

# MULTIPLICATIVE RELATIONS AMONG DIFFERENCES OF SINGULAR MODULI

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*To Jonathan Pila*

ABSTRACT. Let  $n \in \mathbb{Z}_{>0}$ . We prove that there exist a finite set  $V$  and finitely many algebraic curves  $T_1, \dots, T_k$  with the following property: if  $(x_1, \dots, x_n, y)$  is an  $(n+1)$ -tuple of pairwise distinct singular moduli such that  $\prod_{i=1}^n (x_i - y)^{a_i} = 1$  for some  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$ , then  $(x_1, \dots, x_n, y) \in V \cup T_1 \cup \dots \cup T_k$ . Further, the curves  $T_1, \dots, T_k$  may be determined explicitly for a given  $n$ .

## 1. INTRODUCTION

Let  $\mathbb{H}$  denote the complex upper half plane. The modular group  $\mathrm{SL}_2(\mathbb{Z})$  acts on  $\mathbb{H}$  by fractional linear transformations. The modular  $j$ -function  $j: \mathbb{H} \rightarrow \mathbb{C}$  is the unique holomorphic function  $\mathbb{H} \rightarrow \mathbb{C}$  which is invariant under this action of  $\mathrm{SL}_2(\mathbb{Z})$ , has a simple pole at  $i\infty$ , and satisfies  $j(i) = 1728$  and  $j(\rho) = 0$ , where  $\rho = \exp(2\pi i/3)$ .

A singular modulus is a complex number  $j(\tau)$  for some  $\tau \in \mathbb{H}$  such that  $[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$ . For example, 0 and 1728 are both singular moduli. Equivalently, a singular modulus is the  $j$ -invariant of an elliptic curve with complex multiplication. Singular moduli are algebraic integers and generate the ring class fields of imaginary quadratic fields. By Schneider's theorem [29, IIc], if  $\tau \in \mathbb{H}$  is such that both  $\tau, j(\tau) \in \overline{\mathbb{Q}}$ , then  $j(\tau)$  is a singular modulus.

In this paper, we consider multiplicative relations among differences  $x - y$  of singular moduli  $x, y$ . Since 0 is a singular modulus, every singular modulus is equal to the difference of two singular moduli. Our aim is to generalise the following theorem.

**Theorem 1.1.** *Let  $n \in \mathbb{Z}_{>0}$ . Let  $y$  be a singular modulus. Then there exist only finitely many  $n$ -tuples  $(x_1, \dots, x_n)$  of pairwise distinct singular moduli  $x_1, \dots, x_n$  such that  $y \notin \{x_1, \dots, x_n\}$  and there exist  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$  for which*

$$\prod_{i=1}^n (x_i - y)^{a_i} = 1.$$

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Theorem 1.1 was proved by Pila and Tsimerman [24] for  $y = 0$  and by the third author [11] for  $y$  not in the real interval  $(0, 1728)$ . In Section 3, we show that the result of [11] directly implies the remaining case where  $y$  is in the real interval  $(0, 1728)$ .

This paper addresses the case where  $y$  is allowed to vary over all singular moduli. That is, we consider  $(n+1)$ -tuples  $(x_1, \dots, x_n, y)$  of pairwise distinct singular moduli  $x_1, \dots, x_n, y$  such that

$$(1.1) \quad \prod_{i=1}^n (x_i - y)^{a_i} = 1 \text{ for some } a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}.$$

In this setting, one must account for the following situation.

**Definition 1.2** ([4, p. 1052]). A function  $f: \mathbb{H} \rightarrow \mathbb{C}$  is called a  $j$ -map if either there exists a singular modulus  $x$  such that  $f(z) = x$  for every  $z \in \mathbb{H}$ , or there exists  $g \in \mathrm{GL}_2^+(\mathbb{Q})$  such that  $f(z) = j(gz)$  for every  $z \in \mathbb{H}$ . Here  $\mathrm{GL}_2^+(\mathbb{Q})$  acts on  $\mathbb{H}$  by fractional linear transformations.

**Definition 1.3.** Let  $n \in \mathbb{Z}_{>0}$ . Let  $f_1, \dots, f_n, f$  be pairwise distinct  $j$ -maps, at least one of which is non-constant. The set

$$\left\{ (f_1(z), \dots, f_n(z), f(z)) : z \in \mathbb{H} \right\}$$

is called a multiplicative special curve in  $\mathbb{C}^{n+1}$  if there exist  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$  such that, for all  $z \in \mathbb{H}$ ,

$$\prod_{i=1}^n (f_i(z) - f(z))^{a_i} = 1.$$

Note that a multiplicative special curve is always an algebraic curve (see Proposition 5.3). Clearly, any multiplicative special curve contains infinitely many  $(n+1)$ -tuples  $(x_1, \dots, x_n, y)$  of pairwise distinct singular moduli satisfying (1.1). If  $N \in \mathbb{Z}_{>0}$  is not a perfect square, then the modular polynomial  $\Phi_N \in \mathbb{Z}[X, Y]$  gives rise to a multiplicative special curve, as we explain in Section 1.1. Thus one cannot hope to show, for an arbitrary  $n \in \mathbb{Z}_{>0}$ , that there exist only finitely many such  $(n+1)$ -tuples  $(x_1, \dots, x_n, y)$ .

Instead, we prove that the multiplicative special curves arising from the modular polynomials are the only multiplicative special curves. In particular, for a given  $n$ , there are only finitely many multiplicative special curves in  $\mathbb{C}^{n+1}$  and these may be determined effectively.

**Theorem 1.4.** *Let  $n \in \mathbb{Z}_{>0}$ . Then there are only finitely many multiplicative special curves in  $\mathbb{C}^{n+1}$  and these may be determined effectively. If  $n \leq 5$ , then there are no multiplicative special curves in  $\mathbb{C}^{n+1}$ .*

We then prove that, for every  $n \in \mathbb{Z}_{>0}$ , the finitely many multiplicative special curves in  $\mathbb{C}^{n+1}$  account for all but finitely many of the  $(n+1)$ -tuples  $(x_1, \dots, x_n, y)$  of pairwise distinct singular moduli satisfying (1.1).

**Theorem 1.5.** *Let  $n \in \mathbb{Z}_{>0}$ . Then there exist only finitely many  $(n+1)$ -tuples  $(x_1, \dots, x_n, y)$  of pairwise distinct singular moduli  $x_1, \dots, x_n, y$  such that*

$$\prod_{i=1}^n (x_i - y)^{a_i} = 1$$

for some  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$  and  $(x_1, \dots, x_n, y)$  does not belong to one of the finitely many multiplicative special curves in  $\mathbb{C}^{n+1}$ .

Since there are no multiplicative special curves in  $\mathbb{C}^{n+1}$  for  $n \leq 5$ , one immediately obtains the following corollary. Since (see Example 1.7) there exists a multiplicative special curve in  $\mathbb{C}^7$ , the bound of  $n \leq 5$  in this corollary is sharp.

**Corollary 1.6.** *Let  $n \in \{1, \dots, 5\}$ . There exist only finitely many  $(n + 1)$ -tuples  $(x_1, \dots, x_n, y)$  of pairwise distinct singular moduli  $x_1, \dots, x_n, y$  such that*

$$\prod_{i=1}^n (x_i - y)^{a_i} = 1$$

for some  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$ .

The proof of Theorem 1.5 uses o-minimality and is ineffective. Recently, Li [19] has proved that the difference of two singular moduli is never a unit (in the ring of algebraic integers). Hence, there are no distinct singular moduli  $x, y$  such that  $(x - y)^a = 1$  for some  $a \in \mathbb{Z} \setminus \{0\}$ .

**1.1. Modular polynomials and multiplicative special curves.** For background on modular polynomials, see [8, §11]. For  $N \in \mathbb{Z}_{>0}$ , let

$$C(N) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad = N, a > 0, 0 \leq b < d, \gcd(a, b, d) = 1 \right\}.$$

There exists [8, (11.15)] a polynomial  $\Phi_N \in \mathbb{Z}[X, Y]$  with the property that

$$\Phi_N(X, j(z)) = \prod_{g \in C(N)} (X - j(gz))$$

for all  $z \in \mathbb{H}$ . The polynomial  $\Phi_N$  is called the  $N$ th modular polynomial.

For  $N > 1$ , let  $F_N \in \mathbb{Z}[X]$  be defined by  $F_N(X) = \Phi_N(X, X)$ . Then  $F_N$  is a non-constant polynomial (the explicit formula in [8, Proposition 13.8] in fact implies that  $\deg F_N \geq 2N$ ). The roots of  $F_N$  are all singular moduli (see Corollary 2.4). If  $N$  is not a perfect square, then, by [8, Theorem 11.18], the polynomial  $F_N$  has leading coefficient  $\pm 1$ .

Suppose then that  $N \in \mathbb{Z}_{>1}$  is such that the leading coefficient of  $F_N$  is  $\pm 1$  (e.g. take  $N$  not a perfect square). Write  $\alpha_1, \dots, \alpha_k$  for the distinct roots of  $F_N$  and  $a_i$  for their multiplicities. Write  $g_1, \dots, g_l$  for the elements of  $C(N)$ . Since

$$F_N(j(z)) = \prod_{i=1}^l (j(z) - j(g_i z))$$

for all  $z \in \mathbb{H}$ , one thus obtains (doubling the exponents to eliminate a potential factor of  $-1$ ) that

$$\prod_{i=1}^k (j(z) - \alpha_i)^{2a_i} = \prod_{i=1}^l (j(z) - j(g_i z))^2$$

for all  $z \in \mathbb{H}$ , and hence, for all  $z \in \mathbb{H}$ ,

$$(1.2) \quad \prod_{i=1}^k (j(z) - \alpha_i)^{2a_i} \prod_{i=1}^l (j(z) - j(g_i z))^{-2} = 1.$$

In particular, the set

$$\left\{ (\alpha_1, \dots, \alpha_k, j(g_1 z), \dots, j(g_l z), j(z)) : z \in \mathbb{H} \right\}$$

is a multiplicative special curve in  $\mathbb{C}^{k+l+1}$ .

Further examples of multiplicative special curves may be generated by multiplying together integer powers of relations of the form (1.2) coming from different  $F_N$ . In this case, one must also consider polynomials  $F_N$  with leading coefficient not equal to  $\pm 1$ , because these leading coefficients may cancel with one another. For example,  $-2$  is the leading coefficient of both  $F_4$  and  $F_{16}$ . Theorem 4.1 will show that all the multiplicative special curves arise from the polynomials  $F_N$  in this way.

**Example 1.7.** The modular polynomial  $\Phi_2$  is

$$\begin{aligned} \Phi_2(X, Y) = & -X^2 Y^2 + X^3 + Y^3 + 1488(X^2 Y + XY^2) \\ & - 162000(X^2 + Y^2) + 40773375XY \\ & + 8748000000(X + Y) - 157464000000000. \end{aligned}$$

Thus,

$$\begin{aligned} F_2(X) = & \Phi_2(X, X) \\ = & -X^4 + 2978X^3 + 40449375X^2 + 17496000000X \\ & - 157464000000000 \\ = & -(X - 1728)(X + 3375)^2(X - 8000). \end{aligned}$$

Note that 1728,  $-3375$ , 8000 are singular moduli of discriminant  $-4$ ,  $-7$ ,  $-8$  respectively. The discriminant of a singular modulus  $j(\tau)$  is  $b^2 - 4ac$ , where  $a, b, c \in \mathbb{Z}$ , not all zero, are such that  $a\tau^2 + b\tau + c = 0$  and  $\gcd(a, b, c) = 1$ .

Observe that

$$C(2) = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \right\}.$$

Thus,

$$\Phi_2(X, j(z)) = \left( X - j(2z) \right) \left( X - j\left(\frac{z}{2}\right) \right) \left( X - j\left(\frac{z+1}{2}\right) \right).$$

Hence, for all  $z \in \mathbb{H}$ ,

$$(1.3) \quad \begin{aligned} & - (j(z) - 1728)(j(z) + 3375)^2(j(z) - 8000) \\ = & \left( j(z) - j(2z) \right) \left( j(z) - j\left(\frac{z}{2}\right) \right) \left( j(z) - j\left(\frac{z+1}{2}\right) \right). \end{aligned}$$

The set

$$\left\{ \left( 1728, -3375, 8000, j(2z), j\left(\frac{z}{2}\right), j\left(\frac{z+1}{2}\right), j(z) \right) : z \in \mathbb{H} \right\}$$

is thus a multiplicative special curve in  $\mathbb{C}^7$ .

For an example of a 7-tuple of singular moduli lying on this curve, take

$$z = \frac{-1 + \sqrt{163i}}{2}.$$

Then

$$j(z) = -262537412640768000 = -2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3,$$

which we denote by  $k$ , is a singular modulus of discriminant  $-163$ . In this case,  $j(2z), j(z/2), j((z+1)/2)$  are the three singular moduli of discriminant  $-652$ . These are respectively the roots  $r, s, \bar{s}$  of the irreducible polynomial

$$\begin{aligned} &X^3 - 68925893036109279891085639286946000X^2 \\ &+ 102561728837719322645921325412908000000X \\ &- 1809562562166552295369395087267520089269224800000000, \end{aligned}$$

where  $r \in \mathbb{R}$  and  $s, \bar{s}$  are complex conjugate with  $s \in \mathbb{H}$ . In this case, (1.3) yields that

$$(1.4) \quad -(k - 1728)(k + 3375)^2(k - 8000) = (k - r)(k - s)(k - \bar{s}).$$

The prime factorisation of the two sides of (1.4) is given by

$$\begin{aligned} &- 2^{12} \cdot 3^{22} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 31^2 \cdot 37 \cdot 101 \cdot 103^2 \cdot 127^2 \\ &\cdot 157 \cdot 163 \cdot 229^2 \cdot 277 \cdot 283^2 \cdot 317. \end{aligned}$$

**1.2. Multiplicative properties of differences of singular moduli.** The study of the multiplicative properties of differences of singular moduli goes back at least as far as Berwick [1], who in 1927 determined the factorisations of  $x$  and  $x - 1728$  for all singular moduli  $x$  such that  $[\mathbb{Q}(x) : \mathbb{Q}] \leq 3$ .

The differences of singular moduli are highly divisible numbers, in the sense that they tend to have relatively small prime factors. For example,

$$j\left(\frac{-1 + \sqrt{163i}}{2}\right) - j\left(\frac{-1 + \sqrt{67i}}{2}\right) = -2^{15} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 139 \cdot 331.$$

Example 1.7 gives another illustration of this tendency. This observation led Gross and Zagier [13] to prove a formula for the prime ideal factorisations of differences of singular moduli, subject to some restrictions on the discriminants of the singular moduli considered. A version of their result for arbitrary discriminants has since been proved by Lauter and Viray [18].

Recent work on multiplicative relations among singular moduli, for example the proof of Theorem 1.1 by Pila and Tsimerman [24] and the third author [11], has been motivated by connections to the Zilber–Pink conjecture on atypical intersections.

Effective results on multiplicative relations among singular moduli in low dimensions have also been studied extensively [2, 5, 10, 12, 28] as special cases of the André–Oort conjecture for  $\mathbb{C}^n$ , which was proved ineffectively by Pila [22]. In particular, for  $n \leq 3$ , explicit bounds on multiplicatively dependent  $n$ -tuples of pairwise distinct singular moduli are known [2, 28].

For differences of singular moduli, the most general effective result we are aware of is Li’s result [19] that the difference of two singular moduli is never an algebraic unit. Li’s result generalised Bilu, Habegger, and Kühne’s theorem [3] that no singular modulus is a unit. Work on this topic was

prompted by a question of Masser, answered affirmatively by Habegger [14], as to whether only finitely many singular moduli are algebraic units.

**1.3. Structure of this paper.** In Section 2, we give some of the basic results we will need for this paper. Section 3 completes the proof of Theorem 1.1. The proof of Theorem 1.4 is in Section 4. Section 5 contains the functional transcendence results which are required for the proof of Theorem 1.5, which is then carried out in Section 6. Finally, the connection to the Zilber–Pink conjecture is considered in Section 7.

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## 2. PRELIMINARIES

**2.1. The fundamental domain.** The group  $\mathrm{SL}_2(\mathbb{Z})$  is generated by the matrices corresponding to the transformations  $T: z \mapsto z + 1$  and  $S: z \rightarrow -1/z$ . Let  $\mathfrak{F}_j$  be the fundamental domain for the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathbb{H}$  given by

$$\left\{ z \in \mathbb{H} : \operatorname{Re} z \in \left[ -\frac{1}{2}, \frac{1}{2} \right), |z| \geq 1, \text{ and if } |z| = 1, \text{ then } \operatorname{Re} z \in \left[ -\frac{1}{2}, 0 \right] \right\}.$$

This is a hyperbolic triangle with corners at  $\rho, -\bar{\rho}, i\infty$ . The  $j$ -function restricts to a bijection  $j: \mathfrak{F}_j \rightarrow \mathbb{C}$ .

The  $j$ -function has a series expansion

$$j(z) = e^{-2\pi iz} + 744 + \sum_{n=1}^{\infty} c(n)e^{2n\pi iz},$$

where the coefficients  $c(n) \in \mathbb{Z}$ . It follows immediately that the  $j$ -function is real valued on  $\mathfrak{F}_j$  only along the boundary of  $\mathfrak{F}_j$  and on the imaginary axis. Further, the image under  $j$  of the set  $\{z \in \mathfrak{F}_j : |z| = 1\}$  is the real interval  $[0, 1728]$ .

**Proposition 2.1.** *Let  $z_0 \in \mathfrak{F}_j$ . If  $\operatorname{Re} z_0 \neq -1/2$ , then the  $\mathrm{SL}_2(\mathbb{Z})$ -orbit of  $z_0$  is equal to*

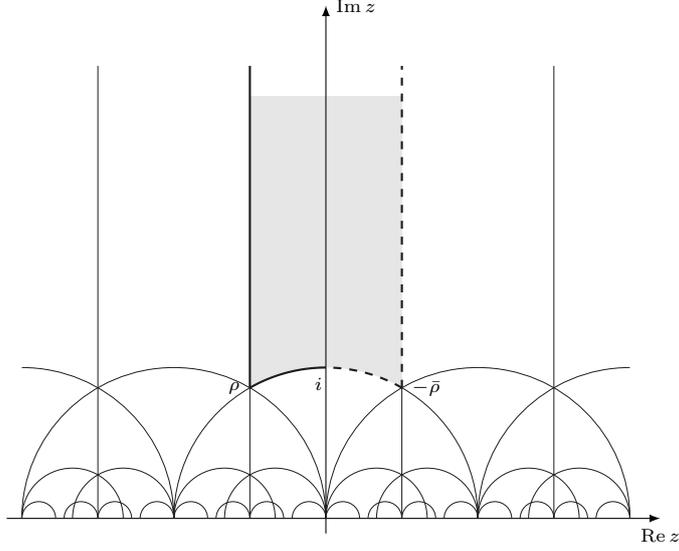
$$\begin{aligned} & \{z_0 + k : k \in \mathbb{Z}\} \cup \left\{ \frac{-1}{z_0} + k : k \in \mathbb{Z} \right\} \\ & \cup \left\{ w \in \mathbb{H} : w \in \operatorname{Orbit}(z_0) \text{ and } \operatorname{Im} w < \operatorname{Im} \frac{-1}{z_0} \right\}. \end{aligned}$$

*If  $\operatorname{Re} z_0 = -1/2$ , then the  $\mathrm{SL}_2(\mathbb{Z})$ -orbit of  $z_0$  is equal to*

$$\begin{aligned} & \{z_0 + k : k \in \mathbb{Z}\} \cup \left\{ \frac{-1}{z_0} + k : k \in \mathbb{Z} \right\} \cup \left\{ \frac{-1}{z_0 + 1} + k : k \in \mathbb{Z} \right\} \\ & \cup \left\{ w \in \mathbb{H} : w \in \operatorname{Orbit}(z_0) \text{ and } \operatorname{Im} w < \operatorname{Im} \frac{-1}{z_0} \right\}. \end{aligned}$$

*Proof.* First, we claim that the following algorithm applied to a point  $z \in \mathbb{H}$  will output the unique point in  $\mathfrak{F}_j \cap \operatorname{Orbit}(z)$ .

- (1) If  $z \in \mathfrak{F}_j$ , then output  $z$ . Otherwise proceed to step (2).
- (2) Replace  $z$  with  $z + k$ , where  $k \in \mathbb{Z}$  is such that  $\operatorname{Re}(z + k) \in [-1/2, 1/2)$ .


 FIGURE 1. The fundamental domain  $\mathfrak{F}_j$ 

- (3) If  $z \in \mathfrak{F}_j$ , then output  $z$ . Otherwise proceed to step (4).  
 (4) Replace  $z$  with  $-1/z$ . Return to step (1).

Clearly, if this algorithm terminates, then it outputs the unique point in  $\mathfrak{F}_j$  in the same  $\mathrm{SL}_2(\mathbb{Z})$ -orbit as the initial input. We claim that this algorithm always terminates. To prove this, note that  $S: z \mapsto -1/z$  sends  $z$  to

$$-\frac{\operatorname{Re} z}{|z|^2} + \frac{\operatorname{Im} z}{|z|^2}i.$$

In particular, if  $|z| < 1$ , then the imaginary part of  $-1/z$  is strictly larger than  $\operatorname{Im} z$ . Now every application of step (4) is performed on some  $z$  with  $|z| \leq 1$ . And if  $|z| = 1$ , then applying (4) immediately yields a point in  $\mathfrak{F}_j$ .

Hence, it suffices to prove that, given  $z \in \mathbb{H}$  with  $\operatorname{Re} z \in [-1/2, 1/2)$ , there are only finitely many  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  with  $\operatorname{Re} \gamma z \in [-1/2, 1/2)$  and  $\operatorname{Im} \gamma z > \operatorname{Im} z$ . Write

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$\operatorname{Im} \gamma z = \operatorname{Im} \frac{az + b}{cz + d} = \frac{1}{|cz + d|^2} \operatorname{Im} z,$$

and so  $\operatorname{Im} \gamma z > \operatorname{Im} z$  implies that

$$|cz + d|^2 < 1.$$

Hence,

$$(c \operatorname{Re} z + d)^2 + (c \operatorname{Im} z)^2 < 1.$$

Hence, there are only finitely many possibilities for  $c$ , and for each such  $c$ , only finitely many possibilities for  $d$ . So we may assume that  $c, d$  are fixed.

We now show that, for the pair  $(c, d)$ , there exists a unique pair  $(a, b)$  such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and

$$\mathrm{Re} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} z \right) \in \left[ -\frac{1}{2}, \frac{1}{2} \right).$$

Suppose  $a, b, a', b'$  are such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

So  $ad - bc = 1$  and  $a'd - b'c = 1$ . Hence, by Bézout's Lemma, there exists  $k \in \mathbb{Z}$  such that

$$(a', b') = (a + kc, b + kd).$$

Thus,

$$\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} z = \frac{(a + kc)z + (b + kd)}{cz + d} = \frac{az + b}{cz + d} + k.$$

In particular, there is a unique  $k \in \mathbb{Z}$  (and hence a unique pair  $(a', b')$ ) such that

$$\mathrm{Re} \left( \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} z \right) \in \left[ -\frac{1}{2}, \frac{1}{2} \right).$$

Thus, the algorithm always terminates. We may now complete the proof of the proposition. Let  $z_0 \in \mathfrak{F}_j$ . The proposition amounts to classifying all the  $w \in \mathrm{Orbit}(z_0)$  such that

$$\mathrm{Im} w \geq \mathrm{Im} \frac{-1}{z_0}.$$

Suppose first that  $\mathrm{Re} z_0 \neq -1/2$ . Let  $w_0 \in \mathrm{Orbit}(z_0)$  be such that

$$w_0 \notin \{z_0 + k : k \in \mathbb{Z}\} \cup \left\{ \frac{-1}{z_0} + k : k \in \mathbb{Z} \right\}.$$

We claim that  $\mathrm{Im} w_0 < \mathrm{Im}(-1/z_0)$ . Applying the above algorithm to  $w_0$ , we must obtain  $z_0$  after finitely many steps. The last transformation applied is either  $z \mapsto -1/z$  or  $z \mapsto z + k$  for some  $k \in \mathbb{Z} \setminus \{0\}$ . Recall that  $z \mapsto -1/z$  is its own inverse.

If the last transformation applied is  $z \mapsto -1/z$ , then the algorithm applied to  $w_0$  must pass through  $-1/z_0$ . By assumption on  $w_0$ , this must happen after an application of  $z \mapsto -1/z$ , which must have strictly increased the imaginary part.

If the last transformation applied is  $z \mapsto z + k$  with  $k \neq 0$ , then the transformation prior to that must have been  $z \mapsto -1/z$ . We thus must have that

$$\mathrm{Im} w_0 \leq \mathrm{Im} \frac{-1}{z_0 - k} = \frac{1}{|z_0 - k|^2} \mathrm{Im} z_0.$$

Then

$$|z_0 - k| > |z_0| \geq 1,$$

since  $\mathrm{Re} z_0 \in (-1/2, 1/2)$  and  $k \neq 0$ . Hence,

$$\mathrm{Im} w_0 < \frac{1}{|z_0|^2} \mathrm{Im} z_0 = \mathrm{Im} \frac{-1}{z_0} \leq \mathrm{Im} z_0.$$

Now let  $z_0 \in \mathfrak{F}_j$  be such that  $\operatorname{Re} z_0 = -1/2$ . Let  $w_0 \in \operatorname{Orbit}(z_0)$  be such that

$$w_0 \notin \{z_0 + k : k \in \mathbb{Z}\} \cup \left\{ \frac{-1}{z_0} + k : k \in \mathbb{Z} \right\} \cup \left\{ \frac{-1}{z_0 + 1} + k : k \in \mathbb{Z} \right\}.$$

We claim that  $\operatorname{Im} w_0 < \operatorname{Im}(-1/z_0)$ . To show this, we repeat the above argument. The only place where  $\operatorname{Re} z \neq -1/2$  was used was to obtain the inequality

$$|z_0 - k| > |z_0|.$$

If  $k \neq -1$ , then this inequality still holds and the above argument works. So assume  $k = -1$ . Then, by the assumption that

$$w_0 \notin \left\{ \frac{-1}{z_0 + 1} + k : k \in \mathbb{Z} \right\},$$

there must have been an application of  $z \mapsto -1/z$  prior to passing through  $-1/(z_0 + 1)$  and this application must have strictly increased the imaginary part. Hence,

$$\operatorname{Im} w_0 < \frac{1}{|z_0 + 1|^2} \operatorname{Im} z_0 = \operatorname{Im} \frac{-1}{z_0} \leq \operatorname{Im} z_0.$$

This completes the proof.  $\square$

**2.2. Singular moduli.** Let  $\tau \in \mathbb{H}$  be such that  $[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$ . So

$$a\tau^2 + b\tau + c = 0$$

for some  $(a, b, c) \in \mathbb{Z}^3 \setminus \{(0, 0, 0)\}$  with  $\gcd(a, b, c) = 1$ . The discriminant of the singular modulus  $j(\tau)$  is defined to be

$$b^2 - 4ac.$$

This depends only on the value of  $j(\tau)$  and not on the choice of  $\tau$ . For a singular modulus  $x$ , write  $\Delta(x)$  for the discriminant of  $x$ . The singular moduli of a given discriminant  $\Delta$  form a complete set of  $\mathbb{Q}$ -conjugates [8, Proposition 13.2].

**Lemma 2.2.** *Suppose that  $z \in \mathbb{H}$  is such that  $gz = z$  for some  $g \in \operatorname{M}_2(\mathbb{Z})$  such that  $\det g > 0$  and  $\lambda g \neq \operatorname{Id}_2$  for every  $\lambda \in \mathbb{Q}^\times$ . Then  $j(z)$  is a singular modulus and  $|\Delta(j(z))| \leq 4 \det g$ .*

*Proof.* Let  $N = \det g$ . Write

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

So  $ad - bc = N$ . Since  $z$  is a fixed point, we have that

$$z = \frac{az + b}{cz + d}.$$

Thus,

$$cz^2 + (d - a)z - b = 0.$$

If  $b = c = d - a = 0$ , then  $N = a^2$  and  $g = a\operatorname{Id}_2$ , which is excluded. So some coefficient of this quadratic equation is non-zero. Hence,  $j(z)$  is a singular modulus. Let  $h = \gcd(c, -b, (d - a))$ . Then

$$\Delta(j(z)) = \left( \frac{d - a}{h} \right)^2 + \frac{4bc}{h^2}.$$

Since  $bc = ad - N$ , we have that

$$\Delta(j(z)) = \frac{1}{h^2}((a+d)^2 - 4N).$$

In particular,  $|\Delta(j(z))| \leq 4N$ .  $\square$

**Proposition 2.3.** *Suppose that  $z \in \mathbb{H}$  is such that*

$$j(z) = j(gz)$$

*for some  $g \in \mathrm{M}_2(\mathbb{Z})$  such that  $\det g > 0$  and  $\lambda g \notin \mathrm{SL}_2(\mathbb{Z})$  for every  $\lambda \in \mathbb{Q}^\times$ . Then  $j(z)$  is a singular modulus and  $|\Delta(j(z))| \leq 4 \det g$ .*

*Proof.* Since

$$j(z) = j(gz),$$

there exists  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  such that

$$\gamma z = gz.$$

In particular,  $z$  is a fixed point for the action of the integer matrix  $\gamma^{-1}g$  on  $\mathbb{H}$ . Apply Lemma 2.2 to  $\gamma^{-1}g$ .  $\square$

**Corollary 2.4.** *Let  $x \in \mathbb{C}$  be a root of the polynomial  $F_N$  for some  $N \in \mathbb{Z}_{>1}$ . Then  $x$  is a singular modulus and  $|\Delta(x)| \leq 4N$ . Further, if  $N \in \mathbb{Z}_{>1}$ , then every singular modulus of discriminant  $-4N$  is a root of  $F_N$ .*

*Proof.* Suppose  $N \in \mathbb{Z}_{>1}$  and  $x \in \mathbb{C}$  are such that  $F_N(x) = 0$ . Recall that

$$F_N(j(z)) = \prod_{g \in C(N)} (j(z) - j(gz)).$$

The  $j$ -function is surjective, so there exists  $z_0 \in \mathbb{H}$  and  $g \in C(N)$  such that  $j(z_0) = j(gz_0) = x$ . Then, by Proposition 2.3, we have that  $x$  is a singular modulus and  $|\Delta(x)| \leq 4N$ .

For the second part, note that for  $N \in \mathbb{Z}_{>1}$ ,

$$\begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} \in C(N).$$

Since the  $j$ -function is invariant under  $z \mapsto -1/z$ , we have that

$$j(\sqrt{N}i) = j\left(\frac{1}{N}\sqrt{N}i\right).$$

So  $F_N(j(\sqrt{N}i)) = 0$ . Clearly,  $j(\sqrt{N}i)$  is a singular modulus of discriminant  $-4N$ . Recall that the singular moduli of discriminant  $-4N$  are all conjugate over  $\mathbb{Q}$ . Thus, every singular modulus of discriminant  $-4N$  is a root of  $F_N$ , since  $F_N$  has coefficients in  $\mathbb{Z}$ .  $\square$

**Corollary 2.5.** *Let  $N_1, \dots, N_k \in \mathbb{Z}_{>1}$  be pairwise distinct. Let  $b_1, \dots, b_k \in \mathbb{Z} \setminus \{0\}$ . Then*

$$\prod_{i=1}^k F_{N_i}(X)^{b_i}$$

*is a non-constant rational function of  $X$ .*

*Proof.* Each  $F_{N_i}(X)$  is a non-constant polynomial in  $X$ . Hence, the rational function

$$\prod_{i=1}^k F_{N_i}(X)^{b_i}$$

is not constantly zero. Without loss of generality, we may assume that  $N_k > N_1, \dots, N_{k-1}$  and  $b_k > 0$ . By Corollary 2.4, the polynomial  $F_{N_i}(X)$  vanishes at a singular modulus of discriminant  $-4N_k$  if and only if  $i = k$ . Thus the rational function

$$\prod_{i=1}^k F_{N_i}(X)^{b_i}$$

vanishes at every singular modulus of discriminant  $-4N_k$ ; in particular, the function is non-constant.  $\square$

**Proposition 2.6.** *Suppose that  $g_1, g_2 \in M_2(\mathbb{Z})$  are such that  $\det g_1, \det g_2 > 0$  and  $g_1 \neq \lambda \gamma g_2$  for every  $\lambda \in \mathbb{Q}^\times$  and  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ . If  $z \in \mathbb{H}$  is such that  $j(g_1 z) = j(g_2 z)$ , then  $j(z)$  is a singular modulus and  $|\Delta(j(z))| \leq 4 \det(g_1) \det(g_2)$ .*

*Proof.* Since  $j(g_1 z) = j(g_2 z)$ , there exists  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  such that  $\gamma g_1 z = g_2 z$ . Hence,  $z$  is a fixed point for the action of  $g = g_2^{-1} \gamma g_1 \in \mathrm{GL}_2^+(\mathbb{Q})$  on  $\mathbb{H}$ . Multiplying the entries of  $g$  by  $\det(g_2)$ , we may assume that  $g \in M_2(\mathbb{Z})$  and  $\det g \leq \det(g_1) \det(g_2)$ . Since  $g_1 \neq \lambda \gamma' g_2$  for every  $\lambda \in \mathbb{Q}^\times$  and  $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$ ,  $g$  is not a rational scalar multiple of  $\mathrm{Id}_2$ . The desired result thus follows from Lemma 2.2.  $\square$

**Proposition 2.7.** *For every  $\epsilon > 0$ , there exist an ineffective constant  $c_1(\epsilon) > 0$  and an effective constant  $c_2(\epsilon) > 0$ , such that if  $x$  is a singular modulus of discriminant  $\Delta$ , then*

$$[\mathbb{Q}(x) : \mathbb{Q}] \geq c_1(\epsilon) |\Delta|^{1/2-\epsilon}$$

and

$$[\mathbb{Q}(x) : \mathbb{Q}] \leq c_2(\epsilon) |\Delta|^{1/2+\epsilon}.$$

*Proof.* The ineffective lower bound is due to Siegel [30]. The upper bound is [21, Proposition 2.2].  $\square$

Since the restriction  $j: \mathfrak{F}_j \rightarrow \mathbb{C}$  of the  $j$ -function to the fundamental domain is bijective, the map

$$(a, b, c) \mapsto j\left(\frac{-b + |\Delta|^{1/2}i}{2a}\right)$$

is a bijection between the set

$$T_\Delta = \left\{ (a, b, c) \in \mathbb{Z}^3 : \Delta = b^2 - 4ac, \gcd(a, b, c) = 1, \right.$$

$$\left. \text{and either } -a < b \leq a < c \text{ or } 0 \leq b \leq a = c \right\}$$

and the singular moduli of discriminant  $\Delta$ . Observe that, for each discriminant  $\Delta$ , there is a unique triple  $(a, b, c) \in T_\Delta$  with  $a = 1$ . This triple is

given by  $(1, k, (k^2 - \Delta)/4)$ , where  $k = 0$  if  $\Delta$  is even and  $k = 1$  if  $\Delta$  is odd. The corresponding singular modulus has preimage

$$\frac{-k + |\Delta|^{1/2}i}{2} \in \mathfrak{F}_j,$$

which has imaginary part strictly greater than the preimage of any other singular modulus of discriminant  $\Delta$  and of any singular modulus of discriminant  $\Delta'$  with  $|\Delta'| < |\Delta|$ .

For  $\alpha \in \overline{\mathbb{Q}}$ , write  $H(\alpha)$  for the absolute multiplicative height of  $\alpha$  and  $h(\alpha)$  for the absolute logarithmic height (see e.g. [6, §1.5]).

**Proposition 2.8.** *Let  $x$  be a singular modulus of discriminant  $\Delta$ . Let  $\tau \in \mathfrak{F}_j$  be such that  $j(\tau) = x$ . Then*

$$H(\operatorname{Re} \tau) \leq \frac{2|\Delta|^{1/2}}{\sqrt{3}} \text{ and } H(\operatorname{Im} \tau) \leq \frac{4|\Delta|}{3}.$$

*Proof.* By the above characterisation of the singular moduli of discriminant  $\Delta$ , we have that

$$\tau = \frac{-b + |\Delta|^{1/2}i}{2a}$$

for some  $(a, b, c) \in \mathbb{Z}^3$  with  $0 \leq |b| \leq a \leq c$  and  $b^2 - 4ac = \Delta$ . Hence, by e.g. [6, Propositions 1.6.5 and 1.6.6],

$$H(\operatorname{Re} \tau) = \max\{|b|, 2a\} = 2a \text{ and } H(\operatorname{Im} \tau) = \max\{|\Delta|, 4a^2\}.$$

The desired inequality follows, since  $3a^2 \leq 4ac - b^2 = |\Delta|$ . Finally, observe that if  $\Delta = -3$ , then  $\tau = (-1 + \sqrt{3}i)/2$  and both bounds are achieved.  $\square$

**Proposition 2.9** ([15, Lemma 4.3]). *For every  $\epsilon > 0$ , there exists an ineffective constant  $c(\epsilon) > 0$  such that if  $x$  is a singular modulus of discriminant  $\Delta$ , then*

$$h(x) \leq c(\epsilon)|\Delta|^\epsilon.$$

**2.3. Properties of  $j$ -maps.** Let  $f$  be a non-constant  $j$ -map. Then [4, Proposition 7.1] there exist  $r, s \in \mathbb{Q}$  such that  $r > 0$  and  $0 \leq s < 1$  such that  $f(z) = j(rz + s)$  for all  $z \in \mathbb{H}$ . Two non-constant  $j$ -maps are equal if and only if the corresponding pairs  $(r, s)$  are equal.

Recall that, for  $N \in \mathbb{Z}_{>0}$ , we define

$$C(N) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \operatorname{M}_2(\mathbb{Z}) : ad = N, a > 0, 0 \leq b < d, \gcd(a, b, d) = 1 \right\}.$$

**Proposition 2.10.** *Let  $g \in \operatorname{GL}_2^+(\mathbb{Q})$ . Then there exist a unique  $N \in \mathbb{Z}_{>0}$  and a unique  $g' \in C(N)$  such that  $j(gz) = j(g'z)$  for all  $z \in \mathbb{H}$ .*

*Proof.* Let  $g \in \operatorname{GL}_2^+(\mathbb{Q})$ . Then there exist  $r, s \in \mathbb{Q}$  with  $r > 0$  and  $0 \leq s < 1$  such that  $f(z) = j(rz + s)$  for all  $z \in \mathbb{H}$ . Further, the pair  $(r, s)$  is unique. Let  $\lambda \in \mathbb{Q}_{>0}$  be such that

$$\lambda \begin{pmatrix} r & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

for some  $a, b, d \in \mathbb{Z}$  with  $\gcd(a, b, d) = 1$ . Since  $0 \leq s < 1$ , we have that  $0 \leq b < d$ . Let  $N = ad$ . Then

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in C(N),$$

and

$$j(gz) = j\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} z\right)$$

for all  $z \in \mathbb{H}$ .

Suppose

$$\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \in C(M)$$

were also such that

$$j\left(\frac{a}{d}z + \frac{b}{d}\right) = j\left(\frac{a'}{d'}z + \frac{b'}{d'}\right)$$

for all  $z \in \mathbb{H}$ . Then, by the uniqueness of the representation of a  $j$ -map in terms of  $r$  and  $s$ , we would have that

$$\frac{a}{d} = \frac{a'}{d'} \text{ and } \frac{b}{d} = \frac{b'}{d'}.$$

Hence,

$$\frac{a}{a'} = \frac{d}{d'},$$

and either  $b = b' = 0$  or

$$\frac{b}{b'} = \frac{d}{d'}.$$

So one matrix is just the rescaling of the other. Since the entries of each of the two matrices are coprime integers and  $a, a' > 0$ , the two matrices are in fact identical and  $M = N$ .  $\square$

**Proposition 2.11.** *Let  $N \in \mathbb{Z}_{>0}$ .*

- (1) *For every  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  and  $g \in C(N)$ , there exists  $h \in C(N)$  such that  $j(g\gamma z) = j(hz)$ .*
- (2) *For every  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  and  $g, h \in C(N)$ , if  $g \neq h$ , then  $j(g\gamma z) \neq j(h\gamma z)$ .*
- (3) *For every  $g, h \in C(N)$ , there exists  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  such that  $j(g\gamma z) = j(hz)$ .*

*Proof.* Let  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  and  $g \in C(N)$ . The entries of  $g$  are coprime integers and  $\det g = N$ . Since  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ , the entries of  $g\gamma$  are coprime integers and  $\det g\gamma = N$ . Write  $g\gamma$  as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for  $a, b, c, d \in \mathbb{Z}$  with  $\gcd(a, b, c, d) = 1$  and  $N = ad - bc$ . Let  $\mu = \gcd(a, c)$  and  $m, n \in \mathbb{Z}$  be such that  $\mu = ma + nc$ . Then

$$\underbrace{\begin{pmatrix} m & n \\ -c/\mu & a/\mu \end{pmatrix}}_{\in \mathrm{SL}_2(\mathbb{Z})} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mu & mb + nd \\ 0 & -bc/\mu + ad/\mu \end{pmatrix},$$

which is upper triangular with coprime integer entries and has determinant  $N$ .

Let  $p = mb + nd$  and  $q = N/\mu$ . Then

$$j(g\gamma z) = j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = j\left(\begin{pmatrix} \mu & p \\ 0 & q \end{pmatrix} z\right) = j\left(\begin{pmatrix} \mu & p+kq \\ 0 & q \end{pmatrix} z\right)$$

for all  $z \in \mathbb{H}$ , where  $k \in \mathbb{Z}$  is the unique integer such that  $0 \leq p+kq < q$ . This proves (1), since

$$\begin{pmatrix} \mu & p+kq \\ 0 & q \end{pmatrix} \in C(N).$$

Given Proposition 2.10, (2) follows immediately by making the change of variables  $w = \gamma^{-1}z$ .

Now we prove (3). Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

Let

$$\sigma_N = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}.$$

Observe that  $\sigma_N \in C(N)$ . By [8, Lemma 11.11], the map

$$g \mapsto \sigma_N^{-1} \mathrm{SL}_2(\mathbb{Z}) g \cap \mathrm{SL}_2(\mathbb{Z})$$

gives a bijection between the elements  $g \in C(N)$  and the right cosets of  $\Gamma_0(N)$  in  $\mathrm{SL}_2(\mathbb{Z})$ . In particular, for every  $g \in C(N)$ , the set  $\sigma_N^{-1} \mathrm{SL}_2(\mathbb{Z}) g \cap \mathrm{SL}_2(\mathbb{Z})$  is non-empty.

Let  $g, h \in C(N)$ . Let  $\gamma_1 \in \sigma_N^{-1} \mathrm{SL}_2(\mathbb{Z}) h \cap \mathrm{SL}_2(\mathbb{Z})$  and  $\gamma_2 \in \sigma_N^{-1} \mathrm{SL}_2(\mathbb{Z}) g \cap \mathrm{SL}_2(\mathbb{Z})$ . Let  $\gamma_{1,1}, \gamma_{2,1} \in \mathrm{SL}_2(\mathbb{Z})$  be such that

$$\gamma_1 = \sigma_N^{-1} \gamma_{1,1} h$$

and

$$\gamma_2 = \sigma_N^{-1} \gamma_{2,1} g.$$

Then

$$j(g\gamma_2^{-1}\gamma_1 z) = j(gg^{-1}\gamma_{2,1}^{-1}\sigma_N\sigma_N^{-1}\gamma_{1,1}hz) = j(\gamma_{2,1}^{-1}\gamma_{1,1}hz) = j(hz)$$

for all  $z \in \mathbb{H}$ . So we may take  $\gamma = \gamma_2^{-1}\gamma_1$  in (3).  $\square$

### 3. COMPLETING THE PROOF OF THEOREM 1.1

**Definition 3.1.** Let  $n \in \mathbb{Z}_{>0}$  and  $x_1, \dots, x_n \in \mathbb{C}^\times$  be pairwise distinct. The set  $\{x_1, \dots, x_n\}$  is multiplicatively dependent if there exist  $a_1, \dots, a_n \in \mathbb{Z}$ , not all zero, such that

$$\prod_{i=1}^n x_i^{a_i} = 1.$$

The set  $\{x_1, \dots, x_n\}$  is minimally multiplicatively dependent if  $\{x_1, \dots, x_n\}$  is multiplicatively dependent and no non-empty proper subset of  $\{x_1, \dots, x_n\}$  is multiplicatively dependent.

**Theorem 3.2.** Let  $y \in \mathbb{C}$  be such that  $y \notin (0, 1728)$ . Let  $n \in \mathbb{Z}_{>0}$ . Then there exist only finitely many  $n$ -tuples  $(x_1, \dots, x_n)$  of singular moduli  $x_1, \dots, x_n$  such that  $x_1, \dots, x_n, y$  are pairwise distinct and  $\{x_1-y, \dots, x_n-y\}$  is minimally multiplicatively dependent.

We do not need to assume that  $y$  is a singular modulus in Theorem 3.2, because the same proof works for all  $y$  outside the real interval  $(0, 1728)$ .

*Proof.* Let  $f(z) = j(z) - y$ . Then the only zero of  $f$  in  $\mathfrak{F}_j$  is at the unique  $\tau \in \mathfrak{F}_j$  such that  $j(\tau) = y$ . Since  $y \notin (0, 1728)$ , this point  $\tau$  does not lie on the arc of the circle  $|z| = 1$  strictly between  $i$  and  $\rho$ . So

$$\operatorname{Im} \frac{-1}{\tau} < \operatorname{Im} \tau,$$

and, by Proposition 2.1,  $f(\tau + s) \neq 0$  for all  $s \in (0, 1)$ . Thus,  $f$  satisfies the ‘‘divisor condition’’ of [11, Definition 1.3], and hence [11, Theorem 1.6] implies the desired result.  $\square$

**Theorem 3.3.** *Let  $y$  be a singular modulus. Let  $n \in \mathbb{Z}_{>0}$ . There exist only finitely many  $n$ -tuples  $(x_1, \dots, x_n)$  such that  $x_1, \dots, x_n, y$  are pairwise distinct singular moduli and  $\{x_1 - y, \dots, x_n - y\}$  is minimally multiplicatively dependent.*

*Proof.* By Theorem 3.2, we may assume that  $y \in (0, 1728)$ . Let  $\Delta = \Delta(y)$ . Note that  $|\Delta| > 4$ , since  $0, 1728$  are the only singular moduli with discriminant in the set  $\{-3, -4\}$ . In particular,  $y$  has the  $\mathbb{Q}$ -conjugate

$$y' = j\left(\frac{-k + |\Delta|^{1/2}i}{2}\right),$$

where  $k = 0$  if  $\Delta$  is even and  $k = 1$  if  $\Delta$  is odd. Since

$$\frac{|\Delta|^{1/2}}{2} > 1,$$

we have that  $y' \notin (0, 1728)$ . Thus, Theorem 3.2 holds for  $y'$ , and so Theorem 3.2 for  $y$  follows since  $y, y'$  are conjugate over  $\mathbb{Q}$ .  $\square$

Theorem 1.1 seems stronger than Theorem 3.3, since the former does not require the multiplicative dependence to be minimal, only that all the exponents are non-zero. In fact, we may deduce Theorem 1.1 from Theorem 3.3 by the following formal argument.

**Proposition 3.4.** *Let  $\mathcal{S} \subset \mathbb{C}^\times$ . Let  $n \in \mathbb{Z}_{>0}$ . Suppose, for every  $k \in \{1, \dots, n\}$  there are only finitely many  $k$ -tuples  $(s_1, \dots, s_k) \in \mathcal{S}^k$  such that  $s_1, \dots, s_k$  are pairwise distinct and  $\{s_1, \dots, s_k\}$  is minimally multiplicatively dependent. Then there are only finitely many  $n$ -tuples  $(s_1, \dots, s_n) \in \mathcal{S}^n$  such that  $s_1, \dots, s_n$  are pairwise distinct and*

$$\prod_{i=1}^n s_i^{a_i} = 1$$

for some  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$ .

*Proof.* Let  $(s_1, \dots, s_n) \in \mathcal{S}^n$  be such that  $s_1, \dots, s_n$  are pairwise distinct and

$$\prod_{i=1}^n s_i^{a_i} = 1$$

for some  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$ . The set  $\{s_1, \dots, s_n\}$  is thus multiplicatively dependent. For each  $i \in \{1, \dots, n\}$ , there exists [11, Lemma 5.9] a minimally multiplicatively dependent subset  $S_i \subset \mathcal{S}$  such that  $s_i \in S_i$ . In particular,

$s_1, \dots, s_n$  all belong to the set consisting of, for each  $k \in \{1, \dots, n\}$ , all the coordinates of tuples  $(s'_1, \dots, s'_k) \in \mathcal{S}^k$  such that  $s'_1, \dots, s'_k$  are pairwise distinct and the set  $\{s'_1, \dots, s'_k\}$  is minimally multiplicatively dependent. By assumption, this set is finite and hence there are only finitely many possibilities for  $(s_1, \dots, s_n)$ .  $\square$

*Proof of Theorem 1.1.* Apply Proposition 3.4 to Theorem 3.3 with

$$\mathcal{S} = \{x - y : x \text{ is a singular modulus and } x \neq y\}. \quad \square$$

#### 4. MULTIPLICATIVE SPECIAL CURVES

In this section, we prove Theorem 1.4. To do this, we first prove the following result.

**Theorem 4.1.** *Let  $n \in \mathbb{Z}_{>0}$ . Suppose that  $T \subset \mathbb{C}^{n+1}$  is a multiplicative special curve. Then there exist  $k \in \{1, \dots, n\}$ ,  $b_1, \dots, b_k \in \mathbb{Z} \setminus \{0\}$ , and pairwise distinct  $N_1, \dots, N_k \in \mathbb{Z}_{>1}$  such that, after reordering the first  $n$  coordinates, we have that*

$$T = \{(\alpha_1, \dots, \alpha_m, j(g_1 z), \dots, j(g_l z), j(z)) : z \in \mathbb{H}\},$$

where

(1)  $\alpha_1, \dots, \alpha_m$  are pairwise distinct and such that

$$\{\alpha_1, \dots, \alpha_m\} = \{\alpha \in \mathbb{C} : \alpha \text{ is either a zero or a pole of } \prod_{i=1}^k F_{N_i}(X)^{b_i}\};$$

(2)  $g_1, \dots, g_l \in \mathrm{GL}_2^+(\mathbb{Q})$  are pairwise distinct and such that

$$\{g_1, \dots, g_l\} = \bigcup_{i=1}^k C(N_i).$$

This follows immediately from the following result, which we will prove in Section 4.2. Throughout this paper, by a change of variables we mean replacing a variable  $z$  by  $gz$  for some  $g \in \mathrm{GL}_2^+(\mathbb{Q})$ .

**Theorem 4.2.** *Let  $n \in \mathbb{Z}_{>0}$ . Let  $f_1, \dots, f_n, f$  be pairwise distinct  $j$ -maps, at least one of which is non-constant. Suppose that  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$  and  $c \in \mathbb{C}^\times$  are such that*

$$(4.1) \quad \prod_{i=1}^n (f_i(z) - f(z))^{a_i} = c$$

for all  $z \in \mathbb{H}$ . Then, after a change of variables,  $f(z) = j(z)$  and there exist  $k \in \{1, \dots, n\}$ ,  $N_1, \dots, N_k \in \mathbb{Z}_{>1}$  pairwise distinct, and  $b_1, \dots, b_k \in \mathbb{Z} \setminus \{0\}$  such that

$$\{f_i : f_i \text{ is non-constant}\} = \{j(gz) : g \in C(N_i), i = 1, \dots, k\},$$

and, for all  $z \in \mathbb{H}$ ,

$$\prod_{\substack{i \in \{1, \dots, n\} \text{ s.t.} \\ f_i \text{ non-constant}}} (f_i(z) - f(z))^{a_i} = \prod_{i=1}^k \left( \prod_{g \in C(N_i)} (j(gz) - j(z)) \right)^{b_i}$$

and

$$\prod_{\substack{i \in \{1, \dots, n\} \text{ s.t.} \\ f_i \text{ constant}}} (f_i(z) - f(z))^{a_i} = c \prod_{i=1}^k F_{N_i}(j(z))^{-b_i}.$$

**4.1. Functional independence modulo constants.** Before proving Theorem 4.2, we first prove some propositions using ideas from [11, §2]. These will allow us to show that if

$$(4.2) \quad \{(f_1(z), \dots, f_n(z), f(z)) : z \in \mathbb{H}\}$$

is a multiplicative special curve, then we must be in the situation that some  $f_i$  is non-constant,  $f$  is non-constant, and some  $f_i$  is constant.

**Definition 4.3.** Functions  $f_1, \dots, f_n: \mathbb{H} \rightarrow \mathbb{C}$  are called multiplicatively independent modulo constants if, whenever  $a_1, \dots, a_n \in \mathbb{Z}$  are not all zero, the function  $F: \mathbb{H} \rightarrow \mathbb{C}$  defined by

$$F(z) = \prod_{i=1}^n f_i(z)^{a_i}$$

is non-constant.

**Proposition 4.4.** *Let  $n \in \mathbb{Z}_{>0}$ . Let  $f$  be a non-constant  $j$ -map. Let  $\alpha_1, \dots, \alpha_n \in \mathbb{C}$  be pairwise distinct. Then the functions  $h_i(z) = f(z) - \alpha_i$  are multiplicatively independent modulo constants.*

*Proof.* By changing variables, we may assume that  $f(z) = j(z)$ . The result is then immediate since  $j$  is a transcendental function.  $\square$

Thus, a multiplicative special curve as in (4.2) must have at least one of  $f_1, \dots, f_n$  non-constant.

**Proposition 4.5.** *Let  $n \in \mathbb{Z}_{>0}$ . Let  $f_1, \dots, f_n$  be pairwise distinct non-constant  $j$ -maps. Let  $\alpha$  be a singular modulus. Then the functions  $h_i(z) = f_i(z) - \alpha$  are multiplicatively independent modulo constants.*

*Proof.* Suppose, for contradiction, that  $c \in \mathbb{C}^\times$  and  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$  are such that

$$(4.3) \quad \prod_{i=1}^n (f_i(z) - \alpha)^{a_i} = c$$

for all  $z \in \mathbb{H}$ . If  $[\mathbb{Q}(z) : \mathbb{Q}] = 2$ , then  $f_i(z)$  is a singular modulus and so  $f_i(z) \in \overline{\mathbb{Q}}$ . Hence,  $c \in \overline{\mathbb{Q}}$ . Let  $K = \mathbb{Q}(\alpha, c)$ .

We may write  $f_i(z) = j(r_i z + s_i)$  for some  $r_i, s_i \in \mathbb{Q}$  with  $r_i > 0$ ,  $s_i \in [0, 1)$ , and the pairs  $(r_i, s_i)$  all distinct. Re-indexing and making a change of variables, we may assume that  $f_1(z) = j(z)$  and  $r_i \geq 1$  for  $i \geq 2$ .

For  $k \in \mathbb{Z}_{>0}$ , let  $z_k = \sqrt{-k}$ . Then  $j(z_k)$  is a singular modulus of discriminant  $-4k$  and every preimage under  $j$  of every singular modulus of discriminant  $-4k$  has imaginary part  $\leq \sqrt{k}$  with equality precisely at the preimages of  $j(z_k)$  itself which have the form  $z_k + l$  for  $l \in \mathbb{Z}$ . This follows from the characterisation of the preimages of the singular moduli of a given discriminant which follows Proposition 2.7 and the properties of the fundamental domain in Proposition 2.1.

For  $i > 1$ , we thus have that  $f_i(z_k)$  is a singular modulus with discriminant not equal to  $-4k$ . Also, the  $f_i(z_k)$  are all pairwise distinct, by Proposition 2.1, and, if  $k$  is large enough, not equal to  $\alpha$ . One thus has that

$$(j(z_k) - \alpha)^{a_1} \prod_{i=2}^n (x_i - \alpha)^{a_i} = c$$

for some singular moduli  $x_2, \dots, x_n$  of discriminants not equal to  $4k$ .

For all  $k$  large enough, Proposition 2.7 implies that the tuple

$$(j(z_k), x_2, \dots, x_n)$$

has some Galois conjugate over  $K$  of the form

$$(\beta, x'_2, \dots, x'_n),$$

where  $\beta \neq j(z_k)$ . Note that

$$(\beta - \alpha)^{a_1} \prod_{i=2}^n (x'_i - \alpha)^{a_i} = c.$$

Thus,

$$(j(z_k) - \alpha)^{a_1} \prod_{i=2}^n (x_i - \alpha)^{a_i} = (\beta - \alpha)^{a_1} \prod_{i=2}^n (x'_i - \alpha)^{a_i}.$$

The only singular moduli of discriminant  $-4k$  in this relation are  $j(z_k)$  and  $\beta$ , and they are distinct. Hence, at least the terms  $(j(z_k) - \alpha)$  and  $(\beta - \alpha)$  in the above relation do not cancel.

Grouping the terms where  $x_i = x'_k$ , which we then cancel if  $a_i = a_k$ , we obtain, for some  $m \in \{2, \dots, 2n\}$ , an  $m$ -tuple

$$(j(z_k), \beta, y_1, \dots, y_{m-2})$$

of singular moduli such that  $j(z_k), \beta, y_1, \dots, y_{m-2}, \alpha$  are pairwise distinct and

$$(j(z_k) - \alpha)^{e_1} (\beta - \alpha)^{e_2} \prod_{i=1}^{m-2} (y_i - \alpha)^{e_{i+2}} = 1$$

for some  $e_1, \dots, e_m \in \mathbb{Z} \setminus \{0\}$ . Further, the tuples that arise in this way for different  $k$  are all distinct, since the  $j(z_k)$  are all distinct.

By the pigeonhole principle, there is thus some  $m \in \{2, \dots, 2n\}$  for which there exist infinitely many  $m$ -tuples  $(w_1, \dots, w_m)$  of singular moduli such that  $w_1, \dots, w_m, \alpha$  are pairwise distinct and

$$\prod_{i=1}^m (w_i - \alpha)^{b_i} = 1$$

for some  $b_1, \dots, b_m \in \mathbb{Z} \setminus \{0\}$ . This contradicts Theorem 1.1 and so we are done.  $\square$

Hence a multiplicative special curve as in (4.2) must have  $f$  non-constant.

**Proposition 4.6.** *Let  $n \in \mathbb{Z}_{>0}$ . Let  $f_1, \dots, f_n, f$  be pairwise distinct, non-constant  $j$ -maps. Then the functions  $h_1, \dots, h_n: \mathbb{H} \rightarrow \mathbb{C}$  defined by  $h_i(z) = f_i(z) - f(z)$  are multiplicatively independent modulo constants.*

*Proof.* We will find some  $z \in \mathbb{H}$  where precisely one of the functions  $h_i$  vanishes. By a change of variables, we may assume that  $f(z) = j(z)$  and  $f_i(z) = j(r_i z + s_i)$  for some  $r_i, s_i \in \mathbb{Q}$  such that  $r_i > 0$  and  $0 \leq s_i < 1$ . Note that  $(r_i, s_i) \neq (1, 0)$  since  $f_i \neq f$ . We may and do assume that the pairs  $(r_i, s_i)$  are strictly increasing when ordered lexicographically.

Suppose first that  $r_1 \geq 1$ . Let

$$z_0 = -\frac{s_1}{2r_1} + \frac{\sqrt{4r_1 - s_1^2}}{2r_1}i,$$

so that

$$\frac{-1}{r_1 z_0 + s_1} = z_0.$$

Observe that  $|r_1 z_0 + s_1| = \sqrt{r_1} \geq 1$ . If  $r_1 > 1$ , then  $r_1 z_0 + s_1 \in \mathfrak{F}_j$ . If  $r_1 = 1$ , then  $s_1 > 0$  and  $z_0$  is on the left hand side of the lower boundary of  $\mathfrak{F}_j$  and  $r_1 z_0 + s_1$  is the reflection of  $z_0$  in the imaginary axis.

Since  $j(-1/z) = j(z)$ , we have that  $f_1(z_0) = f(z_0)$ . If  $r_i > r_1$ , then, by Proposition 2.1,  $\text{Im}(r_i z_0 + s_i) > \text{Im}(r_1 z_0 + s_1)$  and hence  $j(r_i z_0 + s_i) \neq j(r_1 z_0 + s_1)$ . If  $r_i = r_1$  for  $i \geq 2$ , then  $s_1 < s_i < 1$  and  $j(r_i z_0 + s_i) \neq j(r_1 z_0 + s_1)$  by Proposition 2.1 again. Thus,  $f_i(z_0) = f(z_0)$  if and only if  $i = 1$  and we are done.

Now suppose that  $r_1 < 1$ . Let  $k \in \mathbb{Z}$  be such that  $0 \leq kr_1 - s_1 < r_1$ . Let

$$z_1 = -\frac{k}{2} - \frac{s_1}{2r_1} + \frac{\sqrt{4r_1 - (kr_1 - s_1)^2}}{2r_1}i,$$

so that

$$\frac{-1}{z_1 + k} = r_1 z_1 + s_1.$$

Hence,  $f_1(z_1) = f(z_1)$ . Observe also that  $|z_1 + k| = 1/\sqrt{r_1}$  and  $z_1 + k \in \mathfrak{F}_j \setminus \partial \mathfrak{F}_j$  and so  $r_1 z_1 + s_1 \in S\mathfrak{F}_j \setminus \partial(S\mathfrak{F}_j)$ , where  $S$  denotes the transformation  $z \mapsto -1/z$ . Thus, by Proposition 2.1, the points in the  $\text{SL}_2(\mathbb{Z})$ -orbit of  $z_1$  with imaginary part  $\geq \text{Im}(r_1 z_1 + s_1)$  are the elements of

$$\{z_1 + m : m \in \mathbb{Z}\} \cup \{r_1 z_1 + s_1 + l : l \in \mathbb{Z}\}.$$

In particular,  $r_i z_1 + s_i$  is not in the  $\text{SL}_2(\mathbb{Z})$ -orbit of  $z_1$  if  $i \geq 2$ , since  $(r_i, s_i) \neq (1, 0)$ . Hence,  $f_i(z_1) = f(z_1)$  if and only if  $i = 1$ . The proof is thus complete.  $\square$

Therefore a multiplicative special curve as in (4.2) must have some  $f_i$  constant.

## 4.2. The shape of multiplicative special curves.

*Proof of Theorem 4.2.* Let  $n \in \mathbb{Z}_{>0}$ . Let  $f_1, \dots, f_n, f$  be pairwise distinct  $j$ -maps, at least one of which is non-constant. Suppose that  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$  and  $c \in \mathbb{C}^\times$  are such that

$$(4.4) \quad \prod_{i=1}^n (f_i(z) - f(z))^{a_i} = c$$

for all  $z \in \mathbb{H}$ .

By Proposition 4.5, the  $j$ -map  $f$  must be non-constant. Thus, by Proposition 4.4, at least one of the  $j$ -maps  $f_1, \dots, f_n$  must be non-constant.

By Proposition 4.6, at least one of the  $j$ -maps  $f_1, \dots, f_n$  is constant. After relabelling, we thus have that

$$\prod_{i \in I_1} (f_i - f)^{a_i} \prod_{i \in I_2} (f_i - f)^{a_i} = c$$

for all  $z \in \mathbb{H}$ , where the  $j$ -map  $f$  is non-constant and  $I_1, I_2$  are non-empty index sets such that  $I_1 \cup I_2 = \{1, \dots, n\}$  and the  $j$ -map  $f_i$  is constant if  $i \in I_1$  and non-constant if  $i \in I_2$ .

By a change of variables, we may write  $f(z) = j(z)$ . For  $i \in I_1$ , let  $\alpha_i$  be the singular modulus such that  $f_i = \alpha_i$ . Note that the  $\alpha_i$  must be pairwise distinct, since the  $f_i$  are. For  $i \in I_2$ , there is, by Proposition 2.10, a unique  $N_i \in \mathbb{Z}_{>0}$  and  $g_i \in C(N_i)$  such that  $f_i(z) = j(g_i z)$  and  $N_i > 1$  since  $f_i(z) \neq j(z)$ . Rearrange to obtain that

$$(4.5) \quad c' \prod_{i \in I_1} (j(z) - \alpha_i)^{a_i} = \prod_{i \in I_2} (j(z) - j(g_i z))^{-a_i}$$

for all  $z \in \mathbb{H}$ , where

$$c' = \frac{(-1)^{a_1 + \dots + a_n}}{c}.$$

We will show that the right hand side of (4.5) must be a product of powers of functions

$$\prod_{g \in C(N_i)} (j(z) - j(gz)).$$

Rewrite the right hand side of (4.5) by grouping factors with the same  $N_i$  to obtain that

$$(4.6) \quad c' \prod_{i \in I_1} (j(z) - \alpha_i)^{a_i} = \prod_{i \in I_3} \prod_{g \in S_i} (j(z) - j(gz))^{a_i(g)}$$

for all  $z \in \mathbb{H}$ , where  $I_3$  is a new index set and, for each  $i \in I_3$ ,  $S_i \subset C(M_i)$  is non-empty, the  $M_i \in \mathbb{Z}_{>1}$  are pairwise distinct, and the  $a_i(g)$  belong to  $\mathbb{Z} \setminus \{0\}$ . We will show that, for each  $i \in I_3$ , we have that  $S_i = C(M_i)$  and the  $a_i(g)$  are equal for every  $g \in C(M_i)$ .

Suppose then that there is  $i_0 \in I_3$  with the property that there exist  $g_0 \in S_{i_0}$  and  $h_0 \in C(M_{i_0})$  such that either  $h_0 \notin S_{i_0}$  or  $h_0 \in S_{i_0}$  but  $a_{i_0}(h_0) \neq a_{i_0}(g_0)$ . By Proposition 2.11, there exists  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  such that  $j(g_0 \gamma z) = j(h_0 z)$ .

Since the function  $j(z)$  is invariant under the map  $z \mapsto \gamma z$ , we obtain from (4.6) that

$$(4.7) \quad \prod_{i \in I_3} \prod_{g \in S_i} (j(z) - j(gz))^{a_i(g)} = \prod_{i \in I_3} \prod_{g \in S_i} (j(z) - j(g\gamma z))^{a_i(g)}$$

for all  $z \in \mathbb{H}$ . Now, by Proposition 2.11, the factor  $j(z) - j(h_0 z)$  appears on the right hand side of (4.7) with exponent  $a_{i_0}(g_0)$ , and either does not appear on the left hand side (if  $h_0 \notin S_{i_0}$ ) or appears on the left hand side with exponent equal to  $a_{i_0}(h_0)$ , which is not equal to  $a_{i_0}(g_0)$ , otherwise.

The equation (4.7) thus implies that there exist  $l \in \mathbb{Z}_{>0}$  and non-constant  $j$ -maps  $f_1, \dots, f_l, f$  with  $f(z) = j(z)$  and  $f_1(z) = j(h_0 z)$  such that the functions  $v_i$  for  $i = 1, \dots, l$  defined by  $v_i(z) = f_i(z) - f(z)$  are multiplicatively dependent modulo constants. This though contradicts Proposition 4.6.

Therefore, in (4.6), we must have, for each  $i \in I_3$ , that  $S_i = C(M_i)$  and that the  $a_i(g)$  are equal for every  $g \in C(M_i)$ . The right hand side of (4.6) may thus be rewritten to obtain that

$$(4.8) \quad c' \prod_{i \in I_1} (j(z) - \alpha_i)^{a_i} = \prod_{i \in I_3} \prod_{g \in C(M_i)} (j(z) - j(gz))^{b_i}$$

for all  $z \in \mathbb{H}$ , for some  $b_i \in \mathbb{Z} \setminus \{0\}$ , and pairwise distinct  $M_i \in \mathbb{Z}_{>1}$ . The right hand side is thus equal to the function

$$\prod_{i \in I_3} F_{M_i}(j(z))^{b_i},$$

the zeros and poles of which are thus equal to the  $\alpha_i$  on the left hand side of (4.8).  $\square$

**4.3. Determining the multiplicative special curves.** We now complete the proof of Theorem 1.4. Let  $n \in \mathbb{Z}_{>0}$ . We will show that there are only finitely many multiplicative special curves in  $\mathbb{C}^{n+1}$  and these may be determined effectively.

*Proof of Theorem 1.4.* Suppose that

$$T = \{(f_1(z), \dots, f_n(z), f(z)) : z \in \mathbb{H}\}$$

is a multiplicative special curve in  $\mathbb{C}^{n+1}$ . Then, by Theorem 4.1, we may reorder the first  $n$  coordinates of  $T$  in such a way that

$$T = \{(\alpha_1, \dots, \alpha_m, j(g_1 z), \dots, j(g_l z), j(z)) : z \in \mathbb{H}\},$$

where

(1)  $\alpha_1, \dots, \alpha_m$  are pairwise distinct and such that

$$\{\alpha_1, \dots, \alpha_m\} = \{\alpha \in \mathbb{C} : \alpha \text{ is either a zero or a pole of } \prod_{i=1}^k F_{N_i}(X)^{b_i}\};$$

(2)  $g_1, \dots, g_l \in \text{GL}_2^+(\mathbb{Q})$  are pairwise distinct and such that

$$\{g_1, \dots, g_l\} = \bigcup_{i=1}^k C(N_i);$$

for some  $k \in \mathbb{Z}_{>0}$ ,  $b_1, \dots, b_k \in \mathbb{Z} \setminus \{0\}$ , and pairwise distinct  $N_1, \dots, N_k \in \mathbb{Z}_{>1}$ . In particular,

$$m + l = n.$$

Also,

$$l = \sum_{i=1}^k \#C(N_i).$$

Since ([17, p. 53])

$$\#C(N_i) = N_i \prod_{p|N_i} \left(1 + \frac{1}{p}\right),$$

we have that  $\#C(N_i) \geq N_i + 1$ .

Corollary 2.5 implies that

$$\prod_{i=1}^k F_{N_i}(X)^{b_i}$$

is non-constant. Hence,  $m \geq 1$ . Thus we must have that

$$\sum_{i=1}^k \#C(N_i) \leq n - 1.$$

So  $\max\{N_1, \dots, N_k\} \leq n - 2$ . Since  $N_1, \dots, N_k$  are pairwise distinct and  $\geq 2$ , we must have that

$$\sum_{i=2}^{k+1} (i + 1) \leq n - 1.$$

Thus

$$\frac{1}{2}k(k + 5) \leq n - 1,$$

and hence

$$k \leq \frac{1}{2}(\sqrt{8n + 17} - 5).$$

In particular, there are only finitely many possibilities for  $k, N_1, \dots, N_k$  and these may be computed.

Let  $k, N_1, \dots, N_k$  be such a possible choice for a multiplicative special curve in  $\mathbb{C}^{n+1}$ . Compute

$$l = \sum_{i=1}^k \#C(N_i).$$

The corresponding polynomials  $F_{N_i}$  may also be computed [8, §13B]. Let  $\beta_1, \dots, \beta_r$  be pairwise distinct and such that

$$\{\beta_1, \dots, \beta_r\} = \{\beta \in \mathbb{C} : F_{N_i}(\beta) = 0 \text{ for some } i = 1, \dots, k\}.$$

Write  $e_{i,u}$  for the multiplicity of  $\beta_u$  as a root of  $F_{N_i}$ . Let  $d_i$  be the leading coefficient of  $F_{N_i}$ . Note that  $d_i \in \mathbb{Z} \setminus \{0\}$ . Let  $p_1, \dots, p_t$  be a complete list of the prime factors of  $d_1, \dots, d_k$ . Let  $f_{i,v}$  be the exponent of  $p_v$  occurring in the prime factorisation of  $d_i$ .

The choice  $k, N_1, \dots, N_k$  then gives rise to a multiplicative special curve in  $\mathbb{C}^{n+1}$  if and only if there exist  $b_1, \dots, b_k \in \mathbb{Z} \setminus \{0\}$  such that

$$\sum_{i=1}^k b_i f_{i,v} = 0$$

for every  $v \in \{1, \dots, t\}$  and

$$\sum_{i=1}^k b_i e_{i,u} = 0$$

for exactly  $n - l$  choices of  $u \in \{1, \dots, r\}$ . This condition may be checked effectively. Consequently, there are only finitely many multiplicative special curves in  $\mathbb{C}^{n+1}$  and these may be determined effectively.

Now suppose that  $n \leq 5$ . Then

$$k \leq \frac{1}{2}(\sqrt{57} - 5) < \frac{3}{2}.$$

So  $k = 1$  is the only possibility. And

$$N_1 \leq 3.$$

So the only possible multiplicative special curves in  $\mathbb{C}^{n+1}$  arise with  $k = 1$  and  $N_1 \in \{2, 3\}$ . If  $N_1 = 2$ , then  $l = 3$  and so one needs  $m \leq 2$ , which is impossible since  $F_2$  has three distinct roots (see Example 1.7). If  $N_1 = 3$ , then  $l = 4$  and so one needs  $m \leq 1$ , but the polynomial

$$F_3(X) = -X(X - 8000)^2(X + 32768)^2(X - 54000)$$

has four distinct roots. Thus, there are no multiplicative special curves in  $\mathbb{C}^{n+1}$  for  $n \in \{1, \dots, 5\}$ .  $\square$

Finally, we remark that there does exist a multiplicative special curve in  $\mathbb{C}^7$ , namely that given in Example 1.7.

## 5. WEAKLY SPECIAL SUBVARIETIES AND AX-LINDEMANN

**5.1. Weakly special subvarieties.** For the proof of Theorem 1.5, we will need the notion of (weakly) special subvarieties. Varieties and subvarieties are always irreducible over  $\mathbb{C}$ .

**Definition 5.1.** Let  $m, n \in \mathbb{Z}_{>0}$ .

- (1) A weakly special subvariety of  $\mathbb{C}^m$  is an irreducible component of a subvariety of  $\mathbb{C}^m$  defined by equations of the form  $\Phi_N(x_i, x_k) = 0$  and  $x_l = c$  for  $N \in \mathbb{Z}_{>0}$  and  $c \in \mathbb{C}$ .
- (2) A special point of  $\mathbb{C}^m$  is a point  $(x_1, \dots, x_m) \in \mathbb{C}^m$  such that  $x_1, \dots, x_m$  are singular moduli.
- (3) A special subvariety of  $\mathbb{C}^m$  is a weakly special subvariety of  $\mathbb{C}^m$  which contains a special point of  $\mathbb{C}^m$ . Equivalently, a weakly special subvariety for which any constant coordinates are singular moduli.
- (4) A weakly special subvariety of  $(\mathbb{C}^\times)^n$  is a coset of a subtorus (i.e. a coset of an irreducible algebraic subgroup of  $(\mathbb{C}^\times)^n$ ).
- (5) A special point of  $(\mathbb{C}^\times)^n$  is a point  $(\zeta_1, \dots, \zeta_n) \in (\mathbb{C}^\times)^n$  such that  $\zeta_1, \dots, \zeta_n$  are roots of unity.
- (6) A special subvariety of  $(\mathbb{C}^\times)^n$  is a weakly special subvariety of  $(\mathbb{C}^\times)^n$  which contains a special point of  $(\mathbb{C}^\times)^n$ .
- (7) A (weakly) special subvariety of  $\mathbb{C}^m \times (\mathbb{C}^\times)^n$  is a product  $M \times T$ , where  $M \subset \mathbb{C}^m$  is a (weakly) special subvariety of  $\mathbb{C}^m$  and  $T \subset (\mathbb{C}^\times)^n$  is a (weakly) special subvariety of  $(\mathbb{C}^\times)^n$ .

It follows from this definition that a weakly special subvariety  $T \subset (\mathbb{C}^\times)^n$  is defined by equations of the form

$$t_1^{a_1} \dots t_n^{a_n} = c$$

for some  $c \in \mathbb{C}^\times$  and  $a_1, \dots, a_n \in \mathbb{Z}$  not all zero. Also,  $T$  is a special subvariety if and only if  $T$  may be defined by equations of this kind with the additional property that every such  $c$  is a root of unity. See, for example, [31, Remark 1.0.1].

It follows from the above description that special subvarieties of  $\mathbb{C}^m$  and  $(\mathbb{C}^\times)^n$  are defined over  $\overline{\mathbb{Q}}$ . Special subvarieties of  $\mathbb{C}^m$  have the following useful properties.

**Proposition 5.2** ([4, Proposition 2.1]). *Let  $m \in \mathbb{Z}_{>0}$ . Let  $M \subset \mathbb{C}^m$  be a positive-dimensional special subvariety. Then  $M$  contains a Zariski-dense union of special subvarieties of  $\mathbb{C}^m$  of dimension 1.*

**Proposition 5.3** ([4, Proposition 2.3]). *Let  $m \in \mathbb{Z}_{>0}$ . Let  $M \subset \mathbb{C}^m$ . Then  $M$  is a special subvariety of dimension 1 if and only if there exist  $j$ -maps  $f_1, \dots, f_n$ , at least one of which is non-constant, such that*

$$M = \{(f_1(z), \dots, f_n(z)) : z \in \mathbb{H}\}.$$

In particular, a multiplicative special curve in  $\mathbb{C}^{n+1}$  is a special subvariety of  $\mathbb{C}^{n+1}$  of dimension 1.

**5.2. Ax–Lindemann.** Now we come to the functional transcendence result of Pila [22] which we will apply in the proof of Theorem 1.5.

For  $m, n \in \mathbb{Z}_{>0}$ , let

$$X = \mathbb{C}^m \times (\mathbb{C}^\times)^n$$

and

$$U = \mathbb{H}^m \times \mathbb{C}^n.$$

Define  $e: \mathbb{C} \rightarrow \mathbb{C}^\times$  by  $e(t) = \exp(2\pi it)$ . Define  $\pi: U \rightarrow X$  by

$$\pi(z_1, \dots, z_m, t_1, \dots, t_n) = (j(z_1), \dots, j(z_m), e(t_1), \dots, e(t_n)).$$

We make the following definitions.

**Definition 5.4** ([22, Definition 6.1]). Let  $Z \subset U$  be a complex analytic subset. A complex algebraic component of  $Z$  is a positive-dimensional connected component  $Y \subset W \cap U$  for some algebraic subvariety  $W \subset \mathbb{C}^{m+n}$  such that  $Y \subset Z$ . Here,  $W \cap U$  is considered as a complex analytic set. A maximal complex algebraic component of  $Z$  is a complex algebraic component of  $Z$  which is not contained in any complex algebraic component of  $Z$  of strictly larger dimension.

**Definition 5.5** ([22, Definition 6.5]). Let  $m, n \in \mathbb{Z}_{>0}$ .

- (1) A weakly special subvariety of  $\mathbb{H}^m$  is (the intersection with  $\mathbb{H}^m$  of) a subvariety defined by equations of the forms  $z_i = g_{i,k} z_k$  and  $z_l = c_l$  for some matrices  $g_{i,k} \in \mathrm{GL}_2^+(\mathbb{Q})$  and constants  $c_l \in \mathbb{H}$ .
- (2) A weakly special subvariety for  $e$  of  $\mathbb{C}^n$  is a subvariety of the form  $b + L$  for some  $b \in \mathbb{C}^n$  and linear subspace  $L \subset \mathbb{C}^n$  defined over  $\mathbb{Q}$ .
- (3) A weakly special subvariety of  $U = \mathbb{H}^m \times \mathbb{C}^n$  is a product  $M \times T$ , where  $M \subset \mathbb{H}^m$  is a weakly special subvariety of  $\mathbb{H}^m$  and  $T \subset \mathbb{C}^n$  is a weakly special subvariety for  $e$  of  $\mathbb{C}^n$ .

In particular, if  $W$  is a weakly special subvariety of  $U$ , then  $\pi(W)$  is a weakly special subvariety of  $X$ . The slightly cumbersome terminology of weakly special subvariety for  $e$  of  $\mathbb{C}^n$  is chosen to avoid confusion with the definition of a weakly special subvariety of  $\mathbb{C}^n$  in Definition 5.1(1).

The functional transcendence result we require is the following statement, which Pila calls an Ax–Lindemann result. Note that Pila [22] formulates his result with the ordinary complex exponential function  $\exp: \mathbb{C} \rightarrow \mathbb{C}^\times$  in place of the function  $e$ , but this difference is of no consequence for our purposes.

**Theorem 5.6** ([22, Theorem 6.8]). *Let  $V \subset X$  be an algebraic subvariety. If  $Y$  is a maximal complex algebraic component of  $\pi^{-1}(V)$ , then  $Y$  is a weakly special subvariety of  $U$ .*

## 6. THE PROOF OF THEOREM 1.5

We will prove Theorem 1.5 by applying the so-called Pila–Zannier strategy of o-minimal point counting. This strategy was proposed by Zannier and was first used by Pila and Zannier [26] to give a new proof of the Manin–Mumford conjecture. The approach used here is similar to that employed in [11, 24]. For background on o-minimality and on the Pila–Zannier method, see Pila’s book [23].

**6.1. The counting theorem for semirational points.** We will use an extension, due to Habegger and Pila [16, Corollary 7.2], of the Pila–Wilkie o-minimal counting theorem [25]. We will always work in the o-minimal structure  $\mathbb{R}_{\text{an,exp}}$ ; see [23, p. 77] for details of this structure. Definable will mean definable with parameters in  $\mathbb{R}_{\text{an,exp}}$ . Complex numbers, when considered as elements of definable sets, will be identified with their real and imaginary parts. Throughout this section, constants  $c = c(\dots)$  will be positive and have only the indicated dependencies.

To state Habegger and Pila’s result, we need to define the  $k$ -height of a real number. Let  $k \in \mathbb{Z}_{>0}$ . For  $y \in \mathbb{R}$ , define the  $k$ -height of  $y$  by

$$H_k(y) = \min\{\max\{|a_0|, \dots, |a_k|\} : a_0, \dots, a_k \text{ are coprime integers, not all zero, such that } a_k y^k + \dots + a_0 = 0\},$$

with the convention that  $\min \emptyset = \infty$ . Note that  $y \in \mathbb{R}$  thus has  $H_k(y) < \infty$  if and only if  $[\mathbb{Q}(y) : \mathbb{Q}] \leq k$ . For  $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ , define

$$H_k(y) = \max\{H_k(y_1), \dots, H_k(y_n)\}.$$

The  $k$ -height is related to the multiplicative height in the following way.

**Proposition 6.1.** *Let  $d \in \mathbb{Z}_{>0}$ . There exists a constant  $c(d) > 0$  with the property that if  $\alpha \in \overline{\mathbb{Q}}$  is such that  $[\mathbb{Q}(\alpha) : \mathbb{Q}] = d$ , then*

$$H_d(\alpha) \leq c(d)H(\alpha)^d.$$

*Proof.* Let

$$c(d) = \binom{d}{\lfloor d/2 \rfloor}.$$

Suppose that

$$f(t) = a_d t^d + \dots + a_0$$

is a minimal polynomial over  $\mathbb{Z}$  of some  $\alpha \in \overline{\mathbb{Q}}$ . Then

$$H_d(\alpha) \leq \max\{|a_0|, \dots, |a_d|\} \leq c(d)M(f) = c(d)H(\alpha)^d,$$

where the second inequality is [6, Lemma 1.6.7] and the final equality is [6, Proposition 1.6.6]. Here  $M(f)$  denotes the Mahler measure of  $f$ .  $\square$

Habegger and Pila’s point counting result is the following.

**Theorem 6.2** ([16, Corollary 7.2]). *Let  $F \subset \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n$  be a definable family parametrised by  $\mathbb{R}^l$ . Let  $\epsilon > 0$  and  $k \in \mathbb{Z}_{>0}$ . Let  $\pi_1: \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $\pi_2: \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the projection maps. Then there exists a constant  $c = c(F, k, \epsilon) > 0$  with the following property.*

*Let  $x \in \mathbb{R}^l$ . Write  $F_x \subset \mathbb{R}^m \times \mathbb{R}^n$  for the fibre of  $F$  over  $x$ . If  $T \geq 1$  and*

$$\Sigma \subset \{(y, z) \in F_x : H_k(y) \leq T\}$$

*is such that  $\#\pi_2(\Sigma) > cT^\epsilon$ , then there exists a continuous, definable function  $\beta: [0, 1] \rightarrow F_x$  such that:*

- (1) *The composition  $\pi_1 \circ \beta: [0, 1] \rightarrow \mathbb{R}^m$  is semialgebraic and its restriction to  $(0, 1)$  is real analytic.*
- (2) *The composition  $\pi_2 \circ \beta: [0, 1] \rightarrow \mathbb{R}^n$  is non-constant.*
- (3)  *$\pi_2(\beta(0)) \in \pi_2(\Sigma)$ .*
- (4) *The restriction of  $\beta$  to  $(0, 1)$  is real analytic.*

The constant  $c = c(F, k, \epsilon)$  here is not effective. For (4), we use the fact that  $\mathbb{R}_{\text{an,exp}}$  admits analytic cell decomposition [9, Theorem 8.8].

We will also require the following bound on the size of the exponents in a multiplicative dependency.

**Proposition 6.3.** *Let  $n \in \mathbb{Z}_{>0}$ . There exist constants  $c_1(n), c_2(n) > 0$  with the following property. Let  $L$  be a number field and  $d = [L : \mathbb{Q}]$ . If  $\alpha_1, \dots, \alpha_n \in L^\times$  are pairwise distinct and such that*

$$\prod_{i=1}^n \alpha_i^{a_i} = 1$$

*for some  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$ , then there exist  $b_1, \dots, b_n \in \mathbb{Z} \setminus \{0\}$  such that*

$$\prod_{i=1}^n \alpha_i^{b_i} = 1$$

*and, for every  $i \in \{1, \dots, n\}$ ,*

$$|b_i| \leq c_1(n)(d \max\{1, h(\alpha_1), \dots, h(\alpha_n)\})^{c_2(n)}.$$

This will follow from the following bound, which covers the case where the multiplicative dependency is minimal.

**Lemma 6.4.** *Let  $n \in \mathbb{Z}_{>0}$ . There exists an explicit constant  $c(n) > 0$  with the following property. Let  $L$  be a number field and  $d = [L : \mathbb{Q}]$ . If  $\alpha_1, \dots, \alpha_n \in L^\times$  are pairwise distinct and the set  $\{\alpha_1, \dots, \alpha_n\}$  is minimally multiplicatively dependent, then there exist  $b_1, \dots, b_n \in \mathbb{Z} \setminus \{0\}$  such that*

$$\prod_{i=1}^n \alpha_i^{b_i} = 1$$

*and*

$$|b_i| \leq c(n)d^{n+1}(1 + \log d) \prod_{\substack{k=1 \\ k \neq i}}^n h(\alpha_k).$$

*Proof.* If  $n = 1$ , then  $\alpha_1$  is a root of unity of degree  $\leq d$ . Hence,  $\alpha_1$  is a primitive  $N$ th root of unity for some  $N$  with  $\phi(N) \leq d$ , where  $\phi$  denotes Euler's totient function. The desired result then follows from the elementary bound

$$\phi(N) \geq \sqrt{\frac{N}{2}}.$$

For  $n \geq 2$ , this is a result of Yu [20, Corollary 3.2]. The version stated in [20] has  $d^n \log d$  in place of the  $d^{n+1}(1 + \log d)$  here; the slight weakening here allows one to state a uniform result for all  $d, n \geq 1$  which still suffices for our purposes.  $\square$

Proposition 6.3 follows from Lemma 6.4 via the following elementary lemma.

**Lemma 6.5.** *Let  $n \in \mathbb{Z}_{>1}$ . Let  $v, w \in \mathbb{Z}^n$ . Suppose that, for some  $k \in \{1, \dots, n-1\}$ , we have that*

$$v = (v_1, \dots, v_k, 0, \dots, 0),$$

where  $v_1, \dots, v_k \neq 0$ , and that

$$w = (w_1, \dots, w_n)$$

with  $w_{k+1} \neq 0$ . Let

$$\lambda = 1 + \max\{|v_1|, \dots, |v_k|, |w_1|, \dots, |w_n|\}.$$

Let

$$u = v + \lambda w,$$

and write

$$u = (u_1, \dots, u_n).$$

Then  $u_1, \dots, u_{k+1} \neq 0$  and

$$|u_i| \leq 2\lambda^2 \text{ for } i = 1, \dots, n.$$

*Proof.* For  $i = 1, \dots, k$ , if  $w_i \neq 0$ , then  $|w_i| \geq 1$  and so  $\lambda|w_i| > |v_i|$ . The result follows immediately since  $m^2 + m \leq 2m^2$  for all  $m \in \mathbb{Z}$ .  $\square$

*Proof of Proposition 6.3.* Let  $L$  be a number field and  $d = [L : \mathbb{Q}]$ . Suppose that  $\alpha_1, \dots, \alpha_n \in L^\times$  are pairwise distinct and such that

$$\prod_{i=1}^n \alpha_i^{a_i} = 1$$

for some  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$ . The set

$$\mathcal{S} = \{\alpha_1, \dots, \alpha_n\}$$

is thus multiplicatively dependent, but not necessarily minimally multiplicatively dependent. For each  $i = 1, \dots, n$  though, there exists  $S_i \subset \mathcal{S}$  such that  $\alpha_i \in S_i$  and  $S_i$  is minimally multiplicatively dependent, see e.g. [11, Lemma 5.9]. We will apply Lemma 6.4 to each set  $S_i$ .

By Lemma 6.4, there exist constants  $c_1(n), c_2(n) > 0$  such that, for each  $i = 1, \dots, n$ , there exist  $b_{i,k} \in \mathbb{Z} \setminus \{0\}$  for  $k \in S_i$  with

$$\prod_{k \in S_i} \alpha_k^{b_{i,k}} = 1$$

and

$$|b_{i,k}| \leq c_1(n)(d \max\{1, h(\alpha_1), \dots, h(\alpha_n)\})^{c_2(n)}.$$

Now let  $v_i \in \mathbb{Z}^n$  be the vector with  $k$ th coordinate  $v_{i,k}$  equal to  $b_{i,k}$  if  $k \in S_i$  and 0 otherwise. Hence, for  $i = 1, \dots, n$ , we have that  $v_{i,i} \neq 0$  and

$$\prod_{k=1}^n \alpha_k^{v_{i,k}} = 1.$$

Let

$$\mu = 1 + \max\{|v_{i,k}| : i, k \in \{1, \dots, n\}\}.$$

Apply Lemma 6.5 inductively to  $v_1, \dots, v_n$  to obtain a vector

$$w = (w_1, \dots, w_n) \in \mathbb{Z}^n$$

which is a  $\mathbb{Z}$ -linear combination of  $v_1, \dots, v_n$  and such that  $w_1, \dots, w_n \neq 0$  and

$$|w_i| \leq c_3(n)\mu^{c_4(n)} \text{ for } i = 1, \dots, n$$

for some constants  $c_3(n), c_4(n) > 0$ . In particular, there are constants  $c_5(n), c_6(n) > 0$  such that

$$|w_i| \leq c_5(n)(d \max\{1, h(\alpha_1), \dots, h(\alpha_n)\})^{c_6(n)} \text{ for } i = 1, \dots, n.$$

Since  $w$  is a  $\mathbb{Z}$ -linear combination of  $v_1, \dots, v_n$ , one has that

$$\prod_{i=1}^n \alpha_i^{w_i} = 1,$$

as required. □

**6.2. Completing the proof of Theorem 1.5.** Now we come to the proof of Theorem 1.5. Fix  $n \in \mathbb{Z}_{>0}$ . In the proof,  $c_1, c_2, \dots$  will denote positive constants which depend only on  $n$ . Any other dependencies among constants will be explicitly indicated.

By Theorem 1.4, there are only finitely many multiplicative special curves in  $\mathbb{C}^{n+1}$ . Since every multiplicative special curve is defined over  $\overline{\mathbb{Q}}$ , we may fix some number field  $K$  over which all the multiplicative special curves in  $\mathbb{C}^{n+1}$  are defined.

Define the complexity  $\Delta$  of an  $(n+1)$ -tuple  $(x_1, \dots, x_n, y)$  of singular moduli  $x_1, \dots, x_n, y$  by

$$\Delta = \max\{|\Delta(x_1)|, \dots, |\Delta(x_n)|, |\Delta(y)|\}.$$

Recall that  $e: \mathbb{C} \rightarrow \mathbb{C}^\times$  is given by  $e(z) = \exp(2\pi iz)$ . Let  $\mathfrak{F}_e = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z < 1\}$ , so that  $e$  restricted to  $\mathfrak{F}_e$  is a bijection.

Recall that by definable we always mean definable (with parameters) in the structure  $\mathbb{R}_{\text{an,exp}}$ . For the purposes of definability, we identify subsets of  $\mathbb{C}^n$  with subsets of  $\mathbb{R}^{2n}$  in the obvious way. In particular, the usual field operations on  $\mathbb{C}$  are thus definable. We will use the fact that the restricted functions  $e: \mathfrak{F}_e \rightarrow \mathbb{C}^\times$  and  $j: \mathfrak{F}_j \rightarrow \mathbb{C}$  are both definable. The restriction of  $e$  is definable using restricted  $\sin$  and  $\cos$  and the unrestricted real exponential function. The definability of  $j$  may then be deduced from its  $q$ -expansion, see [31, Example 4.14] for the details.

Let

$$Y = \left\{ (z_1, \dots, z_n, z, w_1, \dots, w_n, w, u_1, \dots, u_n, r_1, \dots, r_n, s) \right. \\ \left. \in \mathfrak{F}_j^{2(n+1)} \times \mathfrak{F}_e^n \times \mathbb{R}^{n+1} : \sum_{i=1}^n r_i u_i = s, w = z, \text{ and} \right. \\ \left. w_i = z_i \text{ and } e(u_i) = j(z_i) - j(z) \text{ for } i = 1, \dots, n \right\}$$

and

$$Z = \left\{ (z_1, \dots, z_n, z, r_1, \dots, r_n, s) \in \mathfrak{F}_j^{n+1} \times \mathbb{R}^{n+1} : \right. \\ \left. \exists (u_1, \dots, u_n) \in \mathfrak{F}_e^n \text{ such that} \right. \\ \left. (z_1, \dots, z_n, z, z_1, \dots, z_n, z, u_1, \dots, u_n, r_1, \dots, r_n, s) \in Y \right\}.$$

The sets  $Y, Z$  are both definable.

Suppose that  $(x_1, \dots, x_n, y)$  is an  $(n+1)$ -tuple of pairwise distinct singular moduli  $x_1, \dots, x_n, y$  such that

$$\prod_{i=1}^n (x_i - y)^{a_i} = 1$$

for some  $a_i \in \mathbb{Z} \setminus \{0\}$ . Let  $\Delta$  be the complexity of this tuple. Let

$$d = [\mathbb{Q}(x_1, \dots, x_n, y) : \mathbb{Q}].$$

By Proposition 6.3, we may assume that

$$|a_i| \leq c_1 (d \max\{1, h(x_1), \dots, h(x_n), h(y)\})^{c_2}$$

for  $i \in \{1, \dots, n\}$ . Then apply Proposition 2.7 (with  $\epsilon = 1/4$  say) to give an upper bound on  $d$  and Proposition 2.9 to bound the logarithmic heights of the singular moduli. One thereby obtains that, for  $i \in \{1, \dots, n\}$ ,

$$|a_i| \leq c_3 \Delta^{c_4}.$$

Let

$$(\tau_1, \dots, \tau_n, \tau, \nu_1, \dots, \nu_n) \in \mathfrak{F}_j^{n+1} \times \mathfrak{F}_e^n$$

be the preimage of

$$(x_1, \dots, x_n, y, x_1 - y, \dots, x_n - y)$$

with respect to the map  $(j, e): \mathfrak{F}_j^{n+1} \times \mathfrak{F}_e^n \rightarrow \mathbb{C}^{n+1} \times (\mathbb{C}^\times)^n$ . Note that  $\tau_1, \dots, \tau_n, \tau$  are all quadratic, since they are the preimages for  $j$  of singular moduli. By Proposition 2.8, the real and imaginary parts of  $\tau_1, \dots, \tau_n, \tau$  all have multiplicative height  $\leq 4\Delta/3$ . Observe also that

$$\sum_{i=1}^n a_i \nu_i \in \mathbb{Z},$$

since

$$\prod_{i=1}^n e(\nu_i)^{a_i} = 1.$$

Let  $b = \sum_{i=1}^n a_i \nu_i$ . Then

$$|b| \leq \sum_{i=1}^n |a_i|,$$

since  $\nu_1, \dots, \nu_n$  all have real part in the interval  $[0, 1)$ . In particular,

$$|b| \leq c_5 \Delta^{c_4}.$$

The tuple  $(x_1, \dots, x_n, y)$  thus gives rise to the point

$$(\tau_1, \dots, \tau_n, \tau, a_1, \dots, a_n, b) \in Z,$$

which is quadratic in the  $\tau_i, \tau$  coordinates and integral in the  $a_i, b$  coordinates. Further, the 2-height of this point is  $\leq c_6 \Delta^{c_7}$  by Proposition 6.1 and the above bound on the multiplicative height.

Every Galois conjugate  $(x'_1, \dots, x'_n, y')$  of  $(x_1, \dots, x_n, y)$  over  $K$  satisfies the multiplicative relation

$$\prod_{i=1}^n (x'_i - y')^{a_i} = 1,$$

where  $a_1, \dots, a_n$  are the same integers as before. The conjugate  $(x'_1, \dots, x'_n, y')$  thus gives rise, in the same way as  $(x_1, \dots, x_n, y)$  did, to a point

$$(\tau'_1, \dots, \tau'_n, \tau', a_1, \dots, a_n, b') \in Z,$$

where the  $\tau'_i, \tau'$  are quadratic and of multiplicative height  $\leq 4\Delta/3$ , the  $a_i$  are the same integers as before, and  $b'$  is an integer such that  $|b'| \leq c_5 \Delta^{c_4}$ . Note that  $b'$  is not necessarily the same as  $b$ . In particular, the point  $(\tau'_1, \dots, \tau'_n, \tau', a_1, \dots, a_n, b')$  also has 2-height  $\leq c_6 \Delta^{c_7}$ . Further, the corresponding points of  $Z$  arising from distinct  $K$ -conjugates of  $(x_1, \dots, x_n, y)$  are always distinct in the  $\mathfrak{F}_j^{n+1}$  coordinates. By Proposition 2.7 with  $\epsilon = 1/4$ , there are at least  $c_8 \Delta^{1/4}$  distinct  $K$ -conjugates of  $(x_1, \dots, x_n, y)$ , each of which gives rise to a distinct point of  $Z$  in the above way.

View  $Y$  as a definable family of sets fibred over the  $(r_1, \dots, r_n)$  coordinates. Each of the points

$$(\tau'_1, \dots, \tau'_n, \tau', a_1, \dots, a_n, b') \in Z$$

described above is the projection of a point

$$(\tau'_1, \dots, \tau'_n, \tau', \tau'_1, \dots, \tau'_n, \tau', \nu'_1, \dots, \nu'_n, a_1, \dots, a_n, b') \in Y.$$

Note that the  $Y$ -points arising in this way from distinct conjugates over  $K$  of  $(x_1, \dots, x_n, y)$  are distinct in their  $(\tau'_1, \dots, \tau'_n, \tau')$  coordinates.

These points  $(\tau'_1, \dots, \tau'_n, \tau', \tau'_1, \dots, \tau'_n, \tau', \nu'_1, \dots, \nu'_n, b')$  all lie on the fibre  $Y_{(a_1, \dots, a_n)}$  of  $Y$  over  $(a_1, \dots, a_n)$ . Let

$$\pi_1: Y_{(a_1, \dots, a_n)} \rightarrow \mathfrak{F}_j^{n+1} \times \mathbb{R} \text{ and } \pi_2: Y_{(a_1, \dots, a_n)} \rightarrow \mathfrak{F}_j^{n+1} \times \mathfrak{F}_e^n$$

be the projection maps sending

$$(z_1, \dots, z_n, z, w_1, \dots, w_n, w, u_1, \dots, u_n, s) \mapsto (z_1, \dots, z_n, z, s)$$

and

$$(z_1, \dots, z_n, z, w_1, \dots, w_n, w, u_1, \dots, u_n, s) \mapsto (w_1, \dots, w_n, w, u_1, \dots, u_n)$$

respectively. Observe that  $\pi_2$  is injective. Let  $\Sigma \subset Y_{(a_1, \dots, a_n)}$  be the set consisting of all the points arising in the way described above from the  $K$ -conjugates of  $(x_1, \dots, x_n, y)$ . Then  $\pi_1(\Sigma)$  contains only algebraic points of degree at most 2 and which have 2-height  $\leq c_6 \Delta^{c_7}$ . Also,  $\#\pi_2(\Sigma) > c_8 \Delta^{1/4}$ .

Now let  $C_{HP} > 0$  be the constant given by Theorem 6.2 applied to  $Y$  with  $k = 2$  and  $\epsilon = (8c_7)^{-1}$ . Note that  $c_7$  depends only on  $n$ , which is fixed, and  $C_{HP}$  depends only on  $Y, k, \epsilon$ , which are all fixed. In particular,  $C_{HP}$  is independent of  $(x_1, \dots, x_n, y)$  and  $a_1, \dots, a_n$ . Let  $T = c_6 \Delta^{c_7}$ . Then  $\pi_1(\Sigma)$  contains only algebraic points of degree  $\leq 2$  and 2-height  $\leq T$  and

$$\#\pi_2(\Sigma) > c_8 \Delta^{1/4}.$$

In particular, if  $\Delta > (c_6 C_{HP} / c_8)^8$ , then

$$\#\pi_2(\Sigma) > C_{HP} T^\epsilon$$

(here we assume without loss of generality that  $c_6, c_7 \geq 1$ , so  $c_6^\epsilon \leq c_6$ ).

Suppose then that  $\Delta > (c_6 C_{HP} / c_8)^8$ . Theorem 6.2 implies that there exists a continuous, definable function  $\beta: [0, 1] \rightarrow Y_{(a_1, \dots, a_n)}$  with the following properties:

- (1) The composition  $\pi_1 \circ \beta: [0, 1] \rightarrow \mathfrak{F}_j^{n+1} \times \mathbb{R}$  is semialgebraic and its restriction to  $(0, 1)$  is real analytic.
- (2) The composition  $\pi_2 \circ \beta: [0, 1] \rightarrow \mathfrak{F}_j^{n+1} \times \mathfrak{F}_e^n$  is non-constant.
- (3)  $\pi_2(\beta(0)) \in \pi_2(\Sigma)$ .
- (4) The restriction of  $\beta$  to  $(0, 1)$  is real analytic.

Note that, by the construction of the set  $Y$ , property (2) implies that  $\pi_1 \circ \beta$  composed with projection to the  $\mathfrak{F}_j^{n+1}$  coordinates is non-constant. Since  $\pi_2$  is injective, we have that  $\beta(0) \in \Sigma$ , i.e.  $\beta(0)$  is a point of  $Y_{(a_1, \dots, a_n)}$  arising from some  $K$ -conjugate  $(x'_1, \dots, x'_n, y')$  of  $(x_1, \dots, x_n, y)$ .

Let

$$V_{(a_1, \dots, a_n)} = \left\{ (z_1, \dots, z_n, z, w) \in \mathbb{C}^{n+1} \times \mathbb{C}^\times : \prod_{i=1}^n (z_i - z)^{a_i} = w \right\}.$$

Let  $\mathcal{V} = \pi^{-1}(V_{(a_1, \dots, a_n)})$ , where  $\pi = (j, \dots, j, e)$  is the map defined in Section 5.2. Observe that  $(\pi_1 \circ \beta)([0, 1])$  is a connected, positive-dimensional semialgebraic set contained in  $\mathcal{V}$ .

Note that  $V_{(a_1, \dots, a_n)}$  is an algebraic subvariety of  $\mathbb{C}^{n+1} \times \mathbb{C}^\times$ . Hence we may apply [22, Proposition 6.2] to  $\mathcal{V}$  and thereby obtain a complex algebraic component  $W$  of  $\mathcal{V}$  such that  $\pi_1(\beta(0)) \in W$ . Note that  $W$  is, by definition, positive-dimensional. Enlarging  $W$  as necessary, we may assume that  $W$  is a maximal complex algebraic component of  $\mathcal{V}$ .

The Ax–Lindemann result of Theorem 5.6 thus implies that  $W$  is a weakly special subvariety of  $\mathbb{H}^{n+1} \times \mathbb{C}$ . So  $W = W_1 \times W_2$ , where  $W_1$  is a weakly special subvariety of  $\mathbb{H}^{n+1}$  and  $W_2$  is a weakly special subvariety for  $e$  of  $\mathbb{C}$ . Since  $W \subset \mathcal{V}$  and the projection  $\mathcal{V} \rightarrow \mathbb{H}^{n+1}$  has discrete fibres, the weakly special subvariety  $W_2$  must just be a point. Hence,  $W_2$  is equal to the projection of  $\pi_1(\beta(0))$ , which is  $\{b'\}$  for some  $b' \in \mathbb{Z}$ . Moreover,  $W_1$  must be positive-dimensional.

Also,  $W_1$  contains the preimage in  $\mathfrak{F}_j^{n+1}$  of the  $K$ -conjugate  $(x'_1, \dots, x'_n, y')$  of  $(x_1, \dots, x_n, y)$ , since  $\pi_1(\beta(0)) \in W$ . Finally, note that

$$(6.1) \quad W_1 \subset \left\{ (z_1, \dots, z_n, z) : \prod_{i=1}^n (j(z_i) - j(z))^{a_i} = 1 \right\},$$

since  $W_2 = \{b'\}$  and  $b' \in \mathbb{Z}$ .

The image  $j(W_1)$  is therefore a positive-dimensional weakly special subvariety of  $\mathbb{C}^{n+1}$  which contains  $(x'_1, \dots, x'_n, y')$ . In particular,  $j(W_1)$  is in fact a special subvariety of  $\mathbb{C}^{n+1}$ , because  $j(W_1)$  contains the special point  $(x'_1, \dots, x'_n, y')$ .

Let  $l = \dim j(W_1)$ . A special subvariety of  $\mathbb{C}^{n+1}$  of dimension  $l$  is equal, up to reordering coordinates, to some Cartesian product  $M_1 \times \dots \times M_l$ , where  $M_1 \subset \mathbb{C}^{m_1}, \dots, M_l \subset \mathbb{C}^{m_l}$  are one-dimensional special subvarieties and  $m_1, \dots, m_l \in \mathbb{Z}_{>0}$ , see e.g. [23, pp. 33–34]. Hence, after reordering only the first  $n$  coordinates of  $j(W_1)$ , we have that

$$j(W_1) = M_1 \times \dots \times M_l$$

for some one-dimensional special subvarieties  $M_1 \subset \mathbb{C}^{m_1}, \dots, M_l \subset \mathbb{C}^{m_l}$ . By Proposition 5.3, for each  $i \in \{1, \dots, l\}$ , there exist  $j$ -maps  $f_{i,1}, \dots, f_{i,m_i}$ , which are not all constant, such that

$$M_i = \{(f_{i,1}(z_i), \dots, f_{i,m_i}(z_i)) : z_i \in \mathbb{H}\}.$$

By (6.1), we have that

$$(6.2) \quad \prod_{i=1}^l \prod_{\substack{k=1 \\ (i,k) \neq (l,m_l)}}^{m_i} (f_{i,k}(z_i) - f_{l,m_l}(z_l))^{a_{i,k}} = 1$$

for all  $(z_1, \dots, z_l) \in \mathbb{H}^l$ , where the  $a_{i,k}$  are the appropriate re-indexing of the  $a_i$  in (6.1). For  $i \in \{1, \dots, l-1\}$ , let  $\tau_i \in \mathbb{H}$  be such that

$$(f_{i,1}(\tau_i), \dots, f_{i,m_i}(\tau_i)) = \pi_{M_i}((x'_1, \dots, x'_n, y')),$$

where  $\pi_{M_i}: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{m_i}$  denotes the projection map onto the coordinates corresponding to  $M_i$ . Such a  $\tau_i$  exists since  $(x'_1, \dots, x'_n, y') \in j(W_1)$ . Let

$$M_i(\tau_i) = \{(f_{i,1}(\tau_i), \dots, f_{i,m_i}(\tau_i))\}.$$

Now let

$$M = \prod_{i=1}^{l-1} M_i(\tau_i) \times M_l.$$

Then  $M$  is a one-dimensional special subvariety of  $\mathbb{C}^{n+1}$ . By Proposition 5.3, there exist  $j$ -maps  $f_1, \dots, f_n, f$  such that

$$M = \{(f_1(z), \dots, f_n(z), f(z)) : z \in \mathbb{H}\}.$$

Note that at least one of  $f_1, \dots, f_n, f$  is non-constant. By (6.2), we have that

$$\prod_{i=1}^n (f_i(z) - f(z))^{a_i} = 1 \text{ for all } z \in \mathbb{H}.$$

After reordering only the first  $n$  coordinates of  $M$ , we may assume that  $(x'_1, \dots, x'_n, y') \in M$ . Since  $x'_1, \dots, x'_n, y'$  are pairwise distinct, the  $j$ -maps

$f_1, \dots, f_n, f$  are therefore pairwise distinct. Therefore,  $M$  is a multiplicative special curve.

Thus,  $M$  is one of the finitely many multiplicative special curves in  $\mathbb{C}^{n+1}$  given by Theorem 1.4. In particular,  $M$  is defined over  $K$ . Thus,  $(x_1, \dots, x_n, y) \in M$ , since  $M$  contains the  $K$ -conjugate  $(x'_1, \dots, x'_n, y')$  of  $(x_1, \dots, x_n, y)$ .

We have therefore shown that, for  $(x_1, \dots, x_n, y)$  an  $(n+1)$ -tuple of pairwise distinct singular moduli  $x_1, \dots, x_n, y$  of complexity  $\Delta$  such that

$$\prod_{i=1}^n (x_i - y)^{a_i} = 1$$

for some  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$ , if  $\Delta > (c_6 C_{HP}/c_8)^8$ , then  $(x_1, \dots, x_n, y)$  belongs to one of the finitely many multiplicative special curves in  $\mathbb{C}^{n+1}$ . Hence, the complexity of every such  $(n+1)$ -tuple which does not lie on a multiplicative special curve in  $\mathbb{C}^{n+1}$  is  $\leq (c_6 C_{HP}/c_8)^8$ . In particular, there are only finitely many such  $(n+1)$ -tuples. This completes the proof of Theorem 1.5. Corollary 1.6 follows immediately.

## 7. THE ZILBER–PINK CONNECTION

Let  $m, n \in \mathbb{Z}_{>0}$ . Let

$$X_{m,n} = \mathbb{C}^m \times (\mathbb{C}^\times)^n.$$

Recall the definition of a special subvariety of  $X_{m,n}$  from Definition 5.1.

**Definition 7.1.** Let  $V \subset X_{m,n}$  be a subvariety. A subvariety  $W \subset V$  is called an atypical component of  $V$  in  $X_{m,n}$  if there exists a special subvariety  $T \subset X_{m,n}$  such that  $W$  is an irreducible component of  $V \cap T$  and

$$\dim W > \dim V + \dim T - \dim X_{m,n}.$$

An atypical component  $W$  of  $V$  in  $X_{m,n}$  is a maximal atypical component of  $V$  in  $X_{m,n}$  if there does not exist any atypical component  $W'$  of  $V$  in  $X_{m,n}$  such that  $W \subsetneq W'$ .

The Zilber–Pink conjecture was formulated independently in different contexts by Zilber [32], Pink [27], and Bombieri, Masser, and Zannier [7]. The conjecture is wide open; see [23, Part IV] for more details. In our context, the Zilber–Pink conjecture is the following statement.

**Conjecture 7.2** (Zilber–Pink conjecture). *Let  $V \subset X_{m,n}$  be a subvariety. Then there are only finitely many maximal atypical components of  $V$  in  $X_{m,n}$ .*

In the remainder of this section, we show that, in light of Theorem 1.4, Theorem 1.5 would follow from Conjecture 7.2. For  $n \in \mathbb{Z}_{>0}$ , we define  $V_n \subset X_{n+1,n}$  by

$$V_n = \{(w_1, \dots, w_n, w, t_1, \dots, t_n) \in X_{n+1,n} : t_i = w_i - w \text{ for } i = 1, \dots, n\}.$$

Note that  $\dim X_{n+1,n} = 2n + 1$  and  $\dim V_n = n + 1$ .

**Proposition 7.3.** *Let  $n \in \mathbb{Z}_{>0}$ . Suppose that  $x_1, \dots, x_n, y$  are singular moduli such that  $x_i \neq y$  for  $i \in \{1, \dots, n\}$  and*

$$\prod_{i=1}^n (x_i - y)^{a_i} = 1$$

for some  $a_1, \dots, a_n \in \mathbb{Z}$  which are not all zero. Then

$$\{(x_1, \dots, x_n, y, x_1 - y, \dots, x_n - y)\}$$

is an atypical component of  $V_n$  in  $X_{n+1,n}$ .

*Proof.* Let

$$\sigma = (x_1, \dots, x_n, y, x_1 - y, \dots, x_n - y).$$

Observe that  $\sigma \in V_n$ . Since  $x_1, \dots, x_n, y$  are singular moduli, the set

$$\{(x_1, \dots, x_n, y)\}$$

is a special subvariety of  $\mathbb{C}^{n+1}$  of dimension 0. Write  $M$  for this special subvariety. Since  $x_i \neq y$  and

$$\prod_{i=1}^n (x_i - y)^{a_i} = 1,$$

the point

$$(x_1 - y, \dots, x_n - y)$$

is contained in a special subvariety  $T \subset (\mathbb{C}^\times)^n$  of dimension at most  $n - 1$ . Hence,  $M \times T$  is a special subvariety of  $X_{n+1,n}$  of dimension  $\leq n - 1$ . Thus,

$$\dim V_n + \dim(M \times T) - \dim X_{n+1,n} \leq (n + 1) + (n - 1) - (2n + 1) < 0.$$

Thus,  $\{\sigma\} \subset V_n \cap (M \times T)$  is an atypical component of  $V_n$  in  $X_{n+1,n}$ .  $\square$

**Proposition 7.4.** *Let  $n \in \mathbb{Z}_{>0}$ . Suppose that  $x_1, \dots, x_n, y$  are pairwise distinct singular moduli such that the set  $\{x_1 - y, \dots, x_n - y\}$  is minimally multiplicatively dependent. Then either  $(x_1, \dots, x_n, y)$  lies on a multiplicative special curve in  $\mathbb{C}^{n+1}$  or*

$$\{(x_1, \dots, x_n, y, x_1 - y, \dots, x_n - y)\}$$

is a maximal atypical component of  $V_n$  in  $X_{n+1,n}$ .

*Proof.* Suppose that  $x_1, \dots, x_n, y$  are pairwise distinct singular moduli such that the set  $\{x_1 - y, \dots, x_n - y\}$  is minimally multiplicatively dependent. Let

$$\sigma = (x_1, \dots, x_n, y, x_1 - y, \dots, x_n - y).$$

Then, by Proposition 7.3,  $\{\sigma\}$  is an atypical component of  $V_n$  in  $X_{n+1,n}$ .

Suppose then that  $\{\sigma\}$  is not a maximal atypical component of  $V_n$  in  $X_{n+1,n}$ . Then there exist special subvarieties  $M \subset \mathbb{C}^{n+1}$  and  $T \subset (\mathbb{C}^\times)^n$  and an irreducible component  $W \subset V_n \cap (M \times T)$  such that  $\dim W > 0$ ,  $\sigma \in W$ , and

$$\dim W > \dim V_n + \dim(M \times T) - \dim X_{n+1,n} = \dim M + \dim T - n.$$

If  $T = \mathbb{G}_m^n$ , then

$$V_n \cap (M \times T) = \{(w_1, \dots, w_n, w, w_1 - w, \dots, w_n - w) : (w_1, \dots, w_n, w) \in M\}$$

and hence any component of this intersection clearly has dimension  $\leq \dim M$  and so cannot be an atypical component. Similarly, if  $M = Y(1)^{n+1}$ , then any component of the intersection  $V_n \cap (M \times T)$  has dimension  $\leq \dim T + 1$  and hence is not an atypical component. We may thus assume that  $M, T$  are both proper subvarieties.

If  $T$  was defined by two independent multiplicative conditions, then two independent multiplicative relations would hold on the set

$$\{x_1 - y, \dots, x_n - y\},$$

and thus some proper subset would be multiplicatively dependent, a contradiction. So  $T$  must be defined by one independent multiplicative condition.

Hence, for  $M \times T$  to intersect  $V_n$  atypically, one must have that

$$(\alpha_1 - \beta, \dots, \alpha_n - \beta) \in T$$

for every  $(\alpha_1, \dots, \alpha_n, \beta) \in M$ . Thus, by Proposition 5.3, if  $M_0 \subset M$  is a special subvariety of  $\mathbb{C}^{n+1}$  such that  $\dim M_0 = 1$  and no two coordinates of  $M_0$  are identically equal, then  $M_0$  is a multiplicative special curve in  $\mathbb{C}^{n+1}$ .

Suppose that  $\dim M > 1$ . Since  $(x_1, \dots, x_n, y) \in M$  and  $x_1, \dots, x_n, y$  are pairwise distinct, the locus in  $M$  where some two coordinates are equal is a Zariski-closed proper subset of  $M$ . Thus, by Proposition 5.2,  $M$  must contain infinitely many multiplicative special curves in  $\mathbb{C}^{n+1}$ . However, there are only finitely many multiplicative special curves in  $\mathbb{C}^{n+1}$  by Theorem 1.4. So we must have that  $\dim M = 1$ , and so  $M$  itself is a multiplicative special curve in  $\mathbb{C}^{n+1}$ . Since  $(x_1, \dots, x_n, y) \in M$ , the proof is complete.  $\square$

**Proposition 7.5.** *Assume Conjecture 7.2. Then Theorem 1.4 implies Theorem 1.5.*

*Proof.* Fix  $n \in \mathbb{Z}_{>0}$ . We will show that there exists a constant  $C > 0$ , depending only on  $n$ , with the following property. Suppose that  $(x_1, \dots, x_n, y)$  is an  $(n+1)$ -tuple of pairwise distinct singular moduli  $x_1, \dots, x_n, y$  such that

$$\prod_{i=1}^n (x_i - y)^{a_i} = 1$$

for some  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$ . Then either  $|\Delta(y)| \leq C$  or  $(x_1, \dots, x_n, y)$  lies on a multiplicative special curve in  $\mathbb{C}^{n+1}$ . By Theorem 1.1, this suffices to prove Theorem 1.5. In what follows, we let  $c_1, c_2, \dots$  denote positive constants which depend only on  $n$ .

Suppose that  $(x_1, \dots, x_n, y)$  is an  $(n+1)$ -tuple of pairwise distinct singular moduli  $x_1, \dots, x_n, y$  such that

$$\prod_{i=1}^n (x_i - y)^{a_i} = 1$$

for some  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$ . For every  $k \in \{1, \dots, n\}$ , there exists, by [11, Lemma 5.9], a set  $I_k \subset \{1, \dots, n\}$  such that  $k \in I_k$  and the set

$$\{x_i - y : i \in I_k\}$$

is minimally multiplicatively dependent. Let  $m_k = \#I_k$ . Let  $i_{k,1}, \dots, i_{k,m_k} \in \{1, \dots, n\}$  be pairwise distinct and such that  $I_k = \{i_{k,1}, \dots, i_{k,m_k}\}$ .

Conjecture 7.2 implies that, for every  $k \in \{1, \dots, n\}$ , there are only finitely many maximal atypical components of  $V_{m_k}$  in  $X_{m_k+1, m_k}$ . Therefore, there exists a constant  $c_1$  with the property that if there exists  $k$  such that

$$\{(x_{i_{k,1}}, \dots, x_{i_{k,m_k}}, y, x_{i_{k,1}} - y, \dots, x_{i_{k,m_k}} - y)\}$$

is a maximal atypical component of  $V_{m_k}$  in  $X_{m_k+1, m_k}$ , then  $|\Delta(y)| \leq c_1$ . We may therefore assume that, for every  $k \in \{1, \dots, n\}$ , the point

$$\{(x_{i_{k,1}}, \dots, x_{i_{k,m_k}}, y, x_{i_{k,1}} - y, \dots, x_{i_{k,m_k}} - y)\}$$

is not a maximal atypical component of  $V_{m_k}$  in  $X_{m_k+1, m_k}$ .

Proposition 7.4 then implies that, for every  $k \in \{1, \dots, n\}$ , there exists a multiplicative special curve  $M_k \subset \mathbb{C}^{m_k+1}$  such that

$$(x_{i_{k,1}}, \dots, x_{i_{k,m_k}}, y) \in M_k.$$

By Theorem 4.1, there exist  $j$ -maps  $f_{k,1}, \dots, f_{k,m_k}$  such that

$$M_k = \{(f_{k,1}(z), \dots, f_{k,m_k}(z), j(z)) : z \in \mathbb{H}\}.$$

Theorem 1.4 implies that there exists a constant  $c_2$  such that:

- (1) if  $f_{k,r}$  is a constant  $j$ -map whose value is a singular modulus of discriminant  $\Delta$ , then  $|\Delta| \leq c_2$ ,
- (2) if  $f_{k,r}$  is a non-constant  $j$ -map, then there exist  $N \in \mathbb{Z}_{>0}$  and  $g \in C(N)$  such that  $N \leq c_2$  and  $f_{k,r}(z) = j(gz)$ .

If  $f_{k,r}$  is a constant  $j$ -map, then  $f_{k,r} = x_{i_{k,r}}$  and hence  $|\Delta(x_{i_{k,r}})| \leq c_2$ .

Let the constant  $c_3$  be such that  $|\Delta(x)| \leq c_3$  for every singular modulus  $x$  belonging to the set

$$\{z \in \mathbb{C} : \text{there exist } M \in \mathbb{Z}_{>0} \text{ and a singular modulus } w \\ \text{such that } \max\{M, |\Delta(w)|\} \leq c_2 \text{ and } \Phi_M(w, z) = 0\}.$$

Such a constant exists since this set is clearly finite. For  $N \in \mathbb{Z}_{>0}$ , denote by  $\mathbb{V}(\Phi_N)$  the vanishing locus of the polynomial  $\Phi_N(X, Y)$  in  $\mathbb{C}^2$ . The sets  $\mathbb{V}(\Phi_N)$  are pairwise distinct, irreducible plane curves. So if  $M \neq N$ , then the intersection  $\mathbb{V}(\Phi_M) \cap \mathbb{V}(\Phi_N)$  is a finite set. There thus exists a constant  $c_4$  with the property that if  $z, w$  are singular moduli such that

$$(z, w) \in \bigcup_{\substack{M, N \in \mathbb{Z}_{>0} \\ M \neq N \text{ and } M, N \leq c_2}} (\mathbb{V}(\Phi_M) \cap \mathbb{V}(\Phi_N)),$$

then  $\max\{|\Delta(w)|, |\Delta(z)|\} \leq c_4$ .

We now claim that, for every  $r \in \{1, \dots, n\}$ , either

$$\{f_{k,s} : i_{k,s} = r\} = \{x_r\}$$

or there exists  $N \leq c_2$  such that

$$\{f_{k,s}(z) : i_{k,s} = r\} \subset \{j(gz) : g \in C(N)\}.$$

It suffices to show that if  $r \in \{1, \dots, n\}$  is such that  $r = i_{k,l} = i_{k',l'}$ , then either  $f_{k,l} = f_{k',l'} = x_r$  or there exist  $N \leq c_2$  and  $g, h \in C(N)$  such that  $f_{k,l}(z) = j(gz)$  and  $f_{k',l'}(z) = j(hz)$ . Relabelling as necessary, we may assume that  $r = i_{k,1} = i_{k',1}$ .

If  $f_{k,1}, f_{k',1}$  are both constant, then  $f_{k,1} = f_{k',1} = x_r$ . Next, consider the case that the  $j$ -map  $f_{k,1}$  is constant and the  $j$ -map  $f_{k',1}$  is non-constant. So

$f_{k,1} = x_r$  and hence  $|\Delta(x_r)| \leq c_2$ . Also, there exist  $N \in \mathbb{Z}_{>0}$  and  $g \in C(N)$  such that  $N \leq c_2$  and  $f_{k',1}(z) = j(gz)$ . Since  $(x_{i_{k',1}}, \dots, x_{i_{k',m_{k'}}}, y) \in M_{k'}$ , there exists some  $z_0 \in \mathbb{H}$  such that  $(j(gz_0), j(z_0)) = (x_r, y)$ . In particular,  $\Phi_N(x_r, y) = 0$ . Therefore,  $|\Delta(y)| \leq c_3$ .

Finally, suppose that the  $j$ -maps  $f_{k,1}, f_{k',1}$  are both non-constant. Then there exist  $M, N \in \mathbb{Z}_{>0}$  and  $g \in C(M), h \in C(N)$  such that  $f_{k,1}(z) = j(gz)$  and  $f_{k',1}(z) = j(hz)$ . Since

$$(x_{i_{k,1}}, \dots, x_{i_{k,m_k}}, y) \in M_k \text{ and } (x_{i_{k',1}}, \dots, x_{i_{k',m_{k'}}}, y) \in M_{k'},$$

we have that

$$(x_r, y) \in \{(j(gz), j(z)) : z \in \mathbb{H}\} \cap \{(j(hw), j(w)) : w \in \mathbb{H}\}.$$

Hence,  $(x_r, y) \in \mathbb{V}(\Phi_M) \cap \mathbb{V}(\Phi_N)$ , and thus  $|\Delta(y)| \leq c_4$  if  $M \neq N$ . The claim therefore holds if  $|\Delta(y)| \geq \max\{c_3, c_4\}$ .

Suppose then that  $|\Delta(y)| \geq \max\{c_3, c_4\}$ . The proved claim and the properties of multiplicative special curves given in Theorem 4.1 together imply that there exists a partition  $P_0, \dots, P_l$  of  $\{1, \dots, n\}$ , where  $l \geq 1$ , with the following properties:

- (1) The  $j$ -map  $f_{k,r}$  is constant if and only if  $i_{k,r} \in P_0$ , in which case  $f_{k,r} = x_{i_{k,r}}$ .
- (2) For  $u \in \{1, \dots, l\}$  and  $k \in \{1, \dots, n\}$ , if  $P_u \cap I_k \neq \emptyset$ , then  $P_u \subset I_k$ .
- (3) For  $u \in \{1, \dots, l\}$ , there exists  $N_u \in \mathbb{Z}_{>0}$  such that writing  $C(N_u) = \{g_1, \dots, g_s\}$  with  $g_1, \dots, g_s$  pairwise distinct, if  $k \in \{1, \dots, n\}$  is such that  $P_u \subset I_k$ , then

$$\begin{aligned} & \{f_{k,v}(z) : v \in \{1, \dots, m_k\} \text{ such that } i_{k,v} \in P_u\} \\ &= \{j(g_1z), \dots, j(g_sz)\}. \end{aligned}$$

We will now define  $j$ -maps  $f_1, \dots, f_n$ . For  $m \in P_0$ , let  $f_m = x_m$ . For  $m \in I_1 \setminus P_0$ , let  $f_m(z) = f_{1,v}(z)$ , where  $v$  is the unique integer such that  $i_{1,v} = m$ . Now let  $z_0 \in \mathbb{H}$  be such that  $j(z_0) = y$  and  $f_m(z_0) = x_m$  for every  $m \in I_1 \cup P_0$ . Such a  $z_0$  exists by the definition of  $M_1$  and  $P_0$ .

Let  $r \geq 1$  be such that  $P_r \not\subset I_1$ . Write  $C(N_r) = \{g_1, \dots, g_s\}$  with  $g_1, \dots, g_s$  pairwise distinct. Note that there exists  $w_0 \in \mathbb{H}$  such that

$$j(w_0) = y \text{ and } \{j(g_1w_0), \dots, j(g_sw_0)\} = \{x_i : i \in P_r\}.$$

This follows from property (3) of the constructed partition by considering the multiplicative special curve  $M_k$  for some  $k$  such that  $P_r \subset I_k$ . In particular,  $y = j(z_0) = j(w_0)$  and so there exists  $\gamma \in \text{SL}_2(\mathbb{Z})$  such that  $\gamma z_0 = w_0$ . Note that the functions  $j(g_1\gamma z), \dots, j(g_s\gamma z)$  are just a permutation of the functions  $j(g_1z), \dots, j(g_sz)$ . Hence, we also have that

$$\{j(g_1z_0), \dots, j(g_sz_0)\} = \{x_i : i \in P_r\}.$$

Since the  $x_i$  are pairwise distinct, we may then, for  $m \in P_r$ , define  $f_m(z) = j(g_vz)$ , where  $v \in \{1, \dots, s\}$  is the unique integer  $v$  such that  $j(g_vz_0) = x_m$ .

Since the  $P_i$  partition  $\{1, \dots, n\}$ , we may define in this way  $j$ -maps  $f_1, \dots, f_n$ . By construction, we have that

$$(x_1, \dots, x_n, y) = (f_1(z_0), \dots, f_n(z_0), j(z_0)).$$

In particular, the functions  $f_1, \dots, f_n, j$  are pairwise distinct. For  $k \in \{1, \dots, n\}$ , observe that

$$\{f_i(z) : i \in I_k\} = \{f_{k,r}(z) : r \in \{1, \dots, m_k\}\}.$$

Since  $M_k$  is a multiplicative special curve and this property is preserved under permutation of the first  $m_k$  coordinates of  $M_k$ , there therefore exist  $a_{k,i} \in \mathbb{Z} \setminus \{0\}$  such that

$$\prod_{i \in I_k} (f_i(z) - j(z))^{a_{k,i}} = 1$$

for every  $k \in \{1, \dots, n\}$  and  $z \in \mathbb{H}$ . Since  $k \in I_k$  for every  $k$ , Lemma 6.5 implies that there exist  $b_1, \dots, b_n \in \mathbb{Z} \setminus \{0\}$  such that

$$\prod_{i=1}^n (f_i(z) - j(z))^{b_i} = 1$$

for all  $z \in \mathbb{H}$ . Thus, the set

$$\{(f_1(z), \dots, f_n(z), j(z)) : z \in \mathbb{H}\}$$

is a multiplicative special curve in  $\mathbb{C}^{n+1}$  which contains  $(x_1, \dots, x_n, y)$ .

We have thus shown that if  $(x_1, \dots, x_n, y)$  is an  $(n+1)$ -tuple of pairwise distinct singular moduli  $x_1, \dots, x_n, y$  such that

$$\prod_{i=1}^n (x_i - y)^{a_i} = 1$$

for some  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$ , then either  $|\Delta(y)| \leq \max\{c_1, c_3, c_4\}$  or  $(x_1, \dots, x_n, y)$  lies on a multiplicative special curve in  $\mathbb{C}^{n+1}$ . Theorem 1.5 thus follows from Theorem 1.1.  $\square$

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