BOUNDARY HARDY INEQUALITY ON FUNCTIONS OF BOUNDED VARIATION

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Abstract. Classical boundary Hardy inequality states that if $1 < p < \infty$ and Ω is a bounded Lipschitz domain, then for all $u \in C_c^\infty(\Omega)$,

$$\int_\Omega \frac{|u(x)|^p}{\delta_\Omega^p(x)} dx \leq C \int_\Omega |\nabla u(x)|^p dx,$$

where $\delta_\Omega(x)$ is the distance function from the boundary of Ω. In this article, we address the open question on the case $p = 1$ by establishing appropriate boundary Hardy inequalities in the space of functions of bounded variation. We first establish appropriate Hardy inequalities on fractional Sobolev spaces $W^{s,1}(\Omega)$ and then Dávila's result on limiting behaviour of fractional Sobolev spaces as $s \to 1-$ plays an important role in the proof. Moreover, we also derive an infinite series Hardy inequality for the case $p = 1$.

1. Introduction

The classical Hardy inequality for the local case is given by

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^p} dx \leq \left| \frac{p}{p-d} \right|^p \int_{\mathbb{R}^d} |\nabla u(x)|^p dx,$$

for all $u \in C_c^\infty(\mathbb{R}^d)$ if $1 < p < d$ and for all $u \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ if $p > d$. Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with $0 \in \Omega$, we have

$$\int_\Omega \frac{|u(x)|^p}{|x|^p} dx \leq \left(\frac{p}{d-p} \right)^p \int_\Omega |\nabla u(x)|^p dx, \tag{1.1}$$

for all $u \in C_c^\infty(\Omega)$ if $1 < p < d$ and the constant $\left(\frac{p}{d-p}\right)^p$ is sharp but never achieved. The inequality analogous to (1.1) for the case $p = d = 2$ was explored by Leray in [21], and it has been extended to $p = d \geq 2$ by [1, 7, 8]. It can be formulated as follows: Let $\Omega \subset \mathbb{R}^d$, where $d \geq 2$, be a bounded domain. Then, there exists a constant $C = C(d, \Omega, R)$ such that for any $u \in W_0^{1,d}(\Omega)$,

$$\int_\Omega |\nabla u(x)|^d dx \geq C \int_\Omega \frac{|u(x)|^d}{|x|^d} \left(\ln \frac{R}{|x|} \right)^{-d} dx,$$

where $R \geq \sup_\Omega (|x| e^{\frac{2}{p}})$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and $1 < p < \infty$. The boundary Hardy inequality for the local case (cf. [22]) states that there exists a constant $C = C(d, p, \Omega) > 0$ such that

$$\int_\Omega \frac{|u(x)|^p}{\delta_\Omega^p(x)} dx \leq C \int_\Omega |\nabla u(x)|^p dx, \quad \text{for all } u \in C_c^\infty(\Omega), \tag{1.2}$$

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where δ_Ω is the distance function from the boundary of Ω, defined by

$$\delta_\Omega(x) := \min_{y\in\partial\Omega}|x-y|.$$

Several generalizations and extensions of the above inequality have been made over the last three and a half decades. We refer to some of the works in this direction [7, 8, 25]. To the best of our knowledge, none of the works in the literature address the case $p=1$ in (1.2). The aim of this article is to establish appropriate inequalities for the case $p=1$. In this article, our objective is to derive a boundary Hardy-type inequality within the space $BV(\Omega)$, where Ω is a bounded Lipschitz domain. Our approach initially involves establishing a fractional boundary Hardy inequality for the case $p=1$ and $s\geq \frac{1}{2}$. Later, we utilize the well-known result of Dávila, as presented in [14] (also see [10] for closely related work), to obtain the Hardy inequality on functions of bounded variation. The exact version of their result required for our purposes will be recalled in the next section.

Define the functions:

$$\mathcal{L}_1(t) := \frac{1}{1-\ln t}, \quad \forall\, t\in(0,1), \tag{1.3}$$

and recursively

$$\mathcal{L}_m(t) := \mathcal{L}_1(\mathcal{L}_{m-1}(t)), \quad \forall\, m\geq 2. \tag{1.4}$$

Let us define the space of functions of bounded variation (also see [16, 18]):

Definition 1. Let $\Omega\subset\mathbb{R}^d$ be an open set. A function $u\in L^1(\Omega)$ has bounded variation in Ω if

$$[u]_{BV(\Omega)} := \sup\left\{\int_\Omega u(x)\mathrm{div}(\phi(x))dx : \phi\in C_c^1(\Omega;\mathbb{R}^d),\ |\phi(x)|\leq 1 \text{ on } \Omega\right\} < \infty.$$

We denote $BV(\Omega)$ the space of functions of bounded variation in Ω with the norm $\|.\|_{BV(\Omega)}$ on $BV(\Omega)$ as $\|u\|_{BV(\Omega)} := [u]_{BV(\Omega)} + \|u\|_{L^1(\Omega)}$. Throughout this article $(u)_\Omega$ will denote the average of u over Ω which is given by

$$(u)_\Omega := \frac{1}{|\Omega|}\int_\Omega u(y)dy.$$

The following theorem is the main result of this article.

Theorem 1. *Let $\Omega\subset\mathbb{R}^d$ be a bounded Lipschitz domain such that $\delta_\Omega(x) < R$ for all $x\in\Omega$, for some $R>0$ and $m\geq 2$ be a positive integer. Then there exists a constant $C=C(d,\Omega)>0$ such that for all $u\in BV(\Omega)$,*

$$\int_\Omega \frac{|u(x)-(u)_\Omega|}{\delta_\Omega(x)}\mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right)\mathcal{L}_m^2\left(\frac{\delta_\Omega(x)}{R}\right)dx \leq C2^m[u]_{BV(\Omega)}. \tag{1.5}$$

Furthermore, for any $0<\alpha<\frac{1}{2}$, there exists a constant $C=C(d,\Omega)$ such that for all $u\in BV(\Omega)$,

$$\sum_{m=2}^{\infty}\alpha^m\int_\Omega \frac{|u(x)-(u)_\Omega|}{\delta_\Omega(x)}\mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right)\mathcal{L}_m^2\left(\frac{\delta_\Omega(x)}{R}\right)dx \leq C\left(\frac{4\alpha^2}{1-2\alpha}\right)[u]_{BV(\Omega)}. \tag{1.6}$$

The above inequality fails when $\alpha\geq 1$.

The following identity (see [7, Section 2, equation (2.1)])

$$\frac{d}{dt}\mathcal{L}_m(t) = \frac{1}{t}\mathcal{L}_1(t)\cdots\mathcal{L}_{m-1}(t)\mathcal{L}_m^2(t), \quad \text{where } m \geq 2, \tag{1.7}$$

plays an important role in establishing the previous theorem, among other key ingredients. The constant appearing in the previous result may not be sharp. Since constant functions belong to $BV(\Omega)$, this justifies the presence of $(u)_\Omega$ on the left-hand side of (1.5). Moreover, $W^{1,1}(\Omega) \subset BV(\Omega)$ (see [16, Chapter 5]), and

$$\int_\Omega |\nabla u(x)|dx = [u]_{BV(\Omega)}, \quad \forall\, u \in W^{1,1}(\Omega).$$

Therefore, Theorem 1 holds true for any $u \in W^{1,1}(\Omega)$.

Let $m \geq 1$, $\beta > 1$ and R be as in the previous theorem, then there exists a constant $C = C(\beta) > 0$ (see (6.1), Appendix 6) such that

$$\mathcal{L}_m^\beta\left(\frac{\delta_\Omega(x)}{R}\right) \leq C\mathcal{L}_m\left(\frac{\delta_\Omega(x)}{R}\right)\mathcal{L}_{m+1}^2\left(\frac{\delta_\Omega(x)}{R}\right), \quad \forall\, x \in \Omega. \tag{1.8}$$

Therefore, by applying Theorem 1 with $m+1 \geq 2$ and utilizing the above inequality, we obtain the following immediate corollary, which enhances the previous theorem:

Corollary 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain such that $\delta_\Omega(x) < R$ for all $x \in \Omega$, for some $R > 0$, $\beta > 1$ and $m \geq 1$ be a positive integer. Then there exists a constant $C = C(d, \Omega, \beta) > 0$ such that for all $u \in BV(\Omega)$,*

$$\int_\Omega \frac{|u(x) - (u)_\Omega|}{\delta_\Omega(x)}\mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right)\mathcal{L}_m^\beta\left(\frac{\delta_\Omega(x)}{R}\right)dx \leq C2^m[u]_{BV(\Omega)}, \tag{1.9}$$

with the convention that for $m = 1$, $\mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right)\mathcal{L}_m^\beta\left(\frac{\delta_\Omega(x)}{R}\right) = \mathcal{L}_1^\beta\left(\frac{\delta_\Omega(x)}{R}\right)$. Furthermore, for any $0 < \alpha < \frac{1}{2}$, there exists a constant $C = C(d, \Omega, \beta)$ such that for all $u \in BV(\Omega)$,

$$\sum_{m=2}^{\infty} \alpha^m \int_\Omega \frac{|u(x) - (u)_\Omega|}{\delta_\Omega(x)}\mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right)\mathcal{L}_m^\beta\left(\frac{\delta_\Omega(x)}{R}\right)dx \leq C\left(\frac{4\alpha^2}{1-2\alpha}\right)[u]_{BV(\Omega)}. \tag{1.10}$$

The above inequality fails whenever $\alpha \geq 1$ or $0 < \beta \leq 1$.

The above corollary does not hold true when $\beta = 1$, as illustrated by considering a non-constant function $u \in BV(\Omega)$ such that u takes a non-zero constant value near the boundary of Ω, resulting in the left-hand side of the inequality in the corollary becoming infinite. This failure highlights the optimality of the aforementioned corollary with respect to the choice of β. Furthermore, it can be easily verified that the constant in the corollary, denoted as $C = C(d, \Omega, \beta)$, approaches ∞ as $\beta \to 1$. This is because $C = C(\beta)$, defined in (1.8), tends to ∞ as $\beta \to 1$ (see (6.2) with $\theta = \beta - 1$, Appendix 6).

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\beta > 1$, we define a space of functions

$$X = \left\{u \in L^1(\Omega) : \int_\Omega \frac{|u(x)|}{\delta_\Omega(x)}\mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right)\mathcal{L}_m^\beta\left(\frac{\delta_\Omega(x)}{R}\right)dx < \infty\right\}.$$

By applying Corollary 1.1, we can deduce that the operator $T : BV(\Omega) \to X$, defined by

$$T(u) = u \tag{1.11}$$

is a bounded linear operator. In Subsection 5.1, we prove that the operator T is compact. To achieve this result, we utilized the compactness of the identity operator $I : BV(\Omega) \to L^1(\Omega)$ (see [16, Theorem 5.5]). This compactness result for BV functions enables us to present the following remark:

Remark 1. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then, the bounded linear operator T, defined in (1.11), is compact for every $\beta > 1$.

Let $\rho : (0,1) \to \mathbb{R}$ be a measurable function satisfying $\rho > 0$, $\rho(t) \to 0$ as $t \to 0$, $\beta > 1$ and for some constant $C > 0$,

$$\mathcal{L}_m^{1+\rho(t)}(t) \le C\mathcal{L}_m(t)\,\mathcal{L}_{m+1}^{\beta}(t), \quad \forall\, t \in (0,1). \tag{1.12}$$

Using the definition of $\mathcal{L}_m$, for any $m \ge 1$, and define $\mathcal{L}_0(t) := t$, we obtain

$$(1 - \ln(\mathcal{L}_m(t)))^{\beta} \le C\,(1 - \ln(\mathcal{L}_{m-1}(t)))^{\rho(t)}.$$

Then, taking ln both sides, we obtain

$$\rho^*(t) := \frac{\beta \ln(1 - \ln(\mathcal{L}_m(t)))}{\ln(1 - \ln(\mathcal{L}_{m-1}(t)))} \le \rho(t) + \frac{\ln C}{\ln(1 - \ln(\mathcal{L}_{m-1}(t)))}.$$

Since $\frac{\ln C}{\ln(1-\ln(\mathcal{L}_{m-1}(t)))} \to 0$ as $t \to 0$, it is not necessary to consider this term, or we can assume $C = 1$. We also observe that $\rho^*(t) \to 0$ as $t \to 0$, and

$$\mathcal{L}_m^{1+\rho^*(t)}(t) = \mathcal{L}_m(t)\,\mathcal{L}_{m+1}^{\beta}(t), \quad \forall\, t \in (0,1).$$

This implies that the function ρ^* is optimal in the inequality (1.12) with the choice of ρ. Therefore, we again present the following corollary which is a consequence and improvement of the previous corollary:

Corollary 1.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain such that $\delta_\Omega(x) < R$ for all $x \in \Omega$, for some $R > 0$, $\beta > 1$ and $m \ge 1$ be a positive integer. Then there exists a constant $C = C(d, \Omega, \beta) > 0$ such that for all $u \in BV(\Omega)$,*

$$\int_\Omega \frac{|u(x) - (u)_\Omega|}{\delta_\Omega(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^{1+\rho^*\left(\frac{\delta_\Omega(x)}{R}\right)}\left(\frac{\delta_\Omega(x)}{R}\right) dx \le C 2^m [u]_{BV(\Omega)}. \tag{1.13}$$

Furthermore, the above inequality fails when $0 < \beta \le 1$ in the definition of ρ^*.

We can illustrate the failure of the above corollary for $0 < \beta \le 1$ in the definition of ρ^* by selecting a non-zero function $u \in BV(\Omega)$ that remains constant near the boundary $\partial\Omega$. This choice makes the left-hand side of the inequality in the above corollary infinite, while the right-hand side remains finite

We also establish a similar type of Hardy inequality in fractional Sobolev space when $p = 1$. The next theorem can be treated as an independent result in its own right and serves as a crucial component in establishing Theorem 1. In particular, we prove the following theorem: Let $\Omega \subset \mathbb{R}^d$ be an open set, $s \in (0,1)$, and $p \ge 1$. We define the Gagliardo fractional seminorm as

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} dxdy \right)^{\frac{1}{p}}. \tag{1.14}$$

Theorem 2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain such that $\delta_\Omega(x) < R$ for all $x \in \Omega$, for some $R > 0$, $\frac{1}{2} \le s < 1$ and $m \ge 2$ be a positive integer. Then there exists a constant $C = C(d, \Omega) > 0$ such that*

$$\begin{aligned} &\int_\Omega \frac{|u(x)|}{\delta_\Omega^s(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^2\left(\frac{\delta_\Omega(x)}{R}\right) dx \\ &\qquad\qquad \le C 2^m (1-s)[u]_{W^{s,1}(\Omega)} + C 2^m \|u\|_{L^1(\Omega)}, \quad \forall\, u \in W^{s,1}(\Omega). \end{aligned} \tag{1.15}$$

Furthermore, for any $0 < \alpha < \frac{1}{2}$, there exists a constant $C = C(d, \Omega)$ such that

$$\begin{aligned} &\sum_{m=2}^{\infty} \alpha^m \int_\Omega \frac{|u(x)|}{\delta_\Omega^s(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^2\left(\frac{\delta_\Omega(x)}{R}\right) dx \\ &\qquad\qquad \le C\left(\frac{4\alpha^2}{1-2\alpha}\right)\left\{(1-s)[u]_{W^{s,1}(\Omega)} + \|u\|_{L^1(\Omega)}\right\}, \quad \forall\, u \in W^{s,1}(\Omega). \end{aligned} \tag{1.16}$$

Also, the above inequality fails when $\alpha \ge 1$.

The inequalities (1.8) and (1.12) can also be applied in Theorem 2 to obtain similar types of corollaries as those obtained for Theorem 1 in Corollary 1.1 and Corollary 1.2.

The result that comes closer to our present work is that of Barbatis, Filippas, and Tertikas in [7], where they obtained a series expansion for L^p Hardy inequalities in $\mathbb{R}^d$, $p > 1$, involving the distance function from the boundary of the domain $\Omega \subset \mathbb{R}^d$. For more literature on Hardy-type inequalities, we refer to [1, 2, 3, 4, 5, 6, 7, 12, 13, 17, 19, 24] and to the works mentioned therein.

The article is organized in the following way: In Section 2, we present preliminary lemmas and notation that will be utilized to prove Theorem 1 and Theorem 2. In section 3, we prove Theorem 1 and Theorem 2 in dimension one. Section 4 contains the proof of the main theorem, Theorem 1 in dimension $d \ge 2$ which follows from Theorem 2 and utilizes Dávila's result (see Lemma 2.2). In Section 5, a counterexample is provided to illustrate that (1.6) in Theorem 1 and (1.16) in Theorem 2 fail for $\alpha \ge 1$.

2. Notation and Preliminaries

In this section, we introduce the notation and preliminary lemmas that will be used in proving Theorem 1 and Theorem 2. All the lemmas proved in this section are essentially known results in the literature. Throughout this article, we shall use the following notation:

- $\mathbb{R}^d$ will denote the Euclidean space of dimension d.
- the parameter s will always be understood to be in $(0, 1)$.
- we denote $|\Omega|$ the Lebesgue measure of $\Omega \subset \mathbb{R}^d$.
- for any $f,\ g : \Omega(\subset \mathbb{R}^d) \to \mathbb{R}$, we denote $f \sim g$ if there exist $C_1, C_2 > 0$ such that $C_1 g(x) \le f(x) \le C_2 g(x)$ for all $x \in \Omega$.
- $C > 0$ will denote a generic constant that may change from line to line.

Let $\Omega \subset \mathbb{R}^d$ be an open set and $s \in (0, 1)$. For any $p \in [1, \infty)$, define the fractional Sobolev space

$$W^{s,p}(\Omega) := \left\{u \in L^p(\Omega) : [u]^p_{W^{s,p}(\Omega)} < \infty\right\},$$

where $[u]_{W^{s,p}(\Omega)}$ is defined in (1.14). This space is equipped with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left([u]^p_{W^{s,p}(\Omega)} + \|u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}.$$

Let $W_0^{s,1}(\Omega)$ denotes the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|.\|_{W^{s,1}(\Omega)}$.

Inclusion of $BV(\Omega)$ in $W^{s,1}(\Omega)$: Let Ω be an open Lipschitz domain and $s \in (0,1)$. It is a well-known fact that $BV(\Omega) \subset W^{s,1}(\Omega)$. To understand this, take $u \in BV(\Omega)$. From [16, Theorem 5.3], we know there is a sequence of functions $\{u_n\}_{n=1}^\infty \subset C^\infty(\Omega) \cap BV(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$, and $[u_n]_{BV(\Omega)} \to [u]_{BV(\Omega)}$ as $n \to \infty$. Also, for all n, $[u_n]_{BV(\Omega)} = \|\nabla u_n\|_{L^1(\Omega)}$. Using these facts, along with [15, Proposition 2.2] and Fatou's lemma, we can write

$$\|u\|_{W^{s,1}(\Omega)} \le C\|u\|_{BV(\Omega)}, \tag{2.1}$$

where $C = C(d,s,\Omega) > 0$. This shows that $BV(\Omega) \subset W^{s,1}(\Omega)$.

A bounded domain with Lipschitz boundary: Let Ω be a bounded Lipschitz domain. Then, for each $x \in \partial\Omega$, there exist $r'_x > 0$, an isometry T_x of $\mathbb{R}^d$ and a Lipschitz function $\phi_x : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$T_x(\Omega) \cap B_{r'_x}(T_x(x)) = \{\xi : \xi_d > \phi_x(\xi')\} \cap B_{r'_x}(T_x(x)).$$

The next lemma proves the fractional Poincaré inequality with a specific constant (see [11, page no. 80 ("fact")]) for any cube of side length $\lambda > 0$. A more general version of this lemma is also available in [23, Corollary 1]. This lemma is useful in proving Lemma 3.3 and Lemma 4.1.

Lemma 2.1. *Let $d \ge 1$, $\frac{1}{2} \le s < 1$ and Ω_λ be any cube of side length $\lambda > 0$ in $\mathbb{R}^d$. Then, there exists a constant $C_{d,Poin} = C_{d,Poin}(d) > 0$ such that*

$$⨍_{\Omega_\lambda} |u(x) - (u)_{\Omega_\lambda}|dx \le C_{d,Poin}\lambda^{s-d}(1-s)\int_{\Omega_\lambda}\int_{\Omega_\lambda} \frac{|u(x)-u(y)|}{|x-y|^{d+s}}dxdy, \quad \forall\ u \in W^{s,1}(\Omega_\lambda). \tag{2.2}$$

Proof. Let Ω be any unit cube. Then, from [11, page no. 80 ("fact")], we have

$$⨍_{\Omega} |u(x) - (u)_{\Omega}|dx \le C_{d,Poin}(1-s)\int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|}{|x-y|^{d+s}}dxdy,$$

where $C_{d,Poin}$ is the best fractional Poincaré constant. Let us apply the above inequality to $u(\lambda x)$ instead of $u(x)$. This gives

$$⨍_{\Omega} \left|u(\lambda x) - ⨍_{\Omega} u(\lambda x)dx\right|dx \le C_{d,Poin}(1-s)\int_{\Omega}\int_{\Omega} \frac{|u(\lambda x)-u(\lambda y)|}{|x-y|^{d+s}}dxdy.$$

Using the fact

$$⨍_{\Omega} u(\lambda x)dx = ⨍_{\Omega_\lambda} u(x)dx,$$

we have

$$⨍_{\Omega} |u(\lambda x) - (u)_{\Omega_\lambda}|\ dx \le C_{d,Poin}(1-s)\int_{\Omega}\int_{\Omega} \frac{|u(\lambda x)-u(\lambda y)|}{|x-y|^{d+s}}dxdy.$$

By changing the variable $X = \lambda x$ and $Y = \lambda y$, we obtain

$$⨍_{\Omega_\lambda} |u(x) - (u)_{\Omega_\lambda}|dx \le C_{d,Poin}\lambda^{s-d}(1-s)\int_{\Omega_\lambda}\int_{\Omega_\lambda} \frac{|u(x)-u(y)|}{|x-y|^{d+s}}dxdy.$$

This finishes the proof of the lemma. □

The next lemma is a well-known result from [14], which we crucially use.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let $u \in BV(\Omega)$. Then,*
$$\lim_{s\to 1-}(1-s)[u]_{W^{s,1}(\Omega)} = C_{BV,d}[u]_{BV(\Omega)},$$
where $C_{BV,d} = C_{BV,d}(d) > 0$.

Proof. See [14, Theorem 1] for the proof. □

The following lemma establishes a connection to the average of u over two disjoint sets. This technical step is very crucial in the proof of Lemma 3.3 and Lemma 4.1.

Lemma 2.3. *Let E and F be measurable disjoint bounded sets in $\mathbb{R}^d$ and G be a cube of side length $\lambda > 0$ such that $E \cup F \subset G$. Then,*
$$|(u)_E - (u)_F| \le C_{d,Poin}\lambda^{s-d}(1-s)\left(\frac{|G|}{\min\{|E|,|F|\}}\right)\int_G\int_G \frac{|u(x)-u(y)|}{|x-y|^{d+s}}dxdy, \tag{2.3}$$
where $C_{d,Poin}$ is best fractional Poincaré constant for unit cube (the same $C_{d,Poin}$ as in Lemma 2.1).

Proof. Let us consider:
$$\begin{aligned}|(u)_E - (u)_F| &\le |(u)_E - (u)_G| + |(u)_F - (u)_G| \\ &\le \fint_E |u(x)-(u)_G|dx + \fint_F |u(x)-(u)_G|dx \\ &\le \frac{1}{\min\{|E|,|F|\}}\int_{E\cup F}|u(x)-(u)_G|dx.\end{aligned}$$
In the second inequality above, we have used triangle inequality for the integrals. Given that $E \cup F \subset G$, it follows that
$$|(u)_E - (u)_F| \le \frac{1}{\min\{|E|,|F|\}}\int_G |u(x)-(u)_G|dx.$$
Using Lemma 2.1, we have
$$\begin{aligned}|(u)_E - (u)_F| &\le \left(\frac{|G|}{\min\{|E|,|F|\}}\right)\fint_G |u(x)-(u)_G|dx \\ &\le C_{d,Poin}\lambda^{s-d}(1-s)\left(\frac{|G|}{\min\{|E|,|F|\}}\right)\int_G\int_G \frac{|u(x)-u(y)|}{|x-y|^{d+s}}dxdy.\end{aligned}$$
This finishes the proof of the lemma. □

The following lemma establishes a Poincaré type inequality for functions of bounded variation $BV(\Omega)$, where Ω is a bounded Lipschitz domain. This lemma is useful in establishing our main result.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then there exists a constant $C_{BV,Poin} = C_{BV,Poin}(\Omega) > 0$ such that*
$$\int_\Omega |u(x)-(u)_\Omega|dx \le C_{BV,Poin}[u]_{BV(\Omega)}, \quad \forall\ u \in BV(\Omega). \tag{2.4}$$

Proof. See [9, Theorem 3.2] for the proof. □

The next lemma establishes an inequality when any function $u \in W^{s,p}(\Omega)$ is multiplied by a test function. This lemma plays a crucial role in establishing Theorem 2. We denote by $C^{0,1}(\Omega)$ the class of bounded Lipschitz functions $u : \Omega \to \mathbb{R}$ (see [16, Chapter 3, Definition 3.1]).

Lemma 2.5. *Let Ω be an open set in $\mathbb{R}^d$. Let us consider $u \in W^{s,p}(\Omega)$ and $\xi \in C^{0,1}(\Omega)$, $0 \le \xi \le 1$. Then, $\xi u \in W^{s,p}(\Omega)$, and for some constant $C = C(d,p,s,\Omega) > 0$,*

$$\|\xi u\|_{W^{s,p}(\Omega)} \le C\|u\|_{W^{s,p}(\Omega)}. \tag{2.5}$$

Proof. See [15, Lemma 5.3] for the proof. □

3. Proofs of the main results in dimension one

In this section, we present the proof of Theorem 1 and Theorem 2 in dimension one. Establishing our main results in the one-dimensional case ($d = 1$) builds the foundation for extending the proof to higher dimensions, as all the major ideas can be explained more easily in this case. Extending the results to higher dimensions involves additional technicalities. Also, we present quantitative estimates of the constants involved in this case. For simplicity, we first establish the main results for the domain $\Omega = (0,2)$. For any other general domain ($\Omega = (0,2D), D > 0$), the results can be obtained through translation and dilation of the domain Ω.

The strategy is as follows: The proof of Theorem 1 for $\Omega = (0,2D), D > 0$, presented in subsection 3.3, follows from the proof of Theorem 2 for $\Omega = (0,2D), D > 0$, which is given in subsection 3.2. The first part of the proof of Theorem 2 for $\Omega = (0,2)$ follows easily from Lemma 3.3, presented in subsection 3.1. Lemma 3.1 and Lemma 3.2 are basic inequalities that will be used to prove Lemma 3.3.

For each $k \in \mathbb{Z}$, $k \le -1$, and $d = 1$, define

$$A_k := \{x : 3^k \le x < 3^{k+1}\}.$$

The next lemma establishes a basic inequality for each A_k. It gives a basic relation between each $\mathcal{L}_m$ and $x \in A_k$. This lemma is helpful in proving Lemma 3.3 and Lemma 4.1.

Lemma 3.1. *For any A_k, $R > 1$ and $x \in A_k$, we have*

$$\mathcal{L}_1\left(\frac{x}{R}\right) < \frac{1}{-k} =: \mathcal{Y}_1(k), \tag{3.1}$$

and for any $m \ge 2$,

$$\mathcal{L}_m\left(\frac{x}{R}\right) < \frac{1}{1 - \ln(\mathcal{Y}_{m-1}(k))} =: \mathcal{Y}_m(k). \tag{3.2}$$

Proof. Let $x \in A_k$. Then $x < 3^{k+1}$, which implies

$$\ln\left(\frac{x}{R}\right) = \ln(x) - \ln R < (k+1)\ln 3 - \ln R < (k+1)\ln 3.$$

Therefore, we have

$$1 - \ln\left(\frac{x}{R}\right) > 1 - (k+1)\ln 3 > 1 - (k+1) = -k.$$

From the definition of $\mathcal{L}_1$, we obtain

$$\mathcal{L}_1\left(\frac{x}{R}\right) = \frac{1}{1 - \ln\left(\frac{x}{R}\right)} < \frac{1}{-k} = \mathcal{Y}_1(k).$$

Using the above inequality, we have

$$\ln\left(\mathcal{L}_1\left(\frac{x}{R}\right)\right) < \ln(\mathcal{Y}_1(k)).$$

So, from the definition of $\mathcal{L}_2(x)$, we have

$$\mathcal{L}_2\left(\frac{x}{R}\right) = \mathcal{L}_1\left(\mathcal{L}_1\left(\frac{x}{R}\right)\right) = \frac{1}{1-\ln\left(\mathcal{L}_1\left(\frac{x}{R}\right)\right)} < \frac{1}{1-\ln\left(\mathcal{Y}_1(k)\right)} = \mathcal{Y}_2(k).$$

Therefore, recursively, we obtain for any $m \geq 2$,

$$\mathcal{L}_m\left(\frac{x}{R}\right) < \frac{1}{1-\ln\left(\mathcal{Y}_{m-1}(k)\right)} = \mathcal{Y}_m(k).$$

This proves the lemma. □

The next lemma establishes a basic inequality for $k \leq -1$. This lemma is helpful in the proof of Lemma 3.3 and Lemma 4.1.

Lemma 3.2. *For all $k \leq -1$, we have*

$$\mathcal{Y}_m(k) - \mathcal{Y}_m(k-1) \geq \frac{\mathcal{Y}_1(k)\cdots\mathcal{Y}_{m-1}(k)\mathcal{Y}_m^2(k)}{2^{m+1}}. \tag{3.3}$$

Proof. Let $f : [0,1] \to [0,\infty)$ be a differentiable function on $(0,1)$ such that

$$f(x) = \mathcal{Y}_m(k-1+x), \quad \text{for some } k \leq -1.$$

By the mean value theorem, there exists $\gamma \in (0,1)$ such that

$$f'(\gamma) = f(1) - f(0) = \mathcal{Y}_m(k) - \mathcal{Y}_m(k-1).$$

Also,

$$f'(\gamma) = \mathcal{Y}_1(k-1+\gamma)\cdots\mathcal{Y}_{m-1}(k-1+\gamma)\mathcal{Y}_m^2(k-1+\gamma)$$

follows easily from direct computations or using an induction argument. Therefore, combining the above two inequalities, we have

$$\mathcal{Y}_m(k) - \mathcal{Y}_m(k-1) = \mathcal{Y}_1(k-1+\gamma)\cdots\mathcal{Y}_{m-1}(k-1+\gamma)\mathcal{Y}_m^2(k-1+\gamma). \tag{3.4}$$

Since, $\left(1 - \frac{1}{k} + \frac{\gamma}{k}\right) \leq 2$ for all $k \leq -1$. Therefore, we have

$$\mathcal{Y}_1(k-1+\gamma) = \frac{1}{-k+1-\gamma} = \frac{1}{(-k)\left(1-\frac{1}{k}+\frac{\gamma}{k}\right)} \geq \frac{1}{2(-k)} = \frac{\mathcal{Y}_1(k)}{2}.$$

From the above inequality, we have $\ln\left(\mathcal{Y}_1(k-1+\gamma)\right) \geq \ln\left(\frac{\mathcal{Y}_1(k)}{2}\right)$. Using this in the definition of $\mathcal{Y}_2$, we obtain

$$\mathcal{Y}_2(k-1+\gamma) = \frac{1}{1-\ln\left(\mathcal{Y}_1(k-1+\gamma)\right)} \geq \frac{1}{1-\ln\left(\frac{\mathcal{Y}_1(k)}{2}\right)}.$$

But we have

$$\frac{1}{1-\ln\left(\frac{\mathcal{Y}_1(k)}{2}\right)} = \frac{1}{1-\ln\left(\mathcal{Y}_1(k)\right)-\ln\left(\frac{1}{2}\right)} > \frac{1}{2}\left(\frac{1}{1-\ln\left(\mathcal{Y}_1(k)\right)}\right) = \frac{\mathcal{Y}_2(k)}{2}.$$

Here, we have used $2\left(1-\ln\left(\mathcal{Y}_1(k)\right)\right) > 1-\ln\left(\mathcal{Y}_1(k)\right)-\ln\left(\frac{1}{2}\right)$. Therefore, by combining the above two inequalities, we have

$$\mathcal{Y}_2(k-1+\gamma) \geq \frac{\mathcal{Y}_2(k)}{2}.$$

From the definition of $\mathcal{Y}_m$, and using recursively, we obtain

$$\mathcal{Y}_m(k-1+\gamma) \geq \frac{\mathcal{Y}_m(k)}{2}, \quad \forall\ m \geq 2.$$

Hence, from (3.4), we have

$$\mathcal{Y}_m(k) - \mathcal{Y}_m(k-1) \geq \frac{\mathcal{Y}_1(k)\cdots\mathcal{Y}_{m-1}(k)\mathcal{Y}_m^2(k)}{2^{m+1}}.$$

This proves the lemma. □

The next lemma is the main step towards the proof of Theorem 2 for $\Omega = (0,2)$. Once the next lemma is established, Theorem 2 for $\Omega = (0,2)$ follows fairly easily.

Lemma 3.3. *Let $R > 1$, $\frac{1}{2} \leq s < 1$, and $m \geq 2$ be a positive integer. Then there exists a constant $C_{1,Poin} > 0$ such that for all $u \in W^{s,1}((0,1))$,*

$$\int_0^1 \frac{|u(x)|}{x^s}\mathcal{L}_1\left(\frac{x}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{x}{R}\right)\mathcal{L}_m^2\left(\frac{x}{R}\right)dx \\ \leq C_{1,Poin}(2^{3s+m+2}+2^s)(1-s)[u]_{W^{s,1}((0,1))} + 2^{m+1}3^s\|u\|_{L^1((0,1))}. \tag{3.5}$$

Proof. It is well known that $W^{s,1}((0,1)) = W_0^{s,1}((0,1))$ (see [20, Theorem 6.78], as $sp<1$ with $p=1$ in this case). Therefore, it suffices to establish (3.5) for any $u \in C_c^1((0,1))$. Let $u \in C_c^1((0,1))$ and fix any A_k. Applying Lemma 2.1 with $\frac{1}{2} \leq s < 1$, $\Omega = (\frac{1}{2}, \frac{3}{2})$, and $\lambda = 2 \times 3^k$, we have

$$⨍_{A_k} |u(x) - (u)_{A_k}|dx \leq C_{1,Poin}2^{s-1}3^{k(s-1)}(1-s)[u]_{W^{s,1}(A_k)},$$

where $C_{1,Poin} > 0$ is as in Lemma 2.1. For $x \in A_k$, one has $\frac{1}{x} \leq \frac{1}{3^k}$ which implies

$$\int_{A_k}\frac{|u(x)|}{x^s}dx \leq \frac{1}{3^{ks}}\int_{A_k}|u(x)-(u)_{A_k}+(u)_{A_k}|dx \\ \leq \frac{1}{3^{ks}}\int_{A_k}|u(x)-(u)_{A_k}|dx + \frac{1}{3^{ks}}\int_{A_k}|(u)_{A_k}|dx.$$

Now, using the previous inequality, we obtain

$$\int_{A_k}\frac{|u(x)|}{x_d^s}dx \leq \frac{|A_k|}{3^{ks}}⨍_{A_k}|u(x)-(u)_{A_k}|dx + \frac{|A_k|}{3^{ks}}|(u)_{A_k}| \\ \leq C_{1,Poin}2^{s-1}\frac{2\times 3^k}{3^{ks}}3^{k(s-1)}(1-s)[u]_{W^{s,1}(A_k)} + 2\times 3^{k(1-s)}|(u)_{A_k}| \\ \leq C_{1,Poin}2^s(1-s)[u]_{W^{s,1}(A_k)} + 2\times 3^{k(1-s)}|(u)_{A_k}|.$$

From Lemma 3.1, and using $\mathcal{Y}_1(k)\cdots\mathcal{Y}_{m-1}(k)\mathcal{Y}_m^2(k) \leq 1$, we have

$$\int_{A_k}\frac{|u(x)|}{x^s}\mathcal{L}_1\left(\frac{x}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{x}{R}\right)\mathcal{L}_m^2\left(\frac{x}{R}\right)dx \\ \leq C_{1,Poin}2^s(1-s)[u]_{W^{s,1}(A_k)} \\ + 2\times 3^{k(1-s)}\mathcal{Y}_1(k)\cdots\mathcal{Y}_{m-1}(k)\mathcal{Y}_m^2(k)|(u)_{A_k}|.$$

Summing the above inequality from $k = \ell \in \mathbb{Z}^-$ to -1, we get

$$\sum_{k=\ell}^{-1} \int_{A_k} \frac{|u(x)|}{x^s} \mathcal{L}_1\left(\frac{x}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{x}{R}\right) \mathcal{L}_m^2\left(\frac{x}{R}\right) dx$$

$$\leq C_{1,Poin} 2^s (1-s) \sum_{k=\ell}^{-1} [u]_{W^{s,1}(A_k)} \tag{3.6}$$

$$+ 2 \sum_{k=\ell}^{-1} 3^{k(1-s)} \mathcal{Y}_1(k) \cdots \mathcal{Y}_{m-1}(k) \mathcal{Y}_m^2(k) |(u)_{A_k}|.$$

Independently, using triangle inequality, we have

$$|(u)_{A_k}| \leq |(u)_{A_{k+1}}| + |(u)_{A_k} - (u)_{A_{k+1}}|.$$

Since $A_k \cup A_{k+1}$ is an interval of length $2 \times 3^{k+1} + 2 \times 3^k$, using Lemma 2.3 with $G = A_k \cup A_{k+1}$ and $\lambda = 8 \times 3^k$, we have:

$$\begin{aligned} |(u)_{A_k}| &\leq |(u)_{A_{k+1}}| + C_{1,Poin}(8^{s-1} 3^{k(s-1)} 2^2)(1-s)[u]_{W^{s,1}(A_k \cup A_{k+1})} \\ &\leq |(u)_{A_{k+1}}| + C_{1,Poin} 2^{3s-1} 3^{k(s-1)} (1-s)[u]_{W^{s,1}(A_k \cup A_{k+1})}. \end{aligned}$$

Multiplying the above inequality by $3^{k(1-s)}$, and using the trivial estimate that $3^{1-s} > 1$, we get

$$3^{k(1-s)} |(u)_{A_k}| \leq 3^{(k+1)(1-s)} |(u)_{A_{k+1}}| + C_{1,Poin} 2^{3s-1} (1-s)[u]_{W^{s,1}(A_k \cup A_{k+1})}.$$

Multiplying the above inequality with $\mathcal{Y}_m(k)$, and using $\mathcal{Y}_m(k) \leq 1$ for all $k \leq -1$, we obtain

$$3^{k(1-s)} \mathcal{Y}_m(k) |(u)_{A_k}| \leq 3^{(k+1)(1-s)} \mathcal{Y}_m(k) |(u)_{A_{k+1}}| + C_{1,Poin} 2^{3s-1} (1-s)[u]_{W^{s,1}(A_k \cup A_{k+1})}.$$

Summing the above inequality from $k = \ell \in \mathbb{Z}^-$ to -2, we get

$$\sum_{k=\ell}^{-2} 3^{k(1-s)} \mathcal{Y}_m(k) |(u)_{A_k}| \leq \sum_{k=\ell}^{-2} 3^{(k+1)(1-s)} \mathcal{Y}_m(k) |(u)_{A_{k+1}}|$$

$$+ C_{1,Poin} 2^{3s-1} (1-s) \sum_{k=\ell}^{-2} [u]_{W^{s,1}(A_k \cup A_{k+1})}.$$

By changing sides, rearranging, and re-indexing, we get

$$3^{\ell(1-s)} \mathcal{Y}_m(\ell) |(u)_{A_\ell}| + \sum_{k=\ell+1}^{-2} 3^{k(1-s)} \left\{ \mathcal{Y}_m(k) - \mathcal{Y}_m(k-1) \right\} |(u)_{A_k}|$$

$$\leq 3^{(-1)(1-s)} \mathcal{Y}_m(-2) |(u)_{A_{-1}}| + C_{1,Poin} 2^{3s-1} (1-s) \sum_{k=\ell}^{-2} [u]_{W^{s,1}(A_k \cup A_{k+1})}.$$

Using the asymptotics (see Lemma 3.2),

$$\mathcal{Y}_m(k) - \mathcal{Y}_m(k-1) \geq \frac{\mathcal{Y}_1(k) \cdots \mathcal{Y}_{m-1}(k) \mathcal{Y}_m^2(k)}{2^{m+1}},$$

choose $-\ell$ large enough such that $|(u)_{A_\ell}| = 0$ (as u is assumed to be compactly supported), we obtain

$$\sum_{k=\ell}^{-2} \frac{3^{k(1-s)}}{2^{m+1}} \mathcal{Y}_1(k) \cdots \mathcal{Y}_{m-1}(k) \mathcal{Y}_m^2(k) |(u)_{A_k}| \\ \leq \mathcal{Y}_m(-2) 3^{s-1} |(u)_{A_{-1}}| + C_{1,Poin} 2^{3s-1}(1-s) \sum_{k=\ell}^{-2} [u]_{W^{s,1}(A_k \cup A_{k+1})}.$$

Adding $\frac{3^{(-1)(1-s)}}{2^{m+1}} \mathcal{Y}_1(-1) \cdots \mathcal{Y}_{m-1}(-1) \mathcal{Y}_m^2(-1) |(u)_{A_{-1}}|$ on both sides of the above inequality, we obtain

$$\begin{aligned} \sum_{k=\ell}^{-1} \frac{3^{k(1-s)}}{2^{m+1}} \mathcal{Y}_1(k) \cdots \mathcal{Y}_{m-1}(k) \mathcal{Y}_m^2(k) |(u)_{A_k}| \\ &\leq 3^{s-1} \left\{ \mathcal{Y}_m(-2) + \frac{1}{2^{m+1}} \mathcal{Y}_1(-1) \cdots \mathcal{Y}_m(-1) \mathcal{Y}_m^2(-1) \right\} |(u)_{A_{-1}}| \\ &\quad + C_{1,Poin} 2^{3s-1} (1-s) \sum_{k=\ell}^{-2} [u]_{W^{s,1}(A_k \cup A_{k+1})} \\ &\leq 3^{s-1} |(u)_{A_{-1}}| + C_{1,Poin} 2^{3s-1} (1-s) \sum_{k=\ell}^{-2} [u]_{W^{s,1}(A_k \cup A_{k+1})}. \end{aligned}$$

In the last inequality, we have used Lemma 3.2 with $k = -1$. Therefore, we have

$$\begin{aligned} \sum_{k=\ell}^{-1} 3^{k(1-s)} \mathcal{Y}_1(k) \cdots \mathcal{Y}_{m-1}(k) \mathcal{Y}_m^2(k) |(u)_{A_k}| \\ \leq 2^{m+1} 3^{s-1} |(u)_{A_{-1}}| + C_{1,Poin} 2^{3s-1} 2^{m+1} (1-s) \sum_{k=\ell}^{-2} [u]_{W^{s,1}(A_k \cup A_{k+1})}. \end{aligned} \tag{3.7}$$

Combining (3.6), (3.7) together (6.4) (see Appendix 6) with $d = 1$, yields

$$\begin{aligned} \sum_{k=\ell}^{-1} \int_{A_k} \frac{|u(x)|}{x^s} \mathcal{L}_1\left(\frac{x}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{x}{R}\right) \mathcal{L}_m^2\left(\frac{x}{R}\right) dx \\ &\leq C_{1,Poin} 2^s (1-s) [u]_{W^{s,1}((0,1))} \\ &\quad + 2 \left\{ 2^{m+1} 3^{s-1} |(u)_{A_{-1}}| + C_{1,Poin} 2^{3s-1} 2^{m+1} (1-s) \sum_{k=\ell}^{-2} [u]_{W^{s,1}(A_k \cup A_{k+1})} \right\} \\ &\leq C_{1,Poin} (2^{3s+m+2} + 2^s)(1-s) [u]_{W^{s,1}((0,1))} + 2^{m+2} 3^{s-1} |(u)_{A_{-1}}|. \end{aligned}$$

Using $|(u)_{A_{-1}}| \leq (3/2) \|u\|_{L^1((0,1))}$, we have

$$\int_0^1 \frac{|u(x)|}{x^s} \mathcal{L}_1\left(\frac{x}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{x}{R}\right) \mathcal{L}_m^2\left(\frac{x}{R}\right) dx \\ \leq C_{1,Poin} (2^{3s+m+2} + 2^s)(1-s) [u]_{W^{s,1}((0,1))} + 2^{m+1} 3^s \|u\|_{L^1((0,1))}.$$

This proves the lemma. ∎

3.1. **Proof Theorem 2 for $\Omega = (0,2)$.** From Lemma 3.3, $m \geq 2$, $R > 1$, and $u \in W^{s,1}((0,2))$, we obtain

$$\begin{aligned}\int_0^1 \frac{|u(x)|}{x^s}\mathcal{L}_1\left(\frac{x}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{x}{R}\right)\mathcal{L}_m^2\left(\frac{x}{R}\right)dx \\ \leq C_{1,Poin}\left(2^{3s+m+2}+2^s\right)(1-s)[u]_{W^{s,1}((0,1))}+2^{m+1}3^s\|u\|_{L^1((0,1))}.\end{aligned} \tag{3.8}$$

In the previous step, we used the fact that the restriction of any $W^{s,1}((0,2))$ function to the interval $(0,1)$ is again a $W^{s,1}((0,1))$ function. Now, since

$$\delta_{(0,2)}(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 \leq x < 2, \end{cases}$$

we have

$$\begin{aligned}&\int_0^2 \frac{|u(x)|}{\delta_{(0,2)}^s(x)}\mathcal{L}_1\left(\frac{\delta_{(0,2)}(x)}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{\delta_{(0,2)}(x)}{R}\right)\mathcal{L}_m^2\left(\frac{\delta_{(0,2)}(x)}{R}\right)dx \\ &= \int_0^1 \frac{|u(x)|}{x^s}\mathcal{L}_1\left(\frac{x}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{x}{R}\right)\mathcal{L}_m^2\left(\frac{x}{R}\right)dx \\ &\quad + \int_1^2 \frac{|u(x)|}{(2-x)^s}\mathcal{L}_1\left(\frac{2-x}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{2-x}{R}\right)\mathcal{L}_m^2\left(\frac{2-x}{R}\right)dx.\end{aligned}$$

Using the change of variable $2-x=z$ in the last integral of the above equation, and applying (3.8), we obtain

$$\begin{aligned}&\int_0^2 \frac{|u(x)|}{\delta_{(0,2)}^s(x)}\mathcal{L}_1\left(\frac{\delta_{(0,2)}(x)}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{\delta_{(0,2)}(x)}{R}\right)\mathcal{L}_m^2\left(\frac{\delta_{(0,2)}(x)}{R}\right)dx \\ &\leq C_{1,Poin}\left(2^{3s+m+3}+2^{s+1}\right)(1-s)[u]_{W^{s,1}((0,2))}+2^{m+2}3^s\|u\|_{L^1((0,2))}.\end{aligned}$$

This finishes the proof of the first part. Now, let $\alpha < \frac{1}{2}$ and summing from $m=2$ to ∞, we have

$$\begin{aligned}&\sum_{m=2}^{\infty}\alpha^m\int_0^2 \frac{|u(x)|}{\delta_{(0,2)}^s(x)}\mathcal{L}_1\left(\frac{\delta_{(0,2)}(x)}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{\delta_{(0,2)}(x)}{R}\right)\mathcal{L}_m^2\left(\frac{\delta_{(0,2)}(x)}{R}\right)dx \\ &\leq \left\{2^{3s+3}\frac{4\alpha^2}{1-2\alpha}+2^{s+1}\frac{\alpha^2}{1-\alpha}\right\}C_{1,Poin}(1-s)[u]_{W^{s,1}((0,2))}+2^2\times 3^s\frac{4\alpha^2}{1-2\alpha}\|u\|_{L^1((0,2))} \\ &=: \mathcal{A}(s,\alpha)C_{1,Poin}(1-s)[u]_{W^{s,1}((0,2))}+\mathcal{B}(s,\alpha)\|u\|_{L^1((0,2))}.\end{aligned}$$

This proves the theorem for $\Omega = (0,2)$.

3.2. **Proof Theorem 2 for general domain in dimension one.** Without any loss of generality assume $\Omega = (0,2D)$ for $D > 0$. Scaling appropriately the results in the last subsection it is easy to that for $R > D$, clearly

$$\begin{aligned}&\int_0^{2D} \frac{|u(x)|}{\delta_{(0,2D)}^s(x)}\mathcal{L}_1\left(\frac{\delta_{(0,2D)}(x)}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{\delta_{(0,2D)}(x)}{R}\right)\mathcal{L}_m^2\left(\frac{\delta_{(0,2D)}(x)}{R}\right)dx \\ &\leq C_{1,Poin}\left(2^{3s+m+3}+2^{s+1}\right)(1-s)[u]_{W^{s,1}((0,2D))}+\frac{2^{m+2}3^s}{D^s}\|u\|_{L^1((0,2D))}.\end{aligned}$$

The proof finishes the first part. Let $\alpha < \frac{1}{2}$ and summing from $m = 2$ to ∞, we have

$$\sum_{m=2}^{\infty} \alpha^m \int_0^{2D} \frac{|u(x)|}{\delta^s_{(0,2D)}(x)} \mathcal{L}_1 \left(\frac{\delta_{(0,2D)}(x)}{R} \right) \cdots \mathcal{L}_{m-1} \left(\frac{\delta_{(0,2D)}(x)}{R} \right) \mathcal{L}^2_m \left(\frac{\delta_{(0,2D)}(x)}{R} \right) dx$$
$$\leq \left\{ 2^{3s+3} \frac{4\alpha^2}{1-2\alpha} + 2^{s+1} \frac{\alpha^2}{1-\alpha} \right\} C_{1,Poin}(1-s)[u]_{W^{s,1}((0,2D))}$$
$$+ \left(2^2 \times 3^s \frac{4\alpha^2}{1-2\alpha} \right) \frac{1}{D^s} \|u\|_{L^1((0,2D))}$$
$$= \mathcal{A}(s,\alpha) C_{1,Poin}(1-s)[u]_{W^{s,1}((0,2D))} + \frac{\mathcal{B}(s,\alpha)}{D^s} \|u\|_{L^1((0,2D))}.$$

3.3. **Proof of Theorem 1 for general domain in dimension one.** From Theorem 2 with $d = 1$, $\Omega = (0, 2D)$, $D > 0$, and $R > D$, we have

$$\int_0^{2D} \frac{|u(x)|}{\delta^s_{(0,2D)}(x)} \mathcal{L}_1 \left(\frac{\delta_{(0,2D)}(x)}{R} \right) \cdots \mathcal{L}_{m-1} \left(\frac{\delta_{(0,2D)}(x)}{R} \right) \mathcal{L}^2_m \left(\frac{\delta_{(0,2D)}(x)}{R} \right) dx$$
$$\leq C_{1,Poin} \left(2^{3s+m+3} + 2^{s+1} \right) (1-s)[u]_{W^{s,1}((0,2D))} + \frac{2^{m+2} 3^s}{D^s} \|u\|_{L^1((0,2D))}.$$

Using Fatou's lemma, we have

$$\int_0^{2D} \frac{|u(x)|}{\delta_{(0,2D)}(x)} \mathcal{L}_1 \left(\frac{\delta_{(0,2D)}(x)}{R} \right) \cdots \mathcal{L}_{m-1} \left(\frac{\delta_{(0,2D)}(x)}{R} \right) \mathcal{L}^2_m \left(\frac{\delta_{(0,2D)}(x)}{R} \right) dx$$
$$\leq \liminf_{s \to 1} \int_0^{2D} \frac{|u(x)|}{\delta^s_{(0,2D)}(x)} \mathcal{L}_1 \left(\frac{\delta_{(0,2D)}(x)}{R} \right) \cdots \mathcal{L}_{m-1} \left(\frac{\delta_{(0,2D)}(x)}{R} \right) \mathcal{L}^2_m \left(\frac{\delta_{(0,2D)}(x)}{R} \right) dx$$
$$\leq \liminf_{s \to 1} \left\{ C_{1,Poin} \left(2^{3s+m+3} + 2^{s+1} \right) (1-s)[u]_{W^{s,1}((0,2D))} + \frac{2^{m+2} 3^s}{D^s} \|u\|_{L^1((0,2D))} \right\}.$$

From Lemma 2.2, we have

$$\int_0^{2D} \frac{|u(x)|}{\delta_{(0,2D)}(x)} \mathcal{L}_1 \left(\frac{\delta_{(0,2D)}(x)}{R} \right) \cdots \mathcal{L}_{m-1} \left(\frac{\delta_{(0,2D)}(x)}{R} \right) \mathcal{L}^2_m \left(\frac{\delta_{(0,2D)}(x)}{R} \right) dx$$
$$\leq C_{1,Poin} C_{BV,1} \left(2^{m+6} + 2^2 \right) [u]_{BV((0,2D))} + \frac{3 \times 2^{m+2}}{D} \|u\|_{L^1((0,2D))}.$$

Therefore, from Lemma 2.4 and using $[u - (u)_{(0,2D)}]_{BV((0,2D))} = [u]_{BV((0,2D))}$, we have

$$\int_0^{2D} \frac{|u(x) - (u)_{(0,2D)}|}{\delta_{(0,2D)}(x)} \mathcal{L}_1 \left(\frac{\delta_{(0,2D)}(x)}{R} \right) \cdots \mathcal{L}_{m-1} \left(\frac{\delta_{(0,2D)}(x)}{R} \right) \mathcal{L}^2_m \left(\frac{\delta_{(0,2D)}(x)}{R} \right) dx$$
$$\leq \left\{ C_{1,Poin} C_{BV,1} \left(2^{m+6} + 2^2 \right) + C_{BV,Poin} \frac{3 \times 2^{m+2}}{D} \right\} [u]_{BV((0,2D))}.$$

Let $\alpha < \frac{1}{2}$ and summing from $m = 2$ to ∞, we have

$$\sum_{m=2}^{\infty} \alpha^m \int_0^{2D} \frac{|u(x) - (u)_{(0,2D)}|}{\delta_{(0,2D)}(x)} \mathcal{L}_1 \left(\frac{\delta_{(0,2D)}(x)}{R} \right) \cdots \mathcal{L}_{m-1} \left(\frac{\delta_{(0,2D)}(x)}{R} \right) \mathcal{L}^2_m \left(\frac{\delta_{(0,2D)}(x)}{R} \right) dx$$
$$\leq \left\{ C_{1,Poin} C_{BV,1} \left(\frac{2^8 \times \alpha^2}{1-2\alpha} + \frac{2^2 \times \alpha^2}{1-\alpha} \right) + C_{BV,Poin} \frac{3 \times 2^4 \times \alpha^2}{D(1-2\alpha)} \right\} [u]_{BV((0,2D))}.$$

This proves the Theorem 1 in dimension 1.

4. Proof of the main results in dimension $d \geq 2$

In this section, we prove Theorem 1 and Theorem 2 in dimension $d \geq 2$. Initially, we establish Theorem 2 for the flat boundary case, as shown in Lemma 4.1. Subsequently, in subsection 4.1, we employ patching techniques to prove Theorem 2. Finally, we establish our main result, Theorem 1, in subsection 4.2 using Theorem 2 and Lemma 2.2.

Let $\Omega_n = (-n,n)^{d-1} \times (0,1)$, where $n \in \mathbb{N}$. For each $k \in \mathbb{Z}$ and $k \leq -1$, set

$$A_k := \{(x', x_d) : \ x' \in (-n,n)^{d-1}, \ 3^k \leq x_d < 3^{k+1}\}.$$

Then, we have $\Omega_n = \bigcup_{k=-\infty}^{-1} A_k$. Again, we further divide each A_k into disjoint cubes each of side length 2×3^k (say A_k^i). Then,

$$A_k = \bigcup_{i=1}^{\sigma_k} A_k^i,$$

where $\sigma_k := 3^{(-k)(d-1)} n^{d-1}$.

The next lemma proves Theorem 2 when the domain is $\mathbb{R}^d_+$ and the test functions are supported on $\Omega_n = (-n,n)^{d-1} \times (0,1)$, where $n \in \mathbb{N}$.

Lemma 4.1. *Let $\Omega_n = (-n,n)^{d-1} \times (0,1)$ for some $n \in \mathbb{N}$, $R > 1$, $\frac{1}{2} \leq s < 1$ and $m \geq 2$ be a positive integer. Then, there exists a constant $C = C(d) > 0$ such that for all $u \in W^{s,1}(\Omega_n)$,*

$$\begin{aligned} \int_{\Omega_n} \frac{|u(x)|}{x_d^s} \mathcal{L}_1\left(\frac{x_d}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{x_d}{R}\right) \mathcal{L}_m^2\left(\frac{x_d}{R}\right) dx & \\ \leq C 2^m \left\{(1-s)[u]_{W^{s,1}(\Omega_n)} + \|u\|_{L^1(\Omega_n)}\right\}. & \end{aligned} \tag{4.1}$$

Proof. Since $W^{s,1}(\Omega_n) = W_0^{s,1}(\Omega_n)$ (see [20, Theorem 6.78]), it is sufficient to establish (4.1) for any $u \in C_c^1(\Omega_n)$. Let $u \in C_c^1(\Omega_n)$ and fix any A_k^i. Then, A_k^i is a translation of $(3^k, 3^{k+1})^d$. Applying Lemma 2.1 with $\frac{1}{2} \leq s < 1$, $\Omega = \left(\frac{1}{2}, \frac{3}{2}\right)^d$, and $\lambda = 2 \times 3^k$, and using translation invariance, we have

$$⨍_{A_k^i} |u(x) - (u)_{A_k^i}| dx \leq C_{d,Poin} 2^{s-d} 3^{k(s-d)} (1-s)[u]_{W^{s,1}(A_k^i)},$$

where $C_{d,Poin} > 0$ is a constant as in Lemma 2.1. Let $x = (x', x_d) \in A_k^i$. Then, $x_d \geq 3^k$, which implies $\frac{1}{x_d} \leq \frac{1}{3^k}$. Therefore, we have

$$\begin{aligned} \int_{A_k^i} \frac{|u(x)|}{x_d^s} dx &\leq \frac{1}{3^{ks}} \int_{A_k^i} |u(x) - (u)_{A_k^i} + (u)_{A_k^i}| dx \\ &\leq \frac{1}{3^{ks}} \int_{A_k^i} |u(x) - (u)_{A_k^i}| dx + \frac{1}{3^{ks}} \int_{A_k^i} |(u)_{A_k^i}| dx. \end{aligned}$$

Now, using the previous inequality, we obtain

$$\begin{aligned} \int_{A_k^i} \frac{|u(x)|}{x_d^s} dx &\leq \frac{|A_k^i|}{3^{ks}} ⨍_{A_k^i} |u(x) - (u)_{A_k^i}| dx + \frac{|A_k^i|}{3^{ks}} |(u)_{A_k^i}| \\ &\leq C_{d,Poin} 2^{s-d} \frac{2^d 3^{kd}}{3^{ks}} 3^{k(s-d)} (1-s)[u]_{W^{s,1}(A_k^i)} + 2^d 3^{k(d-s)} |(u)_{A_k^i}| \\ &= C_{d,Poin} 2^s (1-s)[u]_{W^{s,1}(A_k^i)} + 2^d 3^{k(d-s)} |(u)_{A_k^i}|. \end{aligned}$$

From Lemma 3.1, and using $\mathcal{Y}_1(k)\cdots\mathcal{Y}_{m-1}(k)\mathcal{Y}_m^2(k) \leq 1$, we have

$$\begin{aligned}\int_{A_k^i} \frac{|u(x)|}{x_d^s}\mathcal{L}_1\left(\frac{x_d}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{x_d}{R}\right)\mathcal{L}_m^2\left(\frac{x_d}{R}\right)dx \\ \leq C_{d,Poin}2^s(1-s)[u]_{W^{s,1}(A_k^i)} \\ + 2^d3^{k(d-s)}\mathcal{Y}_1(k)\cdots\mathcal{Y}_{m-1}(k)\mathcal{Y}_m^2(k)|(u)_{A_k^i}|.\end{aligned}$$

Summing the above inequality from $i=1$ to σ_k, we obtain

$$\begin{aligned}\int_{A_k} \frac{|u(x)|}{x_d^s}\mathcal{L}_1\left(\frac{x_d}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{x_d}{R}\right)\mathcal{L}_m^2\left(\frac{x_d}{R}\right)dx \\ \leq C_{d,Poin}2^s(1-s)[u]_{W^{s,1}(A_k)} \\ + 2^d3^{k(d-s)}\mathcal{Y}_1(k)\cdots\mathcal{Y}_{m-1}(k)\mathcal{Y}_m^2(k)\sum_{i=1}^{\sigma_k}|(u)_{A_k^i}|.\end{aligned}$$

Summing the above inequality from $k=\ell\in\mathbb{Z}^-$ to -1, we get

$$\begin{aligned}\sum_{k=\ell}^{-1}\int_{A_k} \frac{|u(x)|}{x_d^s}\mathcal{L}_1\left(\frac{x_d}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{x_d}{R}\right)\mathcal{L}_m^2\left(\frac{x_d}{R}\right)dx \\ \leq C_{d,Poin}2^s(1-s)\sum_{k=\ell}^{-1}[u]_{W^{s,1}(A_k)} \\ + 2^d\sum_{k=\ell}^{-1}3^{k(d-s)}\mathcal{Y}_1(k)\cdots\mathcal{Y}_{m-1}(k)\mathcal{Y}_m^2(k)\sum_{i=1}^{\sigma_k}|(u)_{A_k^i}|.\end{aligned} \tag{4.2}$$

Let A_{k+1}^j be the cube that lies above the cube A_k^i and share a face with A_k^i. Independently, using triangle inequality, we have

$$|(u)_{A_k^i}| \leq |(u)_{A_{k+1}^j}| + |(u)_{A_k^i} - (u)_{A_{k+1}^j}|.$$

Choose a cube G_{k+1}^j of side length $2\times3^{k+1}+2\times3^k$ such that $A_k^i\cup A_{k+1}^j\subset G_{k+1}^j$ and $G_{k+1}^j\subset A_k\cup A_{k+1}$. Therefore, using Lemma 2.3 with $E=A_k^i,\ F=A_{k+1}^j$ and $G=G_{k+1}^j$ with $\lambda=8\times3^k$, we have

$$|(u)_{A_k^i}| \leq |(u)_{A_{k+1}^j}| + C_{d,Poin}4^d8^{s-d}3^{k(s-d)}(1-s)[u]_{W^{s,1}(G_{k+1}^j)}.$$

Multiplying the above inequality by $3^{k(d-s)}$, we get

$$3^{k(d-s)}|(u)_{A_k^i}| \leq 3^{k(d-s)}|(u)_{A_{k+1}^j}| + C_{d,Poin}2^{3s-d}(1-s)[u]_{W^{s,1}(G_{k+1}^j)}.$$

There are 3^{d-1} such A_k^i's cubes lies below the cube A_{k+1}^j. Therefore, summing the above inequality from $i=3^{d-1}(j-1)+1$ to $3^{d-1}j$, and using $3^{d-1}\leq3^{d-s}$, we obtain

$$\begin{aligned}3^{k(d-s)}\sum_{i=3^{d-1}(j-1)+1}^{3^{d-1}j}|(u)_{A_k^i}| &\leq 3^{d-1}3^{k(d-s)}|(u)_{A_{k+1}^j}| + 3^{d-1}C_{d,Poin}2^{3s-d}(1-s)[u]_{W^{s,1}(G_{k+1}^j)} \\ &\leq 3^{(k+1)(d-s)}|(u)_{A_{k+1}^j}| + C_{d,Poin}2^{3s-d}3^{d-1}(1-s)[u]_{W^{s,1}(G_{k+1}^j)}.\end{aligned}$$

Again, summing the above inequality from $j=1$ to σ_{k+1}, using the fact that

$$\sum_{j=1}^{\sigma_{k+1}}\left(\sum_{i=3^{d-1}(j-1)+1}^{3^{d-1}j}|(u)_{A_k^i}|\right)=\sum_{i=1}^{\sigma_k}|(u)_{A_k^i}|,$$

and (6.6) (See Appendix 6), we obtain

$$\begin{aligned}3^{k(d-s)}\sum_{i=1}^{\sigma_k}|(u)_{A_k^i}| &\le 3^{(k+1)(d-s)}\sum_{j=1}^{\sigma_{k+1}}|(u)_{A_{k+1}^j}|+C_{d,Poin}2^{3s-d}3^{d-1}(1-s)\sum_{j=1}^{\sigma_{k+1}}[u]_{W^{s,1}(G_{k+1}^j)}\\ &\le 3^{(k+1)(d-s)}\sum_{j=1}^{\sigma_{k+1}}|(u)_{A_{k+1}^j}|+C_{d,Poin}2^{3s-d+1}3^{d-1}(1-s)[u]_{W^{s,1}(A_k\cup A_{k+1})}.\end{aligned}$$

Multiplying the above inequality with $\mathcal{Y}_m(k)$, and using $\mathcal{Y}_m(k)\le 1$ for all $k\le -1$, we obtain

$$\begin{aligned}3^{k(d-s)}\mathcal{Y}_m(k)\sum_{i=1}^{\sigma_k}|(u)_{A_k^i}| &\le 3^{(k+1)(d-s)}\mathcal{Y}_m(k)\sum_{j=1}^{\sigma_{k+1}}|(u)_{A_{k+1}^j}|\\ &\quad +C_{d,Poin}2^{3s-d+1}3^{d-1}(1-s)[u]_{W^{s,1}(A_k\cup A_{k+1})}.\end{aligned}$$

For simplicity, let $a_k=\sum_{i=1}^{\sigma_k}|(u)_{A_k^i}|$. Then, the above inequality will become

$$3^{k(d-s)}\mathcal{Y}_m(k)a_k\le 3^{(k+1)(d-s)}\mathcal{Y}_m(k)a_{k+1}+C_{d,Poin}2^{3s-d+1}3^{d-1}(1-s)[u]_{W^{s,1}(A_k\cup A_{k+1})}.$$

Summing the above inequality from $k=\ell\in\mathbb{Z}^-$ to -2, we get

$$\begin{aligned}\sum_{k=\ell}^{-2}3^{k(d-s)}\mathcal{Y}_m(k)a_k &\le \sum_{k=\ell}^{-2}3^{(k+1)(d-s)}\mathcal{Y}_m(k)a_{k+1}\\ &\quad +C_{d,Poin}2^{3s-d+1}3^{d-1}(1-s)\sum_{k=\ell}^{-2}[u]_{W^{s,1}(A_k\cup A_{k+1})}.\end{aligned}$$

By changing sides, rearranging, and re-indexing, we get

$$\begin{aligned}&3^{\ell(d-s)}\mathcal{Y}_m(\ell)a_\ell+\sum_{k=\ell+1}^{-2}3^{k(d-s)}\left\{\mathcal{Y}_m(k)-\mathcal{Y}_m(k-1)\right\}a_k\\ &\qquad\le 3^{(-1)(d-s)}\mathcal{Y}_m(-2)a_{-1}+C_{d,Poin}2^{3s-d+1}3^{d-1}(1-s)\sum_{k=\ell}^{-2}[u]_{W^{s,1}(A_k\cup A_{k+1})}.\end{aligned}$$

Using the asymptotics (see Lemma 3.2),

$$\mathcal{Y}_m(k)-\mathcal{Y}_m(k-1)\ge\frac{\mathcal{Y}_1(k)\cdots\mathcal{Y}_{m-1}(k)\mathcal{Y}_m^2(k)}{2^{m+1}},$$

choose $-\ell$ large enough such that $|(u)_{A_\ell^j}|=0$ for all $j\in\{1,\cdots,\sigma_\ell\}$, we obtain

$$\begin{aligned}\sum_{k=\ell}^{-2}\frac{3^{k(d-s)}}{2^{m+1}}\mathcal{Y}_1(k)\cdots\mathcal{Y}_{m-1}(k)\mathcal{Y}_m^2(k)a_k &\le 3^{s-d}\mathcal{Y}_m(-2)a_{-1}\\ &\quad +C_{d,Poin}2^{3s-d+1}3^{d-1}(1-s)\sum_{k=\ell}^{-2}[u]_{W^{s,1}(A_k\cup A_{k+1})}.\end{aligned}$$

Adding $\frac{3^{(-1)(d-s)}}{2^{m+1}}\mathcal{Y}_1(-1)\cdots\mathcal{Y}_{m-1}(-1)\mathcal{Y}_m^2(-1)a_{-1}$ on both sides of the above inequality, we obtain

$$\begin{aligned}\sum_{k=\ell}^{-1}\frac{3^{k(d-s)}}{2^{m+1}}&\mathcal{Y}_1(k)\cdots\mathcal{Y}_{m-1}(k)\mathcal{Y}_m^2(k)a_k\\ &\leq 3^{s-d}\left\{\mathcal{Y}_m(-2)+\frac{1}{2^{m+1}}\mathcal{Y}_1(-1)\cdots\mathcal{Y}_{m-1}(-1)\mathcal{Y}_m^2(-1)\right\}a_{-1}\\ &\quad+C_{d,Poin}2^{3s-d+1}3^{d-1}(1-s)\sum_{k=\ell}^{-2}[u]_{W^{s,1}(A_k\cup A_{k+1})}\\ &\leq 3^{s-d}a_{-1}+C_{d,Poin}2^{3s-d+1}3^{d-1}(1-s)\sum_{k=\ell}^{-2}[u]_{W^{s,1}(A_k\cup A_{k+1})}.\end{aligned}$$

In the last inequality, we have used Lemma 3.2 with $k=-1$. Therefore, we have

$$\begin{aligned}\sum_{k=\ell}^{-1}&3^{k(d-s)}\mathcal{Y}_1(k)\cdots\mathcal{Y}_{m-1}(k)\mathcal{Y}_m^2(k)a_k\\ &\leq 2^{m+1}3^{s-d}a_{-1}+C_{d,Poin}2^{3s-d+m+2}3^{d-1}(1-s)\sum_{k=\ell}^{-2}[u]_{W^{s,1}(A_k\cup A_{k+1})}.\end{aligned}$$

Putting the value of a_k in the above inequality, we obtain

$$\begin{aligned}\sum_{k=\ell}^{-1}&3^{k(d-s)}\mathcal{Y}_1(k)\cdots\mathcal{Y}_{m-1}(k)\mathcal{Y}_m^2(k)\sum_{i=1}^{\sigma_k}|(u)_{A_k^i}|\\ &\leq 2^{m+1}3^{s-d}\sum_{j=1}^{\sigma_{-1}}|(u)_{A_{-1}^j}|+C_{d,Poin}2^{3s-d+m+2}3^{d-1}(1-s)\sum_{k=\ell}^{-2}[u]_{W^{s,1}(A_k\cup A_{k+1})}.\end{aligned}\tag{4.3}$$

Combining (4.2) and (4.3) together (6.4) (see Appendix 6), yields

$$\begin{aligned}\sum_{k=\ell}^{-1}\int_{A_k}\frac{|u(x)|}{x_d^s}&\mathcal{L}_1\left(\frac{x_d}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{x_d}{R}\right)\mathcal{L}_m^2\left(\frac{x_d}{R}\right)dx\leq C_{d,Poin}2^s(1-s)[u]_{W^{s,1}(\Omega_n)}\\ &+2^d\left\{2^{m+1}3^{s-d}\sum_{j=1}^{\sigma_{-1}}|(u)_{A_{-1}^j}|+C_{d,Poin}2^{3s-d+m+2}3^{d-1}(1-s)\sum_{k=\ell}^{-2}[u]_{W^{s,1}(A_k\cup A_{k+1})}\right\}\\ &\leq C_{d,Poin}\left(2^s+2^{3s+m+3}3^{d-1}\right)(1-s)[u]_{W^{s,1}(\Omega_n)}+2^{m+d+1}3^{s-d}\sum_{j=1}^{\sigma_{-1}}|(u)_{A_{-1}^j}|.\end{aligned}$$

Also,

$$\sum_{j=1}^{\sigma_{-1}}|(u)_{A_{-1}^j}|\leq\left(\frac{3}{2}\right)^d\|u\|_{L^1(\Omega_n)}.$$

Therefore, we have

$$\begin{aligned}\int_{\Omega_n}&\frac{|u(x)|}{x_d^s}\mathcal{L}_1\left(\frac{x_d}{R}\right)\cdots\mathcal{L}_{m-1}\left(\frac{x_d}{R}\right)\mathcal{L}_m^2\left(\frac{x_d}{R}\right)dx\\ &\leq C_{d,Poin}\left(2^s+2^{3s+m+3}3^{d-1}\right)(1-s)[u]_{W^{s,1}(\Omega_n)}+2^{m+1}3^s\|u\|_{L^1(\Omega_n)}.\end{aligned}$$

This proves the lemma. $\square$

4.1. **Proof of Theorem 2.** Let Ω be a bounded Lipschitz domain. Consider the definition of bounded Lipschitz domain defined in Section 2. For simplicity, let T_x be the identity map. Then,

$$\Omega \cap B_{r'_x}(x) = \{\xi = (\xi', \xi_d) : \xi_d > \phi_x(\xi')\} \cap B_{r'_x}(x),$$

and $\partial\Omega \subset \cup_{x\in\partial\Omega} B_{r'_x}(x)$. Choose $0 < r_x < 1$ such that $r_x \leq r'_x$, and for all $y \in \Omega \cap B_{r_x}(x)$, there exists $z \in \partial\Omega \cap B_{r_x}(x)$ satisfying $\delta_\Omega(y) = |y - z|$. Then, $\partial\Omega \subset \cup_{x\in\partial\Omega} B_{r_x}(x)$. Since $\partial\Omega$ is compact, there exist $x_1, \ldots, x_n \in \partial\Omega$ such that

$$\partial\Omega \subset \bigcup_{i=1}^{n} B_{r_i}(x_i),$$

where $r_{x_i} = r_i$.

Let $u \in W^{s,1}(\Omega)$. Let $\Omega \subset \cup_{i=0}^{n} \Omega_i$, where $\overline{\Omega}_0 \subset \Omega$, and $\Omega_i = B_{r_i}(x_i)$ for all $1 \leq i \leq n$. Let $\{\eta_i\}_{i=0}^{n}$ be the associated partition of unity. Then,

$$u = \sum_{i=0}^{n} u_i, \quad \text{where } u_i = \eta_i u.$$

From Lemma 2.5, we have

$$\|u_i\|_{W^{s,1}(\Omega\cap\Omega_i)} \leq C\|u\|_{W^{s,1}(\Omega\cap\Omega_i)}, \quad \forall\ 0 \leq i \leq 1.$$

Therefore, it is sufficient to prove Theorem 2 for all $u_i,\ 0 \leq i \leq n$. Given that $\operatorname{supp}(u_0) \subset \Omega_0$, and for all $x \in \Omega_0$,

$$C_1 \leq \delta_\Omega(x) \leq C_2 \quad \text{for some } C_1, C_2 > 0.$$

Therefore, using $\mathcal{L}_m\left(\frac{\delta_\Omega(x)}{R}\right) \leq 1$ for all $m \geq 1$, we have

$$\int_{\Omega_0} \frac{|u_0(x)|}{\delta_\Omega^s(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^2\left(\frac{\delta_\Omega(x)}{R}\right) dx \leq C \int_{\Omega_0} |u_0(x)| dx.$$

For $1 \leq i \leq n,\ \operatorname{supp}(u_i) \subset \Omega \cap \Omega_i$. Consider the transformation $F : \mathbb{R}^d \to \mathbb{R}^d$ such that $F(x', x_d) = (x', x_d - \phi_{x_i}(x'))$ and $G = F^{-1}$ (see subsection 6.1, Appendix 6), then

$$\delta_\Omega(x) \sim \xi_d \quad \text{for all } x \in \Omega \cap \Omega_i,$$

where $F(x) = (\xi_1, \cdots, \xi_d)$. Therefore, from Lemma 4.1, we have

$$\begin{aligned}
&\int_{\Omega\cap\Omega_i} \frac{|u_i(x)|}{\delta_\Omega^s(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^2\left(\frac{\delta_\Omega(x)}{R}\right) dx \\
&\qquad \sim \int_{F(\Omega\cap\Omega_i)} \frac{|u_i \circ G(\xi)|}{\xi_d^s} \mathcal{L}_1\left(\frac{\xi_d}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\xi_d}{R}\right) \mathcal{L}_m^2\left(\frac{\xi_d}{R}\right) d\xi \\
&\qquad \leq C2^m(1-s)[u_i \circ G]_{W^{s,1}(F(\Omega\cap\Omega_i))} + C2^m\|u_i \circ G\|_{L^1(F(\Omega\cap\Omega_i))} \\
&\qquad = C2^m(1-s)[u_i]_{W^{s,1}(\Omega\cap\Omega_i)} + C2^m\|u_i\|_{L^1(\Omega\cap\Omega_i)}.
\end{aligned}$$

Hence, combining all the above cases, we obtain the following inequality:

$$\begin{aligned}
&\int_{\Omega} \frac{|u(x)|}{\delta_\Omega^s(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^2\left(\frac{\delta_\Omega(x)}{R}\right) dx \\
&\qquad \leq C2^m(1-s)[u]_{W^{s,1}(\Omega)} + C2^m\|u\|_{L^1(\Omega)}.
\end{aligned}$$

Let $\alpha < \frac{1}{2}$, and summing from $m = 2$ to ∞, we have

$$\sum_{m=2}^{\infty} \alpha^m \int_{\Omega} \frac{|u(x)|}{\delta_{\Omega}^s(x)} \mathcal{L}_1 \left(\frac{\delta_{\Omega}(x)}{R}\right) \cdots \mathcal{L}_{m-1} \left(\frac{\delta_{\Omega}(x)}{R}\right) \mathcal{L}_m^2 \left(\frac{\delta_{\Omega}(x)}{R}\right) dx$$
$$\leq C \left(\frac{4\alpha^2}{1-2\alpha}\right) \left\{(1-s)[u]_{W^{s,1}(\Omega)} + \|u\|_{L^1(\Omega)}\right\}.$$

This completes the proof of Theorem 2.

4.2. **Proof of Theorem 1.** Let $u \in BV(\Omega)$. From Theorem 2 and noting that $BV(\Omega) \subset W^{s,1}(\Omega)$ (refer to (2.1)), it follows that

$$\int_{\Omega} \frac{|u(x)|}{\delta_{\Omega}^s(x)} \mathcal{L}_1 \left(\frac{\delta_{\Omega}(x)}{R}\right) \cdots \mathcal{L}_{m-1} \left(\frac{\delta_{\Omega}(x)}{R}\right) \mathcal{L}_m^2 \left(\frac{\delta_{\Omega}(x)}{R}\right) dx$$
$$\leq C2^m (1-s)[u]_{W^{s,1}(\Omega)} + C2^m \|u\|_{L^1(\Omega)}.$$

Using Fatou's lemma, we have

$$\int_{\Omega} \frac{|u(x)|}{\delta_{\Omega}(x)} \mathcal{L}_1 \left(\frac{\delta_{\Omega}(x)}{R}\right) \cdots \mathcal{L}_{m-1} \left(\frac{\delta_{\Omega}(x)}{R}\right) \mathcal{L}_m^2 \left(\frac{\delta_{\Omega}(x)}{R}\right) dx$$
$$\leq \liminf_{s\to 1} \int_{\Omega} \frac{|u(x)|}{\delta_{\Omega}^s(x)} \mathcal{L}_1 \left(\frac{\delta_{\Omega}(x)}{R}\right) \cdots \mathcal{L}_{m-1} \left(\frac{\delta_{\Omega}(x)}{R}\right) \mathcal{L}_m^2 \left(\frac{\delta_{\Omega}(x)}{R}\right) dx$$
$$\leq C2^m \liminf_{s\to 1} (1-s)[u]_{W^{s,1}(\Omega)} + C2^m \|u\|_{L^1(\Omega)}.$$

From Lemma 2.2, we have

$$\int_{\Omega} \frac{|u(x)|}{\delta_{\Omega}(x)} \mathcal{L}_1 \left(\frac{\delta_{\Omega}(x)}{R}\right) \cdots \mathcal{L}_{m-1} \left(\frac{\delta_{\Omega}(x)}{R}\right) \mathcal{L}_m^2 \left(\frac{\delta_{\Omega}(x)}{R}\right) dx \leq C2^m \left([u]_{BV(\Omega)} + \|u\|_{L^1(\Omega)}\right).$$

Therefore, from Lemma 2.4, and using $[u-(u)_{\Omega}]_{BV(\Omega)} = [u]_{BV(\Omega)}$, we have

$$\int_{\Omega} \frac{|u(x)-(u)_{\Omega}|}{\delta_{\Omega}(x)} \mathcal{L}_1 \left(\frac{\delta_{\Omega}(x)}{R}\right) \cdots \mathcal{L}_{m-1} \left(\frac{\delta_{\Omega}(x)}{R}\right) \mathcal{L}_m^2 \left(\frac{\delta_{\Omega}(x)}{R}\right) dx \leq C2^m [u]_{BV(\Omega)}.$$

Let $\alpha < \frac{1}{2}$, and summing from $m = 2$ to ∞, we have

$$\sum_{m=2}^{\infty} \alpha^m \int_{\Omega} \frac{|u(x)-(u)_{\Omega}|}{\delta_{\Omega}(x)} \mathcal{L}_1 \left(\frac{\delta_{\Omega}(x)}{R}\right) \cdots \mathcal{L}_{m-1} \left(\frac{\delta_{\Omega}(x)}{R}\right) \mathcal{L}_m^2 \left(\frac{\delta_{\Omega}(x)}{R}\right) dx$$
$$\leq C \left(\frac{4\alpha^2}{1-2\alpha}\right) [u]_{BV(\Omega)}.$$

This proves our main result, Theorem 1.

5. Failure for $\alpha \geq 1$ in Theorem 1 and Theorem 2 and proof of Remark 1

In this section, we prove the failure of Theorem 1 and 2 for $\alpha \geq 1$. First, we establish the failure in Theorem 1, and then we establish the failure in Theorem 2 using Theorem 1. To prove the failure of our main results when $\alpha \geq 1$, it is sufficient to establish for the domain $\Omega = (-2n, 2n)^{d-1} \times (0, 2)$, where $n \in \mathbb{N}$, and a function supported on $\Omega_n = (-n, n)^{d-1} \times (0, 1)$. Let $u' \in C_c^{\infty}((-n, n)^{d-1})$, and $u_d(x) = 1$ for all $x \in (0, 2)$. For any $x = (x', x_d) \in \Omega$, define

$$u(x) = u'(x')u_d(x_d) = u'(x'). \tag{5.1}$$

Then $u \in BV(\Omega)$, and for any $x \in \Omega_n$, we have $\delta_\Omega(x) = x_d$. From (1.7), we obtain

$$\begin{aligned}
&\int_{\Omega_n} \frac{|u(x)|}{x_d} \mathcal{L}_1\left(\frac{x_d}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{x_d}{R}\right) \mathcal{L}_m^2\left(\frac{x_d}{R}\right) dx \\
&\qquad = \frac{1}{R} \int_{\Omega_n} |u(x)| \frac{d}{dx_d} \mathcal{L}_m\left(\frac{x_d}{R}\right) dx' dx_d \\
&\qquad = \frac{1}{R} \int_{(-n,n)^{d-1}} |u'(x')| dx' \int_0^1 \frac{d}{dx_d} \mathcal{L}_m\left(\frac{x_d}{R}\right) dx_d \\
&\qquad = \frac{1}{R} \left(\mathcal{L}_m\left(\frac{1}{R}\right) - \mathcal{L}_m(0) \right) \int_{(-n,n)^{d-1}} |u'(x')| dx'.
\end{aligned} \tag{5.2}$$

From the definition of $\mathcal{L}_m$, and using (6.3) (see Appendix 6), we have $\mathcal{L}_m\left(\frac{1}{R}\right) \geq \frac{1}{(m+1)R}$ and $\mathcal{L}_m(0) = 0$. Therefore, from above inequality, we have

$$\int_{\Omega_n} \frac{|u(x)|}{x_d} \mathcal{L}_1\left(\frac{x_d}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{x_d}{R}\right) \mathcal{L}_m^2\left(\frac{x_d}{R}\right) dx \geq \frac{1}{(m+1)R^2} \int_{(-n,n)^{d-1}} |u'(x')| dx'.$$

For any $\alpha \geq 1$, we have

$$\begin{aligned}
&\sum_{m=2}^{\infty} \alpha^m \int_{\Omega_n} \frac{|u(x)|}{x_d} \mathcal{L}_1\left(\frac{x_d}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{x_d}{R}\right) \mathcal{L}_m^2\left(\frac{x_d}{R}\right) dx \\
&\qquad \geq \frac{1}{R^2} \int_{(-n,n)^{d-1}} |u'(x')| dx' \sum_{m=2}^{\infty} \frac{\alpha^m}{m+1} = \infty.
\end{aligned} \tag{5.3}$$

This proves that Theorem 1 fails when $\alpha \geq 1$.

We will now establish that the Theorem 2 fails when $\alpha \geq 1$. We will prove by using contradiction. Let $u \in BV(\Omega)$ be a function defined in (5.1). Since $BV(\Omega) \subset W^{s,1}(\Omega)$ (see (2.1)), we have $u \in W^{s,1}(\Omega)$. Assume there exists a constant $C = C(\Omega, d, \alpha) > 0$ and $\alpha \geq 1$ such that

$$\begin{aligned}
&\sum_{m=2}^{\infty} \alpha^m \int_{\Omega} \frac{|u(x)|}{\delta_\Omega^s(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^2\left(\frac{\delta_\Omega(x)}{R}\right) dx \\
&\qquad \leq C(1-s)[u]_{W^{s,1}(\Omega)} + C\|u\|_{L^1(\Omega)}.
\end{aligned}$$

Using Fatou's lemma, and Lemma 2.2, we have for all $m_0 > 2$,

$$\begin{aligned}
&\sum_{m=2}^{m_0} \alpha^m \int_{\Omega} \frac{|u(x)|}{\delta_\Omega(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^2\left(\frac{\delta_\Omega(x)}{R}\right) dx \\
&\quad \leq \sum_{m=2}^{m_0} \alpha^m \liminf_{s \to 1} \int_{\Omega} \frac{|u(x)|}{\delta_\Omega^s(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^2\left(\frac{\delta_\Omega(x)}{R}\right) dx \\
&\quad \leq \liminf_{s \to 1} \sum_{m=2}^{\infty} \alpha^m \int_{\Omega} \frac{|u(x)|}{\delta_\Omega^s(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^2\left(\frac{\delta_\Omega(x)}{R}\right) dx \\
&\quad \leq \liminf_{s \to 1} C(1-s)[u]_{W^{s,1}(\Omega)} + C\|u\|_{L^1(\Omega)} \leq C[u]_{BV(\Omega)} + C\|u\|_{L^1(\Omega)},
\end{aligned}$$

which is a contradiction (see (5.3)). This proves that Theorem 2 fails when $\alpha \geq 1$.

5.1. **Proof of Compactness of operator T.** In this subsection, we prove that the operator T defined in (1.11) is compact. Let $\{u_k\}_{k=1}^{\infty}$ be a bounded sequence of functions in $BV(\Omega)$. Since, the identity operator $I : BV(\Omega) \to L^1(\Omega)$ is compact (see [16, Theorem 5.5]). Therefore, there exists a subsequence of $\{u_k\}_{k=1}^{\infty}$ which is convergent in $L^1(\Omega)$. Therefore, without loss of generality, we assume $\{u_k\}_{k=1}^{\infty}$ be any bounded sequence of functions in $BV(\Omega)$ such that $\|u_k\|_{L^1(\Omega)} \to 0$ as $k \to \infty$. Let $\epsilon > 0$, and $1 < \beta_1 < \beta$, there exists a constant $A_1 = A_1(\epsilon) > 0$ such that for any $x \in \Omega$ with $\delta_\Omega(x) < A_1$, we have $\mathcal{L}_m^{\beta-\beta_1}\left(\frac{\delta_\Omega(x)}{R}\right) < \epsilon$. Then, using this and $\mathcal{L}_m \leq 1$, for all $m \geq 1$, we have

$$
\begin{aligned}
&\int_\Omega \frac{|u_k(x)|}{\delta_\Omega(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^{\beta}\left(\frac{\delta_\Omega(x)}{R}\right) dx \\
&\quad = \int_\Omega \frac{|u_k(x)|}{\delta_\Omega(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^{\beta-\beta_1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^{\beta_1}\left(\frac{\delta_\Omega(x)}{R}\right) dx \\
&\quad \leq \epsilon \int_{\{x\in\Omega \,:\, \delta_\Omega(x)<A_1\}} \frac{|u_k(x)|}{\delta_\Omega(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^{\beta_1}\left(\frac{\delta_\Omega(x)}{R}\right) dx \\
&\qquad + C_1(\epsilon) \int_{\{x\in\Omega \,:\, \delta_\Omega(x)\geq A_1\}} |u_k(x)| dx \\
&\quad \leq \epsilon \int_\Omega \frac{|u_k(x)|}{\delta_\Omega(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^{\beta_1}\left(\frac{\delta_\Omega(x)}{R}\right) dx + C_1(\epsilon) \int_\Omega |u_k(x)| dx.
\end{aligned}
$$

In the above inequality, we used that for any $x \in \Omega$ with $\delta_\Omega(x) \geq A_1$, we have $\frac{1}{\delta_\Omega(x)} \leq C_1(\epsilon)$, for some $C_1(\epsilon) > 0$ depending only on ϵ. Using Corollary 1.1 with $\beta_1 > 1$ in the above inequality, and $[u_k]_{BV(\Omega)} \leq C$ for all $k \in \mathbb{N}$ and for some $C > 0$, we obtain

$$
\begin{aligned}
\int_\Omega \frac{|u_k(x)|}{\delta_\Omega(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^{\beta}\left(\frac{\delta_\Omega(x)}{R}\right) dx &\leq C\epsilon [u_k]_{BV(\Omega)} \\
&\quad + C_1(\epsilon) \|u_k\|_{L^1(\Omega)} \\
&\leq C\epsilon + C_1(\epsilon) \|u_k\|_{L^1(\Omega)},
\end{aligned}
$$

where $C > 0$ is a constant does not depend on k and ϵ. Taking the limit as $k \to \infty$, and then $\epsilon \to 0$, we obtain

$$
\int_\Omega \frac{|u_k(x)|}{\delta_\Omega(x)} \mathcal{L}_1\left(\frac{\delta_\Omega(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_\Omega(x)}{R}\right) \mathcal{L}_m^{\beta}\left(\frac{\delta_\Omega(x)}{R}\right) dx \to 0.
$$

This proves that the operator T (defined in (1.11)) is compact.

6. Appendix

6.1. **Domain above the graph of a Lipschitz function.** In this section, we will prove that if Ω is a domain above the graph of a Lipschitz function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$, and $F : \mathbb{R}^d \to \mathbb{R}^d$ is defined by $F(x) = (\xi', \xi_d)$, where $\xi' = x'$ and $\xi_d = x_d - \gamma(x')$. Then,

$$
\delta_\Omega(x) \sim \xi_d,
$$

holds for all $x \in \Omega$. Specifically, there exist constants $C_1, C_2 > 0$ such that

$$
C_1 \xi_d \leq \delta_D(x) \leq C_2 \xi_d.
$$

Let $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ be a Lipschitz function, and let $M > 0$ be such that for $x', y' \in \mathbb{R}^{d-1}$, we have

$$
|\gamma(x') - \gamma(y')| \leq M|x' - y'|.
$$

Let $F:\mathbb{R}^d\to\mathbb{R}^d$ be defined as $F(x)=(F_1(x),\dots,F_d(x))=(x',x_d-\gamma(x'))$, where $x'=(x_1,\dots,x_{d-1})$. Then,

$$\begin{aligned}|F(x)-F(y)|^2 &= |x'-y'|^2+|x_n-y_n-\gamma(x')+\gamma(y')|^2\\ &\le |x'-y'|^2+|x_d-y_d|^2+|\gamma(x')-\gamma(y')|^2+2<x_d-y_d,\gamma(y')-\gamma(x')>\\ &\le |x-y|^2+M^2|x'-y'|^2+|x_d-y_d|^2+|\gamma(x')-\gamma(y')|^2\\ &\le |x-y|^2+2M^2|x'-y'|^2+|x_d-y_d|^2\le (2M^2+2)|x-y|^2.\end{aligned}$$

Let $C=(2M^2+2)^{1/2}$. Then, $|F(x)-F(y)|\le C|x-y|$. Define $G(\xi)=F^{-1}(\xi)=(\xi',\xi_d+\gamma(\xi'))$. Then, G is Lipschitz, and $|G(\xi)-G(\eta)|\le (2M^2+2)^{1/2}|\xi-\eta|$. Hence, there exists $C>0$ such that

$$\frac{1}{C}|x-y|\le |F(x)-F(y)|\le C|x-y|.$$

Let $\Omega=\{x\in\mathbb{R}^d: x_d>\gamma(x')\}$ and $\partial\Omega=\{x\in\mathbb{R}^d: x_d=\gamma(x')\}$. Then, $F(\Omega)=\mathbb{R}^d_+$ and $F(\partial\Omega)=\partial\mathbb{R}^d_+$. Let $x\in\Omega$ and $y\in\partial\Omega$ be such that

$$\delta_\Omega(x)=|x-y|=\inf\{|x-\eta|:\eta\in\partial\Omega\}.$$

Then, $\delta_\Omega(x)=|x-y|\le|x-\eta|$ for all $\eta\in\partial\Omega$. Therefore,

$$\delta_\Omega(x)=|x-y|\le C|F(x)-F(\eta)|\le C|F(x)-\xi|,$$

for all $\xi\in\partial\mathbb{R}^d_+$. So, $\delta_\Omega(x)\le C\inf_{\xi\in\partial\mathbb{R}^d_+}|F(x)-\xi|=CF_d(x)$. Let $F(x)=(\xi',\xi_d)$. Then, we have $\delta_\Omega(x)\le C\xi_d$. Similarly, considering G, we get $C_1\xi_d\le\delta_\Omega(x)$. Therefore, $C_1\xi_d\le\delta_\Omega(x)\le C\xi_d$.

6.2. **Some estimates.** **(1)** Let $\theta>0$ and for any $m\ge1$, we establish that there exists a constant $C=C(\theta)>0$ such that

$$\mathcal{L}_m^\theta(t)\le C\mathcal{L}^2_{m+1}(t),\quad \forall\, t\in(0,1). \tag{6.1}$$

First, assume $0<\theta\le1$, and let $\mathcal{L}_m(t)=e^{-x}$. Then, if $t=0$, $x\to\infty$, and if $t=1$, $x=0$. Define

$$g_\theta(x):=\frac{e^{\theta x}}{(1+x)^2}=\left(\frac{1}{1-\ln(\mathcal{L}_m(t))}\right)^2\frac{1}{\mathcal{L}_m^\theta(t)}=\frac{\mathcal{L}^2_{m+1}(t)}{\mathcal{L}^\theta_m(t)},\quad\forall\, x\in(0,\infty).$$

Clearly,

$$\frac{\mathcal{L}^2_{m+1}(t)}{\mathcal{L}^\theta_m(t)}\ge\min_{x\in(0,\infty)}g_\theta(x)=g_\theta\left(-1+\frac{2}{\theta}\right)=\left(\frac{\theta}{2}\right)^2e^{2-\theta}.$$

Therefore, we have

$$\mathcal{L}^\theta_m(t)\le\left(\frac{2}{\theta}\right)^2e^{\theta-2}\mathcal{L}^2_{m+1}(t),\quad\forall\, t\in(0,1). \tag{6.2}$$

Now, assume $\theta>1$. Then, $\theta=n_1+r$, where $r\in(0,1]$. Therefore, using $\mathcal{L}_m(t)\le1$ and above inequality, we have

$$\mathcal{L}^\theta_m(t)=\mathcal{L}^{n_1}_m(t)\mathcal{L}^r_m(t)\le\mathcal{L}^r_m(t)\le\left(\frac{2}{r}\right)^2e^{r-2}\mathcal{L}^2_{m+1}(t)$$

This establishes the required inequality for any $\theta>0$.

(2) Let $\mathcal{L}_m$ be defined in the introduction section, and let $R>1$. We prove that

$$\mathcal{L}_m\left(\frac{1}{R}\right)\ge\frac{1}{(m+1)R}. \tag{6.3}$$

Since $1 - \ln\left(\frac{1}{R}\right) = 1 + \ln R \leq 2R$, we have

$$\mathcal{L}_1\left(\frac{1}{R}\right) = \frac{1}{1 - \ln\left(\frac{1}{R}\right)} \geq \frac{1}{2R}.$$

From above inequality, we have $\ln\left(\mathcal{L}_1\left(\frac{1}{R}\right)\right) \geq \ln\left(\frac{1}{2R}\right)$. Therefore, we get

$$1 - \ln\left(\mathcal{L}_1\left(\frac{1}{R}\right)\right) \leq 1 - \ln\left(\frac{1}{2R}\right) = 1 + \ln(2R) \leq 3R.$$

Using the definition of $\mathcal{L}_2$, we have

$$\mathcal{L}_2\left(\frac{1}{R}\right) = \frac{1}{1 - \ln\left(\mathcal{L}_1\left(\frac{1}{R}\right)\right)} \geq \frac{1}{3R}.$$

From the definition of $\mathcal{Y}_m$, and using recursively, we obtain

$$\mathcal{L}_m\left(\frac{1}{R}\right) \geq \frac{1}{(m+1)R}.$$

This establishes the inequality.

(3) Let $\Omega_n = (-n, n)^{d-1} \times (0, 1)$, and A_k as defined in Lemma 4.1. We aim to show that

$$\sum_{k=\ell}^{-2} [u]_{W^{s,1}(A_k \cup A_{k+1})} \leq 2[u]_{W^{s,1}(\Omega_n)}. \tag{6.4}$$

Consider two families of sets:

$$\mathcal{E} := \{A_k \cup A_{k+1} : -k \text{ is even and } k \leq -1\}$$

and

$$\mathcal{O} := \{A_k \cup A_{k+1} : -k \text{ is odd and } k \leq -1\}.$$

Then $\mathcal{E}$ and $\mathcal{O}$ are collections of mutually disjoint sets respectively. Define

$$\mathcal{F}_e := \bigcup_{\substack{k=\ell \\ -k \text{ is even}}}^{-2} A_k \cup A_{k+1} \quad \text{and} \quad \mathcal{F}_o := \bigcup_{\substack{k=\ell \\ -k \text{ is odd}}}^{-2} A_k \cup A_{k+1}.$$

From the definition of A_k, we have $\mathcal{F}_e \subset \Omega_n$ and $\mathcal{F}_o \subset \Omega_n$. Therefore, we have

$$\begin{aligned} \sum_{k=\ell}^{-2} [u]_{W^{s,1}(A_k \cup A_{k+1})} &= \sum_{\substack{k=\ell \\ -k \text{ is even}}}^{-2} [u]_{W^{s,1}(A_k \cup A_{k+1})} + \sum_{\substack{k=\ell \\ -k \text{ is odd}}}^{-2} [u]_{W^{s,1}(A_k \cup A_{k+1})} \\ &\leq [u]_{W^{s,1}(\mathcal{F}_e)} + [u]_{W^{s,1}(\mathcal{F}_o)} \leq 2[u]_{W^{s,1}(\Omega_n)}. \end{aligned} \tag{6.5}$$

This establishes the desired inequality.

(4) Let A_k^i and A_{k+1}^j be the cubes defined in Lemma 4.1 such that A_k^i lies below the cube A_{k+1}^j. Let G_{k+1}^j be a cube of side length $2 \times 3^{k+1} + 2 \times 3^k$ such that $A_k^i \cup A_{k+1}^j \subset G_{k+1}^j$ and $G_{k+1}^j \subset A_k \cup A_{k+1}$. Also, there are 3^{d-1} cubes of side length 2×3^k (like A_k^i) lies below the cube A_{k+1}^j and the same cube G_{k+1}^j will work for all 3^{d-1} such cubes (like A_k^i). Therefore, we will establish:

$$\sum_{j=1}^{\sigma_{k+1}} [u]_{W^{s,1}(G_{k+1}^j)} \leq 2[u]_{W^{s,1}(A_k \cup A_{k+1})}. \tag{6.6}$$

According to the construction of G^j_{k+1}, the families of sets $\{G^j_{k+1} :\ j$ is even and $1 \leq j \leq \sigma_{k+1}\}$ and $\{G^j_{k+1} :\ j$ is odd and $1 \leq j \leq \sigma_{k+1}\}$ are collections of mutually disjoint sets respectively. Therefore, similarly as the previous case, we have

$$\sum_{j=1}^{\sigma_{k+1}} [u]_{W^{s,1}(G^j_{k+1})} = \sum_{\substack{j=1 \\ j \text{ is even}}}^{\sigma_{k+1}} [u]_{W^{s,1}(G^j_{k+1})} + \sum_{\substack{j=1 \\ j \text{ is odd}}}^{\sigma_{k+1}} [u]_{W^{s,1}(G^j_{k+1})} \leq 2[u]_{W^{s,1}(A_k \cup A_{k+1})}.$$

This establishes our inequality.

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