

The L^p - L^q maximal regularity for the Beris-Edwards model in the half-space

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Abstract

In this paper, we consider the model describing viscous incompressible liquid crystal flows, which is called the Beris-Edwards model in the half-space. This model is a coupled system by the Navier-Stokes equations with the evolution equation of the director fields Q . The purpose of this paper is to prove the linearized problem has a unique solution satisfying the maximal L^p - L^q regularity estimates, which is essential for the study of quasi-linear parabolic or parabolic-hyperbolic equations. Our method relies on the \mathcal{R} -boundedness of the solution operator families to the resolvent problem in order to apply operator-valued Fourier multiplier theorems. Consequently, we also have the local well-posedness for the Beris-Edwards model with small initial data.

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1 Introduction

In this paper we study the Beris-Edwards model for nematic liquid crystals:

$$\begin{cases} (\partial_t - \Delta)u + \nabla p + \beta \operatorname{Div}(\Delta - a)Q = f(u, Q) & \text{in } (0, T) \times \mathbb{R}_+^N \\ (\partial_t - \Delta + a)Q - \beta D(u) = G(u, Q) & \text{in } (0, T) \times \mathbb{R}_+^N \\ \operatorname{div} u = 0 & \text{in } (0, T) \times \mathbb{R}_+^N \\ u = h, \quad D_N Q = H & \text{on } (0, T) \times \mathbb{R}_0^N \\ u(0) = u_0, \quad Q(0) = Q_0 & \text{in } \mathbb{R}_+^N, \end{cases} \quad (1)$$

with

$$(\operatorname{Div} A)_k = \sum_{j=1}^N \partial_j A_{k,j} \quad \forall A: \mathbb{R}^N \rightarrow \mathbb{R}^{N^2}, \quad k = 1, \dots, N.$$

Liquid crystals are a state of matter intermediate between the solid state and the liquid state: such substances flow like liquids but they are strongly anisotropic. As the name suggested, the model was introduced by Beris and Edward in [4]. Here $u: (0, T) \times \mathbb{R}_+^N \rightarrow \mathbb{R}^N$ and $p: (0, T) \times \mathbb{R}_+^N \rightarrow \mathbb{R}$ for $T > 0$ are respectively the velocity field of the particles and the pressure of the material, while $Q: (0, T) \times \mathbb{R}_+^N \rightarrow S_0(N, \mathbb{R})$ was introduced by [8] in order to measure the anisotropy of the substance, where

$$S_0(N, \mathbb{R}) := \left\{ A \in \mathbb{R}^{N^2} \mid A^T = A, \quad \operatorname{tr}(A) = 0 \right\},$$

where A^T and $\operatorname{tr} A$ are respectively the transpose and the trace of a matrix A

$$(A^T)_{j,k} = A_{k,j} \quad (j, k = 1, \dots, N), \quad \operatorname{tr} A = \sum_{j=1}^N A_{j,j}$$

and where \mathbb{R}_+^N and \mathbb{R}_0^N are respectively the half-space and its boundary, i.e.

$$\mathbb{R}_+^N := \left\{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid x_N > 0 \right\},$$

$$\mathbb{R}_0^N := \left\{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid x_N = 0 \right\}.$$

Moreover, $\xi, a, b, c \in \mathbb{R}$, $\beta = \frac{2\xi}{N}$, $h: (0, T) \times \mathbb{R}_+^N \rightarrow \mathbb{R}^N$, $H: (0, T) \times \mathbb{R}_+^N \rightarrow \mathbb{R}^{N^2}$,

$$f(u, Q) = -(u \cdot \nabla)u + \text{Div} \left[2\xi \mathbb{H} : Q \left(Q + \frac{Id}{N} \right) - (\xi + 1)\mathbb{H}Q + (1 - \xi)Q\mathbb{H} - \nabla Q \odot \nabla Q \right] - \beta \text{Div} \mathcal{L}[\mathcal{F}(Q)],$$

$$G(u, Q) = -(u \cdot \nabla)Q + \xi(D(u)Q + QD(u)) + W(u)Q - QW(u) - 2\xi \left(Q + \frac{Id}{N} \right) Q : \nabla u + \mathcal{L}[\mathcal{F}(Q)],$$

$$W(u) = \frac{1}{2} (\nabla u - \nabla^T u), \quad D(u) = \frac{1}{2} (\nabla u + \nabla^T u), \quad \mathcal{F}(Q) = bQ^2 - c|Q|^2Q,$$

$$[\nabla Q \odot \nabla Q]_{jk} = \sum_{\alpha, \beta=1}^N \partial_j Q_{\alpha\beta} \partial_k Q_{\alpha\beta} \quad j, k = 1, \dots, N,$$

$$\mathbb{H} = \Delta Q - aQ + b\mathcal{L}[Q^2] - c|Q|^2Q,$$

where Id is the identity matrix of \mathbb{R}^{N^2} , $|A|$ and $A : B$ of two symmetric matrices $A, B \in \mathbb{R}^{N^2}$ are respectively the Frobenius norm and his associated scalar product

$$A : B := \text{tr}(B^T A) = \sum_{i,j=1}^N B_{ji} A_{ji},$$

$$|A| := \sqrt{A : A} = \sqrt{\sum_{i,j=1}^N A_{ij}^2},$$

and where

$$\mathcal{L}[A] = A - \text{tr}(A) \frac{Id}{N}.$$

The Beris-Edwards model was mathematically studied by several authors. Concerning the case $\xi = 0$, the first result was obtained by Paicu and Zarnescu [17]. They proved the existence of global weak solutions in \mathbb{R}^N with $N = 2, 3$ as well as weak-strong uniqueness for $N = 2$. An improved result of [17] in \mathbb{R}^2 was established in [7]. Huang and Ding [13] proved the existence of global weak solutions with a more general energy functional in \mathbb{R}^3 . Abels, Dolzmann, and Liu [2] proved that the classical Beris-Edwards model, fluid viscosity depends on the Q -tensor, has a unique local solution in a bounded domain with Dirichlet boundary conditions. The global well-posedness was proved by Luo, Li, and Zhao [15] in a bounded with Dirichlet boundary conditions under the assumption viscosity is sufficiently large. Xiao [25] proved the global well-posedness in a bounded domain. The author constructed a strong solution in the L^p - L^q maximal regularity class.

On the other hand, concerning the model with general parameter ξ , Abels, Dolzmann, and Liu [1] showed the unique existence of a strong local solution and global weak solutions with higher regularity in time in the case of inhomogeneous mixed Dirichlet/Neumann boundary conditions in a bounded domain. Liu and Wang [14] improved the spatial regularity of solutions obtained in [1] and generalized their result to the case of anisotropic elastic energy. The global well-posedness and long-time behavior of the model in the two-dimensional periodic case was investigated by Cavaterra et al. [6]. In [18] Schonbek and Shibata proved the global well-posedness and the decay properties in the L^p - L^q maximal regularity class for the simplified

model, which means that the linear terms $\Delta Q - aQ$ are removed from the first equation of (1). Shibata and the second author obtained in [16] the unique existence and the decay properties of a strong global solution in the same solution spaces as [18] for the Beris-Edwards model.

As far as we know, there is no result relating to the well-posedness for boundary value problems in unbounded domains even if $\xi = 0$. In this paper, we prove the L^p - L^q maximal regularity for the linearized system in \mathbb{R}_+^N :

$$\begin{cases} \partial_t u - \Delta u + \nabla p + \beta \operatorname{Div}(\Delta - a)Q = f & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^N \\ \partial_t Q - (\Delta - a)Q - \beta D(u) = G & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^N \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^N \\ u = h, \quad D_N Q = H & \text{on } \mathbb{R}_+ \times \mathbb{R}_0^N \\ u(0) = u_0, \quad Q(0) = Q_0 & \text{in } \mathbb{R}_+^N. \end{cases}$$

For this purpose, \mathcal{R} -boundedness of the solution operator families to the resolvent problem is a key issue. Moreover, we prove the L^p - L^q maximal regularity yields the local well-posedness for the system (1) with small initial data in \mathbb{R}_+^N .

1.1 Notations

In this section, we summarize the symbols and functional spaces used through the paper.

Let $\theta \in (0, \frac{\pi}{2})$ and $r > 0$, then we can define

$$\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} \mid |\operatorname{Arg}(z)| < \pi - \theta\}$$

and

$$\Sigma_{\theta,r} := \{z \in \Sigma_\theta \mid |z| > r\}.$$

We denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0, +\infty)$ and $\mathbb{R}_- = (-\infty, 0)$. For any $q \in (1, \infty)$ we denote the dual exponent $q' = \frac{q}{q-1}$. For any multi-index $\alpha \in \mathbb{N}_0^N$ we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_N,$$

$$D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}.$$

For any $k \in \mathbb{N}_0$, for any $\Omega \subseteq \mathbb{R}^N$ open set and for any function $f: \Omega \rightarrow \mathbb{R}$, $g: \Omega \rightarrow \mathbb{R}^N$ and $A: \Omega \rightarrow \mathbb{R}^{N^2}$ we denote

$$\nabla^k f = (D^\alpha f \mid |\alpha| = k), \quad \nabla^k g = (D^\alpha g_j \mid |\alpha| = k, \quad j = 1, \dots, N),$$

$$\nabla^k A = (D^\alpha A_{\ell,j} \mid |\alpha| = k, \quad \ell, j = 1, \dots, N).$$

We also denote $C^\infty(\Omega)$ the space of infinitely differentiable functions in Ω and $C_c^\infty(\Omega)$ the $C^\infty(\Omega)$ -functions with compact support.

Let \mathcal{F} and \mathcal{F}^{-1} denote the Fourier transform and the Fourier inverse transform, respectively, which are defined by setting

$$\widehat{f}(\tau) = \mathcal{F}[f](\tau) = \int_{\mathbb{R}} e^{-it\tau} f(t) dt, \quad \mathcal{F}^{-1}[f](t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\tau} f(\tau) d\tau.$$

Let X, Y be two Banach spaces, then we denote with $\mathcal{L}(X; Y)$ the linear bounded operators between X and Y . We write $\mathcal{L}(X)$ when $Y = X$.

Let $p, q \in (1, \infty)$, $m \in \mathbb{N}_0$ and $s \in \mathbb{R}$, then we denote $L^q(\Omega)$, $W^{m,q}(\Omega)$ and $B_{q,p}^s(\Omega)$ respectively the Lebesgue, the Sobolev and the Besov spaces and we denote $\|\cdot\|_{L^q(\Omega)}$, $\|\cdot\|_{W^{m,q}(\Omega)}$ and $\|\cdot\|_{B_{q,p}^s(\Omega)}$ their norms. We denote $H^m(\Omega) := W^{m,2}(\Omega)$.

Let $s \in (0, 1)$ and $p \in (1, \infty)$, then we recall the definition of

$$H_p^s(\mathbb{R}) := \left\{ v \in L^p(\mathbb{R}) \mid \mathcal{F}^{-1}[(1 + |\tau|^2)^{s/2} \mathcal{F}[v]] \in L^p(\mathbb{R}) \right\}$$

with the norm

$$\|v\|_{H_p^s(\mathbb{R})} := \left\| \mathcal{F}^{-1} \left[(1 + |\tau|^2)^{s/2} \widehat{v} \right] \right\|_{L^p(\mathbb{R})}.$$

Let now $A \subseteq \mathbb{R}$ open, then we define

$$H_p^s(A) := \left\{ v \in L^p(A) \mid \exists \tilde{v} \in H_p^s(\mathbb{R}) \text{ such that } \tilde{v}|_A = v \right\},$$

with the norm

$$\|v\|_{H_p^s(A)} := \inf_{\tilde{v}|_A = v} \|\tilde{v}\|_{H_p^s(\mathbb{R})}.$$

Moreover, let X be a Banach space, then we denote $L^p((a, b); X)$, $W^{m,p}((a, b); X)$ and $H^s((a, b); X)$ the previous spaces function for X -valued functions for any $(a, b) \subseteq \mathbb{R}$.

Let $N \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^N$ open set, $0 \leq k < s < m$ and $p, q \in [1, \infty]$, then we recall the definition of the Besov spaces

$$B_{p,q}^s(\Omega) := (W^{k,p}(\Omega), W^{m,p}(\Omega))_{\theta,q},$$

with $s = (1 - \theta)k + \theta m$.

Let $q \in (1, \infty)$ then we denote

$$J_q(\mathbb{R}_+^N) := \left\{ f \in L^q(\mathbb{R}_+^N; \mathbb{R}^N) \mid \langle f, \nabla \varphi \rangle = 0, \quad \forall \varphi \in \widehat{H}_q^1(\mathbb{R}_+^N) \right\},$$

$$\widehat{H}_q^1(\mathbb{R}_+^N) := \left\{ \varphi \in L_{loc}^q(\mathbb{R}_+^N) \mid \nabla \varphi \in L^q(\mathbb{R}_+^N; \mathbb{R}^N) \right\}.$$

Finally, in the paper we use C to indicate a constant which depends on the parameters of the problem. In the statements we use $C(a, b, \dots)$ to underline the dependence from a, b, \dots , otherwise we use the symbols

$$f(x) \lesssim g(x) \Leftrightarrow \exists C \text{ s.t. } f(x) \leq Cg(x)$$

$$f(x) \gtrsim g(x) \Leftrightarrow \exists C \text{ s.t. } f(x) \geq Cg(x)$$

$$f(x) \simeq g(x) \Leftrightarrow \exists C \text{ s.t. } f(x) = Cg(x).$$

1.2 \mathcal{R} -boundedness and main results

The main purpose of the paper is to prove the L^p - L^q maximal regularity for the linearized system

$$\begin{cases} \partial_t u - \Delta u + \nabla p + \beta \operatorname{Div}(\Delta - a)Q = f & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^N \\ \partial_t Q - (\Delta - a)Q - \beta D(u) = G & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^N \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^N \\ u = h, \quad D_N Q = H & \text{on } \mathbb{R}_+ \times \mathbb{R}_0^N \\ u(0) = u_0, \quad Q(0) = Q_0 & \text{in } \mathbb{R}_+^N, \end{cases} \quad (2)$$

with

$$h, f: \mathbb{R}_+ \times \mathbb{R}_+^N \rightarrow \mathbb{R}^N, \quad G, H: \mathbb{R}_+ \times \mathbb{R}_+^N \rightarrow S_0(N, \mathbb{R}),$$

$$u_0: \mathbb{R}_+^N \rightarrow \mathbb{R}^N, \quad Q_0: \mathbb{R}_+^N \rightarrow S_0(N, \mathbb{R})$$

in some suitable function spaces. We start from the study of the resolvent system:

$$\begin{cases} (\lambda - \Delta)u + \nabla p + \beta \operatorname{Div}(\Delta - a)Q = f & \text{in } \mathbb{R}_+^N \\ (\lambda + a - \Delta)Q - \beta D(u) = G & \text{in } \mathbb{R}_+^N \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+^N \\ u = h, \quad D_N Q = H & \text{on } \mathbb{R}_0^N. \end{cases} \quad (3)$$

As is done in [19], [20], [21], [18] and [16], we need to introduce the notion of \mathcal{R} -boundedness:

Definition 1.2.1.

Let X and Y be two Banach spaces, then we say that a family $\mathcal{S} \subseteq \mathcal{L}(X, Y)$ is \mathcal{R} -bounded if there is $C > 0$ and $p \in [1, \infty)$ such that, for any $m \in \mathbb{N}$, for any $T_j \in \mathcal{S}$, for any $x_j \in X$ with $j = 1, \dots, m$ and for any sequence $\{r_j(z)\}_{j=1}^m$ of independent, symmetric, random $\{-1, 1\}$ -valued variables on $[0, 1]$ it holds

$$\int_0^1 \left\| \sum_{j=1}^m r_j(z) T_j(x_j) \right\|_Y^p dz \leq C \int_0^1 \left\| \sum_{j=1}^m r_j(z) x_j \right\|_X^p dz.$$

The minimal C it is called \mathcal{R} -bound of \mathcal{S} and it is denoted by $\mathcal{R}(\mathcal{S})$.

We prove in the case $f = G = 0$ that the solutions for (3) can be written as $(u, p, Q) = \phi_\lambda(h, H)$, with

$$\{(\tau \partial_\tau)^\ell \phi_\lambda \mid \lambda \in \Sigma_{\theta, r}\} \quad \ell = 0, 1$$

\mathcal{R} -bounded for any $\theta \in (\theta_0, \frac{\pi}{2})$ and $r > 0$ for some $\theta_0 > 0$. In fact, the \mathcal{R} -boundedness is crucial not only for the study of solution for the resolvent system (3), but also for the existence of the linear evolution system (2).

We are now ready to state the main result of the paper, that is the L^p - L^q maximal regularity result for the linearized system (2):

Theorem 1.2.2. *Let $N \geq 2$, $a > 0$, $\beta \in \mathbb{R}$ and $p, q \in (1, +\infty)$ with $\frac{2}{p} + \frac{1}{q} < 2$, then there is $\gamma_0 > 0$ such that for any $\gamma \geq \gamma_0$, for any f, G, h, H with $h_N = 0$ on \mathbb{R}_0^N and*

$$e^{-\gamma t} f \in L^p(\mathbb{R}_+; L^q(\mathbb{R}_+^N; \mathbb{R}^N)), \quad e^{-\gamma t} G \in L^p(\mathbb{R}_+; W^{1,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))),$$

$$e^{-\gamma t} h \in \bigcap_{l=0}^2 H_p^{l/2}(\mathbb{R}_+; W^{2-l,q}(\mathbb{R}_+^N; \mathbb{R}^N)), \quad e^{-\gamma t} H \in \bigcap_{l=0}^2 H_p^{l/2}(\mathbb{R}_+; W^{2-l,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))),$$

and $u_0 \in B_{q,p}^{2(1-1/p)}(\mathbb{R}_+^N; \mathbb{R}^N) \cap J_q(\mathbb{R}_+^N)$, $Q_0 \in B_{q,p}^{3-2/p}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))$ such that

$$u_0 - h(0) = D_N Q_0 - H(0) = 0 \quad \text{on } \mathbb{R}_0^N,$$

there is a solution (u, p, Q) for (2), unique up to additive functions $c(t)$ on the pressure term p , with $p(t) \in \widehat{H}_q^1(\mathbb{R}_+^N)$ for a.e. $t > 0$ and

$$e^{-\gamma t} u \in \bigcap_{l=0}^2 H_p^{l/2}(\mathbb{R}_+; W^{2-l,q}(\mathbb{R}_+^N; \mathbb{R}^N)), \quad e^{-\gamma t} Q \in \bigcap_{l=0}^2 H_p^{l/2}(\mathbb{R}_+; W^{3-l,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))),$$

$$e^{-\gamma t} \nabla p \in L^p(\mathbb{R}_+; L^q(\mathbb{R}_+^N; \mathbb{R}^N)),$$

with

$$\sum_{l=0}^2 \|e^{-\gamma t} u\|_{H_p^{l/2}(\mathbb{R}_+; W^{2-l,q}(\mathbb{R}_+^N))} + \sum_{l=0}^2 \|e^{-\gamma t} Q\|_{H_p^{l/2}(\mathbb{R}_+; W^{3-l,q}(\mathbb{R}_+^N))} + \|e^{-\gamma t} \nabla_x p\|_{L^p(\mathbb{R}_+; L^q(\mathbb{R}_+^N))} \leq$$

$$\leq C \left[\sum_{l=0}^2 \|e^{-\gamma t}(h, H)\|_{H_p^{l/2}(\mathbb{R}_+; W^{2-k, q}(\mathbb{R}_+^N))} + \|e^{-\gamma t}f\|_{L^p(\mathbb{R}_+; L^q(\mathbb{R}_+^N))} + \|e^{-\gamma t}G\|_{L^p(\mathbb{R}_+; W^{1, q}(\mathbb{R}_+^N))} + \|u_0\|_{B_{q, p}^{2(1-1/p)}(\mathbb{R}_+^N)} + \|Q_0\|_{B_{q, p}^{3-2/p}(\mathbb{R}_+^N)} \right],$$

for some $C = C(a, \beta, p, q, N) > 0$.

The conditions

$$u_0 - h(0) = D_N Q_0 - H(0) = 0 \quad \text{on } \mathbb{R}_0^N$$

are called compatibility conditions, while $h_N = 0$ on \mathbb{R}_0^N follows by the divergence-free condition of u . We notice that it is reasonable to take the trace for u_0 and $D_N Q_0$ thanks to the condition $\frac{2}{p} + \frac{1}{q} < 2$ (see Theorem 6.6.1 of [3]). As we will see later in the paper, the resolvent estimate follows from the proof of Theorem 1.2.2 and, in particular, from the \mathcal{R} -boundedness of the map ϕ_λ we introduced before:

Theorem 1.2.3. *Let $N \geq 2$, $a, r > 0$, $\beta \in \mathbb{R}$, $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, let $q \in (1, \infty)$, let $f \in L^q(\mathbb{R}_+^N; \mathbb{R}^N)$, $G \in W^{1, q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))$, $h \in W^{2, q}(\mathbb{R}_+^N; \mathbb{R}^N)$ with $h_N = 0$ on \mathbb{R}_0^N and $H \in W^{2, q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))$, then for any $\lambda \in \Sigma_{\theta, r}$ there is a solution (u, p, Q) for (3), unique up to additive constants on the pressure term p , with $u \in W^{2, q}(\mathbb{R}_+^N; \mathbb{R}^N)$, $p \in \widehat{H}_q^1(\mathbb{R}_+^N)$ and $Q \in W^{3, q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))$, moreover there is $C = C(a, \beta, \theta, r, q, N) > 0$ such that*

$$\begin{aligned} & \left\| \left(|\lambda|u, |\lambda|^{\frac{1}{2}}\nabla u, D^2u \right) \right\|_{L^q(\mathbb{R}_+^N)} + \|\nabla p\|_{L^q(\mathbb{R}_+^N)} + \left\| \left(|\lambda|^{\frac{3}{2}}Q, |\lambda|\nabla Q, |\lambda|^{\frac{1}{2}}D^2Q, D^3Q \right) \right\|_{L^q(\mathbb{R}_+^N)} \leq \\ & \leq C \left[\|f\|_{L^q(\mathbb{R}_+^N)} + \left\| \left(|\lambda|^{\frac{1}{2}}G, \nabla G \right) \right\|_{L^q(\mathbb{R}_+^N)} + \left\| \left(|\lambda|(h, H), |\lambda|^{\frac{1}{2}}\nabla(h, H), D^2(h, H) \right) \right\|_{L^q(\mathbb{R}_+^N)} \right]. \end{aligned}$$

Later in the paper, as an application of Theorem 1.2.2, we prove the local well-posedness for the system (1) with small initial data:

Theorem 1.2.4. *Let $N \geq 2$, $\xi, b, c \in \mathbb{R}$, $a > 0$, let $p \in (2, \infty)$ and $q \in (N, \infty)$, then we can find $\varepsilon_0 > 0$ and $T = T(\varepsilon_0) > 0$ such that for any $\varepsilon < \varepsilon_0$, for any h, H with $h_N = 0$ on \mathbb{R}_0^N such that*

$$h \in \bigcap_{l=0}^2 H_p^{l/2}((0, T); W^{2-l, q}(\mathbb{R}_+^N; \mathbb{R}^N)), \quad H \in \bigcap_{l=0}^2 H_p^{l/2}((0, T); W^{2-l, q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))),$$

for any $u_0 \in J_q(\mathbb{R}_+^N) \cap B_{q, p}^{2(1-1/p)}(\mathbb{R}_+^N; \mathbb{R}^N)$, $Q_0 \in B_{q, p}^{3-2/p}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))$ and

$$u_0 - h(0) = D_N Q_0 - H(0) = 0 \quad \text{on } \mathbb{R}_0^N,$$

with

$$\|u_0\|_{B_{q, p}^{2(1-1/p)}(\mathbb{R}_+^N)} + \|Q_0\|_{B_{q, p}^{3-2/p}(\mathbb{R}_+^N)} \leq \varepsilon,$$

we can find a solution (u, p, Q) for (1), unique up to additive functions $c(t)$ on the pressure term p , with $p(t) \in \widehat{H}_q^1(\mathbb{R}_+^N)$ for a.e. $t \in (0, T)$ and

$$u \in \bigcap_{l=0}^2 H_p^{l/2}((0, T); W^{2-l, q}(\mathbb{R}_+^N; \mathbb{R}^N)), \quad \nabla p \in L^p((0, T); L^q(\mathbb{R}_+^N; \mathbb{R}^N)),$$

$$Q \in \bigcap_{l=0}^2 H_p^{l/2}(\mathbb{R}_+; W^{3-l, q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))),$$

moreover we can find $C = C(\xi, a, b, c, p, q, N) > 0$ such that

$$\begin{aligned} & \|u\|_{H_p^1((0,T);L^q(\mathbb{R}_+^N))} + \|u\|_{L^p((0,T);W^{2,q}(\mathbb{R}_+^N))} + \\ & + \|Q\|_{H_p^1((0,T);W^{1,q}(\mathbb{R}_+^N))} + \|Q\|_{L^p((0,T);W^{3,q}(\mathbb{R}_+^N))} + \|\nabla p\|_{L^p((0,T);L^q(\mathbb{R}_+^N))} \leq \\ & \leq C \left[\sum_{l=0}^2 \|h\|_{H_p^{l/2}((0,T);W^{2-l,q}(\mathbb{R}_+^N))} + \|H\|_{H_p^{l/2}((0,T);W^{2-l,q}(\mathbb{R}_+^N))} + \|u_0\|_{B_{q,p}^{2(1-1/p)}(\mathbb{R}_+^N)} + \|Q_0\|_{B_{q,p}^{3-2/p}(\mathbb{R}_+^N)} \right]. \end{aligned}$$

The paper is organized as follows: in Section 2 we study the existence of a solution for the system (2) composed with the partial Fourier Transformation in \mathbb{R}_+^N ; in Section 3 we prove the \mathcal{R} -boundedness of the solution for (2) and the consequent proof of Theorem 1.2.3; in Section 4 we prove Theorem 1.2.2 and finally in Section 5 we prove Theorem 1.2.4.

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2 Existence and uniqueness of the solution for the Fourier System

2.1 Solution formula for the Fourier System

As we said in the introduction, we firstly focus on the resolvent system (3) with $f = G = 0$, that is

$$\begin{cases} (\lambda - \Delta)u + \nabla p + \beta \operatorname{Div}(\Delta - a)Q = 0 & \text{in } \mathbb{R}_+^N \\ (\lambda + a - \Delta)Q - \beta D(u) = 0 & \text{in } \mathbb{R}_+^N \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+^N \\ u = h, \quad D_N Q = H & \text{on } \mathbb{R}_0^N. \end{cases} \quad (4)$$

Let us take the partial Fourier Transformation

$$\widehat{v}(\xi', x_N) := \int_{\mathbb{R}^{N-1}} e^{-ix' \cdot \xi'} v(x', x_N) dx' \quad \xi' \in \mathbb{R}^{N-1} \quad x_N > 0$$

of the system (4):

$$\begin{cases} (\lambda + |\xi'|^2 - D_N^2) \widehat{u}_j + i\xi_j \widehat{p} + \beta \sum_{k=1}^{N-1} i\xi_k (-|\xi'|^2 + D_N^2 - a) \widehat{Q}_{jk} \\ + \beta D_N (-|\xi'|^2 + D_N^2 - a) \widehat{Q}_{jN} = 0 \quad (j = 1, \dots, N-1), \\ (\lambda + |\xi'|^2 - D_N^2) \widehat{u}_N + D_N \widehat{p} + \beta \sum_{k=1}^{N-1} i\xi_k (-|\xi'|^2 + D_N^2 - a) \widehat{Q}_{Nk} \\ + \beta D_N (-|\xi'|^2 + D_N^2 - a) \widehat{Q}_{NN} = 0, \end{cases} \quad (5)$$

$$\sum_{k=1}^{N-1} i\xi_k \hat{u}_k + D_N \hat{u}_N = 0, \quad (6)$$

$$\begin{cases} (\lambda + |\xi'|^2 - D_N^2 + a) \hat{Q}_{jk} - \frac{\beta}{2} (i\xi_j \hat{u}_k + i\xi_k \hat{u}_j) = 0 & (j, k = 1, \dots, N-1), \\ (\lambda + |\xi'|^2 - D_N^2 + a) \hat{Q}_{jN} - \frac{\beta}{2} (i\xi_j \hat{u}_N + D_N \hat{u}_j) = 0 & (j = 1, \dots, N-1), \\ (\lambda + |\xi'|^2 - D_N^2 + a) \hat{Q}_{NN} - \beta D_N \hat{u}_N = 0, \end{cases} \quad (7)$$

$$\hat{u}_k(0) = \hat{h}_k, \quad D_N \hat{Q}_{jk}(0) = \hat{H}_{jk} \quad (j, k = 1, \dots, N). \quad (8)$$

In order to find the solution formula of the systems (5), (6) and (7), we multiply the first equation of (5) by $i\xi_j$ and sum with respect to j :

$$\begin{aligned} & \sum_{j=1}^{N-1} (\lambda + |\xi'|^2 - D_N^2) i\xi_j \hat{u}_j - |\xi'|^2 \hat{p} \\ & + \beta \sum_{j,k=1}^{N-1} i\xi_j i\xi_k (-|\xi'|^2 + D_N^2 - a) \hat{Q}_{jk} + \beta \sum_{j=1}^{N-1} i\xi_j D_N (-|\xi'|^2 + D_N^2 - a) \hat{Q}_{jN} = 0. \end{aligned} \quad (9)$$

Applying D_N to the second equation of (5), we have

$$\begin{aligned} & (\lambda + |\xi'|^2 - D_N^2) D_N \hat{u}_N + D_N^2 \hat{p} \\ & + \beta \sum_{k=1}^{N-1} i\xi_k (-|\xi'|^2 + D_N^2 - a) D_N \hat{Q}_{Nk} + \beta D_N^2 (-|\xi'|^2 + D_N^2 - a) \hat{Q}_{NN} = 0. \end{aligned} \quad (10)$$

The summation of (9) and (10) gives us

$$\begin{aligned} & (-|\xi'|^2 + D_N^2) \hat{p} + \beta \sum_{j,k=1}^{N-1} i\xi_j i\xi_k (-|\xi'|^2 + D_N^2 - a) \hat{Q}_{jk} \\ & + 2\beta \sum_{j=1}^{N-1} i\xi_j D_N (-|\xi'|^2 + D_N^2 - a) \hat{Q}_{jN} \\ & + \beta D_N^2 (-|\xi'|^2 + D_N^2 - a) \hat{Q}_{NN} = 0, \end{aligned} \quad (11)$$

where we have used (6) and the fact that Q is a symmetric matrix. Then applying $(\lambda + |\xi'|^2 - D_N^2 + a)$ to (11) and using (6) and (7), we get

$$(-|\xi'|^2 + D_N^2)(\lambda + |\xi'|^2 - D_N^2 + a) \hat{p} = 0. \quad (12)$$

Thanks to (6), (7), and (12), \hat{p} and \hat{Q} can be eliminated from (5):

$$(-|\xi'|^2 + D_N^2) L(D_N) \hat{u} = 0, \quad (13)$$

where

$$L(t) = (\lambda + |\xi'|^2 - t^2)(\lambda + a + |\xi'|^2 - t^2) + \frac{\beta^2}{2} ((t^2 - |\xi'|^2)^2 - a(t^2 - |\xi'|^2)).$$

Then applying $(-|\xi'|^2 + D_N^2) L(D_N)$ to (7), we have

$$(\lambda + |\xi'|^2 - D_N^2 + a)(-|\xi'|^2 + D_N^2) L(D_N) \hat{Q} = 0. \quad (14)$$

In term of (12), (13), and (14), we prove that

$$\begin{aligned} \hat{u}_j &= A_j^0 e^{-Ax_N} + A_j^1 e^{-L_1 x_N} + A_j^2 e^{-L_2 x_N} \quad (j = 1, \dots, N) \\ \hat{Q}_{jk} &= A_{jk} e^{-Ax_N} + P_{jk} e^{-B_a x_N} + Q_{jk}^1 e^{-L_1 x_N} + Q_{jk}^2 e^{-L_2 x_N} \quad (j, k = 1, \dots, N) \\ \hat{p} &= C e^{-Ax_N} + D e^{-B_a x_N}, \end{aligned} \quad (15)$$

are solutions of the systems (5), (6) and (7), where

$$A = |\xi'|, \quad B_a = \sqrt{\lambda + a + |\xi'|^2},$$

and $L_{1,2}$ are the roots of $L(t)$ with $\text{Re}(L_{1,2}) > 0$. It can be seen that

$$[L_{1,2}(\lambda, \xi')]^2 = |\xi'|^2 + z_{1,2}(\lambda),$$

where $z_{1,2}$ are the roots of

$$L(z) = (\lambda - z)(\lambda + a - z) + \frac{\beta^2}{2}(z^2 - az).$$

In particular,

$$\begin{aligned} z_{1,2} &= \frac{2\lambda + a(1 + \beta^2/2) \pm \sqrt{(2\lambda + a(1 + \beta^2/2))^2 - 4\lambda(\lambda + a)(1 + \beta^2/2)}}{2(1 + \beta^2/2)} \\ &= \frac{2\lambda + a(1 + \beta^2/2) \pm \sqrt{a^2(1 + \beta^2/2)^2 - 2\lambda^2\beta^2}}{2(1 + \beta^2/2)}. \end{aligned} \quad (16)$$

Moreover, the coefficients of (15) satisfy the following relationships:

$$\begin{cases} A_j^0 = -\lambda^{-1}i\xi_j C & j = 1, \dots, N-1 \\ A_N^0 = \lambda^{-1}AC \end{cases} \quad (17)$$

$$\begin{aligned} &(L_2 - L_1) \{B_a^3(L_1 + L_2) - A^2B_a^2 - A^2L_1L_2\} A_j^1 \\ &= -\left\{ (B_a^2 - L_2^2)E_j - L_2(B_aL_2 - A^2)(\hat{h}_j - A_j^0) \right\} (B_a^2 - L_1^2) \quad j = 1, \dots, N-1 \end{aligned} \quad (18)$$

$$A_N^1 = -\frac{A}{\lambda} \frac{L_2 - A}{L_2 - L_1} C - \frac{1}{L_2 - L_1} i\xi' \cdot \hat{h}' \quad (19)$$

$$\begin{aligned} &(L_2 - L_1) \{B_a^3(L_1 + L_2) - A^2B_a^2 - A^2L_1L_2\} A_j^2 \\ &= \left\{ (B_a^2 - L_1^2)E_j - L_1(B_aL_1 - A^2)(\hat{h}_j - A_j^0) \right\} (B_a^2 - L_2^2) \quad j = 1, \dots, N-1 \end{aligned} \quad (20)$$

$$A_N^2 = \frac{A}{\lambda} \frac{L_1 - A}{L_2 - L_1} C + \frac{1}{L_2 - L_1} i\xi' \cdot \hat{h}' \quad (21)$$

$$\begin{cases} A_{jk} = -\beta\lambda^{-1}(\lambda + a)^{-1}i\xi_j i\xi_k C & j, k = 1, \dots, N-1 \\ A_{Nk} = \beta\lambda^{-1}(\lambda + a)^{-1}Ai\xi_k C & k = 1, \dots, N-1 \\ A_{NN} = -\beta\lambda^{-1}(\lambda + a)^{-1}A^2C \end{cases} \quad (22)$$

$$\begin{cases} Q_{jk}^1 = \frac{\beta(i\xi_k A_j^1 + i\xi_j A_k^1)}{2(B_a^2 - L_1^2)} & j, k = 1, \dots, N-1 \\ Q_{Nk}^1 = \frac{\beta(i\xi_k A_N^1 - L_1 A_k^1)}{2(B_a^2 - L_1^2)} & k = 1, \dots, N-1 \\ Q_{NN}^1 = -\frac{\beta L_1 A_N^1}{B_a^2 - L_1^2} \end{cases} \quad (23)$$

$$\begin{cases} Q_{jk}^2 = \frac{\beta(i\xi_k A_j^2 + i\xi_j A_k^2)}{2(B_a^2 - L_2^2)} & j, k = 1, \dots, N-1 \\ Q_{Nk}^2 = \frac{\beta(i\xi_k A_N^2 - L_2 A_k^2)}{2(B_a^2 - L_2^2)} & k = 1, \dots, N-1 \\ Q_{NN}^2 = -\frac{\beta L_2 A_N^2}{B_a^2 - L_2^2} \end{cases} \quad (24)$$

$$P_{jk} = -\frac{1}{B_a} \left(AA_{jk} + L_1 Q_{jk}^1 + L_2 Q_{jk}^2 + \hat{H}_{jk} \right) \quad (25)$$

$$\left(\mathcal{I}_1 + \frac{\mathcal{I}_2}{L_2 - L_1}\right) \frac{A}{\lambda} C = \frac{\hbar}{L_2 - L_1} + \mathcal{H}_1, \quad (26)$$

$$D = \frac{\beta\lambda}{B_a} \left(\sum_{k=1}^{N-1} i\xi_k P_{Nk} - B_a P_{NN} \right), \quad (27)$$

where

$$E_j = \frac{2i\xi_j B_a}{\beta^2\lambda} D - \frac{2A}{\beta} \left(\sum_{k=1}^{N-1} i\xi_k A_{jk} - B_a A_{jN} \right) + i\xi_j \sum_{\alpha=1,2} \frac{L_\alpha A_N^\alpha}{B_a + L_\alpha} - \frac{2}{\beta} \left(\sum_{k=1}^{N-1} \xi_k \widehat{H}_{jk} - B_a \widehat{H}_{jN} \right), \quad (28)$$

$$\mathcal{I}_1 = \beta \frac{A^2}{B_a^2 - A^2} \left\{ 2A^2 - \frac{A(B_a^2 + A^2)}{B_a} \right\}, \quad (29)$$

$$\mathcal{I}_2 = -\beta \frac{L_1(L_2 - A)}{B_a^2 - L_1^2} \left\{ 2A^2 L_1 - \frac{(B_a^2 + A^2)(L_1^2 + A^2)}{2B_a} \right\} + \beta \frac{L_2(L_1 - A)}{B_a^2 - L_2^2} \left\{ 2A^2 L_2 - \frac{(B_a^2 + A^2)(L_2^2 + A^2)}{2B_a} \right\}, \quad (30)$$

$$\begin{aligned} \hbar = & \left[\beta \frac{L_1}{B_a^2 - L_1^2} \left\{ 2A^2 L_1 - \frac{(B_a^2 + A^2)(L_1^2 + A^2)}{2B_a} \right\} \right. \\ & \left. - \beta \frac{L_2}{B_a^2 - L_2^2} \left\{ 2A^2 L_2 - \frac{(B_a^2 + A^2)(L_2^2 + A^2)}{2B_a} \right\} \right] i\xi' \cdot \widehat{h}', \end{aligned} \quad (31)$$

$$\mathcal{H}_1 = A^2 \widehat{H}_{NN} - \frac{B_a^2 + A^2}{B_a} \sum_{j=1}^{N-1} i\xi_j \widehat{H}_{jN} + \sum_{j,k=1}^{N-1} i\xi_j i\xi_k \widehat{H}_{jk}, \quad (32)$$

It is possible to write explicitly the value of the coefficients of (15), anyway these formulas are complicated and, in the paper, we use just the relations from (17) to (31).

Our next task is to prove, as we anticipated, that

$$(u, p, Q) = \phi_\lambda(h, H),$$

with $\{\phi_\lambda\}_\lambda$ \mathcal{R} -bounded. In order to do so, we need some estimates over the coefficients of (15). We do it more specifically in Section 3. In this section, we prove that the relations from (17) to (31) give us a solution for the Fourier system from (5) to (8). This is not so easy for the presence of the functions:

$$\begin{aligned} C_a(\lambda, \xi') &= \frac{1}{\lambda} \left(\mathcal{I}_1 + \frac{\mathcal{I}_2}{L_2 - L_1} \right), \\ A_a(\lambda, \xi') &= B_a^3(L_1 + L_2) - A^2 B_a^2 - A^2 L_1 L_2. \end{aligned}$$

As it can be seen from the relations from (17) to (31), it is crucial that these two quantities are different from zero in order to have a well-defined solution. This is the main aim of the section. Before going on, we notice that, if $\beta = 0$, then $L_1(\lambda) \neq L_2(\lambda)$ for any $\lambda \in \Sigma_\theta$, otherwise

$$L_1(\lambda) = L_2(\lambda) \Leftrightarrow \lambda = \pm \frac{a(1 + \beta^2/2)}{\sqrt{2}|\beta|}.$$

The value $\eta := \frac{a(1 + \beta^2/2)}{\sqrt{2}|\beta|}$, is a positive real number, so it belongs to Σ_θ for any $\theta \in (0, \frac{\pi}{2})$. For this reason, in the following, we separate the case $\lambda \neq \eta$ and the case $\lambda = \eta$.

2.2 The case $\lambda \neq \eta$

We want to prove that \mathcal{C}_a and \mathcal{A}_a do not vanish for any $\lambda \in \Sigma_{\theta,r}$ and $\xi' \in \mathbb{R}^{N-1}$ for some $\theta \in (0, \frac{\pi}{2})$ and $r \geq 0$. The idea is the following: let us suppose $h = H = 0$ and let λ and ξ' such that $\mathcal{C}_a(\lambda, \xi') = 0$, then (26) gives no conditions over the coefficient C . In other words, we expect the value of C to be not unique for such a choice of λ and ξ' . For this reason, we prove a uniqueness result for the Fourier system:

Lemma 2.2.1. *Let*

$$\widehat{u}(\xi', \cdot), \widehat{Q}(\xi', \cdot), \widehat{p}(\xi', \cdot) \in C^\infty((0, +\infty)) \cap H^2((0, +\infty))$$

be a solution for the systems (5), (6) and (7) with $a \geq 0$, $\beta \in \mathbb{R}$ and with initial conditions

$$\widehat{u}(0) = 0, \quad D_N \widehat{Q}(0) = 0,$$

let $\theta \in [\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, let $\lambda \in \overline{\Sigma_\theta}$, then $\widehat{u} = \widehat{Q} \equiv 0$. Moreover $\widehat{p}(\xi') = 0$ for any $\xi' \neq 0$.

Proof.

Thanks to (7) and (6), for any $k = 1, \dots, N$ we have that

$$\begin{aligned} \sum_{j=1}^{N-1} i\xi_j (-|\xi'|^2 - a + D_N^2) \widehat{Q}_{kj} + D_N (-|\xi'|^2 - a + D_N^2) \widehat{Q}_{kN} = \\ = \lambda \left(\sum_{j=1}^{N-1} i\xi_j \widehat{Q}_{kj} + D_N \widehat{Q}_{kN} \right) + \frac{\beta}{2} (|\xi'|^2 - D_N^2) \widehat{u}_k. \end{aligned} \quad (33)$$

Moreover, we know from the hypothesis that \widehat{Q} solves the system (7). Recalling that

$$D(u) = \frac{1}{2} (\nabla u + \nabla^T u),$$

the system (7) can be rewritten as

$$(\lambda + a + |\xi'|^2 - D_N^2) \widehat{Q}_{jk} = \beta \widehat{D(u)}_{jk} \quad \forall j, k = 1, \dots, N.$$

Moreover, $D_N \widehat{Q}(0) = 0$ for any $j, k = 1, \dots, N$ by hypothesis, so

$$\widehat{Q}_{kj} = \beta (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} (\widehat{D(u)})_{kj} \quad \forall j, k = 1, \dots, N,$$

where $(\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} f$ is the only solution in $H^2((0, +\infty))$ of the system

$$\begin{cases} (\lambda + a + |\xi'|^2 - D_N^2) v = f & x_N \in (0, +\infty) \\ D_N v(0) = 0. \end{cases}$$

It is easy to see that

$$v = (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} f = (\lambda + a + |\xi'|^2 - D_N^2)^{-1} E_{\text{even}}[f]|_{(0, +\infty)}, \quad (34)$$

where $(\lambda + a + |\xi'|^2 - D^2)^{-1}$ is the resolvent for D^2 in \mathbb{R} and $E_{\text{even}}[f]$ is the even extension of f . In particular, it can be seen that $(\lambda + a + |\xi'|^2 - D^2)^{-1} E_{\text{even}}[f]$ is still an even function in

x_N . If we use (33) in (5) we gain

$$\begin{aligned}
 & \left[\lambda + \left(1 + \frac{\beta^2}{2} \right) (|\xi'|^2 - D_N^2) \right] \widehat{u}_k + i\xi_k \widehat{p} + \beta^2 \lambda \sum_{j=1}^{N-1} i\xi_j (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} \widehat{D}(u)_{kj} + \\
 & + \beta^2 \lambda D_N (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} \widehat{D}(u)_{kN} = 0 \quad (k = 1, \dots, N-1) \\
 & \left[\lambda + \left(1 + \frac{\beta^2}{2} \right) (|\xi'|^2 - D_N^2) \right] \widehat{u}_N + D_N \widehat{p} + \beta^2 \lambda \sum_{j=1}^{N-1} i\xi_j (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} \widehat{D}(u)_{Nj} + \\
 & + \beta^2 \lambda D_N (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} \widehat{D}(u)_{NN} = 0 \quad (k = N).
 \end{aligned} \tag{35}$$

Thanks to the divergence free condition, we also have that

$$(|\xi'|^2 - D_N^2) \widehat{u} = -2 \text{Div}(\widehat{D}(u)). \tag{36}$$

Let us multiply the k -th row of (35) for \widehat{u}_k in the sense of $L^2((0, +\infty))$ for any $k = 1, \dots, N$ and then we sum the rows:

$$\begin{aligned}
 & \lambda \int_0^\infty |\widehat{u}|^2 dx_N + \left(1 + \frac{\beta^2}{2} \right) \int_0^\infty |\xi'|^2 |\widehat{u}|^2 + |D_N \widehat{u}|^2 dx_N + \\
 & + \sum_{k=1}^{N-1} \int_0^\infty i\xi_k \widehat{p} \widehat{u}_k dx_N + \int_0^\infty D_N \widehat{p} \widehat{u}_N dx_N + \\
 & + \lambda \beta^2 \sum_{j=1}^{N-1} \int_0^\infty i\xi_j (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} \widehat{D}(u)^j \cdot \widehat{u} dx_N + \\
 & + \lambda \beta^2 \int_0^\infty D_N (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} \widehat{D}(u)^N \cdot \widehat{u} dx_N.
 \end{aligned} \tag{37}$$

Firstly, we notice that

$$\int_0^\infty \sum_{k=1}^{N-1} i\xi_k \widehat{p} \widehat{u}_k + D_N \widehat{p} \widehat{u}_N dx_N = - \int_0^\infty \widehat{p} \left(\sum_{k=1}^{N-1} i\xi_k \widehat{u}_i + D_N \widehat{u}_N \right) dx_N = 0, \tag{38}$$

where we've used that $\widehat{u}(0) = 0$ and (6). Let us pass to the last terms of (37):

$$\int_0^\infty i\xi_j (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} \widehat{D}(u)^j \cdot \widehat{u} dx_N = - \int_0^\infty (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} \widehat{D}(u)^j \cdot \widehat{\partial_j u} dx_N.$$

Analogously, using the condition $\widehat{u}(0) = 0$, we achieve

$$\int_0^\infty D_N (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} \widehat{D}(u)^N \cdot \widehat{u} dx_N = - \int_0^\infty (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} \widehat{D}(u)^N \cdot \widehat{D_N u} dx_N.$$

Therefore

$$\begin{aligned}
 & \int_0^\infty i\xi_j (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} \widehat{D}(u)^j \cdot \widehat{u} + D_N (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} \widehat{D}(u)^N \cdot \widehat{u} dx_N = \\
 & = - \int_0^\infty (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} \widehat{D}(u) : \widehat{\nabla u} dx_N,
 \end{aligned}$$

where we recall $A : B = \text{tr}(B^T A)$ for any $A, B \in \mathbb{R}^{N^2}$. It is easy to check that $A : B = 0$ when A is symmetric and B is anti-symmetric, so

$$= - \int_0^\infty (\lambda + a + |\xi'|^2 - D_{N|N\text{eu}}^2)^{-1} \widehat{D}(u) : \widehat{\overline{D}(u)} dx_N.$$

Now, thanks to (34) and the Plancherel Identity applied in x_N , we have that

$$\begin{aligned}
 &= -\frac{1}{2} \int_{\mathbb{R}} (\lambda + a + |\xi'|^2 - D_N^2)^{-1} E_{\text{even}}[\widehat{D(u)}] : \overline{E_{\text{even}}[\widehat{D(u)}]} dx_N = \\
 &= -\frac{1}{2} \sum_{j,k=1}^N \int_{\mathbb{R}} (\lambda + a + |\xi'|^2 - D_N^2)^{-1} E_{\text{even}}[\widehat{D(u)}]_{jk} \overline{E_{\text{even}}[\widehat{D(u)}]_{jk}} dx_N = \\
 &= -\frac{1}{2} \sum_{j,k=1}^N \int_{\mathbb{R}} \mathcal{F}_N \left[(\lambda + a + |\xi'|^2 - D_N^2)^{-1} E_{\text{even}}[\widehat{D(u)}]_{jk} \right] \overline{\mathcal{F}_N \left[E_{\text{even}}[\widehat{D(u)}]_{jk} \right]} dx_N,
 \end{aligned}$$

where $\mathcal{F}_N(f)$ is the Fourier Transformation in x_N . Since

$$\mathcal{F}_N[(\mu - \partial_{x_N}^2)^{-1}v](\xi_N) = \frac{\mathcal{F}_N[v](\xi_N)}{\mu + \xi_N^2},$$

for any $v \in L^2(\mathbb{R})$ and for any $\mu \in \mathbb{C} \setminus \mathbb{R}_-$, then

$$\begin{aligned}
 &-\frac{1}{2} \sum_{j,k=1}^N \int_{\mathbb{R}} \mathcal{F}_N \left[(\lambda + a + |\xi'|^2 - D_N^2)^{-1} E_{\text{even}}[\widehat{D(u)}]_{jk} \right] \overline{\mathcal{F}_N \left[E_{\text{even}}[\widehat{D(u)}]_{jk} \right]} dx_N = \\
 &= -\frac{1}{2} \int_{\mathbb{R}} \frac{\left| \mathcal{F}_N \left[E_{\text{even}}[\widehat{D(u)}] \right] \right|^2}{\lambda + a + |\xi'|^2 + \xi_N^2} d\xi_N.
 \end{aligned}$$

On the other hand, thanks to (36), we can repeat the same argument for the free gradient part:

$$\begin{aligned}
 &\int_0^\infty (|\xi'|^2 - D_N^2) \widehat{u} \cdot \widehat{u} dx_N = -2 \int_0^\infty \text{Div}(\widehat{D(u)}) \cdot \widehat{u} dx_N = \\
 &= 2 \int_0^\infty \widehat{D(u)} : \overline{\widehat{D(u)}} dx_N = \int_{\mathbb{R}} |E_{\text{even}}[\widehat{D(u)}]|^2 dx_N = \int_{\mathbb{R}} \left| \mathcal{F}_N \left[E_{\text{even}} \left[\widehat{D(u)} \right] \right] \right|^2 d\xi_N.
 \end{aligned}$$

Therefore, if we come back to (37), we gain that

$$\lambda \int_0^\infty |\widehat{u}|^2 dx_N + \int_{\mathbb{R}} \left[\left(1 + \frac{\beta^2}{2} \right) - \frac{\lambda \beta^2}{2(\lambda + a + |\xi'|^2 + \xi_N^2)} \right] \left| \mathcal{F}_N \left[E_{\text{even}} \left[\widehat{D(u)} \right] \right] \right|^2 d\xi_N = 0. \quad (39)$$

Now we notice that

$$1 + \frac{\beta^2}{2} - \frac{\beta^2 \lambda}{2(\lambda + a + |\xi'|^2 + \xi_N^2)} = 1 + \frac{\beta^2(a + |\xi'|^2 + \xi_N^2)}{2(\lambda + a + |\xi'|^2 + \xi_N^2)}.$$

If $\lambda \in \mathbb{R}$, then $\lambda > 0$ and (39) implies $\widehat{u} = 0$ a.e. in $(0, +\infty)$. Otherwise, we can take the imaginary part of (39) we gain

$$\text{Im} \lambda \int_0^\infty |\widehat{u}|^2 dx_N = \int_{\mathbb{R}} \text{Im} \left(\frac{\lambda \beta^2}{2(\lambda + a + |\xi'|^2 + \xi_N^2)} \right) \left| \mathcal{F}_N \left[E_{\text{even}} \left[\widehat{D(u)} \right] \right] \right|^2 d\xi_N. \quad (40)$$

It can be seen that

$$\text{Im} \left(\frac{\lambda}{\lambda + a + |\xi'|^2 + \xi_N^2} \right) = \frac{\text{Im} \lambda (a + |\xi'|^2 + \xi_N^2)}{|\lambda + a + |\xi'|^2 + \xi_N^2|^2},$$

therefore

$$\int_0^\infty |\widehat{u}|^2 dx_N = \int_{\mathbb{R}} \frac{\beta^2(a + |\xi'|^2 + \xi_N^2)}{2|\lambda + a + |\xi'|^2 + \xi_N^2|^2} \left| \mathcal{F}_N \left[E_{\text{even}} \left[\widehat{D(u)} \right] \right] \right|^2 d\xi_N. \quad (41)$$

Applying (41) to (39) we get

$$\int_{\mathbb{R}} \left[\left(1 + \frac{\beta^2}{2} \right) - \frac{|\lambda|^2 \beta^2}{2|\lambda + a + |\xi'|^2 + \xi_N^2|^2} \right] \left| \mathcal{F}_N \left[E_{\text{even}} \left[\widehat{D(u)} \right] \right] \right|^2 d\xi_N = 0.$$

Let us suppose for the moment that

$$\left(1 + \frac{\beta^2}{2}\right) - \frac{|\lambda|^2 \beta^2}{2|\lambda + a + |\xi'|^2 + \xi_N^2|^2} > 0 \quad \forall \xi = (\xi', \xi_N) \in \mathbb{R}^N. \quad (42)$$

In this case, it has to happen that

$$\mathcal{F}_N[\widehat{E_{even}[D(\widehat{u})]}](\xi) = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^N,$$

and, thanks to (41), we get also in this case $\widehat{u} = 0$ a.e. on $(0, \infty)$. On the other hand, we have now that for any $k, j = 1, \dots, N$ the function \widehat{Q}_{kj} resolves

$$\begin{cases} (\lambda + a + |\xi'|^2 - D_N^2) \widehat{Q}_{kj} = 0 & \text{in } (0, +\infty) \\ D_N \widehat{Q}_{ij}(0) = 0 \end{cases}$$

and it is well-known that it implies $\widehat{Q}_{kj} \equiv 0$.

Finally, if $\xi' \neq 0$, we can find $\xi_j \neq 0$ for some $j = 1, \dots, N-1$. In this case, if we turn back to the j -th row of (5) we have that $p(\xi') = 0$ for any $\xi' \neq 0$. Finally, in order to verify property (42) we can notice that, if we denote

$$f(x) := \frac{1}{|\lambda + a + x|^2} = \frac{1}{(\operatorname{Re}\lambda + a + x)^2 + \operatorname{Im}\lambda^2} \quad x \in \mathbb{R},$$

then

$$\max_{x \geq 0} f(x) = \begin{cases} \frac{1}{|\lambda + a|^2} & \operatorname{Re}\lambda \geq -a \\ \frac{1}{\operatorname{Im}\lambda^2} & \operatorname{Re}\lambda \leq -a. \end{cases}$$

By standard analytic arguments and the condition $\tan \theta \geq \frac{|\beta|}{\sqrt{2}}$ it can be seen that

$$\frac{\beta^2 |\lambda|^2}{2} \max_{\xi \in \mathbb{R}^N} f(|\xi|^2) \leq 1 + \frac{\beta^2}{2}.$$

□

We are now ready to prove that $\mathcal{C}_a, \mathcal{A}_a \neq 0$ for the case $\lambda \neq \eta$:

Proposition 2.2.2. *Let $a \geq 0$, $\beta \neq 0$ and $\theta \in (0, \frac{\pi}{2})$ with $\tan \theta \geq \frac{|\beta|}{\sqrt{2}}$, let us suppose that the coefficients of (15) satisfy (17), (19), (21), conditions from (23) to (25) and (27), then $(\widehat{u}, \widehat{p}, \widehat{Q})$ defined as in (15) is the only solution of the systems from (5) to (8) and*

$$\mathcal{A}_a(\lambda, \xi'), \mathcal{C}_a(\lambda, \xi') \neq 0 \quad \forall \xi' \in \mathbb{R}^{N-1}, \quad \forall \lambda \in \overline{\Sigma_\theta} \setminus \{\eta\},$$

where $\eta = \frac{a(1+\beta^2/2)}{\sqrt{2}|\beta|}$.

Proof.

Let $\widehat{h} = \widehat{H} \equiv 0$ and let us suppose by contradiction that there are $\lambda \in \overline{\Sigma_\theta} \setminus \{\eta\}$ and $\xi' \in \mathbb{R}^{N-1}$ such that $\mathcal{A}_a(\lambda, \xi') = 0$ or $\mathcal{C}_a(\lambda, \xi') = 0$. Our claim is that we can find a solution of (5), (6), (7) and (8) different from 0. If we prove it, we get the contradiction with Lemma 2.2.1.

In our first calculation, we consider C, A_k^1 and A_k^2 for $k = 1, \dots, N-1$ as free variables and we substitute the other parameters according to the relationships in the hypothesis. With this partial substitution, many of the equations of the systems (5), (6), (7) and (8) are satisfied. In

particular, we want to find C , A_k^1 and A_k^2 for $k = 1, \dots, N-1$ which solve the linear system

$$\begin{cases} i\xi_k D + \lambda\beta \sum_{j=1}^{N-1} i\xi_j P_{kj} - \lambda\beta B_a P_{kN} = 0 & k = 1, \dots, N-1 \\ \sum_{k=1}^{N-1} i\xi_k A_k^1 - L_1 A_N^1 = 0 \\ \sum_{k=1}^{N-1} i\xi_k A_k^2 - L_2 A_N^2 = 0 \\ A_k^0 + A_k^1 + A_k^2 = 0 & k = 1, \dots, N-1. \end{cases} \quad (43)$$

Let us see that the third equation follows from the fourth equation of (43) and the definition of A_N^j for $j = 0, 1, 2$ in (17), (19) and (21):

$$\begin{aligned} \sum_{k=1}^{N-1} i\xi_k A_k^2 - L_2 A_N^2 &= - \sum_{k=1}^{N-1} i\xi_k A_k^0 - \sum_{k=1}^{N-1} i\xi_k A_k^1 + L_2 A_N^0 + L_2 A_N^1 = \\ &= (L_2 - A) A_N^0 + (L_2 - L_1) A_N^1 = \left[\frac{A(L_2 - A)}{\lambda} - \frac{A(L_2 - A)}{\lambda} \right] C = 0. \end{aligned}$$

So we reduce to the system

$$\begin{cases} i\xi_k D + \lambda\beta \sum_{j=1}^{N-1} i\xi_j P_{kj} - \lambda\beta B_a P_{kN} = 0 & k = 1, \dots, N-1 \\ \sum_{k=1}^{N-1} i\xi_k A_k^1 - L_1 A_N^1 = 0 \\ A_k^0 + A_k^1 + A_k^2 = 0 & k = 1, \dots, N-1. \end{cases} \quad (44)$$

Thanks to the relationships (17), (19), (21), from (23) to (25) and (27), we can rewrite the system (44) in the following way:

$$\begin{pmatrix} \alpha_1 Id_{N-1} & \alpha_2 Id_{N-1} & i\alpha_C \xi' \\ i(\xi')^T & \underline{0}_{N-1}^T & \Lambda \\ Id_{N-1} & Id_{N-1} & -i\lambda^{-1} \xi' \end{pmatrix} \begin{pmatrix} A_k^1 \\ A_k^2 \\ C \end{pmatrix} = \underline{0}_{2N-1},$$

where $\underline{0}_d$ is the 0 vector of \mathbb{R}^d for any $d \in \mathbb{N}$ and

$$\begin{aligned} \alpha_C &= \frac{A\beta^2}{B_a^2} \left[\frac{A(B_a - A)^2}{B_a^2 - A^2} + \frac{L_2(A^2 + L_2^2 - 3B_a L_2 + B_a^2)(L_1 - A)}{2(B_a^2 - L_2^2)(L_2 - L_1)} - \frac{L_1(A^2 + L_1^2 - 3B_a L_1 + B_a^2)(L_2 - A)}{2(B_a^2 - L_1^2)(L_2 - L_1)} \right], \\ \alpha_1 &= \frac{\lambda\beta^2 L_1(A^2 - B_a L_1)}{2B_a(B_a^2 - L_1^2)}, \quad \alpha_2 = \frac{\lambda\beta^2 L_2(A^2 - B_a L_2)}{2B_a(B_a^2 - L_2^2)}, \quad \Lambda = \frac{L_1 A(L_2 - A)}{\lambda(L_2 - L_1)}. \end{aligned}$$

Let us call M the previous matrix. By a computation we get

$$\det M = (-1)^{N-1} (\alpha_1 - \alpha_2)^{N-2} \left[A^2 \left(\frac{\alpha_2}{\lambda} + \alpha_C \right) + \Lambda(\alpha_1 - \alpha_2) \right]. \quad (45)$$

It can be seen that:

$$\begin{aligned} \alpha_1 - \alpha_2 &= \frac{\beta^2 \lambda (L_2 - L_1) \mathcal{A}_a}{2B_a(B_a^2 - L_1^2)(B_a^2 - L_2^2)}, \\ \frac{\alpha_2}{\lambda} + \alpha_C &= \frac{\beta^2 L_2(A^2 - B_a L_2)}{2B_a(B_a^2 - L_2^2)} + \end{aligned}$$

$$+ \frac{\beta^2}{B_a^2} \left[\frac{A^2(B_a - A)^2}{B_a^2 - A^2} + \frac{AL_2(A^2 + L_2^2 - 3B_aL_2 + B_a^2)(L_1 - A)}{2(B_a^2 - L_2^2)(L_2 - L_1)} - \frac{AL_1(A^2 + L_1^2 - 3B_aL_1 + B_a^2)(L_2 - A)}{2(B_a^2 - L_1^2)(L_2 - L_1)} \right].$$

Finally

$$A^2 \left(\frac{\alpha_2}{\lambda} + \alpha_C \right) + \frac{L_1 A (L_2 - A)}{\lambda (L_2 - L_1)} (\alpha_1 - \alpha_2) = -\frac{\beta A}{B_a} \left(\mathcal{I}_1 + \frac{\mathcal{I}_2}{L_2 - L_1} \right).$$

This tell us that, if $\mathcal{A}_a = 0$ or $\mathcal{C}_a = 0$, we can find C , A_k^1 and A_k^2 different from 0 which resolves the systems (5), (6), (7) and (8), which contradicts Lemma 2.2.1. \square

2.3 The case $\lambda = \eta$

In this chapter, we suppose $a > 0$. In order to understand what happens for $\lambda = \eta$ we need the following remark: for any $\lambda \neq \eta$ we have already proved that the functions $\hat{u}, \hat{p}, \hat{Q}$ defined in (15) solves the systems from (5) to (8). The representation formula (15) for $\lambda \neq \eta$ can be rewritten as follows:

$$\begin{aligned} \hat{u}_k &= A_k^0 e^{-Ax_N} + (A_k^1 + A_k^2) e^{-L_1 x_N} + A_k^2 (L_2 - L_1) \mathcal{M}(L_2, L_1, x_N), \\ \hat{Q}_{kj} &= A_{kj} e^{-Ax_N} + P_{kj} e^{-B_a x_N} + (Q_{kj}^1 + Q_{kj}^2) e^{-L_1 x_N} + Q_{kj}^2 (L_2 - L_1) \mathcal{M}(L_2, L_1, x_N), \\ \hat{p} &= C e^{-Ax_N} + D e^{-B_a x_N}, \end{aligned} \quad (46)$$

where

$$\mathcal{M}(\gamma_1, \gamma_2, t) = \frac{e^{-\gamma_1 t} - e^{-\gamma_2 t}}{\gamma_1 - \gamma_2} \quad \forall \gamma_{1,2} \in \mathbb{C}, \quad \forall t \in \mathbb{R}.$$

Let us call $L_0 := L_1(\eta) = L_2(\eta)$, then we claim that the functions

$$\begin{aligned} \tilde{u}_k &= \tilde{A}_k^0 e^{-Ax_N} + \tilde{A}_k^1 e^{-L_0 x_N} + \tilde{A}_k^2 x_N e^{-L_0 x_N}, \\ \tilde{Q}_{kj} &= \tilde{A}_{kj} e^{-Ax_N} + \tilde{P}_{kj} e^{-B_a x_N} + \tilde{Q}_{kj}^1 e^{-L_0 x_N} + \tilde{Q}_{kj}^2 x_N e^{-L_0 x_N}, \\ \tilde{p} &= \tilde{C} e^{-Ax_N} + \tilde{D} e^{-B_a x_N}. \end{aligned} \quad (47)$$

are the solutions for the system (5) to (8) for $\lambda = \eta$, where

$$\tilde{A}_k^0 := \lim_{\lambda \rightarrow \eta} A_k^0(\lambda) = A_k^0(\eta), \quad (48)$$

$$\tilde{A}_k^1 := \lim_{\lambda \rightarrow \eta} A_k^1(\lambda) + A_k^2(\lambda) = \begin{cases} \hat{h}_k + \frac{i\xi_k}{\eta} \tilde{C} & k < N \\ -\frac{A}{\eta} \tilde{C} & k = N, \end{cases} \quad (49)$$

$$\begin{aligned} \tilde{A}_k^2 &:= -\lim_{\lambda \rightarrow \eta} (L_2(\lambda) - L_1(\lambda)) A_k^2(\lambda) = \\ &= \begin{cases} -\tilde{A}_a^{-1} \left[(B_a^2 - L_0^2) \left\{ (B_a^2 - L_0^2) \tilde{E}_k - L_0 (B_a L_0 - A^2) (\hat{h}_k - \tilde{A}_k^0) \right\} \right] & k < N \\ -\frac{A(L_0 - A)}{\eta} \tilde{C} - i\xi' \cdot \hat{h}' & k = N, \end{cases} \end{aligned} \quad (50)$$

$$\tilde{A}_{jk} := \lim_{\lambda \rightarrow \eta} A_{jk}(\lambda) = A_{jk}(\eta), \quad (51)$$

$$\tilde{Q}_{jk}^1 := \lim_{\lambda \rightarrow \eta} Q_{jk}^1(\lambda) + Q_{jk}^2(\lambda) = \begin{cases} \frac{\beta(i\xi_k \tilde{A}_j^1 + i\xi_j \tilde{A}_k^1)}{2(B_a^2 - L_0^2)} - \frac{\beta L_0 (i\xi_k \tilde{A}_j^2 + i\xi_j \tilde{A}_k^2)}{(B_a^2 - L_0^2)^2} & j, k < N \\ \frac{\beta(i\xi_k \tilde{A}_N^1 - L_0 \tilde{A}_k^1)}{2(B_a^2 - L_0^2)} + \frac{\beta[(B_a^2 + L_0^2) \tilde{A}_k^2 - 2i\xi_k L_0 \tilde{A}_N^2]}{2(B_a^2 - L_0^2)^2} & k < j = N \\ -\frac{\beta L_0 \tilde{A}_N^1}{B_a^2 - L_0^2} + \frac{\beta(B_a^2 + L_0^2) \tilde{A}_N^2}{(B_a^2 - L_0^2)^2} & k = j = N, \end{cases} \quad (52)$$

$$\tilde{Q}_{jk}^2 := -\lim_{\lambda \rightarrow \eta} (L_2(\lambda) - L_1(\lambda)) Q_{jk}^2(\lambda) = \begin{cases} \frac{\beta(i\xi_k \tilde{A}_j^2 + i\xi_j \tilde{A}_k^2)}{2(B_a^2 - L_0^2)} & j, k < N \\ \frac{\beta(i\xi_k \tilde{A}_N^2 - L_0 \tilde{A}_k^2)}{2(B_a^2 - L_0^2)} & k < j = N \\ -\frac{\beta L_2 \tilde{A}_N^2}{B_a^2 - L_0^2} & k = j = N, \end{cases} \quad (53)$$

$$\tilde{P}_{jk} := \lim_{\lambda \rightarrow \eta} P_{jk}(\lambda) = P_{jk}(\eta), \quad (54)$$

$$\tilde{D} := \lim_{\lambda \rightarrow \eta} D(\lambda) = D(\eta), \quad (55)$$

and where

$$\tilde{\mathcal{A}}_a := \lim_{\lambda \rightarrow \eta} \mathcal{A}_a = 2B_a^3 L_0 - A^2 B_a^2 - A^2 L_0^2. \quad (56)$$

$$\tilde{E}_k = \frac{2i\xi_k B_a}{\beta^2 \eta} \tilde{D} - \frac{2A}{\beta} \left(\sum_{j=1}^{N-1} i\xi_j \tilde{A}_{kj} - B_a \tilde{A}_{kN} \right) + i\xi_k \frac{L_0 \tilde{A}_N^1}{B_a + L_0} - \frac{2}{\beta} \left(\sum_{k=1}^{N-1} \xi_j \hat{H}_{kj} - B_a \hat{H}_{jN} \right), \quad (57)$$

and \tilde{C} satisfies the identity

$$\tilde{C}_a \tilde{C} = A^{-1} [\tilde{h} + \tilde{\mathcal{H}}_1], \quad (58)$$

with

$$\tilde{C}_a := \lim_{\lambda \rightarrow \eta} C_a(\lambda), \quad (59)$$

$$\tilde{h} := \lim_{\lambda \rightarrow \eta} \frac{\tilde{h}(\lambda)}{L_2(\lambda) - L_1(\lambda)}, \quad \tilde{\mathcal{H}}_1 = \lim_{\lambda \rightarrow \eta} \mathcal{H}_1(\lambda) = \mathcal{H}_1(\eta) \quad (60)$$

where we recall the definition of $\tilde{h}(\lambda)$ and $\mathcal{H}_1(\lambda)$ respectively in (31) and (32). We notice that \tilde{C}_a and \tilde{h} are well-defined: it can be seen that

$$\begin{aligned} \frac{\mathcal{I}_2}{L_2 - L_1} &= \frac{4\beta A^2 B_a [L_1 L_2 (B_a^2 + L_1 L_2) - AB_a^2 (L_1 + L_2)]}{2B_a (B_a^2 - L_1^2) (B_a^2 - L_2^2)} + \\ &- \frac{\beta (B_a^2 + A^2) [(B_a^2 + A^2) L_1 L_2 (L_1 + L_2) - AB_a^2 (L_1^2 + L_1 L_2 + L_2^2 + A^2) + AL_1 L_2 (L_1 L_2 - A^2)]}{2B_a (B_a^2 - L_1^2) (B_a^2 - L_2^2)}; \\ \frac{\tilde{h}}{L_2 - L_1} &= -\frac{2\beta A^2 B_a^2 (L_1 + L_2)}{(B_a^2 - L_1^2) (B_a^2 - L_2^2)} + \frac{\beta (B_a^2 + A^2) [B_a^2 (L_1^2 + L_1 L_2 + L_2^2) + A^2 B_a^2 - L_1^2 L_2^2 + A^2 L_1 L_2]}{2B_a (B_a^2 - L_1^2) (B_a^2 - L_2^2)}. \end{aligned}$$

If we manage to prove that

$$\tilde{C} = \lim_{\lambda \rightarrow \eta} C(\lambda),$$

we would have that the formula (47) can be obtained by passing to the limit as $\lambda \rightarrow \eta$ in the formula (46). However, as before, we need to check that $\tilde{C}_a(\xi') \neq 0$ for any $\xi' \in \mathbb{R}^{N-1}$. On the other hand $L_0^2 = A^2 + z$, where

$$z := z_1(\eta) = z_2(\eta) = \frac{2\eta + a(1 + \beta^2/2)}{2(1 + \beta^2/2)} = \frac{a}{2} + \frac{\eta}{1 + \beta^2/2} \in (0, a + \eta).$$

Therefore $A < L_0 < B_a$ and, in particular, if we turned back to the definition of $\tilde{\mathcal{A}}_a$ in (56) we get that

$$\tilde{\mathcal{A}}_a = B_a^2 (B_a L_0 - A^2) + L_0 (B_a^3 - A^2 L_0) > 0.$$

Let us prove that $\tilde{C}_a \neq 0$ using again Lemma 2.2.1:

Proposition 2.3.1. *Let $a > 0$, $\beta \neq 0$, then $\tilde{C}_a(\xi') \neq 0$ for any $\xi' \in \mathbb{R}^{N-1}$ and the functions given by (47) are solution of the system from (5) to (8) for $\lambda = \eta$.*

Proof.

As in the previous section, we want to use the relations from (48) to (57) in order to rewrite the system from (5) to (8) as

$$\widetilde{M} \cdot (\widetilde{C}, \widetilde{A}_k^1, \widetilde{A}_k^2)^T = V(\widehat{h}, \widehat{H}) \quad k = 1, \dots, N-1,$$

with \widetilde{M} and $V(\widehat{h}, \widehat{H})$ respectively a matrix and a vector of suitable dimensions depending on $\xi' \in \mathbb{R}^{N-1}$. The strategy is similar to what we did in Proposition 2.2.2: firstly we prove that, if by contradiction, $\widetilde{C}_a(\xi') = 0$ for some $\xi' \in \mathbb{R}^{N-1}$, then we can find a solution different from zero which solves the system from (5) to (8) even if $\widehat{h} = \widehat{H} = 0$, which contradicts Lemma 2.2.1. Once we proved that $\widetilde{C}_a(\xi') \neq 0$ for any $\xi' \in \mathbb{R}^{N-1}$, we are able to show that the functions defined in (47) are solutions of the system from (5) to (8).

Let us suppose then $\widehat{h} = \widehat{H} = 0$. Let us substitute the functions $(\widetilde{u}, \widetilde{p}, \widetilde{Q})$ in the systems from (5) to (8). As for Proposition 2.2.2, it can be seen that, thanks to the conditions from (48) to (57), many equations of the problem are satisfied and what remains is the following system in $(\widetilde{C}, \widetilde{A}_k^1, \widetilde{A}_k^2)$ for $k = 1, \dots, N-1$:

$$\begin{cases} i\xi_k \widetilde{\alpha}_C \widetilde{C} + \widetilde{\alpha}_1 \widetilde{A}_k^1 + \widetilde{\alpha}_2 \widetilde{A}_k^2 = 0 & k = 1, \dots, N-1 \\ \sum_{k=1}^{N-1} i\xi_k \widetilde{A}_k^1 - L_0 \widetilde{A}_N^1 + \widetilde{A}_N^2 = 0 \\ \sum_{k=1}^{N-1} i\xi_k \widetilde{A}_k^2 - L_0 \widetilde{A}_N^2 = 0. \end{cases} \quad (61)$$

where

$$\widetilde{\alpha}_C := \lim_{\lambda \rightarrow \eta} \alpha_C(\lambda), \quad \widetilde{\alpha}_1 := \lim_{\lambda \rightarrow \eta} \alpha_1(\lambda), \quad \widetilde{\alpha}_2 := - \lim_{\lambda \rightarrow \eta} \frac{\alpha_2(\lambda) - \alpha_1(\lambda)}{L_2 - L_1},$$

and where $\alpha_1, \alpha_2, \alpha_C$ are the same of the proof of Proposition 2.2.2. Let us see that the second equation of (61) depends linearly by the other conditions:

$$\begin{aligned} & \sum_{k=1}^{N-1} i\xi_k \widetilde{A}_k^1 - L_0 \widetilde{A}_N^1 + \widetilde{A}_N^2 = \\ & = - \sum_{k=1}^{N-1} i\xi_k \widetilde{A}_k^0 + L_0 \widetilde{A}_N^0 - \frac{A(L_0 - A)}{\eta} \widetilde{C} = -A \widetilde{A}_N^0 + L_0 \widetilde{A}_N^0 - \frac{A(L_0 - A)}{\eta} \widetilde{C} = 0. \end{aligned}$$

Finally, if we also substitute the formulas (49) and (50) in (61), our system becomes

$$\begin{cases} i\xi_k \left(\widetilde{\alpha}_C + \frac{\widetilde{\alpha}_1}{\eta} \right) \widetilde{C} + \widetilde{\alpha}_2 \widetilde{A}_k^2 = 0 & k = 1, \dots, N-1 \\ \sum_{k=1}^{N-1} i\xi_k \widetilde{A}_k^2 + \frac{L_0 A(L_0 - A)}{\eta} \widetilde{C} = 0. \end{cases} \quad (62)$$

The matrix associated to (62) is

$$\widetilde{M} := \begin{pmatrix} \widetilde{\alpha}_2 Id_{N-1} & i\xi' \left(\widetilde{\alpha}_C + \frac{\widetilde{\alpha}_1}{\eta} \right) \\ i(\xi')^T & \frac{L_0 A(L_0 - A)}{\eta} \end{pmatrix}.$$

If we turn back to the matrix M of Proposition 2.2.2, we can notice that $\det \widetilde{M} = (-1)^{N-1} \det M$, with

$$\begin{cases} \alpha_1 = \widetilde{\alpha}_2, \\ \alpha_2 = 0, \\ \alpha_C = \widetilde{\alpha}_C + \frac{\widetilde{\alpha}_1}{\eta}, \\ \Lambda = \frac{L_0 A(L_0 - A)}{\eta}, \\ \lambda = \eta. \end{cases}$$

So, using the formula (45), we get that

$$\det \widetilde{M} = \widetilde{\alpha}_2^{N-2} \left[A^2 \left(\widetilde{\alpha}_C + \frac{\widetilde{\alpha}_1}{\eta} \right) + \frac{L_0 A (L_0 - A)}{\eta} \widetilde{\alpha}_2 \right].$$

Now we notice that

$$\lim_{\lambda \rightarrow \eta} \alpha_2(\lambda) = \lim_{\lambda \rightarrow \eta} \frac{\alpha_2(\lambda) - \alpha_1(\lambda)}{L_2(\lambda) - L_1(\lambda)} (L_2(\lambda) - L_1(\lambda)) + \alpha_1(\lambda) = \widetilde{\alpha}_1.$$

So

$$\begin{aligned} & A^2 \left(\widetilde{\alpha}_C + \frac{\widetilde{\alpha}_1}{\eta} \right) + \frac{L_0 A (L_0 - A)}{\eta} \widetilde{\alpha}_2 = \\ &= \lim_{\lambda \rightarrow \eta} A^2 \left(\frac{\alpha_2(\lambda)}{\lambda} + \alpha_C(\lambda) \right) + \frac{L_1 A (L_2 - A)}{\lambda (L_2(\lambda) - L_1(\lambda))} (\alpha_1(\lambda) - \alpha_2(\lambda)) = \\ &= \lim_{\lambda \rightarrow \eta} -\frac{\beta A \lambda}{B_a} C_a(\lambda) = -\frac{\beta A \eta}{B_a} \widetilde{C}_a. \end{aligned}$$

We are now ready to conclude: let us suppose by contradiction that exists $\xi' \in \mathbb{R}^{N-1}$ s.t. $\widetilde{C}_a(\xi') = 0$, then by the calculation we have just done, we have that \widetilde{M} is a singular matrix and, in particular, we can find $(\widetilde{C}, \widetilde{A}_k^2)^T \in \ker \widetilde{M}$. This means that we can find \widetilde{C} and \widetilde{A}_k^2 not zero which solve the system (61) with $\widehat{h} = \widehat{H} = 0$. Then, again by the calculation we have made so far, we can find a solution $(\widehat{u}, \widehat{Q}, \widehat{p})$ of systems from (5) to (8) with $\widehat{h} = \widehat{H} = 0$ and this contradicts Lemma 2.2.1. Therefore $\widetilde{C}_a(\xi') \neq 0$ for any $\xi' \in \mathbb{R}^{N-1}$.

In order to conclude, it remains to prove that the functions $(\widetilde{u}, \widetilde{p}, \widetilde{Q})$ defined in (47) are solutions for the system from (5) to (8). As we mentioned above, since now we know that $\widetilde{C}_a \neq 0$, the functions $(\widetilde{u}, \widetilde{p}, \widetilde{Q})$ are the limit of $(\widehat{u}, \widehat{p}, \widehat{Q})$ defined in (15) for $\lambda \rightarrow \eta$. Moreover, it is easy to see that, for any $\alpha \in \mathbb{N}$, it holds

$$\begin{aligned} D_N^\alpha \lim_{\lambda \rightarrow \eta} A_k^0(\lambda) e^{-A x_N} &= \lim_{\lambda \rightarrow \eta} D_N^\alpha A_k^0(\lambda) e^{-A x_N} \\ D_N^\alpha \lim_{\lambda \rightarrow \eta} A_{jk}(\lambda) e^{-A x_N} &= \lim_{\lambda \rightarrow \eta} D_N^\alpha A_{jk}(\lambda) e^{-A x_N} \\ D_N^\alpha \lim_{\lambda \rightarrow \eta} P_{jk}(\lambda) e^{-B_a x_N} &= \lim_{\lambda \rightarrow \eta} D_N^\alpha P_{jk}^0(\lambda) e^{-B_a x_N} \\ D_N^\alpha \lim_{\lambda \rightarrow \eta} D(\lambda) e^{-B_a x_N} &= \lim_{\lambda \rightarrow \eta} D_N^\alpha D(\lambda) e^{-B_a x_N} \\ D_N^\alpha \left(\lim_{\lambda \rightarrow \eta} (A_k^1(\lambda) + A_k^2(\lambda)) e^{-L_1 x_N} \right) &= \lim_{\lambda \rightarrow \eta} D_N^\alpha (A_k^1(\lambda) + A_k^2(\lambda)) e^{-L_1 x_N} \\ D_N^\alpha \left(\lim_{\lambda \rightarrow \eta} (Q_{jk}^1(\lambda) + Q_{jk}^2(\lambda)) e^{-L_1 x_N} \right) &= \lim_{\lambda \rightarrow \eta} D_N^\alpha (Q_{jk}^1(\lambda) + Q_{jk}^2(\lambda)) e^{-L_1 x_N} \\ D_N^\alpha \left(\lim_{\lambda \rightarrow \eta} A_k^2(\lambda) (e^{-L_2 x_N} - e^{-L_1 x_N}) \right) &= \lim_{\lambda \rightarrow \eta} D_N^\alpha A_k^2(\lambda) (e^{-L_1 x_N} - e^{-L_2 x_N}) \\ D_N^\alpha \left(\lim_{\lambda \rightarrow \eta} Q_{jk}^2(\lambda) (e^{-L_2 x_N} - e^{-L_1 x_N}) \right) &= \lim_{\lambda \rightarrow \eta} D_N^\alpha Q_{jk}^2(\lambda) (e^{-L_1 x_N} - e^{-L_2 x_N}). \end{aligned} \tag{63}$$

We already know that the functions $(\widehat{u}, \widehat{p}, \widehat{Q})$ are solutions of the system from (5) to (8) for $\lambda \neq \eta$ so, thanks to (63), it is easy to conclude that $(\widetilde{u}, \widetilde{p}, \widetilde{Q})$ are solutions for the systems from (5) to (7) for $\lambda = \eta$. \square

3 Resolvent Estimate

3.1 Lower bound for \mathcal{C}_a and \mathcal{A}_a

In this section, we start the proof of Theorem 1.2.3, in particular we want to prove that $(u, p, Q) = \phi_\lambda(h, H)$ with $\{\phi_\lambda\}_\lambda$ a \mathcal{R} -bounded family of functions. Firstly, we need an estimate for the functions \mathcal{C}_a and \mathcal{A}_a extended to the case $\lambda = \eta$, that is

$$\mathcal{C}_a = \begin{cases} \frac{1}{\lambda} \left(\mathcal{I}_1 + \frac{\mathcal{I}_2}{L_2 - L_1} \right) & \lambda \neq \eta \\ \lim_{\lambda \rightarrow \eta} \frac{1}{\lambda} \left(\mathcal{I}_1 + \frac{\mathcal{I}_2}{L_2 - L_1} \right) & \lambda = \eta, \end{cases}$$

$$\mathcal{A}_a = \begin{cases} B_a^3(L_1 + L_2) - A^2 B_a^2 - A^2 L_1 L_2 & \lambda \neq \eta \\ \lim_{\lambda \rightarrow \eta} B_a^3(L_1 + L_2) - A^2 B_a^2 - A^2 L_1 L_2 & \lambda = \eta. \end{cases}$$

Let us start from the estimates of \tilde{z}_j for $j = 1, 2$:

Lemma 3.1.1. *Let $a \geq 0$ and $\beta \in \mathbb{R}$, let $\tilde{z}_j := \frac{z_j}{\lambda + a}$ for $j = 1, 2$, then*

$$\tilde{z}_1 \xrightarrow{|\lambda| \rightarrow +\infty} z_- := \frac{2 - i|\beta|\sqrt{2}}{2\left(1 + \frac{\beta^2}{2}\right)}, \quad \tilde{z}_2 \xrightarrow{|\lambda| \rightarrow +\infty} z_+ := \frac{2 + i|\beta|\sqrt{2}}{2\left(1 + \frac{\beta^2}{2}\right)}.$$

Moreover, let $\theta \in (0, \frac{\pi}{2})$ and $r > 0$, there are $K_m = K_m(a, \beta, r) > 0$ and $K_M = K_M(a, \beta)$ such that

$$K_m \leq |\tilde{z}_j(\lambda)| \leq K_M \quad \forall \lambda \in \overline{\Sigma_{\theta, r}} \quad j = 1, 2.$$

Proof.

It is easy to prove that $\tilde{z}_1 \rightarrow z_-$ and $\tilde{z}_2 \rightarrow z_+$ as $|\lambda| \rightarrow +\infty$. Now we notice that $z_j \neq 0$ for any $\lambda \in \overline{\Sigma_{\theta, r}}$: by definition of z_j we have that

$$(\lambda - z_j)(\lambda + a - z_j) + \frac{\beta^2}{2}(z_j^2 - az_j) = 0 \quad \forall \lambda \in \overline{\Sigma_{\theta, r}} \quad j = 1, 2. \quad (64)$$

Let us suppose by contradiction that exists $\lambda \in \overline{\Sigma_{\theta, r}}$ such that $z_j(\lambda) = 0$ for some $j = 1, 2$. Therefore, turning back to (64), we should have that

$$\lambda(\lambda + a) = 0,$$

which is impossible if $\lambda \in \overline{\Sigma_{\theta, r}}$. Moreover, since the limits are different from 0 for any $\beta \in \mathbb{R}$, by continuity in λ we can conclude. \square

In the previous section, we have proved that \mathcal{C}_a and \mathcal{A}_a are always different from 0. Since these two functions are also regular, it means that they assume values far from 0 in every compact of $\overline{\Sigma_{\theta, r}} \times \mathbb{R}^{N-1}$. Now we want to study the behaviour when λ or ξ' go to infinity. In order to do so, we use a Laurent series with a special kind of rest term:

Definition 3.1.2.

Let $\alpha \in \mathbb{Z}$ and let $A \subseteq \mathbb{C}$, then we denote

$$o_\infty(A, t^{-\alpha}) := \left\{ g \left| \sup_{\lambda \in A} |g(\lambda, t)| |t|^\alpha \xrightarrow{|t| \rightarrow +\infty} 0 \right. \right\},$$

$$o_0(A, t^\alpha) := \left\{ g \left| \sup_{\lambda \in A} |g(\lambda, t)| |t|^{-\alpha} \xrightarrow{|t| \rightarrow 0} 0 \right. \right\}.$$

It is easy to see that such a symbols satisfy the rules of standard Landau Symbols:

Lemma 3.1.3. *Let $\alpha, \beta \in \mathbb{N}_0$ and $A, B \subseteq \mathbb{C}$ then*

- 1) *If $\beta \geq \alpha$, then $o_\infty(A, t^{-\beta}) \subseteq o_\infty(A, t^{-\alpha})$ and $o_0(A, t^\beta) \subseteq o_0(A, t^\alpha)$;*
- 2) *If $B \subseteq A$, then $o_\infty(A, t^{-\alpha}) \subseteq o_\infty(B, t^{-\alpha})$ and $o_0(A, t^\alpha) \subseteq o_0(B, t^\alpha)$;*
- 3) *We have that*

$$t^\beta o_\infty(A, t^{-\alpha}) = o_\infty(A, t^{-\alpha+\beta}), \quad t^\beta o_0(A, t^\alpha) = o_0(A, t^{\alpha+\beta});$$

4) Let $f(\lambda)$ be s.t. $\sup_{\lambda \in A} |f(\lambda)| \leq C$, then

$$f(\lambda)o_\infty(A, t^{-\alpha}) \subseteq o_\infty(A, t^{-\alpha}), \quad f(\lambda)o_0(A, t^\alpha) \subseteq o_0(A, t^\alpha);$$

5) We have that

$$o_\infty(A, t^{-\alpha}) + o_\infty(A, t^{-\beta}) \subseteq o_\infty(A, t^{-\min\{\alpha, \beta\}}), \quad o_0(A, t^\alpha) + o_0(A, t^\beta) \subseteq o_0(A, t^{\min\{\alpha, \beta\}});$$

6) We have that

$$f(t) \in o_0(A, t^\alpha) \Leftrightarrow f(1/t) \in o_\infty(A, t^{-\alpha}).$$

The idea is the following: we rewrite \mathcal{C}_a and \mathcal{A}_a in dependence of λ and $t = t(\lambda, \xi')$. We take the Laurent series with respect with t using the rest expressions we have just introduced:

Proposition 3.1.4. Let $a, r > 0$, $\beta \neq 0$ and $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, we can find $M = M(a, \beta, \theta, r) > 0$ such that

$$\inf_{\xi' \in \mathbb{R}^{N-1}, \lambda \in \Sigma_{\theta, r}} |\mathcal{C}_a(\lambda, \xi')| \geq M.$$

Proof.

We have already notice that

$$\begin{aligned} \frac{\mathcal{I}_2}{L_2 - L_1} &= \frac{4\beta A^2 B_a [L_1 L_2 (B_a^2 + L_1 L_2) - AB_a^2 (L_1 + L_2)]}{2B_a (B_a^2 - L_1^2)(B_a^2 - L_2^2)} + \\ &- \frac{\beta(B_a^2 + A^2)[(B_a^2 + A^2)L_1 L_2 (L_1 + L_2) - AB_a^2 (L_1^2 + L_1 L_2 + L_2^2 + A^2) + AL_1 L_2 (L_1 L_2 - A^2)]}{2B_a (B_a^2 - L_1^2)(B_a^2 - L_2^2)}. \end{aligned}$$

Therefore it can be seen that, for any $\lambda \in \Sigma_\theta$ and for any $\xi' \in \mathbb{R}^{N-1}$, we have

$$\mathcal{I}_1(\lambda, \xi') = \beta(\lambda + a) \mathcal{F}_1 \left(\lambda, \frac{|\xi'|}{\sqrt{\lambda + a}} \right), \quad (65)$$

$$\left(\frac{\mathcal{I}_2}{L_2 - L_1} \right) (\lambda, \xi') = \beta(\lambda + a) [\mathcal{F}_2^1 + \mathcal{F}_2^2 + \mathcal{F}_2^3 + \mathcal{F}_2^4] \left(\lambda, \frac{|\xi'|}{\sqrt{\lambda + a}} \right), \quad (66)$$

where

$$\begin{aligned} \mathcal{F}_1(\lambda, t) &:= t^2 \left(2t^2 - t\sqrt{1+t^2} - \frac{t^3}{\sqrt{1+t^2}} \right), \\ \mathcal{F}_2^1(\lambda, t) &:= \frac{2t^2 [\sqrt{t^2 + \tilde{z}_1} \sqrt{t^2 + \tilde{z}_2} (t^2 + 1 + \sqrt{t^2 + \tilde{z}_1} \sqrt{t^2 + \tilde{z}_2}) - t(t^2 + 1)(\sqrt{t^2 + \tilde{z}_1} + \sqrt{t^2 + \tilde{z}_2})]}{(1 - \tilde{z}_1)(1 - \tilde{z}_2)}, \\ \mathcal{F}_2^2(\lambda, t) &:= -\frac{(2t^2 + 1)^2 \sqrt{t^2 + \tilde{z}_1} \sqrt{t^2 + \tilde{z}_2} (\sqrt{t^2 + \tilde{z}_1} + \sqrt{t^2 + \tilde{z}_2})}{2\sqrt{t^2 + 1}(1 - \tilde{z}_1)(1 - \tilde{z}_2)}, \\ \mathcal{F}_2^3(\lambda, t) &:= \frac{t(2t^2 + 1)(t^2 + 1)(3t^2 + \tilde{z}_1 + \tilde{z}_2 + \sqrt{t^2 + \tilde{z}_1} \sqrt{t^2 + \tilde{z}_2})}{2\sqrt{t^2 + 1}(1 - \tilde{z}_1)(1 - \tilde{z}_2)}, \\ \mathcal{F}_2^4(\lambda, t) &:= -\frac{t(2t^2 + 1)\sqrt{t^2 + \tilde{z}_1} \sqrt{t^2 + \tilde{z}_2} (\sqrt{t^2 + \tilde{z}_1} \sqrt{t^2 + \tilde{z}_2} - t^2)}{2\sqrt{t^2 + 1}(1 - \tilde{z}_1)(1 - \tilde{z}_2)}, \end{aligned}$$

where we recall that $\tilde{z}_j = \frac{z_j}{\lambda + a}$. Let us call $t(\lambda, \xi') := \frac{|\xi'|}{\sqrt{\lambda + a}}$. The idea of the proof is to study the behaviour of \mathcal{C}_a when $|t(\lambda, \xi')| \rightarrow +\infty$ and when $t(\lambda, \xi')$ is bounded. For what concerns the first part, it is sufficient to take the Laurent expression of \mathcal{F}_1 and \mathcal{F}_2^j for $j = 1, 2, 3, 4$. In fact

$$\sqrt{t^2 + \tilde{z}_j} = \tilde{z}_j^{1/2} \sqrt{1 + \frac{t^2}{\tilde{z}_j}} = \tilde{z}_j^{1/2} \left(\sum_{l=0}^N C_l \left(\frac{t}{\tilde{z}_j^{1/2}} \right)^{1-2l} + f \left(\frac{t}{\tilde{z}_j^{1/2}} \right) \right) \quad \forall N \in \mathbb{N}_0,$$

with $C_l \in \mathbb{R}$ and $f(x) \in o_\infty(|x|^{-2N})$. Now we notice that

$$f\left(\frac{t}{\tilde{z}_j^{1/2}}\right) \in o_\infty(\Sigma_{\theta,r}, t^{-2N}) \quad \forall \lambda \in \Sigma_{\theta,r} :$$

let $\varepsilon > 0$, then we can find $M > 0$ such that

$$\left|\frac{t}{\tilde{z}_j^{1/2}}\right|^{2N} \left|f\left(\frac{t}{\tilde{z}_j^{1/2}}\right)\right| \leq \varepsilon \quad \forall \left|\frac{t}{\tilde{z}_j^{1/2}}\right| \geq M.$$

Then, for any $|t| \geq MK_M^{1/2}$, we have that

$$\sup_{\lambda \in \Sigma_{\theta,r}} |t|^{2N} \left|f\left(\frac{t}{\tilde{z}_j^{1/2}}\right)\right| = \sup_{\lambda \in \Sigma_{\theta,r}} |\tilde{z}_j|^N \left|\frac{t}{\tilde{z}_j^{1/2}}\right|^{2N} \left|f\left(\frac{t}{\tilde{z}_j^{1/2}}\right)\right| \leq K_M^N \varepsilon.$$

Moreover, thanks to point 4 of Lemma 3.1.3, we have that

$$\tilde{z}_j^{\frac{1}{2}} o_\infty(\Sigma_{\theta,r}, t^{-2N}) \subseteq o_\infty(\Sigma_{\theta,r}, t^{-2N}) \quad \forall N \in \mathbb{N}_0.$$

Therefore

$$\sqrt{t^2 + \tilde{z}_j} = \sum_{l=1}^N C_l \left(\frac{t^{1-2l}}{\tilde{z}_j^{-l}}\right) + o_\infty(\Sigma_{\theta,r}, t^{-2N}) \quad \forall N \in \mathbb{N}_0.$$

Similarly it can be done the same for \mathcal{F}_1 and \mathcal{F}_2^j with $j = 1, 2, 3, 4$ and it can be seen that

$$\mathcal{F}_1 + \sum_{j=1}^4 \mathcal{F}_2^j = -\frac{\tilde{z}_1 \tilde{z}_2}{2(1-\tilde{z}_1)(1-\tilde{z}_2)} + o_\infty(\Sigma_{\theta,r}, 1). \quad (67)$$

We have already noticed that $\mathcal{C}_a(\lambda, \xi') = \frac{\beta(\lambda+a)}{\lambda} F_a(\lambda, t(\lambda, \xi'))$ where

$$F_a(\lambda, t) := \mathcal{F}_1(\lambda, t) + \sum_{j=1}^4 \mathcal{F}_2^j(\lambda, t) \quad \forall \lambda, t \in \mathbb{C}.$$

In particular

$$\inf_{\xi' \in \mathbb{R}^{N-1}, \lambda \in \Sigma_{\theta,r}} |\mathcal{C}_a(\lambda, \xi')| \geq \inf_{t \in \mathbb{C}, \lambda \in \Sigma_{\theta,r}} \left| \frac{\beta(\lambda+a)}{\lambda} F_a(\lambda, t) \right| \geq K(a, \beta, \theta, r) \inf_{t \in \mathbb{C}, \lambda \in \Sigma_{\theta,r}} |F_a(\lambda, t)|.$$

Let

$$M := \inf_{t \in \mathbb{C}, \lambda \in \Sigma_{\theta,r}} |F_a(\lambda, t)|,$$

so, it is sufficient to prove that $M > 0$. Let us suppose by contradiction that $M = 0$. In this case we can find $\{\lambda_n, t_n\}$ such that $F_a(\lambda_n, t_n) \rightarrow 0$ as $n \rightarrow +\infty$. Thanks to (67)

$$\sup_{\lambda \in \Sigma_{\theta,r}} \left| F_a(\lambda, t) + \frac{\tilde{z}_1 \tilde{z}_2}{2(1-\tilde{z}_1)(1-\tilde{z}_2)} \right| \xrightarrow{|t| \rightarrow +\infty} 0.$$

In particular, for any $\varepsilon > 0$, we can find $0 < R_1 = R_1(\varepsilon)$ such that

$$\sup_{|t| \geq R_1, \lambda \in \Sigma_{\theta,r}} \left| F_a(\lambda, t) + \frac{\tilde{z}_1 \tilde{z}_2}{2(1-\tilde{z}_1)(1-\tilde{z}_2)} \right| \leq \varepsilon.$$

Thanks to Lemma 3.1.1 it is easy to see that $1 \lesssim |1 - \tilde{z}_j| \lesssim 1$ and $1 \lesssim |\tilde{z}_j| \lesssim 1$ for any $\lambda \in \Sigma_{\theta,r}$ and for $j = 1, 2$. Therefore, choosing ε sufficiently small, we can find $R_1 > 0$ such that

$$\inf_{\lambda \in \Sigma_{\theta,r}} |F_a(\lambda, t)| > 0 \quad \forall t \in B_{R_1}^c.$$

Since $M = 0$, it has to happen that $|t_n| \leq R_1$ for any $n \in \mathbb{N}_0$. Less then subsequences, we can suppose $t_n \rightarrow \hat{t} \in \mathbb{C}$. We can distinguish two cases:

- If $|\lambda_n| \leq R_2$ for some R_2 for every $n \in \mathbb{N}_0$, we can suppose that

$$\lambda_n \rightarrow \widehat{\lambda} \in \overline{\Sigma_{\theta,r}},$$

less then subsequences. In particular, by continuity of F , we should get

$$F_a(\widehat{\lambda}, \widehat{t}) = \lim_{n \rightarrow +\infty} F_a(\lambda_n, t_n) = \inf_{t \in \mathbb{C}, \lambda \in \Sigma_{\theta,r}} F_a(\lambda, t) = 0,$$

which contradicts Propositions 2.2.2 (when $\widehat{\lambda} \neq \eta$) or 2.3.1 (when $\widehat{\lambda} = \eta$).

- Let us suppose, less then subsequences, that $|\lambda_n| \rightarrow +\infty$ as $n \rightarrow +\infty$. Since the dependence of F_a from λ and a derives from \widetilde{z}_1 and \widetilde{z}_2 , we can also write

$$F_a(\lambda, t) = \widetilde{F}(\widetilde{z}_1(a, \lambda_n), \widetilde{z}_2(a, \lambda_n), t)$$

for a suitable function \widetilde{F} . We have already pointed out in Lemma 3.1.1 that $\widetilde{z}_l \rightarrow z_{\pm} := \frac{2 \pm i\sqrt{2}|\beta|}{2(1+\beta^2/2)}$ as $|\lambda| \rightarrow +\infty$ so, by continuity of \widetilde{F} , we have that

$$\widetilde{F}(\widetilde{z}_1(a, \lambda_n), \widetilde{z}_2(a, \lambda_n), t_n) \xrightarrow{n \rightarrow +\infty} \widetilde{F}(z_-, z_+, \widehat{t}).$$

Now we notice that $\widetilde{F}(z_-, z_+, t) = F_0(1, t)$ for any $t \in \mathbb{C}$. The case $a = 0$ and $\lambda = 1$ is still described in Proposition 2.2.2, so we get a contradiction as before. □

Proposition 3.1.5. *Let $a, r > 0$, $\beta \neq 0$ and $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, we can find $M = M(a, \beta, \theta, r) > 0$ such that*

$$\inf_{\xi' \in \mathbb{R}^{N-1}, \lambda \in \Sigma_{\theta,r}} |\mathcal{A}_a(\lambda, \xi')| \geq M.$$

Proof.

The idea of the proof is the same as before: it can be seen that

$$\mathcal{A}_a(\lambda, \xi') = (\lambda + a)^2 G_a(\lambda, t(\lambda, \xi'))$$

with $t(\lambda, \xi') = \frac{|\xi'|}{\sqrt{\lambda+a}}$ and

$$G_a(\lambda, t) = (t^2 + 1)^{\frac{3}{2}} (\sqrt{t^2 + \widetilde{z}_1} + \sqrt{t^2 + \widetilde{z}_2}) - t^2(t^2 + 1) - t^2 \sqrt{t^2 + \widetilde{z}_1} \sqrt{t^2 + \widetilde{z}_2}.$$

Taking the Laurent expression for G , we get as before that

$$G_a(\lambda, t) = 2t^2 + o_{\infty}(\Sigma_{\theta,r}, t).$$

Therefore, we have that $|G_a(\lambda, t)| \rightarrow +\infty$ as $|t| \rightarrow +\infty$. In particular, it remains far from 0 uniformly in $\lambda \in \Sigma_{\theta,r}$. On the other hand, thanks to Propositions 2.2.2 and 2.3.1, we know that $\mathcal{A}_a(\lambda, \xi') \neq 0$ for any $\lambda \in \overline{\Sigma_{\theta,r}}$ and $\xi' \in \mathbb{R}^{N-1}$. Finally, we can conclude as in the previous proposition. □

3.2 Fourier Multipliers

In order to get the L^p - L^q maximal regularity, we need to know the behaviour of the derivatives of the coefficients in ξ' and λ . For this reason, we introduce these two classes of Fourier Multipliers (see [21]):

Definition 3.2.1.

Let $\theta \in (0, \frac{\pi}{2})$, let $r \geq 0$, let $m(\xi', \lambda): \mathbb{R}^{N-1} \setminus \{0\} \times \Sigma_{\theta,r} \rightarrow \mathbb{C}$ be a C^∞ function in ξ' and a C^1 function in $\tau \in \mathbb{R} \setminus \{0\}$, where $\lambda = \gamma + i\tau$ with $\gamma \in \mathbb{R}$ such that $\lambda \in \Sigma_{\theta,r}$. If there is $s \in \mathbb{R}$ such that, for any $\xi' \in \mathbb{R}^{N-1} \setminus \{0\}$ and for any $\lambda \in \Sigma_{\theta,r}$, it holds

$$\begin{cases} |D_{\xi'}^\alpha m(\xi', \lambda)| \lesssim (|\lambda|^{\frac{1}{2}} + 1 + |\xi'|)^{s-|\alpha|} & \forall \alpha \in \mathbb{N}_0^{N-1}, \\ |D_{\xi'}^\alpha \tau \partial_\tau m(\xi', \lambda)| \lesssim (|\lambda|^{\frac{1}{2}} + 1 + |\xi'|)^{s-|\alpha|} & \forall \alpha \in \mathbb{N}_0^{N-1}, \end{cases}$$

then we say that $m(\xi', \lambda)$ is a **Fourier Multiplier of Order s with type 1**. Conversely, if for any $\xi' \in \mathbb{R}^{N-1} \setminus \{0\}$ and for any $\lambda \in \Sigma_{\theta,r}$, it holds

$$\begin{cases} |D_{\xi'}^\alpha m(\xi', \lambda)| \lesssim (|\lambda|^{\frac{1}{2}} + 1 + |\xi'|)^s |\xi'|^{-|\alpha|} & \forall \alpha \in \mathbb{N}_0^{N-1}, \\ |D_{\xi'}^\alpha \tau \partial_\tau m(\xi', \lambda)| \lesssim (|\lambda|^{\frac{1}{2}} + 1 + |\xi'|)^s |\xi'|^{-|\alpha|} & \forall \alpha \in \mathbb{N}_0^{N-1}, \end{cases}$$

then we say that $m(\xi', \lambda)$ is a **Fourier Multiplier of Order s with type 2**. We denote $\mathcal{M}_{s,i,\theta,r}$ the set of the Fourier multipliers of order s and type i .

Remark 3.2.2. In this chapter we follow the approach used in [21]. In this paper, the definition of Fourier Multiplier is stated with $|\lambda|^{\frac{1}{2}} + |\xi'|$ as right-hand side of the inequality. Anyway, in the following, we consider the case of $\lambda \in \Sigma_{\theta,r}$ with $r > 0$ so, it is easy to see that $|\lambda|^{\frac{1}{2}} \sim |\lambda|^{\frac{1}{2}} + 1$.

The Fourier Multipliers satisfy the following algebraic rules:

Lemma 3.2.3. *Let $s_1, s_2 \in \mathbb{R}$, $i = 1, 2$, $\theta \in (0, \frac{\pi}{2})$ and $r \geq 0$, then*

1. *If $m_i \in \mathcal{M}_{s_i,1,\theta,r}$, then $m_1 m_2 \in \mathcal{M}_{s_1+s_2,1,\theta,r}$;*
2. *If $m_i \in \mathcal{M}_{s_i,i,\theta,r}$, then $m_1 m_2 \in \mathcal{M}_{s_1+s_2,2,\theta,r}$;*
3. *If $m_i \in \mathcal{M}_{s_i,2,\theta,r}$, then $m_1 m_2 \in \mathcal{M}_{s_1+s_2,2,\theta,r}$.*

The proof follows by the definition of Fourier Multipliers. From now on, our aim is to rewrite the coefficients of the solution formula of $(\widehat{u}, \widehat{p}, \widehat{Q})$ in (15) as a combination of Fourier Multipliers. Some results are already known:

Lemma 3.2.4. *Let $a > 0$, $\theta \in (0, \frac{\pi}{2})$, then*

1. *$B_a^s \in \mathcal{M}_{s,1,\theta,0}$ for any $s \in \mathbb{R}$;*
2. *$A^s \in \mathcal{M}_{s,2,\theta,0}$ for any $s \geq 0$.*

The proof of this lemma can be found in [21] (Lemma 5.2) changing λ with $\lambda + a$ and using the next well-known lemma:

Lemma 3.2.5. *Let $\theta \in (0, \frac{\pi}{2})$, $\alpha \geq 0$ and $\lambda \in \Sigma_\theta$, then*

$$|\lambda + \alpha| \geq C(\theta)(|\lambda| + \alpha).$$

Now we need to understand that L_j for $j = 1, 2$ is a Fourier Multiplier. In order to do so, we need some lemma. The first is a technical one:

Lemma 3.2.6. *Let $z \in \mathbb{C}$ with $\text{Re}(z) \geq 0$, let $\alpha \geq 0$, then*

$$|z + \alpha| \gtrsim |z| + \alpha.$$

Proof.

We can divide the proof in two cases:

- If $\text{Arg}(z) \leq \frac{\pi}{4}$, then $z = |z|(\cos \theta + i \sin \theta)$ with $\cos \theta \geq \frac{\sqrt{2}}{2}$. Therefore

$$|z + \alpha| = \sqrt{(|z| \cos \theta + \alpha)^2 + |z|^2 \sin^2 \theta} \geq |z| \cos \theta + \alpha \gtrsim |z| + \alpha.$$

- If $\text{Arg}(z) > \frac{\pi}{4}$, then $\sin \theta > \frac{\sqrt{2}}{2}$ and $\text{Re} z \geq 0$, so

$$|z + \alpha| = \sqrt{(\text{Re} z + \alpha)^2 + |z|^2 \sin^2 \theta} \geq \sqrt{\alpha^2 + |z|^2 \sin^2 \theta} \gtrsim \alpha + |z| \sin \theta \gtrsim |z| + \alpha.$$

□

In the paper, we need not only to prove that L_j is a Fourier Multiplier, we also need an estimate for L_j^s with $s \in \mathbb{R}$. In particular, for the case $s < 0$, we need an estimate from below of L_j . The strategy is the same used for $\mathcal{C}_a, \mathcal{A}_a$: we need firstly to prove that $L_j(\lambda, \xi') \neq 0$ for any $\lambda \in \overline{\Sigma_{\theta, r}}$ and $\xi' \in \mathbb{R}^{N-1}$:

Lemma 3.2.7. *Let $a \geq 0, r > 0, \beta \in \mathbb{R}, \theta \in (0, \frac{\pi}{2})$ and $\lambda \in \overline{\Sigma_{\theta, r}}$, if $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$ or if $\text{Re} \lambda > -\frac{a}{2}$, then $z_j \notin \mathbb{R}_-$ for $j = 1, 2$.*

Proof.

Let us suppose by contradiction that $z_j = -\alpha^2$ for some $\alpha \in \mathbb{R}$ and $j = 1, 2$ and for some $\lambda \in \overline{\Sigma_{\theta, r}}$. So, by definition of z_j , we have that

$$(\lambda + \alpha^2)(\lambda + a + \alpha^2) + \frac{\beta^2}{2}(\alpha^4 + a\alpha^2) = 0. \quad (68)$$

If we take the imaginary part of (68) we get

$$\text{Im} \lambda (\text{Re} \lambda + a + \alpha^2) + (\text{Re} \lambda + \alpha^2) \text{Im} \lambda = \text{Im} \lambda (2\alpha^2 + 2\text{Re} \lambda + a) = 0.$$

If $\text{Im} \lambda = 0$, then $\text{Re} \lambda = \lambda > 0$ and we have a contradiction with (68). So, if $\text{Re} \lambda > -\frac{a}{2}$, we have a contradiction. Otherwise, let us suppose

$$\text{Re} \lambda = -\alpha^2 - \frac{a}{2}.$$

Let us take now the real part of (68):

$$\alpha^4 \left(1 + \frac{\beta^2}{2}\right) + \alpha^2 \left(2\text{Re} \lambda + a \left(1 + \frac{\beta^2}{2}\right)\right) + \text{Re} \lambda (\text{Re} \lambda + a) - (\text{Im} \lambda)^2 = 0.$$

Therefore

$$(\text{Im} \lambda)^2 = \alpha^4 \left(1 + \frac{\beta^2}{2}\right) + \alpha^2 \left[-2\alpha^2 + \frac{\beta^2}{2}a\right] + \left(-\alpha^2 - \frac{a}{2}\right) \left(-\alpha^2 + \frac{a}{2}\right) = \frac{\beta^2}{2} [\alpha^4 + a\alpha^2] - \frac{a^2}{4}.$$

Combining these two information and with standard analytic arguments, we get that

$$\tan^2 \theta \leq \frac{|\text{Im} \lambda|^2}{|\text{Re} \lambda|^2} = \frac{\frac{\beta^2}{2} [\alpha^4 + a\alpha^2] - \frac{a^2}{4}}{\alpha^4 + a\alpha^2 + \frac{a^2}{4}} \leq \frac{\beta^2}{2},$$

which contradicts the hypothesis on θ . □

As a consequence, we get that $L_j = \sqrt{|\xi'|^2 + z_j} \neq 0$.

Lemma 3.2.8. *Let $a, r > 0$, $\beta \in \mathbb{R}$ and $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$ and $\lambda \in \Sigma_{\theta, r}$, then*

$$\operatorname{Re}(L_j) \geq C(a, \beta, \theta, r) \left(|\lambda|^{\frac{1}{2}} + 1 + |\xi'| \right) \quad j = 1, 2.$$

Proof.

Let $\xi' \in \mathbb{R}^{N-1}$ and $\lambda \in \Sigma_{\theta, r}$, then

$$\operatorname{Re} \left(\sqrt{|\xi'|^2 + z_j} \right) = \frac{1}{\sqrt{2}} \sqrt{|\xi'|^2 + z_j| + |\xi'|^2 + \operatorname{Re} z_j}.$$

Let $t := \frac{|\xi'|}{|z_j|^{\frac{1}{2}}}$, then

$$\begin{aligned} \operatorname{Re} \left(\sqrt{|\xi'|^2 + z_j} \right) &= \frac{|z_j|^{\frac{1}{2}}}{\sqrt{2}} \sqrt{\left| t^2 + \frac{z_j}{|z_j|} \right| + t^2 + \frac{\operatorname{Re} z_j}{|z_j|}} = \\ &= \frac{|z_j|^{\frac{1}{2}}}{\sqrt{2}} \sqrt{\sqrt{\left(t^2 + \frac{\operatorname{Re} z_j}{|z_j|} \right)^2 + \left| \frac{\operatorname{Im} z_j}{|z_j|} \right|^2} + t^2 + \frac{\operatorname{Re} z_j}{|z_j|}}. \end{aligned}$$

Taking the Laurent expression in t , we get

$$\operatorname{Re} \left(\sqrt{|\xi'|^2 + z_j} \right) = |z_j|^{\frac{1}{2}} t + o_{\infty}(\Sigma_{\theta, r}, 1).$$

On the other hand

$$|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}} + |\xi'| = |z_j|^{\frac{1}{2}} \left(\frac{|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}}}{|z_j|^{\frac{1}{2}}} + t \right).$$

Moreover, from Lemma 3.1.1 and 3.2.5, we have

$$\frac{|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}}}{|z_j|^{\frac{1}{2}}} = \frac{|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}}}{|\lambda + a|^{\frac{1}{2}}} \cdot |\tilde{z}_j|^{-\frac{1}{2}} \lesssim 1.$$

Therefore

$$\frac{\operatorname{Re}(L_j)}{|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}} + |\xi'|} = G(\lambda, t(\lambda, \xi')),$$

with

$$G(\lambda, t) := \frac{\frac{1}{\sqrt{2}} \sqrt{\left| t^2 + \frac{z_j}{|z_j|} \right| + t^2 + \frac{\operatorname{Re} z_j}{|z_j|}}}{\frac{|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}}}{|z_j|^{\frac{1}{2}}} + t} = 1 + o_{\infty}(\Sigma_{\theta, r}, 1). \quad (69)$$

Let us see that $\operatorname{Re} L_j \neq 0$ for any $\xi' \in \mathbb{R}^{N-1}$ and $\lambda \in \overline{\Sigma_{\theta, r}}$: if $\operatorname{Re} L_j = 0$ for some $j = 0, 1$ then

$$z_j = -|\xi'|^2 - \alpha^2 \quad \text{for some } \alpha \in \mathbb{R}$$

and this contradicts Lemma 3.2.7. Finally, we can conclude: let us suppose by contradiction that

$$\inf_{\lambda \in \Sigma_{\theta, r}, t \in \mathbb{C}} |G(\lambda, t)| = 0.$$

Thanks to (69), we know that G remains far from 0 uniformly in $\lambda \in \Sigma_{\theta, r}$ as $|t| \rightarrow +\infty$. On the other hand we have just seen that $\operatorname{Re}(L_j) \neq 0$ for any $\xi' \in \mathbb{R}^{N-1}$ and $\lambda \in \overline{\Sigma_{\theta, r}}$. Finally, we can get the contradiction as in the proof of the Proposition (3.1.4) (to be noticed that Lemma 3.2.7 is true also when $a = 0$). In this way we proved that

$$|\operatorname{Re} L_j| \geq C(a, \beta, \theta, r) \left(|\lambda|^{\frac{1}{2}} + 1 + |\xi'| \right).$$

On the other hand, by definition of L_j , $\operatorname{Re}(L_j) > 0$ for any choice of $\lambda \in \Sigma_{\theta}$ and $j = 1, 2$. Therefore we conclude. \square

We are now ready to prove the estimate for L_j :

Lemma 3.2.9. *Let $a, r > 0$, $\beta \in \mathbb{R}$ and $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, then*

1. *For any $\lambda \in \Sigma_{\theta, r}$ we can find $C_1, C_2 > 0$ depending on a, β, θ, r such that*

$$C_1(|\lambda| + 1) \leq |z_j(\lambda)| \leq C_2(|\lambda| + 1) \quad j = 1, 2;$$

2. *For any $\lambda \in \Sigma_{\theta, r}$ we can find $C_1, C_2 > 0$ depending on a, β, θ, r such that*

$$C_1(|\lambda| + 1) \leq |\lambda + a - z_j| \leq C_2(|\lambda| + 1) \quad j = 1, 2;$$

3. *For any $\xi' \in \mathbb{R}^{N-1}$, for any $\lambda \in \Sigma_{\theta, r}$, for any $s \in \mathbb{R}$ and for any $\alpha \in \mathbb{N}_0^{N-1}$ we have*

$$|D_{\xi'}^\alpha L_j^s| \leq C(a, \beta, \theta, r) \left(|\lambda|^{\frac{1}{2}} + 1 + |\xi'| \right)^{s-|\alpha|};$$

4. *For any $\xi' \in \mathbb{R}^{N-1}$, for any $\lambda \in \Sigma_{\theta, r}$ and for any $\alpha \in \mathbb{N}_0^{N-1}$ we have*

$$|D_{\xi'}^\alpha (\tau \partial_\tau L_j)| \leq C(a, \beta, \theta, r) \left(|\lambda|^{\frac{1}{2}} + 1 + |\xi'| \right)^{1-|\alpha|}.$$

Proof.

For the first point, it is sufficient to notice that, thanks to Lemma 3.2.5

$$\left| \frac{z_j}{|\lambda| + 1} \right| \sim |\tilde{z}_j|$$

and from Lemma 3.1.1 we know that $1 \lesssim |\tilde{z}_j| \lesssim 1$. For what concerns the second point, it is sufficient to see that

$$\lambda + a - z_j = (\lambda + a)(1 - \tilde{z}_j)$$

and using the limits for \tilde{z}_j from Lemma 3.1.1 we get also that $1 \lesssim |1 - \tilde{z}_j| \lesssim 1$. So using Lemma 3.2.5 we conclude also the proof of the second point. In order to prove the third point, we can use Bell's Formula:

$$|D_{\xi'}^\alpha L_j^s| \lesssim \sum_{l=1}^{|\alpha|} |f^{(l)}(|\xi'|^2 + z_j)| \sum_{\alpha_1 + \dots + \alpha_l = \alpha, |\alpha_l| \geq 1} |D_{\xi'}^{\alpha_1}(|\xi'|^2 + z_j)| \cdots |D_{\xi'}^{\alpha_l}(|\xi'|^2 + z_j)|, \quad (70)$$

where $f(x) := x^{s/2}$. We know from Lemma 3.2.8 that

$$|L_j| \geq \operatorname{Re}(L_j) \gtrsim |\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}} + |\xi'|.$$

Moreover, it's easy to see that

$$|D_{\xi'}^\alpha(|\xi'|^2 + z_j)| \lesssim (|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}} + |\xi'|)^{2-|\alpha|} \quad \forall \alpha \in \mathbb{N}_0^{N-1}.$$

In fact, the inequality is obvious for $|\alpha| > 2$, while the other cases follow by a simple computation and from the estimate of z_j we have just proved. Finally, turning back to (70), we have that

$$\begin{aligned} |D_{\xi'}^\alpha L_j^s| &\lesssim \sum_{l=1}^{|\alpha|} |L_j|^{s-2l} \sum_{\alpha_1 + \dots + \alpha_l = \alpha, |\alpha_l| \geq 1} (|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}} + |\xi'|)^{2-|\alpha_1|} \cdots (|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}} + |\xi'|)^{2-|\alpha_l|} \leq \\ &\leq \sum_{l=1}^{|\alpha|} (|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}} + |\xi'|)^{s-2l} (|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}} + |\xi'|)^{2l-|\alpha|} = (|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}} + |\xi'|)^{s-|\alpha|}. \end{aligned}$$

Finally, let us study $\tau \partial_\tau L_j$ for $j = 1, 2$:

$$\tau \partial_\tau L_j = \frac{\tau \partial_\tau z_j}{2L_j} = \frac{i\tau z_j'}{2L_j} = \frac{Im\lambda z_j'}{2L_j}.$$

From the previous point we have then

$$|D_{\xi'}^\alpha(\tau \partial_\tau L_j)| \lesssim (|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}} + |\xi'|)^{-1-|\alpha|} |Im\lambda z_j'|. \quad (71)$$

So, in order to conclude, we need to estimate the last term uniformly in λ :

$$z_j' = \frac{1}{2(1 + \beta^2/2)} \left[2 \pm \frac{2\lambda\beta^2}{\sqrt{a^2(1 + \beta^2/2)^2 - 2\lambda^2\beta^2}} \right].$$

If $\beta = 0$, then $|\tau \partial_\tau z_j| \lesssim |\lambda|$. Otherwise,

$$z_j' = \frac{1}{2(1 + \beta^2/2)} \left[2 - \frac{|\beta|\sqrt{2}\lambda}{(\eta^2 - \lambda^2)^{\frac{1}{2}}} \right],$$

where $\eta := \frac{a(1+\beta^2/2)}{|\beta|\sqrt{2}} \in \mathbb{R}$. We can now notice that

$$Im\lambda = -Im(\eta - \lambda).$$

Therefore

$$|Im\lambda z_j'| \leq 2|Im\lambda| + \frac{\sqrt{2}|\beta||Im\lambda||\lambda|}{|\eta^2 - \lambda^2|^{\frac{1}{2}}} \lesssim |\lambda| + \frac{|Im\lambda|^{\frac{1}{2}}|\lambda|}{|\eta + \lambda|^{\frac{1}{2}}}.$$

Since $\eta > 0$, we can use Lemma 3.2.5. So, for any $\beta \in \mathbb{R}$

$$|\tau \partial_\tau z_j| \lesssim |\lambda| \leq (|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}} + |\xi'|)^2 \quad \forall j = 1, 2. \quad (72)$$

Finally, applying the previous estimate in (71), we conclude. \square

Now, we need to rewrite \mathcal{C}_a in a different way from the previous chapters: we already know that

$$\mathcal{C}_a = \frac{1}{\lambda} \left(\mathcal{I}_1 + \frac{\mathcal{I}_2}{L_2 - L_1} \right),$$

with

$$\begin{aligned} \mathcal{I}_1 &= \beta \frac{A^2}{B_a^2 - A^2} \left\{ 2A^2 - \frac{A(B_a^2 + A^2)}{B_a} \right\}, \\ \mathcal{I}_2 &= -\beta \frac{L_1(L_2 - A)}{B_a^2 - L_1^2} \left\{ 2A^2 L_1 - \frac{(B_a^2 + A^2)(L_1^2 + A^2)}{2B_a} \right\} + \beta \frac{L_2(L_1 - A)}{B_a^2 - L_2^2} \left\{ 2A^2 L_2 - \frac{(B_a^2 + A^2)(L_2^2 + A^2)}{2B_a} \right\}. \end{aligned}$$

It will be useful in the following to underline the differences hidden inside \mathcal{I}_1 and \mathcal{I}_2 . It can be seen by a calculation that

$$\begin{aligned} \mathcal{I}_1 + \frac{\mathcal{I}_2}{L_2 - L_1} &= -\frac{\beta A^3(B_a^2 - A^2)}{B_a(B_a + A)^2} + \\ &+ \frac{\beta A B_a(L_1 - A)(L_2 - A)[(L_1 A - B_a^2)(L_1 - A) + L_2(A^2 - B_a^2)]}{B_a(B_a^2 - L_1^2)(B_a^2 - L_2^2)} + \\ &+ \frac{\beta(B_a - A)^2[(A^2 B_a^2 + A^2 L_1 L_2 + B_a^2 L_1 L_2 + L_2^2 B_a^2)(A - L_1) + A L_1(A + L_1)(B_a^2 - L_2^2)]}{2B_a(B_a^2 - L_1^2)(B_a^2 - L_2^2)}. \end{aligned} \quad (73)$$

As a consequence of Lemma 3.2.4 and Lemma 3.2.9, we have that many factors of (73) are Fourier Multipliers.

Lemma 3.2.10. *Let $a, r > 0$, $\beta \in \mathbb{R}$, $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, let $j = 1, 2$ then*

1. $(A + B_a)^{-1} \in \mathcal{M}_{-1,2,\theta,r}$;
2. $(L_j + A)^{-1} \in \mathcal{M}_{-1,2,\theta,r}$;
3. $(L_j + B_a)^{-1} \in \mathcal{M}_{-1,1,\theta,r}$;
4. $\frac{L_j - A}{B_a^2 - L_k^2}, \frac{L_j - B_a}{B_a^2 - L_k^2}, \frac{L_j - A}{B_a^2 - A^2}, \frac{L_j - B_a}{B_a^2 - A^2} \frac{B_a - A}{B_a^2 - L_k^2} \in \mathcal{M}_{-1,1,\theta,r}$ for any $j, k = 1, 2$;
5. $\left(\frac{S_1 - S_2}{\sqrt{\lambda + a}}\right)^s, \left(\frac{\lambda + a}{\lambda}\right)^s \in \mathcal{M}_{0,1,\theta,r}$ for any $s = 1, -1$ and for any $S_1, S_2 = B_a, L_1, L_2$;
6. $\frac{S - A}{\sqrt{\lambda + a}} \in \mathcal{M}_{0,2,\theta,r}$ for any $S = B_a, L_1, L_2$.

Proof.

The first point is proved in Lemma 5.2 of [21]. The proof of the second and third points is almost the same, using that

$$|L_j + A| \geq \operatorname{Re}(L_j + A) \geq \operatorname{Re}(L_j) \gtrsim |\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}} + |\xi'|.$$

For what concerns the fourth point, let $j, k = 1, 2$, then

$$\frac{L_j - A}{B_a^2 - L_k^2} = \frac{L_j^2 - A^2}{B_a^2 - L_k^2} \frac{1}{L_j + A}.$$

We already know that $(L_j + A)^{-1} \in \mathcal{M}_{-1,1,\theta,r}$. On the other hand

$$\frac{L_j^2 - A^2}{B_a^2 - L_k^2} = \frac{z_j}{\lambda + a - z_k}$$

and thanks to Lemma 3.2.9, we have $\left|\frac{L_j^2 - A^2}{B_a^2 - L_k^2}\right| \lesssim 1$ for any $\lambda \in \Sigma_{\theta,r}$. Moreover

$$\tau \partial_\tau \left(\frac{L_j^2 - A^2}{B_a^2 - L_k^2} \right) = \frac{\tau \partial_\tau z_j}{B_a^2 - L_k^2} - \frac{(i\tau - \tau \partial_\tau z_k)(L_j^2 - A^2)}{(B_a^2 - L_k^2)^2}.$$

We have already seen in (72) that

$$|\tau \partial_\tau z_j| \lesssim |\lambda| \quad \forall j = 1, 2.$$

Therefore, as before, we have that $\left|\tau \partial_\tau \left(\frac{L_j^2 - A^2}{B_a^2 - L_k^2}\right)\right| \lesssim 1$. The other results can be proved in the same way. The last two points are obvious. \square

Thanks to the previous lemma and (73) we can prove that \mathcal{C}_a^{-1} is a Fourier Multiplier:

Lemma 3.2.11. *Let $a, r > 0$, $\beta \neq 0$ let $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, then we can find $L > 0$ and $\rho_j(\xi', \lambda) \in \mathcal{M}_{-1,1,\theta,r}$ and $m_j \in \mathcal{M}_{1,2,\theta,r}$ for $j = 1, \dots, L$ such that*

$$\mathcal{C}_a = \frac{\lambda + a}{\lambda} \sum_{j=1}^L \rho_j(\xi', \lambda) m_j(\xi', \lambda).$$

Moreover, $\mathcal{C}_a^{-1} \in \mathcal{M}_{0,2,\theta,r}$.

Proof.

The first part of the result comes from the decomposition of \mathcal{C}_a we have made in (73) and the Lemmas 3.2.3, 3.2.4 and 3.2.10. In particular, $\mathcal{C}_a \in \mathcal{M}_{0,2,\theta,r}$. Let now $s \in \mathbb{N}$, then by Bell's Formula we have

$$|D_{\xi'}^\alpha \mathcal{C}_a^{-s}| \lesssim \sum_{l=1}^{|\alpha|} |\mathcal{C}_a|^{-s-l} \sum_{\alpha_1+\dots+\alpha_l=\alpha, |\alpha_k|\geq 1} |D_{\xi'}^{\alpha_1} \mathcal{C}_a| \cdots |D_{\xi'}^{\alpha_l} \mathcal{C}_a|.$$

Now, using that $\mathcal{C}_a \in \mathcal{M}_{0,2,\theta,r}$ and Proposition 3.1.4, we get that

$$|D_{\xi'}^\alpha \mathcal{C}_a^{-s}| \lesssim |\xi'|^{-|\alpha|} \quad \forall s \in \mathbb{N}, \quad \forall \alpha \in \mathbb{N}_0^{N-1}.$$

Now we notice that

$$\tau \partial_\tau \mathcal{C}_a^{-1} = -\frac{\tau \partial_\tau \mathcal{C}_a}{\mathcal{C}_a^2}.$$

Therefore, using again that $\mathcal{C}_a \in \mathcal{M}_{0,2,\theta,r}$, the previous estimate with $s = 2$ and the Leibniz Formula we get

$$|D_{\xi'}^\alpha (\tau \partial_\tau \mathcal{C}_a^{-1})| \lesssim \sum_{\beta+\gamma=\alpha} |D_{\xi'}^\beta \tau \partial_\tau \mathcal{C}_a| |D_{\xi'}^\gamma \mathcal{C}_a^{-2}| \lesssim |\xi'|^{-|\alpha|}.$$

□

Let us pass to \mathcal{A}_a . As for \mathcal{C}_a , it is convenient to rewrite \mathcal{A}_a in order to highlight the cancellations:

$$\begin{aligned} \mathcal{A}_a &= B_a^2(B_a L_1 - A^2) + L_2(B_a^3 - A^2 L_1) = \\ &= B_a^3(L_1 - B_a) + B_a^2(B_a^2 - A^2) + L_2 B_a(B_a^2 - A^2) + L_2 A^2(B_a - L_1) = \\ &= (B_a^2 - A^2)(B_a^2 + L_2 B_a) + (B_a - L_1)[A^2(L_2 - B_a) + B_a(A^2 - B_a^2)] = \\ &= B_a(B_a^2 - A^2)(L_1 + L_2) - A^2(B_a - L_1)(B_a - L_2). \end{aligned}$$

Lemma 3.2.12. *Let $a, r > 0$, $\beta \neq 0$ and $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, then $(\lambda + a)^{-1} \mathcal{A}_a \in \mathcal{M}_{2,1,\theta,r}$ and $(\lambda + a) \mathcal{A}_a^{-1} \in \mathcal{M}_{-2,1,\theta,r}$.*

Proof.

Thanks to the decomposition we just made

$$\frac{\mathcal{A}_a}{\lambda + a} = B_a(L_1 + L_2) - A^2 \cdot \frac{(B_a - L_1)(B_a - L_2)}{\lambda + a}.$$

We know that $B_a, L_1 + L_2 \in \mathcal{M}_{1,1,\theta,r}$, so their product is in $\mathcal{M}_{2,1,\theta,r}$. On the other hand, $A^2 \in \mathcal{M}_{2,1,\theta,r}$, while from Lemma 3.2.10 $\frac{B_a - L_1}{\sqrt{\lambda + a}}, \frac{B_a - L_2}{\sqrt{\lambda + a}} \in \mathcal{M}_{0,1,\theta,r}$. In particular, the decomposition tells us that $\frac{\mathcal{A}_a}{\lambda + a} \in \mathcal{M}_{2,1,\theta,r}$. Thanks to Lemma 3.2.4, we have that $\frac{\mathcal{A}_a}{(\lambda + a) B_a^2} \in \mathcal{M}_{0,1,\theta,r}$. Moreover, we know from Proposition 3.1.5, that

$$\inf_{\lambda \in \Sigma_{\theta,r}, \xi' \in \mathbb{R}^{N-1}} \left| \frac{\mathcal{A}_a(\lambda, t)}{(\lambda + a) B_a^2} \right| > 0.$$

Let now $s \in \mathbb{N}$, then by Bell's Formula

$$\begin{aligned} \left| D_{\xi'}^\alpha \left(\frac{(\lambda + a) B_a^2}{\mathcal{A}_a} \right)^s \right| &= \left| D_{\xi'}^\alpha \left(\frac{\mathcal{A}_a}{(\lambda + a) B_a^2} \right)^{-s} \right| \lesssim \\ &\lesssim \sum_{l=1}^{|\alpha|} \frac{|\mathcal{A}_a|^{-(s+l)}}{B_a^{-2(s+l)} |\lambda + a|^{-(s+l)}} \sum_{\alpha_1+\dots+\alpha_l=\alpha, |\alpha_k|\geq 1} \left| D_{\xi'}^{\alpha_1} \left(\frac{\mathcal{A}_a}{B_a^2(\lambda + a)} \right) \right| \cdots \left| D_{\xi'}^{\alpha_l} \left(\frac{\mathcal{A}_a}{B_a^2(\lambda + a)} \right) \right| \lesssim \end{aligned}$$

$$\lesssim (|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}} + |\xi'|)^{-|\alpha|}.$$

Now we notice that

$$\tau \partial_\tau \left(\frac{\mathcal{A}_a}{(\lambda + a)B_a^2} \right) = B_a^{-2} \tau \partial_\tau \left(\frac{\mathcal{A}_a}{\lambda + a} \right) - \frac{i\tau \mathcal{A}_a}{(\lambda + a)B_a^4}.$$

Therefore, using that $\frac{\mathcal{A}_a}{\lambda + a} \in \mathcal{M}_{2,1,\theta,r}$ and the Leibniz Formula, we get

$$\begin{aligned} & \left| D_{\xi'}^\alpha \left(\tau \partial_\tau \left(\frac{\mathcal{A}_a}{(\lambda + a)B_a^2} \right) \right) \right| \lesssim \\ & \lesssim \sum_{\beta + \gamma = \alpha} |D_{\xi'}^\beta B_a^{-2}| \left| D_{\xi'}^\gamma \tau \partial_\tau \left(\frac{\mathcal{A}_a}{\lambda + a} \right) \right| + |\lambda| \left| D_{\xi'}^\beta \left(\frac{\mathcal{A}_a}{\lambda + a} \right) \right| |D_{\xi'}^\gamma B_a^{-4}| \lesssim (|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}} + |\xi'|)^{-|\alpha|}. \end{aligned}$$

Now we notice that

$$\tau \partial_\tau \left(\frac{(\lambda + a)B_a^2}{\mathcal{A}_a} \right) = -\tau \partial_\tau \left(\frac{\mathcal{A}_a}{(\lambda + a)B_a^2} \right) \frac{(\lambda + a)^2 B_a^4}{\mathcal{A}_a^2}.$$

Finally, using Leibniz Formula, we get

$$\begin{aligned} & \left| D_{\xi'}^\alpha \tau \partial_\tau \left(\frac{(\lambda + a)B_a^2}{\mathcal{A}_a} \right) \right| \lesssim \\ & \lesssim \sum_{\beta + \gamma = \alpha} \left| D_{\xi'}^\beta \left(\tau \partial_\tau \left(\frac{\mathcal{A}_a}{(\lambda + a)B_a^2} \right) \right) \right| \left| D_{\xi'}^\gamma \left(\frac{(\lambda + a)^2 B_a^4}{\mathcal{A}_a^2} \right) \right| \lesssim (|\lambda|^{\frac{1}{2}} + a^{\frac{1}{2}} + |\xi'|)^{-|\alpha|}. \end{aligned}$$

Therefore $\frac{(\lambda + a)B_a^2}{\mathcal{A}_a} \in \mathcal{M}_{0,1,\theta,r}$. This concludes the proof, because

$$\frac{\lambda + a}{\mathcal{A}_a} = \frac{(\lambda + a)B_a^2}{\mathcal{A}_a} \cdot B_a^{-2} \in \mathcal{M}_{-2,1,\theta,r}.$$

□

Also the term E_k deserves a special treatment: it can be seen by a calculation that

$$E_k = E_k^h i\xi' \cdot \widehat{h}' + \sum_{j,l=1}^N E_k^{H_{jl}} \widehat{H}_{jl}, \quad (74)$$

with $\mathcal{E}_h i\xi' \cdot \widehat{h}' := \frac{\hbar}{L_2 - L_1}$ and

$$\mathcal{E}_h = \frac{\beta}{L_2 - L_1} \left[\frac{L_1}{B_a^2 - L_1^2} \left(2A^2 L_1 - \frac{(B_a^2 + A^2)(L_1^2 + A^2)}{2B_a} \right) - \frac{L_2}{B_a^2 - L_2^2} \left(2A^2 L_2 - \frac{(B_a^2 + A^2)(L_2^2 + A^2)}{2B_a} \right) \right];$$

$$\begin{aligned} E_k^h & := i\xi_k \left\{ \frac{\mathcal{E}_h}{\lambda \mathcal{C}_a} \left[\frac{2AB_a}{(B_a + A)^2} + \right. \right. \\ & \quad \left. \left. - \frac{2[(L_2 B_a^2 - A^2 B_a)(L_1 - A) + (AL_1 L_2 + AB_a L_1)(A - B_a)]}{(B_a^2 - A^2)(B_a + L_1)(B_a + L_2)} - \frac{AB_a + L_1 L_2}{(B_a + L_1)(B_a + L_2)} \right] + \right. \\ & \quad \left. - \frac{2(B_a^2(L_1 + L_2) - A^2 B_a + L_1 L_2 B_a)}{(B_a^2 - A^2)(B_a + L_1)(B_a + L_2)} + \frac{B_a}{(B_a + L_1)(B_a + L_2)} \right\}. \end{aligned}$$

$$E_k^{H_{NN}} := \frac{i\xi_k \mathcal{B}}{\lambda \mathcal{C}_a} A^2 + \frac{2i\xi_k B_a^2}{\beta(B_a^2 - A^2)} \quad \forall k = 1, \dots, N - 1;$$

$$E_k^{H_{jN}} := -\frac{(B_a^2 + A^2)i\xi_k i\xi_j \mathcal{B}}{B_a \lambda \mathcal{C}_a} - \frac{4i\xi_j i\xi_k B_a}{\beta(B_a^2 - A^2)} + \frac{2B_a}{\beta} \delta_{jk} \quad \forall j, k = 1, \dots, N - 1;$$

$$E_k^{H,jl} = \frac{i\xi_j i\xi_k i\xi_l \mathcal{B}}{\lambda \mathcal{C}_a} + \frac{2i\xi_j i\xi_k i\xi_l}{\beta(B_a^2 - A^2)} - \frac{2}{\beta} i\xi_j \delta_{kl} - \frac{2}{\beta} i\xi_l \delta_{jk} \quad \forall j, k, l = 1, \dots, N-1;$$

$$\mathcal{B} := \frac{2A^2}{(B_a + A)^2} + \frac{2[(L_2 B_a^2 - A^2 B_a)(L_1 - A) + (AL_1 L_2 + AB_a L_1)(A - B_a)]}{(B_a + L_1)(B_a + L_2)(B_a^2 - A^2)} + \frac{AB_a + L_1 L_2}{(B_a + L_1)(B_a + L_2)}.$$

Moreover, for what concerns \mathcal{E}_h , the internal parenthesis are the same of $\frac{\mathcal{I}_2}{L_2 - L_1}$, so

$$\mathcal{E}_h = \frac{\beta A \{(B_a - L_2)[2B_a L_1(A - L_1) + L_1^2(B_a - L_2) + L_1^2(B_a - L_1)] - L_1 L_2(B_a - A)^2 - B_a^2(L_2 - A)^2\}}{(B_a^2 - L_1^2)(B_a^2 - L_2^2)} + \frac{\beta(B_a - A)^2[-A^2 B_a^2 - L_1 L_2 A^2 - B_a^2(L_1^2 + L_1 L_2 + L_2^2) + L_1^2 L_2^2]}{2B_a(B_a^2 - L_1^2)(B_a^2 - L_2^2)}.$$

Lemma 3.2.13. *Let $a, r > 0$, $\beta \neq 0$ and $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, then*

1. $\mathcal{E}_h = Am_0 + m_1$ with $m_0 \in \mathcal{M}_{0,1,\theta,r}$ and $m_1 \in \mathcal{M}_{1,1,\theta,r}$;
2. $\mathcal{B} \in \mathcal{M}_{0,2,\theta,r}$

The proof follows from the previous decomposition and Lemmas 3.2.3, 3.2.4 and 3.2.10.

3.3 Setting and technical lemmas

We are near to the proof of the main theorem of the resolvent part. Let us start writing the solutions in a different way: for any $k = 1, \dots, N$, we have

$$\begin{aligned} \widehat{u}_k &= A_k^0 e^{-Ax_N} + A_k^1 e^{-L_1 x_N} + A_k^2 e^{-L_2 x_N} = \\ &= A_k^0 e^{-Ax_N} + (A_k^1 + A_k^2) e^{-L_1 x_N} + A_k^2 (L_2 - L_1) \mathcal{M}(L_2, L_1, x_N) = \\ &= (A_k^0 + A_k^1 + A_k^2) e^{-Ax_N} + (A_k^1 + A_k^2)(L_1 - A) \mathcal{M}(L_1, A, x_N) + A_k^2 (L_2 - L_1) \mathcal{M}(L_2, L_1, x_N), \end{aligned}$$

where

$$\mathcal{M}(\gamma_1, \gamma_2, x_N) := \frac{e^{-\gamma_1 x_N} - e^{-\gamma_2 x_N}}{\gamma_1 - \gamma_2} \quad \forall \gamma_1, \gamma_2 \in \mathbb{C}.$$

Thanks to the boundary conditions, we have that

$$A_k^0 + A_k^1 + A_k^2 = \widehat{h}_k \quad \forall k = 1, \dots, N.$$

Therefore

$$\begin{aligned} \widehat{u}_k &= \widehat{h}_k e^{-Ax_N} + (\widehat{h}_k - A_k^0)(L_1 - A) \mathcal{M}(L_1, A, x_N) + A_k^2 (L_2 - L_1) \mathcal{M}(L_2, L_1, x_N) = \\ &= \widehat{h}_k (e^{-Ax_N} + \mathcal{M}(L_1, A, x_N)(L_1 - A)) - A_k^0 (L_1 - A) \mathcal{M}(L_1, A, x_N) + A_k^2 (L_2 - L_1) \mathcal{M}(L_2, L_1, x_N) = \\ &= \widehat{h}_k e^{-L_1 x_N} - A_k^0 (L_1 - A) \mathcal{M}(L_1, A, x_N) + A_k^2 (L_2 - L_1) \mathcal{M}(L_2, L_1, x_N). \end{aligned}$$

Now we use the Volhevic trick (see [23]): as an example, let us take $v: \mathbb{R}_+^N \rightarrow \mathbb{R}$ such that

$$v(x) = \mathcal{F}^{-1} \left[M(\xi', \lambda) g(\xi', 0) e^{-\gamma(\xi', \lambda) x_N} \right],$$

with $Re(\gamma) > 0$, where \mathcal{F}^{-1} is the Fourier inverse transform in the tangential components.

Then we can rewrite v as

$$v(x) = \mathcal{F}^{-1} \left[\int_0^\infty \partial_{y_N} \left(M(\xi', \lambda) g(\xi', y_N) e^{-\gamma(\xi', \lambda)(x_N + y_N)} \right) dy_N \right].$$

In this section we will see that these kind of functions are strongly related with the \mathcal{R} -boundedness of the solution:

Lemma 3.3.1. *Let $a, r > 0$, $\beta \in \mathbb{R}$ and $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, then for any $\ell = 0, 1$, $j = 1, 2$, $\alpha \in \mathbb{N}_0^{N-1}$ and for any $\lambda \in \Sigma_{\theta, r}$ it can be found $d > 0$ such that*

$$\begin{aligned} |D_{\xi'}^\alpha (\tau \partial_\tau)^\ell e^{-L_j x_N}| &\lesssim \left(|\lambda|^{\frac{1}{2}} + 1 + |\xi'| \right)^{-|\alpha|} e^{-d(|\lambda|^{\frac{1}{2}} + 1 + |\xi'|) x_N}; \\ |D_{\xi'}^\alpha (\tau \partial_\tau)^\ell \mathcal{M}(L_1, A, x_N)| &\lesssim \left((|\lambda|^{-\frac{1}{2}} + 1) \text{ or } x_N \right) |\xi'|^{-|\alpha|} e^{-d(|\lambda|^{\frac{1}{2}} + 1 + |\xi'|)}; \\ |D_{\xi'}^\alpha (\tau \partial_\tau)^\ell \mathcal{M}(L_2, L_1, x_N)| &\lesssim \left((|\lambda|^{-\frac{1}{2}} + 1) \text{ or } x_N \right) \left(|\lambda|^{\frac{1}{2}} + 1 + |\xi'| \right)^{-|\alpha|} e^{-d(|\lambda|^{\frac{1}{2}} + 1 + |\xi'|)}. \end{aligned}$$

The proof of these estimates, can be gained using Lemma 3.2.9 and following the proof of Lemma 5.3 of [21].

Lemma 3.3.2. *Let $a, r > 0$, $\beta \in \mathbb{R}$, $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, let $q \in (1, \infty)$, let $m_1 \in \mathcal{M}_{0,1,\theta,r}$, $m_2 \in \mathcal{M}_{0,2,\theta,r}$ and $m_3 \in \mathcal{M}_{1,2,\theta,r}$, let us define the following operators in $\mathcal{L}(L^q(\mathbb{R}_+^N))$:*

$$\begin{aligned} (K_1(\lambda)g)(x) &:= \int_0^\infty \mathcal{F}_{\xi'}^{-1} [m_1(\lambda, \xi') (|\lambda|^{\frac{1}{2}} + 1) e^{-L_1(x_N + y_N)} \widehat{g}(\xi', y_N)](x') dy_N \\ (K_2(\lambda)g)(x) &:= \int_0^\infty \mathcal{F}_{\xi'}^{-1} [m_2(\lambda, \xi') A e^{-L_1(x_N + y_N)} \widehat{g}(\xi', y_N)](x') dy_N \\ (K_3(\lambda)g)(x) &:= \int_0^\infty \mathcal{F}_{\xi'}^{-1} [m_2(\lambda, \xi') A^2 \mathcal{M}(L_1, A, x_N + y_N) \widehat{g}(\xi', y_N)](x') dy_N \\ (K_4(\lambda)g)(x) &:= \int_0^\infty \mathcal{F}_{\xi'}^{-1} [m_2(\lambda, \xi') A (|\lambda|^{\frac{1}{2}} + 1) \mathcal{M}(L_1, A, x_N + y_N) \widehat{g}(\xi', y_N)](x') dy_N \\ (K_5(\lambda)g)(x) &:= \int_0^\infty \mathcal{F}_{\xi'}^{-1} [m_2(\lambda, \xi') A^2 \mathcal{M}(L_2, L_1, x_N + y_N) \widehat{g}(\xi', y_N)](x') dy_N \\ (K_6(\lambda)g)(x) &:= \int_0^\infty \mathcal{F}_{\xi'}^{-1} [m_1(\lambda, \xi') (|\lambda| + 1) \mathcal{M}(L_2, L_1, x_N + y_N) \widehat{g}(\xi', y_N)](x') dy_N \\ (K_7(\lambda)g)(x) &:= \int_0^\infty \mathcal{F}_{\xi'}^{-1} [m_2(\lambda, \xi') (|\lambda|^{\frac{1}{2}} + 1) A \mathcal{M}(L_2, L_1, x_N + y_N) \widehat{g}(\xi', y_N)](x') dy_N \\ (K_8(\lambda)g)(x) &:= \int_0^\infty \mathcal{F}_{\xi'}^{-1} [m_3(\lambda, \xi') A \mathcal{M}(L_2, L_1, x_N + y_N) \widehat{g}(\xi', y_N)](x') dy_N. \end{aligned}$$

Then, for $\ell = 0, 1$, for $h = 1, 2$ and for $j, s = 1, \dots, N$ the sets

$$\{(\tau \partial_\tau)^\ell (K_h(\lambda)) \mid \lambda \in \Sigma_{\theta, r}\}$$

are \mathcal{R} -bounded, where the \mathcal{R} -bound doesn't exceed a constant $C = C(\theta, r, N, q)$.

The proof can be done following the one of Lemma 5.4 of [21].

Lemma 3.3.3. *Let $a, r > 0$, $\beta \in \mathbb{R}$, $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, let $q \in (1, \infty)$, let $k_1 \in \mathcal{M}_{-1,1,\theta,r}$, $k_2 \in \mathcal{M}_{-2,2,\theta,r}$, $k_3 \in \mathcal{M}_{-1,2,\theta,r}$ and $k_4 \in \mathcal{M}_{0,1,\theta,r}$, let us define the following operators in $\mathcal{L}(L^q(\mathbb{R}_+^N))$:*

$$\begin{aligned} (L_1(\lambda)g)(x) &:= \int_0^\infty \mathcal{F}_{\xi'}^{-1} [k_1(\lambda, \xi') e^{-L_1(x_N + y_N)} \widehat{g}(\xi', y_N)](x') dy_N \\ (L_2(\lambda)g)(x) &:= \int_0^\infty \mathcal{F}_{\xi'}^{-1} [k_2(\lambda, \xi') A e^{-L_1(x_N + y_N)} \widehat{g}(\xi', y_N)](x') dy_N \\ (L_3(\lambda)g)(x) &:= \int_0^\infty \mathcal{F}_{\xi'}^{-1} [k_2(\lambda, \xi') A^2 \mathcal{M}(L_1, A, x_N + y_N) \widehat{g}(\xi', y_N)](x') dy_N \\ (L_4(\lambda)g)(x) &:= \int_0^\infty \mathcal{F}_{\xi'}^{-1} [k_2(\lambda, \xi') A |\lambda|^{\frac{1}{2}} \mathcal{M}(L_1, A, x_N + y_N) \widehat{g}(\xi', y_N)](x') dy_N \\ (L_5(\lambda)g)(x) &:= \int_0^\infty \mathcal{F}_{\xi'}^{-1} [k_4(\lambda, \xi') \mathcal{M}(L_2, L_1, x_N + y_N) \widehat{g}(\xi', y_N)](x') dy_N \\ (L_6(\lambda)g)(x) &:= \int_0^\infty \mathcal{F}_{\xi'}^{-1} [k_3(\lambda, \xi') A \mathcal{M}(L_2, L_1, x_N + y_N) \widehat{g}(\xi', y_N)](x') dy_N. \end{aligned}$$

Then, for $\ell = 0, 1$, for $k = 1, \dots, 6$ and for $j, s = 1, \dots, N$ the sets

$$\{(\tau \partial_\tau)^\ell (\lambda L_k(\lambda)) \mid \lambda \in \Sigma_{\theta, r}\}, \quad \{(\tau \partial_\tau)^\ell (\gamma L_k(\lambda)) \mid \lambda \in \Sigma_{\theta, r}\},$$

$$\{(\tau \partial_\tau)^\ell (|\lambda|^{\frac{1}{2}} D_j L_k(\lambda)) \mid \lambda \in \Sigma_{\theta, r}\}, \quad \{(\tau \partial_\tau)^\ell (D_j D_s L_k(\lambda)) \mid \lambda \in \Sigma_{\theta, r}\},$$

are \mathcal{R} -bounded, where the \mathcal{R} -bound doesn't exceed a constant $C = C(\theta, r, N, q)$.

The proof can be done following the proof of Lemma 5.6 of [21] and using Lemma 3.3.2. It is now clear the reason beyond the Volhevic trick and the study of the Fourier Multipliers. In the proof of Theorem 1.2.3 we verify that our solution is a combination of functions like the one of Lemma 3.3.3.

3.4 Proof of Theorem 1.2.3

We are ready to prove the existence of the solution by \mathcal{R} -boundedness. Before, we state Proposition 3.4 of [9] and Theorem 3.3 of [10]:

Lemma 3.4.1. *Let X, Y, Z be three Banach spaces,*

1. *Let \mathcal{T} and \mathcal{S} be two \mathcal{R} -bounded families of functions in $\mathcal{L}(X; Y)$, then $\mathcal{T} + \mathcal{S}$ is still an \mathcal{R} -bounded family of functions in $\mathcal{L}(X; Y)$ with*

$$\mathcal{R}_{\mathcal{L}(X; Y)}(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}_{\mathcal{L}(X; Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X; Y)}(\mathcal{S});$$

2. *Let \mathcal{T} and \mathcal{S} be two \mathcal{R} -bounded families of functions in $\mathcal{L}(X; Y)$ and $\mathcal{L}(Y; Z)$ respectively, then $\mathcal{S}\mathcal{T}$ is still an \mathcal{R} -bounded family of functions in $\mathcal{L}(X; Z)$ with*

$$\mathcal{R}_{\mathcal{L}(X; Z)}(\mathcal{S}\mathcal{T}) \leq \mathcal{R}_{\mathcal{L}(Y; Z)}(\mathcal{S})\mathcal{R}_{\mathcal{L}(X; Y)}(\mathcal{T}).$$

Lemma 3.4.2. *Let $m \in \mathcal{M}_{0,2,\theta,0}$ in \mathbb{R}^N for any $\theta \in (0, \frac{\pi}{2})$, then*

$$\mathcal{R}_{\mathcal{L}(W^{q,n}(\mathbb{R}^N))}(L(\lambda)) \leq C(q, N, \theta) \quad n = 0, 1,$$

where

$$L(\lambda)[f] := \mathcal{F}^{-1} \left(m(\lambda, \xi) \widehat{f}(\xi) \right).$$

We have already noticed that the case $\beta = 0$ is special for our problem: when $\beta = 0$, the system (4) can be perfectly splitted in a Stokes system for u and a Laplace problem for Q and it is easy to prove with the same approach the \mathcal{R} -boundedness results for these problems:

Proposition 3.4.3. *Let $N \geq 2$, $q \in (1, \infty)$, $r > 0$, $\theta \in (0, \frac{\pi}{2})$, $\lambda \in \Sigma_{\theta,r}$ and let $h \in W^{2,q}(\mathbb{R}_+^N; \mathbb{R}^N)$ with $h_N = 0$ on \mathbb{R}_0^N , then we can find a unique solution (u, p) of*

$$\begin{cases} (\lambda - \Delta)u + \nabla p = 0 & \text{in } \mathbb{R}_+^N \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+^N \\ u = h & \text{on } \mathbb{R}_0^N \end{cases}$$

with $u \in W^{2,q}(\mathbb{R}_+^N; \mathbb{R}^N)$ and $p \in \widehat{H}_q^1(\mathbb{R}_+^N)$ and

$$u = \mathcal{A}(\lambda) \left[D^2 h, |\lambda|^{\frac{1}{2}} \nabla h, |\lambda| h \right],$$

with

$$\mathcal{A}(\lambda) \in \operatorname{Hol}(\Sigma_{\theta,r}; \mathcal{L}(W^{2,q}(\mathbb{R}_+^N; \mathbb{R}^N)))$$

and we can find $K > 0$ such that

$$\mathcal{R}_{\mathcal{L}(L^q(\mathbb{R}_+^N)^3)} \left(\left\{ (\tau \partial_\tau)^\ell \mathcal{S}(\lambda) \mathcal{A}(\lambda) \mid \tau \in \mathbb{R}, \lambda = \gamma + i\tau \right\} \right) \leq K \quad \ell = 0, 1$$

where $\mathcal{S}(\lambda) = (D^2, \lambda^{\frac{1}{2}} \nabla, \lambda)$.

Proposition 3.4.4. *Let $N \geq 2$, $q \in (1, \infty)$, $r > 0$, $\theta \in (0, \frac{\pi}{2})$, $\lambda \in \Sigma_{\theta,r}$ and let $H \in W^{2,q}(\mathbb{R}_+^N)$, then we can find a unique solution v of*

$$\begin{cases} (\lambda - \Delta)v = 0 & \text{in } \mathbb{R}_+^N \\ D_N v = H & \text{on } \mathbb{R}_0^N \end{cases}$$

with $v \in W^{3,q}(\mathbb{R}_+^N)$ and

$$v = \mathcal{A}(\lambda) \left[D^2 H, |\lambda|^{\frac{1}{2}} \nabla H, |\lambda| H \right],$$

with

$$\mathcal{A}(\lambda) \in \text{Hol}(\Sigma_{\theta,r}; \mathcal{L}(W^{2,q}(\mathbb{R}_+^N); W^{3,q}(\mathbb{R}_+^N)))$$

and we can find $K > 0$ such that

$$\mathcal{R}_{\mathcal{L}(L^q(\mathbb{R}_+^N)^3)} \left(\left\{ (\tau \partial_\tau)^\ell \mathcal{S}(\lambda) \mathcal{A}(\lambda) \mid \tau \in \mathbb{R}, \lambda = \gamma + i\tau \right\} \right) \leq K \quad \ell = 0, 1$$

where $\mathcal{S}(\lambda) = (D^3, \lambda^{\frac{1}{2}} D^2, \lambda \nabla, \lambda^{\frac{3}{2}})$.

We are finally ready to prove the main theorem: firstly we prove the existence result for the case $f = G = 0$:

Theorem 3.4.5. *Let $a, r > 0$, $\beta \in \mathbb{R}$, $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, let $q \in (1, \infty)$, let $h \in W^{2,q}(\mathbb{R}_+^N; \mathbb{R}^N)$ and $H \in W^{2,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))$, then we can find (u, p, Q) which solves*

$$\begin{cases} (\lambda - \Delta)u + \nabla p + \beta \text{Div}(\Delta - a)Q = 0 & \text{in } \mathbb{R}_+^N \\ (\lambda + a - \Delta)Q - \beta D(u) = 0 & \text{in } \mathbb{R}_+^N \\ \text{div} u = 0 & \text{in } \mathbb{R}_+^N \\ u = h, \quad D_N Q = H & \text{on } \mathbb{R}_0^N, \end{cases}$$

with $u \in W^{2,q}(\mathbb{R}_+^N; \mathbb{R}^N)$, $p \in \widehat{H}_q^1(\mathbb{R}_+^N)$ and $Q \in W^{3,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))$; moreover, let us call

$$X := L^q \left(\mathbb{R}_+^N; \mathbb{R}^{N^3+N^4} \right) \times L^q \left(\mathbb{R}_+^N; \mathbb{R}^{N^2+N^3} \right) \times L^q \left(\mathbb{R}_+^N; \mathbb{R}^{N+N^2} \right),$$

then

$$(u, Q) = (\mathcal{A}(\lambda), \mathcal{B}(\lambda)) \left[D^2(h, H), |\lambda|^{\frac{1}{2}} \nabla(h, H), |\lambda|(h, H) \right],$$

with

$$\mathcal{A}(\lambda) \in \text{Hol}(\Sigma_{\theta,r}; \mathcal{L}(X; W^{2,q}(\mathbb{R}_+^N; \mathbb{R}^N)))$$

$$\mathcal{B}(\lambda) \in \text{Hol}(\Sigma_{\theta,r}; \mathcal{L}(X; S_0(N, \mathbb{R}))),$$

and we can find $K_1, K_2 > 0$ such that

$$\mathcal{R}_{\mathcal{L}(X; Y_1)} \left(\left\{ (\tau \partial_\tau)^\ell \mathcal{S}_1(\lambda) \mathcal{A}(\lambda) \mid \tau \in \mathbb{R}, \lambda = \gamma + i\tau \right\} \right) \leq K_1 \quad \ell = 0, 1$$

$$\mathcal{R}_{\mathcal{L}(X; Y_2)} \left(\left\{ (\tau \partial_\tau)^\ell \mathcal{S}_2(\lambda) \mathcal{B}(\lambda) \mid \tau \in \mathbb{R}, \lambda = \gamma + i\tau \right\} \right) \leq K_2 \quad \ell = 0, 1,$$

where $\mathcal{S}_1(\lambda) = (D^2, \lambda^{\frac{1}{2}} \nabla, \lambda)$, $\mathcal{S}_2(\lambda) = (D^3, \lambda^{\frac{1}{2}} D^2, \lambda \nabla, \lambda^{\frac{3}{2}})$ and

$$Y_1 = L^q \left(\mathbb{R}_+^N; \mathbb{R}^{N^3} \right) \times L^q \left(\mathbb{R}_+^N; \mathbb{R}^{N^2} \right) \times L^q \left(\mathbb{R}_+^N; \mathbb{R}^N \right)$$

$$Y_2 = L^q \left(\mathbb{R}_+^N; \mathbb{R}^{N^5} \right) \times L^q \left(\mathbb{R}_+^N; \mathbb{R}^{N^4} \right) \times L^q \left(\mathbb{R}_+^N; \mathbb{R}^{N^3} \right) \times L^q \left(\mathbb{R}_+^N; \mathbb{R}^{N^2} \right).$$

Proof.

From Propositions 3.4.3 and 3.4.4 we have already the result for $\beta = 0$. So, from now on, we suppose $\beta \neq 0$. Let $k < N$, then we have already seen that

$$\widehat{u}_k = \widehat{h}_k e^{-L_1 x_N} - A_k^0 (L_1 - A) \mathcal{M}(L_1, A, x_N) + A_k^2 (L_2 - L_1) \mathcal{M}(L_2, L_1, x_N).$$

It can be seen by (20) that

$$A_k^2 (L_2 - L_1) = \frac{\left\{ (B_a^2 - L_1^2) E_k - L_1 (B_a L_1 - A^2) (\widehat{h}_k - A_k^0) \right\} (B_a^2 - L_2^2)}{B_a^3 (L_1 + L_2) - A^2 B_a^2 - L_1 L_2}.$$

Using the Volhevic trick (see [23])

$$\begin{aligned}
 u_k &= \sum_{m=1}^{14} \mathcal{U}_k^m := - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\partial_{y_N} \left(\widehat{h}(\xi', y_N)' e^{-L_1(x_N + y_N)} \right) \right] dy_N + \\
 &\quad - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\partial_{y_N} \left(\frac{i\xi_k \mathcal{E}_h(L_1 - A)}{\lambda \mathcal{C}_a A} i\xi' \cdot \widehat{h}(\xi', y_N)' \mathcal{M}(L_1, A, x_N + y_N) \right) \right] dy_N + \\
 &\quad - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\partial_{y_N} \left(\frac{i\xi_k}{\lambda \mathcal{C}_a A} A^2 (L_1 - A) \widehat{H}_{NN}(\xi', y_N) \mathcal{M}(L_1, A, x_N + y_N) \right) \right] dy_N + \\
 &\quad - \sum_{j=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\partial_{y_N} \left(\frac{i\xi_k}{\lambda \mathcal{C}_a A} \frac{(L_1 - A)(B_a^2 + A^2)}{B_a} i\xi_j \widehat{H}_{jN}(\xi', y_N) \mathcal{M}(L_1, A, x_N + y_N) \right) \right] dy_N + \\
 &\quad - \sum_{j,l=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\partial_{y_N} \left(\frac{i\xi_k}{\lambda \mathcal{C}_a A} (L_1 - A) i\xi_j i\xi_l \widehat{H}_{jl}(\xi', y_N) \mathcal{M}(L_1, A, x_N + y_N) \right) \right] dy_N + \\
 &\quad - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\partial_{y_N} \left(\frac{(B_a^2 - L_1^2)(B_a^2 - L_2^2)}{\mathcal{A}_a} E_k^h i\xi' \cdot \widehat{h}(\xi', y_N) \mathcal{M}(L_2, L_1, x_N + y_N) \right) \right] dy_N + \\
 &\quad - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\partial_{y_N} \left(\frac{(B_a^2 - L_1^2)(B_a^2 - L_2^2)}{\mathcal{A}_a} E_k^{H_{NN}} \widehat{H}_{NN}(\xi', y_N) \mathcal{M}(L_2, L_1, x_N + y_N) \right) \right] dy_N + \\
 &\quad - \sum_{j=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\partial_{y_N} \left(\frac{(B_a^2 - L_1^2)(B_a^2 - L_2^2)}{\mathcal{A}_a} E_k^{H_{jN}} \widehat{H}_{jN}(\xi', y_N) \mathcal{M}(L_2, L_1, x_N + y_N) \right) \right] dy_N + \\
 &\quad - \sum_{j,l=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\partial_{y_N} \left(\frac{(B_a^2 - L_1^2)(B_a^2 - L_2^2)}{\mathcal{A}_a} E_k^{H_{jl}} \widehat{H}_{jl}(\xi', y_N) \mathcal{M}(L_2, L_1, x_N + y_N) \right) \right] dy_N + \\
 &\quad + \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\partial_{y_N} \left(\frac{L_1(B_a L_1 - A^2)(B_a^2 - L_2^2)}{\mathcal{A}_a} \widehat{h}_k(\xi', y_N) \mathcal{M}(L_2, L_1, x_N + y_N) \right) \right] dy_N + \\
 &\quad - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\partial_{y_N} \left(\frac{i\xi_k \mathcal{E}_h L_1 (B_a L_1 - A^2)(B_a^2 - L_2^2)}{\lambda \mathcal{C}_a A \mathcal{A}_a} i\xi' \cdot \widehat{h}(\xi', y_N)' \mathcal{M}(L_2, L_1, x_N + y_N) \right) \right] dy_N + \\
 &\quad - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\partial_{y_N} \left(\frac{i\xi_k}{\lambda \mathcal{C}_a A} \frac{L_1 (B_a L_1 - A^2)(B_a^2 - L_2^2) A^2}{\mathcal{A}_a} \widehat{H}_{NN}(\xi', y_N) \mathcal{M}(L_2, L_1, x_N + y_N) \right) \right] dy_N + \\
 &\quad - \sum_{j=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\partial_{y_N} \left(\frac{i\xi_k}{\lambda \mathcal{C}_a A} \frac{L_1 (B_a L_1 - A^2)(B_a^2 - L_2^2)}{\mathcal{A}_a} \frac{(B_a^2 + A^2)}{B_a} i\xi_j \widehat{H}_{jN}(\xi', y_N) \mathcal{M}(L_2, L_1, x_N + y_N) \right) \right] dy_N + \\
 &\quad - \sum_{j,l=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\partial_{y_N} \left(\frac{i\xi_k}{\lambda \mathcal{C}_a A} \frac{L_1 (B_a L_1 - A^2)(B_a^2 - L_2^2)}{\mathcal{A}_a} i\xi_j i\xi_l \widehat{H}_{jl}(\xi', y_N) \mathcal{M}(L_2, L_1, x_N + y_N) \right) \right] dy_N.
 \end{aligned}$$

The idea is to write every \mathcal{U}_k^m with $m = 1, \dots, 14$ as one of the operators of Lemma 3.3.3. The proof is almost the same for every term, so we see just few of them:

$$\mathcal{U}_k^1 = - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\partial_{y_N} \widehat{h}_k(\xi', y_N) e^{-L_1(x_N + y_N)} - L_1 \widehat{h}_k e^{-L_1(x_N + y_N)} \right] dy_N.$$

Using the identity $1 = \frac{\lambda + a + A^2}{B_a^2}$, we have that

$$\begin{aligned}
 \mathcal{U}_k^1 &= - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{\sqrt{\lambda + a}}{B_a^2} \sqrt{\lambda + a} \partial_{y_N} \widehat{h}_k(\xi', y_N) e^{-L_1(x_N + y_N)} - \sum_{j=1}^{N-1} \frac{i\xi_j}{AB_a^2} i\xi_j \partial_{y_N} \widehat{h}_k(\xi', y_N) A e^{-L_1(x_N + y_N)} \right] dy_N + \\
 &\quad + \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{L_1}{B_a^2} (\lambda + a) \widehat{h}_k e^{-L_1(x_N + y_N)} + \frac{L_1}{B_a^2} A^2 \widehat{h}_k e^{-L_1(x_N + y_N)} \right] dy_N.
 \end{aligned}$$

We already that

$$\frac{\sqrt{\lambda+a}}{B_a^2}, \frac{L_1}{B_a^2} \in \mathcal{M}_{-1,1,\theta,r}, \quad \frac{i\xi_j}{A} \in \mathcal{M}_{0,2,\theta,r}, \quad B_a^{-2} \in \mathcal{M}_{-2,1,\theta,r}.$$

Therefore, using Lemma 3.3.3 we get the \mathcal{R} -boundedness for \mathcal{U}_k^1 .

We can repeat a similar argument for all the terms:

$$\begin{aligned} \mathcal{U}_k^2 &= - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_k \mathcal{E}_h(L_1 - A)}{\lambda \mathcal{C}_a A} i\xi' \cdot \partial_{y_N} \widehat{h}(\xi', y_N) \mathcal{M}(L_1, A, x_N + y_N) \right] dy_N + \\ &- \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_k \mathcal{E}_h(L_1 - A)}{\lambda \mathcal{C}_a A} i\xi' \cdot \widehat{h}(\xi', y_N) \left(-e^{-L_1(x_N + y_N)} - A \mathcal{M}(L_1, A, x_N + y_N) \right) \right] dy_N, \end{aligned}$$

where we used the identity

$$\partial_{x_N} \mathcal{M}(L_1, A, x_N) = -e^{-L_1 x_N} - A \mathcal{M}(L_1, A, x_N).$$

Now we notice that

$$\frac{L_1 - A}{\lambda} = \frac{L_1 - A}{\lambda + a} \frac{\lambda + a}{\lambda} \in \mathcal{M}_{-1,1,\theta,r}. \quad (75)$$

Therefore:

$$\begin{aligned} \mathcal{U}_k^2 &= - \sum_{j=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_k i\xi_j \mathcal{E}_h(L_1 - A) \sqrt{\lambda+a}}{\lambda \mathcal{C}_a A^2 |\lambda|^{\frac{1}{2}} B_a^2} \sqrt{\lambda+a} \partial_{y_N} \widehat{h}_j(\xi', y_N) A |\lambda|^{\frac{1}{2}} \mathcal{M}(L_1, A, x_N + y_N) \right] dy_N + \\ &- \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_k \mathcal{E}_h(L_1 - A)}{\lambda \mathcal{C}_a A B_a^2} i\xi' \cdot \partial_{y_N} \widehat{h}'(\xi', y_N) A^2 \mathcal{M}(L_1, A, x_N + y_N) \right] dy_N + \\ &+ \sum_{j=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_k i\xi_j \mathcal{E}_h(L_1 - A)}{\lambda \mathcal{C}_a A^2 B_a^2} [\lambda + a + A^2] \widehat{h}_j(\xi', y_N) \left(A e^{-L_1(x_N + y_N)} + A^2 \mathcal{M}(L_1, A, x_N + y_N) \right) \right] dy_N. \end{aligned}$$

Using (75) with Lemmas 3.2.4, 3.2.11 and 3.2.13, we have that

$$\frac{i\xi_k}{A}, \mathcal{C}_a^{-1}, \frac{\mathcal{E}_h(L_1 - A)}{\lambda}, \frac{\sqrt{\lambda+a}}{|\lambda|^{\frac{1}{2}}} \in \mathcal{M}_{0,2,\theta,r}, \quad B_a^{-2} \in \mathcal{M}_{-2,1,\theta,r}.$$

Therefore we can apply Lemma 3.3.3. From the definition of E_k^h and Lemma 3.2.9

$$E_k^h = \frac{i\xi_k \mathcal{E}_h}{\lambda \mathcal{C}_a} m_0(\lambda, \xi') + i\xi_k \frac{m_1(\lambda, \xi')}{B_a^2 - A^2},$$

with $m_0 \in \mathcal{M}_{0,2,\theta,r}$ and $m_1 \in \mathcal{M}_{1,2,\theta,r}$. Therefore

$$\begin{aligned} \mathcal{U}_k^6 &= - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_k(\lambda+a)(B_a^2 - L_2^2)(B_a^2 - L_1^2) \mathcal{E}_h m_0(\lambda, \xi')}{A \mathcal{A}_a \lambda \mathcal{C}_a (\lambda+a)} i\xi' \cdot \partial_{y_N} \widehat{h}'(\xi', y_N) A \mathcal{M}(L_2, L_1, x_N + y_N) \right] dy_N + \\ &- \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_k(\lambda+a)(B_a^2 - L_2^2)(B_a^2 - L_1^2) m_1(\lambda, \xi')}{A \mathcal{A}_a (B_a^2 - A^2) (\lambda+a)} i\xi' \cdot \partial_{y_N} \widehat{h}'(\xi', y_N) A \mathcal{M}(L_2, L_1, x_N + y_N) \right] dy_N + \\ &+ \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_k(\lambda+a)^{\frac{3}{2}} (B_a^2 - L_2^2)(B_a^2 - L_1^2) \mathcal{E}_h m_0(\lambda, \xi')}{B_a^2 \mathcal{A}_a \lambda \mathcal{C}_a (\lambda+a) A} \sqrt{\lambda+a} i\xi' \cdot \widehat{h}'(\xi', y_N) A e^{-L_2(x_N + y_N)} \right] dy_N + \\ &- \sum_{j=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_k(\lambda+a)(B_a^2 - L_2^2)(B_a^2 - L_1^2) \mathcal{E}_h m_0(\lambda, \xi') i\xi_j}{B_a^2 \mathcal{A}_a \lambda \mathcal{C}_a (\lambda+a) A} i\xi_j i\xi' \cdot \widehat{h}'(\xi', y_N) A e^{-L_2(x_N + y_N)} \right] dy_N + \\ &+ \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_k(\lambda+a)^{\frac{3}{2}} (B_a^2 - L_2^2)(B_a^2 - L_1^2) m_1(\lambda, \xi')}{B_a^2 \mathcal{A}_a (B_a^2 - A^2) (\lambda+a) A} \sqrt{\lambda+a} i\xi' \cdot \widehat{h}'(\xi', y_N) A e^{-L_2(x_N + y_N)} \right] dy_N + \end{aligned}$$

$$\begin{aligned}
 & - \sum_{j=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_k(\lambda+a)(B_a^2-L_2^2)(B_a^2-L_1^2)m_1(\lambda,\xi')i\xi_j}{B_a^2\mathcal{A}_a(B_a^2-A^2)(\lambda+a)A} i\xi_j i\xi' \cdot \widehat{h}'(\xi', y_N) A e^{-L_2(x_N+y_N)} \right] dy_N + \\
 & + \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_k(\lambda+a)^{\frac{3}{2}}(B_a^2-L_2^2)(B_a^2-L_1^2)\mathcal{E}_h L_1 m_0 \sqrt{\lambda+a} i\xi' \cdot \widehat{h}'(\xi', y_N) |\lambda|^{\frac{1}{2}} A \mathcal{M}(L_2, L_1, x_N+y_N)}{B_a^2 \mathcal{A}_a \lambda \mathcal{C}_a(\lambda+a) A |\lambda|^{\frac{1}{2}}} \right] dy_N + \\
 & + \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{(\lambda+a)(B_a^2-L_2^2)(B_a^2-L_1^2)\mathcal{E}_h L_1 m_0}{B_a^2 \mathcal{A}_a \lambda \mathcal{C}_a(\lambda+a)} i\xi_k i\xi' \cdot \widehat{h}'(\xi', y_N) A^2 \mathcal{M}(L_2, L_1, x_N+y_N) \right] dy_N + \\
 & + \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_k(\lambda+a)^{\frac{3}{2}}(B_a^2-L_2^2)(B_a^2-L_1^2)L_1 m_1 \sqrt{\lambda+a} i\xi' \cdot \widehat{h}'(\xi', y_N) |\lambda|^{\frac{1}{2}} A \mathcal{M}(L_2, L_1, x_N+y_N)}{B_a^2 \mathcal{A}_a (B_a^2-A^2)(\lambda+a) A |\lambda|^{\frac{1}{2}}} \right] dy_N + \\
 & + \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{(\lambda+a)(B_a^2-L_2^2)(B_a^2-L_1^2)L_1 m_1}{B_a^2 \mathcal{A}_a (B_a^2-A^2)(\lambda+a)} i\xi_k i\xi' \cdot \widehat{h}'(\xi', y_N) A^2 \mathcal{M}(L_2, L_1, x_N+y_N) \right] dy_N +
 \end{aligned}$$

Thanks to Lemmas 3.2.12 and 3.2.13 we have that

$$\begin{aligned}
 & \frac{i\xi_k}{A}, \frac{B_a^2-L_1^2}{\lambda}, \frac{B_a^2-L_2^2}{\lambda+a}, \mathcal{C}_a^{-1}, m_0, \frac{(\lambda+a)L_1\mathcal{E}_h}{\mathcal{A}_a}, \frac{(\lambda+a)i\xi_j\mathcal{E}_h}{\mathcal{A}_a}, \frac{(\lambda+a)L_1m_1}{\mathcal{A}_a}, \frac{(\lambda+a)m_1i\xi_j}{\mathcal{A}_a} \in \mathcal{M}_{0,2,\theta,r}, \\
 & \frac{\sqrt{\lambda+a}}{|\lambda|^{\frac{1}{2}}}, \frac{(\lambda+a)^{\frac{3}{2}}\mathcal{E}_h}{\mathcal{A}_a}, \frac{(\lambda+a)^{\frac{3}{2}}m_1}{\mathcal{A}_a}, \frac{(\lambda+a)^{\frac{3}{2}}i\xi_j}{\mathcal{A}_a}, \frac{(\lambda+a)^{\frac{3}{2}}L_1}{\mathcal{A}_a} \in \mathcal{M}_{0,2,\theta,r}, \\
 & \frac{(\lambda+a)\mathcal{E}_h}{\mathcal{A}_a}, \frac{(\lambda+a)m_1}{\mathcal{A}_a} \in \mathcal{M}_{-1,2,\theta,r} \quad B_a^{-2} \in \mathcal{M}_{-2,1,\theta,r}.
 \end{aligned}$$

Then we can apply Lemma 3.3.3. We can divide $\mathcal{U}_k^8 = \mathcal{U}_k^{8,1} + \mathcal{U}_k^{8,2} + \mathcal{U}_k^{8,3}$. The terms $\mathcal{U}_k^{8,1}$ and $\mathcal{U}_k^{8,2}$ can be treated as before. For what concerns $\mathcal{U}_k^{8,3}$

$$\begin{aligned}
 \mathcal{U}_k^{8,3} &= - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{2(\lambda+a)^{\frac{3}{2}}(B_a^2-L_1^2)(B_a^2-L_2^2) \sqrt{\lambda+a} \partial_{y_N} \widehat{H}_{kN}(\xi', y_N) \mathcal{M}(L_2, L_1, x_N+y_N)}{\beta B_a(\lambda+a)\mathcal{A}_a} \right] dy_N + \\
 & + \sum_{l=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{2i\xi_l(\lambda+a)(B_a^2-L_1^2)(B_a^2-L_2^2)}{\beta A B_a(\lambda+a)\mathcal{A}_a} i\xi_l \partial_{y_N} \widehat{H}_{kN}(\xi', y_N) A \mathcal{M}(L_2, L_1, x_N+y_N) \right] dy_N + \\
 & + \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{2(\lambda+a)(B_a^2-L_1^2)(B_a^2-L_2^2)}{\beta(\lambda+a)B_a\mathcal{A}_a} B_a^2 \widehat{H}_{jN}(\xi', y_N) \left(e^{-L_2(x_N+y_N)} + L_1 \mathcal{M}(L_2, L_1, x_N+y_N) \right) \right] dy_N.
 \end{aligned}$$

Moreover

$$\frac{B_a^2-L_1^2}{\lambda+a}, \frac{(\lambda+a)(B_a^2-L_2^2)}{\mathcal{A}_a}, \frac{L_1}{B_a} \in \mathcal{M}_{0,1,\theta,r}, \quad \frac{i\xi_l}{A} \in \mathcal{M}_{0,2,\theta,r}, \quad B_a^{-1} \in \mathcal{M}_{-1,1,\theta,r}.$$

Then we can apply Lemma 3.3.3.

When $k=N$ then $\widehat{h}_N=0$ and

$$\begin{aligned}
 \widehat{u}_N &= -A_N^0(L_1-A)\mathcal{M}(L_1, A, x_N) + A_N^2(L_2-L_1)\mathcal{M}(L_2, L_1, x_N) = \\
 &= -\frac{AC(L_1-A)}{\lambda}\mathcal{M}(L_1, A, x_N) + \frac{A(L_1-A)}{\lambda}C\mathcal{M}(L_2, L_1, x_N) + i\xi' \cdot \widehat{h}'\mathcal{M}(L_2, L_1, x_N)
 \end{aligned}$$

and we can conclude as before. Let us pass now to Q : Q solves the system

$$\begin{cases} (\lambda+a-\Delta)Q = \beta D(u) & \text{in } \mathbb{R}_+^N \\ D_N Q = H & \text{on } \mathbb{R}_0^N. \end{cases}$$

Then using Proposition 3.4.4 and the \mathcal{R} -boundedness of u that we have just proved, we conclude. \square

Remark 3.4.6. Considering the proof just done, it can be seen that $\mathcal{S}_j(\lambda)$ can be switched with $\mathcal{S}_j(\gamma)$ for $j = 1, 2$, where $\lambda = \gamma + i\tau$ (look at the proof of Lemma 5.6 of [21]).

Let us focus on the resolvent estimate for the system

$$\begin{cases} (\lambda - \Delta)u + \nabla p + \beta \text{Div}(\Delta - a)Q = f & \text{in } \mathbb{R}_+^N \\ (\lambda + a - \Delta)Q - \beta D(u) = G & \text{in } \mathbb{R}_+^N \\ \text{div} u = 0 & \text{in } \mathbb{R}_+^N \\ u|_{x_N=0} = h, \quad D_N Q|_{x_N=0} = H & \text{on } \mathbb{R}_0^N. \end{cases}$$

The resolvent estimate for the case $f = G = 0$ follows just from the \mathcal{R} -boundedness of the solution (u, p, Q) (see $m = 1$ of the definition 1.2.1). The case with $f, G \neq 0$ is a consequence of Theorem 3.4.5:

Corollary 3.4.7. *Let $a, r > 0$, $\beta \in \mathbb{R}$, $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, let $q \in (1, \infty)$, let $f \in L^q(\mathbb{R}_+^N; \mathbb{R}^N)$ and $G \in W^{1,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))$, then for any $\lambda \in \Sigma_{\theta, r}$ we can find $u \in W^{2,q}(\mathbb{R}_+^N; \mathbb{R}^N)$, $p \in \widehat{H}_q^1(\mathbb{R}_+^N)$ and $Q \in W^{3,q}(\mathbb{R}_+^N, S_0(N, \mathbb{R}))$ solution of the system*

$$\begin{cases} (\lambda - \Delta)u + \nabla p + \beta \text{Div}(\Delta - a)Q = f & \text{in } \mathbb{R}_+^N \\ (\lambda + a - \Delta)Q - \beta D(u) = G & \text{in } \mathbb{R}_+^N \\ \text{div} u = 0 & \text{in } \mathbb{R}_+^N \\ u|_{x_N=0} = 0, \quad D_N Q|_{x_N=0} = 0 & \text{on } \mathbb{R}_0^N, \end{cases} \quad (76)$$

moreover

$$\begin{aligned} & \left\| \left(|\lambda|u, |\lambda|^{\frac{1}{2}}\nabla u, D^2u \right) \right\|_{L^q(\mathbb{R}_+^N)} + \|\nabla p\|_{L^q(\mathbb{R}_+^N)} + \left\| \left(|\lambda|^{\frac{3}{2}}Q, |\lambda|\nabla Q, |\lambda|^{\frac{1}{2}}D^2Q, D^3Q \right) \right\|_{L^q(\mathbb{R}_+^N)} \leq \\ & \leq C(a, \beta, \theta, r, q, N) \left[\|f\|_{L^q(\mathbb{R}_+^N)} + \left\| \left(|\lambda|^{\frac{1}{2}}G, \nabla G \right) \right\|_{L^q(\mathbb{R}_+^N)} \right]. \end{aligned}$$

Proof.

From [16] we know that (76) admits a solution in \mathbb{R}^N . Let

$$\lambda Id - \mathcal{A}: L^q(\mathbb{R}^N; \mathbb{R}^N) \times W^{1,q}(\mathbb{R}^N; S_0(N, \mathbb{R})) \rightarrow W^{2,q}(\mathbb{R}^N; \mathbb{R}^N) \times \widehat{H}_q^1(\mathbb{R}^N; \mathbb{R}) \times W^{3,q}(\mathbb{R}^N; S_0(N, \mathbb{R})),$$

be the operator correspondent to the system (76) in \mathbb{R}^N . Then our solution (u, p, Q) is the summation of (v_1, ρ_1, V_1) and (v_2, ρ_2, V_2) , where

$$(v_1, \rho_1, V_1) = (\lambda - \mathcal{A})^{-1}(E_v[f], E_M[G]),$$

where E_v, E_M are suitable extensions from \mathbb{R}_+^N to \mathbb{R}^N and (v_2, ρ_2, V_2) is the solution of

$$\begin{cases} (\lambda - \Delta)v_2 + \nabla \rho_2 + \beta \text{Div}(\Delta - a)V_2 = 0 & \text{in } \mathbb{R}_+^N \\ (\lambda + a - \Delta)V_2 - \beta D(v_2) = 0 & \text{in } \mathbb{R}_+^N \\ \text{div} v_2 = 0 & \text{in } \mathbb{R}_+^N \\ v_2 = -v_1|_{x_N=0}, \quad D_N V_2 = -D_N V_1|_{x_N=0} & \text{on } \mathbb{R}_0^N. \end{cases}$$

It can be seen that, if choose E_v and E_M as it follows

$$E_v[f_k] = \begin{cases} E_{\text{even}}[f_k] & k = 1, \dots, N-1 \\ E_{\text{odd}}[f_N] & k = N \end{cases}$$

$$E_M[G_{jk}] = \begin{cases} E_{\text{even}}[G_{jk}] & j, k = 1, \dots, N-1 \\ E_{\text{odd}}[G_{Nk}] & j = N, k = 1, \dots, N-1 \\ E_{\text{odd}}[G_{jN}] & j = 1, \dots, N-1, k = N \\ E_{\text{even}}[G_{NN}] & j = k = N, \end{cases}$$

then $(v_1 \cdot e_N)_{x_N=0} = 0$, so we can really find (v_2, ρ_2, V_2) which resolves the previous system. By construction (u, p, Q) solves (76) in \mathbb{R}_+^N and thanks to the resolvent estimate for $(\lambda - \mathcal{A})^{-1}$ in [16] we conclude. \square

Finally, the uniqueness is a consequence of the existence result:

Corollary 3.4.8. *Let $a, r > 0$, $\beta \in \mathbb{R}$ $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, let $q \in (1, \infty)$, let $f \in L^q(\mathbb{R}_+^N; \mathbb{R}^N)$ and $G \in W^{1,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))$, then for any $\lambda \in \Sigma_{\theta,r}$ there are $u \in W^{2,q}(\mathbb{R}_+^N; \mathbb{R}^N)$, $p \in \widehat{H}_q^1(\mathbb{R}_+^N)$ and $Q \in W^{3,q}(\mathbb{R}_+^N, S_0(N, \mathbb{R}))$ unique solution of (76), up to additive constants on the pressure term p .*

Proof.

For the uniqueness we use a duality argument: we can rewrite our system in the following way:

$$\begin{cases} \left(\lambda - \left(1 + \frac{\beta^2}{2}\right) \Delta \right) u + \nabla p + \beta \lambda \operatorname{Div} Q = f + \beta \operatorname{Div} G & \text{in } \mathbb{R}_+^N \\ (\lambda + a - \Delta) Q - \beta D(u) = G & \text{in } \mathbb{R}_+^N \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+^N \\ u|_{x_N=0} = 0, \quad D_N Q|_{x_N=0} = 0 & \text{on } \mathbb{R}_0^N. \end{cases}$$

It can be seen now that the adjoint system is

$$\begin{cases} \left(\bar{\lambda} - \left(1 + \frac{\beta^2}{2}\right) \Delta \right) u + \nabla p + \beta \operatorname{Div} Q = f + \beta \operatorname{Div} G & \text{in } \mathbb{R}_+^N \\ (\bar{\lambda} + a - \Delta) Q - \beta \bar{\lambda} D(u) = G & \text{in } \mathbb{R}_+^N \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+^N \\ u|_{x_N=0} = 0, \quad D_N Q|_{x_N=0} = 0 & \text{on } \mathbb{R}_0^N. \end{cases}$$

which can be rewritten as

$$\begin{cases} (\bar{\lambda} - \Delta) u + \nabla p + \frac{\beta}{\bar{\lambda}} \operatorname{Div}(\Delta - a) Q = f & \text{in } \mathbb{R}_+^N \\ (\bar{\lambda} + a - \Delta) Q - \beta \bar{\lambda} D(u) = \bar{\lambda} G & \text{in } \mathbb{R}_+^N \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+^N \\ u|_{x_N=0} = 0, \quad D_N Q|_{x_N=0} = 0 & \text{on } \mathbb{R}_0^N. \end{cases} \quad (77)$$

Now we notice that (77) admits a solution: the previous system is equivalent to

$$\begin{cases} (\bar{\lambda} - \Delta) u + \nabla p + \beta \operatorname{Div}(\Delta - a) \left(\frac{Q}{\bar{\lambda}} \right) = f & \text{in } \mathbb{R}_+^N \\ (\bar{\lambda} + a - \Delta) \left(\frac{Q}{\bar{\lambda}} \right) - \beta D(u) = G & \text{in } \mathbb{R}_+^N \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+^N \\ u|_{x_N=0} = 0, \quad D_N Q|_{x_N=0} = 0 & \text{on } \mathbb{R}_0^N. \end{cases}$$

So, $(u, p, Q/\bar{\lambda})$ is the solution of (76) with (f, G) . Since (77) admits a solution, we can conclude with the uniqueness: let (u, p, Q) be a solution of (76) with $f = G \equiv 0$, let $\varphi \in C_c^\infty(\mathbb{R}_+^N; \mathbb{R}^N)$ and $\phi \in C_c^\infty(\mathbb{R}^N; S_0(N, \mathbb{R}^N))$ and let (v, ρ, V) be the solution of (77) with $f = \varphi$ and $G = \phi$. Therefore,

$$\begin{aligned} \int_{\mathbb{R}_+^N} u \cdot \bar{\varphi} dx + \int_{\mathbb{R}_+^N} Q : \bar{\phi} dx &= \int_{\mathbb{R}_+^N} u \cdot \overline{(\bar{\lambda} - (1 + \beta^2/2)\Delta)v + \nabla \rho + \beta \operatorname{Div} V} dx + \\ &+ \int_{\mathbb{R}_+^N} Q : \overline{(\bar{\lambda} + a - \Delta)V - \beta \bar{\lambda} D(v)} dx. \end{aligned}$$

Thanks to the boundary conditions on u and Q and thanks to $\operatorname{div} u = 0$ we get

$$= \int_{\mathbb{R}_+^N} (\lambda - (1 + \beta^2/2)\Delta) u \cdot \bar{v} dx - \beta \int_{\mathbb{R}_+^N} D(u) : \bar{V} dx +$$

$$+ \int_{\mathbb{R}_+^N} (\lambda + a - \Delta)Q : \bar{V} dx + \beta \lambda \int_{\mathbb{R}_+^N} \text{Div}Q \cdot \bar{v} dx.$$

Using the system solved by (u, p, Q) and the property $\text{div}v = 0$ we get

$$= - \int_{\mathbb{R}_+^N} \beta \lambda \text{Div}Q \cdot \bar{v} dx - \beta \int_{\mathbb{R}_+^N} D(u) : \bar{V} dx + \beta \int_{\mathbb{R}_+^N} D(u) : \bar{V} dx + \beta \lambda \int_{\mathbb{R}_+^N} \text{Div}Q \cdot \bar{v} dx = 0.$$

So, $u, Q = 0$ a.e. and from the first equation of (76) it follows that $\nabla p = 0$. \square

Finally, thanks to Theorem 3.4.5 and Corollaries 3.4.7 and 3.4.8, we get Theorem 1.2.3: we have already proved the estimate for u and Q . For what concerns p , it is sufficient to see that

$$\nabla p = (\Delta - \lambda)u + \beta \text{Div}(a - \Delta)Q = (\Delta - \lambda)u - \beta \lambda \text{Div}Q + \frac{\beta^2}{2} \Delta u \quad \text{in } \mathbb{R}_+^N,$$

therefore

$$\|\nabla p\|_{L^q(\mathbb{R}_+^N)} \lesssim |\lambda| \|u\|_{L^q(\mathbb{R}_+^N)} + \|D^2 u\|_{L^q(\mathbb{R}_+^N)} + |\lambda| \|\nabla Q\|_{L^q(\mathbb{R}_+^N)}.$$

4 L^p - L^q maximal regularity

4.1 Semigroup Setting

Let us pass to the evolution problem. We focus on the linear one

$$\begin{cases} \partial_t u - \Delta u + \nabla p + \beta \text{Div}(\Delta - a)Q = f & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^N \\ \partial_t Q - (\Delta - a)Q - \beta D(u) = G & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^N \\ \text{div}u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^N \\ u = 0, \quad D_N Q = 0 & \text{on } \mathbb{R}_+ \times \mathbb{R}_0^N \\ u(0) = u_0, \quad Q(0) = Q_0 & \text{in } \mathbb{R}_+^N, \end{cases} \quad (78)$$

The first standard step for the study of a linear evolution system, is to prove the existence of the semigroup corresponding to the operator of the system. The linear resolvent system

$$\begin{cases} (\lambda - \Delta)u + \nabla p + \beta \text{Div}(\Delta - a)Q = f & \text{in } \mathbb{R}_+^N \\ (\lambda + a - \Delta)Q - \beta D(u) = G & \text{in } \mathbb{R}_+^N \\ \text{div}u = 0 & \text{in } \mathbb{R}_+^N \\ u = 0, \quad D_N Q = 0 & \text{on } \mathbb{R}_0^N. \end{cases} \quad (79)$$

is not written in the semigroup setting: we need to express p as a bounded linear operator with respect to u and Q and we also need to erase the divergence-free condition. From (79) we have that

$$\langle \nabla p, \nabla \varphi \rangle = \langle -\lambda u + \Delta u - \beta \text{Div}(\Delta - a \text{Id})Q + f, \nabla \varphi \rangle \quad \forall \varphi \in \widehat{H}_q^1(\mathbb{R}_+^N).$$

Thanks to divergence-free condition on u , we have then

$$\langle \nabla p, \nabla \varphi \rangle = \langle \Delta u - \beta \text{Div}(\Delta - a \text{Id})Q + f, \nabla \varphi \rangle \quad \forall \varphi \in \widehat{H}_{q'}^1(\mathbb{R}_+^N).$$

Then we can use Theorem 2 and 3 of [11]:

Theorem 4.1.1.

Let $q \in (1, +\infty)$, let $N \geq 2$, then for any $f \in L^q(\mathbb{R}_+^N; \mathbb{R}^N)$ there is a unique $\pi \in \widehat{H}_q^1(\mathbb{R}_+^N)$ which satisfies the Neumann weak problem

$$\langle \nabla \pi, \nabla \varphi \rangle = \langle f, \nabla \varphi \rangle \quad \forall \varphi \in \widehat{H}_{q'}^1(\mathbb{R}_+^N),$$

with

$$\|\nabla \pi\|_{L^q(\mathbb{R}_+^N)} \lesssim \|f\|_{L^q(\mathbb{R}_+^N)}.$$

As a consequence, we get a representation formula for ∇p :

Corollary 4.1.2. *Let $q \in (1, +\infty)$, $N \geq 2$, $a, r > 0$, $\beta \in \mathbb{R}$, $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, $\lambda \in \Sigma_{\theta, r}$ and let*

$$(u, p, Q) \in W^{2,q}(\mathbb{R}_+^N; \mathbb{R}^N) \times \widehat{H}_q^1(\mathbb{R}_+^N) \times W^{3,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R})),$$

be the solution of (79) with $f \in L^q(\mathbb{R}_+^N; \mathbb{R}^N)$ and $G \in W^{1,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))$, then $\nabla p = \nabla K_A(u, Q) + \nabla K_e(f)$, where $\nabla K_A(u, Q)$ is the only solution in $\widehat{H}_q^1(\mathbb{R}_+^N)$ of

$$\langle \nabla K_A(u, Q), \nabla \varphi \rangle = \langle \Delta u - \beta \operatorname{Div}(\Delta - a \operatorname{Id})Q, \nabla \varphi \rangle \quad \forall \varphi \in \widehat{H}_{q'}^1(\mathbb{R}_+^N),$$

and $\nabla K_e(f)$ is the only solution in $\widehat{H}_q^1(\mathbb{R}_+^N)$ of

$$\langle \nabla K_e(f), \nabla \varphi \rangle = \langle f, \nabla \varphi \rangle \quad \forall \varphi \in \widehat{H}_{q'}^1(\mathbb{R}_+^N).$$

Proof.

By hypothesis, we know that there is $\tilde{p} \in \widehat{H}_q^1(\mathbb{R}_+^N)$ which resolves (79) with $f - \nabla K_e(f)$ in the place of f . If we multiply the first equation of (79) by $\nabla \varphi$ with $\varphi \in \widehat{H}_{q'}^1(\mathbb{R}_+^N)$, we get

$$\langle (\lambda - \Delta)u + \nabla \tilde{p} + \beta \operatorname{Div}(\Delta - a \operatorname{Id})Q, \nabla \varphi \rangle = \langle f - \nabla K_e(f), \nabla \varphi \rangle = 0.$$

Therefore

$$\langle \nabla \tilde{p}, \nabla \varphi \rangle = \langle \Delta u - \beta \operatorname{Div}(\Delta - a \operatorname{Id})Q, \nabla \varphi \rangle.$$

So, thanks to Theorem 4.1.1, $\nabla \tilde{p} = \nabla K_A(u, Q)$. On the other hand, we also know that the solution of (79) is unique, so $\nabla p = \nabla \tilde{p} + \nabla K_e(f)$. \square

We can now consider the reduced system

$$\begin{cases} (\lambda - \Delta)u + \nabla K_A(u, Q) + \beta \operatorname{Div}(\Delta - a)Q = f - \nabla K_e(f) & \text{in } \mathbb{R}_+^N \\ (\lambda + a - \Delta)Q - \beta D(u) = G & \text{in } \mathbb{R}_+^N \\ u = 0, \quad D_N Q = 0 & \text{on } \mathbb{R}_0^N, \end{cases} \quad (80)$$

This system is written in the semigroup setting, but before we need to prove that it is equivalent to the system (78):

Corollary 4.1.3. *Let $q \in (1, +\infty)$, $N \geq 2$, $a, r > 0$, $\beta \in \mathbb{R}$, $\theta \in (\theta_0, \frac{\pi}{2})$ with $\tan \theta_0 \geq \frac{|\beta|}{\sqrt{2}}$, $\lambda \in \Sigma_{\theta, r}$ and $f \in L^q(\mathbb{R}_+^N; \mathbb{R}^N)$ and $G \in W^{1,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))$, then (u, Q) solves (80) if and only if (u, p, Q) solves (79) with $\nabla p = \nabla K_A(u, Q) + \nabla K_e(f)$. In particular, the solution for (80) is unique.*

Proof.

We have already seen that the solution of (79) is a solution for (80). On the other hand, we only need to check that, if (u, Q) is a solution for (80), then u is divergence-free: let us multiply the first equation (80) with $\nabla \varphi$ with $\varphi \in \widehat{H}_{q'}^1(\mathbb{R}_+^N)$

$$\langle (\lambda - \Delta)u + \nabla K_A(u, Q) + \beta \operatorname{Div}(\Delta - a \operatorname{Id})Q, \nabla \varphi \rangle = \langle f - K_e(f), \nabla \varphi \rangle.$$

By definition of $K_A(u, Q)$ and $K_e(f)$ we have then

$$\lambda \langle u, \nabla \varphi \rangle = 0 \quad \forall \varphi \in \widehat{H}_{q'}^1(\mathbb{R}_+^N).$$

Which implies (thanks to the Dirichlet boundary conditions on u) that $\operatorname{div} u = 0$. \square

We are now ready to introduce the semigroup:

Proposition 4.1.4. *Let us call \mathcal{A} the operator which defines the resolvent system (80) for $a > 0$ and $\beta \in \mathbb{R}$, let $q \in (1, \infty)$ and let us define the space*

$$X_q := J_q(\mathbb{R}_+^N) \times W^{1,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R})),$$

then $(\mathcal{A}, D(\mathcal{A}))$ is the generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ on X_q , where

$$D(\mathcal{A}) = \mathcal{D}_1(\mathcal{A}) \times \mathcal{D}_2(\mathcal{A}),$$

with

$$\begin{aligned} \mathcal{D}_1(\mathcal{A}) &= J_q(\mathbb{R}_+^N) \cap \{v \in W^{2,q}(\mathbb{R}_+^N; \mathbb{R}^N) \mid v = 0 \text{ on } \mathbb{R}_0^N\} \\ \mathcal{D}_2(\mathcal{A}) &= \{V \in W^{3,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R})) \mid D_N V = 0 \text{ on } \mathbb{R}_0^N\}. \end{aligned}$$

Moreover, there is $\gamma_0 > 0$ such that for any $t > 0$ it holds

$$\begin{aligned} \|T(t)(u_0, Q_0)\|_{X_q} &\leq C(q, N, \gamma_0) e^{\gamma_0 t} \|(u_0, Q_0)\|_{X_q} & (u_0, Q_0) \in X_q \\ \|\partial_t T(t)(u_0, Q_0)\|_{X_q} &\leq C(q, N, \gamma_0) t^{-1} e^{\gamma_0 t} \|(u_0, Q_0)\|_{X_q} & (u_0, Q_0) \in X_q \\ \|\partial_t T(t)(u_0, Q_0)\|_{X_q} &\leq C(q, N, \gamma_0) e^{\gamma_0 t} \|(u_0, Q_0)\|_{D(\mathcal{A})} & (u_0, Q_0) \in D(\mathcal{A}) \end{aligned} \quad (81)$$

The proof follows from Corollary 4.1.2 and standard semigroup arguments.

4.2 Linear Evolution problem

We recall the evolution problem

$$\begin{cases} \partial_t u - \Delta u + \nabla p + \beta \text{Div}(\Delta - a)Q = f & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^N \\ \partial_t Q - (\Delta - a)Q - \beta D(u) = G & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^N \\ \text{div} u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^N \\ u = h, \quad D_N Q = H & \text{on } \mathbb{R}_+ \times \mathbb{R}_0^N \\ u(0) = u_0, \quad Q(0) = Q_0 & \text{in } \mathbb{R}_+^N. \end{cases} \quad (82)$$

The aim of this section is to find a solution for (82). In order to find it, we consider the following systems:

$$\begin{cases} \partial_t u_1 - \Delta u_1 + \nabla p_1 + \beta \text{Div}(\Delta - a)Q_1 = f_1 & \text{in } \mathbb{R}_+ \times \mathbb{R}^N \\ \partial_t Q_1 - (\Delta - a)Q_1 - \beta D(u_1) = G_1 & \text{in } \mathbb{R}_+ \times \mathbb{R}^N \\ \text{div} u_1 = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^N \\ u_1(0) = 0, \quad Q_1(0) = 0 & \text{in } \mathbb{R}_+^N. \end{cases} \quad (83)$$

$$\begin{cases} \partial_t u_2 - \Delta u_2 + \nabla p_2 + \beta \text{Div}(\Delta - a)Q_2 = 0 & \text{in } \mathbb{R} \times \mathbb{R}_+^N \\ \partial_t Q_2 - (\Delta - a)Q_2 - \beta D(u_2) = 0 & \text{in } \mathbb{R} \times \mathbb{R}_+^N \\ \text{div} u_2 = 0 & \text{in } \mathbb{R} \times \mathbb{R}_+^N \\ u_2 = h_2, \quad D_n Q_2 = H_2 & \text{on } \mathbb{R} \times \mathbb{R}_0^N; \end{cases} \quad (84)$$

$$\begin{cases} \partial_t u_3 - \Delta u_3 + \nabla p_3 + \beta \text{Div}(\Delta - a)Q_3 = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^N \\ \partial_t Q_3 - (\Delta - a)Q_3 - \beta D(u_3) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^N \\ \text{div} u_3 = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^N \\ u_3 = 0, \quad D_N Q_3 = 0 & \text{on } \mathbb{R}_+ \times \mathbb{R}_0^N \\ u_3(0) = u_3^0, \quad Q_3(0) = Q_3^0 & \text{in } \mathbb{R}_+^N. \end{cases} \quad (85)$$

Firstly we find a solution for all these systems and then we use the results to get the existence for the general linear problem (82). Let us start from the second system. We need to introduce the Laplace Transformation:

$$\mathcal{L}[f](\lambda) = \int_{\mathbb{R}} e^{-\lambda t} f(t) dt, \quad \mathcal{L}^{-1}[f](t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\lambda t} f(\tau) d\tau, \quad \lambda = \gamma + i\tau.$$

Remark 4.2.1. It follows from the definition that

$$\mathcal{L}[f](\lambda) = \mathcal{F}[e^{-\gamma t} f(t)](\tau); \quad \mathcal{L}^{-1}[f](t) = e^{\gamma t} \mathcal{F}^{-1}[f](t),$$

where \mathcal{F} is the 1-dimensional Fourier Transformation. Moreover

$$\begin{aligned} \mathcal{L}[\partial_t f](\lambda) &= \int_{\mathbb{R}} e^{-\lambda t} \partial_t f(t) dt = \lambda \int_{\mathbb{R}} e^{-\lambda t} f(t) dt, \\ \partial_t \mathcal{L}^{-1}[f](t) &= \frac{1}{2\pi} \int_{\mathbb{R}} \lambda e^{\lambda t} f(\lambda) d\lambda = \mathcal{L}^{-1}[\lambda f(\lambda)](t). \end{aligned}$$

Thanks to this remark, applying the Laplace Transformation to the system (84), we get that $\mathcal{L}_\lambda[(u, p, Q)]$ solve the resolvent system (3). We already know that such a system admits a solution, but we want to transfer the resolvent estimate we got in the previous section to the linear evolution estimate we need. In order to do so, we use again the \mathcal{R} -boundedness and the following Theorem:

Definition 4.2.2.

We say that a Banach space X is a UMD Space, if the Hilbert Transformation H is bounded on $L^p(\mathbb{R}; X)$ for some $p \in (1, \infty)$, where

$$H(f)(t) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0^+} \int_{|t-s|>\varepsilon} \frac{f(s)}{t-s} ds \quad t \in \mathbb{R}.$$

Theorem 4.2.3. *Weis' Theorem [24]*

Let X, Y be two UMD spaces and $p \in (1, \infty)$, let $m \in C^1(\mathbb{R} \setminus \{0\}; \mathcal{L}(X; Y))$ be such that

$$\mathcal{R}_{\mathcal{L}(X; Y)}(\{(\tau \partial_\tau)^\ell m \mid \tau \in \mathbb{R} \setminus \{0\}\}) < k_\ell \quad \ell = 0, 1,$$

let $T_m : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}; X) \rightarrow \mathcal{S}'(\mathbb{R}; Y)$ defined as

$$T_m \phi := \mathcal{F}^{-1}[m \mathcal{F}[\phi]],$$

where \mathcal{D} and \mathcal{S} are respectively the distributional and the tempered functions spaces, then T_m can be extended as an operator from $L^p(\mathbb{R}; X)$ to $L^p(\mathbb{R}; Y)$ with

$$\|T_m f\|_{L^p(\mathbb{R}; Y)} \leq C(p)(k_0 + k_1) \|f\|_{L^p(\mathbb{R}; X)}.$$

In the following, we use these operators:

$$\Lambda_{\gamma, k} f = \mathcal{L}^{-1} \left[|\lambda|^{\frac{k}{2}} \mathcal{L}[f](\lambda) \right] = e^{\gamma t} \mathcal{F}_\tau \left[|\lambda|^{\frac{k}{2}} \mathcal{F}^{-1}[e^{-\gamma t}] \right] \quad k \in \mathbb{N}_0.$$

By the Mihlin theorem it can be easily seen that

$$\|e^{-\gamma t} \Lambda_{\gamma, k} f\|_{L^p(\mathbb{R})} \simeq \|e^{-\gamma t} f\|_{H_p^{k/2}(\mathbb{R})} \quad k \in \mathbb{N}.$$

Finally, we define

$$H_{p, \gamma}^s(A) := \{v \in L_{loc}^p(A) \mid e^{-\gamma t} v \in H_p^s(A)\} \quad \forall s \geq 0, \quad p \in (1, \infty),$$

with the norm

$$\|v\|_{H_{p, \gamma}^s(A)} := \|e^{-\gamma t} v\|_{H_p^s(A)}$$

for any $A \subseteq \mathbb{R}$ open set. We are now ready to prove the existence and the estimate for the system (84):

Proposition 4.2.4. *Let $a, \beta, \gamma > 0$ and $p, q \in (1, +\infty)$ then for any h_2, H_2 such that $h_2 \cdot e_N = 0$ in \mathbb{R}_0^N , $h_2(t) = H_2(t) = 0$ when $t < 0$ and*

$$h_2 \in \bigcap_{l=0}^2 H_{p,\gamma}^{l/2}(\mathbb{R}; W^{2-l,q}(\mathbb{R}_+^N; \mathbb{R}^N)), \quad H_2 \in \bigcap_{l=0}^2 H_{p,\gamma}^{l/2}(\mathbb{R}; W^{2-l,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))),$$

then we can find (u_2, p_2, Q_2) solution for (84) with $p_2(t) \in \widehat{H}_q^1(\mathbb{R}_+^N)$ for a.e. $t > 0$ and

$$u_2 \in \bigcap_{l=0}^2 H_{p,\gamma}^{l/2}(\mathbb{R}; W^{2-l,q}(\mathbb{R}_+^N; \mathbb{R}^N)), \quad Q_2 \in \bigcap_{l=0}^3 H_{p,\gamma}^{l/2}(\mathbb{R}; W^{3-l,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))),$$

$$\nabla_x p_2 \in L_\gamma^p(\mathbb{R}; L^q(\mathbb{R}_+^N; \mathbb{R}^N)),$$

such that

$$\begin{aligned} & \sum_{l=0}^2 \|u_2\|_{H_{p,\gamma}^{l/2}(\mathbb{R}; W^{2-l,q}(\mathbb{R}_+^N))} + \sum_{l=0}^3 \|Q_2\|_{H_{p,\gamma}^{l/2}(\mathbb{R}; W^{3-l,q}(\mathbb{R}_+^N))} + \\ & + \sum_{l=0}^2 \|\gamma^{l/2} u_2\|_{L_\gamma^p(\mathbb{R}; W^{2-l,q}(\mathbb{R}_+^N))} + \sum_{l=0}^3 \|\gamma^{l/2} Q_2\|_{L_\gamma^p(\mathbb{R}; W^{3-l,q}(\mathbb{R}_+^N))} + \|\gamma^{\frac{1}{2}} \partial_t Q_2\|_{L_\gamma^p(\mathbb{R}; L^q(\mathbb{R}_+^N))} + \\ & + \|\nabla_x p_2\|_{L_\gamma^p(\mathbb{R}; L^q(\mathbb{R}_+^N))} \leq C \sum_{l=0}^2 \|(h_2, H_2)\|_{H_{p,\gamma}^{l/2}(\mathbb{R}; W^{2-l,q}(\mathbb{R}_+^N))}, \end{aligned}$$

for some $C > 0$.

Proof.

If we apply the Laplace transformation to the system (84), thanks to Remark 4.2.1, we get that

$$\begin{cases} (\lambda - \Delta)\mathcal{L}[u_2](\lambda) + \nabla\mathcal{L}[p_2](\lambda) + \beta\text{Div}(\Delta - a)\mathcal{L}[Q_2](\lambda) = 0 & \text{in } \mathbb{R}_+^N \\ (\lambda + a - \Delta)\mathcal{L}[Q_2](\lambda) - \beta D(\mathcal{L}[u_2](\lambda)) = 0 & \text{in } \mathbb{R}_+^N \\ \text{div}\mathcal{L}[u_2](\lambda) = 0 & \text{in } \mathbb{R}_+^N \\ \mathcal{L}[u_2](\lambda) = \mathcal{L}[h_2](\lambda), \quad D_N\mathcal{L}[Q_2](\lambda) = \mathcal{L}[H_2](\lambda) & \text{on } \mathbb{R}_0^N. \end{cases}$$

If we call $G := (D^2(h_2, H_2), \Lambda_{\gamma,1}\nabla(h_2, H_2), \Lambda_{\gamma,2}(h_2, H_2))$, then we know from Theorem 3.4.5 that for any $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > 0$,

$$(\mathcal{L}[u_2](\lambda), \mathcal{L}[Q_2](\lambda)) = (\mathcal{A}(\lambda), \mathcal{B}(\lambda))\mathcal{L}[G](\lambda) \quad \forall \lambda \in \Sigma_{\theta,r}.$$

We define then

$$(u_2, Q_2) := \mathcal{L}^{-1}[(\mathcal{A}(\lambda), \mathcal{B}(\lambda))\mathcal{L}[G](\lambda)]$$

for $\text{Re}\lambda > 0$. Thanks to Remark 4.2.1

$$(u_2, Q_2) = e^{\gamma t} \mathcal{F}_\tau^{-1} [(\mathcal{A}(\lambda), \mathcal{B}(\lambda))\mathcal{F}[e^{-\gamma t}(D^2(h_2, H_2), \Lambda_{\gamma,1}\nabla(h_2, H_2), \Lambda_{\gamma,2}(h_2, H_2))]]].$$

Moreover, since $h_2 = H_2 = 0$ for $t < 0$, then $\mathcal{L}[G]$ is holomorphic for $\text{Re}\lambda > 0$ and, by Cauchy's Integral Theorem, we get that for any fixed $\gamma_0 > 0$

$$(u_2, Q_2)(\gamma_1 + i\tau) = (u_2, Q_2)(\gamma_2 + i\tau) \quad \forall \gamma_1, \gamma_2 > \gamma_0.$$

So, using Theorem 3.4.5, we can apply Theorem 4.2.3:

$$\|(D_x^2 u_2, \Lambda_{\gamma,1}\nabla_x u_2, \Lambda_{\gamma,2} u_2)\|_{L_\gamma^p(\mathbb{R}; L^q(\mathbb{R}_+^N))} +$$

$$+ \|(D_x^3 Q_2, \Lambda_{\gamma,1} D_x^2 Q_2, \Lambda_{\gamma,2} \nabla Q_2, \Lambda_{\gamma,3} Q_2)\|_{L_\gamma^p(\mathbb{R}; L^q(\mathbb{R}_+^N))} \leq C \|G\|_{L_\gamma^p(\mathbb{R}; L^q(\mathbb{R}_+^N))},$$

with $C > 0$ which depends on γ_0 . Analogously, thanks to Remark 3.4.6, we have that

$$\begin{aligned} & \|(D_x^2 u_2, \gamma^{1/2} \nabla_x u_2, \gamma u_2)\|_{L_\gamma^p(\mathbb{R}; L^q(\mathbb{R}_+^N))} + \\ & + \|(D_x^3 Q_2, \gamma^{1/2} D_x^2 Q_2, \gamma \nabla Q_2, \gamma^{\frac{3}{2}} Q_2, \gamma^{\frac{1}{2}} \partial_t Q_2)\|_{L_\gamma^p(\mathbb{R}; L^q(\mathbb{R}_+^N))} \leq C \|G\|_{L_\gamma^p(\mathbb{R}; L^q(\mathbb{R}_+^N))}. \end{aligned}$$

Finally, we define $p_2(x, t) := K_A(u_2(t), Q_2(t))$, where for a.e. $t > 0$

$$\langle \nabla K_A(u_2(t), Q_2(t)), \nabla \varphi \rangle = \langle \Delta u_2(t) - \beta \operatorname{Div}(\Delta - a) Q_2(t), \nabla \varphi \rangle \quad \forall \varphi \in \widehat{H}_{q'}^1(\mathbb{R}_+^N).$$

Then, by construction, (u_2, p_2, Q_2) solves (84) and

$$\begin{aligned} \|\nabla p_2\|_{L_\gamma^p(\mathbb{R}; L^q(\mathbb{R}_+^N))} & \leq C(\beta) \|(\partial_t u_2, D_x^2 u_2)\|_{L_\gamma^p(\mathbb{R}; L^q(\mathbb{R}_+^N))} + \\ & + \|\partial_t \nabla_x Q_2\|_{L_\gamma^p(\mathbb{R}; L^q(\mathbb{R}_+^N))} \lesssim \|G\|_{L_\gamma^p(\mathbb{R}; L^q(\mathbb{R}_+^N))}. \end{aligned}$$

We conclude here the estimate, because, as we have noticed before, we can estimate the L^p -norm of $e^{-\gamma t} \Lambda_{\gamma,k} G$ with the $H_{p,\gamma}^{k/2}$ -norm of G and the same we can do for the norms of u and Q . \square

Let us pass to the first system (83). Here the result is already known:

Theorem 4.2.5. (Theorem 2.1, [16])

Let $a > 0$, $\beta \in \mathbb{R}$, $\gamma_0 \geq 1$, $p, q \in (1, \infty)$, let

$$\begin{aligned} f & \in L_\gamma^p(\mathbb{R}_+; L^q(\mathbb{R}^N; \mathbb{R}^N)), \quad G \in L_\gamma^p(\mathbb{R}_+; W^{1,q}(\mathbb{R}^N; \mathbb{R}^{N^2})), \\ v_0 & \in B_{q,p}^{2(1-1/p)}(\mathbb{R}^N; \mathbb{R}^N) \cap J_q(\mathbb{R}^N), \quad V_0 \in B_{q,p}^{3-2/p}(\mathbb{R}^N; \mathbb{R}^{N^2}), \end{aligned}$$

then we can find a unique (v, ρ, V) which solves

$$\begin{cases} (\partial_t - \Delta)v + \nabla \rho + \beta \operatorname{Div}(\Delta V - a(Id - \frac{Id}{N} \operatorname{tr}(V))) = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^N \\ (\partial_t - \Delta)V + a(V - \frac{Id}{N} \operatorname{tr}(V)) - \beta D(v) = G & \text{in } \mathbb{R}_+ \times \mathbb{R}^N \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^N \\ v(0) = v_0, \quad V(0) = V_0 & \text{in } \mathbb{R}^N \end{cases} \quad (86)$$

with $\rho(t) \in \widehat{H}_q^1(\mathbb{R}^N)$ for a.e. $t > 0$ and

$$v \in \bigcap_{l=0}^2 H_{p,\gamma}^{l/2}(\mathbb{R}_+; W^{2-l,q}(\mathbb{R}^N; \mathbb{R}^N)), \quad V \in \bigcap_{l=0}^2 H_{p,\gamma}^{l/2}(\mathbb{R}_+; W^{3-l,q}(\mathbb{R}^N; \mathbb{R}^{N^2})),$$

such that

$$\begin{aligned} & \sum_{l=0}^2 \|v\|_{H_{p,\gamma}^{l/2}(\mathbb{R}_+; W^{2-l,q}(\mathbb{R}^N))} + \sum_{l=0}^2 \|V\|_{H_{p,\gamma}^{l/2}(\mathbb{R}_+; W^{3-l,q}(\mathbb{R}^N))} + \|\nabla \rho\|_{L_\gamma^p(\mathbb{R}_+; L^q(\mathbb{R}^N))} \lesssim \\ & \lesssim \|f\|_{L_\gamma^p(\mathbb{R}_+; L^q(\mathbb{R}^N))} + \|G\|_{L_\gamma^p(\mathbb{R}_+; W^{1,q}(\mathbb{R}^N))} + \|v_0\|_{B_{q,p}^{2(1-1/p)}(\mathbb{R}^N)} + \|V_0\|_{B_{q,p}^{3-2/p}(\mathbb{R}^N)}. \end{aligned}$$

Remark 4.2.6. We notice that, if $\operatorname{tr}(G) = 0$ and $\operatorname{tr}(V_0) = 0$, then $\operatorname{tr}(V)$ solves

$$\begin{cases} (\partial_t - \Delta) \operatorname{tr}(V) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^N \\ \operatorname{tr}(V)(0) = 0 & \text{in } \mathbb{R}^N. \end{cases}$$

Therefore, $\operatorname{tr}(V) \equiv 0$.

Finally, we pass to the third system (85). We already know that there is a unique solution $(u_3, Q_3) = T(t)(u_0^3, Q_0^3)$, so we just need to prove that the solution satisfies the estimate wanted. Firstly we need an interpolation lemma:

Lemma 4.2.7. *Let $\Omega \subseteq \mathbb{R}^N$ a uniform C^2 open set, then*

$$\begin{aligned} \|f\|_{H_p^{1/2}(\mathbb{R}; W^{1,q}(\Omega))} &\lesssim \|f\|_{L^p(\mathbb{R}; W^{2,q}(\Omega))} + \|\partial_t f\|_{L^p(\mathbb{R}; L^q(\Omega))}, \\ \|g\|_{H_p^{1/2}(\mathbb{R}; W^{2,q}(\Omega))} &\lesssim \|g\|_{L^p(\mathbb{R}; W^{3,q}(\Omega))} + \|\partial_t g\|_{L^p(\mathbb{R}; W^{1,q}(\Omega))}. \end{aligned}$$

The proof of the first estimate comes from Proposition 1 of [19]. The other one can be proven similarly. We are now ready to prove the estimate for (u_3, Q_3) :

Proposition 4.2.8. *Let $a > 0$ and $\beta \in \mathbb{R}$ and $p, q \in (1, \infty)$, let $(u_3, Q_3) = T(t)(u_0^3, Q_0^3)$ be the solution of (85), let us call*

$$\mathcal{D}_{q,p} := (X_q, D(\mathcal{A}))_{1-\frac{1}{p}, p},$$

let $(u_0^3, Q_0^3) \in \mathcal{D}_{q,p}$, then for any $\gamma \geq 2\gamma_0$ we have that

$$\sum_{l=0}^2 \|u_3\|_{H_{p,\gamma}^{l/2}(\mathbb{R}_+; L^q(\mathbb{R}_+^N))} + \sum_{l=0}^2 \|Q_3\|_{H_{p,\gamma}^{l/2}(\mathbb{R}_+; W^{3-l,q}(\mathbb{R}_+^N))} + \|\nabla_x p_3\|_{L^p(\mathbb{R}; L^q(\mathbb{R}_+^N))} \lesssim \|(u_0^3, Q_0^3)\|_{\mathcal{D}_{p,q}},$$

where X_q and $\mathcal{D}(\mathcal{A})$ are defined in Proposition 4.1.4 and γ_0 comes from Proposition 4.1.4.

The proof is the same of Theorem 3.9 of [20]. Now we state this lemma which follows from the theory of [22]:

Lemma 4.2.9. *Let X_1, X_2 be two Banach spaces such that X_2 is dense in X_1 , then*

$$L^p(\mathbb{R}_+; X_2) \cap W^{1,p}(\mathbb{R}_+; X_1) \subseteq C\left([0, \infty); (X_1, X_2)_{1-1/p, p}\right)$$

with

$$\sup_{t \in \mathbb{R}_+} \|u(t)\|_{(X_1, X_2)_{1-1/p, p}} \lesssim \|u\|_{L^p(\mathbb{R}_+; X_2)} + \|u\|_{W^{1,p}(\mathbb{R}_+; X_1)}.$$

We are now ready to prove the existence and the estimate for (82), i.e. Theorem 1.2.2:

Proof of Theorem 1.2.2.

We write the solution as

$$(u, p, Q) = (u_1, p_1, Q_1) + (u_2, p_2, Q_2) + (u_3, p_3, Q_3),$$

where (u_1, p_1, Q_1) is the solution of (83) with $(f_1, G_1) = (E_v[f], E_M[G])$, where E_v and E_M are the extension operators defined as in the proof of Corollary 3.4.7, (u_2, p_2, Q_2) is the solution of (84) with $(h_2, H_2) = (E_0[h - u_1], E_0[H - D_N Q_1])$, where E_0 is the 0-extension on $t \leq 0$ and where (u_3, p_3, Q_3) is the solution of (85) with $(u_0^3, Q_0^3) = (u_0 - u_2(0), Q_0 - Q_2(0))$. We notice that:

- The existence of (u_1, p_1, Q_1) follows from Theorem 4.2.5 and Remark 4.2.6;
- Thanks to the choice of f_1 and G_1 , we have that $\operatorname{div} u_1 = 0$ for a.e. $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$. Therefore, the existence of (u_2, p_2, Q_2) comes from the fact that the N -component of u_1 is equal to 0 on \mathbb{R}_0^N ;

- The existence of (u_3, p_3, Q_3) comes from Proposition 4.2.8: from Lemma 4.2.9 we know that $(u_2 e^{-\gamma t})(0) = u_2(0) \in B_{q,p}^{2(1-1/p)}(\mathbb{R}_+^N)$ and $(Q_2 e^{-\gamma t})(0) = Q_2(0) \in B_{q,p}^{3-2/p}(\mathbb{R}_+^N)$; moreover, if we call

$$B_{q,p,0}^{2(1-1/p)}(\mathbb{R}_+^N) := \{v \in B_{q,p}^{2(1-1/p)}(\mathbb{R}_+^N) \mid v = 0 \text{ on } \mathbb{R}_0^N\},$$

$$B_{q,p,N}^{3-2/p}(\mathbb{R}_+^N) := \{V \in B_{q,p}^{3-2/p}(\mathbb{R}_+^N) \mid D_N V = 0 \text{ on } \mathbb{R}_0^N\},$$

then it is well-known (see Theorem 2.7 of [12]) that

$$(u_0 - u_2(0), Q_0 - Q_2(0)) \in \left(J_q(\mathbb{R}_+^N) \cap B_{q,p,0}^{2(1-1/p)}(\mathbb{R}_+^N) \right) \times B_{q,p,N}^{3-2/p}(\mathbb{R}_+^N) \subseteq \mathcal{D}_{q,p}.$$

Let us call now

$$I_k := \sum_{l=0}^2 \|u_k\|_{H_{p,\gamma}^{l/2}(\mathbb{R}_+; W^{2-l,q}(\mathbb{R}_+^N))} + \sum_{l=0}^2 \|Q_k\|_{H_{p,\gamma}^{l/2}(\mathbb{R}_+; W^{3-l,q}(\mathbb{R}_+^N))} + \|\nabla_x p_k\|_{L_\gamma^p(\mathbb{R}_+; L^q(\mathbb{R}_+^N))}$$

for $k = 1, 2, 3$. From Theorem 4.2.5 we know that

$$I_1 \lesssim \|f\|_{L_\gamma^p(\mathbb{R}_+; L^q(\mathbb{R}_+^N))} + \|G\|_{L_\gamma^p(\mathbb{R}_+; W^{1,q}(\mathbb{R}_+^N))}.$$

From Proposition 4.2.4 we get that

$$I_2 \lesssim \sum_{l=0}^2 \|(h - u_1, H - D_N Q_1)\|_{H_{p,\gamma}^{l/2}(\mathbb{R}_+; W^{2-l,q}(\mathbb{R}_+^N))} \leq I_1 + \sum_{l=0}^2 \|(h, H)\|_{H_{p,\gamma}^{l/2}(\mathbb{R}_+; W^{2-l,q}(\mathbb{R}_+^N))}.$$

For what concerns the third term, thanks to Proposition 4.2.8 we have that

$$I_3 \lesssim \|(u_0 - u_2(0), Q_0 - Q_2(0))\|_{\mathcal{D}_{q,p}}.$$

From Lemma 4.2.9 we also have that

$$\begin{aligned} & \|u_2(0)\|_{B_{q,p}^{2(1-1/p)}(\mathbb{R}_+^N)} + \|Q_2(0)\|_{B_{q,p}^{3-2/p}(\mathbb{R}_+^N)} \lesssim \\ & \lesssim \|e^{-\gamma t} u_2\|_{L^p(\mathbb{R}_+; W^{2,q}(\mathbb{R}_+^N; \mathbb{R}^N))} + \|e^{-\gamma t} (\gamma + \partial_t) u_2\|_{L^p(\mathbb{R}_+; L^q(\mathbb{R}_+^N; \mathbb{R}^N))} + \\ & + \|e^{-\gamma t} Q_2\|_{L^p(\mathbb{R}_+; W^{3,q}(\mathbb{R}_+^N; \mathbb{R}^N))} + \|e^{-\gamma t} (\gamma + \partial_t) Q_2\|_{L^p(\mathbb{R}_+; W^{1,q}(\mathbb{R}_+^N; \mathbb{R}^N))}. \end{aligned}$$

Now we notice that

$$\begin{aligned} & \|e^{-\gamma t} u_2\|_{L^p(\mathbb{R}_+; L^q(\mathbb{R}_+^N))} \leq \frac{1}{\gamma_0} \|e^{-\gamma t} \gamma u_2\|_{L^p(\mathbb{R}_+; L^q(\mathbb{R}_+^N))} \lesssim \\ & \lesssim \sum_{l=0}^2 \|(h, H)\|_{H_{p,\gamma}^{l/2}(\mathbb{R}_+; W^{2-l,q}(\mathbb{R}_+^N))} + \|f\|_{L_\gamma^p(\mathbb{R}_+; L^q(\mathbb{R}_+^N))} + \|G\|_{L_\gamma^p(\mathbb{R}_+; W^{1,q}(\mathbb{R}_+^N))}, \end{aligned}$$

where we have used Proposition 4.2.4. In the same way we can get the other estimates. Finally, we conclude by the uniqueness: since (82) is a linear system, it is sufficient to see that $(u, p, Q) = 0$ when $f = G = h = H = u_0 = Q_0 = 0$. This follows by the semigroup theory:

$$(u, Q) = T(t)(u_0, Q_0) = 0 \Rightarrow \nabla p = K_A(u, Q) = 0.$$

□

5 Proof of Theorem 1.2.4

Finally, we conclude with an application of Theorem 1.2.2: let us consider the general Beris-Edwards model:

$$\begin{cases} (\partial_t - \Delta)u + \nabla p + \beta \operatorname{Div}(\Delta - a)Q = f(u, Q) & \text{in } (0, T) \times \mathbb{R}_+^N \\ (\partial_t - \Delta + a)Q - \beta D(u) = G(u, Q) & \text{in } (0, T) \times \mathbb{R}_+^N \\ \operatorname{div} u = 0 & \text{in } (0, T) \times \mathbb{R}_+^N \\ u = h, \quad D_N Q = H & \text{on } (0, T) \times \mathbb{R}_0^N \\ u(0) = u_0, \quad Q(0) = Q_0 & \text{in } \mathbb{R}_+^N, \end{cases} \quad (87)$$

where

$$f(u, Q) = -(u \cdot \nabla)u + \operatorname{Div} \left[2\xi \mathbb{H} : Q \left(Q + \frac{Id}{N} \right) - (\xi + 1)\mathbb{H}Q + (1 - \xi)Q\mathbb{H} - \nabla Q \odot \nabla Q \right] - \beta \operatorname{Div} \mathcal{L}[\mathcal{F}(Q)];$$

$$G(u, Q) = -(u \cdot \nabla)Q + \xi(D(u)Q + QD(u)) + W(u)Q - QW(u) - 2\xi \left(Q + \frac{Id}{N} \right) Q : \nabla u + \mathcal{L}[\mathcal{F}(Q)],$$

$$W(u) = \frac{1}{2} (\nabla u - \nabla^T u), \quad [\nabla Q \odot \nabla Q]_{jk} = \sum_{\alpha, \beta=1}^N \partial_j Q_{\alpha\beta} \partial_k Q_{\alpha\beta} \quad j, k = 1, \dots, N,$$

$$\mathbb{H} = \Delta Q - aQ + b\mathcal{L}[Q^2] - c|Q|^2 Q, \quad \mathcal{F} = bQ^2 - c|Q|^2 Q,$$

where $\xi, a, b, c \in \mathbb{R}$, $\beta = \frac{2\xi}{N}$ and

$$\mathcal{L}[A] = A - \operatorname{tr}(A) \frac{Id}{N} \quad \forall A \in \mathbb{R}^{N^2}.$$

We recall the following two results from Theorem 2.1 and 2.2 of [5]:

Theorem 5.0.1.

Let $N \in \mathbb{N}$, $p \in [1, \infty]$, $q, q_1 \in [1, \infty]$ and $s > s_1$, then

$$B_{p,q}^s(\mathbb{R}^N) \hookrightarrow B_{p,q_1}^{s_1}(\mathbb{R}^N).$$

Moreover, if $m \in \mathbb{N}$, then

$$B_{p,1}^m(\mathbb{R}^N) \hookrightarrow W^{m,p}(\mathbb{R}^N) \hookrightarrow B_{p,\infty}^m(\mathbb{R}^N).$$

We are now ready to prove Theorem 1.2.4:

Proof of Theorem 1.2.4.

The strategy is to use the Contraction theorem: let us define

$$\begin{aligned} \|(u, Q)\|_T &:= \|u\|_{H_p^1((0,T);L^q(\mathbb{R}_+^N))} + \|u\|_{L^p((0,T);W^{2,q}(\mathbb{R}_+^N))} + \\ &+ \|Q\|_{H_p^1((0,T);W^{1,q}(\mathbb{R}_+^N))} + \|Q\|_{L^p((0,T);W^{3,q}(\mathbb{R}_+^N))}. \end{aligned}$$

Let $T \in (0, 1]$, then we can define also

$$Y_1 := H_p^1((0, T); L^q(\mathbb{R}_+^N; \mathbb{R}^N)) \cap L^p((0, T); W^{2,q}(\mathbb{R}_+^N; \mathbb{R}^N));$$

$$Y_2 := H_p^1((0, T); W^{1,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))) \cap L^p((0, T); W^{3,q}(\mathbb{R}_+^N; S_0(N, \mathbb{R}))).$$

Let $\omega > 0$. We apply the theorem on the space

$$Y_\omega := \{(v, W) \in Y_1 \times Y_2 \mid \|(v, W)\|_T \leq \omega; v|_{t=0} = u_0, W|_{t=0} = Q_0\}.$$

We consider the function $\phi: (v, W) \in Y_\omega \mapsto (u, Q) \in Y_1 \times Y_2$ which solves

$$\begin{cases} (\partial_t - \Delta)u + \nabla p + \beta \operatorname{Div}(\Delta - a)Q = f(\mathcal{E}_1(v), \mathcal{E}_2(W)) & \text{in } (0, T) \times \mathbb{R}_+^N \\ (\partial_t - \Delta + a)Q - \beta D(u) = G(\mathcal{E}_1(v), \mathcal{E}_2(W)) & \text{in } (0, T) \times \mathbb{R}_+^N \\ \operatorname{div} u = 0 & \text{in } (0, T) \times \mathbb{R}_+^N \\ u = \mathcal{E}_3(h), \quad D_N Q = \mathcal{E}_4(H) & \text{on } (0, T) \times \mathbb{R}_0^N \\ u(0) = u_0, \quad Q(0) = Q_0 & \text{in } \mathbb{R}_+^N, \end{cases}$$

with $p(t) = K_A(u(t), Q(t))$ and

$$\mathcal{E}_1(v) = E_T[v - T_1(t)(u_0, Q_0)] + \psi(t)T_1(|t|)(u_0, Q_0),$$

$$\mathcal{E}_2(W) = E_T[W - T_2(t)(u_0, Q_0)] + \psi(t)T_2(|t|)(u_0, Q_0),$$

$$\mathcal{E}_3(h) = E_T[h - T_1(t)(u_0, Q_0)] + \psi(t)T_1(|t|)(u_0, Q_0),$$

$$\mathcal{E}_4(H) = E_T[H - D_N T_2(t)(u_0, Q_0)] + \psi(t)D_N T_2(|t|)(u_0, Q_0),$$

where $\psi(t)$ is a $C^\infty(\mathbb{R})$ function equal to 0 for $t < -2$ and equal to 1 for $t > -1$, $(T_1(t), T_2(t)) := T(t)$ is the semigroup associated with the linearized problem (85) and

$$E_T(f) = \begin{cases} 0 & t < 0 \\ f(t) & t \in (0, T) \\ f(2T - t) & t \in (T, 2T) \\ 0 & t > 2T. \end{cases}$$

We have taken the extension E_T of $(v, W) - T(t)(u_0, Q_0)$ and of $(h, H) - (T(t), D_N T(t))(u_0, Q_0)$ because when the function f we are extending satisfies $f(0) = 0$, then $E_T[f] \in H_p^1(\mathbb{R})$ with

$$\partial_t E_T(f) = \begin{cases} 0 & t < 0 \\ \partial_t f(t) & t \in (0, T) \\ -\partial_t f(2T - t) & t \in (T, 2T) \\ 0 & t > 2T. \end{cases}$$

In order to find such (u, p, Q) we have to check that $f(\mathcal{E}_1(v), \mathcal{E}_2(W))$, $G(\mathcal{E}_1(v), \mathcal{E}_2(W))$, $\mathcal{E}_3(h)$ and $\mathcal{E}_4(H)$ satisfies the conditions of Theorem 1.2.2: let us start seeing that

$$\begin{aligned} & \|e^{-\gamma t} \mathcal{E}_3(h)\|_{L^p(\mathbb{R}; W^{2,q}(\mathbb{R}_+^N))} + \|e^{-\gamma t} \mathcal{E}_3(h)\|_{H_p^1(\mathbb{R}; L^q(\mathbb{R}_+^N))} \lesssim \\ & \lesssim \|h\|_{L^p((0,T); W^{2,q}(\mathbb{R}_+^N))} + \|h\|_{H_p^1((0,T); L^q(\mathbb{R}_+^N))} + \\ & + \|e^{-\gamma t} T_1(t)(u_0, Q_0)\|_{L^p(\mathbb{R}_+; W^{2,q}(\mathbb{R}_+^N))} + \|e^{-\gamma t} T_1(t)(u_0, Q_0)\|_{H_p^1(\mathbb{R}_+; L^q(\mathbb{R}_+^N))}. \end{aligned}$$

Thanks to Proposition 4.2.8 we get that

$$\begin{aligned} & \|e^{-\gamma t} T_1(t)(u_0, Q_0)\|_{L^p(\mathbb{R}_+; W^{2,q}(\mathbb{R}_+^N))} + \|e^{-\gamma t} T_1(t)(u_0, Q_0)\|_{H_p^1(\mathbb{R}_+; L^q(\mathbb{R}_+^N))} \lesssim \\ & \lesssim \|u_0\|_{B_{q,p}^{2(1-1/p)}(\mathbb{R}_+^N)} + \|Q_0\|_{B_{q,p}^{3-2/p}(\mathbb{R}_+^N)}. \end{aligned}$$

Finally, thanks to Lemma 4.2.7, for $\ell = 0, 1, 2$, we have that

$$\|e^{-\gamma t} \mathcal{E}_3(h)\|_{H_p^{\ell/2}(\mathbb{R}; W^{2-\ell,q}(\mathbb{R}_+^N))} \lesssim \|h\|_{H_p^{\ell/2}((0,T); W^{2-\ell,q}(\mathbb{R}_+^N))} + \varepsilon. \quad (88)$$

On the other hand

$$\|e^{-\gamma t} \mathcal{E}_4(H)\|_{L^p(\mathbb{R}; W^{2,q}(\mathbb{R}_+^N))} + \|e^{-\gamma t} \mathcal{E}_4(H)\|_{H_p^1(\mathbb{R}; L^q(\mathbb{R}_+^N))} \lesssim$$

$$\begin{aligned} &\lesssim \|H\|_{L^p((0,T);W^{2,q}(\mathbb{R}_+^N))} + \|H\|_{H_p^1((0,T);L^q(\mathbb{R}_+^N))} + \\ &+ \|e^{-\gamma t} D_N T_2(t)(u_0, Q_0)\|_{L^p(\mathbb{R}_+;W^{2,q}(\mathbb{R}_+^N))} + \|e^{-\gamma t} D_N T_2(t)(u_0, Q_0)\|_{H_p^1(\mathbb{R}_+;L^q(\mathbb{R}_+^N))}. \end{aligned}$$

Again, thanks to Proposition 4.2.8, we get that

$$\begin{aligned} &\|e^{-\gamma t} T_2(t)(u_0, Q_0)\|_{L^p(\mathbb{R}_+;W^{3,q}(\mathbb{R}_+^N))} + \|e^{-\gamma t} T_2(t)(u_0, Q_0)\|_{H_p^1(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))} \lesssim \\ &\lesssim \|u_0\|_{B_{q,p}^{2(1-1/p)}(\mathbb{R}_+^N)} + \|Q_0\|_{B_{q,p}^{3-2/p}(\mathbb{R}_+^N)}. \end{aligned}$$

So

$$\|e^{-\gamma t} \mathcal{E}_4(H)\|_{H_p^{\ell/2}(\mathbb{R};W^{2-\ell,q}(\mathbb{R}_+^N))} \lesssim \|H\|_{H_p^{\ell/2}((0,T);W^{2-\ell,q}(\mathbb{R}_+^N))} + \varepsilon. \quad (89)$$

Now, we list the main non-linearities that arise of $f(u, Q)$: let $j, k, \ell = 1, \dots, N$

$$\begin{aligned} I_1(u) &= (u \cdot \nabla)u, \quad I_2(Q) = \partial_{jk}Q\partial_\ell Q, \quad I_3(Q) = \partial_jQQ\partial_{k\ell}Q \\ I_4(Q) &= \partial_jQQ \quad I_5(Q) = \partial_jQQ^2, \quad I_6(Q) = \partial_jQQ^3, \quad I_7(Q) = \partial_jQQ^4 \\ I_8(Q) &= Q\partial_{jkl}Q, \quad I_9(Q) = Q^2\partial_{jkl}Q; \end{aligned}$$

Next, we list the main non-linearities of $G(u, Q)$: let $j, k, \ell = 1, \dots, N$

$$II_1(u, Q) = \partial_jQu, \quad II_2(u, Q) = Q\partial_ju, \quad II_3(u, Q) = Q^2\partial_ju, \quad II_4(Q) = Q^2, \quad II_5(Q) = Q^3.$$

Therefore, we need to estimate

$$\begin{aligned} &\|e^{-\gamma t} I_k(\mathcal{E}_1(v), \mathcal{E}_2(W))\|_{L^p(\mathbb{R}_+;L^q(\mathbb{R}_+^N))} \quad k = 1, \dots, 8; \\ &\|e^{-\gamma t} II_\ell(\mathcal{E}_1(v), \mathcal{E}_2(W))\|_{L^p(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))} \quad \ell = 1, \dots, 5. \end{aligned}$$

We will write in details just some of the estimates: let us start with the estimates of I_1 :

$$\|\mathcal{E}_1(v)\nabla\mathcal{E}_1(v)\|_{L^q(\mathbb{R}_+^N)} \leq \|\mathcal{E}_1(v)\|_{L^\infty(\mathbb{R}_+^N)}\|\mathcal{E}_1(v)\|_{W^{1,q}(\mathbb{R}_+^N)} \lesssim \|\mathcal{E}_1(v)\|_{W^{1,q}(\mathbb{R}_+^N)}^2,$$

where we have used Sobolev embedding with $q > N$. Therefore

$$\|e^{-\gamma t}\mathcal{E}_1(v)\nabla\mathcal{E}_1(v)\|_{L^p(\mathbb{R}_+;L^q(\mathbb{R}_+^N))} \lesssim \|e^{-\gamma t}\|\mathcal{E}_1(v)\|_{W^{1,q}(\mathbb{R}_+^N)}^2\|_{L^p(\mathbb{R}_+)} = \|e^{-\frac{\gamma}{2}t}\mathcal{E}_1(v)\|_{L^{2p}(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))}^2.$$

By definition of \mathcal{E}_1 we have that

$$\|e^{-\frac{\gamma}{2}t}\mathcal{E}_1(v)\|_{L^{2p}(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))}^2 \lesssim \|v\|_{L^{2p}((0,T);W^{1,q}(\mathbb{R}_+^N))}^2 + \|e^{-\frac{\gamma}{2}t}T_1(t)(u_0, Q_0)\|_{L^{2p}(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))}^2.$$

For what concerns the first term:

$$\begin{aligned} \|v\|_{L^{2p}((0,T);W^{1,q}(\mathbb{R}_+^N))}^2 &\leq T^{\frac{1}{p}}\|v\|_{L^\infty((0,T);W^{1,q}(\mathbb{R}_+^N))}^2 \lesssim T^{\frac{1}{p}}\|e^{-\frac{\gamma}{2}t}\mathcal{E}_1(v)\|_{L^\infty(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))}^2 \lesssim \\ &\lesssim T^{\frac{1}{p}}\left(\|e^{-\frac{\gamma}{2}t}\mathcal{E}_1(v)\|_{L^p(\mathbb{R};W^{2,q}(\mathbb{R}_+^N))}^2 + \|e^{-\frac{\gamma}{2}t}\mathcal{E}_1(v)\|_{H_p^1(\mathbb{R};L^q(\mathbb{R}_+^N))}^2\right), \end{aligned}$$

where we have used Lemma 4.2.9 and the fact that, since $p > 2$, by Theorem 5.0.1 then

$$(L^q(\mathbb{R}_+^N), W^{2,q}(\mathbb{R}_+^N))_{1-1/p,p} = B_{q,p}^{2(1-1/p)}(\mathbb{R}_+^N) \hookrightarrow B_{q,1}^1(\mathbb{R}_+^N) \hookrightarrow W^{1,q}(\mathbb{R}_+^N).$$

On the other hand

$$\|e^{-\frac{\gamma}{2}t}T_1(t)(u_0, Q_0)\|_{L^{2p}(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))}^2 \lesssim \|e^{-\frac{\gamma}{4}t}T_1(t)(u_0, Q_0)\|_{L^\infty(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))}^2 \lesssim \varepsilon^2,$$

where we have used Lemma 4.2.9 and Proposition 4.2.8. Finally, we get

$$\|e^{-\gamma t}I_1(\mathcal{E}_1(v))\|_{L^p(\mathbb{R};L^q(\mathbb{R}_+^N))} \lesssim \varepsilon^2 + T^{\frac{1}{p}}(\omega^2 + \varepsilon^2). \quad (90)$$

Let us pass to I_2 : as before

$$\begin{aligned} & \|e^{-\gamma t} \partial_{jk} \mathcal{E}_2(W) \partial_\ell \mathcal{E}_2(W)\|_{L^p(\mathbb{R}_+; L^q(\mathbb{R}_+^N))} \lesssim \|e^{-\frac{\gamma}{2} t} \mathcal{E}_2(W)\|_{L^{2p}(\mathbb{R}_+; W^{2,q}(\mathbb{R}_+^N))}^2 \lesssim \\ & \lesssim T^{1/p} \|e^{-\frac{\gamma}{2} t} \mathcal{E}_2(W)\|_{L^\infty(\mathbb{R}_+; W^{2,q}(\mathbb{R}_+^N))}^2 + \|e^{-\frac{\gamma}{4} t} T_2(t)(u_0, Q_0)\|_{L^\infty(\mathbb{R}_+; W^{2,q}(\mathbb{R}_+^N))}^2. \end{aligned}$$

We use Lemma 4.2.7 noticing as before that, since $p > 2$, by Theorem 5.0.1 we have the embedding

$$(W^{1,q}(\mathbb{R}_+^N), W^{3,q}(\mathbb{R}_+^N))_{1-1/p, p} = B_{q,p}^{3-2/p}(\mathbb{R}_+^N) \hookrightarrow W^{2,q}(\mathbb{R}_+^N).$$

So we get

$$\|e^{-\gamma t} I_2(\mathcal{E}_2(W))\|_{L^p(\mathbb{R}_+; L^q(\mathbb{R}_+^N))} \lesssim \varepsilon^2 + T^{\frac{1}{p}} (\omega^2 + \varepsilon^2). \quad (91)$$

The estimate for I_3 is similar: with the same strategy we get that

$$\begin{aligned} & \|e^{-\gamma t} \partial_j \mathcal{E}_2(W) \mathcal{E}_2(W) \partial_{k\ell} \mathcal{E}_2(W)\|_{L^p(\mathbb{R}_+; L^q(\mathbb{R}_+^N))} \lesssim \|e^{-\frac{\gamma}{3} t} \mathcal{E}_2(W)\|_{L^{3p}(\mathbb{R}_+; W^{2,q}(\mathbb{R}_+^N))}^3 \lesssim \\ & \lesssim T^{1/p} \|e^{-\frac{\gamma}{3} t} \mathcal{E}_2(W)\|_{L^\infty(\mathbb{R}_+; W^{2,q}(\mathbb{R}_+^N))}^3 + \|e^{-\frac{\gamma}{6} t} T_2(t)(u_0, Q_0)\|_{L^\infty(\mathbb{R}_+; W^{2,q}(\mathbb{R}_+^N))}^3. \end{aligned}$$

Then we repeat the argument for I_2 and I_3 . The estimate for I_ℓ with $\ell = 4, 5, 6, 7$ is the same as before. Therefore, let us pass to I_8 :

$$\begin{aligned} & \|e^{-\gamma t} \mathcal{E}_2(W) \partial_{jkl} \mathcal{E}_2(W)\|_{L^p(\mathbb{R}_+; L^q(\mathbb{R}_+^N))}^p \leq \int_0^\infty e^{-\gamma pt} \|\mathcal{E}_2(W)\|_{W^{1,q}(\mathbb{R}_+^N)}^p \|\mathcal{E}_2(W)\|_{W^{3,q}(\mathbb{R}_+^N)}^p dt \leq \\ & \leq \|e^{-\frac{\gamma}{2} t} \mathcal{E}_2(W)\|_{L^\infty(\mathbb{R}_+; W^{1,q}(\mathbb{R}_+^N))}^p \|e^{-\frac{\gamma}{2} t} \mathcal{E}_2(W)\|_{L^p(\mathbb{R}_+; W^{3,q}(\mathbb{R}_+^N))}^p. \end{aligned}$$

Since $p > 2$, we can use Sobolev embeddings:

$$\begin{aligned} & \|e^{-\frac{\gamma}{2} t} \mathcal{E}_2(W)\|_{L^\infty(\mathbb{R}_+; W^{1,q}(\mathbb{R}_+^N))} \lesssim \|e^{-\frac{\gamma}{2} t} \mathcal{E}_2(W)\|_{W^{1/2,p}(\mathbb{R}_+; W^{1,q}(\mathbb{R}_+^N))} \lesssim \\ & \lesssim \|e^{-\frac{\gamma}{2} t} E_T[W - T_2(t)(u_0, Q_0)]\|_{W^{1/2,p}(\mathbb{R}_+; W^{1,q}(\mathbb{R}_+^N))} + \|e^{-\frac{\gamma}{2} t} T_2(t)(u_0, Q_0)\|_{W^{1/2,p}(\mathbb{R}_+; W^{1,q}(\mathbb{R}_+^N))}. \end{aligned}$$

For what concerns the second part, we can just use the inequality

$$\|e^{-\frac{\gamma}{2} t} T_2(t)(u_0, Q_0)\|_{W^{1/2,p}(\mathbb{R}_+; W^{1,q}(\mathbb{R}_+^N))} \leq \|e^{-\frac{\gamma}{2} t} T_2(t)(u_0, Q_0)\|_{H_p^1(\mathbb{R}_+; W^{1,q}(\mathbb{R}_+^N))} \lesssim \varepsilon.$$

From the definition of E_T we have that

$$\|E_T[f]\|_{L^p(\mathbb{R}_+; W^{1,q}(\mathbb{R}_+^N))} \lesssim \|f\|_{L^p((0,T); W^{1,q}(\mathbb{R}_+^N))}, \quad \|E_T[f]\|_{H_p^1(\mathbb{R}_+; W^{1,q}(\mathbb{R}_+^N))} \lesssim \|f\|_{H_p^1((0,T); W^{1,q}(\mathbb{R}_+^N))}.$$

So, since

$$W^{1/2,p}(\mathbb{R}_+) = (L^p(\mathbb{R}_+), H_p^1(\mathbb{R}_+))_{1/2,p},$$

By interpolation we have that

$$\begin{aligned} & \|e^{-\frac{\gamma}{2} t} E_T[W - T_2(t)(u_0, Q_0)]\|_{W^{1/2,p}(\mathbb{R}_+; W^{1,q}(\mathbb{R}_+^N))} \lesssim \|W - T_2(t)(u_0, Q_0)\|_{W^{1/2,p}((0,T); W^{1,q}(\mathbb{R}_+^N))} \leq \\ & \leq \|W\|_{W^{1/2,p}((0,T); W^{1,q}(\mathbb{R}_+^N))} + \|T_2(t)(u_0, Q_0)\|_{W^{1/2,p}((0,T); W^{1,q}(\mathbb{R}_+^N))}. \end{aligned}$$

In particular

$$\begin{aligned} & \|W\|_{W^{1/2,p}((0,T); W^{1,q}(\mathbb{R}_+^N))}^p = \\ & = \int_0^T \int_0^T \frac{\|W(t) - W(s)\|_{W^{1,q}(\mathbb{R}_+^N)}^p}{|t-s|^{p/2+1}} dt ds = \int_0^T \int_0^T \frac{\|\mathcal{E}_2(W)(t) - \mathcal{E}_2(W)(s)\|_{W^{1,q}(\mathbb{R}_+^N)}^p}{|t-s|^{p/2+1}} dt ds. \end{aligned}$$

Since $p > 1$, we have that $\mathcal{E}_2(W) \in C^{0,1-1/p}(\mathbb{R}_+; W^{1,q}(\mathbb{R}_+^N; \mathbb{R}^{N^2}))$, so

$$\|W\|_{W^{1/2,p}((0,T);W^{1,q}(\mathbb{R}_+^N))}^p \lesssim \|\mathcal{E}_2(W)\|_{H_p^1(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))}^p \int_0^T \int_0^T |t-s|^{p/2-2} dt ds.$$

We also know that $p > 2$, so $p/2 - 2 > -1$ and

$$\int_0^T \int_0^T |t-s|^{p/2-2} dt ds = \int_0^T \int_{-s}^{T-s} |t|^{p/2-2} dt ds \simeq \int_0^T [(T-s)^{p/2-1} - (-s)^{p/2-1}] ds \simeq T^{p/2}.$$

so

$$\|W\|_{W^{1/2,p}((0,T);W^{1,q}(\mathbb{R}_+^N))}^p \lesssim T^{p/2} \|\mathcal{E}_2(W)\|_{H_p^1(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))}^p.$$

Similarly,

$$\|T_2(t)(u_0, Q_0)\|_{W^{1/2,p}((0,T);W^{1,q}(\mathbb{R}_+^N))} \lesssim T^{1/2} \|T_2(t)(u_0, Q_0)\|_{H_p^1(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))}.$$

Therefore,

$$\|e^{-\gamma t} I_8(\mathcal{E}_2(W))\|_{L^p(\mathbb{R}_+;L^q(\mathbb{R}_+^N))} \lesssim \varepsilon^2 + T^{\frac{1}{2}} (\omega^2 + \varepsilon^2). \quad (92)$$

The estimate for I_9 is the same, so we pass to the II_1 :

$$\begin{aligned} & \|e^{-\gamma t} \partial_j \mathcal{E}_2(W) \mathcal{E}_1(v)\|_{L^p(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))}^p \lesssim \\ & \lesssim \sum_{k=1}^N \|e^{-\gamma t} \partial_{jk} \mathcal{E}_2(W) \mathcal{E}_1(v)\|_{L^p(\mathbb{R}_+;L^q(\mathbb{R}_+^N))}^p + \|e^{-\gamma t} \partial_j \mathcal{E}_2(W) \partial_k \mathcal{E}_1(v)\|_{L^p(\mathbb{R}_+;L^q(\mathbb{R}_+^N))}^p \lesssim \\ & \lesssim \int_0^\infty e^{-\gamma pt} \|\mathcal{E}_2(W)\|_{W^{2,q}(\mathbb{R}_+^N)}^p \|\mathcal{E}_1(v)\|_{W^{1,q}(\mathbb{R}_+^N)}^p dt. \end{aligned}$$

So we can repeat the argument for I_7 and we get

$$\|e^{-\gamma t} II_1(\mathcal{E}_1(v), \mathcal{E}_2(W))\|_{L^p(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))} \lesssim \varepsilon^2 + T^{\frac{1}{p}} (\omega^2 + \varepsilon^2). \quad (93)$$

Let us pass to II_2 :

$$\begin{aligned} & \|e^{-\gamma t} \mathcal{E}_2(W) \partial_j \mathcal{E}_1(v)\|_{L^p(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))} \leq \|e^{-\gamma t} \mathcal{E}_2(W) \partial_j \mathcal{E}_1(v)\|_{L^p(\mathbb{R}_+;L^q(\mathbb{R}_+^N))} + \\ & + \sum_{k=1}^N \|e^{-\gamma t} \mathcal{E}_2(W) \partial_{jk} \mathcal{E}_1(v)\|_{L^p(\mathbb{R}_+;L^q(\mathbb{R}_+^N))} + \|e^{-\gamma t} \partial_k \mathcal{E}_2(W) \partial_j \mathcal{E}_1(v)\|_{L^p(\mathbb{R}_+;L^q(\mathbb{R}_+^N))}. \end{aligned}$$

For what concerns the second term, we can repeat the argument of II_1 . For the other one

$$\|e^{-\gamma t} \mathcal{E}_2(W) \partial_{jk} \mathcal{E}_1(v)\|_{L^p(\mathbb{R}_+;L^q(\mathbb{R}_+^N))} \lesssim \|e^{-\frac{\gamma}{2} t} \mathcal{E}_2(W)\|_{L^\infty(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))} \|e^{-\frac{\gamma}{2} t} \mathcal{E}_1(v)\|_{L^p(\mathbb{R}_+;W^{2,q}(\mathbb{R}_+^N))}.$$

Then we can repeat the argument of I_5 , so

$$\|e^{-\gamma t} II_2(\mathcal{E}_1(v), \mathcal{E}_2(W))\|_{L^p(\mathbb{R}_+;W^{1,q}(\mathbb{R}_+^N))} \lesssim \varepsilon^2 + T^{\frac{1}{p}} (\omega^2 + \varepsilon^2). \quad (94)$$

Repeating this kind of arguments also with II_3 , II_4 and II_5 , if $\varepsilon \leq 1$, using (88), (89) and (90) to (94), we get

$$\|\phi(v, W)\|_T \leq C \left(\sum_{\ell=0}^2 \|(h, H)\|_{H_p^{\ell/2}((0,T);W^{2-\ell}(\mathbb{R}_+^N))} + \varepsilon + T^{\frac{1}{p}} (\omega^2 + \omega^3 + \omega^4 + \omega^5 + \varepsilon) \right),$$

with $C > 0$ which doesn't depend on ω, ε and T . So, if we take $\omega \in \mathbb{R}$ such that

$$C \left(\sum_{\ell=0}^2 \|(h, H)\|_{H_p^{\ell/2}((0,T);W^{2-\ell}(\mathbb{R}_+^N))} + \varepsilon \right) \leq \frac{\omega}{2},$$

and we choose $T > 0$ sufficiently small such that it holds

$$CT^{\frac{1}{p}} (\omega^2 + \omega^3 + \omega^4 + \omega^5 + \varepsilon) \leq \frac{\omega}{2},$$

then the function ϕ is well-defined and $\phi: Y_\omega \rightarrow Y_\omega$. Now, if we take $(v_1, W_1), (v_2, W_2) \in Y_\omega$, it can be seen as before that exists $M > 0$ independent from ω, ε, T such that

$$\|\phi(v_1, W_1) - \phi(v_2, W_2)\|_T \leq M \left(\varepsilon + T^{\frac{1}{p}} (\omega + \omega^2 + \omega^3 + \omega^4 + \varepsilon) \right) \|(v_1, W_1) - (v_2, W_2)\|_T.$$

So, choosing $\varepsilon, T \ll 1$ sufficiently small, ϕ is a contraction on Y_ω and therefore we have a solution. The uniqueness follows from the contraction theorem. \square

References

- [1] H. Abels, G. Dolzmann and Y. Liu, *Well-posedness of a fully coupled Navier-Stokes/Q-tensor system with inhomogeneous boundary data*, SIAM J. Math. Anal., **46(4)** (2014), 3050-3077.
- [2] H. Abels, G. Dolzmann and Y. Liu, *Strong Solutions for the Beris-Edwards model for nematic liquid crystals with homogeneous Dirichlet boundary conditions*, Adv. Differential Equation, **21(1-2)** (2016), 109-152.
- [3] J. Bergh and J. Löfström, "Interpolation spaces. An introduction. Grundlehren der Mathematischen Wissenschaften", Springer-Verlag, Berlin-New York, 1976.
- [4] A. N. Beris and B. J. Edwards, "Thermodynamics of flowing systems with internal microstructure", Oxford Engineering Science Series, The Clarendon Press, Oxford University Press, New York, 1994.
- [5] P. Brenner, V. Thomée and L. B. Wahlbin, "Besov spaces and applications to difference methods for initial value problems", Springer-Verlag, Berlin-New York, 1975.
- [6] C. Cavaterra, E. Rocca, H. Wu and X. Xu, *Global strong solutions of the full Navier-Stokes and Q-tensor system for nematic liquid crystal flows in two dimensions*, SIAM J. Math. Anal., **48(2)** (2016), 1368-1399.
- [7] F. De Anna, *A global 2D well-posedness result on the order tensor liquid crystal theory*, J. Differential Equations, **262(7)** (2017), 3932-3979.
- [8] P.G. de Gennes and J. Prost, "The Physics of Liquid Crystals", International Series of Monographs on Physics, Clarendon Press, 1993.
- [9] R. Denk, M. Hieber and J. Prüss, *R-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc., **166(788)** (2003), viii+114.
- [10] Y. Enomoto and Y. Shibata, *On the R-sectoriality and the initial boundary value problem for the viscous compressible fluid flow*, Funkcial. Ekvac., **56(3)** (2013), 441-505.
- [11] Y. Giga and A. Novotný, "Handbook of Mathematical Analysis in Mechanics of Viscous Fluids", Springer Cham, 2018.
- [12] D. Guidetti, *On interpolation with boundary conditions*, Math. Z., **207(3)** (1991), 439-460.
- [13] J. Huang and S. Ding, *Global well-posedness for the dynamical Q-tensor model of liquid crystals*, Sci. China Math., **58(6)** (2015), 1349-1366.
- [14] Y. Liu and W. Wang, *On the initial boundary value problem of a Navier-Stokes/Q-tensor model for liquid crystals*, Discrete Contin. Dyn. Syst. Ser. B, **23(9)** (2018), 3879-3899.
- [15] Y. Luo, Z. Li and X. Zhao, *Global existence of strong solutions for Beris-Edwards's liquid crystal system in dimension three*, Mathematics, **7(10)** (2019), 972.
- [16] M. Murata and Y. Shibata, *Global well posedness for a Q-tensor model of nematic liquid crystals*, J. Math. Fluid Mech., **24(2)** (2022), 34.
- [17] M. Paicu and A. Zarnescu, *Energy dissipation and regularity for a coupled Navier-Stokes and Q-tensor system*, Arch. Ration. Mech. Anal., **203(1)** (2012), 45-67.

- [18] M. Schonbek and Y. Shibata, *Global well-posedness and decay for a Q-tensor model of incompressible nematic liquid crystals in \mathbb{R}^N* , J. Differential Equations, **266(6)** (2019), 3034-3065.
- [19] Y. Shibata *On the local wellposedness of free boundary problem for the Navier-Stokes equations in an exterior domain*, Commun. Pure Appl. Anal., **17(4)** (2018), 1681-1721.
- [20] Y. Shibata and S. Shimizu *On the $L^p - L^q$ maximal regularity of the Neumann problem for the Stokes equations in a bounded domain*, J. Reine Angew. Math., **615** (2008), 157-209.
- [21] Y. Shibata and S. Shimizu, *On the maximal $L^p - L^q$ regularity of the Stokes problem with first order boundary condition; model problems*, J. Math. Soc. Japan, **64(2)** (2012), 561-626.
- [22] H. Tanabe, "Functional analytic methods for partial differential equations", Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 1997.
- [23] L. R. Volevič, *Solubility of boundary value problems for general elliptic systems*, Mat. Sb. (N.S.), **68 (110)** (1965), 73-416.
- [24] L. Weis, *Operator-valued Fourier multiplier theorems and maximal L^p -regularity*, Math. Ann., **319(4)** (2001), 735-758.
- [25] Y. Xiao, *Global strong solution to the three-dimensional liquid crystal flows of Q-tensor model*, J. Differential Equations, **262(3)** (2017), 1291-1316.

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