$ or $\phi_n(\ldots, \mu_k, \ldots)$ under these conditions: the most one can get in the first case is $\mu_3'(\phi_2, \phi_2, \phi_2)$ and in the second case $\phi_2(\mu_3, \mathrm{id}_H^*(\mathcal{A}), \mathrm{id}_H^*(\mathcal{A}))$, or $\phi_2(\mathrm{id}_H^*(\mathcal{A}), \mu_3, \mathrm{id}_H^*(\mathcal{A}))$, or $\phi_2(\mathrm{id}_H^*(\mathcal{A}), \mathrm{id}_H^*(\mathcal{A}), \mu_3)$. So we only have to check equations (3.1) for $p \leq 6$. Equations (3.1) for $p \leq 2$ are trivially satisfied, while for $p = 3$ we precisely have the equation $d_{\mathrm{Harr}}\phi_2 = \mu_3 - \mu'_3$. So we are left with the three equations corresponding to $p = 4, 5, 6$. For $p = 4$ we have the equation

$$\begin{aligned} &\pm\phi_2(a_1, a_2) \cdot \phi_2(a_3, a_4)) \pm \mu'_3(\phi_2(a_1, a_2), a_3, a_4)) \pm \mu'_3(a_1, \phi_2(a_2, a_3), a_4)) \\ &\qquad \pm \mu'_3(a_1, a_2, \phi_2(a_3, a_4)) \\ &= \pm\phi_2(\mu_3(a_1, a_2, a_3), a_4) \pm \phi_2(a_1, \mu_3(a_2, a_3, a_4)), \end{aligned} \tag{3.2}$$

where we do not make the signs explicit since they are inessential here. If any of the a_i has degree 0, then both the left hand side and the right hand side are zero. Since $\mathcal{A}^*$ is $(r-1)$-connected we can therefore assume that the cohomology classes a_i have degree at least r. The element $\mu_3(a_i, a_j, a_k)$, if nonzero, is in degree $3r - 1$; therefore, the right hand side of (3.2) is always zero. Similarly, the element $\phi_2(a_i, a_j)$, if nonzero, is in degree k with $k \equiv -1 \mod r$; therefore, the only possibly nonzero term in the left hand side of (3.2) is the first one, and so (3.2) reduces to

$$\phi_2(a_1, a_2) \cdot \phi_2(a_3, a_4) = 0. \tag{3.3}$$

This is automatically satisfied unless $\deg(a_i)$ is a strictly positive multiple of r, for any $i = 1, \ldots, 4$. In this case, the left hand side of (3.3) is an element in $H^{kr-2}(\mathcal{A}^*)$ for some $k \geq 4$. Since $\mathcal{A}^*$ is an $(r-1)$-connected Poncaré algebra of degree $4r - 1$ with $r \geq 2$ we have that $H^{kr-2}(\mathcal{A}^*) = 0$ for any $k \geq 4$, and so also in this case equation (3.3) is automatically satisfied.

For $p = 5$ we have the equation

$$\begin{aligned} &\pm\mu'_3(\phi_2(a_1, a_2), \phi_2(a_3, a_4), a_5) \pm \mu'_3(\phi_2(a_1, a_2), a_3, \phi_2(a_4, a_5)) \\ &\quad \pm \mu'_3(a_1, \phi_2(a_2, a_3), \phi_2(a_4, a_5)) = 0. \end{aligned}$$

Since $\phi_2(a_i, a_j)$ can never be a nonzero element of degree r, this equation is automatically satisfied. Finally for $p = 6$ we have the equation

$$\pm\mu'_3(\phi_2(a_1, a_2), \phi_2(a_3, a_4), \phi_2(a_5, a_6)) = 0$$

and this is automatically satisfied by the same argument as for the $p = 5$ case. □

Putting together Lemma 3.2 and Corollary 3.8 we obtain the following.

Theorem 3.9. *Let $r \geq 2$. The graded commutative algebra $(H^*(\mathcal{A}), \cdot)$ and the Harrison cohomology class $[\mu_3]$ are complete invariants of the DGCA homotopy class of an $(r-1)$ connected Poincaré DGCA $\mathcal{A}^*$ of degree n with $n \leq 4r - 1$ admitting a Hodge homotopy, i.e., $\mathcal{A}^*$ and $\mathcal{A}'^*$ are in the same homotopy class if and only if there exists an isomorphism of graded commutative algebras $\phi\colon (H^*(\mathcal{A}), \cdot_{\mathcal{A}}) \xrightarrow{\sim} (H^*(\mathcal{A}'), \cdot_{\mathcal{A}'})$ mapping $[\mu_3]$ to $[\mu'_3]$ via the induced isomorphism of Harrison cohomologies. In particular, $\mathcal{A}^*$ is formal if and only if $[\mu_3] = 0$.*

Proof. The 'only if' part is Lemma 3.2, so we only need to prove the 'if' part. Assume there is an isomorphism ϕ as in the statement. Then ϕ

is a linear C_∞-isomorphism between $(H^*(\mathcal{A}), \cdot_{\mathcal{A}}, \mu_3)$ and the C_∞-algebra $(H^*(\mathcal{A}'), \cdot_{\mathcal{A}'}, \mu_3^\phi)$, where μ_3^ϕ is the degree -1 ternary multiplication

$$\phi \circ \mu_3 \circ (\phi^{-1} \otimes \phi^{-1} \otimes \phi^{-1})\colon H^*(\mathcal{A}')^{\otimes 3} \to H^*(\mathcal{A}')$$

on $H^*(\mathcal{A}')$. By assumption, $[\mu_3^\phi] = [\mu_3']$. Moreover, the C_∞-algebras $H^*(\mathcal{A}'), \cdot_{\mathcal{A}'}, \mu_3^\phi)$ and $H^*(\mathcal{A}'), \cdot_{\mathcal{A}'}, \mu_3')$ satisfy the assumptions of Corollary 3.8. Therefore $H^*(\mathcal{A}'), \cdot_{\mathcal{A}'}, \mu_3^\phi)$ and $H^*(\mathcal{A}'), \cdot_{\mathcal{A}'}, \mu_3')$ are isomorphic C_∞-algebras and so also $H^*(\mathcal{A}), \cdot_{\mathcal{A}}, \mu_3)$ and $H^*(\mathcal{A}'), \cdot_{\mathcal{A}'}, \mu_3')$ are isomorphic.

□

Corollary 3.10. *Let $r \geq 2$. The real homotopy type of an $(r-1)$ connected compact manifold M of dimension n with $n \leq 4r-1$ is completely encoded in the de Rham cohomology algebra $(H^*_{\mathrm{dR}}(M), \wedge)$ of M and in the Harrison cohomology class $[\mu_3] \in \mathrm{HHarr}^{3,-1}(H^*_{\mathrm{dR}}(M), H^*_{\mathrm{dR}}(M))$.*

Remark 3.11. If $n \leq 4r-2$ then $\mu_3 = 0$ and $\mathcal{A}^*$ is formal, see Theorem 2.12. So Theorem 3.9 is actually a statement on $(r-1)$ connected Poincaré DGCAs of degree $4r-1$.

4. Harrison cohomology, symmetric cohomology and the Bianchi-Massey tensor

In this section we relate the Harrison cohomology class $[\mu_3]$ with the Bianchi-Massey tensor defined by Crowley-Nordström in [5] (Theorem 4.4.) The proof of Theorem 4.4 is based on the spectral sequence symmetric-to-Harrison cohomology from [1, 12], giving in characteristic zero a canonical isomorphism

$$H^{\bullet-1}_{\mathrm{Sym}}(H^*(\mathcal{A}), H^*(\mathcal{A})) \cong \mathrm{HHarr}^\bullet(H^*(\mathcal{A}), H^*(\mathcal{A}))$$

relates the Harrison cohomology class $[\mu_3]$ to the Bianchi-Massey tensor from [5]. Let us first recall a few general features of the symmetric-to-Harrison cohomology spectral sequence. For a graded commutative algebra B and a graded B-module N, the 0-th page of the spectral sequence is

$$E_0^{p,q} = \mathrm{CHarr}^{p+1}(S^{q+1}B, N),$$

where S is the (graded) symmetric algebra functor, i.e.,

$$SV = \bigoplus_{k=0}^{\infty} \mathrm{Sym}^k(V)$$

for any (graded) vector space V, and S^n will denote the n-fold composition of S with itself: it should not be confused with the n-th (graded) symmetric power that, in the hope of avoiding confusion, we are denoting with Sym^n. The horizontal differential

$$d_{\mathrm{Harr}}\colon \mathrm{CHarr}^{p+1}(S^{q+1}B, N) \to \mathrm{CHarr}^{p+2}(S^{q+1}B, N)$$

is the Harrison differential, while the vertical differential

$$d_{\mathrm{Sym}}\colon \mathrm{CHarr}^{p+1}(S^{q+1}B, N) \to \mathrm{CHarr}^{p+1}(S^{q+2}B, N)$$

is induced by the multiplication map from the symmetric algebra over a (graded) commutative algebra to the algebra itself. More precisely, the multiplication maps induce maps

$$\partial_i \colon S^{q+2}B = S^{q+1-i}(S(S^iB)) \to S^{q+1-i}(S^iB) = S^{q+1}B,$$

and the alternate sum $d_{S^\bullet B} = \sum_{i=0}^{q+1}(-1)^i\partial_i$ produces a chain complex

$$\cdots \to S^{q+2}B \xrightarrow{d_{S^\bullet B}} S^{q+1}B \xrightarrow{d_{S^\bullet B}} S^qB \to \cdots$$

By applying $\mathrm{CHarr}^{p+1}(-,N)$ we get the vertical differential d_{Sym}. The first page of the spectral sequence obtained by starting with the horizontal differential has

$$({}'E)_1^{p,q} = \mathrm{HHarr}^{p+1}(S^{q+1}B),N)$$

By [1, Proposition 3.1] one has

$$\mathrm{HHarr}^k(SV,M) = \begin{cases} \mathrm{Hom}(V,M) & \text{if } k=1; \\ 0 & \text{if } k>1 \end{cases}$$

for every finitely dimensional graded vector space V and every graded SV-module M, for any $k > 1$. Therefore, for any $q \geq 0$

$${}'E_1^{p,q} = \begin{cases} \mathrm{Hom}(S^qB,N) & \text{if } p=0; \\ 0 & \text{if } p>0 \end{cases}$$

and the page ${}'E_1$ takes the form

$$\begin{array}{c|ccc}
 & \uparrow & & \\
2 & \mathrm{Hom}(S^2B,N) & 0 & 0 \\
 & \uparrow & & \\
1 & \mathrm{Hom}(SB,N) & 0 & 0 \\
 & \uparrow & & \\
0 & \mathrm{Hom}(B,N) & 0 & 0 \\
\hline
 & 0 & 1 & 2
\end{array}$$

Therefore ${}'E_2 = {}'E_\infty$ is

$$
\begin{array}{c|ccc}
2 & H^2_{\mathrm{Sym}}(B,N) & 0 & 0 \\
1 & H^1_{\mathrm{Sym}}(B,N) & 0 & 0 \\
0 & H^0_{\mathrm{Sym}}(B,N) & 0 & 0 \\
\hline
 & 0 & 1 & 2
\end{array}
$$

Now let us consider the spectral sequence starting with the vertical differential. Barr shows that $''E_1$ has the form

$$
\begin{array}{c|ccc}
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathrm{CHarr}^1(B,N) \longrightarrow & \mathrm{CHarr}^2(B,N) \longrightarrow & \mathrm{CHarr}^3(B,N) \longrightarrow \\
\hline
 & 0 & 1 & 2
\end{array}
$$

Therefore $''E_2 = ''E_\infty$ is

$$
\begin{array}{c|ccc}
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathrm{HHarr}^1(B,N) & \mathrm{HHarr}^2(B,N) & \mathrm{HHarr}^3(B,N) \\
\hline
 & 0 & 1 & 2
\end{array}
$$

and by comparing the two E_∞ pages one finds the isomorphism $H^{\bullet-1}_{\mathrm{Sym}}(B,N) \cong \mathrm{HHarr}^\bullet(B,N)$. Moreover, the usual zig-zag argument then tells us that an element $[\xi]$ in $\mathrm{Harr}^3(B,N) = ''E_\infty^{2,0}$, that is naturally represented by an element in $E_0^{2,0}$, is also represented by an element Ξ in $E_0^{1,1}$. The element Ξ is an element in $\mathrm{CHarr}^2(S^2B, N)$ and so a morphism

$$\Xi\colon \mathrm{Sym}^2(S^2B) \to N$$

characterized by the fact that

$$d_{\mathrm{Harr}}\Xi = d_{\mathrm{Sym}}\xi$$

Here we can take as ξ a representative for $[\xi]$ in $''E_1^{2,0} = \mathrm{CHar}^3(B, N)$ and look at it as a representative for $[\xi]$ in $E_0^{2,0} = \mathrm{CHar}^3(SB, N)$ via the multiplication map $\epsilon\colon SB \to B$. The element $d_{\mathrm{Sym}}\xi$ is then the further pullback to $\mathrm{CHarr}^3(S^2B, N)$ via the multiplication map $\epsilon\colon S^2B \to SB$ Since the algebras S^nB are iteratively generated by B, the equation $d_{\mathrm{Harr}}\Xi = d_{\mathrm{Sym}}\xi$ reduces to

$$-x\Xi(y, z) + \Xi(x \odot y, z) - \Xi(x, y \odot z) + \Xi(x, y)z = \xi(x, y, z) \tag{4.1}$$

for any x, y, z in B. We can now prove the main statement of this section. First we need recalling the definition of the Bianchi-Massey tensor of a Poincaré DGCA $\mathcal{A}^*$ endowed with a Hodge homotopy, see [5].

Definition 4.1. Let $(\mathcal{A}^*, d, d^-)$ be a Poincaré DGCA endowed with a Hodge homotopy. Let $\iota\colon H^*(\mathcal{A}) \to \mathcal{A}^*$ be the composition $H^*(\mathcal{A}) \xrightarrow{\sim} \mathcal{H}^* \hookrightarrow \mathcal{A}^*$, where the first isomorphism is the inverse to the projection $\mathcal{H}^* \to H^*(\mathcal{A})$ and the second arrow is the inclusion of $\mathcal{H}^*$ in $\mathcal{A}^*$. Denote by the same symbol ι the extension of ι to a morphism of algebras $S^n(H^*(\mathcal{A})) \to S^n(\mathcal{A}^*)$ for any $n \geq 1$, and let $\epsilon\colon S^n\mathcal{A}^* \to \mathcal{A}^*$ be the iterated multiplication, for any $n \geq 1$. Finally, let

$$\alpha_2\colon \mathrm{Sym}^2(H^*(\mathcal{A})) \to \mathcal{A}^*$$

be the restriction of $\epsilon \circ \iota\colon S^n(H^*(\mathcal{A})) \to \mathcal{A}^*$ to $\mathrm{Sym}^2(H^\bullet(A))$, and let

$$\gamma\colon \mathrm{Sym}^2(H^*(\mathcal{A}^*)) \to \mathcal{A}^{*-1}$$

be the restriction of $d^- \circ \epsilon \circ \iota\colon S^n(H^*(\mathcal{A}^*)) \to \mathcal{A}^{*-1}$ to $\mathrm{Sym}^2(H^*(\mathcal{A}))$. The Bianchi-Massey tensor of $(\mathcal{A}^*, d, d^-)$ is the morphism

$$\Xi_{BM}\colon \mathrm{Sym}^2(\mathrm{Sym}^2(H^*(\mathcal{A}))) \to H^*(\mathcal{A})$$

given by

$$e \odot e' \mapsto [\pi_{\mathcal{H}}(\gamma(e) \cdot \alpha_2(e') + (-1)^{\deg e}\alpha_2(e) \cdot \gamma(e'))] \tag{4.2}$$

Remark 4.2. Notice that the definition of the Bianchi-Massey tensor does not make use of the orthogonality relations (2.4), but only of the fact d^- is a homotopy between the identity of $\mathcal{A}^*$ and $\iota \circ \pi_{\mathcal{H}}$. So it can be defined for any Poincaré DGCA $\mathcal{A}^*$, since algebraic homotopies of this kind always exist. Here we assumed the homotopy is a Hodge one in Definition 4.1 only because that is the setting we are working in in the present article.

Remark 4.3. As for Massey products, one can restrict Ξ_{BM} to the subspace K of $\mathrm{Sym}^2(\mathrm{Sym}^2(H^*(\mathcal{A})))$ spanned by those elements $e \odot e'$ such that both e and e' are in the kernel of the multiplication $\mathrm{Sym}^2(H^*(\mathcal{A})) \to H^*(\mathcal{A})$, i.e., such that $\epsilon(\iota((e))$ and $\epsilon(\iota((e'))$ are exact elements in $\mathcal{A}^*$. For these elements,

$\gamma(e)\cdot\alpha_2(e')+(-1)^{\deg e}\alpha_2(e)\cdot\gamma(e')$ is a closed element in $\mathcal{A}^*$ and so, writing $\mathcal{F}$ for $\Xi_{BM}|_K$ gets the simplified formula $\mathcal{F}(e\odot e')=[\gamma(e)\cdot\alpha_2(e')\pm\alpha_2(e)\cdot\gamma(e')]$, which is the form the Bianchi-Massey tensor originally appears in [5]. Notice that, thanks to Poincaré duality, $\mathcal{F}$ retains all of the information on Ξ_{BM}. Indeed, by Poincar e duality, Ξ_{BM} is determined by its top degree component, i.e., by the elements $\Xi_{BM}(e\odot e')$ with $\deg(e)+\deg(e')=n+1$, where n is the degree of $\mathcal{A}^*$. If a,b are elements in $H^*(\mathcal{A})$, we have

$$d^-(\iota(a)\iota(b)-\iota(ab))=d^-(\iota(a)\iota(b)),$$

since d^- vanishes on $\mathcal{H}^*$. Therefore, if we take e to be the element $e=a\odot b-(ab)\odot 1$ in $\mathrm{Sym}^2(H^*(\mathcal{A}))$, then e is in the kernel of the multiplication $\mathrm{Sym}^2(H^*(\mathcal{A}))\to H^*(\mathcal{A})$ and moreover we have

$$\iota(e)=\iota(a)\odot\iota(b)-\iota(ab)\odot 1$$

and so

$$\gamma(e)=(d^-\circ\epsilon\circ\iota)(e)=d^-(\iota(a)\iota(b))=\gamma(a\odot b).$$

Similarly, let $a',b'\in H^*(\mathcal{A})$ with $\deg(a')+\deg(b')+\deg(a)+\deg(b)=n+1$, and $e'=a'\odot b'-(a'b')\odot 1$. Then

$$\alpha_2(e')=\iota(a)\iota(b)-\iota(ab)=\alpha_2(a'\odot b')-\iota(a'b'),$$

and so

$$[\gamma(e)\alpha_2(e')]=[\gamma(a\odot b)\alpha_2(a'\odot b')]-[d^-(\iota(a)\iota(b))\,\iota(a'b')],$$

where we used that we are in top degree to imply all the expressions above correspond to closed elements. By the orthogonality relations and Poincaré duality we have $[d^-(\iota(a)\iota(b))\,\iota(a'b')]=0$ and so $[\gamma(e)\alpha_2(e')]=[\gamma(a\odot b)\alpha_2(a'\odot b')]$. Switching the role of $a\odot b$ and $a'\odot b'$ one gets the identity

$$\Xi_{BM}((a\odot b)\odot(a'\odot b'))=\Xi_{BM}(e\odot e')$$

with e,e' in the kernel of the multiplication $\mathrm{Sym}^2(H^*(\mathcal{A}))\to H^*(\mathcal{A})$, so that $\Xi_{BM}(e\odot e')=\mathcal{F}(e\odot e')$.

Theorem 4.4. *Let $(\mathcal{A}^*,d,d^-)$ be a Poincaré DGCA endowed with a Hodge homotopy. The Bianchi-Massey tensor Ξ_{BM} (or, more precisely, its extension to $\mathrm{Sym}^2(S^2(H^*(\mathcal{A}^*)))$ is a representative for the distinguished Harrison cohomology class $[\mu_3]$ of $(\mathcal{A}^*,d,d^-)$.*

Proof. Since the ternary multiplication μ_3 on $H^*(\mathcal{A})$ is induced by the ternary multiplication m_3 on $\mathcal{H}^*$ via the linear isomorphism ι, we see from (2.12) that it is explicitly given by

$$\mu_3(a,b,c)=[\pi_{\mathcal{H}}(d^-(\iota(a)\cdot\iota(b))\cdot\iota(c)-(-1)^{\deg a}\iota(a)\cdot d^-(\iota(b)\cdot\iota(c)))].$$

By (4.2) we see that for $x,y\in S^2(H^*(\mathcal{A}))$ we have

$$\Xi_{BM}(x,y)=[\pi_{\mathcal{H}}(\epsilon((d^-\epsilon\iota(x))\odot\epsilon\iota(y))]+(-1)^{\deg x}[\pi_{\mathcal{H}}(\epsilon(\epsilon\iota(x)\odot(d^-\epsilon\iota(y)))]$$

We have to check equation (4.1) for any three a, b, c in $H^*(\mathcal{A})$. Since d^{-1} vanishes on $\mathcal{H}^*$, for any two $x, y \in H^*(\mathcal{A})$ we have $\Xi_{BM}(x, y) = 0$. Therefore, for any three $a, b, c \in A$,

$$\begin{aligned}
&d_{\mathrm{Harr}}\Xi_{BM}(x, y, z)\\
&\quad = -a\,\Xi_{BM}(b, c) + \Xi_{BM}(a \odot b, c) - \Xi_{BM}(a, b \odot c) + \Xi_{BM}(a, b)c\\
&\quad = \Xi_{BM}(a \odot b, c) - \Xi_{BM}(a, b \odot c)\\
&\quad = [\pi_{\mathcal{H}}(\epsilon(d^-(\epsilon\iota(a \odot b)) \odot \epsilon\iota(c))] + (-1)^{\deg a\cdot b}[\pi_{\mathcal{H}}(\epsilon(\epsilon\iota(a \odot b) \odot d^-(\epsilon\iota(c)))]\\
&\qquad\quad - [\pi_{\mathcal{H}}(\epsilon(d^-(\epsilon\iota(a)) \odot \epsilon\iota(b \odot c))] - (-1)^{\deg a}[\pi_{\mathcal{H}}(\epsilon(\epsilon\iota(a) \odot d^-\iota(\epsilon(b \odot c)))]\\
&\quad = [\pi_{\mathcal{H}}(\epsilon(d^-(\epsilon\iota(a \odot b)) \odot \epsilon\iota(c))] - (-1)^{\deg a}[\pi_{\mathcal{H}}(\epsilon(\epsilon\iota(a) \odot d^-\epsilon(\iota(b \odot c)))]\\
&\quad = [\pi_{\mathcal{H}}((d^-(\iota(a)\cdot\iota(b))\cdot\iota(c))] - (-1)^{\deg a}[\pi_{\mathcal{H}}(\iota(a)\cdot d^-(\iota(b)\cdot\iota(c)))]\\
&\quad = \mu_3(a, b, c).
\end{aligned}$$

□

Remark 4.5. (1) In [5] Crowley and Nordström also discussed the relation between their Bianchi-Massey tensor on a Poincaré DGA $(\mathcal{A}^*, d)$ and the associated A_∞-structure on $H^*(\mathcal{A}^*)$ obtained by a homotopy transfer as described in [37]. They noticed that the multiplication μ_3 on the A_∞ algebra $H^*(\mathcal{A}^*)$ defines the Bianchi-Massey tensor uniquely and the Bianchi-Massey tensor defines μ_3 up to a "stabilization" [5, Lemma 2.12]. Refer to [5, §2.5] for a precise formulation of their result.

(2) Despite the Bianchi-Massey tensor Ξ_{BM} (or its restriction $\mathcal{F}$, see Remark 4.3) and the Harrison cohomology class $[\mu_3]$ contain the same amount of information, Ξ_{BM} and $\mathcal{F}$ appear to be more suitable than $[\mu_3]$ for the investigation of questions related to rational and real homotopy types. One reason for this is that $\mathcal{F}$ enjoys a graded version of the symmetries of the Riemann curvature tensor (hence the name "Bianchi", while the name "Massey" comes from its relation with the ternary multiplication μ_3 and so with Massey products), so that in the analysis of $\mathcal{F}$ one can make full use of the vast amount of known algebraic results on the Riemann curvature tensor. For instance, in [5], Crowley and Nordström make a brilliant use of the fact that the Ricci curvature of a manifold of dimension n with $n \leq 3$ determines the full Riemann curvature tensor to prove that if M is a closed $(r-1)$-connected manifold of dimension $n = (4r-1)$ with $b_r(M) \leq 3$ and such that there is a $\phi \in H^{2r-1}_{\mathrm{dR}}(M)$ such that multiplication by ϕ induces a linear isomorphism $H^r_{\mathrm{dR}}(M) \to H^{3r-1}_{\mathrm{dR}}(M)$, then M is intrinsically formal. Another truly remarkable result in [5] is the fact that, by using the Bianchi-Massey tensor, Crowley and Nordström are able to improve the dimension bound $n \leq 4r-1$ in Theorem 3.9 (that was also the bound in the first arXiv version of [5]) to the full range $n \leq 5r-3$ of the vanishing of μ_4. Theorem 3.9 with the optimal bound $n \leq 5r-3$ appears in [11] as Corollary 4.11. Yet, it is fair to say that the proof of Corollary 4.11 in [11] is incomplete and rather corresponds to the proof of Lemma 3.2 in the present article than

to a complete proof of Theorem 3.9 with the optimal bound $n \leq 5r-3$. It would be interesting to understand whether one can give a proof of this result entirely in terms of the Harrison cohomology class $[\mu_3]$ alone, as the proof we present here for $n \leq 4r-1$, without relying on its equivalence with Ξ_{BM} or $\mathcal{F}$.

5. Conclusions and final remarks

(1) In this paper, using Hodge homotopy transfers, we gave a very quick proof of Zhou's result that the real homotopy type of a $(r-1)$-connected compact manifold M of dimension $n \leq \ell(r-1)+2$, with $\ell \geq 4$, is encoded in a minimal unital C_∞-algebra $(H^*(M,\mathbb{R}), \mu_2, \mu_3, \ldots)$ whose multiplication μ_2 is the usual multiplication of the cohomology alegebra $H^*(M,\mathbb{R})$ and μ_k vanishes for $k \geq \ell-1$ (Theorem 2.12). Also, we extended Zhou's result, that is originally given in terms of A_∞-algebras, to C_∞-algebras.

(2) We gave a proof of Zhou's conjecture, stating that the vanishing in Theorem 2.12 also holds if we increase the dimension of M up to 2, under the condition that $b_r = 1$ (Theorem 2.19). If $\ell = 4$, this is a generalization of Cavalcanti's formality result in [3].

(3) Let M be an arbitrary simply connected compact manifold. We showed that the Harrison cohomology class $[\mu_3] \in \mathrm{HHarr}^{3,-1}(H^*(M,\mathbb{R}), H^*(M,\mathbb{R}))$ is an invariant of the homotopy type of M and it is the first obstruction to the formality of M. If M is a $(r-1)$ connected compact manifold of dimension $n \leq 4r-1$ then $[\mu_3]$ is moreover a complete invariant of the real homotopy type of M. These results follow from more general Lemma 3.2 and Theorem 3.9.

(4) We relate our above results with Crowley-Nordström's results on Bianchi-Massey tensor [5] and Nagy-Nordström's results on Bianchi-Massey and pentagon Massey tensor [29] by showing that the Harrison cohomology class $[\mu_3]$ and the Bianchi-Massey tensor define each other uniquely (Theorem 4.4). Using this result, our Theorem 3.9 can be obtained from Crowley-Nordström results [5, Theorems 1, 2], but Theorem 2.12 and Lemma 3.2 don't have direct counterparts in Crowley-Nordström paper [5] and in Nagy-Nordströme paper [29], respectively.

(5) Theorem 4.4 and Crowley-Nordström result [5, Theorem 1.2] suggest that we should be able to improve the proof of Theorem 3.9 to increase the dimension $4r-1$ to $5r-3$.

(6) Taking into account Kadishvili's results on homotopy type of a C_∞-algebra [21, Proposition 17], [19, §4.1], Theorem 2.12 suggests that the real homotopy type of a $(r-1)$ simply connected compact manifold of dimension $n \leq 6r-4$ is encoded in $[\mu_3]$ and a "canonical" cochain in $\mathrm{CHarr}^{4,-2}(H^*(M,\mathbb{R}), H^*(M,\mathbb{R}))$. This conjecture agrees with and extends Nagy-Nordström's conjecture [29, Conjecture 1.9]

that in the case the Bianchi-Massey tensor vanishes, the rational homotopy type of a $(r-1)$-connected compact manifold M of dimension less or equal to $5r-2$ is defined by the pentagon Massey tensor.

Acknowledgement. We would like to thank Johannes Nordström for alerting us about his eprint with Nagy [29], for his inspiring lecture [30], which motivated us to start this project, and for useful comments on a preliminary version of this article. We thank Fernando Muro for having pointed our attention to relevant works concerning the Harrison cohomology class $[\mu_3]$ and its role as an obstruction to formality. We thank the referee for careful reading and precious suggestions. DF is thankful to the Institute of Mathematics of the Czech Academy of Sciences and the Charles University for their hospitality and excellent working conditions during his visiting Prague in August 2023, where a part of this project has been carried out. This paper is part of the activities of the MIUR Excellence Department Projects CUP:B83C23001390001. DF has been partially supported by the PRIN 2017 *Moduli and Lie Theory*, Ref. 2017YRA3LK. He is a member of the *Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni* (GNSAGA-INdAM). Research of HVL was supported by the Institute of Mathematics, Czech Academy of Sciences (RVO 67985840), and GAČR-project GA22-00091S.

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