PSEUDO-DIFFERENTIAL OPERATORS ON RADIAL SECTIONS OF LINE BUNDLES OVER THE POINCARÉ UPPER HALF PLANE

TAPENDU RANA, MICHAEL RUZHANSKY

Abstract. In this article, we explore the boundedness properties of pseudo-differential operators on radial sections of line bundles over the Poincaré upper half plane, even when dealing with symbols of limited regularity in the spatial variable. We first prove the boundedness of these operators when the symbol is smooth. To achieve this, we establish a connection between the operator norm of the local part of our pseudo-differential operators and the corresponding Euclidean pseudo-differential operators. Additionally, we introduce a class of rough symbols that lack any regularity conditions in the space variable and investigate the boundedness properties of the associated pseudo-differential operators. As a crucial component of our proof, we provide asymptotic estimates and functional identities for certain matrix coefficients of the principal and discrete series representations of the group $\mathrm{SL}(2,\mathbb{R})$.

1. Introduction

The study of pseudo-differential operators can be traced back to the 1960s, with pioneering works by Hörmander [24, 25] and Kohn-Nirenberg [32]. Motivated by the deep connection between pseudo-differential operators and elliptic and hypoelliptic equations, they extensively studied the boundedness properties of pseudo-differential operators in classical spaces, which played a pivotal role in determining the regularity of solutions to related equations. By introducing the concept of symbol class and utilizing Fourier analysis, Hörmander provided a powerful framework for analyzing these operators on Euclidean space. Consequently, the study of pseudo-differential operators on $\mathbb{R}^d$, as well as in various other spaces, flourished, becoming an important and active research area in modern harmonic analysis and partial differential equations. In this article, we aim to explore the boundedness properties of pseudo-differential operators on radial sections of line bundles over the Poincaré upper half plane $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, and establish their connection to the classical Euclidean space. We begin by providing some essential background information to facilitate a mathematically rigorous discussion.

A pseudo-differential operator on $\mathbb{R}^d$ associated with a symbol $a(x,\xi)$ is an operator defined using the Fourier inversion formula:

$$a(x,D)f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} a(x,\xi)\mathcal{F}f(\xi)\, d\xi,$$

where f is a function in $C_c^\infty(\mathbb{R}^d)$, $\mathcal{F}f$ is the Fourier transform of f, and the symbol $a(x,\xi)$ belongs to an appropriate class of functions that encodes the behavior of the operator. One

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of the most widely used classes of symbols is $\mathcal{S}^m_{\varrho,\delta}$ introduced by Hörmander [26]. This class consists of all $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with

$$\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-\varrho|\alpha|+\delta|\beta|},$$

for all multi-indices α, β, where $m \in \mathbb{R}$ and $0 \leq \varrho, \delta \leq 1$. Hörmander's work established the boundedness of operators with symbols belonging to $\mathcal{S}^{d(\varrho-1)/2}_{\varrho,\delta}$ on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, where $0 \leq \delta < \varrho < 1$; see [26]. In 1993, E. M. Stein further investigated the weak type and L^p-boundedness problems of pseudo-differential operators on $\mathbb{R}^d$. Specifically, he considered cases where the symbol belongs to the class $\mathcal{S}^0_{1,0}$; see [56, Chap VI, Theorem 1]. Moreover, by closely following the calculations, the aforementioned result can be extended to the following version in the one-dimensional case:

Theorem 1.1. *Let $a(x,D)$ be the pseudo-differential operator associated with the symbol a, which satisfies the following conditions for all $\alpha, \beta \in \{0,1,2\}$:*

$$\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \leq C_{\alpha,\beta}\,(1+|\xi|)^{-\alpha}. \tag{1.1}$$

Then, for $1 < p < \infty$, $a(x,D)$ extends to a bounded operator from $L^p(\mathbb{R})$ to itself.

Remark 1.2. If the symbol $a(x,\xi)$ is independent of the x variable, say $a(x,\xi) = m(\xi)$, then the associated operator is called a multiplier operator. If the multiplier $m : \mathbb{R}^d \to \mathbb{C}$ satisfies the following Hörmander-Mikhlin differential inequalities:

$$\left|\frac{d^\alpha}{d\xi^\alpha} m(\xi)\right| \leq A_\alpha |\xi|^{-|\alpha|},$$

for all $\xi \in \mathbb{R}^d \setminus \{0\}$ and for all multi indices α with $0 \leq |\alpha| \leq [d/2]+1$, then the corresponding multiplier operator is a bounded operator on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. We remark that for a pseudo-differential operator, one cannot weaken the regularity assumption on the symbol $a(x,\xi)$, having singularity near $\xi = 0$. Particularly if the symbol a satisfies the following simpler-looking condition

$$\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \leq C_{\alpha,\beta}\,|\xi|^{-|\alpha|},$$

then the corresponding operator may not be bounded; see [56, p. 267].

It is well known that pseudo-differential operators with symbols belonging to the class $\mathcal{S}^0_{1,0}$ are generalized Calderón-Zygmund operators, meaning that their kernels satisfy Hörmander's condition, which imposes smoothness conditions on the space variable of the symbol. This naturally raises the question of investigating the boundedness problem of pseudo-differential operators with limited regularity in the space variable of the symbol. In this direction, several authors, including H. Kumano-go [35], Hörmander [27], Nagase [43], and R. Coifman and Y. Meyer [10] have studied the L^p-boundedness problem with limited regularity. More recently, in 2007, Kenig and Staubach [31] introduced the class of Ψ-pseudo-differential operators, where the symbols have no regularity assumptions in the space variable. This opens up new possibilities for exploring the boundedness properties of pseudo-differential operators without the usual smoothness conditions on the symbol.

Let $\mathcal{S}^m_{\varrho,\infty}$ be the class of symbols consisting of all $a(x,\cdot) \in C^\infty(\mathbb{R}^d_\xi)$ such that $x \mapsto a(x,\xi)$ is a measurable function in x and

$$\|\partial_\xi^\alpha a(x,\xi)\|_{L^\infty_x} \leq C_\alpha (1+|\xi|)^{m-\varrho|\alpha|}$$

for all multi indices α, where $m \in \mathbb{R}$, $\varrho \leq 1$, and C_α's are constants. The authors Kenig and Staubach [31] established the following result concerning the L^p-boundedness problem of pseudo-differential operators with non-regular symbols.

Theorem 1.3 ([31, Theorem 2.7])**.** *Let $a \in \mathcal{S}^m_{\varrho,\infty}$, $0 \leq \varrho \leq 1$. Suppose that $m < n(\varrho - 1)/2$, then $a(x, D)$ is a bounded operator from $L^p(\mathbb{R}^d)$ to itself for all $p \in [2, \infty]$. For $\varrho = 1$ and $m < 0$ the range of p for which the operator is L^p bounded is $p \in [1, \infty]$.*

Over the past few decades, the theory of pseudo-differential operators on various Lie groups has gained significant popularity and has become a rich field of study with vast literature. For instance, the second author and his collaborators explored a noncommutative analog of the Kohn–Nirenberg quantization [32] for operators on compact Lie groups [50, 51, 49, 8] and introduced the corresponding Hörmander symbol class [52, 53, 14]. Moreover, research on the L^p-boundedness of pseudo-differential operators in the context of compact Lie groups has been undertaken; see [54, 19, 15]. Bernicot and Frey [4] have delved into the study of pseudo-differential operators on homogeneous spaces. Additionally, the theory of pseudo-differential operators has been extended to graded Lie groups, with the authors in [16, 18, 7] introducing the symbol class based on a positive Rockland operator. Further research on pseudo-differential operators in the context of nilpotent Lie groups can be found in [17, 41].

In recent times, there has been a growing interest in the theory of pseudo-differential operators on discrete spaces, primarily due to its connections with quantum ergodicity problems and the discretization of continuous problems; see [5, 36, 47]. Moreover, other noteworthy works on pseudo-differential operators in non-Euclidean settings can be found in [6, 40, 42, 13].

The study of pseudo-differential operators has predominantly occurred in the doubling setting, which relies on suitable covering lemmas. However, when dealing with the Poincaré upper half plane or more general noncompact type symmetric spaces, these spaces exhibit exponential volume growth, leading to the absence of analogues for the Calderon-Zygmund decomposition or useful covering lemmas. Despite these challenges, Clerc and Stein addressed the multiplier problem in their seminal work [9] on general noncompact type symmetric spaces. In the context of the rank one symmetric space $\mathbb{X}$, the authors observed that for a given multiplier m, if the associated multiplier operator T_m is to be bounded on $L^p(\mathbb{X})$ for some $p \in (1, \infty) \setminus \{2\}$, then m must necessarily extend to a bounded holomorphic function in the interior of the strip S_p with

$$S_p := \{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leq \gamma_p|\rho|\}, \quad \text{where} \quad \gamma_p := \left|\frac{2}{p} - 1\right|, \quad \text{for } p \in [1, \infty), \tag{1.2}$$

and ρ is half the sum of all positive roots with multiplicity. Subsequently, the endeavor to extend the classical Hörmander-Mikhlin multiplier theorem to symmetric spaces of noncompact type has attracted the attention of several authors [55, 2, 58, 1, 20, 37, 38]. In 1990, Anker, in his remarkable work [1], improved and generalized the previous results of Clerc and Stein [9], Stanton and Tomas [55], Anker and Lohoue [2], and Taylor [58] by proving the following analogue of the Hörmander-Mikhlin multiplier theorem on noncompact type symmetric spaces of arbitrary rank. However, to avoid introducing additional notation, we present the results here in the context of rank one symmetric spaces.

Theorem 1.4 ([1, Theorem 1])**.** *Let $\mathbb{X}$ be a rank one symmetric space of noncompact type, $1 < p < \infty$, $v = |\frac{1}{p} - \frac{1}{2}|$ and $N = [v \dim \mathbb{X}] + 1$. Assume that $m : \mathbb{R} \to \mathbb{C}$ extends to an even* *holomorphic function on* S_p°*,* $\frac{\partial^\alpha}{\partial\lambda^\alpha}m\,(\alpha = 0, 1, \cdots, N)$ *extend continuously to* S_p *and satisfy*

$$\sup_{\lambda\in S_p}(1+|\lambda|)^\alpha \left|\frac{\partial^\alpha}{\partial\lambda^\alpha}m(\lambda)\right| < \infty,$$

for all $\alpha = 0, 1, \cdots, N$*. Then the associated multiplier operator* T_m *is a bounded operator on* $L^p(\mathbb{X})$ *for all* $p \in (1,\infty)$*.*

We note that Anker also proved the weak type $(1, 1)$-boundedness of the multiplier operator by imposing certain regularity assumptions on the boundary. However, Anker suggested that it might be possible to relax these assumptions and allow m to have a singularity at the boundary points $\pm i\gamma_p|\rho|$ while still being an L^p multiplier on $\mathbb{X}$. Later, Ionescu [29, 30] made further improvements to the theorem by replacing the continuity requirement of the multiplier m on the boundary with a condition related to the singularity at $\pm i\gamma_p|\rho|$. This condition is the best-known sufficient condition of the Hörmander-Mikhlin type on $\mathbb{X}$.

Theorem 1.5 ([29, Theorem 8])**.** *Let* $\mathbb{X}$ *be a rank one symmetric space of noncompact type and* $1 < p < \infty$*. Assume that* $m : \mathbb{R} \to \mathbb{C}$ *extends to an even holomorphic function on* S_p° *and satisfies*

$$\sup_{\lambda\in S_p^\circ}(|\lambda \pm i\gamma_p|\rho||)^\alpha \left|\frac{\partial^\alpha}{\partial\lambda^\alpha}m(\lambda)\right| < \infty,$$

for all $\alpha = 0, 1, \cdots, \left[\frac{\dim \mathbb{X}+1}{2}\right] + 2$*. Then, the associated multiplier operator* T_m *is a bounded operator on* $L^p(\mathbb{X})$*.*

Motivated by the works of Anker [1] and Ionescu [29, 30], the authors Ricci and Wróbel [48] studied the L^p-boundedness $(1 < p < \infty)$ of multiplier operators on the radial sections of line bundles over the Poincaré upper half plane $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. More precisely, they focused on the group $G = \mathrm{SL}(2,\mathbb{R})$, and instead of considering $K = \mathrm{SO}(2)$-biinvariant functions, they explored the problem of the L^p-boundedness for the multiplier operator defined on functions on G satisfying the property:

$$f(k_\theta g k_\vartheta) = e^{in(\theta+\vartheta)}f(g), \tag{1.3}$$

for all $g \in G$, $k_\theta, k_\vartheta \in K$, where $n \in \mathbb{Z}$ is fixed and

$$k_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The functions satisfying (1.3) are called (n, n) type functions on G. In the special case where $n = 0$, their result [48, Theorem 5.3] aligns with the multiplier theorems on symmetric spaces obtained by Anker [1] and Stanton and Tomas [55] in the K-biinvariant case. Moreover, the authors [48] extended the result of Clerc and Stein [9, Theorem 1] in this setting by providing a necessary condition on the multiplier m for the associated multiplier operator to be L^p-bounded for $p \in (1,\infty) \setminus \{2\}$. For more details, please see Theorem 3.1.

Given the progress in studying multiplier operators, the present article continues the research trend by introducing pseudo-differential operators on radial sections of line bundles over the Poincaré upper half plane and investigating their boundedness properties.

In this study, we establish a sufficient condition on the symbol that ensures the L^p-boundedness of the corresponding pseudo-differential operator for (n, n) type functions on G; see Theorem 3.3. This result not only extends the findings in [48] for the multiplier case but also presents an analogous version of Theorem 1.1 in the context of Euclidean space.

Additionally, we present a weak-type version of the generalized transference principle by Coifmann-Weiss. By employing this principle, we establish a connection between the operator norm of the local part of our pseudo-differential operators and the corresponding Euclidean pseudo-differential operators.

Furthermore, we delve into the boundedness problem of pseudo-differential operators associated with non-smooth symbols in the space variables in our setting; see Theorem 3.5. This result is a natural analogue of the boundedness of rough pseudo-differential operators established by Kenig and Staubach in Euclidean spaces Theorem 1.3. To further investigate this area, inspired by the works of Kenig and Staubach [31], we introduce the class $\mathcal{S}^m_{\varrho,\infty}(S_p)$ of rough symbols with no regularity conditions in the space variable and analyze the boundedness properties of the associated pseudo-differential operators for (n, n) type functions on G.

Before we proceed into the specifics of the paper, let us compare our analysis of pseudo-differential operators with that of multiplier operators for (n, n) type functions on G.

In the context of multipliers, the associated operator can be represented as a convolution operator, which allows one to make use of various convolution inequalities like the *Kunze-Stein* phenomenon and Herz majorizing principle available on the group G. However, dealing with pseudo-differential operators does not provide the same convenience of employing these mentioned methods, marking a crucial distinction between the two cases. Additionally, in this group setting, there is an extra discrete component of the Plancherel measure, leading us to decompose our pseudo-differential operator Ψ_σ into discrete and continuous parts; see (6.1). Addressing the discrete part poses a new challenge, where we use an asymptotic estimate of the spherical functions $(\psi^{n,n}_{ik})$, departing from the *Kunze-Stein* phenomenon used in the multiplier case [48]. In fact, we establish a characterization of the matrix coefficients of discrete series representation lying in weak L^p spaces; see Lemma 5.9. This characterization represents the weak-type version of Miličić's result [39, Corollary, p.84] in the context of $\mathrm{SL}(2, \mathbb{R})$, which we employed to establish the boundedness of the discrete part of Ψ_σ.

Moving to the continuous part of Ψ_σ, we first use a functional identity of the spherical functions $(\phi^{n,n}_{\tau,\lambda})$ to rewrite the operator as a kernel integral operator on G. However, considering the exponential growth of the measure on the group, we further decompose the continuous part of Ψ_σ into two parts, namely the local part $(\Psi^{\mathrm{loc}}_\sigma)$ and the global part $(\Psi^{\mathrm{glo}}_\sigma)$; see Section 6.1.

Proving the boundedness of the local part $(\Psi^{\mathrm{loc}}_\sigma)$ constitutes a major challenge. In the case of multipliers, the availability of a transference principle for convolution operators leads to their boundedness. On the other hand, for pseudo-differential operators, we utilize a generalized Coifman-Weiss transference principle for singular integral operators to establish a relation between the operator norm of the local part of our pseudo-differential operators and the corresponding Euclidean pseudo-differential operators. This relation helps us derive the derivative condition on the space variable of the symbol.

The situation is entirely different for the global part $(\Psi^{\mathrm{glo}}_\sigma)$, where the analysis of the Poincaré upper half-plane becomes crucial. While multiplier theory in [48] relies on the Herz majorizing principle to prove the boundedness, we need to adopt a different approach. We use the global expansion of the spherical function and the holomorphicity condition on the dual variable of the symbol to obtain a quantitative estimate of the kernel away from the origin. Finally, by employing a property of the Abel transform and a duality argument, we establish the L^p-boundedness of $\Psi^{\mathrm{glo}}_\sigma$.

A critical element in proving the boundedness of Ψ_σ for a rough symbol σ (Theorem 3.5) is the established relation between the local part of a pseudo-differential operator on radial sections of line bundles over the Poincaré upper half-plane and the corresponding operator in the Euclidean space. Specifically, using the generalized Coifmann-Weiss transference principle, we reduce the L^p-boundedness of the local part of $\Psi_\sigma^{\mathrm{loc}}$ to the L^p-boundedness of certain Euclidean pseudo-differential operators whose symbols depend on σ (as shown in Section 8). This enables us to apply the result of Kenig and Staubach (Theorem 3.5) in our setting and, consequently, establish the boundedness of the operator $\Psi_\sigma^{\mathrm{loc}}$. The boundedness of $\Psi_\sigma^{\mathrm{dis}}$ and $\Psi_\sigma^{\mathrm{glo}}$ follows similarly, as these operators do not require any smoothness condition on the space variable of the symbol σ.

We conclude this section by providing an outline of this article.

In the next section, we present the necessary background on the group $\mathrm{SL}(2,\mathbb{R})$ and establish some results relevant to our study. Then, in Section 3, we introduce pseudo-differential operators for (n,n)-type functions on G and present our main results. In the same section, we introduce the class $\mathcal{S}^m_{\varrho,\infty}(S_p)$.

In Section 4, we delve into the generalized Coifman-Weiss transference principles. Following that, in Section 5, we prove some functional identities and derive asymptotic estimates of the spherical functions, which are crucial for our study.

Next, in Section 6, we represent Ψ_σ as singular integral operators. The subsequent sections, namely Sections 7, 8, and 9, are dedicated to stating and proving the boundedness results of $\Psi_\sigma^{\mathrm{dis}}$, $\Psi_\sigma^{\mathrm{loc}}$, and $\Psi_\sigma^{\mathrm{glo}}$, respectively. The proofs in these sections will lead to the establishment of Theorem 3.3.

Finally, in Section 10, we prove Theorem 3.5, concluding our investigation.

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2. Preliminaries

2.1. **Generalities.** The letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ will respectively denote the set of all natural numbers, the ring of integers, and the fields of real and complex numbers. We denote the set of all non-negative integers and nonzero integers by $\mathbb{Z}_+$ and $\mathbb{Z}^*$, respectively. For $z \in \mathbb{C}$, we use the notations $\operatorname{Re} z$ and $\operatorname{Im} z$ for real and imaginary parts of z, respectively. We shall follow the standard practice of using the letters C, C_1, C_2, etc., for positive constants, whose value may change from one line to another. Occasionally, the constants will be suffixed to show their dependencies on important parameters. We will use $X \lesssim Y$ or $Y \gtrsim X$ to denote the estimate $X \leq CY$ for some absolute constant $C > 0$. We shall also use the notation $X \simeq Y$ for $X \lesssim Y$ and $Y \lesssim X$. For any Lebesgue exponent $p \in [1, \infty)$, let p' denote the conjugate exponent $p/(p-1)$. Throughout this article, we denote

$$S_p = \{z \in \mathbb{C} : |\operatorname{Im} z| \le \gamma_p\} \quad \text{and} \quad \gamma_p = \left|\frac{2}{p} - 1\right|, \quad \text{for } p \in [1, \infty). \tag{2.1}$$

From the above definitions it is evident that $\gamma_p = \gamma_{p'}$ and $S_p = S_{p'}$ for all $p \in [1, \infty)$. We shall henceforth write S_p° and ∂S_p to denote the usual topological interior and the boundary of S_p respectively.

2.2. **Lorentz spaces.** Let (X, m) be a σ-finite measure space. For $f : X \to \mathbb{C}$ a measurable function on X, the distribution function d_f defined on $[0, \infty)$ is given by

$$d_f(\alpha) = m(\{x \in X \mid |f(x)| > \alpha\}).$$

For $p \in [1, \infty), q \in [1, \infty]$, the Lorentz spaces $L^{p,q}(X)$ consist of all measurable functions f on X for which $\|f\|_{p,q}$ is finite, where $\|f\|_{p,q}$ is the Lorentz space norm defined as follows, as in [21, Prop.1.4.9]

$$\|f\|_{p,q} = \begin{cases} p^{\frac{1}{q}} \left(\int_0^\infty \alpha^q d_f(\alpha)^{\frac{q}{p}} \frac{d\alpha}{\alpha} \right)^{\frac{1}{q}} & \text{when } q < \infty, \\ \sup\limits_{\alpha > 0} \alpha^p d_f(\alpha) & \text{when } q = \infty. \end{cases} \tag{2.2}$$

2.3. **The group** $\mathrm{SL}(2, \mathbb{R})$**.** From this section onwards, G will always denote the group $\mathrm{SL}(2, \mathbb{R})$. The Iwasawa decomposition for G gives a diffeomorphism between $K \times A \times N$ and G, where $K = \mathrm{SO}(2)$,

$$A = \left\{ a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\}, \quad \text{and} \quad N = \left\{ n_v := \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} : v \in \mathbb{R} \right\}.$$

That is, by the Iwasawa decomposition any $x \in G$ can be written uniquely as $x = k_\theta a_t n_v$; using this, we write

$$K(x) = k_\theta, \quad H(x) = t, \quad \text{and} \quad N(x) = v.$$

In fact, if $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$, then θ, t and v are given by

$$e^{2t} = a^2 + c^2, \; e^{i\theta} = \frac{a - ic}{\sqrt{a^2 + c^2}}, \text{ and } v = \frac{ab + cd}{\sqrt{a^2 + c^2}}. \tag{2.3}$$

We also have another Iwasawa decomposition

$$G = \overline{N} A K,$$

where

$$\overline{N} = \left\{ \overline{v} := \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} : v \in \mathbb{R} \right\}.$$

Next, let $A^+ = \{a_t \mid t > 0\}$; then the Cartan decomposition for G gives

$$G = K \overline{A^+} K.$$

Using this we define g^+ as the $\mathbb{R}^+$ component in the Cartan decomposition, of the element $g \in G$, that is we denote a_{g^+} as the unique element such that

$$g = k_1 a_{g^+} k_2, \quad \text{for some } k_1, k_2 \in K.$$

A function $f : G \to \mathbb{C}$ is said to be of (n, n) type if

$$f(k_\theta x k_\vartheta) = e_n(k_\theta) f(x) e_n(k_\vartheta) \tag{2.4}$$

for all $k_\theta, k_\vartheta \in K$ and $x \in G$, where $e_n(k_\theta) := e^{in\theta}$. If $n = 0$, we refer to such functions as K-biinvariant functions.

By the Cartan decomposition we can extend a function f defined on $\overline{A^+}$ to an (n, n) type function on G by

$$f(k_\theta a_t k_\vartheta) = e_n(k_\theta) f(a_t) e_n(k_\vartheta), \quad \text{for all } k_\theta, k_\vartheta \in K. \tag{2.5}$$

It is known that the following groups G, $\overline{N}$, and K are unimodular, and we will denote the (left or equivalently right) Haar measures of these groups as dx, $d\overline{v}$ and dk, where $\int_K dk = 1$. Then we have the following integral formulae corresponding to the Iwasawa and Cartan decompositions, which hold for any integrable function f:

$$\int\limits_G f(x)dx = \int_{\overline{N}} \int_{\mathbb{R}} \int_K f(\overline{v} a_t k) e^{2t} d\overline{v} dt dk, \tag{2.6}$$

and

$$\int_G f(x)dx = \int_K \int_{\mathbb{R}^+} \int_K f(k_1 a_t k_2) \Delta(t) dk_1 dt dk_2, \tag{2.7}$$

where $\Delta(t) = 2\sinh 2t$. We recall the following integration formula on K from [22, Lemma 5.19, p.197].

Lemma 2.1. *Let $x \in G$. The mapping $T_x : k \mapsto K(xk)$ is a diffeomorphism of K onto itself and*

$$\int_K F(K(xk))\, dk = \int_K F(k) e^{-2H(x^{-1}k)}\, dk, \quad F \in C(K). \tag{2.8}$$

We require a connection between the Iwasawa decomposition of $G = \overline{N}AK$ and the Cartan decomposition. Strömberg [57] previously employed a similar relation for noncompact-type symmetric spaces. In this context, Ionescu established the following result:

Lemma 2.2. ([29, Lemma 3]) *If $\overline{v} \in \overline{N}$ and $r \geq 0$, then*

$$[\overline{v} a_r]^+ = r + H(\overline{v}) + E(\overline{v}, r), \tag{2.9}$$

where

$$0 \leq E(\overline{v}, r) \leq 2e^{-2r}. \tag{2.10}$$

From the formula (2.3), we have

$$e^{2H(\overline{v})} = \left(1 + v^2\right), \quad \text{for all } v \in \mathbb{R},$$

and so

$$H(\overline{v}) \geq 0, \quad \text{for all } \overline{v} \in \overline{N}. \tag{2.11}$$

Also, it is easy to see for any $\epsilon_0 > 0$ that

$$\int_{\overline{N}} e^{-(1+\epsilon_0)H(\overline{v})} d\overline{v} = C_{\epsilon_0} < \infty. \tag{2.12}$$

It is well known that the abelian group A acts as a dilation on $\overline{N}$, by the mapping

$$\overline{n} \mapsto a\overline{n}a^{-1} \in \overline{N}.$$

Moreover, if δ_r is the dilation of $\overline{N}$, defined by

$$\delta_r(\overline{n}) := a_r \overline{n} a_{-r}, \tag{2.13}$$

then the following is true

$$\int_{\overline{N}} \mathfrak{h}(\delta_r(\overline{n}))\, d\overline{n} = e^{2r} \int_{\overline{N}} \mathfrak{h}(\overline{n})\, d\overline{n}, \tag{2.14}$$

for any integrable function $\mathfrak{h}$ on $\overline{N}$. We equip the set $\overline{N}A$ with the binary operation induced by this conjugation δ_r and refer to it as the semi-direct product $\overline{N} \ltimes A$. With the Iwasawa decomposition, as shown in (2.6), we can establish an identification between K-right invariant functions on G and functions on $\overline{N}A$, and it is known that the corresponding L^p-norms coincide. Additionally, we will make use of the fact that the Abel transform

$$\mathcal{A}\phi(r) := e^r \int_{\overline{N}} \phi(\overline{n} a_r) d\overline{n} \tag{2.15}$$

takes a K-biinvariant function ϕ to an even function on $\mathbb{R}$ (see [23, p. 381]).

2.4. **Spherical analysis of (n, n) type functions.** We are going to define Fourier transforms of suitable (n, n) functions using representations of G (see Chapter 9 of [3]). Let $\widehat{M}$ denote the equivalence classes of irreducible representations of $M = \{\pm I\}$, where I is the identity matrix. Then

$$\widehat{M} = \{\tau_+, \tau_-\},$$

where $\tau_+(\pm I) = 1$ and $\tau_-(\pm I) = \pm 1$. For each $\tau \in \widehat{M}$, $\mathbb{Z}^\tau$ stands for the set of even integers for $\tau = \tau_+$, the set of odd integers for $\tau = \tau_-$ and $-\tau$ will denote the opposite parity of τ. We define

$$\Gamma_n = \begin{cases} \{k : 0 < k < n \text{ and } k \in \mathbb{Z}^{-\tau}\} \text{ if } n > 0 \\ \{k : n < k < 0 \text{ and } k \in \mathbb{Z}^{-\tau}\} \text{ if } n < 0. \end{cases} \tag{2.16}$$

For $\lambda \in \mathbb{C}$ and $\tau \in \widehat{M}$, let $(\pi_{\tau,\lambda}, H_\tau)$ be the principal series representation of G given by

$$(\pi_{\tau,\lambda}(x) e_n)(k) = e^{(i\lambda - 1)H(xk)} e_n \left(K(x^{-1}k^{-1})\right)^{-1} \tag{2.17}$$

for all $x \in G, k \in K$, where H_τ is the subspace of $L^2(K)$ generated by the orthonormal set $\{e_n : n \in \mathbb{Z}^\tau\}$. This representation is unitary if and only if $\lambda \in \mathbb{R}$. For $\lambda = 0$, the representation $\pi_{\tau_-,0}$ has two irreducible subrepresentations, the so called mock discrete series. We will denote them by D_+ and D_-. The representation spaces of D_+ and D_- contain $e_n \in L^2(K)$ respectively for positive odd n's and negative odd n's. For each $k \in \mathbb{Z}^*$ (set of nonzero integers), there is a discrete series representation π_{ik}, which occurs as a subrepresentation of $\pi_{\tau,i|k|}$, $k \in \mathbb{Z} \setminus \mathbb{Z}^\tau$ (see [3, p.19]). We define for $k \in \mathbb{Z}^*$

$$\mathbb{Z}(k) = \begin{cases} \{m \in \mathbb{Z}^\tau : m \geq k+1\} \text{ if } k \geq 1 \\ \{m \in \mathbb{Z}^\tau : m \leq k-1\} \text{ if } k \leq -1. \end{cases}$$

For $n \in \mathbb{Z}^\tau$, the canonical matrix coefficient for the principal series

$$\phi^{n,n}_{\tau,\lambda}(x) := \langle \pi_{\tau,\lambda}(x) e_n, e_n \rangle = \int_K e^{-(i\lambda+1)H(xk)} e_n(k^{-1}) \overline{e_n(K(xk)^{-1})} dk, \tag{2.18}$$

are functions of (n,n) type. For $k \in \mathbb{Z}^*$ and $n \in \mathbb{Z}(k)$ the canonical matrix coefficient of the discrete series representation is

$$\psi_{ik}^{n,n}(x) := \langle \pi_{ik}(x) e_n^k, e_n^k \rangle_k,$$

where e_n^k are the renormalized basis and $\langle \, , \rangle_k$ is the renormalized inner product for π_{ik}, see [3, p. 20] for more details. These functions $\phi_{\tau,\lambda}^{n,n}$ and $\psi_{ik}^{n,n}$ are also known as spherical functions of (n,n) type. We will denote H_k as the Hilbert space generated by $\{e_m^k : m \in \mathbb{Z}(k)\}$. It is known [3, Prop. 7.3] that

$$\psi_{ik}^{n,n} = \psi_{-ik}^{n,n} = \phi_{\tau, i|k|}^{n,n}, \tag{2.19}$$

where $\tau \in \widehat{M}$ is determined by $k \in \mathbb{Z}^{-\tau}$. Additionally, it is worth noting that $\phi_{\tau_+,\lambda}^{0,0}$ corresponds to the elementary spherical function commonly denoted as ϕ_λ.

Let Ω denote the Casimir element on G (see [3, (2.6)]), which acts as a biinvariant differential operator on G. As stated in [3, (4.7)], the smooth eigenfunctions $\phi_{\tau,\lambda}^{n,n}$ of Ω satisfy the differential equation:

$$\Omega f = -\frac{\lambda^2+1}{4} f. \tag{2.20}$$

Furthermore, the spherical functions $\phi_{\tau,\lambda}^{n,n}$ have the following well-known properties:

(1) We have from [46, Section 4.1]

$$\phi_{\tau,\lambda}^{n,n}(a_t) = \phi_{\tau,\lambda}^{-n,-n}(a_t) = \phi_{\tau,\lambda}^{n,n}(a_{-t}), \quad \text{for all } t \in \mathbb{R}.$$

(2) For any fixed $x \in G$, $\lambda \mapsto \phi_{\tau,\lambda}^{n,n}(x)$ is an entire function.

(3) By utilizing the Cartan decomposition of G, it is easy to see that

$$\phi_{\tau,\lambda}^{-n,-n}(x^{-1}) = \phi_{\tau,\lambda}^{n,n}(x) \quad \text{and} \quad \phi_{\tau,\lambda}^{n,n} = \phi_{\tau,-\lambda}^{n,n}, \tag{2.21}$$

for all $x \in G, \lambda \in \mathbb{R}$.

We remark that we use a different parameterization of the representations and spherical functions from Barker [3]. According to our definition in (2.17), the unitary dual of $\mathrm{SL}(2,\mathbb{R})$ is $\mathbb{R}$, while according to Barker's convention, the unitary dual of $\mathrm{SL}(2,\mathbb{R})$ is $i\mathbb{R}$. As a result, our π_λ corresponds to his $\pi_{-i\lambda}$, and similarly, $\phi_{\tau,\lambda}^{n,n}$ and $\psi_{ik}^{n,n}$ are reparametrized accordingly. This choice of parametrization offers a clearer analogy with the general semisimple Lie groups case, making our analysis more transparent.

Let $n \in \mathbb{Z}^\tau$. Then for a smooth compactly supported (n,n) type function f on G, the principal series Fourier transform of f is defined by

$$\widehat{f_H}(\lambda) = \int_G f(x) \phi_{\tau,\lambda}^{n,n}(x^{-1}) dx,$$

for all $\lambda \in \mathbb{R}$, and the discrete series Fourier transform is defined by

$$\widehat{f_B}(ik) = \int_G f(x) \psi_{ik}^{n,n}(x^{-1}) dx,$$

for all $k \in \Gamma_n$. Then the inversion formula is given by [3, Theorem 10.4]:

$$f(x) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \widehat{f}_H(\lambda) \phi_{\tau,\lambda}^{n,n}(x) \left| c_\tau^{n,n}(\lambda) \right|^{-2} d\lambda + \frac{1}{2\pi} \sum_{k \in \Gamma_n} \widehat{f}_B(ik) \psi_{ik}^{n,n}(x) |k|, \tag{2.22}$$

where $\tau \in \widehat{M}$ is determined by $n \in \mathbb{Z}^\tau$, and $c_\tau^{n,n}(\lambda)$ is given by [3, (6.2)]

$$c_\tau^{n,n}(\lambda) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{i\lambda}{2}\right)\Gamma\left(\frac{1+i\lambda}{2}\right)}{\Gamma\left(\frac{1+i\lambda-|n|}{2}\right)\Gamma\left(\frac{1+i\lambda+|n|}{2}\right)}, \tag{2.23}$$

for all $\lambda \in \mathbb{C}$. The functions $c_\tau^{n,n}$ are regarded as meromorphic functions on the complex plane. In the following, we identify the singular points and discuss the estimates of the function $c_\tau^{n,n}(-\lambda)^{-1}$.

Lemma 2.3. *Let $\tau \in \widehat{M}$ and $n \in \mathbb{Z}^\tau$. The function $\lambda \to c_\tau^{n,n}(-\lambda)^{-1}$ is meromorphic on the complex plane and exhibits the following properties:*

(i) *The zeros of $c_\tau^{n,n}(-\cdot)^{-1}$ are simple and occur at $\lambda \in i\mathbb{Z}^{\tau-}$ such that $\operatorname{Im}\lambda \leq 0$.*

(ii) *The function $c_\tau^{n,n}(-\cdot)^{-1}$ has simple poles at $\lambda \in i\mathbb{Z}^{-\tau}$ such that $\operatorname{Im}\lambda < -|n|$ or $0 < \operatorname{Im}\lambda < |n|$.*

(iii) *For $p > 1$ and a fixed $N \in \mathbb{N}$, following estimate holds*

$$\left|\frac{d^\alpha}{d\lambda^\alpha} c_\tau^{n,n}(-\lambda)^{-1}\right| \leq C_\alpha (1+|\lambda|)^{1/2-\alpha}, \tag{2.24}$$

for all integers $\alpha = 0, 1, \ldots N$, and for all $\lambda \in \mathbb{C}$ with $0 \leq \operatorname{Im}\lambda \leq \gamma_p$.

(iv) *Furthermore, if $p = 1$ and $n \in \mathbb{Z}^{\tau-}$, then (2.24) holds true. However, if $n \in \mathbb{Z}^{\tau+}$, then (2.24) is valid for all $\lambda \in \mathbb{C} \setminus \boldsymbol{B}(i)$ with $0 \leq \operatorname{Im}\lambda \leq 1$, where $\boldsymbol{B}(i)$ is any compact neighbourhood of i in the complex plane.*

Proof. We have (i) and (ii) according to [3, Proposition 6.1]. Now, considering $p > 1$, we observe from (2.23) that we may assume $n \geq 0$. Consequently, using the property of the Gamma function, we can write from (2.23):

$$c_\tau^{n,n}(-\lambda)^{-1} = \frac{\lambda + i(n-1)}{\lambda - i(n-1)} c_\tau^{n-2,n-2}(-\lambda)^{-1}.$$

Hence, for $n \geq 2$, using the above formula, we obtain

$$c_\tau^{n,n}(-\lambda)^{-1} = \left(\prod_{k \in \Gamma_n} p_k(\lambda)\right) c_\tau^{m,m}(-\lambda)^{-1},$$

where $p_k(\lambda) = \frac{\lambda+ik}{\lambda-ik}$, and $m = 0$ if $\tau = \tau_+$, $m = 1$ if $\tau = \tau_-$. Using the Leibniz rule, we can easily see that $p_k(\lambda)$ satisfy the following inequalities for all λ, with $0 \leq \operatorname{Im}\lambda \leq \gamma_p$:

$$\left|\frac{d^\alpha}{d\lambda^\alpha} p_k(\lambda)\right| \leq C_\alpha (1+|\lambda|)^{-\alpha}$$

for all integers $\alpha = 0, 1, \ldots N$. With this inequality in mind, it is sufficient to prove (2.24) for $c_\tau^{0,0}(-\lambda)^{-1}$ and $c_\tau^{1,1}(-\lambda)^{-1}$. Finally, the required estimates of $c_\tau^{0,0}(-\lambda)^{-1}$ and its derivatives

$$\left|\frac{d^\alpha}{d\lambda^\alpha} c_\tau^{0,0}(-\lambda)^{-1}\right| \leq C_\alpha (1+|\lambda|)^{1/2-\alpha},$$

follow from [29, (4.2)] (see also [28, (A.2)]), the estimates of $c_\tau^{1,1}(-\lambda)^{-1}$ follows similarly using the Stirling formula [59, Chapter 4]. This, in turn, completes the proof of (iii).

To prove (iv), we note that for $p = 1$, the same proof is applicable when $n \in \mathbb{Z}^{\tau-}$ since, in this situation, the meromorphic function $p_1(\lambda)$ does not occur, and thus, there is no singularity at $\lambda = i$. However, when $n \in \mathbb{Z}^{\tau+}$, we exclude a neighborhood of i in the complex plane to address the singularity of $p_1(\lambda)$ at $\lambda = i$, and obtain (2.24). □

Lemma 2.4 ([55, Lemma 4.2]). *Let N be any fixed natural number. Then we have*

$$\left| \frac{d^\alpha}{d\lambda^\alpha} |c_\tau^{n,n}(\lambda)|^{-2} \right| \leq C_\alpha (1+|\lambda|)^{1-\alpha}, \tag{2.25}$$

for all $\lambda \in \mathbb{R}$ and $\alpha \in [0, N]$.

Proof. The lemma above follows from the explicit formula of $|c_\tau^{n,n}(\lambda)|^{-2}$,

$$|c_\tau^{n,n}(\lambda)|^{-2} = \begin{cases} \frac{\lambda\pi}{2} \tanh(\lambda\pi/2), & \text{if } \tau = \tau_+ \\ \frac{\lambda\pi}{2} \coth(\lambda\pi/2), & \text{if } \tau = \tau_-, \end{cases} \tag{2.26}$$

which can be found in [3, (10.1)]. □

2.5. **Hörmander norm of symbols.** Consider an open subset U of the complex plane $\mathbb{C}$. We use $\mathcal{H}^\infty(U)$ to denote the space of bounded holomorphic functions in U, which is equipped with the supremum norm. Now, let m be a bounded holomorphic function defined on U, which is continuous on the closure $\overline{U}$, including its derivatives up to the kth order. In [18], the authors introduced the Mikhlin-Hörmander norm at infinity of order k on $\overline{U}$, which is given by the following expression:

$$\|m\|_{\mathcal{MH}(\overline{U},k)} = \max_{\alpha \in \{0,\dots,k\}} \sup_{\lambda \in \overline{U}} (1+|\lambda|)^\alpha \left| \frac{\partial^\alpha}{\partial\lambda^\alpha} m(\lambda) \right|. \tag{2.27}$$

Now, let us consider a smooth function $\sigma : \mathbb{R} \times U \to \mathbb{C}$ such that, for each fixed $s \in \mathbb{R}$, the map $\lambda \mapsto \sigma(s, \lambda)$ defines a bounded holomorphic function on U and remains continuous on its closure $\overline{U}$. In this context, we define the Hörmander norm of such symbols σ at infinity of order (j, k) on $\mathbb{R} \times \overline{U}$ as follows:

$$\|\sigma\|_{\mathcal{H}(\overline{U},j,k)} = \max_{\substack{\alpha \in \{0,\dots,k\} \\ \beta \in \{0,\dots,j\}}} \sup_{\lambda \in \overline{U},\, s \in \mathbb{R}} (1+|\lambda|)^\alpha \left| \frac{\partial^\beta}{\partial s^\beta} \frac{\partial^\alpha}{\partial\lambda^\alpha} \sigma(s,\lambda) \right|. \tag{2.28}$$

Slightly abusing this notation, we also write:

$$\|\sigma\|_{\mathcal{H}(\mathbb{R},j,k)} = \max_{\substack{\alpha \in \{0,\dots,k\} \\ \beta \in \{0,\dots,j\}}} \sup_{\lambda \in \mathbb{R},\, s \in \mathbb{R}} (1+|\lambda|)^\alpha \left| \frac{\partial^\beta}{\partial s^\beta} \frac{\partial^\alpha}{\partial\lambda^\alpha} \sigma(s,\lambda) \right|. \tag{2.29}$$

We say σ belongs to $\mathcal{H}(\overline{U}, j, k)$, if $\sigma : \mathbb{R} \times U \to \mathbb{C}$ is a smooth function and $\|\sigma\|_{\mathcal{H}(\overline{U},j,k)} < \infty$.

3. Main results

In this section, we will present our main results concerning the pseudo-differential operators for (n, n) type functions on $G = \mathrm{SL}(2, \mathbb{R})$. Before doing so, we first recall the multiplier results of Ricci and Wróbel in this setting.

Throughout this article, we assume that $n \in \mathbb{Z}$ is a fixed integer. Let $m : \mathbb{R} \cup i\Gamma_n \to \mathbb{C}$ be a bounded measurable function. Then the associated Fourier multiplier operator T_m for (n, n) type functions on G is defined through the inversion formula (see (2.22)) as follows:

$$T_m f(x) = \frac{1}{4\pi^2} \int_{\mathbb{R}} m(\lambda) \widehat{f_H}(\lambda) \phi_{\tau,\lambda}^{n,n}(x) |c_\tau^{n,n}(\lambda)|^{-2} d\lambda + \frac{1}{2\pi} \sum_{k \in \Gamma_n} m(ik) \widehat{f_B}(ik) \psi_{ik}^{n,n}(x) |k|. \tag{3.1}$$

In their work [48], the authors observe that if T_m is bounded on $L^p(G)_{n,n}$ for some $p \in (1,\infty)\setminus\{2\}$, then the multiplier m, which was initially defined on $\mathbb{R}\times i\Gamma_n$, must necessarily extend to a bounded even holomorphic function in S_p°. Moreover, they demonstrated that ([48, Proposition 5.2])

$$\|m\|_{\mathcal{H}^\infty(S_p^\circ)} \leq \|T_m\|_{L^p\to L^p}.$$

With the notations introduced in Section 2.5, the authors proved the following analogue of the Hörmander-Mikhlin multiplier theorem.

Theorem 3.1 ([48, Theorem 5.3])**.** *Fix $1<p<\infty$, $p\neq 2$. Assume that the multiplier $m : S_p \cup i\Gamma_n \to \mathbb{C}$ is a function satisfying the following properties:*

(i) *The multiplier m is a bounded even holomorphic function in S_p°.*
(ii) $\|m\|_{\mathcal{MH}(S_p,2)} < \infty$.

Then the corresponding Fourier multiplier operator T_m is bounded on $L^p(G)_{n,n}$. Furthermore, there exists a constant $C_{p,n} > 0$ such that

$$\|T_m f\|_{L^p(G)_{n,n}} \leq C_{p,n}\left(\|m\|_{\mathcal{MH}(S_p,2)} < \infty + \sum_{k\in\Gamma_n} |k|\,|m(ik)|\right)\|f\|_{L^p},$$

for all $f \in L^p(G)_{n,n}$.

Remark 3.2. It is well-known that multiplier theorems are valuable tools for estimating the L^p behavior of differential operators. As an example in this setting, let us consider the following multiplier

$$m(\lambda) = \left(\frac{\lambda^2+1}{4}\right)^{i\nu}, \qquad \text{for all } \lambda \in S_p \cup i\Gamma_n,$$

where $1 \leq p < \infty$ and $\nu \in \mathbb{R}$. We note that the associated multiplier operator for the (n,n) type function is $(-\Omega)^{i\nu}$, where we recall that Ω (see (2.20)) is the Casimir element of G. Using the aforementioned theorem, we can conclude that this operator is bounded from $L^p(G)_{n,n}$ to itself for all $p \in (1,\infty)$. However, this result does not provide information about the case $p=1$. To address this, we observe that the multiplier m has a zero at $\lambda = i \in S_1$. Building upon this observation, we establish a sufficient condition for a multiplier (or symbol) that guarantees the weak type $(1,1)$-boundedness of the corresponding multiplier operator T_m; please see Theorem 3.5 (2).

Inspired by the results on multipliers discussed above, our interest naturally shifts to exploring the implications of substituting multipliers with symbols of broader generality. Specifically, we aim to determine the conditions on the symbol under which the associated pseudo-differential operator remains bounded. In the Euclidean space, these conditions often involve considering the regularity of the symbol $\sigma(x,\lambda)$ with respect to both x and λ, its growth properties, and its behavior at infinity. Thus, finding the appropriate conditions on $\sigma(x,\lambda)$ becomes a fundamental aspect of our analysis. Within this context, we formally introduce pseudo-differential operators in our framework to present our results.

Let $\sigma : G \times \mathbb{R} \cup i\Gamma_n \to \mathbb{C}$ be a suitable function. We define the associated pseudo-differential operator Ψ_σ for (n,n) type functions on G through the inversion formula (see

(2.22)) as follows:

$$\Psi_\sigma f(x) = \frac{1}{4\pi^2}\int_{\mathbb{R}} \sigma(x,\lambda)\widehat{f_H}(\lambda)\phi^{n,n}_{\tau,\lambda}(x)|c^{n,n}_\tau(\lambda)|^{-2}d\lambda + \frac{1}{2\pi}\sum_{k\in\Gamma_n}\sigma(x,ik)\widehat{f_B}(ik)\psi^{n,n}_{ik}(x)|k|, \quad (3.2)$$

where τ is determined by $n \in \mathbb{Z}^\tau$. We now present one of our main results on pseudo-differential operators alluded to in the introduction.

Theorem 3.3. *Let $p \in (1,\infty)\setminus\{2\}$. Suppose that $\sigma : G \times S_p \cup i\Gamma_n \to \mathbb{C}$ is a function satisfying the following properties:*

(i) *For each $\lambda \in S_p \cup i\Gamma_n$, $x \mapsto \sigma(x,\lambda)$ is a K-biinvariant function on G and $\|\sigma\|_{L^\infty(G\times i\Gamma_n)}$ is finite.*

(ii) *For each $x \in G$, $\lambda \mapsto \sigma(x,\lambda)$ is an even holomorphic function on the strip S_p°.*

(iii) *For each $\overline{v} \in \overline{N}$, the function $(s,\lambda) \mapsto \sigma_{\overline{v}}(s,\lambda) := \sigma(\overline{v}a_s,\lambda)$ belongs to $\mathcal{H}(S_p,2,2)$ and*

$$\sup_{\overline{v}\in\overline{N}} \|\sigma_{\overline{v}}\|_{\mathcal{H}(S_p,2,2)} < \infty,$$

where $\|\sigma\|_{\mathcal{H}(S_p,2,2)}$ is defined as in (2.28).

Then the operator Ψ_σ extends to a bounded operator on $L^p(G)_{n,n}$ to itself. Moreover, there exists a constant $C_{p,n} > 0$ such that

$$\|\Psi_\sigma f\|_{L^p(G)} \leq C_{p,n}\left(\sup_{\overline{v}\in\overline{N}} \|\sigma_{\overline{v}}\|_{\mathcal{H}(S_p,2,2)} + \|\sigma\|_{L^\infty(G\times i\Gamma_n)}\right)\|f\|_{L^p(G)},$$

for all $f \in L^p(G)_{n,n}$.

Remark 3.4. (1) We recall that in the case of multipliers, the holomorphic extension property of the multiplier is necessary for the multiplier operator to be bounded on $L^p(G)_{n,n}$. Taking this into account, along with the results on pseudo-differential operators in Euclidean spaces, it is natural to assume that the symbol satisfies the holomorphicity condition stated in Theorem 3.3 to establish the boundedness of the associated pseudo-differential operator. In particular, when the symbol σ is independent of the space variable, Theorem 3.3 recovers the result for the multiplier case as presented in [48, Theorem 5.3].

(2) Next, we compare Theorem 3.3 with the corresponding result on rank one symmetric spaces of noncompact type. In [45, Theorem 1.6], the authors established the L^p-boundedness (for $p \in (1,\infty)\setminus\{2\}$) of the pseudo-differential operators on symmetric spaces by assuming, among other things, the following condition:

$$\left|\frac{\partial^\beta}{\partial s^\beta}\frac{\partial^\alpha}{\partial\lambda^\alpha}\sigma(ga_s,\lambda)\right| \leq C_{\alpha,\beta}(1+|\lambda|)^{-\alpha}, \quad (3.3)$$

for all $g \in G, s \in \mathbb{R}$ and $\lambda \in S_p$, where the order of the mixed partial derivatives is up to a prescribed order. In our case, due to the K-biinvariant hypothesis on the symbol, we can observe that the corresponding condition (3.3) for the symbol in the symmetric space simplifies to hypothesis (iii) of Theorem 3.3. However, there is a difference in the behavior of our integral kernel $\mathcal{K}$ of the operator Ψ_σ compared to the one in symmetric spaces. Our kernel $\mathcal{K}$ is no longer a K-biinvariant function with respect to the second variable, which means that the argument used in [45] cannot be directly applied in our setting.

(3) We only need a regularity condition on the space variable of the symbol $\sigma(x, \lambda)$ when λ lies on the real line. There is no need for any regularity condition on the space variable x for all $\lambda \in S_p \setminus \mathbb{R}$. Please see Remark 9.2.

(4) As the spherical functions are of (n, n) type, we observe from (3.2) that for any given $f \in C_c^\infty(G)_{n,n}$, if $\sigma(\cdot, \lambda)$ is any fixed K-type function then $\Psi_\sigma(f)$ cannot be of (n, n) type unless $\sigma(\cdot, \lambda)$ is a K-biinvariant function. This demonstrates that the K-biinvariant nature of the symbol is necessary to establish the boundedness of Ψ_σ in $L^p(G)_{n,n}$.

We now shift our focus to the boundedness problem of the pseudo-differential operator Ψ_σ associated with non-smooth symbols in the space variables. Building upon the works of Kenig and Staubach [31], we introduce the symbol class $\mathcal{S}^m_{\varrho,\infty}(S_p)$ for rough symbols in our context.

The symbol class $\mathcal{S}^m_{\varrho,\infty}(S_p)$, where $1 < p < \infty$, $m \in \mathbb{R}$, and $0 \leq \varrho \leq 1$, includes all functions $\sigma : G \times S_p \cup i\Gamma_n \to \mathbb{C}$, that satisfy the following properties:

(1) For each $\lambda \in S_p \cup i\Gamma_n$, the function $x \mapsto \sigma(x, \lambda)$ is K-biinvariant, measurable on G and $\|\sigma\|_{L^\infty(G \times i\Gamma_n)} < \infty$.

(2) For each $x \in G$, the function $\lambda \mapsto \sigma(x, \lambda)$ is an even holomorphic function on S_p° and satisfies the differential inequalities:

$$\left\| \frac{\partial^\alpha}{\partial \lambda^\alpha} \sigma(x, \lambda) \right\|_{L^\infty_x} \leq C_\alpha (1 + |\lambda|)^{m - \varrho\alpha}, \tag{3.4}$$

for all $\lambda \in S_p$, $\alpha \in \mathbb{N}$, where $C_\alpha > 0$ are constants.

Additionally, we define the symbol class $\mathcal{S}^m_{\varrho,\infty}(S_1)$, which consists of functions $\sigma : G \times S_1 \to \mathbb{C}$ that satisfy properties (1) and (2) for $p = 1$, and further have the property $\frac{\partial^\alpha}{\partial \lambda^\alpha} \sigma(x, \lambda)|_{\lambda = i} = 0$, for all $\alpha \in \{0, 1, 2\}$.

Example 3.1. Let $1 < p < \infty$. For $\nu \in \mathbb{C}$, let us define

$$\sigma(g, \lambda) = \left(\lambda^2 + 1\right)^{i[g]^+ - i\nu}, \quad g \in G, \lambda \in S_p \cup i\Gamma_n,$$

where we recall that $[g]^+$ denotes the $\mathbb{R}^+$ component of g in the Cartan decomposition (see Section 2.3). By a simple computation, it follows that σ_ν satisfies (3.4). Now, to check the other condition, we observe that $g \mapsto [g]^+$ is a K-biinvariant function. Thus, we have $\sigma \in \mathcal{S}^{2\,\mathrm{Im}\,\nu}_{1,\infty}(S_p)$, for all $p \in (1, \infty) \setminus \{2\}$. More generally, one can consider the following class of symbols of the form

$$\sigma(g, \lambda) = \left(\lambda^2 + 1\right)^{i\eta(g) - i\nu}, \quad g \in G, \ \lambda \in S_p \cup i\Gamma_n,$$

where η is any real-valued K-biinvariant function on G.

Example 3.2. Let $1 < p < \infty$. For a fixed real number $\zeta \leq 0$, let us define

$$\sigma_\zeta(g, \lambda) = \left(\lambda^2 + 1 + \eta(g)\right)^{\frac{\zeta}{2}}, \quad g \in G, \lambda \in S_p \cup i\Gamma_n,$$

where η is any non-negative K-biinvariant function on G. Then by using $\eta \geq 0$ and $\zeta \leq 0$, it follows that $\sigma_\zeta \in \mathcal{S}^\zeta_{1,\infty}(S_p)$.

We now present our boundedness result for the associated pseudo-differential operator Ψ_σ when $\sigma \in \mathcal{S}^m_{\varrho,\infty}(S_p)$, which serves as an analogue of Theorem 1.3 established by Kenig and Staubach [31] in the Euclidean spaces.

Theorem 3.5. *Let $p \in [1,2) \cup (2,\infty)$. Suppose that $m < 0$ and $\sigma = \sigma(x,\lambda) \in \mathcal{S}^m_{1,\infty}(S_p)$. Then we have the following:*

(1) *The operator Ψ_σ is a bounded operator from $L^p(G)_{n,n}$ to itself for all $p \in (1,2) \cup (2,\infty)$.*
(2) *When $p = 1$ and $n \in \mathbb{Z}^{\tau-} \cup \{0\}$, then Ψ_σ is a bounded operator from $L^1(G)_{n,n}$ to itself. If $n \in \mathbb{Z}^{\tau+} \setminus \{0\}$, Ψ_σ is a weak type $(1,1)$-bounded operator from $L^1(G)_{n,n}$ to $L^{1,\infty}(G)_{n,n}$.*

Remark 3.6. (1) We note that similar to Theorem 1.3, the number of derivatives needed in the dual variable is not infinite. In fact, by following the proof of Theorem 3.5, one can demonstrate that the required number of derivatives is finite, although it does depend on the value of m as in (3.4).

(2) Let $p \in (1,\infty) \setminus \{2\}$. Suppose $\sigma : G \times S_p$ is a function satisfying all the hypotheses of Theorem 3.3 or Theorem 3.5. Then the operator Ψ_σ is bounded from $L^p(G)_{n,n}$ to itself and from $L^{p'}(G)_{n,n}$ to itself. By interpolation, it follows that the operator Ψ_σ is also bounded on $L^2(G)_{n,n}$. However, the condition imposed on the symbol σ for this result, particularly the holomorphicity, is not natural for establishing L^2-boundedness. Further investigation is required to prove the L^2-boundedness of Ψ_σ under the assumption that σ satisfies only the smoothness condition (without holomorphicity) on the real line.

4. Generalized Coifman-Weiss transference principles

Let $\{X, \mathfrak{m}\}$ be a measure space. An operator $\mathcal{B}$ on $L^p(X,\mathfrak{m})$ is said to be of weak-type (p,p) if it maps $\phi \in L^p(X,\mathfrak{m})$ into a measurable function defined on a measure space (Y,ν) in such a way that for each $s > 0$,

$$\nu\{y \in Y : |(\mathcal{B}\,\phi)(y)| > s\} \leq [\mathcal{C}\|\phi\|_{L^p(X)}/s]^p,$$

where $\mathcal{C}$ is independent of $\phi \in L^p(X)$.

Let G be a locally compact group satisfying the following property: Given a compact subset B of G and $\epsilon > 0$, there exists an open neighborhood V of the identity e having finite measure such that

$$\frac{\mu(B^{-1}V)}{\mu(V)} \leq 1 + \epsilon, \tag{4.1}$$

where μ is, say, left Haar measure on the group G. Let us suppose, further, that $\mathcal{R}$ is a representation consisting of measure-preserving transformations of the space X. Since $\mathcal{R}_u$ is measure-preserving

$$\int_X |f(\mathcal{R}_u x)|^p d\mathfrak{m}(x) = \int_X |f(x)|^p d\mathfrak{m}(x) \tag{4.2}$$

for all $u \in G$. The transformation we will consider is of the form

$$(Tf)(x) = \int_G k(x, \mathcal{R}_u x, u) f(\mathcal{R}_u x) d\mu(u), \tag{4.3}$$

where $k(x,y,u)$ is a measurable function on $X \times X \times G$ which is 0 if u does not belong to a compact set $B \subset G$. Moreover, we assume that for each $x \in X$, the kernel

$$k_x(v,u) := k(\mathcal{R}_v x, \mathcal{R}_{u^{-1}}\mathcal{R}_v x, u) = k(\mathcal{R}_v x, \mathcal{R}_{u^{-1}v} x, u)$$

satisfies

$$\left(\int_G \left|\int_G k_x(v,u)h(u^{-1}v)d\mu(u)\right|^p d\mu(v)\right)^{\frac{1}{p}} \leq \mathcal{C}\left(\int_G |h(u)|^p\, d\mu(u)\right)^{\frac{1}{p}}, \tag{4.4}$$

for all $h \in L^p(G)$, where $\mathcal{C}$ is independent of $x \in X$. Then the authors in [11, p. 292, (2.7)] proved that the operator T is bounded from $L^p(X)$ to itself with norm not exceeding $\mathcal{C}$ that is:

$$\left(\int_X |Tf(x)|^p\, dx\right)^{\frac{1}{p}} \leq \mathcal{C}\left(\int_X |f(x)|^p\, dx\right)^{\frac{1}{p}}, \tag{4.5}$$

for all $f \in L^p(X)$. We present the following weak-type version of the transference principle mentioned above. This version also extends the result of [12, Theorem 2.6] and is of independent interest.

Theorem 4.1. *Let $p \in [1,\infty)$. Assume that $k(x,y,u)$ is a measurable function on $X \times X \times G$ as in (4.3), and also satisfies the following for all $s > 0$,*

$$\mu\left\{v \in G : \left|\int_G k_x(v,u)h(u^{-1}v)d\mu(u)\right| > s\right\} \leq \frac{\mathcal{C}^p}{s^p}\|h\|^p_{L^p(G)}, \tag{4.6}$$

where $\mathcal{C}$ is not depending on "x" and $h \in L^p(G)$. Then the operator T defined in (4.3) is of weak type (p,p). Moreover, the following is true,

$$\mathfrak{m}\left\{x \in X : |(Tf)(x)| > s\right\} \leq \frac{\mathcal{C}^p}{s^p}\|f\|^p_{L^p(X)}, \tag{4.7}$$

for all $f \in L^p(X)$.

Proof. Let us define

$$\xi(s) = \left\{x \in X : \left|\int_G k(x, \mathcal{R}_{u^{-1}}x, u) f(\mathcal{R}_{u^{-1}}x)du\right| > s\right\},$$

$$\xi_v(s) = \left\{x \in X : \left|\int_G k(\mathcal{R}_v x, \mathcal{R}_{u^{-1}v}x, u) f(\mathcal{R}_{u^{-1}v}x)du\right| > s\right\},$$

$$\mathcal{F}(s) = \left\{(v,x) \in G \times X : \left|\int_G k(\mathcal{R}_v x, \mathcal{R}_{u^{-1}v}x, u) f(\mathcal{R}_{u^{-1}v}x)du\right| > s\right\}.$$

Observe that $\xi(s) = \xi_e(s)$ and moreover, since $\mathcal{R}_v$ is measure preserving, we have

$$\mathfrak{m}(\xi(s)) = \mathfrak{m}(\mathcal{R}_v\xi(s)) = \mathfrak{m}(\xi_v(s)). \tag{4.8}$$

Let $\chi(v,x)$ be the characteristic function on $\mathcal{F}(s)$ and ψ be the characteristic function of $B^{-1}V$ (thus $\psi(u^{-1}v) = 1$ when $u \in B$ and $v \in V$). We note that if we fix v, then $\chi(v,x)$ is the characteristic function on $\xi_v(s)$. Now, integrating both sides of the equation above and using the Fubini theorem, we get

$$\begin{aligned}
\mathfrak{m}(\xi(s)) &= \frac{1}{\mu(V)}\int_V \mathfrak{m}(\xi_v(s))\, d\mu(v) \\
&= \frac{1}{\mu(V)}\int_V \int_X \chi(v,x)\, d\mathfrak{m}(x)\, d\mu(v) \\
&= \frac{1}{\mu(V)}\int_X \mu\left\{v \in G \cap V : \left|\int_G k(\mathcal{R}_v x, \mathcal{R}_{u^{-1}v}x, u) f(\mathcal{R}_{u^{-1}v}x)d\mu(u)\right| > s\right\} d\mathfrak{m}(x)
\end{aligned}$$

$$= \frac{1}{\mu(V)} \int_X \mu \left\{ v \in G \cap V : \left| \int_G k(\mathcal{R}_v x, \mathcal{R}_{u^{-1}v} x, u) f(\mathcal{R}_{u^{-1}v} x) \psi(u^{-1}v) d\mu(u) \right| > s \right\} \cdot d\mathfrak{m}(x)$$

$$\leq \frac{1}{\mu(V)} \int_X \mu \left\{ v \in G : \left| \int_G k(\mathcal{R}_v x, \mathcal{R}_{u^{-1}v} x, u) f(\mathcal{R}_{u^{-1}v} x) \psi(u^{-1}v) d\mu(u) \right| > s \right\} d\mathfrak{m}(x)$$

$$\leq \frac{1}{\mu(V)} \left(\int_X \frac{\mathcal{C}^p}{s^p} \int_G |f(\mathcal{R}_w x)|^p \psi(w) d\mu(w) \right) d\mathfrak{m}(x) \quad \text{(using (4.6))}$$

$$\leq \frac{1}{\mu(V)} \frac{\mathcal{C}^p}{s^p} \int_G \|f\|^p_{L^p(X)} \psi(w) d\mu(w) \quad \text{(since } \mathcal{R}_w \text{ is measure-preserving)}$$

$$\leq \frac{\mu(B^{-1}V)}{\mu(V)} \frac{\mathcal{C}^p}{s^p} \|f\|^p_{L^p(X)}$$

$$\leq (1+\epsilon) \frac{\mathcal{C}^p}{s^p} \|f\|^p_{L^p(X)},$$

where we used (4.6) by taking $h(u^{-1}v) = f(\mathcal{R}_{u^{-1}v} x)\psi(u^{-1}v)$. Since $\epsilon > 0$ is arbitrary, this concludes the proof of Theorem 4.1. ☐

5. Properties of spherical functions

In this section, we will derive some functional identities and asymptotic estimates of the spherical functions.

5.1. **Functional identities for the spherical functions.** We start with the following formula.

Proposition 5.1. *We have*

$$\phi^{n,n}_{\tau,\lambda}(y^{-1}x) = \int_K e^{-(i\lambda+1)H(y^{-1}k)} e^{(i\lambda-1)H(x^{-1}k)} e_n(K(x^{-1}k)^{-1}) \overline{e_n(K(y^{-1}k)^{-1})}\, dk,$$

for all $x, y \in G$ and $\lambda \in \mathbb{C}$.

Proof. We recall from the Iwasawa decomposition and use it for $xk = K(xk)\exp(H(xk))n_1$. We can write

$$\begin{aligned} y^{-1}xk &= y^{-1}K(xk) exp(H(xk))n_1 \\ &= K(y^{-1}K(xk)) \exp(H(y^{-1}K(xk)))n_2 \exp(H(xk))n_1. \end{aligned} \tag{5.1}$$

Since A normalizes N, we get from (5.1),

$$y^{-1}xk = K(y^{-1}K(xk)) \exp\left(H(y^{-1}K(xk)) + H(xk)\right) n_3,$$

which in turn implies

$$H(y^{-1}xk) = H(y^{-1}K(xk)) + H(xk), \tag{5.2}$$

and

$$K(y^{-1}xk) = K(y^{-1}K(xk)). \tag{5.3}$$

Plugging (5.2) and (5.3) in (2.18), we get

$$\phi^{n,n}_{\tau,\lambda}(y^{-1}x) = \int_K e^{-(i\lambda+1)H(y^{-1}xk)} e_n(k^{-1}) \overline{e_n(K(y^{-1}xk)^{-1})}\, dk$$

$$= \int_K e^{-(i\lambda+1)\left(H(y^{-1}K(xk)+H(xk)\right)} e_n(k^{-1})\,\overline{e_n(K(y^{-1}K(xk))^{-1})}\,dk.$$

But again by (5.2) and (5.3) (putting $y = x$), we get

$$H(xk) = -H(x^{-1}K(xk)) \quad \text{and} \quad k = K(x^{-1}K(xk)), \tag{5.4}$$

whence we obtain

$$\begin{aligned}
&\phi^{n,n}_{\tau,\lambda}(y^{-1}x) \\
&\qquad = \int_K e^{-(i\lambda+1)\left(H(y^{-1}K(xk)-H(x^{-1}K(xk))\right)} e_n(K(x^{-1}K(xk))^{-1})\,\overline{e_n(K(y^{-1}K(xk))^{-1})}\,dk.
\end{aligned}$$

Now we apply Lemma 2.1 to get

$$\begin{aligned}
\phi^{n,n}_{\tau,\lambda}(y^{-1}x) &= \int_K e^{-(i\lambda+1)\left(H(y^{-1}k)-H(x^{-1}k)\right)} e_n(K(x^{-1}k)^{-1})\,\overline{e_n(K(y^{-1}k)^{-1})} e^{-2H(x^{-1}k)}\,dk \\
&= \int_K e^{-(i\lambda+1)H(y^{-1}k)} e^{(i\lambda-1)H(x^{-1}k)} e_n(K(x^{-1}k)^{-1})\,\overline{e_n(K(y^{-1}k)^{-1})}\,dk,
\end{aligned}$$

which concludes the proof of our lemma. □

Lemma 5.2. *The spherical function $\phi^{n,n}_{\tau,\lambda}$ satisfies the following identity*

$$\int_K \phi^{n,n}_{\tau,\lambda}(ykx) e_n(k^{-1})\,dk = \phi^{n,n}_{\tau,\lambda}(y)\phi^{n,n}_{\tau,\lambda}(x), \tag{5.5}$$

for all $\lambda \in \mathbb{R}$ and $x, y \in G$.

Proof. We can write from Proposition 5.1,

$$\phi^{n,n}_{\tau,\lambda}(ykx) = \int_K e^{-(i\lambda+1)H(ykk_1)} e^{(i\lambda-1)H(x^{-1}k_1)} e_n(K(x^{-1}k_1)^{-1})\overline{e_n(K(ykk_1)^{-1})}\,dk_1.$$

Using the formula above, the left-hand side of (5.5) transforms to

$$\int_K \left(\int_K e^{-(i\lambda+1)H(ykk_1)} e^{(i\lambda-1)H(x^{-1}k_1)} e_n(K(x^{-1}k_1)^{-1})\overline{e_n(K(ykk_1)^{-1})}\,dk_1 \right) e_n(k^{-1})\,dk.$$

We now apply Fubini's theorem, followed by the change of variable $kk_1 \to k$ in the expression above, to get

$$\begin{aligned}
&\int_K \left(\int_K e^{-(i\lambda+1)H(yk)} e_n(k^{-1})\overline{e_n(K(yk)^{-1})}\,dk \right) e^{(i\lambda-1)H(x^{-1}k_1)} e_n\left(K(x^{-1}k_1)^{-1}\right) e_n(k_1)\,dk_1 \\
&= \phi^{n,n}_{\tau,\lambda}(y) \int_K e^{(i\lambda-1)H(x^{-1}k_1)} e_n\left(K(x^{-1}k_1)^{-1}\right) e_n(k_1)\,dk_1 \\
&= \phi^{n,n}_{\tau,\lambda}(y)\phi^{-n,-n}_{\tau,-\lambda}(x^{-1}) \\
&= \phi^{n,n}_{\tau,\lambda}(y)\phi^{n,n}_{\tau,\lambda}(x)
\end{aligned}$$

where in the last step, we used (2.21). This completes the proof of the lemma □

5.2. **Asymptotic estimates of spherical functions.** In this section, we will discuss the asymptotic estimate of the spherical functions. We begin by presenting the local and global expansions of $\phi^{n,n}_{\tau,\lambda}$ in terms of well-known special functions. It is important to note that the spherical functions exhibit different behaviors near and away from the identity, and this distinction will be evident in the expansions.

Let $J_\mu(z)$ be the Bessel functions of the first kind, and let

$$\mathcal{J}_\mu(z) = \frac{J_\mu(|z|)}{|z|^\mu}\Gamma(\mu+\frac{1}{2})\Gamma(\frac{1}{2})2^{\mu-1}. \tag{5.6}$$

Using this definition, we present the asymptotic expansion of the spherical function $\phi^{n,n}_{\tau,\lambda}$ near the identity. To deduce this result, we will follow the approach in [48, Lemma 4.5] (see also [55, Theorem 2.1]).

Lemma 5.3. *For $0 \le t \le 1$, the spherical function $\phi^{n,n}_{\tau,\lambda}$ can be decomposed as*

$$\phi^{n,n}_{\tau,\lambda}(a_t) = \left(\frac{t}{\Delta(t)}\right)^{1/2}\sum_{j=0}^{2} t^{2j} b_j^n(t)\mathcal{J}_j(\lambda t) + E_n(\lambda,t), \quad \lambda \ge 0,$$

where $b_0^n(t) \equiv b_0$ is a constant independent of n, while $|b_j^n(t)| \le C(1+|n|)^4$, $j=1,2$, and

$$\int_1^\infty |E_n(\lambda,t)|\lambda d\lambda \le C(1+|n|)^6, \qquad \textit{uniformly in } 0 \le t \le 1.$$

Proof. We recall that $\phi^{n,n}_{\tau,\lambda}$ is a solution of the following differential equation

$$\Omega f = -\frac{(\lambda^2+1)}{4} f,$$

where f is a (n,n) type function on G. Since Ω preserves the types of a function and an (n,n) type function is uniquely determined by its restriction to $(0,\infty)$, we obtain an operator $\Pi_{n,n}(\Omega)$ on $(0,\infty)$ which satisfies

$$\Omega f(a_t) = \Pi_{n,n}(\Omega)(f\mid_{A^+})(a_t).$$

The operator $\Pi_{n,n}(\Omega)$, known as the (n,n)-th radial component of the Casimir, is a second-order differential operator, and its expression is given by [3, Theorem 13.1]:

$$\Pi_{n,n}(\Omega)f(a_t) = \frac{1}{4}\frac{d^2}{dt^2}f(a_t) + \frac{1}{2}\coth 2t\frac{d}{dt}f(a_t) + \frac{1}{4}\frac{n^2}{\cosh^2 t}f(a_t), \quad t>0.$$

We substitute $f(t) = (\cosh t)^n g(t)$ in

$$\Pi_{n,n}(\Omega)f = -\frac{(\lambda^2+1)}{4}f \tag{5.7}$$

to obtain the following,

$$\frac{d^2 g}{dt^2} + ((2n+1)\tanh t + \coth t)\frac{dg}{dt} + (\lambda^2 + (n+1)^2)g = 0. \tag{5.8}$$

After performing the change of variable $z := -\sinh^2 t$, the ODE (5.8) simplifies to the hypergeometric differential equation

$$z(1-z)\frac{d^2g}{dz^2} + (c-(a+b+1)z)\frac{dg}{dz} - \frac{1}{4}abg = 0,$$

with parameters $a = \frac{n+1}{2} - \frac{i\lambda}{2}, b = \frac{n+1}{2} + \frac{i\lambda}{2}, c = 1$. Hence, using the expression from [34, (2.2)], we find that the Jacobi function $\varphi_\lambda^{(0,n)}$ is the unique solution satisfying regularity conditions and equaling 1 at $z = 0$. By employing the integral representation of the Jacobi function from [33, (2.21)], we deduce from (5.7) the following:

$$\phi_{\tau,\lambda}^{n,n}(a_t) = \frac{2^{\frac{3}{2}}}{\pi} \int_0^t \cos(\lambda s) \left(\cosh 2t - \cosh 2s\right)^{-\frac{1}{2}} {}_2F_1 \left(n, -n; 1/2; \frac{\cosh t - \cosh s}{2\cosh t} \right) ds.$$

Subsequently, by performing similar calculations as presented in [48, (4.12)], we can arrive at our lemma. □

The following lemma is a counterpart of Ionescu's result [28, Propositin A.2 (c)] in our context. It provides an estimate for the spherical function away from the identity, which will be utilized in the large-scale analysis of Ψ_σ.

Lemma 5.4. *Suppose that $t \geq 1/10$ and $N \in \mathbb{N}$. Then $\phi_{\tau,\lambda}^{n,n}(a_t)$ can be written in the following form*

$$\phi_{\tau,\lambda}^{n,n}(a_t) = e^{-t} \left(e^{i\lambda t} c_\tau^{n,n}(\lambda)(1 + a(\lambda, t)) + e^{-i\lambda t} c_\tau^{n,n}(-\lambda)(1 + a(-\lambda, t)) \right), \tag{5.9}$$

where the function $a(\lambda, t)$ satisfies the following inequalities,

$$\left| \frac{\partial^\alpha}{\partial \lambda^\alpha} a(\lambda, t) \right| \leq C_{\alpha,n}(1 + |\lambda|)^{-\alpha}, \tag{5.10}$$

for all integers $\alpha \in [0, N]$, and for all λ in the region $0 \leq \operatorname{Im} \lambda \leq 1 + 1/10$.

Remark 5.5. We note that, unlike the local estimate of the spherical function $\phi_{\tau,\lambda}^{n,n}$ in Lemma 5.3, we are unable to maintain the polynomial dependence on n in (5.10). Specifically, following the proof of Lemma 5.4, it can be shown that the constant in (5.10) grows at a rate controlled by $\Gamma(C_\alpha n^2)$ for some constant C_α independent of n. For further details on how the constants involved in inequality (5.10) grow with n, we refer the reader to [48, p. 568].

Proof. From [3, (13.1)], we have for all $t > 0$,

$$\phi_{\tau,\lambda}^{n,n}(a_t) = e^{-t} \left[e^{-i\lambda t} c_\tau^{n,n}(\lambda)(1 + a(\lambda, t)) + e^{i\lambda t} c_\tau^{n,n}(-\lambda)(1 + a(-\lambda, t)) \right],$$

where

$$a(\lambda, t) = \sum_{k=1}^{\infty} a_k^{n,n}(\lambda) e^{-2kt},$$

and the functions $a_k^{n,n}$ satisfy the following recursion relation for $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$,

$$a_k^{n,n}(\lambda) = -\frac{1}{k(k - i\lambda)} \left(\sum_{j=1,odd}^{k} a_{k-j}^{n,n}(\lambda) j n^2 - \sum_{j=2,even}^{k} a_{k-j}^{n,n}(\lambda)(1 - i\lambda + 2k - 2j + jn^2) \right) \tag{5.11}$$

with $a_0^{n,n} \equiv 1$. We define

$$\alpha_j^k(\lambda) = \begin{cases} -\dfrac{jn^2}{k(k - i\lambda)}, & \text{when } j \text{ is odd and } 1 \leq j \leq k \\ \dfrac{1}{k}\left(1 + \dfrac{1 + k - 2j + jn^2}{k - i\lambda}\right), & \text{when } j \text{ is even.} \end{cases}$$

Our aim is to show that for all $\beta \in [0, N]$, there exists constants C_n and A_β such that

$$\left|\frac{\partial^\beta}{\partial\lambda^\beta} a_k^{n,n}(\lambda)\right| \le C_n k^{A_\beta}(1+|\operatorname{Re}\lambda|)^{-\beta} \tag{5.12}$$

for all $k \ge 1$ and all $\lambda \in \mathbb{C}$, with $0 \le \operatorname{Im}\lambda \le 1+1/10$. Using the following inequality above

$$|k-i\lambda|^2 = |k+\operatorname{Im}\lambda - i\operatorname{Re}\lambda|^2 = (k+\operatorname{Im}\lambda)^2 + \operatorname{Re}\lambda^2 \ge \max\{k, |\operatorname{Re}\lambda|\}^2,$$

we can directly say that

$$\left|\frac{\partial^\beta}{\partial\lambda^\beta}\alpha_j^k(\lambda)\right| \le \frac{C_n}{k|k-i\lambda|^\beta} \le \frac{C_n}{k(1+|\operatorname{Re}\lambda|)^\beta} \tag{5.13}$$

for all integers $k \ge 1, j \le k-1$, $\beta \in [0, N]$ and for all $\lambda \in \mathbb{C}$, with $0 \le \operatorname{Im}\lambda \le 1+1/10$. We now prove (5.12) for $\beta = 0$ by induction over $k \ge 1$. For $k = 1$,

$$|a_1^{n,n}(\lambda)| = |\alpha_1^1(\lambda)| \le \frac{n^2}{|1-i\lambda|} \le C_n \text{ (depending on } n).$$

Next, we assume (5.12) holds for all $1 \le j \le k-1$. Then using (5.11), (5.13) and (5.12) we get,

$$\begin{aligned}
|a_k^{n,n}(\lambda)| &\le \left(\sum_{j=1,odd}^{k} |a_{k-j}^{n,n}(\lambda)||\alpha_j^k| + \sum_{j=2,even}^{k} |a_{k-j}^{n,n}(\lambda)||\alpha_j^k|\right)\\
&\le \frac{C_n}{k}\left(\sum_{j=1,odd}^{k} |a_{k-j}^{n,n}(\lambda)| + \sum_{j=2,even}^{k} |a_{k-j}^{n,n}(\lambda)|\right)\\
&\le \frac{C_n}{k}\sum_{j=1}^{k-1}\frac{C_n j^{A_0}}{A_0}\\
&\le \frac{C_n}{k}\frac{C_n k^{A_0+1}}{A_0} = C_n k^{A_0}\frac{C_n}{A_0},
\end{aligned}$$

where we used the fact for all $k \ge 2$, $A \ge 4$,

$$1 + \sum_{j=1}^{k-1} j^A \le \frac{k^{A+1}}{A}.$$

Now if we choose $A_0 = C_n$, then by induction, we have proved (5.12) for $\beta = 0$ and for all $k \ge 1$. Next, to prove (5.12) for arbitrary integer $\beta \le N$, we assume by induction that we found suitable powers A_β, such that (5.12) holds for all $0 \le \alpha \le \beta - 1$ and for all $k \ge 1$. We can also assume $A_0 \le A_1 \le \cdots \le A_{\beta-1}$. Again we apply induction over k for fixed β. Its obvious that $\left|\frac{\partial^\beta}{\partial\lambda^\beta} a_1^{n,n}(\lambda)\right| \le C_n(1+|\operatorname{Re}\lambda|)^{-\beta}$. We assume (5.12) holds for β and for all $j \in \{1, 2.\cdots k-1\}$. Then again (5.11), (5.13) and (5.12) imply

$$\left|\frac{\partial^\beta}{\partial\lambda^\beta} a_k^{n,n}(\lambda)\right| \le 2^\beta\left(\sum_{j=1}^{k-1}\sum_{\alpha=0}^{\beta}\left|\frac{\partial^{\beta-\alpha}}{\partial\lambda^{\beta-\alpha}} a_{k-j}^{n,n}(\lambda)\right|\left|\frac{\partial^\alpha}{\partial\lambda^\alpha}\alpha_j^k\right|\right)$$

$$\leq 2^\beta \sum_{\alpha=0}^{\beta} \left(\sum_{j=1}^{k-1} \frac{C_n}{k(1+|\operatorname{Re}\lambda|)^{\beta-\alpha}} \left| \frac{\partial^\alpha}{\partial \lambda^\alpha} \alpha_j^k \right| \right)$$
$$\leq 2^\beta \sum_{\alpha=0}^{\beta} \frac{C_n}{k(1+|\operatorname{Re}\lambda|)^{\beta-\alpha}} \left(\sum_{j=1}^{k-1} \frac{C_n \max\{j,1\}^{A_\alpha}}{(1+|\operatorname{Re}\lambda|)^{\alpha}} \right)$$
$$\leq 2^\beta \sum_{\alpha=0}^{\beta} \frac{C_n}{(1+|\operatorname{Re}\lambda|)^{\beta}} \frac{C_n k^{A_\beta}}{A_\beta}$$
$$\leq C_n k^{A_\beta} (1+|\operatorname{Re}\lambda|)^{-\beta} \frac{C_n 2^\beta(\beta+1)}{A_\beta}.$$

By taking $A_\beta \geq \max\{A_{\beta-1}, C_n 2^\beta(\beta+1)\}$, the proof follows by induction. In fact one can set $A_\beta = C_n 2^\beta(\beta+1)$ for all integers $\beta \in [0, N]$. □

We now focus on the estimate of the canonical discrete series matrix coefficients $\psi_{ik}^{n,n}$. We recall the following uniform growth properties of $\psi_{ik}^{n,n}$ from [3, Theorem 8.1].

Theorem 5.6. *Fix $l \in \mathbb{N}$. There exist constants $C, r_1, r_2, r_3 \geq 0$ such that*

$$|\psi_{ik}^{n,n}(x)| \leq C(1+|n|)^{r_1}(1+|k|)^{r_2}(1+x^+)^{r_3} \phi_{\tau^+,0}^{0,0}(x)^{(1+l)} \tag{5.14}$$

for all $k \in \mathbb{Z}^*$ for which $|k| \geq l$, and for all $n \in \mathbb{Z}(k)$, for all $t \geq 0$.

We remark that the estimate above is a consequence of a more general result by Trombi and Varadarajan, where they found a necessary condition on a discrete series representation having the K-finite matrix coefficient with a certain rate of decay; see [60, Theorem 8.1]. Later, in 1977, Miličić proved that their condition was sufficient, too, which, in turn, provided a precise characterization of discrete series representations whose K-finite matrix coefficients lie in $L^p(G')$ for $1 \leq p < 2$, where G' is a connected semisimple Lie group with finite center; see [39, Theorem, p.60] for more details. In the setting of SL$(2, \mathbb{R})$ group, the result can be stated as follows:

Corollary 5.7 ([39, Corollary, p.84])**.** *Let $0 < p \leq 2$ and $\gamma_p = (2/p) - 1$. If (π_{ik}, H_k) is a discrete series representation corresponding to $k \in \mathbb{Z}^*$, and ψ is a K-finite matrix coefficient of π_{ik}, then the following conditions are equivalent:*

(1) $|k| > \gamma_p$,
(2) $\psi \in L^p(G)$.

Remark 5.8. It also follows from the estimate in [3, (3.2)] for $\phi_{\tau^+,0}^{0,0}$, which exhibits exponential decay, and (5.14), that the matrix coefficients of the discrete series representation belong to $L^q(G)$ for all $q > 1$. More precisely, for $q > 1, k \in \mathbb{Z}^*$, and $n \in \mathbb{Z}(k)$, we have

$$\|\psi_{ik}^{n,n}\|_{L^q(G)} \leq C_q(1+|n|)^{r_1}(1+|k|)^{r_2}. \tag{5.15}$$

The previous result illustrates that, given a $k \in \mathbb{Z}^*$ and $p \in (0, 2]$, how we can examine whether a K-finite matrix coefficient of π_{ik} belongs to $L^p(G)$ or not. However, it does not say anything about the case $|k| = \gamma_p$. So, to accommodate the case $|k| = \gamma_p$, we provide the following weak-type version of the result [39, Corollary, p. 84] for the (n, n)-th matrix coefficient of π_{ik}.

Lemma 5.9. *Let $0 < p \leq 2$ and $\gamma_p = (2/p)-1$. If (π_{ik}, H_k) is a discrete series representation corresponding to $k \in \mathbb{Z}^*$, and $n \in \mathbb{Z}(k)$, then the following conditions are equivalent:*

(1) $|k| \geq \gamma_p$,
(2) $\psi_{ik}^{n,n} \in L^{p,\infty}(G)$.

Moreover, if $|k| = \gamma_p$, then $\psi_{ik}^{n,n} \in L^{p,q}(G)$, if and only if $q = \infty$.

Before we give the proof of the lemma above, let us first provide an asymptotic estimate for the discrete series coefficient.

Lemma 5.10. *Given $k \in \mathbb{Z}^*$, $n \in \mathbb{Z}(k)$, we have*

$$|\psi_{ik}^{n,n}(a_t)| \simeq e^{-(1+|k|)t}, \quad t \geq 0. \tag{5.16}$$

Proof. Suppose $k \in \mathbb{Z}^*$ and $n \in \mathbb{Z}(k)$, then from [3, (12.2), Chap. 12] we have

$$\psi_{ik}^{n,n}(a_t) = 2e^{-(1+|k|)t} \sum_{l=0}^{\infty} d_l^{n,n}(k)\, e^{-2lt}, \quad t > 0. \tag{5.17}$$

Using the fact that $\psi_{ik}^{n,n} = \phi_{\tau,i|k|}^{n,n}$ (see 2.19), Barker showed that the coefficient $d_l^{n,n}(k)$ as in (5.17) satisfies the following relation (see [3, Proposition 14.2])

$$d_l^{n,n}(k) = \begin{cases} c_\tau^{n,n}(k) a_l^{n,n}(k) & \text{if } k > 0, \\ c_\tau^{n,n}(-k) a_l^{n,n}(-k) & \text{if } k < 0, \end{cases}$$

where $a_l^{n,n}(k)$'s are defined in (5.11). We note that the poles of $\lambda \mapsto c_\tau^{n,n}(\lambda)$ occur only at $\lambda \in i\mathbb{Z}^\sigma$, $\operatorname{Im} \lambda \geq 0$, where σ is determined by $n \in \mathbb{Z}^\sigma$ (see Lemma 2.3). Therefore, using [3, Proposition 13.5], we can say that for a given $k \in \mathbb{Z}^*$, $n \in \mathbb{Z}(k)$, and for any $r > 0$ there exists a constant $C_k > 0$ such that for all $l \in \mathbb{N}$

$$|d_l^{n,n}(k)| \leq C_k e^{rl}.$$

Utilizing the inequality above for $r < 1$, it follows from (5.17) that,

$$\lim_{t \to \infty} \left| e^{(1+|k|)t} \psi_{ik}^{n,n}(a_t) - 2d_0^{n,n}(k) \right| = 0,$$

whence we obtain

$$|\psi_{ik}^{n,n}(a_t)| \simeq e^{-(1+|k|)t}, \quad \text{for all } t \geq 0.$$

□

Proof of Lemma 5.9. Let us define for $k \in \mathbb{N}$,

$$f_k(t) = e^{-(k+1)t} \quad \text{for } t \geq 0,$$

and extend it as a (n,n)-type function on G as shown in (2.5). Then in view of (5.16), the asymptotic estimate of $\psi_{ik}^{n,n}$, it is sufficient to prove that $f_k \in L^{p,\infty}(G//K)$ if and only if $k \geq \gamma_p$. To establish this, we will first determine the distribution function of f_k. Taking into account that m is the Haar measure on G in the polar decomposition and observing that f_k is a K-biinvariant function, we arrive at the following expressions:

$$d_{f_k}(\alpha) = m\left\{t \in [0,\infty) : e^{-(k+1)t} > \alpha\right\} = m\left\{t \in [0,\infty) : t < \frac{1}{k+1} \log \frac{1}{\alpha}\right\}.$$

Clearly, if $\alpha \geq 1$, $d_{f_k}(\alpha) = 0$, hence we only need to consider $\alpha \in (0, 1)$. Let $0 < \alpha < e^{-(k+1)/2}$, then $\frac{1}{k+1} \log \frac{1}{\alpha} > 1/2$. Thus in this range of α, using the asymptotic behaviour of $\Delta(t)$ we can write

$$d_{f_k}(\alpha) \simeq \int_0^{\frac{1}{2}} t\, dt + \int_{\frac{1}{2}}^{\frac{1}{k+1}\log\frac{1}{\alpha}} e^{2t}\, dt \simeq \alpha^{-\frac{2}{k+1}}. \tag{5.18}$$

When $e^{-(k+1)/2} < \alpha < 1$, we can show similarly

$$d_{f_k}(\alpha) \simeq \log\left(\frac{1}{\alpha}\right)^{\frac{2}{k+1}}.$$

Thus, from the definition of Lorentz space (2.2), it follows that

$$\|f\|_{L^{p,\infty}} = \sup_{\alpha>0} \alpha^p\, d_{f_k}(\alpha) \simeq \sup_{0<\alpha<1} \alpha^{p-\frac{2}{k+1}},$$

which in turn implies $\|f_k\|_{L^{p,\infty}} < \infty$ if and only if $p - \frac{2}{k+1} \geq 0$, that is, $k \geq \gamma_p$. Finally, we will complete the proof of the lemma by contradiction. Let us assume that for a given $k = \gamma_p$, $f_k \in L^{p,q}(G)$ for some $q \in (0, \infty)$. Then we have from (5.18) and the definition of Lorentz space (2.2),

$$\|f\|_{L^{p,q}}^q \simeq \int_0^{e^{-(k+1)/2}} \alpha^{q-1}\alpha^{-\frac{2q}{p(k+1)}}\, d\alpha \simeq \int_0^{e^{-(k+1)/2}} \alpha^{q-1}\alpha^{-q} d\alpha = \infty,$$

which contradicts our assumption $f_k \in L^{p,q}(G)$, for $q \in (0, \infty)$, concluding the lemma. ☐

Remark 5.11. By following a similar calculation as presented in Lemma 5.9, we can establish that any matrix coefficient $\psi_{ik}^{m,n}$ of π_{ik}, where $m, n \in \mathbb{Z}(k)$ (see [3, (5.1)] for definition), satisfies the following asymptotic estimate:

$$|\psi_{ik}^{m,n}(a_t)| \simeq e^{-(1+|k|)t}, \quad t \geq 0. \tag{5.19}$$

Since any K-finite matrix coefficient ψ of π_{ik} can be represented as a finite linear combination of $\{\psi_{ik}^{m,n} : m, n \in \mathbb{Z}(k)\}$, the matrix coefficients of π_{ik} (see [3, Corollary 5.5]), we can combine the arguments from the proof of Lemma 5.9 with the estimate in (5.19), to establish the following theorem. This theorem offers a precise characterization of discrete series representations whose K-finite matrix coefficients belong to the space $L^{p,\infty}(G)$ for $0 < p \leq 2$. This characterization serves as a weak-type analog of Miličić's result [39, Corollary, p.84] in the context of $\mathrm{SL}(2, \mathbb{R})$; see also [3, Theorem, 5.3]. For $0 < p \leq 2$, let $E_{p,\infty}(G)$ denote the set of equivalence classes of irreducible representations of G whose K-finite matrix coefficient belongs to $L^{p,\infty}(G)$.

Theorem 5.12. *Let $0 < p \leq 2$ and $\gamma_p = (2/p) - 1$. If (π_{ik}, H_k) is a discrete series representation corresponding to $k \in \mathbb{Z}^*$, then the following conditions are equivalent:*

(1) $|k| \geq \gamma_p$,
(2) (π_k, H_k) *is in* $E_{p,\infty}(G)$.

6. Singular integral realization of Ψ_σ

In studying the boundedness of the pseudo-differential operator Ψ_σ on symmetric spaces, it is customary to analyze the local and global components of the operator separately. However, in this group setting, there is an additional discrete component of the Plancherel measure.

To address this, we will decompose the operator Ψ_σ into continuous and discrete parts. To deal with the continuous part, we will first represent it as a singular integral kernel operator.

Let f be a smooth, compactly supported (n,n) type function on G. Then we recall the definition of Ψ_σ from (3.2)

$$\begin{aligned}\Psi_\sigma f(x) &= \frac{1}{4\pi^2}\int_{\mathbb{R}} \sigma(x,\lambda)\widehat{f_H}(\lambda)\phi^{n,n}_{\tau,\lambda}(x)|c^{n,n}_\tau(\lambda)|^{-2}d\lambda + \frac{1}{2\pi}\sum_{k\in\Gamma_n}\sigma(x,ik)\widehat{f_B}(ik)\psi^{n,n}_{ik}(x)|k| \\ &=: \Psi^{\mathrm{con}}_\sigma f(x) + \Psi^{\mathrm{dis}}_\sigma f(x). \end{aligned} \tag{6.1}$$

By substituting the expression of $\widehat{f_H}(\lambda)$ in the above expression of $\Psi^{\mathrm{con}}_\sigma f$, we get

$$\begin{aligned}\Psi^{\mathrm{con}}_\sigma f(x) &= \frac{1}{4\pi^2}\int_{\mathbb{R}} \sigma(x,\lambda)\left(\int_G f(y)\phi^{n,n}_{\tau,\lambda}(y^{-1})dy\right)\phi^{n,n}_{\tau,\lambda}(x)|c^{n,n}_\tau(\lambda)|^{-2}d\lambda \\ &= \frac{1}{4\pi^2}\int_{\mathbb{R}} \sigma(x,\lambda)\left(\int_G f(y)\phi^{n,n}_{\tau,\lambda}(y^{-1})\phi^{n,n}_{\tau,\lambda}(x)dy\right)|c^{n,n}_\tau(\lambda)|^{-2}d\lambda.\end{aligned}$$

By applying the formula for the spherical function $\phi^{n,n}_{\tau,\lambda}$ from Lemma 5.2, the expression above simplifies to:

$$\Psi^{\mathrm{con}}_\sigma f(x) = \frac{1}{4\pi^2}\int_{\mathbb{R}} \sigma(x,\lambda)\left(\int_G f(y)\int_K \phi^{n,n}_{\tau,\lambda}(y^{-1}kx)e_n(k^{-1})\,dk\,dy\right)|c^{n,n}_\tau(\lambda)|^{-2}d\lambda.$$

Using Fubini's theorem, followed by the change of variable $y\to ky$, and taking into account that f is a (n,n) type function on G and $\int_K dk = 1$, we can further simplify this expression as follows:

$$\Psi^{\mathrm{con}}_\sigma f(x) = \int_G f(y)\mathcal{K}^{\mathrm{con}}(x, y^{-1}x)dy,$$

where

$$\mathcal{K}^{\mathrm{con}}(x,y) := \frac{1}{4\pi^2}\int_{\mathbb{R}} \sigma(x,\lambda)\phi^{n,n}_{\tau,\lambda}(y)|c^{n,n}_\tau(\lambda)|^{-2}d\lambda. \tag{6.2}$$

Similarly, for $\Psi^{\mathrm{dis}}_\sigma f$, we can write

$$\Psi^{\mathrm{dis}}_\sigma f(x) = \int_G f(y)\mathcal{K}^{\mathrm{dis}}(x, y^{-1}x)dy,$$

where

$$\mathcal{K}^{\mathrm{dis}}(x,y) := \frac{1}{2\pi}\sum_{k\in\Gamma_n}\sigma(x,ik)\psi^{n,n}_{ik}(y)|k|.$$

In Section 5, we observed that the spherical function $\phi^{n,n}_{\tau,\lambda}$ behaves differently near the identity element compared to its behavior away from it. To account for this distinction and handle the exponential volume growth of G, we will decompose $\Psi^{\mathrm{con}}_\sigma$ into a sum of local and global parts. To achieve this decomposition, we introduce a smooth, even function $\eta^\circ : \mathbb{R}\to[0,1]$, which is supported on $[-1,1]$ and satisfies $\eta^\circ(t) = 1$ if $|t|\le 1/2$. We define another function $\eta(t) = 1-\eta^\circ(t)$. By utilizing the Cartan decomposition, we extend the functions η° and η to become K-biinvariant functions on G, by

$$\eta^\circ(x) = \eta^\circ(x^+) \quad \text{and} \quad \eta(x) = \eta(x^+), \qquad \text{for all } x\in G. \tag{6.3}$$

Then we can write,

$$\Psi^{\mathrm{con}}_\sigma f(x) = \Psi^{\mathrm{loc}}_\sigma f(x) + \Psi^{\mathrm{glo}}_\sigma f(x), \tag{6.4}$$

where

$$\begin{aligned}\Psi_\sigma^{\mathrm{loc}} f(x) &:= \int_G f(y)\eta^\circ(y^{-1}x)\mathcal{K}^{\mathrm{con}}(x, y^{-1}x)dy,\\ &= \int_G f(y)\mathcal{K}^{\mathrm{loc}}(x, y^{-1}x)dy,\end{aligned} \tag{6.5}$$

and

$$\begin{aligned}\Psi_\sigma^{\mathrm{glo}} f(x) &:= \int_G f(y)\eta(y^{-1}x)\mathcal{K}^{\mathrm{con}}(x, y^{-1}x)dy,\\ &= \int_G f(y)\mathcal{K}^{\mathrm{glo}}(x, y^{-1}x)dy.\end{aligned} \tag{6.6}$$

The decomposition of Ψ_σ, as given by (6.1) and (6.4), simplifies the task of proving the boundedness of Ψ_σ. It reduces the problem to demonstrating the boundedness of the individual operators $\Psi_\sigma^{\mathrm{dis}}$, $\Psi_\sigma^{\mathrm{loc}}$, and $\Psi_\sigma^{\mathrm{glo}}$, each of which requires a distinct approach to establish their boundedness. In the remainder of this article, we will examine and discuss the boundedness properties of these operators, $\Psi_\sigma^{\mathrm{dis}}$, $\Psi_\sigma^{\mathrm{loc}}$, and $\Psi_\sigma^{\mathrm{glo}}$, separately.

7. Discrete analysis of Ψ_σ

In this section, we will demonstrate the boundedness of $\Psi_\sigma^{\mathrm{dis}}$. While the authors [48, Lemma 4.4] employed the convolution estimate from the *Kunze-Stein* phenomenon to establish the L^p-boundedness of multipliers, our approach utilizes the uniform growth properties of the canonical discrete series matrix coefficients $\psi_{ik}^{n,n}$ near infinity (Theorem 5.6). This property of $\psi_{ik}^{n,n}$ enables us to show that for all $k \in \mathbb{Z}^*$ and $n \in \mathbb{Z}(k)$, $\psi_{ik}^{n,n} \in L^p(G)$ for any $p \in (1, \infty)$ (see (5.15)), leading to the L^p-boundedness of the $\Psi_\sigma^{\mathrm{dis}}$ operator.

However, the situation is different for $p = 1$. Corollary 5.7 indicates that a similar technique will not work in this case, as $\psi_{ik}^{n,n}$ does not belong to $L^1(G)$ when $k = 1$ (then $n > 0$ is even). Nevertheless, we utilize Lemma 5.9 to establish the weak type $(1,1)$-boundedness of $\Psi_\sigma^{\mathrm{dis}}$.

Theorem 7.1. *Let $1 \le p < \infty$ and Ψ_σ^{dis} be the operator defined in (6.1). Assume that*

$$\|\sigma\|_{L^\infty(G\times i\Gamma_n)} = \sup_{x\in G,\, k\in\Gamma_n} |\sigma(x, ik)| < \infty.$$

Then the following are true

(1) The operator Ψ_σ^{dis} is a bounded operator from $L^p(G)_{n,n}$ to itself for all $p \in (1,\infty)$.
(2) When $p = 1$ and $n \in \mathbb{Z}^{\tau-} \cup \{0\}$, then Ψ_σ^{dis} is a bounded operator from $L^1(G)_{n,n}$ to itself. If $n \in \mathbb{Z}^{\tau+} \setminus \{0\}$, Ψ_σ^{dis} is a weak type $(1,1)$-bounded operator from $L^1(G)_{n,n}$ to $L^{1,\infty}(G)_{n,n}$.

Proof. First, let us proceed with the case $1 < p < \infty$. Let $f \in L^p(G)_{n,n}$, then using Hölder's inequality and (5.15), we obtain

$$\left|\widehat{f_B}(ik)\right| = \left|\int_G f(x)\psi_{ik}^{n,n}(x^{-1})dx\right| \le \|f\|_{L^p(G)}\|\psi_{ik}^{n,n}\|_{L^{p'}(G)},$$

for all $k \in \Gamma_n$. Therefore, using the inequality above, we can write the following.

$$\|\Psi_\sigma^{\mathrm{dis}} f\|_{L^p(G)}^p \le C_{p,n} \sum_{k\in\Gamma_n} |\widehat{f_B}(ik)|^p \int_G |\sigma(x, ik)\psi_{ik}^{n,n}(x)k|^p dx$$

$$\le C_{p,n}\|\sigma\|^p_{L^\infty(G\times i\Gamma_n)}\sum_{k\in\Gamma_n}\|f\|^p_{L^p(G)}\|\psi^{n,n}_{ik}\|^p_{L^{p'}(G)}\int_G|\psi^{n,n}_{ik}(x)|^p dx$$
$$\le C_{p,n}\|\sigma\|^p_{L^\infty(G\times i\Gamma_n)}\|f\|^p_{L^p(G)}\sum_{k\in\Gamma_n}\|\psi^{n,n}_{ik}\|^p_{L^{p'}(G)}\|\psi^{n,n}_{ik}\|^p_{L^p(G)}$$
$$\le C_{p,n}\|\sigma\|^p_{L^\infty(G\times i\Gamma_n)}\|f\|^p_{L^p(G)}.$$

This establishes the $L^p(G)_{n,n}$-boundedness of Ψ^{dis}_σ for $1<p<\infty$. In the case of $p=1$ and $n\in\mathbb{Z}^{\tau-}\cup\{0\}$, the proof for the $L^1(G)_{n,n}$-boundedness of Ψ^{dis}_σ follows the same procedure as in the previous case. Lastly, for $p=1$ and $n\in\mathbb{Z}^{\tau_+}\setminus\{0\}$, by utilizing Hölder's inequality and Lemma 5.9, we obtain the following for $f\in L^1(G)_{n,n}$:

$$\begin{aligned}\|\Psi^{\text{dis}}_\sigma f\|_{L^{1,\infty}(G)} &\le C_n\sum_{k\in\Gamma_n}|\widehat{f_B}(ik)|\|\sigma(\cdot,ik)\psi^{n,n}_{ik}(\cdot)k\|_{L^{1,\infty}(G)}\\ &\le C_n\|\sigma\|_{L^\infty(G\times i\Gamma_n)}\sum_{k\in\Gamma_n}\|f\|_{L^1(G)}\|\psi^{n,n}_{ik}\|_{L^\infty(G)}\|\psi^{n,n}_{ik}\|_{L^{1,\infty}(G)}\\ &\le C_n\|\sigma\|_{L^\infty(G\times i\Gamma_n)}\|f\|_{L^1(G)}\sum_{k\in\Gamma_n}\|\psi^{n,n}_{ik}\|_{L^\infty(G)}\|\psi^{n,n}_{ik}\|_{L^{1,\infty}(G)}\\ &\le C_n\|\sigma\|_{L^\infty(G\times i\Gamma_n)}\|f\|_{L^1(G)},\end{aligned}$$

whence we obtain the weak type $(1,1)$-boundedness of Ψ^{dis}_σ. □

Remark 7.2. Utilizing the $L^p(G)$ norm estimates of $\psi^{n,n}_{ik}$ for $p>1$, we observe that the $L^p(G)_{n,n}$ operator norm of Ψ^{dis}_σ can be bounded by a polynomial dependence on n. In fact, using a similar calculation as in [48, (4.8)], it follows that

$$\|\Psi^{\text{dis}}_\sigma f\|_{L^p(G)_{n,n}}\le C_p(1+|n|)\left(\sum_{k\in\Gamma_n}|k|\sup_{x\in G}|\sigma(x,ik)|\right)\|f\|_{L^p(G)_{n,n}},$$

where C_p is a constant independent of n. However, for the endpoint case $p=1$, the dependence of $\|\Psi^{\text{dis}}_\sigma f\|_{L^{1,\infty}(G)_{n,n}}$ on n may not be explicit, as our method in Lemma 5.10 does not explicitly determine the n-dependence for the $L^{1,\infty}(G)$ estimate of $\psi^{n,n}_{ik}$.

8. Local analysis of Ψ_σ

In the case of the multiplier operator, the local part can be expressed as a convolution with a compactly supported function. By using a Coifmann-Weiss transference principle for convolution, the authors in [28, 48] related the multiplier operator to the Euclidean multiplier. Then, the boundedness of the multiplier operator follows from the Mikhlin multiplier theorem on $\mathbb{R}$. However, a fundamental difference arises between the multiplier case and our current situation with the pseudo-differential operator. In the case of the multiplier operator, writing it as a convolution operator plays a crucial role, but the presence of an extra variable x in the symbol $\sigma(x,\lambda)$ of the pseudo-differential operator prevents us from using the theory of multipliers.

We establish the boundedness of Ψ^{loc}_σ by employing a generalized Coifman-Weiss transference principle in Section 4 for the kernel integral operator. This principle allows us to establish a connection between the L^p-boundedness of Ψ^{loc}_σ and Euclidean pseudo-differential operators. Our analysis of the local part of Ψ_σ leads to the following result.

Theorem 8.1. *Let Ψ_σ^{loc} be the operator defined in (6.5). Assume that $\sigma : G \times \mathbb{R} \mapsto \mathbb{C}$ is a function satisfying the following properties:*

(1) For each $\lambda \in \mathbb{R}$, $x \mapsto \sigma(x, \lambda)$ is a K-biinvariant function on G.
(2) For each $x \in G$, $\lambda \mapsto \sigma(x, \lambda)$ is an even function on $\mathbb{R}$.
(3) For each $\overline{v} \in \overline{N}$, the function $(s, \lambda) \mapsto \sigma_{\overline{v}}(s, \lambda) := \sigma(\overline{v} a_s, \lambda) \in \mathcal{H}(\mathbb{R}, 2, 2)$ and

$$\sup_{\overline{v} \in \overline{N}} \|\sigma_{\overline{v}}\|_{\mathcal{H}(\mathbb{R},2,2)} < \infty, \tag{8.1}$$

where $\|\sigma\|_{\mathcal{H}(\mathbb{R},2,2)}$ is defined in (2.29).

Then, for $1 < p < \infty$, Ψ_σ^{loc} is bounded from $L^p(G)_{n,n}$ to itself. Moreover, there exists a constant $C_p > 0$ such that

$$\|\Psi_\sigma^{loc} f\|_{L^p(G)} \leq C_p (1 + |n|)^6 \left(\sup_{\overline{v} \in \overline{N}} \|\sigma_{\overline{v}}\|_{\mathcal{H}(\mathbb{R},2,2)} \right) \|f\|_{L^p(G)}, \tag{8.2}$$

for all $f \in L^p(G)_{n,n}$.

Remark 8.2. We note that the norm of $\sigma_{\overline{v}}$ in (8.2) is considered in $\mathcal{H}(\mathbb{R}, 2, 2)$ rather than in $\mathcal{H}(S_p, 2, 2)$. Furthermore, (8.2) illustrates how the bound on the $L^p(G)_{n,n}$ operator norm of Ψ_σ^{loc} exhibits a polynomial dependence on n.

Before delving into the proof of Theorem 8.1, let us first obtain a quantitative estimate of $\|\Psi_\sigma^{\text{loc}} f\|_{L^p(G)}$ for $f \in C_c^\infty(G)_{n,n}$ and $p \in [1, \infty)$. We will use this estimate to establish the L^p-boundedness of the operator Ψ_σ^{loc}. Let us recall the definition (6.5) of Ψ_σ^{loc} and proceed with a change of variable, leading to the following expression:

$$\|\Psi_\sigma^{\text{loc}} f\|_{L^p(G)} = \left(\int_G \left| \int_G f(y) \mathcal{K}^{\text{loc}}(x, y^{-1}x) dy \right|^p dx \right)^{\frac{1}{p}} = \left(\int_G \left| \int_G f(xy) \mathcal{K}^{\text{loc}}(x, y^{-1}) dy \right|^p dx \right)^{\frac{1}{p}}.$$

By writing the integral formula (2.7) corresponding to the Cartan decomposition, the right-hand side of the equation above transforms into

$$\left(\int_G \left| \int_K \int_{\mathbb{R}} f(xka_t) \mathcal{K}^{\text{loc}}(x, (ka_t)^{-1}) |\Delta(t)| dk dt \right|^p dx \right)^{\frac{1}{p}}$$
$$\leq \int_K \left(\int_G \left| \int_{\mathbb{R}} f(xka_t) \mathcal{K}^{\text{loc}}(x, a_t) |\Delta(t)| dt \right|^p dx \right)^{\frac{1}{p}} dk,$$

where in the last step, we employed Minkowski's integral inequality and used the fact that $x \mapsto \mathcal{K}^{\text{loc}}(x, y)$ is an (n, n) type function on G. Next, by the change of variable $xk \to x$, and utilizing the fact that K is a compact subgroup of G with $\int_K dk = 1$, we obtain the following:

$$\|\Psi_\sigma^{\text{loc}} f\|_{L^p(G)} \leq \left(\int_G \left| \int_{\mathbb{R}} f(\mathcal{R}_t x)\, \mathcal{K}^{\text{loc}}(x, a_t) |\Delta(t)| dt \right|^p dx \right)^{\frac{1}{p}}, \tag{8.3}$$

where $\{\mathcal{R}_t : t \in \mathbb{R}\}$ are the representations consisting of measure-preserving transformations of the space G, defined by

$$\mathcal{R}_t x := x a_t, \quad x \in G.$$

In particular, the representations $\{\mathcal{R}_t : t \in \mathbb{R}\}$ preserve the L^p norm of any function on G, that is for any $f \in L^p(G)$, we have

$$\|f(\mathcal{R}_t x)\|_{L^p(G)} = \|f\|_{L^p(G)}, \quad \text{for all } t \in \mathbb{R}.$$

Next, we will apply the generalized Coifman-Weiss transference principle to the integral operator on the right-hand side of (8.3) in order to establish the $L^p(G)_{n,n}$-boundedness of the operator $\Psi_\sigma^{\mathrm{loc}}$.

8.1. **Application of generalized Coifman-Weiss transference principle.** The proof of Theorem 8.1 follows as a consequence of the generalized Coifman-Weiss transference principle in Section 4 and the following corollary. This corollary plays a crucial role in applying the Coifman-Weiss transference principle in our specific context, as it provides the explicit estimate required to use (4.4) in our setting. Let us define the Coifman-Weiss kernel $\mathcal{K}^{\mathrm{CW}} : G \times G \times \mathbb{R} \to \mathbb{C}$ by

$$\mathcal{K}^{\mathrm{CW}}(x, y, t) = \mathcal{K}^{\mathrm{con}}(x, y^{-1}x)\eta^\circ(t)|\Delta(t)|. \tag{8.4}$$

Corollary 8.3. *For $x \in G$, let us recall $\mathcal{R}_s x = x a_s$, $s \in \mathbb{R}$. Assume that there exists a constant $\mathcal{C} > 0$ such that for $1 \le p < \infty$,*

$$\left(\int_{\mathbb{R}} \left|\int_{\mathbb{R}} \mathcal{K}^{CW}(\mathcal{R}_s x, \mathcal{R}_{-t}\mathcal{R}_s x, t) h(s-t) dt\right|^p ds\right)^{\frac{1}{p}} \le \mathcal{C}\|h\|_{L^p(\mathbb{R})} \tag{8.5}$$

for all $h \in L^p(\mathbb{R})$. Then we have

$$\|\Psi_\sigma^{loc} f\|_{L^p(G)} \le \mathcal{C}\|f\|_{L^p(G)},$$

for all $f \in L^p(G)_{n,n}$, where $\mathcal{C}$ is the same constant as in (8.5).

Proof. Let us define the operator

$$(Tf)(x) = \int_{\mathbb{R}} \mathcal{K}^{\mathrm{CW}}(x, \mathcal{R}_t x, t) f(\mathcal{R}_t x)\, dt, \quad \text{for } f \in C_c^\infty(G)_{n,n}. \tag{8.6}$$

As the function $t \mapsto \mathcal{K}^{\mathrm{CW}}(x, \mathcal{R}_t x, t)$ is compactly supported, we can compare (8.6) with the Coifmann-Weiss transference operator in (4.3). Furthermore, equation (8.5) indicates that the kernel of the operator T satisfies (4.4) and the hypothesis of Theorem 4.1. Thus, in view of (8.3) and by applying the generalized Coifman-Weiss transference principle (4.4), we establish our corollary. □

8.2. **Kernel of the Coifman-Weiss transference operator.** The corollary above simplifies the proof of Theorem 8.1, since we now only need to focus on establishing (8.5) with

$$\mathcal{C} = C(1+|n|)^6 \sup_{\overline{v} \in \overline{N}} \|\sigma_{\overline{v}}\|_{\mathcal{H}(\mathbb{R},2,2)}.$$

The remaining part of this section will be dedicated to proving the inequality (8.5), assuming the hypotheses of Theorem 8.1. We observe from the definition (8.4) of the Coifman-Weiss kernel $\mathcal{K}^{\mathrm{CW}}$ that

$$\mathcal{K}^{\mathrm{CW}}(\mathcal{R}_s x, \mathcal{R}_{-t}\mathcal{R}_s x, t) = \mathcal{K}^{\mathrm{con}}(x a_s, a_t)\eta^\circ(t)\Delta(t), \quad t \ge 0.$$

By recalling the expression (6.2) of $\mathcal{K}^{\mathrm{con}}$, we get

$$\mathcal{K}^{\mathrm{CW}}(\mathcal{R}_s x, \mathcal{R}_{-t}\mathcal{R}_s x, t) = \eta^\circ(a_t)\Delta(t) \int_{\mathbb{R}} \sigma(x a_s, \lambda)\phi_{\tau,\lambda}^{n,n}(a_t)|c_\tau^{n,n}(\lambda)|^{-2} d\lambda. \tag{8.7}$$

It is not immediately evident from the given hypotheses that $\mathcal{K}^{\mathrm{CW}}$ exists other than in a distributional sense. Throughout the rest of the paper, we will consistently assume that the symbol satisfying estimates as in Theorem 3.3 are, in fact, rapidly decreasing, although our estimates will not depend on the rate of decrease. Explicitly, we assume $\sigma(\cdot,\lambda)$ is multiplied with a factor of the form $e^{-\epsilon\lambda^2}$, where $0<\epsilon\leq 1$. This assumption will enable us to define pointwise functions and perform various formal manipulations, such as integration by parts. Our estimates are uniform in ϵ. Once we prove suitable uniform estimates, standard limiting arguments allow us to pass to the general case.

We will prove (8.5) in the following steps:

Step 1: Expansion of the Kernel $\mathcal{K}^{\mathrm{CW}}$. We will utilize the asymptotic expansion of $\phi^{n,n}_{\tau,\lambda}$ near the identity to express $\mathcal{K}^{\mathrm{CW}}$ in terms of simpler or well-behaved functions.

Let Φ is a smooth even function on $\mathbb{R}$, with $0\leq\Phi\leq 1$; $\Phi=1$ when $|\lambda|>2$; $\Phi(\lambda)=0$ when $|\lambda|<1$. By extending the approach from [55, Proposition 4.1] and applying Lemma 5.3, we can derive an equivalent result to [45, Proposition 4.3] within our framework. This will enable us to decompose $\mathcal{K}^{\mathrm{CW}}$ as follows:

$$\mathcal{K}^{\mathrm{CW}}(\mathcal{R}_s x,\mathcal{R}_{-t}\mathcal{R}_s x,t)=\mathcal{K}_0(xa_s,t)+\mathcal{K}_{\mathrm{err}}(xa_s,t), \tag{8.8}$$

satisfying the following:

(i) There is some constant $C>0$ (independent of n) and $\kappa\in L^1(\mathbb{R})$, such that

$$|\mathcal{K}_{\mathrm{err}}(x,t)|\leq C(1+|n|)^6\left(\sup_{\overline{v}\in\overline{N}}\|\sigma_{\overline{v}}\|_{\mathcal{H}(\mathbb{R},0,2)}\right)\kappa(t), \tag{8.9}$$

for all $x\in G$, and

(ii) $\mathcal{K}_0(xa_s,t)=C\eta^\circ(a_t)\,\Delta(t)\,\left(\frac{t}{\Delta(t)}\right)^{1/2}\int_0^\infty\Phi(\lambda)\sigma(xa_s,\lambda)\mathcal{J}_0(\lambda t)|c^{n,n}_\tau(\lambda)|^{-2}d\lambda$,

where we recall $\mathcal{J}_0$ is the generalized Bessel function defined as in (5.6).

Step 2: Connection with Euclidean pseudo-differential operator. From (8.9), we notice that we can bound $|\mathcal{K}_{\mathrm{err}}(xa_s,\cdot)|$ with an integrable function κ in $\mathbb{R}$ that remains independent of 'x' and 's'. Consequently, it's evident that $\mathcal{K}_{\mathrm{err}}$ satisfies (8.5). To establish the assumptions of Corollary 8.3, our task is to prove that $\mathcal{K}_0$ also fulfills (8.5). To achieve this, we will demonstrate that the functions $\{\mathcal{K}_0(xa_s,t):x\in G\}$ can act as kernels for Euclidean pseudo-differential operators corresponding to a family of symbols $\{a_x(s,\xi):x\in G\}$. These symbols will satisfy the conditions outlined in Theorem 1.1. By doing so, we can employ Theorem 1.1 to conclude that $\mathcal{K}_0$ satisfies (8.5). In summary, our objective is to prove the following:

$$\left|(1+|\xi|)^\alpha\partial_s^\beta\partial_\xi^\alpha\int_{-\infty}^\infty e^{-2\pi i\xi t}\mathcal{K}_0(xa_s,t)dt\right|\leq C\left(\sup_{x\in G}\|\sigma_x\|_{\mathcal{H}(\mathbb{R},2,2)}\right), \tag{8.10}$$

for all $x\in G$, $s,\xi\in\mathbb{R}$, and $\alpha,\beta\in\{0,1,2\}$, where the constant C is independent of n and $x\in G$. Now for a given $x_0\in G$, utilizing the Iwasawa decomposition and the fact that the abelian group A acts as a dilation on $\overline{N}$, we can find $k_0\in K,\overline{v}\in\overline{N}$, and $s_0\in\mathbb{R}$ such that $x_0=k_0\overline{v}a_{s_0}$. Since by hypothesis of Theorem 8.1, $x\mapsto\sigma(x,\lambda)$ is a K-biinvariant function on G, we observe that it suffices to prove (8.10) for all $x=\overline{v}\in\overline{N}$.

Step 3: Estimate of the Kernel $\mathcal{K}_0$. We recall the definition of $\mathcal{K}_0$ from (8.8):

$$\mathcal{K}_0(\overline{v}a_s,t)=C\eta^\circ(a_t)\,\Delta(t)\,\left(\frac{t}{\Delta(t)}\right)^{1/2}\int_0^\infty\Phi(\lambda)\sigma(\overline{v}a_s,\lambda)\mathcal{J}_0(\lambda t)|c^{n,n}_\tau(\lambda)|^{-2}\,d\lambda, \tag{8.11}$$

where we have the following formula from [44, Eq. 10.9.12, p.224]

$$\mathcal{J}_0(\lambda t) = \frac{2}{\pi}\int_\lambda^\infty (\xi^2-\lambda^2)^{-1/2}\sin\xi t\, d\xi.$$

Let us define

$$\Sigma(x,\lambda) = (\partial_\lambda\cdot(1/\lambda))^{(d-2)/2}\left(\varPhi(\lambda)\sigma(x,\lambda)|c_\tau^{n,n}(\lambda)|^{-2}\right),$$

for all $x\in G$ and $\lambda\in\mathbb{R}$. Then using integration by parts we have,

$$\begin{aligned}\mathcal{K}_0(\overline{v}a_s,t) &= C\eta^\circ(a_t)\left[\frac{\Delta(t)}{t}\right]^{\frac12} t\int_{\mathbb{R}}\Sigma(\overline{v}a_s,\lambda)\mathcal{J}_0(\lambda t)d\lambda\\ &= C\eta^\circ(a_t)\left[\frac{\Delta(t)}{t}\right]^{\frac12} t\int_{\mathbb{R}}\sin\xi t\int_0^\xi\Sigma(\overline{v}a_s,\lambda)(\xi^2-\lambda^2)^{-1/2}\,d\lambda\,d\xi\\ &= C\eta^\circ(a_t)\left[\frac{\Delta(t)}{t}\right]^{\frac12}\int_{\mathbb{R}}\cos\xi t\frac{d}{d\xi}\int_0^\xi\Sigma(\overline{v}a_s,\lambda)(\xi^2-\lambda^2)^{-1/2}\,d\lambda\,d\xi.\end{aligned}$$

Let us define

$$\begin{aligned}g(x,\xi) &:= \int_0^\xi(\xi^2-\lambda^2)^{-1/2}\Sigma(x,\lambda)\,d\lambda,\quad x\in G,\xi\geq 0\\ &= \frac12\int_{-1}^1(1-\lambda^2)^{-1/2}\Sigma(x,\lambda\xi)\,d\lambda\quad x\in G,\quad\text{when }\xi>0,\end{aligned}\tag{8.12}$$

and

$$h(x,\xi) = \left(\frac{\partial}{\partial\xi}g\right)(x,|\xi|),\quad \xi\in\mathbb{R}.$$

We obtain

$$\begin{aligned}\mathcal{K}_0(\overline{v}a_s,t) &= -C\eta^\circ(a_t)\left[\frac{\Delta(t)}{t}\right]^{1/2}\int_0^\infty\frac{d}{d\xi}\,g(\overline{v}a_s,\xi)\cos\xi t\,d\xi\\ &= -C\eta^\circ(a_t)\left[\frac{\Delta(t)}{t}\right]^{1/2}\int_{-\infty}^\infty h(\overline{v}a_s,\xi)e^{i\xi t}d\xi.\end{aligned}$$

Let $\nu(t) = -C\eta^\circ(a_t)\left[\frac{\Delta(t)}{t}\right]^{1/2}$, $t\in\mathbb{R}$. Then

$$\mathcal{F}(\mathcal{K}_0(\overline{v}a_s,\cdot))(\xi) = \left(\mathcal{F}(\nu)*_{\mathbb{R}}h(\overline{v}a_s,\cdot)\right)(\xi).\tag{8.13}$$

First, we claim that

$$\sup_{\xi\in\mathbb{R}}\left(\left|\partial_s^\beta h(\overline{v}a_s,\xi)\right|+\left|(1+\xi)\partial_s^\beta\partial_\xi h(\overline{v}a_s,\xi)\right|\right)\leq C\left(\sup_{\overline{v}\in\overline{N}}\|\sigma_{\overline{v}}\|_{\mathcal{H}(\mathbb{R},2,2)}\right)\qquad\text{for all }\beta\in\{0,1,2\},$$

where the constant C is independent of $\overline{v}\in\overline{N}$. From (8.12) we have

$$h(\overline{v}a_s,\xi) = \frac12\int_{-1}^1(1-\lambda^2)^{-1/2}\lambda\frac{\partial}{\partial(\lambda\xi)}\Sigma(\overline{v},\lambda\xi)\,d\lambda\qquad\text{for }\xi>0.$$

Then using Lemma 2.4 and the hypothesis in (8.1) we get

$$\left|\frac{\partial^\beta}{\partial s^\beta}\frac{\partial^\alpha}{\partial\lambda^\alpha}\Sigma(\overline{v}a_s,\lambda)\right|\leq C\left(\sup_{\overline{v}\in\overline{N}}\|\sigma_{\overline{v}}\|_{\mathcal{H}(\mathbb{R},2,2)}\right)(1+|\lambda|)^{1-\alpha}\quad\text{for all }\alpha,\beta\in\{0,1,2\}.$$

Hence

$$\begin{aligned}|\frac{\partial^\beta}{\partial s^\beta} h(\overline{v}a_s,\xi)| &\leq C\left(\sup_{\overline{v}\in\overline{N}}\|\sigma_{\overline{v}}\|_{\mathcal{H}(\mathbb{R},2,2)}\right)\int_0^1(1-\lambda^2)^{-1/2}\lambda\,d\lambda\\ &\leq C\left(\sup_{\overline{v}\in\overline{N}}\|\sigma_{\overline{v}}\|_{\mathcal{H}(\mathbb{R},2,2)}\right).\end{aligned}$$

Also,

$$\begin{aligned}|(1+\xi)\frac{\partial^\beta}{\partial s^\beta}\frac{\partial}{\partial\xi} h(\overline{v}a_s,\xi)| &\leq C\left(\sup_{\overline{v}\in\overline{N}}\|\sigma_{\overline{v}}\|_{\mathcal{H}(\mathbb{R},2,2)}\right)(1+|\xi|)\int_0^1(1-\lambda^2)^{-1/2}\frac{\lambda^2}{(1+|\lambda\xi|)}d\lambda\\ &\leq C\left(\sup_{\overline{v}\in\overline{N}}\|\sigma_{\overline{v}}\|_{\mathcal{H}(\mathbb{R},2,2)}\right)\left(\int_0^1(1-\lambda^2)^{-1/2}\lambda^2 d\lambda\right.\\ &\qquad\left.+|\xi|\int_0^1(1-\lambda^2)^{-1/2}\frac{\lambda}{|\xi|}d\lambda\right)\\ &\leq C\left(\sup_{\overline{v}\in\overline{N}}\|\sigma_{\overline{v}}\|_{\mathcal{H}(\mathbb{R},2,2)}\right).\end{aligned}$$

Similarly, we can prove that

$$\left|(1+\xi)^\alpha\partial_s^\beta\partial_\xi^\alpha h(\overline{v}a_s,\xi)\right| \leq C\left(\sup_{\overline{v}\in\overline{N}}\|\sigma_{\overline{v}}\|_{\mathcal{H}(\mathbb{R},2,2)}\right)\quad\text{for all } \alpha,\beta\in\{0,1,2\}.$$

By applying the inequality above and taking into account that $\mathcal{F}(\nu)$ is a Schwartz function, we can deduce from (8.13) that $\mathcal{K}_0$ satisfies (8.10). Consequently, This also concludes the proof of our Theorem 8.1.

9. Large-scale analysis of Ψ_σ

In the previous section, we observed how the kernel $\mathcal{K}^{\mathrm{loc}}$ exhibits similar behavior to the kernel of an Euclidean pseudo-differential operator. By using the transference method, we successfully obtained a bound for the L^p operator norm of the local part of Ψ_σ. However, the analysis of the global part undergoes significant changes due to the exponential volume growth of the group G and the entirely different local and global behavior of $\phi^{n,n}_{\tau,\lambda}$.

In multiplier theory, the authors in [48] utilized the Herz majorizing principle theorem for the convolution operator. For cases involving noncompact type symmetric spaces, Ionescu [29] further employed the same principle to estimate the L^p norm of the multiplier operator. In this section, we will establish the L^p bound of $\Psi^{\mathrm{glo}}_\sigma$. To achieve this, we will use the expansion of spherical functions $\phi^{n,n}_{\tau,\lambda}$ away from the identity, and we will see that the global analysis of Ψ_σ has no Euclidean analogue. Our approach in this section follows the general outline of [28]. Finally, Theorem 3.3 will be completed as a consequence of the following theorem. Before stating the main result in this section, let us recall the definition (6.6) of $\Psi^{\mathrm{glo}}_\sigma$,

$$\Psi^{\mathrm{glo}}_\sigma f(x) = \int_G f(y)\mathcal{K}^{\mathrm{glo}}(x, y^{-1}x)\,dy,\quad x\in G, \tag{9.1}$$

where

$$\mathcal{K}^{\mathrm{glo}}(x, y^{-1}x) = \eta(y^{-1}x) \int_{\mathbb{R}} \sigma(x,\lambda)\phi^{n,n}_{\tau,\lambda}(y^{-1}x)|c^{n,n}_{\tau}(\lambda)|^{-2} d\lambda. \tag{9.2}$$

We will complete the proof of Theorem 3.3 by establishing the following theorem.

Theorem 9.1. *Let $p \in [1,2) \cup (2,\infty)$. Suppose that $\sigma : G \times S_p^{\circ} \to \mathbb{C}$ is a function satisfying the following properties:*

(i) For each $\lambda \in S_p^{\circ}$, $x \mapsto \sigma(x,\lambda)$ is a K-biinvariant function on G.

(ii) For each x in G, $\lambda \mapsto \sigma(x,\lambda)$ is an even holomorphic function on S_p° and

$$\sup_{x\in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)} < \infty. \tag{9.3}$$

(iii) Additionally, for $p = 1$ and $n \in \mathbb{Z}^{\tau+} \setminus \{0\}$, $\frac{\partial^\alpha}{\partial\lambda^\alpha}\sigma(x,\lambda)|_{\lambda=i} = 0$, for all $\alpha \in \{0,1,2\}$, $x \in G$.

Then the operator Ψ^{glo}_σ defined as in (9.1), is a bounded operator from $L^p(G)_{n,n}$ to itself. Moreover, there exists a constant $C_{p,n} > 0$ such that

$$\|\Psi^{glo}_\sigma f\|_{L^p(G)} \le C_{p,n} \left(\sup_{x\in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)} \right) \|f\|_{L^p(G)},$$

for all $f \in L^p(G)_{n,n}$.

Remark 9.2. (1) We would like to mention that for $p = 1$ case when $n = 0$ or $n \in \mathbb{Z}^{\tau-}$, we obtain an improvement on the L^1 case. More precisely, for $n = 0$ or $n \in \mathbb{Z}^{\tau-}$, it suffices to assume conditions (i) and (ii) of the theorem above to establish the L^1-boundedness of $\Psi^{\mathrm{glo}}_\sigma$.

(2) When examining the proof of the boundedness of $\Psi^{\mathrm{glo}}_\sigma$ and $\Psi^{\mathrm{dis}}_\sigma$, it becomes apparent that these operators do not rely on any regularity conditions concerning the space variable of the symbol σ. The requirement for a derivative condition on the space variable is solely necessary to establish the boundedness of the local part $\Psi^{\mathrm{loc}}_\sigma$. In fact, to address the boundedness of $\Psi^{\mathrm{loc}}_\sigma$, we needed $\sup_{\overline{v}\in\overline{N}} \|\sigma_{\overline{v}}\|_{\mathcal{H}(\mathbb{R},2,2)} < \infty$ (see (8.2)). Hence, the regularity condition on the space variable of the symbol $\sigma(x,\lambda)$ is essential, but only when λ is along the real line, not for the whole strip S_p.

(3) We would like to remark that, similar to the observation made by the authors in the multiplier case (see [48, (5.18)]), we are unable to preserve the explicit polynomial dependence on n for the $L^p(G)_{n,n}$ operator norm of the global part $\Psi^{\mathrm{glo}}_\sigma$ of the pseudo-differential operator Ψ_σ. This contrasts with the discrete and local parts of Ψ_σ, where such polynomial dependence is maintained. One possible way to achieve a norm estimate for the global part that grows polynomially or remains uniformly bounded in n is to improve the estimate in inequality (5.10) (see Remark 5.5).

9.1. **Estimate of the global kernel $\mathcal{K}^{\mathrm{glo}}$.** As in the Euclidean setting, the estimate of $\mathcal{K}^{\mathrm{glo}}$ plays a crucial role in establishing the boundedness of the pseudo-differential operator. However, unlike in Euclidean space, we will observe the exponential decay instead of polynomial decay in our estimates. This exponential decay is essential to handle the exponential volume growth of the group $\mathrm{SL}(2,\mathbb{R})$. In the following lemma, we will explore how the holomorphicity of the symbol is responsible for the decay property mentioned above.

Lemma 9.3. *Let $\mathcal{K}^{glo}$ be as in (9.2). Assume that σ satisfies the hypothesis of Theorem 9.1. Then the kernel $\mathcal{K}^{glo}$ satisfies the following estimates*

(1) For $1 \le p < 2$, one has

$$|\mathcal{K}^{glo}(x, y^{-1}x)| \le C_{p,n} \left(\sup_{x \in G} \|\sigma(x, \cdot)\|_{\mathcal{MH}(S_p,2)} \right) \frac{\eta(y^{-1}x)}{(1 + [y^{-1}x]^+)^2} e^{-\frac{2}{p}([y^{-1}x]^+)}. \tag{9.4}$$

(2) For $2 < p < \infty$, one has

$$|\mathcal{K}^{glo}(x, y^{-1}x)| \le C_{p,n} \left(\sup_{x \in G} \|\sigma(x, \cdot)\|_{\mathcal{MH}(S_p,2)} \right) \eta(y^{-1}x) \frac{e^{-\frac{2}{p'}([y^{-1}x]^+)}}{(1 + [y^{-1}x]^+)^2}, \tag{9.5}$$

where $C_{p,n} > 0$ is a constant depending only on the symbol and $n \in \mathbb{Z}^\sigma$.

Proof. To begin, we will prove the lemma for $1 < p < 2$. We observe from (9.2) that the function $y \mapsto \mathcal{K}^{\mathrm{glo}}(\cdot, y)$ is supported away from identity, as η is a K-biinvariant function supported on the set

$$\{k_1 a_t k_2 : t \ge 1/2, k_1, k_2 \in K\}.$$

This allows us to apply the Harish-Chandra series expansion of the spherical function $\phi^{n,n}_{\tau,\lambda}$ from Lemma 5.4. Substituting the explicit expression of $\phi^{n,n}_{\tau,\lambda}(y^{-1}x)$ from (5.9) and using the fact that $|c^{n,n}_\tau(\lambda)|^2 = c^{n,n}_\tau(\lambda) c^{n,n}_\tau(-\lambda)$ for all $\lambda \in \mathbb{R}$, we obtain

$$\begin{aligned}\mathcal{K}^{\mathrm{glo}}(x, y^{-1}x) =& \eta(y^{-1}x) e^{-[y^{-1}x]^+} \left(\int_{\mathbb{R}} \sigma(x, \lambda)(1 + a(\lambda, [y^{-1}x]^+)) c^{n,n}_\tau(-\lambda)^{-1} e^{i\lambda([y^{-1}x]^+)} d\lambda \right. \\ & \left. + \int_{\mathbb{R}} \sigma(x, \lambda)(1 + a(-\lambda, [y^{-1}x]^+)) c^{n,n}_\tau(\lambda)^{-1} e^{i\lambda([y^{-1}x]^+)} d\lambda \right).\end{aligned}$$

After the change of variable $\lambda \mapsto -\lambda$ in the second integral and using the fact that $\lambda \mapsto \sigma(x, \lambda)$ is an even function, we get

$$\begin{aligned}\mathcal{K}^{\mathrm{glo}}(x, y^{-1}x) =& C\eta(y^{-1}x) e^{-[y^{-1}x]^+} \int_{\mathbb{R}} \sigma(x, \lambda) c^{n,n}_\tau(-\lambda)^{-1} e^{i\lambda([y^{-1}x]^+)} d\lambda \\ & + C\eta(y^{-1}x) e^{-[y^{-1}x]^+} \int_{\mathbb{R}} \sigma(x, \lambda) a(\lambda, [y^{-1}x]^+) c^{n,n}_\tau(-\lambda)^{-1} e^{i\lambda([y^{-1}x]^+)} d\lambda \\ =& \mathcal{K}^{\mathrm{glo}}_0(x, y^{-1}x) + \mathcal{K}^{\mathrm{glo}}_{\mathrm{error}}(x, y^{-1}x).\end{aligned}$$

First, we will demonstrate that $\mathcal{K}^{\mathrm{glo}}_0(x, y^{-1}x)$ satisfies the estimate (9.4). We observe that the above integrand is a holomorphic function on S_p°. Therefore, by applying Cauchy's integral theorem, we move the integration with respect to λ from $\mathbb{R}$ to $\mathbb{R} + i(\gamma_p - \gamma_p/2[y^{-1}x]^+)$, obtaining

$$\mathcal{K}^{\mathrm{glo}}_0(x, y^{-1}x) = C\, \eta(y^{-1}x) e^{-2/p([y^{-1}x]^+)} \int_{\mathbb{R}} \nu(x, \lambda + i(\gamma_p - \gamma_p/[y^{-1}x]^+)) e^{i\lambda([y^{-1}x]^+)} d\lambda, \tag{9.6}$$

where $\nu(x, \lambda)$ is defined as follows

$$\nu(x, \lambda) = \sigma(x, \lambda)) c^{n,n}_\tau(-\lambda)^{-1}.$$

Consequently, integrating by parts the inner integral of (9.6), we deduce that

$$\mathcal{K}^{\mathrm{glo}}_0(x, y^{-1}x) = C\, \eta(y^{-1}x) \frac{e^{-2/p([y^{-1}x]^+)}}{([y^{-1}x]^+)^2} \int_{\mathbb{R}} \frac{\partial^2}{\partial \lambda^2} \nu(x, \lambda + i(\gamma_p - \gamma_p/[y^{-1}x]^+)) e^{i\lambda([y^{-1}x]^+)} d\lambda.$$

Taking modulus on both sides, using hypothesis (9.3), and the estimate (2.24) of $c_\tau^{n,n}(-\lambda)^{-1}$ we get,

$$\begin{aligned}\left|\mathcal{K}_0^{\rm glo}(x,y^{-1}x)\right| &= C_{p,n}\left(\sup_{x\in G}\|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)}\right)\eta(y^{-1}x)\frac{e^{-2/p(\,[y^{-1}x]^+)}}{([y^{-1}x]^+)^2}\int_{\mathbb{R}}\frac{d\lambda}{(1+|\lambda|)^{3/2}}\\ &\leq C_{p,n}\left(\sup_{x\in G}\|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)}\right)\eta(y^{-1}x)\frac{e^{-2/p(\,[y^{-1}x]^+)}}{([y^{-1}x]^+)^2}.\end{aligned}$$

Next, we shall estimate $\mathcal{K}_{\rm error}^{\rm glo}(x,y)$. Proceeding in a similar way as in the proof of $\mathcal{K}_0^{\rm glo}$, we get

$$\begin{aligned}\mathcal{K}_{\rm error}^{\rm glo}(x,y^{-1}x) = C\,\eta(y^{-1}x)\frac{e^{-2/p(\,[y^{-1}x]^+)}}{([y^{-1}x]^+)^2}\int_{\mathbb{R}}\frac{\partial^2}{\partial\lambda^2}&\big(\nu(x,\lambda+i(\gamma_p-\gamma_p/[y^{-1}x]^+))\\ &\cdot a(\lambda+i(\gamma_p-\gamma_p/[y^{-1}x]^+),[y^{-1}x]^+)\big)\,e^{i\lambda([y^{-1}x]^+)}d\lambda.\end{aligned}$$

We recall from (5.10) that the function $\lambda\mapsto a(\lambda,t)$ satisfies a favorable symbol-type estimate for $t\geq 1/10$. Moreover, by the Leibniz rule, one has

$$\left|\frac{\partial^2}{\partial\lambda^2}\left(\frac{\sigma(x,\lambda)\,a(\lambda,t)}{c_\tau^{n,n}(-\lambda)}\right)\right|\leq\frac{C_n}{(1+|\lambda|)^{3/2}} \tag{9.7}$$

for all $\lambda\in\mathbb{C}$ with $0\leq \operatorname{Im}\lambda\leq\gamma_p$, $t\geq 1/10$ and $x\in G$. By (9.7) in the expression of $\mathcal{K}_{\rm error}^{\rm glo}$, we obtain the required estimate of $\mathcal{K}_{\rm error}^{\rm glo}(x,y^{-1}x)$. This settles the lemma for $1<p<2$.

Now, we handle the $p=1$ case. We recall from Lemma 2.3, for $n\in\mathbb{Z}^{\tau+}\setminus\{0\}$, the function $\lambda\to c_\tau^{n,n}(-\lambda)^{-1}$ has a simple pole at $\lambda=i$ within the boundary of S_1. This prevents us from directly using the estimate (2.24) of $c_\tau^{n,n}(-\lambda)^{-1}$ near $\lambda=i$. To overcome this obstacle, we will utilize the hypothesis on the symbol $\sigma(x,\lambda)$, which has a zero at $\lambda=i$. Exploiting this and the Leibniz rule, we obtain that

$$\left|\frac{\partial^2}{\partial\lambda^2}\left(\frac{\sigma(x,\lambda)}{c_\tau^{n,n}(-\lambda)}\right)\right|\leq\frac{C_n}{(1+|\lambda|)^{3/2}}. \tag{9.8}$$

With this inequality in hand and repeating the previous argument, we can complete the proof of (9.4). When $n\in\mathbb{Z}^{\tau-}\cup\{0\}$, $\lambda\to c_\tau^{n,n}(-\lambda)^{-1}$ does not have any pole in S_1, so the proof follows similarly as in $1<p<2$ case, this also concludes the proof for $p=1$.

Next, we address the case $2<p<\infty$, which follows a similar approach. We note that $\gamma_p=|(2/p-1)|=(2/p'-1)$ for all $p>2$, where $1/p'=1-1/p$. Applying the same argument as before, we obtain (9.6) with $e^{-2/p([y^{-1}x]^+)}$ replaced by $e^{-2/p'([y^{-1}x]^+)}$. Then, proceeding analogously to the previous case, we can establish (9.5). □

We now employ the lemma above to establish the L^p-boundedness of $\Psi_\sigma^{\rm glo}$, and, consequently, complete the proof of Theorem 3.3. Before that, we express $\Psi_\sigma^{\rm glo}$ as a sum of two integral operators on the $\overline{N}A$ group. We then proceed to find their L^p norm estimates.

Let f and φ be two smooth compactly supported (n,n) type functions on the group G. Using the Iwasawa integration formula (2.6), we can write from (9.1) the expression for $\Psi_\sigma^{\rm glo}$ with $y=\overline{m}a_sk_1$ and $x=\overline{n}a_tk_2$ as follows:

$$\begin{aligned}\left\langle \Psi_\sigma^{\mathrm{glo}} f, \varphi \right\rangle &= \int_G \Psi_\sigma^{\mathrm{glo}} f(x) \overline{\varphi(x)} dx \\ &= \int_{\overline{N}} \int_{\overline{N}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\overline{m} a_s) \overline{\varphi(\overline{n} a_t)} \mathcal{K}^{\mathrm{glo}}(\overline{n} a_t, a_{-s} \overline{m}^{-1} \overline{n} a_t) e^{2(s+t)} ds\, d\overline{m}\, dt\, d\overline{n} \qquad (9.9) \\ &= \int_{\overline{N}} \int_{\overline{N}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\overline{m} a_s) \overline{\varphi(\overline{n} a_t)} \mathcal{K}^{\mathrm{glo}}(\overline{n} a_t, \delta_{-s}(\overline{m}^{-1} \overline{n}) a_{t-s}) e^{2(s+t)} ds\, d\overline{m}\, dt\, d\overline{n},\end{aligned}$$

where we recall that δ_{-s} is the dilation on the group $\overline{N}$ defined in (2.13).

We observe that the integrations over the k_2 variable are canceled due to the right K-invariance of the symbol σ with respect to the space variable. If we assume that $x \mapsto \sigma(x, \lambda)$ is any other right l ($\neq 0$)-type, then the above integral will be always zero.

Now, let χ^+ and χ^- be the characteristic functions of the intervals $[0, \infty)$ and $(-\infty, 0)$, respectively. We can now express the above operator as follows:

$$\left\langle \Psi_\sigma^{\mathrm{glo}} f, \varphi \right\rangle = \left\langle \mathcal{I}_\sigma^- f, \varphi \right\rangle + \left\langle \mathcal{I}_\sigma^+ f, \varphi \right\rangle \quad \text{(say)},$$

where $\mathcal{I}_\sigma^\pm f$ are the operators defined on $\overline{N}A$ group by the following formulae

$$\mathcal{I}_\sigma^\pm f(\overline{n} a_t) := \int_{\overline{N}} \int_{\mathbb{R}} f(\overline{m} a_s) \mathcal{K}^{\mathrm{glo}}(\overline{n} a_t, \delta_{-s}(\overline{m}^{-1} \overline{n}) a_{t-s}) \chi^\pm(t-s) e^{2s}\, ds\, d\overline{m}, \qquad (9.10)$$

and

$$\left\langle \mathcal{I}_\sigma^\pm f, \varphi \right\rangle = \int_{\overline{N}} \int_{\mathbb{R}} \int_{\overline{N}} \int_{\mathbb{R}} f(\overline{m} a_s) \overline{\varphi(\overline{n} a_t)} \mathcal{K}^{\mathrm{glo}}(\overline{n} a_t, \delta_{-s}(\overline{m}^{-1} \overline{n}) a_{t-s}) \chi^\pm(t-s) e^{2(s+t)}\, ds\, d\overline{m}\, dt\, d\overline{n}. \qquad (9.11)$$

Consequently, the $L^p(G)_{n,n}$-boundedness of $\Psi_\sigma^{\mathrm{glo}}$ follows from that of the operators $\mathcal{I}_\sigma^\pm$ on $L^p(\overline{N}A)$, which we shall take up separately in the rest of this section.

9.2. L^p **operator norm estimates of** $\mathcal{I}_\sigma^\pm$**.** We first shall establish the L^p-boundedness of the operator $\mathcal{I}_\sigma^-$ by proving the following lemma:

Lemma 9.4. *Let $p \in [1, 2) \cup (2, \infty)$. Suppose that σ satisfies the hypothesis of Theorem 9.1 and $\mathcal{I}_\sigma^-$ be as in (9.10). Then we have $\mathcal{I}_\sigma^-$ is bounded from $L^p(\overline{N}A)$ to itself. Moreover, there exists a constant $C_{p,n} > 0$ such that*

$$\|\mathcal{I}_\sigma^- f\|_{L^p(\overline{N}A)} \leq C_{p,n} \left(\sup_{x \in G} \|\sigma(x, \cdot)\|_{\mathcal{MH}(S_p, 2)} \right) \|f\|_{L^p(\overline{N}A)} \qquad (9.12)$$

for all $f \in L^p(\overline{N}A)$.

Proof. To get the desired result, it suffices to prove that for any smooth compactly supported functions $f, \varphi : \overline{N}A \to \mathbb{C}$, one has

$$\left| \left\langle \mathcal{I}_\sigma^- f, \varphi \right\rangle \right| \leq C_{p,n} \left(\sup_{x \in G} \|\sigma(x, \cdot)\|_{\mathcal{MH}(S_p, 2)} \right) \|f\|_{L^p(\overline{N}A)} \|\varphi\|_{L^{p'}(\overline{N}A)}. \qquad (9.13)$$

We recall the expression of $\langle \mathcal{I}_\sigma^- f, h \rangle$ from (9.11)

$$\left\langle \mathcal{I}_\sigma^- f, \varphi \right\rangle = \int_{\overline{N}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\overline{N}} f(\overline{m} a_s) \overline{\varphi(\overline{n} a_t)} \mathcal{K}^{\mathrm{glo}}(\overline{n} a_t, \delta_{-s}(\overline{m}^{-1} \overline{n}) a_{t-s}) \chi^-(t-s) e^{2(s+t)}\, ds\, d\overline{m}\, dt\, d\overline{n},$$

which after a change of variable $\overline{n} \mapsto \overline{m}^{-1}\overline{n}$ yields

$$\langle \mathcal{I}_\sigma^- f, \varphi \rangle = \int_{\overline{N}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\overline{N}} f(\overline{m}a_s)\overline{\varphi(\overline{m}\,\overline{n}a_t)}\mathcal{K}^{\mathrm{glo}}(\overline{m}\,\overline{n}a_t, \delta_{-s}(\overline{n})a_{t-s})\chi^-(t-s)e^{2(s+t)}d\overline{n}\,ds\,dt\,d\overline{m},$$

We recall that the map $\overline{n} \mapsto \delta_{-s}(\overline{n})$ is a dilation of $\overline{N}$. Hence by using (2.14), we get

$$\langle \mathcal{I}_\sigma^- f, \varphi \rangle = \int_{\overline{N}} \int_{\mathbb{R}} \int_{\overline{N}} \int_{\mathbb{R}} f(\overline{m}a_s)\overline{\varphi(\overline{m}\delta_s(\overline{n})a_t)}\mathcal{K}^{\mathrm{glo}}(\overline{m}\delta_s(\overline{n})a_t, \overline{n}a_{t-s})\chi^-(t-s)e^{2t}d\overline{n}\,ds\,dt\,d\overline{m}.$$

Next, by using Fubini's theorem and the change of variable $t \mapsto s-r$ (for fixed s), it follows that

$$\langle \mathcal{I}_\sigma^- f, \varphi \rangle = \int_{\overline{N}} \int_{\mathbb{R}} \int_{\overline{N}} \int_{r=0}^{\infty} f(\overline{m}a_s)\overline{\varphi(\overline{m}\delta_s(\overline{n})a_{s-r})}\mathcal{K}^{\mathrm{glo}}(\overline{m}\delta_s(\overline{n})a_{s-r}, \overline{n}a_{-r})e^{2(s-r)}dr\,d\overline{n}\,ds\,d\overline{m}. \tag{9.14}$$

To make use of the estimates of $\mathcal{K}^{glo}$, we divide the proof into two parts. First, we will prove the lemma for $1 \leq p < 2$. Taking modulus on both sides of the above expression and plugging the estimate (9.4) of $\mathcal{K}^{glo}$ for $1 \leq p < 2$, gives us

$$\left|\langle \mathcal{I}_\sigma^- f, \varphi \rangle\right| \leq C_{p,n} \left(\sup_{x\in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)}\right) \int_{r=0}^{\infty} \int_{\mathbb{R}} \int_{\overline{N}} \int_{\overline{N}} |f(\overline{m}a_s)||\varphi(\overline{m}\delta_s(\overline{n})a_{s-r})| \cdot \eta(\overline{n}a_{-r})\,\frac{e^{-\frac{2}{p}([\overline{n}a_{-r}]^+)}}{(1+[\overline{n}a_{-r}]^+)^2}e^{2(s-r)}d\overline{m}\,d\overline{n}\,ds\,dr. \tag{9.15}$$

Now let us define,

$$\begin{aligned} F(s) &= \left[\int_{\overline{N}} |f(\overline{m}a_s)|^p d\overline{m}\right]^{\frac{1}{p}}, \\ \Phi(s) &= \left[\int_{\overline{N}} |\varphi(\overline{m}a_s)|^{p'} d\overline{m}\right]^{\frac{1}{p'}}. \end{aligned} \tag{9.16}$$

Applying Hölder's inequality in (9.15) and using (9.16), we get

$$\left|\langle \mathcal{I}_\sigma^- f, \varphi \rangle\right| \leq C_{p,n} \left(\sup_{x\in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)}\right) \int_{r=0}^{\infty} \int_{\mathbb{R}} \int_{\overline{N}} F(s)\Phi(s-r) \cdot \eta(\overline{n}a_{-r})\,\frac{e^{-\frac{2}{p}([\overline{n}a_{-r}]^+)}}{(1+[\overline{n}a_{-r}]^+)^2}e^{2(s-r)}d\overline{n}\,ds\,dr. \tag{9.17}$$

Since the Abel transform of a K-biinvariant function is even, from (2.15) it follows that

$$\int\limits_{\overline{N}} \eta(\overline{n}a_{-r})\,\frac{e^{-\frac{2}{p}([\overline{n}a_{-r}]^+)}}{(1+[\overline{n}a_{-r}]^+)^2}\,d\overline{n} = e^{2r}\int\limits_{\overline{N}} \eta(\overline{n}a_r)\,\frac{e^{-\frac{2}{p}([\overline{n}a_r]^+)}}{(1+[\overline{n}a_r]^+)^2}\,d\overline{n}, \quad \text{for } r \geq 0. \tag{9.18}$$

Putting (9.18) the formula above in (9.17), we obtain

$$\left|\langle \mathcal{I}_\sigma^- f, \varphi \rangle\right| \leq C_{p,n} \left(\sup_{x\in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)}\right) \int_{r=0}^{\infty} \int_{\mathbb{R}} \int_{\overline{N}} F(s)\Phi(s-r) \cdot \eta(\overline{n}a_r)\,\frac{e^{-\frac{2}{p}([\overline{n}a_r]^+)}}{(1+[\overline{n}a_r]^+)^2}e^{2s}d\overline{n}\,ds\,dr.$$

Substituting the explicit expression of $[\overline{n}a_r]^+$ (for $r \geq 0$) from Lemma 2.2 in the inequality above yields

$$\begin{aligned}\left|\left\langle \mathcal{I}_\sigma^- f, \varphi \right\rangle\right| &\leq C_{p,n} \left(\sup_{x \in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)} \right) \int_{r=0}^{\infty} \int_{\mathbb{R}} F(s) \Phi(s-r) \\ &\qquad \cdot \int_{\overline{N}} \eta(\overline{n}a_r)\, e^{-\frac{2}{p}r} \frac{e^{-\frac{2}{p}H(\overline{n})}}{\left(1+H(\overline{n})+r\right)^2} e^{2s} d\overline{n}\, ds\, dr \\ &\leq C_{p,n} \left(\sup_{x \in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)} \right) \left(\int_{r=0}^{\infty} \left(\int_{\mathbb{R}} F(s) e^{\frac{2}{p}s} \Phi(s-r) e^{\frac{2}{p'}(s-r)} ds \right) \right. \\ &\qquad \left. \cdot \int_{\overline{N}} e^{-\frac{2}{p}H(\overline{n})}\, d\overline{n} \frac{e^{(\frac{2}{p'}-\frac{2}{p})r}}{(1+r)^2} dr \right).\end{aligned}$$

Here we are using this fact $H(\overline{n}) \geq 0$ for all $\overline{n} \geq 0$ (see (2.11)). Finally using (2.12) with $1 \leq p < 2$, and applying Hölder's inequality, we conclude that

$$\left|\left\langle \mathcal{I}_\sigma^- f, \varphi \right\rangle\right| \leq C_{p,n} \left(\sup_{x \in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)} \right) \|f\|_{L^p(\overline{N}A)}\, \|\varphi\|_{L^{p'}(\overline{N}A)}.$$

We next turn to prove the estimate (9.13) for $p > 2$, which will complete the proof of this lemma. After taking modulus on both sides of (9.14) and plugging the estimate, (9.5) of $\mathcal{K}^{glo}$ for $2 < p < \infty$, we can write

$$\begin{aligned}\left|\left\langle \mathcal{I}_\sigma^- f, \varphi \right\rangle\right| &\leq C_{p,n} \left(\sup_{x \in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)} \right) \int_{r=0}^{\infty} \int_{\mathbb{R}} \int_{\overline{N}} \int_{\overline{N}} |f(\overline{m}a_s)| |\varphi(\overline{m}\delta_s(\overline{n})a_{s-r})| \\ &\qquad \cdot \eta(\overline{n}a_{-r}) \frac{e^{-\frac{2}{p'}(\,[\overline{n}a_{-r}]^+)}}{\left(1+[\overline{n}a_{-r}]^+\right)^2} e^{2(s-r)} d\overline{m}\, d\overline{n}\, ds\, dr.\end{aligned}$$

Then again, after an application of Hölder's inequality, we use (9.18) to get

$$\begin{aligned}\left|\left\langle \mathcal{I}_\sigma^- f, \varphi \right\rangle\right| &\leq C_{p,n} \left(\sup_{x \in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)} \right) \int_{r=0}^{\infty} \int_{\mathbb{R}} F(s) \Phi(s-r) \\ &\qquad \cdot \int_{\overline{N}} \eta(\overline{n}a_r)\, e^{-\frac{2}{p'}r} \frac{e^{-\frac{2}{p'}H(\overline{n})}}{\left(1+H(\overline{n})+r\right)^2} e^{2s} d\overline{n}\, ds\, dr\end{aligned}$$

Since $2 < p < \infty$, so we have $p' < 2$. Thus we can use (2.12) and write

$$\begin{aligned}\left|\left\langle \mathcal{I}_\sigma^- f, \varphi \right\rangle\right| &\leq C_{p,n} \left(\sup_{x \in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)} \right) \left(\int_{r=0}^{\infty} \left(\int_{\mathbb{R}} F(s) e^{\frac{2}{p}s} \Phi(s-r) e^{\frac{2}{p'}(s-r)} ds \right) \right. \\ &\qquad \left. \cdot \int_{\overline{N}} e^{-\frac{2}{p'}H(\overline{n})}\, d\overline{n} \frac{dr}{(1+r)^2} \right) \\ &\leq C_{p,n} \left(\sup_{x \in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)} \right) \|f\|_{L^p(\overline{N}A)}\, \|\varphi\|_{L^{p'}(\overline{N}A)},\end{aligned}$$

completing the proof. □

Lemma 9.5. *Let $p \in [1,2) \cup (2,\infty)$. Suppose that σ satisfies the hypothesis of Theorem 9.1 and $\mathcal{I}_\sigma^+$ be as in (9.10). Then $\mathcal{I}_\sigma^+$ is bounded from $L^p(\overline{N}A)$ to itself. Moreover, there exists a constant $C_{p,n} > 0$ such that*

$$\|\mathcal{I}_\sigma^+ f\|_{L^p(\overline{N}A)} \leq C_{p,n} \left(\sup_{x\in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)} \right) \|f\|_{L^p(\overline{N}A)} \tag{9.19}$$

for all $f \in L^p(\overline{N}A)$.

Proof. Analysis similar to that in the proof of the previous lemma shows that it is enough to prove $\langle \mathcal{I}_\sigma^+ f, \varphi \rangle$ satisfies the estimate in (9.13) for $1 \leq p < 2$, where we recall the expression of $\langle \mathcal{I}_\sigma^+ f, \varphi \rangle$ from (9.11)

$$\left\langle \mathcal{I}_\sigma^+ f, \varphi \right\rangle = \int_{\overline{N}} \int_{\mathbb{R}} \int_{\overline{N}} \int_{\mathbb{R}} f(\overline{m} a_s) \overline{\varphi(\overline{n} a_t)} \mathcal{K}^{\mathrm{glo}}(\overline{n} a_t, \delta_{-s}(\overline{m}^{-1}\overline{n}) a_{t-s}) \chi^+(t-s) e^{2(s+t)}\, ds\, d\overline{m}\, dt\, d\overline{n},$$

for all smooth compactly supported functions f, φ on $\overline{N}A$. Applying the same change of variables as in the previous lemma, we obtain

$$\left\langle \mathcal{I}_\sigma^+ f, \varphi \right\rangle = \int_{\overline{N}} \int_{\mathbb{R}} \int_{\overline{N}} \int_{\mathbb{R}} f(\overline{m} a_s) \overline{\varphi(\overline{m}\delta_s(\overline{n}) a_t)} \mathcal{K}^{\mathrm{glo}}(\overline{m}\delta_s(\overline{n}) a_t, \overline{n} a_{t-s}) \chi^-(t-s) e^{2t} d\overline{n}\, ds\, dt\, d\overline{m}.$$

Next, by using Fubini's theorem and the change of variable $t \mapsto r + s$ (for fixed s), it follows that

$$\left\langle \mathcal{I}_\sigma^+ f, \varphi \right\rangle = \int_{\overline{N}} \int_{\mathbb{R}} \int_{\overline{N}} \int_{r=0}^{\infty} f(\overline{m} a_s) \overline{\varphi(\overline{m}\delta_s(\overline{n}) a_{r+s})} \mathcal{K}^{\mathrm{glo}}(\overline{m}\delta_s(\overline{n}) a_r, \overline{n} a_r) e^{2(r+s)} dr\, d\overline{n}\, ds\, d\overline{m}.$$

Taking modulus on both sides of the above expression and plugging the estimate (9.4) of $\mathcal{K}^{\mathrm{glo}}$, we get

$$\begin{aligned} \left|\left\langle \mathcal{I}_\sigma^+ f, \varphi \right\rangle\right| \leq C_{p,n} \left(\sup_{x\in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)} \right) \int_{r=0}^{\infty} \int_{\mathbb{R}} \int_{\overline{N}} \int_{\overline{N}} |f(\overline{m} a_s)| |\varphi(\overline{m}\delta_s(\overline{n}) a_{r+s})| \\ \cdot\, \eta(\overline{n} a_r)\, \frac{e^{-\frac{2}{p}([\overline{n} a_r]^+)}}{(1 + [\overline{n} a_r]^+)^2} e^{2(r+s)} d\overline{m}\, d\overline{n}\, ds\, dr. \end{aligned} \tag{9.20}$$

Plugging the explicit expression of $[\overline{n} a_r]^+$ from Lemma 2.2 in the inequality above gives us

$$\begin{aligned} \left|\left\langle \mathcal{I}_\sigma^+ f, \varphi \right\rangle\right| &\leq C_{p,n} \left(\sup_{x\in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)} \right) \int_{r=0}^{\infty} \int_{\mathbb{R}} \int_{\overline{N}} \int_{\overline{N}} |f(\overline{m} a_s)| |\varphi(\overline{m}\delta_s(\overline{n}) a_{r+s})| \\ &\qquad \cdot\, \eta(\overline{n} a_r)\, e^{-\frac{2}{p} r} \frac{e^{-\frac{2}{p} H(\overline{n})}}{(1 + H(\overline{n}) + r)^2} e^{2(r+s)} d\overline{m}\, d\overline{n}\, ds\, dr \\ &\leq C_{p,n} \left(\sup_{x\in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)} \right) \int_{r=0}^{\infty} \int_{\mathbb{R}} F(s) e^{\frac{2}{p} s} \Phi(s+r) e^{\frac{2}{p'}(s+r)} ds \\ &\qquad \cdot \int_{\overline{N}} \eta(\overline{n} a_r)\, \frac{e^{-\frac{2}{p} H(\overline{n})}}{(1+r)^2}\, d\overline{n}\, dr, \end{aligned}$$

where in the last inequality we used Hölder's inequality and the fact $H(\overline{n}) \geq 0$. Finally, using (2.12) and another application of Hölder's inequality gives us

$$\left|\left\langle \mathcal{I}_\sigma^+ f, \varphi \right\rangle\right| \leq C_{p,n} \left(\sup_{x\in G} \|\sigma(x,\cdot)\|_{\mathcal{MH}(S_p,2)} \right) \|f\|_{L^p(\overline{N}A)} \|\varphi\|_{L^{p'}(\overline{N}A)},$$

which concludes our lemma. □

10. Rough analysis of Ψ_σ

This section focuses on extending the boundedness result for pseudo-differential operators Ψ_σ in Theorem 3.3 to accommodate rough symbols σ, which have no regularity condition in the space variable. Specifically, our goal in this section is to prove Theorem 3.5.

To achieve this, we first recall that the boundedness of $\Psi_\sigma^{\mathrm{glo}}$ and $\Psi_\sigma^{\mathrm{dis}}$ does not require any regularity condition on the space variable of the symbol σ, so their proofs follow a similar approach. In proving the boundedness of the local part $\Psi_\sigma^{\mathrm{loc}}$, we crucially utilize the relation between it and Euclidean pseudo-differential operators, employing the result of Kenig and Staubach (Theorem 1.3).

To outline the proof, we start by decomposing the pseudo-differential operator Ψ_σ into discrete, local, and global parts, using the same argument as in Section 6. The required boundedness of the operators $\Psi_\sigma^{\mathrm{dis}}$ and $\Psi_\sigma^{\mathrm{glo}}$ follows using similar methods as observed in Section 7 and 9, respectively.

Next, we proceed to prove the corresponding $L^p(G)_{n,n}$ operator norm estimate for the local part $\Psi_\sigma^{\mathrm{loc}}$. By utilizing the generalized transference principle of Coifmann-Weiss and employing the exact same analysis as in the proof of Theorem 8.1, we observe that it is sufficient to prove the hypothesis (8.5) in Corollary 8.3.

Let us recall from (8.8) that we can write

$$\mathcal{K}^{\mathrm{CW}}(\mathcal{R}_s x, \mathcal{R}_{-t}\mathcal{R}_s x, t) = \mathcal{K}_0(xa_s, t) + \mathcal{K}_{\mathrm{err}}(xa_s, t), \quad x \in G,\, t \geq 0,$$

such that

(i) There is some constant $C > 0$ (depending on n) and $\kappa \in L^1(\mathbb{R})$, such that

$$|\mathcal{K}_{\mathrm{err}}(x,t)| \leq C\,\kappa(t), \text{ for all } x \in G,$$

and

(ii) $\mathcal{K}_0(xa_s, t) = C\eta^\circ(a_t)\,\Delta(t)\,\left(\frac{t}{\Delta(t)}\right)^{1/2} \int_0^\infty \Phi(\lambda)\sigma(xa_s,\lambda)\mathcal{J}_0(\lambda t)|c(\lambda)|^{-2}d\lambda.$

Next for each $x \in G$, we define

$$a_x(s,\xi) := \int_{-\infty}^{\infty} \mathcal{K}_0(xa_s, t)e^{-2\pi i t\xi}dt, \quad s, \xi \in \mathbb{R}.$$

By considering the hypothesis $\sigma(x,\lambda) \in \mathcal{S}^m_{1,\infty}(S_p)$, we can deduce from the calculations in the proof of (8.10) that the family of symbols $\{a_x(s,\xi) : x \in G\}$ satisfies the following estimate:

$$\|\partial_\xi^\alpha a_x(s,\xi)\|_{L^\infty_s} \leq C_{\alpha,n}(1+|\xi|)^{m-\alpha}, \tag{10.1}$$

where the constants $C_{\alpha,n}$ are independent of x. In other words, $a_x(s,\xi) \in \mathcal{S}^m_{1,\infty}$ for all $x \in G$. Consequently, Theorem 1.3 implies that the hypothesis (8.5) in Corollary 8.3 is satisfied. This in turn, establishes the $L^p(G)_{n,n}$-boundedness for $1 \leq p < \infty$ of the operator $\Psi_\sigma^{\mathrm{loc}}$. This completes the proof of Theorem 3.5. □

Final remarks

In conclusion, we would like to offer some observations and draw attention to few open questions that, from our perspective, warrant further investigation.

(1) By employing the Fourier inversion formula [3, (10.4), Theorem 10.4], we can define multiplier operators, or more generally, pseudo-differential operators, for functions of (m,n) type on the group G and investigate their boundedness properties. While we have successfully established the boundedness of both the discrete and global components of the pseudo-differential operator using a similar analytical approach as in this article, addressing the local part poses a challenge. By performing a calculation akin to that in Lemma 5.3, we can express $\phi_{\tau,\lambda}^{m,n}$ in terms of the Bessel function $\mathcal{J}_{\frac{m-n}{2}}$, unlike the (n,n) type case where it was $\mathcal{J}_0$. Consequently, the argument presented in Section 8 may not be applicable unless $m=n$, as $\mathcal{J}_{\frac{m-n}{2}}(t)$ introduces a singularity near $t=0$. Therefore, it is necessary to develop an alternative strategy to handle the local part of the pseudo-differential operator.

(2) Recently, Wróbel [61] presented a multiplier theorem for rank one symmetric spaces, which improves upon the results of both [55] and [28]. It would be interesting to investigate whether a similar result can be obtained in our context.

(3) Our current approach has not yet established the L^2-boundedness of the pseudo-differential operator Ψ_σ. Specifically, the method we use to prove the L^p-boundedness of the global part of Ψ_σ may not be effective for $p=2$ when assuming that the symbol σ satisfies only the smoothness condition (without holomorphicity) on the real line. We view this as an opportunity to explore alternative strategies. In the near future, we plan to investigate the L^2-boundedness of pseudo-differential operators under the assumption that the symbol satisfies only the smoothness condition (without holomorphicity) on the real line, both within our current framework and in the context of rank-one symmetric spaces of noncompact type.

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Tapendu Rana
Department of Mathematics: Analysis, Logic and Discrete Mathematics,
Ghent University,
Krijgslaan 281, Building S8, B 9000 Ghent, Belgium.
Email address: tapendurana@gmail.com, tapendu.rana@ugent.be

Michael Ruzhansky

Department of Mathematics: Analysis, Logic and Discrete Mathematics,
Ghent University,
Krijgslaan 281, Building S8, B 9000 Ghent, Belgium.
and
School of Mathematical Sciences
Queen Mary University of London
United Kingdom
Email address: michael.ruzhansky@ugent.be