SPECTRAL MULTIPLIERS IN A GENERAL GAUSSIAN SETTING

VALENTINA CASARINO, PAOLO CIATTI AND PETER SJÖGREN

Abstract. We investigate a class of spectral multipliers for an Ornstein–Uhlenbeck operator $\mathcal{L}$ in $\mathbb{R}^n$, with drift given by a real matrix B whose eigenvalues have negative real parts. We prove that if m is a function of Laplace transform type defined in the right half-plane, then $m(\mathcal{L})$ is of weak type $(1,1)$ with respect to the invariant measure in $\mathbb{R}^n$. The proof involves many estimates of the relevant integral kernels and also a bound for the number of zeros of the time derivative of the Mehler kernel, as well as an enhanced version of the Ornstein–Uhlenbeck maximal operator theorem.

1. Introduction

Given a measure space (X,μ) and a self-adjoint operator L on $L^2(X,\mu)$, an important issue in harmonic analysis concerns the boundedness of the operator $m(L)$, where $m:\mathbb{R}\to\mathbb{C}$ is a Borel function. If E denotes a spectral resolution of L on $\mathbb{R}$, one can define $m(L)$ for many functions m as

$$m(L) = \int_{\mathbb{R}} m(\nu)\,dE(\nu).$$

Great efforts have been devoted to finding minimal assumptions on the multiplier m that will ensure the boundedness of $m(L)$ on the Lebesgue spaces $L^p(X,\mu)$, both in a strong and in a weak sense, when $p\neq 2$.

A few years ago, the authors started a program concerning harmonic analysis in the Ornstein–Uhlenbeck setting. In this framework, (X,μ) is the Euclidean space $\mathbb{R}^n$ equipped with a Gaussian measure $d\gamma_\infty$, known as the invariant measure and defined in Section 2. Further, L is replaced by the Ornstein–Uhlenbeck operator $\mathcal{L}$, defined as

$$\mathcal{L}f = -\frac{1}{2}\,\mathrm{tr}\left(Q\nabla^2 f\right) - \langle Bx,\nabla f\rangle\,, \qquad f\in\mathcal{S}(\mathbb{R}^n), \tag{1.1}$$

where ∇ and ∇^2 denote the gradient and the Hessian, respectively. In this formula, Q and B are real $n\times n$ matrices; Q is symmetric and positive definite, and the

Date: December 11, 2024.

2000 *Mathematics Subject Classification.* 42B15, 47D03, 15A15.

Key words and phrases. Spectral multipliers, Ornstein–Uhlenbeck operator, Laplace transform type functions, weak-type bounds, invariant measure.

The first and second authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and were partially supported by GNAMPA (Project 2020 "Alla frontiera tra l'analisi complessa in più variabili e l'analisi armonica"). Since this work was mainly carried out during a visit to Padova by the third author, he would like to thank the University of Padova for its generosity.

eigenvalues of B all have negative real parts. The space $L^p(\mathbb{R}^n, d\gamma_\infty)$ will be written simply $L^p(\gamma_\infty)$.

Since in general $\mathcal{L}$ has no self-adjoint or normal extension to $L^2(\gamma_\infty)$, one cannot invoke spectral theory to define $m(\mathcal{L})$. Notice that self-adjointness and normality may fail also for the Ornstein–Uhlenbeck semigroup $(\mathcal{H}_t)_{t>0}$, generated by $\mathcal{L}$, which was first introduced in [34]. The focus in this paper is on multipliers of Laplace transform type. This class of multipliers was introduced some fifty years ago by E. M. Stein in [36], in the context of Littlewood–Paley theory for a sublaplacian on a connected Lie group G.

A function m of a real variable $\lambda > 0$ is said to be of Laplace transform type if

$$m(\lambda) = \lambda \int_0^{+\infty} \varphi(t) e^{-t\lambda}\, dt = -\int_0^{+\infty} \varphi(t)\, \frac{d}{dt}\, e^{-t\lambda}\, dt, \qquad \lambda > 0, \tag{1.2}$$

for some $\varphi \in L^\infty(0, +\infty)$. Observe that such a function m can be extended to an analytic function in the half-plane $\Re z > 0$. Thus we pay the price of a rather strong condition on m, to prove, in return, a multiplier theorem for an operator $\mathcal{L}$ which is not necessarily normal. Observe that one obtains as $m(\mathcal{L})$ the imaginary powers $\mathcal{L}^{i\gamma}$ of $\mathcal{L}$, with $\gamma \in \mathbb{R} \setminus \{0\}$, by choosing $\varphi(t) = \text{const.}\, t^{-i\gamma}$. Other significant examples of functions of Laplace transform type may be found in [41].

The exact definition of $m(\mathcal{L})$ for functions m of this type will be given in Section 3. Here we present only a heuristic deduction of the kernel of $m(\mathcal{L})$. If we simply replace λ by $\mathcal{L}$ in the last expression in (1.2), we will get

$$m(\mathcal{L}) = -\int_0^{+\infty} \varphi(t)\, \frac{d}{dt}\, e^{-t\mathcal{L}}\, dt. \tag{1.3}$$

Here $e^{-t\mathcal{L}} = \mathcal{H}_t$ is the Ornstein-Uhlenbeck semigroup, whose kernel is the Mehler kernel $K_t(x, u)$ described in Section 2. We point out that the term kernel in this paper refers to integration with respect to $d\gamma_\infty$, except in one case as explained in Section 8. Thus for each $f \in \mathcal{S}(\mathbb{R}^n)$ and all $t > 0$

$$\mathcal{H}_t f(x) = \int K_t(x, u)\, f(u)\, d\gamma_\infty(u)\,.$$

This makes it plausible that the off-diagonal kernel of $m(\mathcal{L})$ is

$$\mathcal{M}_\varphi(x, u) = -\int_0^{+\infty} \varphi(t)\, \partial_t K_t(x, u)\, dt. \tag{1.4}$$

We will verify this formula later, though after splitting the integral and under some restrictions. It will lead to an expression for the kernel in terms of Q and B.

From now on, we assume that m is of Laplace transform type.

In the standard case $Q = I$ and $B = -I$, the operator $\mathcal{L}$ is self-adjoint, and the $L^p(d\gamma_\infty)$ boundedness of $m(\mathcal{L})$ follows for all $1 < p < \infty$ from a general result due to Stein [36, Ch. 4]. Moreover, J. García-Cuerva, G. Mauceri, J. L.Torrea and the third author proved in this case the weak type $(1, 1)$ of $m(\mathcal{L})$ with respect to $d\gamma_\infty$; see [24, Theorem 3.8]. For more recent results in the standard case, also involving the Gaussian conical square function, we refer to [26, 27]; see also [39, 40], where the author investigates multiplier theorems for systems of Ornstein–Uhlenbeck operators. Overviews of this topic can be found in Urbina's monograph [38, Chapter 6], Bogachev's survey [6] and the references therein.

In the general case, when $\mathcal{L}$ is given by (1.1), the strong $L^p(\gamma_\infty)$ boundedness of $m(\mathcal{L})$ follows for $1 < p < \infty$ from [8, Prop. 3.8]. In the present paper, we consider the endpoint case $p = 1$, where the strong boundedness does not hold.

Our main result is the following.

Theorem 1.1. *If the function m is of Laplace transform type, then the multiplier operator $m(\mathcal{L})$ associated to a general Ornstein–Uhlenbeck operator $\mathcal{L}$ is of weak type $(1,1)$ with respect to the invariant measure $d\gamma_\infty$.*

Thus we shall prove the inequality

$$\gamma_\infty\{x \in \mathbb{R}^n : m(\mathcal{L})\, f(x) > C\alpha\} \le \frac{C}{\alpha}\, \|f\|_{L^1(\gamma_\infty)}, \qquad \alpha > 0, \tag{1.5}$$

for all functions $f \in L^1(\gamma_\infty)$, with $C = C(n, Q, B)$. Our theorem extends Theorem 3.8 in [24] to the framework of a general, not necessarily normal, Ornstein–Uhlenbeck operator.

In this paper, we do not deal with holomorphic Hörmander-type functional calculus. For results in that context the reader is referred in particular to [8] and to [25]. The literature in this field is vast; good bibliographies are given in [8, 25, 35].

A careful study of other exponential integral inequalities in the Gaussian framework may be found in [17, 18, 19]. For strong and weak bounds of Laplace type multipliers between Lebesgue spaces in contexts different from the Ornstein–Uhlenbeck setting, we refer, in particular, to [5, 21, 37, 33, 4]). The main semigroups and operators introduced in these papers, anyway, are symmetric, so that, unlike our case, a spectral decomposition is allowed. In this regard, it is worth mentioning that several interesting results concerning other issues of harmonic analysis in a nonsymmetric Ornstein–Uhlenbeck context, such as square functions, maximal operators and variational bounds, have recently appeared in [1, 2].

What follows next is a description of the structure of the paper, which also gives a plan of the proof of Theorem 1.1.

In Section 2, we introduce some terminology and recall from the authors' earlier papers [9, 10, 11] a few estimates which are essential in our approach. Section 3 gives a rigorous definition of the multiplier operator, and in Subsection 3.2 we split this operator by splitting the integrals in (1.2) and (1.3) into parts taken over $t < 1$ and $t > 1$. Then in Section 4 the time derivative $\partial_t K_t$ of the Mehler kernel is computed and estimated. This leads in Section 5 to some estimates for the kernels of the different parts of the operator. There we also introduce some technical simplifications that will reduce the complexity of the proof of Theorem 1.1; a further reduction will be presented in Subsection 7.2. This proof is given in the remaining sections, in the following way.

The operator part with $t > 1$ is dealt with in Section 6. The part corresponding to $t < 1$ is further split into a local and a global part in Section 7, and several related estimates are given. Section 8 contains the proof for the local part with standard Calderón-Zygmund techniques. The remaining, global part is more delicate. For its kernel we will have a bound

$$\int_0^1 |\partial_t K_t(x,u)|\, dt \le \sum \left| \int \partial_t K_t(x,u)\, dt \right|,$$

where the integrals in the sum are taken between consecutive zeros of $\partial_t K_t$. Therefore, we will need an estimate of the number of zeros of $\partial_t K_t(x,u)$ as t runs through the interval $(0,1]$. This number turns out to be controlled by a constant depending only on n and B, as verified in Section 9. We can then complete the proof of the weak type $(1,1)$ in Section 10. There we also need an enhanced version of the Ornstein–Uhlenbeck maximal operator theorem from [CCS2, Theorem 1.1]. Its proof is given in the Appendix (Section 11).

We will write $C < \infty$ and $c > 0$ for various constants, all of which depend only on n, Q and B, unless otherwise explicitly stated. If a and b are positive quantities, $a \lesssim b$ or equivalently $b \gtrsim a$ means $a \le Cb$. When $a \lesssim b$ and also $b \lesssim a$, we write $a \simeq b$. By $\mathbb{N}$ we denote the set of all nonnegative integers, and $B(x,r)$ is the open ball with center x and radius r. If A is a real $n \times n$ matrix, we write $\|A\|$ for its operator norm on $\mathbb{R}^n$ with the Euclidean norm $|\cdot|$. We will adopt the dot notation for differentiation with respect to the time variable t, writing $\dot{K}_t = \partial_t K_t$.

The authors would like to thank Andrea Carbonaro for several helpful discussions.

2. Preliminaries

In this section we collect some results from [9, 10, 11] related to the Mehler kernel of a general Ornstein–Uhlenbeck semigroup.

2.1. **Some matrices and estimates.**

In terms of the two real $n \times n$ matrices Q and B introduced in Section 1, we define for $t \in (0, +\infty]$ the matrix

$$Q_t = \int_0^t e^{sB}\, Q\, e^{sB^*}\, ds. \tag{2.1}$$

Since Q is real, symmetric and positive definite and the eigenvalues of B have negative real parts, this integral is convergent and the matrix Q_t is symmetric and positive definite and thus invertible, for all $0 < t \le \infty$.

It will be convenient to write

$$|x|_Q = |Q_\infty^{-1/2}\, x|, \qquad x \in \mathbb{R}^n,$$

which is a norm on $\mathbb{R}^n$, and $|x|_Q \simeq |x|$. Further, we let $R(x)$ denote the (positive definite) quadratic form

$$R(x) = \frac{1}{2}\, |x|_Q^2 = \frac{1}{2}\, \langle Q_\infty^{-1}\, x, x \rangle, \qquad x \in \mathbb{R}^n.$$

The invariant measure is given by

$$d\gamma_\infty(x) = (2\pi)^{-n/2} \big(\det Q_\infty \big)^{-1/2} \exp(-R(x))\, dx.$$

Notice that $d\gamma_\infty$ is normalized.

We will also use the one-parameter group of matrices

$$D_t = Q_\infty \, e^{-tB^*} \, Q_\infty^{-1}, \qquad t \in \mathbb{R}, \tag{2.2}$$

introduced in [10]. From [10, formula (2.3) and Lemma 2.1] we know that

$$D_t = \left(Q_t^{-1} - Q_\infty^{-1}\right)^{-1} Q_t^{-1} \, e^{tB}, \qquad t > 0. \tag{2.3}$$

and

$$D_t = e^{tB} + Q_t \, e^{-tB^*} \, Q_\infty^{-1}, \qquad t > 0. \tag{2.4}$$

By means of a Jordan decomposition of B^*, the following estimates were proved in [10, Lemma 3.1]

$$e^{ct}\,|x| \lesssim |D_t\, x| \lesssim e^{Ct}\,|x| \qquad \text{and} \qquad e^{-Ct}\,|x| \lesssim |D_{-t}\, x| \lesssim e^{-ct}\,|x|,$$

holding for $t > 0$ and all $x \in \mathbb{R}^n$. The same bounds are true with D_t replaced by e^{-tB} or e^{-tB^*}; in particular,

$$e^{ct}\,|x| \lesssim |e^{-tB}\, x| \lesssim e^{Ct}\,|x| \qquad \text{and} \qquad e^{-Ct}\,|x| \lesssim |e^{tB}\, x| \lesssim e^{-ct}\,|x| \tag{2.5}$$

for $t > 0$ and $x \in \mathbb{R}^n$.

From these inequalities one deduces (see [10, Lemma 3.2])

$$\|Q_t^{-1}\| \simeq (\min(1,t))^{-1}, \tag{2.6}$$

$$\|Q_t^{-1} - Q_\infty^{-1}\| \lesssim t^{-1}\, e^{-ct}. \tag{2.7}$$

Finally, we recall the following lemma, proved in [11, Lemma 2.3].

Lemma 2.1. *Let $x \in \mathbb{R}^n$ and $|t| \leq 1$. Then*

$$|x - D_t\, x| \simeq |t|\,|x|.$$

2.2. **Spectrum and generalized eigenspaces of $\mathcal{L}$.**

Let $\lambda_1, \dots, \lambda_r$ be the eigenvalues of B. It is known that the spectrum of $\mathcal{L}$ in $L^p(\gamma_\infty)$, $1 < p < \infty$, is

$$\left\{ -\sum_{i=1}^{r} n_i\, \lambda_i : n_i \in \mathbb{N},\ i = 1, \dots, r \right\} \subset \{z \in \mathbb{C} : \Re z > 0\} \cup \{0\},$$

see [31, Theorem 3.1].

Each point λ in this set is an eigenvalue of $\mathcal{L}$. The corresponding generalized eigenfunctions, i.e., the functions annihilated by $(\mathcal{L} - \lambda)^k$ for some $k \in \mathbb{N}$, are polynomials, see [28, Theorem 9.3.20]. For each λ they form a finite-dimensional space, and these generalized eigenspaces together span a dense subspace of $L^2(\gamma_\infty)$. In particular, 0 is an eigenvalue of $\mathcal{L}$. The corresponding eigenspace, which we denote by $\mathcal{E}_0$, is of dimension 1 and consists of the constant functions. As shown in [12, Lemma 2.1], this eigenspace is orthogonal to all other generalized eigenfunctions of $\mathcal{L}$. We denote by $L^2_0(\gamma_\infty)$ the orthogonal complement of $\mathcal{E}_0$ in $L^2(\gamma_\infty)$.

2.3. **The Mehler kernel.** For $x, u \in \mathbb{R}^n$ and $t > 0$ the Mehler kernel K_t is given by (see [10, formula (2.6)])

$$K_t(x,u) = \Big(\frac{\det\, Q_\infty}{\det\, Q_t}\Big)^{1/2} e^{R(x)}\, \exp\Big[-\frac{1}{2}\left\langle (Q_t^{-1} - Q_\infty^{-1})(u - D_t\, x)\,,\, u - D_t\, x\right\rangle\Big]. \quad (2.8)$$

It is convenient to use this expression for K_t when $t \le 1$. But for $t \ge 1$, we will use the following alternative, which can be obtained from [10, first formula in the proof of Proposition 3.3],

$$K_t(x,u) = \Big(\frac{\det Q_\infty}{\det Q_t}\Big)^{1/2} e^{R(x)}\, \exp\Big[-\frac{1}{2}\left\langle Q_t^{-1} e^{tB}(D_{-t}\, u - x), D_t\,(D_{-t}\, u - x)\right\rangle\Big]. \quad (2.9)$$

For $0 < t \le 1$ we have the following estimates, proved in [11, (2.10)]

$$\frac{e^{R(x)}}{t^{n/2}}\, \exp\Big[-C\, \frac{|u - D_t\, x|^2}{t}\Big] \lesssim K_t(x,u) \lesssim \frac{e^{R(x)}}{t^{n/2}}\, \exp\Big[-c\, \frac{|u - D_t\, x|^2}{t}\Big]. \quad (2.10)$$

When $t \ge 1$ one has instead (see [11, (2.11)])

$$e^{R(x)}\, \exp\Big[-C\, |D_{-t}\, u - x|_Q^2\Big] \lesssim K_t(x,u) \lesssim e^{R(x)}\, \exp\Big[-\frac{1}{2}\, |D_{-t}\, u - x|_Q^2\Big]. \quad (2.11)$$

We will actually use only the upper estimates here.

2.4. **Polar coordinates.** We will use a variant of polar coordinates first introduced in [9]. Fix $\beta > 0$ and consider the ellipsoid

$$E_\beta = \{x \in \mathbb{R}^n : \, R(x) = \beta\}\,.$$

Any $x \in \mathbb{R}^n$, $x \ne 0$, can be written uniquely as

$$x = D_s\, \tilde{x}\,,$$

for some $\tilde{x} \in E_\beta$ and $s \in \mathbb{R}$. We call $(s, \tilde{x})$ the polar coordinates of x.

The Lebesgue measure in $\mathbb{R}^n$ is given in terms of $(s, \tilde{x})$ by

$$dx = e^{-s\,\mathrm{tr}\, B}\, \frac{|Q^{1/2}\, Q_\infty^{-1} \tilde{x}|^2}{2\, |Q_\infty^{-1} \tilde{x}|}\, dS_\beta(\tilde{x})\, ds\,, \quad (2.12)$$

where dS_β denotes the area measure of E_β. See [10, Proposition 4.2] for a proof.

3. Definition and splitting of the multiplier operator

3.1. **Definition of the multiplier operator.** We use the definition described in Cowling et al. [14, Section 2], which goes back to McIntosh [29]. The starting-point in [14] is an operator T defined on a Hilbert (or Banach) space, which in our case will be $L^2_0(\gamma_\infty)$, defined below. This operator is to be densely defined and one-to-one with dense range, and its spectrum must be contained in a closed sector

$$S_\omega = \{z \in \mathbb{C} : |\mathrm{arg} z| \le \omega\} \cup \{0\},$$

for some $\omega \in (0, \pi/2)$. Further, the resolvent of T should satisfy the estimate

$$\|(T - zI)^{-1}\| \le C\, |z|^{-1}, \qquad z \in \mathbb{C} \setminus S_\omega, \quad (3.1)$$

for some constant C, where we refer to the operator norm on the Hilbert space.

In order to apply the construction in [14, Section 2], we denote by $\mathcal{L}_2$ the generator in $L^2(\gamma_\infty)$ of the semigroup $\mathcal{H}_t$. This semigroup is strongly continuous on $L^2(\gamma_\infty)$, and its generator, as defined for instance in [22, Definition II.1.2], is an unbounded, closed operator in $L^2(\gamma_\infty)$, see [22, Theorem II.1.4]. Its domain $\mathcal{D}(\mathcal{L}_2) = W^{2,2}_{\gamma_\infty}$ is a Sobolev space adapted to $d\gamma_\infty$, as verified in [32] or [28, Section 9.3]).

The invariance of the measure $d\gamma_\infty$ means that for any $f \in L^2(\gamma_\infty)$ and any $t > 0$

$$\langle \mathcal{H}_t f, 1\rangle = \langle f, 1\rangle.$$

Thus the orthogonal complement in $L^2(\gamma_\infty)$ of the subspace of constant functions, which we denote by $L^2_0(\gamma_\infty)$, is invariant under the semigroup. The definition of the generator says that for $f \in \mathcal{D}(\mathcal{L}_2)$

$$\mathcal{L}_2 f = \lim_{t\to 0} \frac{1}{t}\left(\mathcal{H}_t f - f\right),$$

where the limit is taken in $L^2(\gamma_\infty)$. It follows that $\langle \mathcal{L}_2 f, 1\rangle = 0$, and we see that the restriction T of $\mathcal{L}_2$ to $L^2_0(\gamma_\infty) \cap D(\mathcal{L}_2)$ is an unbounded, closed operator on $L^2_0(\gamma_\infty)$. Further, T is the generator of the semigroup given by the restriction of each $\mathcal{H}_t$ to $L^2_0(\gamma_\infty)$.

We will prove Theorem 1.1 with $\mathcal{L}_2$ replaced by T. The theorem then follows, since $\mathcal{L}_2$ vanishes on $\mathcal{E}_0$.

From the preceding section, it is clear that T has all the properties required in [14] mentioned above, except possibly the inequality (3.1).

Aiming at (3.1), we invoke [15, Theorem 1 and Remark 6]. This yields the existence of an angle $\theta_2 \in (0, \pi/2)$ such that for each t in the sector S_{θ_2} the operator $\mathcal{H}_t$ is a contraction on $L^2(\gamma_\infty)$, and the map $t \mapsto \mathcal{H}_t$ is analytic as a map from S_{θ_2} to the space of bounded linear operators on $L^2(\gamma_\infty)$. Moreover, the restriction of $\mathcal{H}_t$ to $L^2_0(\gamma_\infty)$ has the same two properties with respect to $L^2_0(\gamma_\infty)$. Then (3.1) follows from well-known arguments for bounded analytic semigroups (see [22, Theorem II.4.6]). Anyway, we give a concise proof of (3.1).

Fix a $\theta \in (0, \theta_2)$; like θ_2 this θ will only depend on n, Q and B. If z is on the negative real axis, we obtain from [20, Theorem 2.8] and the contraction property

$$(T - zI)^{-1} = \int_0^{+\infty} e^{-t(T-zI)}\, dt = e^{i\theta} \int_0^{+\infty} e^{-te^{i\theta}T}\, e^{te^{i\theta}zI}\, dt; \tag{3.2}$$

for the second equality we moved the path of integration to the ray $e^{i\theta}\,\mathbb{R}_+$ in $\mathbb{C}$. Here we want to let $z = re^{i\varphi}$, with $r > 0$ and $\varphi \in (\pi/2 - \theta/2, \pi]$. Then

$$0 < \theta/2 < \theta + \varphi - \pi/2 \;\le\; \theta + \pi/2 < \pi$$

and so

$$\Re(t\, e^{i\theta} z) = t\, r \cos(\theta + \varphi) = -t\, r \sin(\theta + \varphi - \pi/2) < -c\, t\, r.$$

For such z the second integral in (3.2) converges, and by analyticity it equals $e^{-i\theta}\,(T - zI)^{-1}$. Thus

$$\|(T - zI)^{-1}\| \le \int_0^{+\infty} e^{-ctr}\, dt \le \frac{C}{|z|},$$

which proves (3.1) for z in the upper half-plane, with $\omega = \pi/2 - \theta/2$. To deal with the case when z is in the lower half-plane, it is enough to take the complex conjugate of the equation (3.2) and repeat the argument, because T is real. Thus (3.1) is verified.

We now claim that 0 is not in the spectrum of T. This follows from the following facts: first, one observes that the semigroup operator $\mathcal{H}_t$ is compact for each $t > 0$ (see [31, p. 45]). The restriction of $\mathcal{H}_t$ to $L^2_0(\gamma_\infty)$ is compact as well, and [20, Theorem 2.20] implies that the generator T consists only of eigenvalues. As a consequence, 0, which is not an eigenvalue for T, does not belong to the spectrum of T.

The resolvent $(T - zI)^{-1}$ is an analytic function of z in the resolvent set, see e.g. [20, Lemma 2.2] In particular, its norm is bounded in a neighborhood of the origin. This leads to the following improvement of (3.1):

$$\|(T - zI)^{-1}\| \le C\,(1 + |z|)^{-1}, \qquad z \in \mathbb{C} \setminus S^*_\omega, \tag{3.3}$$

where $S^*_\omega = \{w \in S_\omega : \Re w \ge \varepsilon\}$ for some $\varepsilon > 0$.

The function m is of Laplace transform type and thus defined and analytic in the right half-plane. Moreover, it is bounded on any sector S_ϕ with $0 < \phi < \pi/2$. The definition of $m(T)$ in [14] goes via a complex integral involving the resolvent of T. To make this integral convergent, we multiply the function $m(z)$ by $\psi(z) = 1/(1 + z^2)$, following [14]. With $\omega \in (0, \pi/2)$ fulfilling (3.3), we fix a $\nu \in (\omega, \pi/2)$ and let Γ be the path

$$\Gamma(t) = |t|\, e^{i\nu \operatorname{sgn} t}, \qquad -\infty < t < \infty.$$

Now define

$$(\psi m)(T) = \frac{1}{2\pi i} \int_\Gamma \psi(z) m(z)\,(zI - T)^{-1}\, dz,$$

which is a convergent integral because of (3.3), and let

$$m(T) = \psi(T)^{-1} (\psi m)(T). \tag{3.4}$$

Proposition 3.1. *Let $\lambda \neq 0$ be a generalized eigenvalue of T with generalized eigenspace $\mathcal{E}_\lambda$. Then the restriction to $\mathcal{E}_\lambda$ of $m(T)$ (defined above) coincides with the restriction to $\mathcal{E}_\lambda$ of the integral*

$$-\int_0^{+\infty} \varphi(t)\, \frac{d}{dt}\, e^{-tT}\, dt.$$

.

Notice that this is the integral from (1.3), and that its restriction to the finite-dimensional, T-invariant subspace $\mathcal{E}_\lambda$ makes perfect sense. Further, $m(T)$ is determined by these restrictions, since the $\mathcal{E}_\lambda$ together span $L^2_0(\gamma_\infty)$ and $m(T)$ is bounded on $L^2_0(\gamma_\infty)$, as proved by [8, Lemma 3.7].

Proof. Observe first that $T\big|_{\mathcal{E}_\lambda} = \lambda I + R_\lambda$, where R_λ a nilpotent operator on $\mathcal{E}_\lambda$. For $z \in \mathbb{C} \setminus \{\lambda\}$ this leads to

$$(zI - T)^{-1}\big|_{\mathcal{E}_\lambda} = ((z - \lambda)I - R_\lambda)^{-1} = (z - \lambda)^{-1} \left(I - \frac{R_\lambda}{z - \lambda} \right)^{-1}$$

$$= \sum_j \frac{1}{(z-\lambda)^{j+1}} R_\lambda^j,$$

where the sum is finite. Thus

$$\begin{aligned}
(\psi m)(T)\big|_{\mathcal{E}_\lambda} &= \frac{1}{2\pi i} \int_\Gamma \psi(z)\, m(z)\, (zI - T)^{-1}\big|_{\mathcal{E}_\lambda} \, dz \\
&= \frac{1}{2\pi i} \sum_j \int_\Gamma \psi(z)\, m(z)\, \frac{1}{(z-\lambda)^{j+1}} \, dz \, R_\lambda^j \\
&= \sum_j \frac{1}{j!} \, (\psi m)^{(j)}(\lambda)\, R_\lambda^j \\
&= \sum_{i,\,k} \frac{1}{i!\,k!} \, \psi^{(i)}(\lambda)\, m^{(k)}(\lambda)\, R_\lambda^{i+k} \\
&= \sum_i \frac{1}{i!} \, \psi^{(i)}(\lambda)\, R_\lambda^i \sum_k \frac{1}{k!} \, m^{(k)}(\lambda)\, R_\lambda^k \\
&= \psi(T) \sum_k \frac{1}{k!} \, m^{(k)}(\lambda)\, R_\lambda^k.
\end{aligned}$$

From (3.4) and (1.2), we conclude that

$$\begin{aligned}
m(T)\big|_{\mathcal{E}_\lambda} &= \sum_k \frac{1}{k!} \, m^{(k)}(\lambda)\, R_\lambda^k \\
&= -\sum_k \frac{1}{k!} \int_0^{+\infty} \varphi(t) \left(\frac{\partial}{\partial\lambda}\right)^k \frac{d}{dt} \, e^{-\lambda t} \, dt \, R_\lambda^k \\
&= -\int_0^{+\infty} \varphi(t)\, \frac{d}{dt} \sum_k \frac{1}{k!} \left(\frac{\partial^k}{\partial\lambda^k} \, e^{-\lambda t}\right) dt \, R_\lambda^k.
\end{aligned}$$

Here the sum equals

$$\sum_k e^{-\lambda t} \, \frac{1}{k!} \, (-t)^k \, R_\lambda^k = e^{-\lambda t} \, e^{-tR_\lambda} = e^{-tT},$$

and the proposition follows. □

3.2. **Splitting of the multiplier operator.** Given $\varphi \in L^\infty(0, +\infty)$, we will restrict the integral in (1.2) to various intervals. For $\varepsilon > 0$ we let

$$m_\varepsilon(\lambda) = -\int_\varepsilon^{+\infty} \varphi(t)\, \frac{d}{dt} \, e^{-\lambda t} \, dt.$$

But replacing ϵ by 0 we also define, in a slightly inconsistent way,

$$m_0(\lambda) = -\int_0^1 \varphi(t)\, \frac{d}{dt} \, e^{-\lambda t} \, dt,$$

and observe that

$$m(T) = m_1(T) + m_0(T).$$

Then (1.4) hints that $m_\varepsilon(T)$ and $m_0(T)$ should have off-diagonal kernels given by

$$\mathcal{M}_\varepsilon(x,u) = -\int_\varepsilon^{+\infty} \varphi(t)\, \dot{K}_t(x,u)\, dt \tag{3.5}$$

and

$$\mathcal{M}_0(x,u) = -\int_0^1 \varphi(t)\, \dot{K}_t(x,u)\, dt. \tag{3.6}$$

As will be verified in Section 5, $\mathcal{M}_\varepsilon$ is the kernel of $m_\varepsilon(T)$ for any $\varepsilon > 0$, and we use it in Section 6 to control $m_1(T)$. But $\mathcal{M}_0$ is singular and (3.6) is problematical on the diagonal $x = u$. We shall need to consider separately the global and local parts of $m_0(T)$; they will be suitably defined in Section 8.

4. The time derivative of the Mehler kernel

We compute the derivative $\dot{K}_t = \partial_t K_t(x,u)$ and estimate it for small and large t. As a preparation, we work out the t derivatives of some of the matrices introduced in the previous section.

Lemma 4.1. *For all $t > 0$ one has*

$$\dot{Q}_t = e^{tB}\, Q\, e^{tB^*}; \tag{4.1}$$

$$\frac{d}{dt} Q_t^{-1} = -Q_t^{-1}\, \dot{Q}_t\, Q_t^{-1} = -Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\, Q_t^{-1}; \tag{4.2}$$

$$\frac{d}{dt} \det Q_t = \det Q_t\ \mathrm{tr}(Q_t^{-1}\, \dot{Q}_t) = \det Q_t\ \mathrm{tr}(Q_t^{-1}\, e^{tB} Q e^{tB^*}); \tag{4.3}$$

$$\dot{D}_t = -Q_\infty\, B^*\, e^{-tB^*}\, Q_\infty^{-1} = -Q_\infty\, B^*\, Q_\infty^{-1}\, D_t. \tag{4.4}$$

Proof. The equality (4.1) trivially follows from (2.1). To obtain (4.2), one differentiates the equation $Q_t\, Q_t^{-1} = I$ and applies (4.1). Since Q_t is nonsingular, Jacobi's formula implies (4.3) (see [3, Fact 10.11.19]). Finally, we obtain the two equalities in (4.4) from (2.2). ☐

It will be convenient to have two different expressions for the t derivative of the Mehler kernel, as follows.

Lemma 4.2. *For all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$ and $t > 0$, we have*

$$\dot{K}_t(x,u) = K_t(x,u)\, N_t(x,u),$$

where the function N_t is given by

$$N_t(x,u) = -\frac{1}{2}\,\mathrm{tr}\left(Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\right) + \frac{1}{2}\,\left|Q^{1/2}\, e^{tB^*}\, Q_t^{-1}\,(u - D_t\, x)\right|^2 - \left\langle Q_\infty\, B^*\, Q_\infty^{-1}\, D_t\, x\,,\, (Q_t^{-1} - Q_\infty^{-1})\,(u - D_t\, x)\right\rangle, \tag{4.5}$$

and also by

$$N_t(x,u) = -\frac{1}{2}\,\mathrm{tr}\left(Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\right) + \frac{1}{2}\,\left|Q^{1/2}\, e^{tB^*}\, Q_t^{-1}\, e^{tB}\,(D_{-t}\, u - x)\right|^2 - \left\langle Q_t^{-1}\, B\, e^{tB}\,(D_{-t}\, u - x)\,,\, e^{tB}\,(D_{-t}\, u - x)\right\rangle$$

$$
\begin{aligned}
&- \left\langle Q_t^{-1}\, e^{tB}\, Q_\infty\, B^*\, Q_\infty^{-1}\, D_{-t}\, u\,,\; e^{tB}\,(D_{-t}\, u - x)\right\rangle \\
&- \left\langle B^*\, Q_\infty^{-1}\, D_{-t}\, u\,,\; D_{-t}\, u - x\right\rangle \\
=\!:\; & I_t + II_t(x,u) + III_t(x,u) + IV_t(x,u) + V_t(x,u). \qquad (4.6)
\end{aligned}
$$

Proof. Differentiating (2.8) with respect to t and applying Lemma 4.1, one obtains

$$
\begin{aligned}
&\dot K_t(x,u) = \\
&K_t(x,u)\Big[-\frac12 \operatorname{tr}\left(Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\right) + \frac12 \left\langle Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\, Q_t^{-1}(u - D_t\, x)\,,\; u - D_t\, x\right\rangle \\
&\qquad - \left\langle (Q_t^{-1} - Q_\infty^{-1})\, Q_\infty\, B^*\, Q_\infty^{-1}\, D_t\, x\,,\; (u - D_t\, x)\right\rangle\Big],
\end{aligned}
$$

from which (4.5) follows.

Next, we differentiate (2.9), applying (4.3) to the first factor, and then use (2.4) to rewrite the matrix D_t in the exponent. The result will be

$$
\begin{aligned}
\dot K_t(x,u) &= K_t(x,u)\Big\{-\frac12 \operatorname{tr}\left(Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\right) \\
&+ \frac{d}{dt}\Big[-\frac12 \left\langle Q_t^{-1}\, e^{tB}\,(D_{-t}\, u - x)\,,\; (e^{tB} + Q_t\, e^{-tB^*}\, Q_\infty^{-1})(D_{-t}\, u - x)\right\rangle\Big]\Big\}. \qquad (4.7)
\end{aligned}
$$

The derivative here consists of two terms, the first term being

$$
\begin{aligned}
&\frac{d}{dt}\Big[-\frac12 \left\langle Q_t^{-1}\, e^{tB}\,(D_{-t}\, u - x)\,,\; e^{tB}\,(D_{-t}\, u - x)\right\rangle\Big] \\
&= \frac12 \left\langle Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\, Q_t^{-1}\, e^{tB}\,(D_{-t}\, u - x)\,,\; e^{tB}\,(D_{-t}\, u - x)\right\rangle \\
&- \left\langle Q_t^{-1}\, B\, e^{tB}\,(D_{-t}\, u - x)\,,\; e^{tB}\,(D_{-t}\, u - x)\right\rangle \\
&- \left\langle Q_t^{-1}\, e^{tB}\, Q_\infty\, B^*\, Q_\infty^{-1}\, D_{-t}\, u\,,\; e^{tB}\,(D_{-t}\, u - x)\right\rangle,
\end{aligned}
$$

where we applied (4.2) and (4.4) with t replaced by $-t$. Notice that we have arrived at the terms II_t, III_t and IV_t in (4.6).

In the second term coming from the derivative in (4.7), we observe some cancellation; the term equals

$$
\frac{d}{dt}\Big[-\frac12 \left\langle D_{-t}\, u - x\,,\; Q_\infty^{-1}\,(D_{-t}\, u - x)\right\rangle\Big] = -\left\langle D_{-t}\, u - x\,,\; B^*\, Q_\infty^{-1}\, D_{-t}\, u)\right\rangle = V_t(x,u),
$$

where we used again (4.4). Summing up, we obtain (4.6), and the lemma is proved. □

Lemma 4.3. *Let $x, u \in \mathbb{R}^n$. Then for $0 < t \le 1$*

$$
|N_t(x,u)| \lesssim \frac1t + \frac{|u - D_t\, x|^2}{t^2} + |x|\,\frac{|u - D_t\, x|}{t} \qquad (4.8)
$$

and for $t \ge 1$

$$
|N_t(x,u)| \lesssim |D_{-t}\, u - x|\,|D_{-t}\, u| + e^{-ct}\,|D_{-t}\, u - x|^2 + e^{-ct}. \qquad (4.9)
$$

Proof. For $0 < t \leq 1$, (4.8) follows from (4.5), by means of (2.6) and (2.7).

When $t \geq 1$ we get, starting from (4.6) and using (2.5) and (2.6),

$$|I_t| = \frac{1}{2} \left| \operatorname{tr}(Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}) \right| \lesssim e^{-ct}.$$

Similarly, we have

$$|II_t(x,u)| = \frac{1}{2} \left| Q^{1/2}\, e^{tB^*}\, Q_t^{-1}\, e^{tB}\, (D_{-t}\, u - x) \right|^2 \lesssim e^{-ct}\, |D_{-t}\, u - x|^2,$$

and also

$$|III_t(x,u)| \lesssim e^{-ct}\, |D_{-t}\, u - x|^2 .$$

Proceeding as above, we further obtain

$$|IV_t(x,u)| + |V_t(x,u)| \lesssim |D_{-t}u - x|\, |D_{-t}u|,$$

and (4.9) is proved. □

5. On the multiplier kernel

In this section, we estimate some parts of the multiplier kernel and verify their relevance for the corresponding parts of the operator. We also state some facts that will simplify the proofs to come.

5.1. **Estimates of kernels.**

Without loss of generality, it will be assumed from now on that

$$\|\varphi\|_\infty \leq 1.$$

We first invoke a lemma from [11, Lemma 5.1 and Remark 5.5].

Lemma 5.1. *Let $\delta > 0$. For $\sigma \in \{1,2,3\}$ and $x, u \in \mathbb{R}^n$, one has*

$$\int_1^{+\infty} \exp\left(-\delta\, |D_{-t}\, u - x|^2\right) \left|D_{-t}\, u\right|^\sigma\, dt \lesssim 1 + |x|^{\sigma - 1},$$

where the implicit constant may depend on δ, in addition to n, Q and B.

Proposition 5.2. *(i) The integral (3.5) defining $\mathcal{M}_\varepsilon$ converges absolutely for any $\varepsilon > 0$ and all $x, u \in \mathbb{R}^n$. Moreover,*

$$|\mathcal{M}_1(x,u)| \lesssim e^{R(x)}, \qquad x, u \in \mathbb{R}^n, \tag{5.1}$$

and for $0 < \varepsilon < 1$

$$|\mathcal{M}_\varepsilon(x,u)| \lesssim \varepsilon^{-C}\, e^{R(x)}\, (1 + |x|), \qquad x, u \in \mathbb{R}^n. \tag{5.2}$$

(ii) For any $\varepsilon > 0$, any $f \in L^2_0(\gamma_\infty)$ and a.a. $x \in \mathbb{R}^n$,

$$m_\varepsilon(T) f(x) = \int \mathcal{M}_\varepsilon(x,u)\, f(u)\, d\gamma_\infty(u). \tag{5.3}$$

Proof. Aiming at *(i)*, we begin by estimating the kernel $\dot{K}_t(x,u) = K_t(x,u)N_t(x,u)$. For $1 < t < +\infty$ we use (2.11) and (4.9). Then we can neglect the factors $|D_{-t}\,u - x|$ in $N_t(x,u)$ by also reducing slightly the positive coefficient in front of the same factor in the exponent in (2.11). As a result,

$$|\dot{K}_t(x,u)| \lesssim e^{R(x)} \exp\left(-c\,|D_{-t}\,u - x|^2\right)(|D_{-t}\,u| + e^{-ct}), \qquad t > 1. \tag{5.4}$$

Lemma 5.1 now implies (5.1).

For $0 < t < 1$ we use instead (2.10) and (4.8), and now we can neglect all powers of $|u - D_t\,x|^2/t$ in $N_t(x,u)$. This leads to

$$|\dot{K}_t(x,u)| \lesssim e^{R(x)}\, t^{-n/2} \exp\left(-c\,\frac{|u - D_t\,x|^2}{t}\right)\left(t^{-1} + |x|t^{-1/2}\right) \lesssim e^{R(x)}\,(1+|x|)\,t^{-n/2-1}. \tag{5.5}$$

Integrating over $\varepsilon < t < 1$ and combining the result with (5.1), we arrive at (5.2). The claimed convergence is now clear, so *(i)* is verified.

For item *(ii)*, we need the following lemma.

Lemma 5.3. *Let $f \in L^2_0(\gamma_\infty)$ and $x \in \mathbb{R}^n$. Then for any $t > 0$*

$$\partial_t \int K_t(x,u)\, f(u)\, d\gamma_\infty(u) = \int \dot{K}_t(x,u)\, f(u)\, d\gamma_\infty(u). \tag{5.6}$$

Proof. Given $t > 0$, we replace t by τ in the right-hand side of (5.6), and then integrate $d\tau$ over the interval $t_0 < \tau < t$, for some fixed $t_0 \in (0,t)$. If we can swap the order of integration in the resulting double integral, we will get

$$\int_{t_0}^{t} \int \dot{K}_\tau(x,u)\, f(u)\, d\gamma_\infty(u)\, d\tau = \int (K_t(x,u) - K_{t_0}(x,u))\, f(u)\, d\gamma_\infty(u).$$

Differentiating this equation with respect to t, we then obtain the lemma.

To see that Fubini's theorem justifies this swap, it is enough to verify that

$$\sup_{u\in\mathbb{R}^n} \int_\varepsilon^\infty |\dot{K}_\tau(x,u)|\, d\tau < \infty \tag{5.7}$$

for each x and each $\varepsilon > 0$. But this follows from (5.5) for the integral over $\varepsilon < \tau < 1$ and from (5.4) and Lemma 5.1 for that over $\tau > 1$. The lemma is proved. □

To verify item *(ii)* in the proposition, we observe that because of (5.2), the right-hand side of (5.3) defines for each x a functional on $L^2_0(\gamma_\infty)$, whose norm is locally uniformly bounded for $x \in \mathbb{R}^n$. Further, the operator $m_\varepsilon(T)$ is bounded on the same space (see [8, Lemma 3.7]). Since the generalized eigenspaces $\mathcal{E}_\lambda$ together span $L^2_0(\gamma_\infty)$, it is enough to verify (5.3) on each $\mathcal{E}_\lambda$.

So let $f \in \mathcal{E}_\lambda$ for some λ. Since $e^{-tT} f(x) = \int K_t(x,u) f(u)\, d\gamma_\infty(u)$, Proposition 3.1 and Lemma 5.3 imply

$$\begin{aligned} m_\varepsilon(T)f(x) &= -\int_\varepsilon^\infty \varphi(t)\, \partial_t \int K_t(x,u) f(u)\, d\gamma_\infty(u)\, dt \\ &= -\int_\varepsilon^\infty \varphi(t) \int \dot{K}_t(x,u) f(u)\, d\gamma_\infty(u)\, dt. \end{aligned}$$

Switching the order of integration, again by means of (5.7), we conclude the proof of *(ii)*. □

Proposition 5.4. *(i) For $0 < t \le 1$ and all $x \ne u$,*

$$K_t(x,u) \lesssim e^{R(x)}\,(1+|x|)^C\,|x-u|^{-C} \tag{5.8}$$

and

$$\left|\dot K_t(x,u)\right| \lesssim e^{R(x)}\,(1+|x|)^C\,|x-u|^{-C}. \tag{5.9}$$

(ii) The integral (3.6) *defining $\mathcal{M}_0$ converges for all $x \ne u$, and*

$$|\mathcal{M}_0(x,u)| \lesssim e^{R(x)}\,(1+|x|)^C\,|x-u|^{-C}, \qquad x \ne u. \tag{5.10}$$

(iii) For any $f \in L^2_0(\gamma_\infty)$ and a.a. $x \notin \operatorname{supp} f$,

$$m_0(T)f(x) = \int \mathcal{M}_0(x,u)\,f(u)\,d\gamma_\infty(u).$$

Proof. To verify *(i)*, we start with the upper estimate in (2.10)

$$K_t(x,u) \lesssim \frac{e^{R(x)}}{t^{n/2}}\exp\left(-c\,\frac{|u-D_t\,x|^2}{t}\right)$$

and the first inequality in (5.5), which implies

$$|\dot K_t(x,u)| \lesssim e^{R(x)}\,(1+|x|)\,t^{-1-n/2}\,\exp\left(-c\,\frac{|u-D_t\,x|^2}{t}\right). \tag{5.11}$$

Notice that this last estimate is also satisfied by $K_t(x,u)$.

We have $|u-D_t\,x| \ge |u-x|-|x-D_t\,x|$, and Lemma 2.1 says that $|x-D_t\,x| \simeq t|x|$. Thus $|u-D_t\,x| \ge |u-x|/2$ if $t < c|u-x|/|x|$ for some $c>0$, and we conclude that for $0 < t < 1 \wedge c|u-x|/|x|$ the right-hand side of (5.11) is then controlled by

$$e^{R(x)}\,(1+|x|)\,t^{-1-n/2}\,\exp\left(-c\,\frac{|u-x|^2}{t}\right) \lesssim e^{R(x)}\,(1+|x|)\,|u-x|^{-2-n}.$$

If instead $c|u-x|/|x| < t \le 1$, it is enough to estimate the right-hand side of (5.11) by constant times

$$e^{R(x)}\,(1+|x|)\,t^{-1-n/2} \lesssim e^{R(x)}\,(1+|x|)^C\,|u-x|^{-C}.$$

This proves both (5.8) and (5.9).

Part *(ii)* follows immediately from (5.9), since $|\mathcal{M}_0(x,u)| \le \int_0^1 \left|\dot K_t(x,u)\right|\,dt$.

To prove *(iii)*, we can assume that φ vanishes for $t \ge 1$. Then (5.9) leads to

$$|\mathcal{M}_\varepsilon(x,u)| \le \int_\varepsilon^1 \left|\dot K_t(x,u)\right|\,dt \lesssim e^{R(x)}\,(1+|x|)^C\,|x-u|^{-C}, \qquad x \ne u \tag{5.12}$$

and this is uniform in ε.

We will let $\varepsilon \to 0$ in (5.3), with $x \notin \operatorname{supp} f$ and a fixed $f \in L^2_0(\gamma_\infty)$. Consider first the right-hand side of (5.3).

Because of (5.9), we see from (3.5) and (3.6) that, with φ supported in $[0,1]$, one has $\mathcal{M}_\varepsilon(x,u) \to \mathcal{M}_0(x,u)$ as $\varepsilon \to 0$, for any $x \neq u$. In the integral in the right-hand side of (5.3), we thus have pointwise convergence, and $|f(u)|\, d\gamma_\infty(u)$ is a finite measure. The estimate (5.12) allows us to apply bounded convergence, and conclude that

$$\int \mathcal{M}_\varepsilon(x,u) f(u)\, d\gamma_\infty(u) \to \int \mathcal{M}_0(x,u) f(u)\, d\gamma_\infty(u), \qquad \varepsilon \to 0,$$

for $x \notin \operatorname{supp} f$. Moreover, the left-hand integral here is a function of x which stays locally bounded in the complement of $\operatorname{supp} f$, uniformly in ε. So we also have convergence in the sense of distributions in $\mathbb{R}^n \setminus \operatorname{supp} f$.

To deal with the left-hand side of (5.3), we claim that $m_\varepsilon(T)f \to m_0(T)f$ in the sense of distributions in $\mathbb{R}^n$, as $\varepsilon \to 0$. This will end the proof of *(ii)*.

With ν, Γ and $\psi(z) = 1/(1+z^2)$ as in Section 3, we have

$$m_\varepsilon(T) = (1+T^2)\, \frac{1}{2\pi i} \int_\Gamma \frac{m_\varepsilon(z)}{1+z^2}\, (zI - T)^{-1}\, dz.$$

To prove the claim, we let $g \in C_0^\infty(\mathbb{R}^n)$. It is enough to verify that

$$\langle m_\varepsilon(T) f,\, g\rangle \to \langle m(T) f,\, g\rangle, \qquad \varepsilon \to 0,$$

with the scalar products taken in $L^2(\gamma_\infty)$. Notice that it does not matter whether we consider the convergence of the functions $m_\varepsilon(T)f$ or the measures $m_\varepsilon(T)f\, d\gamma_\infty$. We have

$$\begin{aligned} \langle m_\varepsilon(T) f,\, g\rangle &= \left\langle (1+T^2)\, \frac{1}{2\pi i} \int_\Gamma \frac{m_\varepsilon(z)}{1+z^2}\, (zI - T)^{-1} f\, dz,\, g \right\rangle \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{m_\varepsilon(z)}{1+z^2}\, \big\langle (zI-T)^{-1} f\,,\, (1+(T^*)^2)\, g \big\rangle\, dz, \end{aligned} \tag{5.13}$$

where T^* is the adjoint of T in $L^2(\gamma_\infty)$, so that $(1+(T^*)^2)\, g$ is another test function in $C_0^\infty(\mathbb{R}^n)$. Now $m_\varepsilon(z) = z \int_\varepsilon^\infty \varphi(t)\, e^{-tz}\, dt$ tends to $m(z)$ for each nonzero $z \in \Gamma$. For such z we also have the bound $|m_\varepsilon(z)| \le \|\varphi\|_\infty\, |z|/\Re z \lesssim 1$. In the last integral in (5.13), the integrand thus converges pointwise, and it is also dominated by constant times

$$\frac{1}{1+|z|^2}\, \|(zI-T)^{-1} f\|_{L^2(\gamma_\infty)}\, \|(1+(T^*)^2)\, g\|_{L^2(\gamma_\infty)},$$

which is integrable along Γ because of (3.3). The dominated convergence theorem now implies the claim and completes the proof of *(iii)* and that of Proposition 5.4. □

5.2. **Simplifications.** The preceding estimates allow some preliminary observations that will simplify the proof of Theorem 1.1.

In (1.5) we take f such that $\|f\|_{L^1(\gamma_\infty)} = 1$. We can then assume that α in the same estimate is large, in particular $\alpha > 2$, since $d\gamma_\infty$ is finite.

Further, we can focus mainly on points x in the ellipsoidal annulus

$$\mathcal{C}_\alpha = \left\{ x \in \mathbb{R}^n : \frac{1}{2} \log \alpha \le R(x) \le 2 \log \alpha \right\}.$$

To justify this, we will follow closely the arguments in [11, Section 6]. The first observation is that the set of points x for which $R(x) > 2\log\alpha$ can be neglected, because its $d\gamma_\infty$ measure is no larger than C/α.

The next proposition deals with the remaining part of the complement of $\mathcal{C}_\alpha$ when $t > 1$ and follows immediately from (5.1).

Proposition 5.5. *Let $x \in \mathbb{R}^n$ satisfy $R(x) < \frac{1}{2}\log\alpha$, where $\alpha > 2$. Then for all $u \in \mathbb{R}^n$*

$$\big|\mathcal{M}_1(x,u)\big| \lesssim \alpha.$$

A further simplification will be introduced in Subsection 7.2.

6. The weak type $(1,1)$ for large t

Proposition 6.1. *For any $f \in L^1(\gamma_\infty)$ such that $\|f\|_{L^1(\gamma_\infty)} = 1$ and any $\alpha > 2$,*

$$\gamma_\infty\left\{x \in \mathcal{C}_\alpha : |m_1(T)f(x)| > \alpha\right\} \lesssim \frac{1}{\alpha\sqrt{\log\alpha}}.$$

In particular, the operator $m_1(T)$ is of weak type $(1,1)$ with respect to the invariant measure $d\gamma_\infty$.

The estimate in this proposition means that for large α one has a slightly stronger estimate than the classical weak type $(1,1)$ bound. This phenomenon was already observed for the Ornstein–Uhlenbeck maximal operator in [10, Section 6], for the first-order Riesz transforms in [11, Proposition 7.1]) and for the variation operator of the Ornstein–Uhlenbeck semigroup in dimension one in [13, Proposition 3.1].

Proof. We will first use our polar coordinates to deduce a sharper version of the estimate (5.1) in Proposition 5.2*(i)*. If $x \in \mathcal{C}_\alpha$ and $u \neq 0$, we can write $x = D_s\,\tilde{x}$ and $u = D_\sigma\,\tilde{u}$ with $\tilde{x}$, $\tilde{u} \in E_{(\log\alpha)/2}$ and $s \geq 0$, $\sigma \in \mathbb{R}$.

Let $t \geq 1$. Applying [10, Lemma 4.3 *(i)*], we obtain

$$|D_{-t}\,u - x| = |D_{\sigma-t}\,\tilde{u} - D_s\,\tilde{x}| \gtrsim |\tilde{x} - \tilde{u}|.$$

Thus (2.11) implies

$$K_t(x,u) \lesssim e^{R(x)}\,\exp\Big(-c\,|\tilde{x}-\tilde{u}|^2\Big)\,\exp\Big(-c\,|D_{-t}\,u - x|^2\Big),$$

for some c.

Using this estimate instead of (2.11), one can follow the proof of (5.1) with an extra factor $\exp(-c\,|\tilde{x}-\tilde{u}|^2)$. The result will be

$$\big|\mathcal{M}_1(x,u)\big| \lesssim e^{R(x)}\,\exp\big(-c\,\big|\tilde{x}-\tilde{u}\big|^2\big), \qquad x \in \mathcal{C}_\alpha.$$

We can now finish the proof of Proposition 6.1 by means of the following lemma, which is the case $\sigma = 1$ of [11, Lemma 7.2].

Lemma 6.2. *Let $f \geq 0$ be normalized in $L^1(\gamma_\infty)$. For $\alpha > 2$*

$$\gamma_\infty\left\{x = D_s\,\tilde{x} \in \mathcal{C}_\alpha :\ e^{R(x)}\int \exp\big(-c\,\big|\tilde{x}-\tilde{u}\big|^2\big)\,f(u)\,d\gamma_\infty(u) > \alpha\right\} \lesssim \frac{C}{\alpha\sqrt{\log\alpha}}.$$

□

7. Localization

In order to study the weak type $(1,1)$ for small t, we need a further splitting of the operator $m_0(T)$. Localization means considering that part of $m_0(T)f(x)$ which depends on the values of f in a ball of center x and radius essentially $1/(1+|x|)$. We start by "filling" $\mathbb{R}^n$ with balls of this type.

7.1. **Local and global parts.** With $x_0 = 0$, we select a sequence $(x_j)_0^\infty$ of points such that the open balls $B_j = B(x_j, 1/(1+|x_j|)),\ j = 0, 1, \dots$, are pairwise disjoint, and such that the family $(B_j)_0^\infty$is maximal with respect to this property.

In the sequel, we will often use notations like $3B_j$ in the sense of concentric scaling.

We first verify that the balls $3B_j,\ j = 0, 1, \dots$, cover $\mathbb{R}^n$. To verify that a given point x is in some $3B_j$, we can assume that $|x| \geq 3$, since otherwise $x \in 3B_0$. The maximality property of the B_j implies that the ball $B(x, 1/(1+|x|))$ must intersect some B_j, necessarily with $j > 0$. Then

$$|x - x_j| < \frac{1}{1+|x|} + \frac{1}{1+|x_j|}, \tag{7.1}$$

and so

$$|x_j| < |x| + \frac{1}{1+|x|} + \frac{1}{1+|x_j|} \leq |x| + \frac{1}{4} + \frac{1}{2} \leq |x| + \frac{3}{4}\frac{|x|}{3} = \frac{5}{4}|x|.$$

Thus $1 + |x_j| < 5(1+|x|)/4$, and (7.1) implies

$$|x - x_j| < \frac{5}{4}\frac{1}{1+|x_j|} + \frac{1}{1+|x_j|} < \frac{3}{1+|x_j|},$$

so that $x \in 3B_j$.

We will need another property of these balls: for any j and any x

$$x \in 6B_j \qquad \Rightarrow \qquad \frac{1}{7} < \frac{1+|x|}{1+|x_j|} < 7. \tag{7.2}$$

Indeed, if $|x - x_j| < 6/(1+|x_j|)$ we have

$$1 + |x_j| < 1 + |x| + 6/(1+|x_j|) \leq 7 + |x| \leq 7(1+|x|),$$

and the other inequality follows in the same way.

This allows us to show that the balls $(6B_j)_0^\infty$ have bounded overlap, as follows. Let $x \in 6B_j$, so that $x_j \in B(x, 6/(1+|x_j|))$. Then

$$B_j = B(x_j, 1/(1+|x_j|) \subset B(x, 7/(1+|x_j|)) \subset B(x, 49/(1+|x|)),$$

the last step by (7.2). Comparing Lebesgue measures, we have $|B_j| \gtrsim (1+|x|)^{-n}$ and $|B(x, 49/(1+|x|))| \lesssim (1+|x|)^{-n}$. Since the B_j are pairwise disjoint, this can occur for at most a bounded number of j. The bounded overlap is verified.

We now introduce functions supported in some of these balls, with which the local part of $m_0(T)$ will be defined. Let first ρ_i for each $i = 0, 1, \dots$ be a nonnegative, smooth function supported in $4B_i$ such that $\rho_i = 1$ in $3B_i$. Its gradient should satisfy $|\nabla \rho_i(x)| \lesssim 1 + |x|$. The sum $\sum_0^\infty \rho_i$ is locally finite and satisfies $1 \leq \sum_0^\infty \rho_i \lesssim 1$ and also $|\nabla \sum_0^\infty \rho_i(x)| \lesssim 1 + |x|$.

The functions we will use are

$$r_j = \frac{\rho_j}{\sum_0^\infty \rho_i}.$$

Clearly, r_j is nonnegative, smooth and supported in $4B_j$, and $\sum_0^\infty r_j = 1$. For the gradient, one has

$$|\nabla r_j(x)| \lesssim 1 + |x|. \tag{7.3}$$

We will also need smooth functions $\widetilde{r}_j,\ j = 0, 1, \ldots,$ again with values in $[0, 1]$ but having larger supports. They shall satisfy $\widetilde{r}_j = 1$ in $5B_j$ and $\operatorname{supp} \widetilde{r}_j \subseteq 6B_j$, and like the r_j they shall also verify

$$\left|\nabla \widetilde{r}_j(x)\right| \lesssim 1 + |x|. \tag{7.4}$$

We observe that the functions $\widetilde{r}_j$ have bounded overlap and that

$$e^{-R(x)} \simeq e^{-R(x_j)} \qquad \text{for} \qquad x \in \operatorname{supp} \widetilde{r}_j, \tag{7.5}$$

which follows from [11, formula (2.9)] with $y = x - x_j$ when $|x|$ is large and is trivial in the opposite case.

Let

$$\eta(x, u) = \sum_{j=0}^{\infty} \widetilde{r}_j(x)\, r_j(u),$$

and note that for all $x, u \in \mathbb{R}^n$

$$0 \le \eta(x, u) \le 1. \tag{7.6}$$

The following two lemmas express that $\eta(x, u)$ indicates how close the points x and u are to each other, and also give an estimate of the gradient of $\eta(x, u)$.

Lemma 7.1. *(i) If $\eta(x, u) > 0$ for some points x and u, then $|x - u| \lesssim \frac{1}{1+|x|}$.*
(ii) For any points with $x \neq u$

$$\left|\nabla_x \eta(x, u)\right| + \left|\nabla_u \eta(x, u)\right| \lesssim |x - u|^{-1}.$$

Proof. (i) Since $\eta(x, u) > 0$, there exists a j such that $\widetilde{r}_j(x) > 0$ and $r_j(u) > 0$. Thus $x \in 6B_j$ so that $1 + |x| \simeq 1 + |x_j|$ by (7.2), and $u \in 4B_j$. We get

$$|x - u| \le |x - x_j| + |u - x_j| < \frac{6}{1 + |x_j|} + \frac{4}{1 + |x_j|} \lesssim \frac{1}{1 + |x|}.$$

(ii) We can assume that (x, u) is in the support of η, and by continuity the conclusion of (i) still holds. The inequality for $\nabla_x \eta$ follows from (7.4) and (i). For $\nabla_u \eta$ we apply (7.2) to x and u with j as in (i), to get $1 + |x| \simeq 1 + |u|$. Now we can use (7.3) and (i). □

Lemma 7.2. *If x and u are two points with $|x - u| \le \frac{1}{3} \frac{1}{1+|x|}$, then*

$$r_j(u) > 0 \qquad \Rightarrow \qquad \widetilde{r}_j(x) = 1, \qquad j = 0, 1, \ldots, \tag{7.7}$$

and $\eta(x, u) = 1$.

Proof. The last conclusion is a consequence of (7.7) and the definition of η. The implication (7.7) follows from

$$u \in 4B_j \qquad \Rightarrow \qquad x \in 5B_j,$$

or equivalently,

$$|u - x_j| < \frac{4}{1+|x_j|} \qquad \Rightarrow \qquad |x - x_j| < \frac{5}{1+|x_j|}.$$

But

$$|x - x_j| \le |x - u| + |u - x_j| < \frac{1}{3}\frac{1}{1+|x|} + \frac{4}{1+|x_j|},$$

So it is enough to show that

$$\frac{1}{3}\frac{1}{1+|x|} \le \frac{1}{1+|x_j|},$$

which is obvious if $|x_j| \le 2$. In case $|x_j| > 2$, we have

$$1+|x_j| \le 1+|u-x_j|+|x-u|+|x| \le 1+\frac{4}{1+|x_j|}+\frac{1}{3}+|x| < 1+\frac{4}{3}+\frac{1}{3}+|x| < 3(1+|x|).$$

□

We now split the multiplier operator $m_0(T)$ in a local and a global part. The local part is defined by

$$m_0(T)^{\text{loc}} f(x) = \sum_{j=0}^{\infty} \widetilde{r}_j(x)\, m_0(T)\,(f r_j)\,(x), \qquad f \in L^1(\gamma_\infty). \tag{7.8}$$

This sum is locally finite and so well defined.

It was proved in Proposition 5.4(*iii*) that the off-diagonal kernel of $m_0(T)$ is $\mathcal{M}_0(x,u)$. To find that of $m_0(T)^{\text{loc}}$, take $f \in L^1(\gamma_\infty)$. For almost all points $x \notin \operatorname{supp} f$, thus not in $\operatorname{supp} f r_j$ for any j, we have

$$m_0(T)^{\text{loc}} f(x) = \sum_{j=0}^{\infty} \widetilde{r}_j(x) \int \mathcal{M}_0(x,u) f(u)\, r_j(u)\, d\gamma_\infty(u),$$

where the sum is again locally finite. As a consequence,

$$m_0(T)^{\text{loc}} f(x) = \int \mathcal{M}_0(x,u)\, \eta(x,u)\, f(u)\, d\gamma_\infty(u),$$

and so the off-diagonal kernel of $m_0(T)^{\text{loc}}$ is

$$\mathcal{M}_0^{\text{loc}}(x,u) = \mathcal{M}_0(x,u)\, \eta(x,u). \tag{7.9}$$

We also define

$$m_0(T)^{\text{glob}} = m_0(T) - m_0(T)^{\text{loc}}.$$

Its off-diagonal kernel is

$$\mathcal{M}_0^{\text{glob}}(x,u) = \mathcal{M}_0(x,u)\,(1 - \eta(x,u)). \tag{7.10}$$

Moreover, the next lemma says that $m_0(T)^{\text{glob}}$ is completely given by this kernel.

Lemma 7.3. *For any $f \in L^1(\gamma_\infty)$ and a.a. x, one has*

$$m_0(T)^{\text{glob}} f(x) = \int \mathcal{M}_0(x,u)\,(1-\eta(x,u))\,f(u)\,d\gamma_\infty(u). \tag{7.11}$$

Proof. Our strategy will be to take a point $\xi \in \mathbb{R}^n$ and verify (7.11) a.e. in the ball

$$B := B\left(\xi, \frac{1}{8}\,\frac{1}{1+|\xi|}\right).$$

If $x,\ u \in B$ then

$$|x-u| < \frac{1}{4}\,\frac{1}{1+|\xi|},$$

and we have

$$1+|x| \le 1+|\xi| + |x-\xi| < 1+|\xi| + \frac{1}{8}\,\frac{1}{1+|\xi|} \le \frac{9}{8}\,(1+|\xi|).$$

Combining these two estimates, we get

$$|x-u| < \frac{1}{4}\,\frac{9}{8}\,\frac{1}{1+|x|} < \frac{1}{3}\,\frac{1}{1+|x|}. \tag{7.12}$$

Thus Lemma 7.2 applies. We compute $m_0(T)^{\text{loc}}(f\chi_B)(x)$ for $x \in B$, by means of the definition (7.8) of $m_0(T)^{\text{loc}}$. If for some j, the product $f\chi_B\, r_j$ does not vanish identically, there exists a point $u \in B$ with $r_j(u) > 0$. Then Lemma 7.2 says that $\widetilde{r}_j(x) = 1$, and it follows that for $x \in B$

$$\begin{aligned} m_0(T)^{\text{loc}}(f\chi_B)(x) = \sum_j m_0(T)(f\chi_B\, r_j)(x) &= m_0(T)^{\text{loc}}\left(\sum_j f\chi_B\, r_j\right)(x) \\ &= m_0(T)(f\chi_B)(x), \end{aligned}$$

since in the sums here, only a finite number of terms are nonzero The equality obtained implies that $m_0(T)^{\text{glob}}(f\chi_B)(x) = 0$.

When we next apply the operator $m_0(T)^{\text{glob}}$ to $f(1-\chi_B)$, we can use the off-diagonal kernel in (7.10). As a result, we have for a.a. $x \in B$

$$\begin{aligned} m_0(T)^{\text{glob}} f(x) &= m_0(T)^{\text{glob}}\big(f(1-\chi_B)\big)(x) = \\ &\int \mathcal{M}_0(x,u)\,(1-\eta(x,u))\,f(u)\,(1-\chi_B(u))\,d\gamma_\infty(u) = \\ \int \mathcal{M}_0(x,u)\,(1-\eta(x,u))\,f(u)\,d\gamma_\infty(u) &- \int \mathcal{M}_0(x,u)\,(1-\eta(x,u))\,f(u)\,\chi_B(u)\,d\gamma_\infty(u). \end{aligned}$$

The last integral is 0, since here $\eta(x,u) = 1$ in view of (7.12) and Lemma 7.2. We have verified (7.11) a.e. in the ball B and thus almost everywhere. $\square$

7.2. **A further simplification.** The following proposition is complementary to Proposition 5.5 and deals with the interior of $\mathcal{C}_\alpha$ when $t < 1$.

Proposition 7.4. *Let $x \in \mathbb{R}^n$ satisfy $R(x) < \frac{1}{2}\log\alpha$, where $\alpha > 2$. Then for all $u \in \mathbb{R}^n$*

$$\mathcal{M}_0^{\text{glob}}(x,u) \lesssim \alpha.$$

Proof. If $\mathcal{M}_0^{\text{glob}}(x,u) \neq 0$ then $\eta(x,u) < 1$, and Lemma 7.2 implies that $|x-u| \gtrsim \frac{1}{1+|x|_Q}$. From (5.10) we then obtain

$$\mathcal{M}_0^{\text{glob}}(x,u) \lesssim e^{R(x)}\,(1+|x|)^C \lesssim \alpha,$$

and the proposition is verified. □

Thus we need to take the region $\{x : R(x) < \frac{1}{2}\log\alpha\}$ into account only when considering $m_0(T)^{\text{loc}}$.

8. The local region

In this section we shall prove the weak type (1,1) of the operator $m_0(T)^{\text{loc}}$.

In order to apply Calderón–Zygmund theory, we pass to Lebesgue measure. Set

$$q_T f(x) = e^{-R(x)}\, m_0(T)^{\text{loc}}\big(f(\cdot)\, e^{R(\cdot)}\big)(x). \tag{8.1}$$

The relationship between q_T and $m_0(T)^{\text{loc}}$ is clarified by the following result.

Proposition 8.1. *If q_T is of weak type $(1,1)$ with respect to Lebesgue measure, then $m_0(T)^{\text{loc}}$ is of weak type $(1,1)$ with respect to the invariant measure.*

Proof. By (7.8) and the bounded overlap of the $\widetilde{r}_j$, we have

$$\begin{aligned}\|m_0(T)^{\text{loc}} f\|_{L^{1,\infty}(\gamma_\infty)} &= \Big\|\sum_{j=0}^{\infty} \widetilde{r}_j m_0(T)\,(f r_j)\Big\|_{L^{1,\infty}(\gamma_\infty)}\\ &\lesssim \sum_{j=0}^{\infty} \|\widetilde{r}_j\, m_0(T)\,(f r_j)\|_{L^{1,\infty}(\gamma_\infty)}.\end{aligned}$$

The last sum may be rewritten as

$$\begin{aligned}&\sum_{j=0}^{\infty} \big\|e^{R(x)}\, e^{-R(x)}\, \widetilde{r}_j(x)\, m_0(T)\,(f r_j)\,(x)\big\|_{L^{1,\infty}(\gamma_\infty)}\\ &= \sum_{j=0}^{\infty} \big\|e^{R(x)}\, \widetilde{r}_j(x)\, q_T\big(f\, r_j\, e^{-R(\cdot)}\big)\,(x)\big\|_{L^{1,\infty}(\gamma_\infty)}\\ &\simeq \sum_{j=0}^{\infty} e^{R(x_j)}\, \big\|\widetilde{r}_j\, q_T\big(f\, r_j\, e^{-R(\cdot)}\big)\big\|_{L^{1,\infty}(\gamma_\infty)} \simeq \sum_{j=0}^{\infty} \big\|\widetilde{r}_j\, q_T\big(f\, r_j\, e^{-R(\cdot)}\big)\big\|_{L^{1,\infty}(dx)};\end{aligned}$$

in the last two steps we used (7.5). The hypothesis of the proposition implies that the last sum is dominated by constant times

$$\sum_{j=0}^{\infty} \left\| f\, r_j\, e^{-R(\cdot)} \right\|_{L^1(du)} = \sum_{j=0}^{\infty} \| f\, r_j \|_{L^1(\gamma_\infty)} = \| f \|_{L^1(\gamma_\infty)},$$

which proves the assertion. □

We will now apply Calderón-Zygmund theory to the operator q_T, in order to prove its weak type $(1,1)$ with respect to Lebesgue measure. First we verify the L^2 boundedness.

Lemma 8.2. *The operator q_T is bounded on $L^2(dx)$.*

Proof. Starting with (8.1) and (7.8), we then apply the bounded overlap of the $\widetilde{r}_j$ and (7.5). We get

$$\begin{aligned}
\int |q_T f(x)|^2 dx &= \int \left| \sum_{j=0}^{\infty} e^{-R(x)}\, \widetilde{r}_j(x) m_0(T) \left(f\, r_j\, e^{R(\cdot)} \right)(x) \right|^2 dx \\
&\lesssim \sum_{j=0}^{\infty} \int \left| e^{-R(x)}\, \widetilde{r}_j(x) m_0(T) \left(f\, r_j\, e^{R(\cdot)} \right)(x) \right|^2 dx \\
&\lesssim \sum_{j=0}^{\infty} e^{-R(x_j)} \int \left| \widetilde{r}_j(x) m_0(T) \left(f\, r_j\, e^{R(\cdot)} \right)(x) \right|^2 d\gamma_\infty(x) \\
&\lesssim \sum_{j=0}^{\infty} e^{-R(x_j)} \int \left| m_0(T) \left(f\, r_j\, e^{R(\cdot)} \right)(x) \right|^2 d\gamma_\infty(x).
\end{aligned}$$

For $m_0(T)$, which is of Laplace transform type, the L^2 boundedness with respect to the invariant measure follows from [8, Lemma 3.7] (we remark that this boundedness is also a consequence of some results in [15] and [16], which can be applied here since [32, Lemma 2.2] exhibits a linear change of coordinates in $\mathbb{R}^n$ reducing the setting to the case where $Q = I$ and Q_∞ is a diagonal matrix). As a consequence of the above, we have

$$\begin{aligned}
\int |q_T f(x)|^2\, dx &\lesssim \sum_{j=0}^{\infty} e^{-R(x_j)} \int \left| f(u)\, r_j(u)\, e^{R(u)} \right|^2 d\gamma_\infty(u) \\
&\simeq \sum_{j=0}^{\infty} \int \left| f(u)\, r_j(u) \right|^2 du \le \int |f(u)|^2\, du,
\end{aligned}$$

concluding the proof. □

We need a lemma from [11].

Lemma 8.3. *Let $p,\, r \ge 0$ with $p + r/2 > 1$. Assume that $\eta(x,u) > 0$ and $x \neq u$. Then for $\delta > 0$*

$$\int_0^1 t^{-p} \exp\left(-\delta\, \frac{|u - D_t\, x|^2}{t}\right) |x|^r\, dt \le C\, |u - x|^{-2p-r+2}.$$

Here the constant C may depend on δ, p and r, in addition to n, Q and B.

Proof. To see that this follows from [11, Lemma 8.1], it is enough to observe that Lemma 7.1(i) leads to $|u - x| \lesssim 1/(1 + |x|)$, i.e., $(x, u) \in L_A$ in the terminology of [11]. □

We shall now find the off-diagonal kernel $\mathcal{Q}(x,u)$ of the operator q_T from (8.1), defined for integration against Lebesgue measure, that is,

$$q_T f(x) = \int \mathcal{Q}(x,u)\, f(u)\, du, \qquad x \notin \operatorname{supp} f.$$

It will be convenient to introduce another kernel

$$\begin{aligned}\mathcal{K}_t(x,u) &:= e^{-R(x)}\, K_t(x,u)\\ &= \Big(\frac{\det Q_\infty}{\det Q_t}\Big)^{1/2} \exp\Big[-\frac{1}{2}\big\langle (Q_t^{-1} - Q_\infty^{-1})(u - D_t\, x)\,,\, u - D_t\, x\big\rangle\Big].\end{aligned}$$

We recall from (7.9) and Proposition 5.4 that the off-diagonal kernel of $m_0(T)^{\mathrm{loc}}$ is

$$\mathcal{M}_0^{\mathrm{loc}}(x,u) = \mathcal{M}_0(x,u)\, \eta(x,u) = -\int_0^1 \varphi(t)\, \dot K_t(x,u)\, dt\; \eta(x,u).$$

From this and (8.1) it follows that

$$\mathcal{Q}(x,u) = e^{-R(x)}\, \mathcal{M}_0^{\mathrm{loc}}(x,u) = -\int_0^1 \varphi(t)\, \dot{\mathcal{K}}_t(x,u)\, dt\; \eta(x,u). \tag{8.2}$$

We will need expressions for some derivatives of $\mathcal{K}_t$; for similar results about the derivatives of K_t we refer to [11, Lemma 4.1].

Using (2.3), one sees that

$$\partial_{x_\ell}\, \mathcal{K}_t(x,u) = \mathcal{K}_t(x,u)\, \mathcal{P}_\ell(t,x,u),$$

where

$$\mathcal{P}_\ell(t,x,u) = \big\langle Q_t^{-1}\, e^{tB}\, e_\ell\,,\, u - D_t\, x\big\rangle. \tag{8.3}$$

Similarly, or as an immediate consequence of [11, formula (4.2)],

$$\partial_{u_\ell}\, \mathcal{K}_t(x,u) = -\mathcal{K}_t(x,u)\, \big\langle Q_t^{-1}\, e^{tB}\, (D_{-t}\, u - x)\,,\, e_\ell\big\rangle. \tag{8.4}$$

The following three technical lemmata give expressions and estimates for derivatives of $\dot{\mathcal{K}}_t = \partial \mathcal{K}_t/\partial t$. Before stating them, we notice that

$$\dot{\mathcal{K}}_t(x,u) = e^{-R(x)}\, \dot K_t(x,u) = e^{-R(x)}\, K_t(x,u)\, N_t(x,u) = \mathcal{K}_t(x,u)\, N_t(x,u), \tag{8.5}$$

with $N_t(x,u)$ from Lemma 4.2.

Lemma 8.4. *For $x,\, u \in \mathbb{R}^n$ and $t > 0$, one has*

$$(i) \qquad \partial_{x_\ell} \dot{\mathcal{K}}_t(x,u) = \mathcal{K}_t(x,u)\, \mathcal{S}_\ell(t,x,u);$$
$$(ii) \qquad \partial_{u_\ell} \dot{\mathcal{K}}_t(x,u) = \mathcal{K}_t(x,u)\, \mathcal{R}_\ell(t,x,u),$$

where the factors $\mathcal{S}_\ell(t,x,u)$ and $\mathcal{R}_\ell(t,x,u)$ are given by

$$\mathcal{S}_\ell(t,x,u) = N_t(x,u)\, \mathcal{P}_\ell(t,x,u) - \left\langle Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\, Q_t^{-1}\, e^{tB}\, e_\ell\,,\, u - D_t\, x \right\rangle$$
$$+ \left\langle Q_t^{-1}\, B\, e^{tB}\, e_\ell\,,\, u - D_t\, x \right\rangle + \left\langle Q_t^{-1}\, e^{tB}\, e_\ell\,,\, Q_\infty\, B^*\, e^{-tB^*}\, Q_\infty^{-1}\, x \right\rangle, \qquad (8.6)$$

and

$$\mathcal{R}_\ell(t,x,u) = -N_t(x,u) \left\langle Q_t^{-1}\, e^{tB}\, (D_{-t}\, u - x)\,,\, e_\ell \right\rangle$$
$$+ \left\langle Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\, Q_t^{-1}\, e^{tB}\, (D_{-t}\, u - x)\,,\, e_\ell \right\rangle$$
$$- \left\langle Q_t^{-1}\, B\, e^{tB}\, (D_{-t}\, u - x)\,,\, e_\ell \right\rangle - \left\langle Q_t^{-1}\, e^{tB}\, Q_\infty\, B^*\, Q_\infty^{-1}\, D_{-t}\, u\,,\, e_\ell \right\rangle. \quad (8.7)$$

Proof. To prove *(i)*, we start by observing that

$$\partial_{x_\ell} \dot{\mathcal{K}}_t(x,u) = \partial_t \left(\mathcal{K}_t(x,u)\, \mathcal{P}_\ell(t,x,u)\right)$$
$$= \mathcal{K}_t(x,u)\, N_t(x,u)\, \mathcal{P}_\ell(t,x,u) + \mathcal{K}_t(x,u)\, \partial_t \left(\left\langle Q_t^{-1}\, e^{tB}\, e_\ell\,,\, u - D_t\, x \right\rangle \right),$$

where we used (8.3). Applying (4.2) and (4.4) to the last derivative here, one arrives at (8.6), and *(i)* is verified.

To prove *(ii)*, we proceed similarly, using (8.4) to write

$$\partial_{u_\ell} \dot{\mathcal{K}}_t(x,u) = -\mathcal{K}_t\, N_t(x,u) \left\langle Q_t^{-1}\, e^{tB}\, (D_{-t}\, u - x)\,,\, e_\ell \right\rangle$$
$$- \mathcal{K}_t(x,u)\, \partial_t \left(\left\langle Q_t^{-1}\, e^{tB}\, (D_{-t}\, u - x)\,,\, e_\ell \right\rangle \right).$$

As in the case of *(i)*, this leads to (8.7) and *(ii)*. ☐

To bound $\mathcal{S}_\ell$ and $\mathcal{R}_\ell$, one concludes from [11, formula (4.5)] that (notice the distinction between our $\mathcal{P}_\ell$ and the P_j used in [11])

$$|\mathcal{P}_\ell(t,x,u)| \lesssim |u - D_t\, x|/t, \qquad 0 < t \le 1. \qquad (8.8)$$

Lemma 8.5. *One has for $0 < t \le 1$ and all $x,\, u \in \mathbb{R}^n$*

$$|\mathcal{S}_\ell(t,x,u)| \lesssim |x|\, \frac{|u - D_t\, x|^2}{t^2} + \frac{|u - D_t\, x|^3}{t^3} + \frac{|u - D_t\, x|}{t} + \frac{|u - D_t\, x|}{t^2} + \frac{|x|}{t}.$$

Proof. We first bound the product $N_t(x,u)\, \mathcal{P}_\ell(t,x,u)$ appearing in (8.6). Because of (4.8) and (8.8), we have for $0 < t \le 1$

$$\left|N_t(x,u) \mathcal{P}_\ell(t,x,u)\right| \lesssim \left(\frac{1}{t} + \frac{|u - D_t\, x|^2}{t^2} + |x|\, \frac{|u - D_t\, x|}{t} \right) \frac{|u - D_t\, x|}{t}$$
$$\lesssim \frac{|u - D_t\, x|}{t^2} + \frac{|u - D_t\, x|^3}{t^3} + |x|\, \frac{|u - D_t\, x|^2}{t^2}.$$

Estimating also the other terms in (8.6), one arrives at the lemma. ☐

Lemma 8.6. *For $t \in (0,1]$ and all x, $u \in \mathbb{R}^n$*

$$|\mathcal{R}_\ell(t,x,u)| \lesssim |x| \frac{|u-D_t\, x|^2}{t^2} + \frac{|u-D_t\, x|^3}{t^3} + \frac{|u-D_t\, x|}{t^2} + + \frac{|x|}{t}.$$

Proof. For $t \in (0,1]$ we have by (8.7) and (4.8)

$$\begin{aligned}\big|\mathcal{R}_\ell(t,x,u)\big| &\lesssim \big|N_t(x,u)\big| \frac{|u-D_t\, x|}{t} + \frac{|u-D_t\, x|}{t^2} + \frac{|u-D_t\, x|}{t} + \frac{|u|}{t} \\ &\lesssim \left(\frac{1}{t} + \frac{|u-D_t\, x|^2}{t^2} + |x| \frac{|u-D_t\, x|}{t}\right) \frac{|u-D_t\, x|}{t} + \frac{|u-D_t\, x|}{t^2} + \frac{|x|}{t}.\end{aligned}$$

Here we estimated $|u|/t$ by $|u-D_t\, x|/t^2 + |x|/t$. The lemma follows. □

Proposition 8.7. *For all (x,u) such that $\eta(x,u) \neq 0$ and $x \neq u$, one has*

$$\int_0^1 \big|\dot{\mathcal{K}}_t(x,u)\big|\, dt \lesssim |u-x|^{-n}.$$

Proof. From (8.5) (2.10) and (4.8) we obtain

$$\begin{aligned}&\int_0^1 \big|\dot{\mathcal{K}}_t(x,u)\big|\, dt \\ &\quad\lesssim \int_0^1 t^{-\frac{n}{2}} \exp\left(-c\, \frac{|D_t\, x-u|^2}{t}\right) \left(\frac{1}{t} + \frac{|u-D_t\, x|^2}{t^2} + |x| \frac{|u-D_t\, x|}{t}\right) dt \\ &\quad\lesssim \int_0^1 t^{-\frac{n}{2}} \exp\left(-c\, \frac{|D_t\, x-u|^2}{t}\right) \left(\frac{1}{t} + \frac{|x|}{\sqrt{t}}\right) dt.\end{aligned}$$

Because of Lemma 8.3, the last integral is controlled by $|u-x|^{-n}$, and the proposition is proved. □

We are now ready to prove standard Calderón-Zygmund bounds for the off-diagonal kernel of q_T with respect to Lebesgue measure.

Proposition 8.8. *For all (x,u) such that $\eta(x,u) \neq 0$ and $x \neq u$, the following estimates hold:*

$$\big|\mathcal{Q}(x,u)\big| \lesssim |u-x|^{-n}; \tag{8.9}$$

$$\big|\nabla_x\, \mathcal{Q}(x,u)\big| \lesssim |u-x|^{-n-1}; \tag{8.10}$$

$$\big|\nabla_u\, \mathcal{Q}(x,u)\big| \lesssim |u-x|^{-n-1}. \tag{8.11}$$

Proof. In the light of (8.2) one has

$$|\mathcal{Q}(x,u)| = \left|\int_0^1 \varphi(t)\, \dot{\mathcal{K}}_t(x,u)\, dt\right|\, \eta(x,u) \leq \int_0^1 \big|\dot{\mathcal{K}}_t(x,u)\big|\, dt \lesssim |u-x|^{-n},$$

where we used both (7.6) and Proposition 8.7. Thus (8.9) is verified.

In order to prove (8.10), we first observe that

$$\big|\partial_{x_\ell}\, (\mathcal{Q}(x,u))\,\big| \lesssim \int_0^1 \big|\partial_{x_\ell}\, \dot{\mathcal{K}}_t(x,u)\big|\, dt\, \eta(x,u) + \int_0^1 \big|\dot{\mathcal{K}}_t(x,u)\big|\, dt\, \big|\partial_{x_\ell}\, \eta(x,u)\big|. \tag{8.12}$$

The last term here satisfies the desired estimate because of Proposition 8.7 and Lemma 7.1(*ii*).

Using Lemma 8.4(*i*) and then (2.10) and Lemma 8.5, we can estimate the first term in (8.12) by

$$\int_0^1 \left|\mathcal{K}_t(x,u)\,\mathcal{S}_\ell(t,x,u)\right| dt \lesssim \int_0^1 t^{-\frac{n}{2}} \exp\left(-c\,\frac{|u-D_t\,x|^2}{t}\right)$$
$$\times \left(|x|\,\frac{|u-D_tx|^2}{t^2} + \frac{|u-D_tx|^3}{t^3} + \frac{|u-D_tx|}{t} + \frac{|u-D_tx|}{t^2} + \frac{|x|}{t}\right) dt$$
$$\lesssim \int_0^1 t^{-\frac{n}{2}} \exp\left(-c\,\frac{|u-D_t\,x|^2}{t}\right)\left(\frac{|x|}{t} + \frac{1}{t\sqrt{t}}\right) dt.$$

Proposition (8.3) says that the last expression is controlled by $|u-x|^{-n-1}$, so that (8.10) is proved.

The verification of (8.11) is analogous, with $\mathcal{R}_\ell(t,x,u)$ instead of $\mathcal{S}_\ell(t,x,u)$. It is enough to observe that Lemma 8.6 implies that $|\mathcal{R}_\ell(t,x,u)|$ is controlled by the right-hand side in the statement of Lemma 8.5. □

By means of Lemma 8.2, Proposition 8.8 and Proposition 8.1 we finally arrive at the goal of this section.

Proposition 8.9. *The operator $m_0(T)^{\mathrm{loc}}$ is of weak type $(1,1)$ with respect to the invariant measure $d\gamma_\infty$.*

9. An auxiliary bound for $0 < t \le 1$

In this section, we verify a bound on the number of zeros of the t derivative of K_t in the interval $(0,1]$, which will be used in the next section to control the kernel $\mathcal{M}_0^{\mathrm{glob}}$.

Proposition 9.1. *For $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$, the number of zeros in $I = (0,1]$ of the function $t \mapsto \dot{K}_t(x,u)$ is bounded by a positive integer depending only on n and B.*

Proof. Instead of $\dot{K}_t(x,u)$ we consider $\mathcal{N}_t(x,u) = 2(\det Q_t)^2\, N_t(x,u)$, since the three kernels $\dot{K}_t(x,u)$, $N_t(x,u)$ and $\mathcal{N}_t(x,u)$ have exactly the same zeros in I. From (4.5) we have

$$\begin{aligned}\mathcal{N}_t(x,u) = &-(\det Q_t)\,\mathrm{tr}\left((\det Q_t)Q_t^{-1}\,e^{tB}\,Q\,e^{tB^*}\right) \qquad (9.1)\\ &+\left\langle Q\,e^{tB^*}\,(\det Q_t)Q_t^{-1}\,(u-D_t\,x)\,,\,e^{tB^*}\,(\det Q_t)Q_t^{-1}\,(u-D_t\,x)\right\rangle\\ &-2(\det Q_t)\left\langle Q_\infty\,B^*\,Q_\infty^{-1}\,D_t\,x\,,\,\left((\det Q_t)Q_t^{-1}-(\det Q_t)Q_\infty^{-1}\right)(u-D_t\,x)\right\rangle;\end{aligned}$$

notice that here we have placed a factor $\det Q_t$ at each occurrence of Q_t^{-1}.

We denote by $\nu_j,\ j=1,\dots,J$ the eigenvalues of B, and observe that those which are nonreal come in conjugate pairs, and that $\Re\nu_j < 0$ for all j.

Claim 9.2. *The function $t \mapsto \mathcal{N}_t(x,u)$ is a finite linear combination, with coefficients depending on (x,u), of terms which are given by a product of type $\prod_{j=1}^J e^{m_j\nu_j t}$ multiplied by a polynomial in t with complex coefficients. Here $m_j \in \mathbb{Z}$. Further,* *the quantities $|m_j|$ and the degrees of the polynomials are all bounded by a constant depending only on n. This bound also applies to the number of terms.*

Proof. Inspection shows that the last two terms in (9.1) are sums of scalar products of vectors given by multiplying x or u from the left by various combinations of the matrices e^{tB}, e^{tB^*}, D_t, Q_t and $(\det Q_t)Q_t^{-1}$, the constant matrices B^*, Q, Q_∞ and Q_∞^{-1}, and the scalar factor $\det Q_t$. The first term in (9.1) is instead the trace of the product of some of these matrices, multiplied by $\det Q_t$. Let us examine precisely how the matrices listed here depend on t.

We pass from $\mathbb{R}^n$ to $\mathbb{C}^n$ and make a Jordan decomposition of B via a change of coordinates in $\mathbb{C}^n$. Each Jordan block is of the form $\nu_j(I + R)$, where R is a supertriangular and thus nilpotent matrix and I is the identity matrix, of some dimension. Then $\exp(t\nu_j(I+R)) = e^{\nu_j t}P(t)$, where $P(t)$ is a matrix with polynomial entries in t. To arrive at $\exp(tB)$, we put these blocks together and then change coordinates back. The result will be that in the coordinates we had before, each entry of the matrix $\exp(tB)$ is a sum over j of terms of type $e^{\nu_j t}p(t)$, where $p(t)$ is a complex polynomial that may depend on j and on the entry considered. The same will be true for the entries of its adjoint $\exp(tB^*)$. From (2.2) we then see that D_t is of the same form but with $e^{-\nu_j t}$ instead of $e^{\nu_j t}$. Considering the integral in (2.1), we see that the matrix Q_t has similar entries, now with terms $e^{(\nu_j+\nu_{j'})t}p(t)$. Since the entries of the matrix $(\det Q_t)Q_t^{-1}$ are given by minors of Q_t, they will be a sum of terms which are like those described in Claim 9.2. Finally, the scalar $\det Q_t$ also has the same structure.

Claim 9.2 now follows, since the bound on the $|m_j|$ and the degrees of the polynomials is easily verified. □

We observe that Claim 9.2 implies that $\mathcal{N}_t(x,u)$ can be extended to an entire function in t, and so the number of zeros in $(0,1]$ is finite.

This claim means that $\mathcal{N}_t(x,u)$ is a sum of terms given by a function of (x,u) times an expression of type

$$\exp\left(\sum_j m_j\nu_j t\right) P(t) = \exp\left((\lambda + i\mu)t\right) P(t), \tag{9.2}$$

where we write $\sum_j m_j\nu_j = \lambda + i\mu$ and $P(t)$ is a complex polynomial.

We will now find a linear differential operator in t, independent of x and u, that annihilates all these expressions and thus also $\mathcal{N}_t(x,u)$, for all (x,u). For this we denote $D = d/dt$.

If $\mu = 0$ the expression in (9.2) is annihilated by

$$(D-\lambda)^{1+\deg P}.$$

If $\mu \neq 0$, the same expression is annihilated by the operator

$$(D-\lambda-i\mu)^{1+\deg P}.$$

Since $\mathcal{N}_t(x,u)$ coincides with its real part, there is also a term

$$\exp\left((\lambda - i\mu)t\right) \bar{P}(t), \tag{9.3}$$

again multiplied by a function of (x,u), in the sum forming $\mathcal{N}_t(x,u)$. This term is annihilated by

$$(D-\lambda+i\mu)^{1+\deg P}.$$

Clearly both terms (9.2) and (9.3) are annihilated by the product of the two operators, which is

$$\left((D-\lambda)^2+\mu^2)\right)^{1+\deg P}.$$

Consider now all the terms in the sum giving $\mathcal{N}_t(x,u)$. It folllows that $\mathcal{N}_t(x,u)$ is annihilated by a differential operator

$$\mathcal{P}(D)=\prod_{i=1}^{K} P_i(D),$$

where each $P_i(D)$ is of either of the following two types: a first-order operator

$$T_\lambda = D-\lambda$$

or a second-order operator of the form

$$S_{\lambda,\mu}=(D-\lambda)^2+\mu^2.$$

Here $\lambda\in\mathbb{R}$ and $\mu\neq 0$. Clearly all these operators commute, and $\mathcal{P}(D)$ is a polynomial in D with coefficients depending only on n and B, and with leading coefficient 1. The number of factors in $\mathcal{P}(D)$ has a bound also depending only on n and B.

Without restriction, we may assume that $\mu>0$ in each operator $S_{\lambda,\mu}$, and also that the equation $\mathcal{P}(D)\mathcal{N}_t(x,u)=0$ does not allow suppression of any of the factors $P_i(D)$ in the product defining $\mathcal{P}(D)$.

Proposition 9.1 is thus reduced to showing that the number of zeros of a real-valued solution of the equation $\mathcal{P}(D)\phi=0$ in $I=(0,1]$ is bounded by a constant depending only on the polynomial $\mathcal{P}$.

Our next claim deals with one operator T_λ or $S_{\lambda,\mu}$.

Claim 9.3. *Let $\lambda\in\mathbb{R}$ and $\mu>0$, and let $J\subset\mathbb{R}$ be a closed interval of length less than $1/\mu$. Assume that $\phi\in C^2(J)$ is a real-valued function. If $S_{\lambda,\mu}\,\phi$ does not vanish in the interior J° of J, then ϕ has at most two zeros in J. Further, if $S_{\lambda,\mu}\phi$ has at most k zeros in J, then ϕ has at most $2k+2$ zeros in the same interval. The same statements hold with $S_{\lambda,\mu}$ replaced by T_λ.*

Proof. To prove the first assertion about $S_{\lambda,\mu}$, we may take $\lambda=0$ since

$$S_{0,\mu}\,\phi(t)=e^{-\lambda t}\,S_{\lambda,\mu}\big(e^{\lambda t}\,\phi(t)\big),$$

and we will write S_μ for $S_{0,\mu}$. The same trick applies to T_λ.

Since by hypothesis $S_\mu\,\phi\neq 0$ in J°, we may as well take $S_\mu\,\phi>0$ there. We assume by contradiction that $t_1<t_2<t_3$ are three zeros of ϕ in J. Then $\phi''(t_2)=S_\mu\,\phi(t_2)>0$. We can then assume that $\phi'(t_2)\geq 0$, since otherwise we consider instead the function $\phi(-t)$ in the interval $-J$. For $t>t_2$ sufficiently close to t_2 we have

$$\phi(t)=\phi'(t_2)(t-t_2)+\frac{1}{2}\,\phi''(t_2)(t-t_2)^2+o\left((t-t_2)^2\right)>0.$$

Since $\phi(t_3)=0$, the maximal value M of ϕ in the interval $[t_2,t_3]$ must be assumed at some point $t_M\in(t_2,t_3)$. Clearly $M>0$ and $\phi'(t_M)=0$. An integration by parts yields

$$\begin{aligned}M&=\int_{t_2}^{t_M}\phi'(t)\,dt=(t-t_2)\phi'(t)|_{t_2}^{t_M}-\int_{t_2}^{t_M}(t-t_2)\phi''(t)\,dt\\&=-\int_{t_2}^{t_M}(t-t_2)\phi''(t)\,dt.\end{aligned}$$

Since here $-\phi''(t)=\mu^2\phi(t)-S_\mu\,\phi(t)<\mu^2\phi(t)\le\mu^2M$ we conclude that

$$M\le\mu^2M\int_{t_2}^{t_M}(t-t_2)dt=\mu^2M\,\frac{(t_M-t_2)^2}{2}\le\frac{1}{2}\,\mu^2M|J|^2.$$

This leads to the contradiction $|J|\ge\sqrt{2}/\mu$, which proves the first assertion of the claim. The second assertion follows from the first, applied in each of the intervals obtained by deleting from J the zeros of ϕ.

For T_λ it is enough to apply Rolle's theorem to $T_0=D$. □

Conclusion of the proof of Proposition 9.1. We know that $\mathcal{P}(D)\mathcal{N}_t(x,u)=0$, where

$$\mathcal{P}(D)=\prod_{i=1}^{K}P_i(D),$$

and each $P_i(D)$ is T_λ or $S_{\lambda,\mu}$ for some with $\lambda\in\mathbb{R}$, and $\mu>0$ in the case of $S_{\lambda,\mu}$.

Choose a natural number κ larger than all the values of μ appearing here. Then split $[0,1]$ into κ closed subintervals of length $1/\kappa$, and let J be one of these subintervals. Observe that Claim 9.3 applies to each $P_i(D)$ in J, since $1/\kappa<1/\mu$.

Set for $m\in\{2,3,\dots,K\}$

$$\mathcal{N}_t^{(m)}(x,u)=\prod_{i=m}^{K}P_i(D)\,\mathcal{N}_t(x,u),$$

and $\mathcal{N}_t^{(K+1)}(x,u)=\mathcal{N}_t(x,u)$.

We will prove by induction that the function $t\mapsto\mathcal{N}_t^{(m)}(x,u)$ has at most 2^m-2 zeros in J, for $m\in\{2,3,\dots,K+1\}$. Here we fix (x,u). Proposition 9.1 will then follow from the case $m=K+1$.

Starting with $m=2$, we have $P_1(D)\mathcal{N}_t^{(2)}(x,u)=0$, and $\mathcal{N}_t^{(2)}(x,u)$ is not identically 0 for all t. By means of a conjugation with the factor $e^{\lambda t}$ as in the proof of Claim 9.3, we can assume that $P_1(D)$ is either $T_0=D$ or $S_{0,\mu}$. If $P_1(D)=D$, then $\mathcal{N}_t^{(2)}(x,u)$ is a nonzero constant; if $P_1(D)=S_{0,\mu}$ we assume that $t=t_0\in J$ is a zero of $\mathcal{N}_t^{(2)}(x,u)$. Then $\mathcal{N}_t^{(2)}(x,u)$ is proportional to $\sin\left((t-t_0)\mu\right)$ and can have no other zero in J, because $|J|<1/\mu$. The first induction step is verified.

Assume the induction step holds for m. Then $P_m(D)\mathcal{N}_t^{(m+1)}(x,u)=\mathcal{N}_t^{(m)}(x,u)$ has at most 2^m-2 zeros in J, and Claim 9.3 implies that the number of zeros of $\mathcal{N}_t^{(m+1)}(x,u)$ in J is at most $2(2^m-2)+2=2^{m+1}-2$. The induction is complete, and so is the proof of Proposition 9.1. □

10. Estimates in the global region for small t

In this section we estimate the operator $m_0^{\mathrm{glob}}(T)$ with kernel

$$-\int_0^1 \varphi(t)\dot{K}_t(x,u)\big(1-\eta(x,u)\big)\,dt.$$

We shall need the following theorem. In order not to burden the exposition, we postpone its proof to the appendix.

Theorem 10.1. *The maximal operator defined by*

$$S_0^{\mathrm{glob}}f(x)=\int \sup_{0<t\le 1} K_t(x,u)\,\big(1-\eta(x,u)\big)\,|f(u)|\,d\gamma_\infty(u) \tag{10.1}$$

is of weak type $(1,1)$ *with respect to the invariant measure* $d\gamma_\infty$.

This is a sharpened version of the weak type $(1,1)$ estimate for the corresponding part of the maximal operator treated in [10], since the supremum in t is now placed inside the integral. As a consequence, we can prove the following result, which will complete the proof of Theorem 1.1.

Proposition 10.2. *The operator* $m_0^{\mathrm{glob}}(T)$ *is of weak type* $(1,1)$ *with respect to the invariant measure* $d\gamma_\infty$.

Proof of Proposition 10.2. Let $N(x,u)$ be the number of zeros in $(0,1)$ of the function $t\mapsto \dot{K}_t(x,u)$. Proposition 9.1 says that $N(x,u)\le \bar{N}$ for some constant $\bar{N}\ge 1$ that is independent of $(x,u)\in\mathbb{R}^n\times\mathbb{R}^n$ (and dependent only of n and B). We denote these zeros by $t_1(x,u)<\cdots<t_{N(x,u)}(x,u)$, and set $t_0(x,u)=0,\ t_{N(x,u)+1}(x,u)=1$. Since $K_t(x,u)$ vanishes at $t=0$, it follows from the fundamental theorem of calculus that

$$\begin{aligned}\int_0^1 \left|\dot{K}_t(x,u)\right|dt &= \sum_{i=0}^{N(x,u)}\left|\int_{t_i(x,u)}^{t_{i+1}(x,u)}\dot{K}_t(x,u)dt\right|\\ &= \sum_{i=0}^{N(x,u)}\left|K_{t_{i+1}(x,u)}(x,u)-K_{t_i(x,u)}(x,u)\right|\\ &\le 2\sum_{i=0}^{N(x,u)+1}K_{t_i(x,u)}(x,u)\ \lesssim\ \bar{N}\sup_{0<t\le 1}K_t(x,u).\end{aligned}$$

This inequality implies

$$\begin{aligned}\left|m_0^{\mathrm{glob}}(T)f(x)\right| &\le \int\int_0^1\left|\dot{K}_t(x,u)\right|\,dt\ (1-\eta(x,u))\,|f(u)|\,d\gamma_\infty(u)\\ &\lesssim \bar{N}\int \sup_{0<t\le 1}K_t(x,u)\ (1-\eta(x,u))\,|f(u)|\,d\gamma_\infty(u),\end{aligned}$$

and Theorem 10.1 yields

$$\gamma_\infty\left\{x:\left|m_0^{\mathrm{glob}}(T)f(x)\right|>\alpha\right\}\lesssim \frac{1}{\alpha}\int |f(u)|\,d\gamma_\infty(u).$$

□

11. Appendix: Proof of Theorem 10.1

In the proof of this theorem, we take $f \geq 0$ normalized in $L^1(\gamma_\infty)$, and we fix $\alpha > 2$. We claim that we need only consider the intersection of the level set $\{S_0^{\mathrm{glob}} f > \alpha\}$ with the annulus $\mathcal{C}_\alpha$. Clearly, the unbounded component of the complement of $\mathcal{C}_\alpha$ can be neglected as in Subsection 5.2.

To deal with the bounded component, we first observe that the integral in the definition (10.1) of $S_0^{\mathrm{glob}} f(x)$ need only be taken over those u satisfying $\eta(x,u) < 1$. But then Lemma 7.2 shows that $|x-u| > \frac{1}{3}\frac{1}{1+|x|}$. By Lemma 5.4(i), this implies that $K_t(x,u) \lesssim e^{R(x)}(1+|x|)^C$. Thus if x is in the bounded component of the complement of $\mathcal{C}_\alpha$, where $R(x) < \frac{1}{2}\log\alpha$, it follows that $K_t(x,u) \lesssim \alpha$. The claim follows by integration against $f(u)\,d\gamma_\infty(u)$.

Thus we restrict x to $\mathcal{C}_\alpha$; in particular $|x| \gtrsim 1$. We will write x and $u \neq 0$ as $x = D_s\,\tilde{x}$ and $u = D_\sigma\,\tilde{u}$, respectively, where $\tilde{x},\ \tilde{u} \in E_\beta$ with $\beta = (\log\alpha)/2$ and $s \geq 0,\ \sigma \in \mathbb{R}$.

Lemma 11.1. *Let (x,u) be such that $\eta(x,u) < 1$, and let $x \in \mathcal{C}_\alpha$. Then*

$$\sup_{0<t\leq 1} K_t(x,u) \lesssim e^{R(x)} \min\left(|\tilde{u}-\tilde{x}|^{-n},\ |x|^n\right).$$

Proof. For the first bound, we use [10, Lemma 4.3(i)] to get $|D_t\,x - u| \gtrsim |\tilde{x}-\tilde{u}|$, which by (2.10) yields

$$\sup_{0<t\leq 1} K_t(x,u) \lesssim e^{R(x)} \sup_{0<t\leq 1} t^{-n/2} \exp\left(-c\,\frac{|\tilde{x}-\tilde{u}|^2}{t}\right) \lesssim e^{R(x)}\,|\tilde{x}-\tilde{u}|^{-n}.$$

To get the second bound, we use the fact that $|x-u| \gtrsim \frac{1}{1+|x|} \simeq \frac{1}{|x|}$ as seen in the beginning of this section. Applying also Lemma 2.1, we obtain

$$|x|^{-1} \lesssim |x-u| \leq |x - D_t\,x| + |D_t\,x - u| \lesssim t|x| + |D_t\,x - u|.$$

Thus $|x|^{-1} \lesssim t|x|$ or $|x|^{-1} \lesssim |D_t\,x - u|$. In the first case, $t^{-n/2} \lesssim |x|^n$, and the desired estimate is immediate from (2.10). In the second case,

$$K_t(x,u) \lesssim e^{R(x)}\,t^{-\frac{n}{2}} \exp\left(-\frac{c}{t|x|^2}\right) \lesssim e^{R(x)}\,|x|^n.$$

The lemma is proved. □

Continuing the proof of Theorem 10.1, we have from Lemma 11.1 that for $x \in \mathcal{C}_\alpha$

$$S_0^{\mathrm{glob}} f(x) \lesssim e^{R(x)} \int \min\left(|\tilde{u}-\tilde{x}|^{-n},\ |x|^n\right) f(u)\,d\gamma_\infty(u) = A(x) + B(x),$$

where

$$A(x) = |x|^n\,e^{R(x)} \int_{\{u:\,|x|\leq|\tilde{u}-\tilde{x}|^{-1}\}} f(u)\,d\gamma_\infty(u) \tag{11.1}$$

and

$$B(x) = e^{R(x)} \int_{\{u\colon |x|>|\tilde u-\tilde x|^{-1}\}} |\tilde u - \tilde x|^{-n} \, f(u)\, d\gamma_\infty(u).$$

We will show that

$$\gamma_\infty \left\{x \in \mathcal{C}_\alpha : A(x) > \alpha\right\} \lesssim \alpha^{-1} \tag{11.2}$$

and

$$\gamma_\infty \left\{x \in \mathcal{C}_\alpha : B(x) > \alpha\right\} \lesssim \alpha^{-1}. \tag{11.3}$$

Starting with (11.2), we first observe that $A(\tilde x) < \alpha$ for $\tilde x \in E_\beta$ with $\beta = (\log \alpha)/2$, because

$$A(\tilde x) \le |\tilde x|^n \, e^{R(\tilde x)} \int_{\mathbb{R}^n} f(u)\, d\gamma_\infty(u) \lesssim (\log\alpha)^n \sqrt{\alpha} \lesssim \alpha\,.$$

Further, $x = D_s\, \tilde x \in \mathcal{C}_\alpha$ implies $0 < s \lesssim 1$ in view of [10, formula (4.3)]. Let

$$E^0_\beta = \{\tilde x \in E_\beta : A(D_s\, \tilde x) > \alpha \ \text{ for some } s > 0 \text{ with } \ D_s\, \tilde x \in \mathcal{C}_\alpha\},$$

and define for $\tilde x \in E^0_\beta$

$$s_0(\tilde x) = \inf\{s : D_s\, \tilde x \in \mathcal{C}_\alpha \ \text{ and } \ A(D_s\, \tilde x) > \alpha\}.$$

Then $0 < s_0(\tilde x) \lesssim 1$ and $A(D_{s_0(\tilde x)}\, \tilde x) = \alpha$. Moreover, if $A(D_s\, \tilde x) > \alpha$ for some $D_s\, \tilde x \in \mathcal{C}_\alpha$, then $\tilde x \in E^0_\beta$ and $s > s_0(\tilde x)$. In the set $\mathcal{C}_\alpha$, the expression (2.12) for the Lebesgue measure yields $dx \simeq \sqrt{\log\alpha}\, dS_\beta\, ds$, and so

$$\gamma_\infty \left\{x \in \mathcal{C}_\alpha : A(x) > \alpha\right\} \lesssim \sqrt{\log\alpha}\, \int_{E^0_\beta} \int_{s_0(\tilde x)}^{C} e^{-R(D_s\, \tilde x)}\, ds\, dS_\beta(\tilde x).$$

We now write $R(D_s\, \tilde x) = R(D_{s_0(\tilde x)}\, \tilde x) + R(D_s\, \tilde x) - R(D_{s_0(\tilde x)}\, \tilde x)$ and apply the Mean Value Theorem to the difference between the last two terms here, observing that $\partial_s R(D_s\, \tilde x) \simeq |D_s\, \tilde x|^2 \simeq \log\alpha$ because of [10, formula (4.3)]. This leads to

$$\begin{aligned}\gamma_\infty \left\{x \in \mathcal{C}_\alpha :\ A(x) > \alpha\right\} &\lesssim \sqrt{\log\alpha} \int_{E^0_\beta} e^{-R(D_{s_0(\tilde x)}\, \tilde x)} \int_{s_0(\tilde x)}^{\infty} e^{-c(s-s_0(\tilde x))\log\alpha}\, ds\, dS_\beta(\tilde x) \\ &\lesssim \frac{1}{\sqrt{\log\alpha}} \int_{E^0_\beta} e^{-R(D_{s_0(\tilde x)}\, \tilde x)}\, dS_\beta(\tilde x).\end{aligned} \tag{11.4}$$

To deal with the last expression here, we insert the factor $\alpha^{-1} A(D_{s_0(\tilde x)}\tilde x) = 1$ in the integral, using the definition (11.1) of $A(.)$. The two exponentials will then cancel. We also use the fact that $\frac12\sqrt{\log\alpha} \le |D_{s_0(\tilde x)}\tilde x| \le 2\sqrt{\log\alpha}$, both to rewrite the first factor in this definition and to extend the domain of integration in u to $\{u : |\tilde u - \tilde x| \le 2(\log\alpha)^{-1/2}\}$. The result is that the last quantity in (11.4) is at most constant times

$$\frac{1}{\alpha}\, (\log\alpha)^{\frac{n-1}{2}} \int_{E^0_\beta} \int_{\{u\colon |\tilde u-\tilde x|\le 2(\log\alpha)^{-1/2}\}} f(u)\, d\gamma_\infty(u)\, dS_\beta(\tilde x) =$$

$$\frac{1}{\alpha}\, (\log\alpha)^{\frac{n-1}{2}} \int f(u) \int_{\{\tilde x\colon |\tilde u-\tilde x|\le 2(\log\alpha)^{-1/2}\}} dS_\beta(\tilde x)\, d\gamma_\infty(u) \lesssim \frac{1}{\alpha} \int f(u)\, d\gamma_\infty(u) = \frac{1}{\alpha}.$$

This proves (11.2), and we move to (11.3). Here we similarly have $B(\tilde{x}) < \alpha$ for $\tilde{x} \in E_\beta$, and we can define E_β^0 and $s_0(\tilde{x})$ as above, replacing $A(.)$ by $B(.)$. The rest of the argument is only slightly different from that for (11.2); we now have

$$\begin{aligned}\gamma_\infty\{x \in \mathcal{C}_\alpha : B(x) > \alpha\} &\lesssim \frac{1}{\sqrt{\log \alpha}} \int_{E_\beta^0} \exp\left(-R(D_{s_0(\tilde{x})}\,\tilde{x})\right) dS_\beta(\tilde{x}) \\ &\lesssim \frac{1}{\alpha}\,\frac{1}{\sqrt{\log \alpha}} \int_{E_\beta^0} \int_{\{u:\, |\tilde{u}-\tilde{x}|>(\log\alpha)^{-1/2}/2\}} |\tilde{u}-\tilde{x}|^{-n} f(u)\, d\gamma_\infty(u)\, dS_\beta(\tilde{x}) \\ &= \frac{1}{\alpha}\,\frac{1}{\sqrt{\log \alpha}} \int f(u) \int_{\{\tilde{x}:\, |\tilde{u}-\tilde{x}|>(\log\alpha)^{-1/2}/2\}} |\tilde{u}-\tilde{x}|^{-n}\, dS_\beta(\tilde{x})\, d\gamma_\infty(u) \lesssim \frac{1}{\alpha}.\end{aligned}$$

This is (11.3), and Theorem 10.1 is proved.

In order to prove Proposition 10.2, Theorem 10.1 is enough, as we saw in the preceding section. However, we take the opportunity to give the following related result, which strengthens Theorem 10.1 and also Theorem 1.1 in [10] and may be of independent interest.

Theorem 11.2. *The operator S^{glob} defined by*

$$S^{\mathrm{glob}} f(x) = \int \sup_{0<t<\infty} K_t(x,u)\,(1-\eta(x,u))|f(u)|\, d\gamma_\infty(u), \qquad f \in L^1(\mathbb{R}^n),$$

is of weak type $(1,1)$ for the measure $d\gamma_\infty$.

This result is a consequence of Theorem 10.1 and the following proposition.

Proposition 11.3. *The operator S_∞, defined by*

$$S_\infty f(x) = \int \sup_{t\ge 1} K_t(x,u)\, |f(u)|\, d\gamma_\infty(u),$$

satisfies the inequality

$$\gamma_\infty\{x : S_\infty f(x) > \alpha\} \lesssim \frac{1}{\alpha\sqrt{\log\alpha}} \tag{11.5}$$

for all normalized functions f in $L^1(\gamma_\infty)$ and all $\alpha > 2$.

Proof. Let $t \ge 1$. We can again restrict x to $\mathcal{C}_\alpha$, since $K_t(x,u) \lesssim e^{R(x)} < \alpha$ if $R(x) < (\log\alpha)/2$. For $x \in \mathcal{C}_\alpha$, a combination of (2.11) and [10, Lemma 4.3(i)] implies

$$K_t(x,u) \lesssim e^{R(x)} \exp\left(-c\,\left|\tilde{u}-\tilde{x}\right|^2\right),$$

where we use polar coordinates with $\beta = (\log\alpha)/2$. The proposition now follows from Lemma 6.2. □

Remark 11.4. The inequality (11.5), which is sharp as verified in [10, Proposition 6.2], is slightly stronger than the weak type $(1,1)$ estimate in Theorem 11.2. The corresponding estimate for the operator S_0^{glob} is false, since f approximating a point mass at 0 gives a counterexample.

Remark 11.5. In the case $Q = I$ and $B = -I$ an estimate similar to Lemma 11.1 with a kernel $\overline{M}$ controlling from above the Mehler kernel K_t in the global region, has recently been proved in [7] (see, in particular, Definition 3.2 and Proposition 3.4 therein). An earlier result of this type may be found in [30, Proposition 2.1]. These estimates are sharp for significant values of (x, u), whereas our Theorem 11.2 is simpler, and sufficient for our needs. Moreover, Proposition 11.3 is stronger than the analogous bounds in [7] and [30].

References

[1] V. Almeida, J. J. Betancor, P. Quijano and L. Rodríguez-Mesa, Maximal operator, Littlewood–Paley functions and variation operators associated with nonsymmetric Ornstein-Uhlenbeck operators, *Mediterr. J. Math.* **20**, 196 (2023).

[2] V. Almeida, J.J. Betancor, J.C. Fariña, P. Quijano, L. Rodríguez-Mesa, Littlewood–Paley functions associated with general Ornstein-Uhlenbeck semigroups, `arXiv:2202.06136`.

[3] D. S. Bernstein, *Matrix Mathematics. Theory, Facts, and Formulas*, (2009), Second Edition, Princeton University Press.

[4] J. J. Betancor, E. Dalmasso, P. Quijano, BLO spaces associated with Laguerre polynomials expansions, `arXiv:2302.04356v1`.

[5] J. J. Betancor, T. Martínez, L. Rodríguez-Mesa, Laplace transform type multipliers for Hankel transforms, *Canad. Math. Bull.* **51**, (2008), 487–496.

[6] V. I. Bogachev, Ornstein–Uhlenbeck operators and semigroups, *Russ. Math. Surv.* (2018), **73**, 191–260.

[7] T. Bruno, Endpoint results for the Riesz transform of the Ornstein–Uhlenbeck operator, *J. Fourier Anal. Appl.*, (2019), **25**, 1609–1631.

[8] A. Carbonaro and O. Dragičević, Bounded holomorphic functional calculus for nonsymmetric Ornstein-Uhlenbeck operators, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **XIX** (5) (2019), 1497–1533.

[9] V. Casarino, P. Ciatti and P. Sjögren, The maximal operator of a normal Ornstein-Uhlenbeck semigroup is of weak type (1,1), *Ann. Sc. Norm. Sup. Pisa Cl. Sci.* (5) **XXI** (2020), 385-410.

[10] V. Casarino, P. Ciatti and P. Sjögren, On the maximal operator of a general Ornstein–Uhlenbeck semigroup, *Math. Z.* **301** (2022), 2393–2413.

[11] V. Casarino, P. Ciatti and P. Sjögren, Riesz transforms of a general Ornstein–Uhlenbeck semigroup, *Calculus of Variations and Partial Differential Equations*, **60**, 135 (2021).

[12] V. Casarino, P. Ciatti and P. Sjögren, On the orthogonality of generalized eigenspaces for the Ornstein–Uhlenbeck operator, *Arch. Math.* **117** (2021), 547–556.

[13] V. Casarino, P. Ciatti and P. Sjögren, On the variation operator for the Ornstein–Uhlenbeck semigroup in dimension one, *Ann. Mat. Pura e Appl.* **203** (2024), 205–219.

[14] M. Cowling, I. Doust, A. McIntosh and A. Yagi, Banach space operators with a bounded H^∞ functional calculus, *J. Austral. Math. Soc. (Series A)* , **60** (1996), 51–89.

[15] R. Chill, E. Fašangová, G. Metafune and D. Pallara, The sector of analyticity of the Ornstein–Uhlenbeck semigroup in L^p spaces with respect to invariant measure, *J. Lond. Math. Soc.* **71**, (2005), 703–722.

[16] R. Chill, E. Fašangová, G. Metafune and D. Pallara, The sector of analyticity of nonsymmetric submarkovian semigroups generated by elliptic operators, *C. R. Math. Acad. Sci. Paris* **342**, (2006), 909–914.

[17] A. Cianchi, V. Musil and L. Pick, Moser inequalities in Gauss space, *Math. Ann.* (2020), **377**, 1265–1312.

[18] A. Cianchi, V. Musil and L. Pick, On the Existence of Extremals for Moser-Type Inequalities in Gauss Space, *Int. Math. Res. Not.* (2022), 2022, Issue 2, 1494–1537.

[19] A. Cianchi, V. Musil and L. Pick, Sharp exponential inequalities for the Ornstein-Uhlenbeck operator, *J. Funct. Anal.* **281** (2021), no. 11, Paper No. 109217.

[20] E. B. Davies, *One-Parameter Semigroups*, Academic Press, London, 1980.

[21] I. Drelichman, R. G. Durán, P.L. de Nápoli, Multipliers of Laplace transform type for Laguerre and Hermite expansions, *Studia Math.* **203**, (2011), 265–290.

[22] K. J. Engel and R. Nagel, *One parameter semigroups for linear evolution equations*, 194 Springer Graduate Texts in Mathematics, 2000.

[23] J. García-Cuerva, G. Mauceri, P. Sjögren and J. L. Torrea, Higher-order Riesz operators for the Ornstein-Uhlenbeck semigroup, *Potential Anal.*, **10** (1999), 379–407.

[24] J. García-Cuerva, G. Mauceri, P. Sjögren and J. L. Torrea, Spectral multipliers for the Ornstein-Uhlenbeck semigroup, *J. Anal. Math.*, **78** (1999), 281–305.

[25] M. Haase and F. Pannasch, Holomorphic Hörmander-Type Functional Calculus on Sectors and Strips, `arXiv:2204.03936`.

[26] M. Kemppainen, An L^1-estimate for certain spectral multipliers associated with the Ornstein–Uhlenbeck operator, *J. Fourier Anal. Appl.* (2016) **22**(6), 1416–1430.

[27] M. Kemppainen, Admissible decomposition for spectral multipliers on Gaussian L^p, *Math. Z.* (2018) **289**, 983–994.

[28] L. Lorenzi and M. Bertoldi, *Analytical methods for Markov semigroups*, Pure and Applied Mathematics (Boca Raton), 283, Chapman & Hall/CRC, Boca Raton, FL, 2007.

[29] A. McIntosh, Operators which have an H_∞ functional calculus, *Miniconference on operator theory and partial differential equations*, (Proc. Centre Math. Analysis 24, ANU, Canberra, 1986), 210–231.

[30] T. Menárguez, S. Pérez and F. Soria, The Mehler maximal function: a geometric proof of the weak type 1, *J. London Math. Soc.* (2000), **61**, 846–856.

[31] G.Metafune, D. Pallara and E. Priola, Spectrum of Ornstein-Uhlenbeck operators in L^p spaces with respect to invariant measures, *J. Funct. Anal.* **196** (2002), 40–60.

[32] G. Metafune, J. Prüss, A. Rhandi, and R. Schnaubelt, The domain of the Ornstein-Uhlenbeck operator on a L^p-space with invariant measure, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **1**, (2002) 471–487.

[33] A. Nowak and K. Stempak, Potential operators and Laplace type multipliers associated with the twisted laplacian, *Acta Mathematica Scientia* **37**, (2017), 280–292.

[34] L. S. Ornstein and G. E. Uhlenbeck, On the theory of Brownian Motion, *Phys. Rev.* **36**, (1930) 823–841.

[35] F. Pannasch, The Holomorphic Hörmander Functional Calculus. Ph.D. Thesis, Kiel University, 2019, `https:d-nb.info/1202630561/34`.

[36] E. M. Stein, Topics in harmonic analysis related to the Littlewood-Paley theory, Annals of Mathematics Studies, No. 63 Princeton University Press, Princeton, N.J..

[37] T.Z. Szarek, Multipliers of Laplace transform type in certain Dunkl and Laguerre settings, *Bull. Austral. Math. Soc.* **85**, (2012) 177–190.

[38] W. Urbina-Romero, *Gaussian Harmonic Analysis*, Springer Monographs in Mathematics, 2019.

[39] B. Wróbel, Multivariate spectral multipliers for systems of Ornstein-Uhlenbeck operators, *Studia Math.* (2013), **216**, 47–67.

[40] B. Wróbel, Joint Spectral Multipliers for Mixed Systems of Operators, *J.Fourier Anal. Appl.* (2017), **23**, 245–287.

[41] B. Wróbel, Laplace Type Multipliers for Laguerre Expansions of Hermite Type, *Mediterr. J. Math.* **10**, (2013), 1867–1881.

DTG, Università degli Studi di Padova, Stradella san Nicola 3, I-36100 Vicenza, Italy
Email address: valentina.casarino@unipd.it

Dipartimento di Matematica "Tullio Levi Civita", Università degli Studi di Padova, Via Trieste, 63, 35131 Padova,, Italy
Email address: paolo.ciatti@unipd.it

Mathematical Sciences, University of Gothenburg and Mathematical Sciences, Chalmers University of Technology, SE - 412 96 Göteborg, Sweden
Email address: peters@chalmers.se