Approximations in Besov Spaces and Jump Detection of Besov Functions with Bounded Variation

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Abstract

In this paper, we provide a proof that functions belonging to Besov spaces $B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$, $q\in[1,\infty)$, $r\in(0,1)$, satisfy the following formula under a certain condition:

$$\lim_{\epsilon\to 0^+}\frac{1}{|\ln\epsilon|}\left[u_\epsilon\right]^q_{W^{r,q}(\mathbb{R}^N,\mathbb{R}^d)}=N\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N}\frac{1}{\epsilon^N}\int_{B_\epsilon(x)}\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx. \tag{0.1}$$

Here, $[\cdot]_{W^{r,q}}$ represents the Gagliardo seminorm, and u_ϵ denotes the convolution of u with a mollifier $\eta_{(\epsilon)}(x):=\frac{1}{\epsilon^N}\eta\left(\frac{x}{\epsilon}\right)$, $\eta\in W^{1,1}(\mathbb{R}^N)$, $\int_{\mathbb{R}^N}\eta(z)dz=1$. Furthermore, we prove that every function u in $BV(\mathbb{R}^N,\mathbb{R}^d)\cap B^{1/p}_{p,\infty}(\mathbb{R}^N,\mathbb{R}^d), p\in(1,\infty)$, satisfies

$$\begin{aligned}\lim_{\epsilon\to 0^+}\frac{1}{|\ln\epsilon|}\left[u_\epsilon\right]^q_{W^{1/q,q}(\mathbb{R}^N,\mathbb{R}^d)}&=N\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N}\frac{1}{\epsilon^N}\int_{B_\epsilon(x)}\frac{|u(x)-u(y)|^q}{|x-y|}dydx\\&=\left(\int_{S^{N-1}}|z_1|\,d\mathcal{H}^{N-1}(z)\right)\int_{\mathcal{J}_u}\left|u^+(x)-u^-(x)\right|^q d\mathcal{H}^{N-1}(x),\end{aligned} \tag{0.2}$$

for every $1<q<p$. Here u^+,u^- are the one-sided approximate limits of u along the jump set $\mathcal{J}_u$.

Contents

1 Introduction

The so-called 'BBM formula', as presented by Bourgain, Brezis, and Mironescu in [3], provides a characterization of Sobolev functions $W^{1,q}(\Omega)$ for $1 < q < \infty$ and of functions of bounded variation $BV(\Omega)$ using double integrals and mollifiers, where $\Omega \subset \mathbb{R}^N$ is an open and bounded set with a Lipschitz boundary. The full characterization for $BV(\Omega)$ functions is attributed to Dávila [7]. A revised version of the BBM formula was presented in [5]. See also [2, 6] for the Γ-limit treatment of a problem related to the BBM formula.

Before describing it, let's recall some definitions.

Definition 1.1. (Decreasing Support Property)

Let $a \in (0, \infty]$ and $\rho_\varepsilon : (0, \infty) \to [0, \infty), \varepsilon \in (0, a)$, be a family of $\mathcal{L}^1$-measurable functions. We say that the family $\{\rho_\varepsilon\}_{\varepsilon \in (0,a)}$ has the *N-dimensional decreasing support property* if for every $\delta \in (0, \infty)$

$$\lim_{\varepsilon \to 0^+} \int_\delta^\infty \rho_\varepsilon(r) r^{N-1} dr = 0. \tag{1.1}$$

Note that by using polar coordinates (see Proposition 10.4), we obtain an alternative form for (1.1):

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\delta(0)} \rho_\varepsilon(|z|) dz = 0. \tag{1.2}$$

Definition 1.2. (Kernel)

Let $a \in (0, \infty]$. Let $\rho_\varepsilon : (0, \infty) \to [0, \infty), \varepsilon \in (0, a)$, be a family of $\mathcal{L}^1$-measurable functions. We say that the family $\{\rho_\varepsilon\}_{\varepsilon \in (0,a)}$ is a *kernel* if $\int_{\mathbb{R}^N} \rho_\varepsilon(|z|)dz = 1, \forall \varepsilon \in (0, a)$, and it has the decreasing support property as defined in Definition 1.1.

The BBM formula states that for an open and bounded set $\Omega \subset \mathbb{R}^N$ with a Lipschitz boundary, $1 < q < \infty$, and $u \in W^{1,q}(\Omega)$, for every kernel $\{\rho_\varepsilon\}_{\varepsilon \in (0,a)}$ (as defined in Definition 1.2), we have

$$\lim_{\varepsilon \to 0^+} \int_\Omega \left(\int_\Omega \rho_\varepsilon(|x-y|) \frac{|u(x) - u(y)|^q}{|x-y|^q} dy \right) dx = \hat{C}_{q,N} \|\nabla u\|^q_{L^q(\Omega)}. \tag{1.3}$$

Similarly, for $u \in BV(\Omega)$, we have

$$\lim_{\varepsilon \to 0^+} \int_\Omega \left(\int_\Omega \rho_\varepsilon(|x-y|) \frac{|u(x) - u(y)|}{|x-y|} dy \right) dx = \hat{C}_{1,N} \|Du\|(\Omega), \tag{1.4}$$

where $\hat{C}_{q,N} := \fint_{S^{N-1}} |z_1|^q d\mathcal{H}^{N-1}(z)$ for every $q \geq 1$.

In [16], the following question was investigated:

Question 1.1. What does happen if we replace the left-hand side of equation (1.3), where $q > 1$, by the following expression:

$$\lim_{\varepsilon \to 0^+} \int_\Omega \left(\int_\Omega \rho_\varepsilon(|x-y|) \frac{|u(x) - u(y)|^q}{|x-y|} dy \right) dx \tag{1.5}$$

?

Here the limit (1.5) is obtained by replacing $\frac{|u(x)-u(y)|^q}{|x-y|^q}$ in (1.3) by $\frac{|u(x)-u(y)|^q}{|x-y|}$.

Then, the following limit was studied

$$\lim_{\varepsilon \to 0^+} \int_\Omega \left(\int_{\Omega \cap B_\varepsilon(x)} \frac{1}{\mathcal{L}^N(B_1(0))\varepsilon^N} \frac{|u(x) - u(y)|^q}{|x-y|} dy \right) dx, \tag{1.6}$$

for $1 < q < \infty$, $\Omega \subset \mathbb{R}^N$ is an open set with a bounded Lipschitz boundary, and $u \in BV(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d)$. This is a particular case of the expression (1.5) with the specific choice of the kernel $\tilde{\rho}_\varepsilon(r)$ given by

$$\tilde{\rho}_\varepsilon(r) := \begin{cases} \frac{1}{\varepsilon^N \mathcal{L}^N(B_1(0))} & if \quad 0 < r < \varepsilon \\ 0 & if \quad r \geq \varepsilon \end{cases}, \quad \varepsilon \in (0, \infty). \tag{1.7}$$

Here, we refer to such a specific kernel as the 'trivial kernel' (see Definition 5.3). The space $BV^q(\Omega, \mathbb{R}^d)$ was also considered in [16]: we define $u \in BV^q(\Omega, \mathbb{R}^d)$ if and only if $u \in L^q(\Omega, \mathbb{R}^d)$ and

$$\limsup_{\varepsilon \to 0^+} \int_\Omega \left(\int_{\Omega \cap B_\varepsilon(x)} \frac{1}{\varepsilon^N} \frac{|u(x) - u(y)|^q}{|x-y|} dy \right) dx < \infty \tag{1.8}$$

holds. In [16], it was proved that the limit in (1.6) is determined solely by the jump part of the distributional derivative of u, without involving the absolutely continuous and Cantor parts:

Theorem. *(Theorem 1.1 in [16])*

Let $\Omega \subset \mathbb{R}^N$ be an open set with bounded Lipschitz boundary and let $u \in BV(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d)$. Then for every $1 < q < \infty$ we have $u \in BV^q(\Omega, \mathbb{R}^d)$ and

$$C_N \int_{\mathcal{J}_u} |u^+(x) - u^-(x)|^q \, d\mathcal{H}^{N-1}(x) = \lim_{\varepsilon \to 0^+} \int_\Omega \left(\int_{\Omega \cap B_\varepsilon(x)} \frac{1}{\varepsilon^N} \frac{|u(x) - u(y)|^q}{|x-y|} dy \right) dx, \tag{1.9}$$

where

$$C_N := \frac{1}{N} \int_{S^{N-1}} |z_1| \, d\mathcal{H}^{N-1}(z), \quad z := (z_1, ..., z_N). \tag{1.10}$$

Here $\mathcal{J}_u$ is the jump set of the function u, and u^+ and u^- are the one-sided approximate limits of u on $\mathcal{J}_u$.

Recall the definition of Besov space $B^r_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d)$:

Definition 1.3. (Besov spaces)
Let $1 \le q < \infty$ and $r \in (0,1)$. Define

$$B^r_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d) := \left\{ u \in L^q(\mathbb{R}^N, \mathbb{R}^d) : \sup_{h \in \mathbb{R}^N \setminus \{0\}} \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|^q}{|h|^{rq}} dx < \infty \right\}. \tag{1.11}$$

For an open set $\Omega \subset \mathbb{R}^N$, the local space $\left(B^r_{q,\infty}\right)_{\text{loc}}(\Omega, \mathbb{R}^d)$ is defined to be the set of all functions $u \in L^q_{\text{loc}}(\Omega, \mathbb{R}^d)$ such that for every compact $K \subset \Omega$ there exists a function $u_K \in B^r_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d)$ such that $u_K(x) = u(x)$ for $\mathcal{L}^N$-almost every $x \in K$.

The following proposition gives us a connection between Besov functions in $B^{1/q}_{q,\infty}$ and BV^q-functions.

Proposition. *(Proposition 1.1 in [16]) For $1 < q < \infty$ we have:*

$$BV^q(\mathbb{R}^N, \mathbb{R}^d) = B^{1/q}_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d). \tag{1.12}$$

Moreover, for every open set $\Omega \subset \mathbb{R}^N$ we have

$$BV^q_{loc}(\Omega, \mathbb{R}^d) = \left(B^{1/q}_{q,\infty}\right)_{loc}(\Omega, \mathbb{R}^d), \tag{1.13}$$

where the local space $BV^q_{loc}(\Omega, \mathbb{R}^d)$ is defined in a usual way.

A more general result than the proposition above was independently obtained by Brasseur in [4]. For a comprehensive introduction to Besov spaces, a recommended reference is [14].

Next, recall the notion of Gagliardo seminorm:

Definition 1.4. (Gagliardo Seminorm)

Let $1 \le q < \infty$, $E \subset \mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set, $u \in L^q(E, \mathbb{R}^d)$ and $r \in (0,1)$. The *Gagliardo seminorm* of u in E is defined by

$$[u]_{W^{r,q}(E,\mathbb{R}^d)} := \left(\int_E \int_E \frac{|u(x) - u(y)|^q}{|x-y|^{N+rq}} dx dy \right)^{\frac{1}{q}}. \tag{1.14}$$

In [15], the following result was proved: for a Lipschitz domain Ω, $q \in (1, \infty)$, $u \in BV(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d)$, and $\eta \in W^{1,1}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} \eta(z)dz = 1$, if we mollify u by setting for every $\varepsilon > 0$ and $x \in \mathbb{R}^N$

$$u_\varepsilon(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta\Big(\frac{y-x}{\varepsilon}\Big) u(y)dy, \tag{1.15}$$

then

$$\lim_{\varepsilon \to 0^+} \frac{1}{|\ln \varepsilon|} \left[u_\varepsilon\right]^q_{W^{1/q,q}(\Omega,\mathbb{R}^d)} = \tilde{C}_N \int_{\mathcal{J}_u} \Big|u^+(x) - u^-(x)\Big|^q d\mathcal{H}^{N-1}(x), \tag{1.16}$$

with an appropriate dimensional constant $\tilde{C}_N > 0$ (where u in (1.15) is assumed to be continued from Ω to $\mathbb{R}^N$ such that $u \in BV(\mathbb{R}^N, \mathbb{R}^d) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^d)$ and $\|Du\|(\partial\Omega) = 0$). It is worth noting that the particular case of (1.16) with η as the Gaussian, $q = 2$, and $\Omega = \mathbb{R}^N$ was previously proved by Figalli and Jerison in [11] for the characteristic function of a set, and by Hernández in [13] for a general function u. Combining (1.9) and (1.16), we deduce that

$$\lim_{\varepsilon \to 0^+} \frac{1}{|\ln \varepsilon|} \left[u_\varepsilon\right]^q_{W^{1/q,q}(\Omega,\mathbb{R}^d)} = \frac{\tilde{C}_N}{C_N} \lim_{\varepsilon \to 0^+} \int_\Omega \left(\int_{\Omega \cap B_\varepsilon(x)} \frac{1}{\varepsilon^N} \frac{|u(x) - u(y)|^q}{|x-y|} dy \right) dx. \tag{1.17}$$

This naturally leads us to pose the following interesting question: does (1.17) hold also for $u \in BV^q \setminus (BV \cap L^\infty)$?

Our first two main results are related to this question:

Theorem 1.1. *Let $q \in [1, \infty)$ and $r \in (0, 1)$. Suppose $u \in B^r_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d)$, $E \subset \mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set and $\eta \in W^{1,1}(\mathbb{R}^N)$. For each $\varepsilon \in (0, \infty)$ we denote*

$$u_\varepsilon(x) := \int_{\mathbb{R}^N} \eta(z) u(x - \varepsilon z) dz. \tag{1.18}$$

Then,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \eta(z)dz\right|^q \liminf_{\varepsilon \to 0^+} \int_{S^{N-1}} \int_E \chi_E(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^{rq}} dx d\mathcal{H}^{N-1}(n) \\
&\qquad \le \liminf_{\varepsilon \to 0^+} \frac{1}{|\ln \varepsilon|} \left[u_\varepsilon\right]^q_{W^{r,q}(E,\mathbb{R}^d)} \le \limsup_{\varepsilon \to 0^+} \frac{1}{|\ln \varepsilon|} \left[u_\varepsilon\right]^q_{W^{r,q}(E,\mathbb{R}^d)} \\
&\qquad \le \left|\int_{\mathbb{R}^N} \eta(z)dz\right|^q \limsup_{\varepsilon \to 0^+} \int_{S^{N-1}} \int_E \chi_E(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^{rq}} dx d\mathcal{H}^{N-1}(n).
\end{aligned} \tag{1.19}$$

Theorem 1.2. *Let $q \in [1, \infty)$, $r \in (0, 1)$. Let $u \in B^r_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d)$, $E \subset \mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set and $\eta \in W^{1,1}\left(\mathbb{R}^N\right)$. For each $\varepsilon \in (0, \infty)$ we denote $u_\varepsilon(x) := \int_{\mathbb{R}^N} \eta(z) u(x - \varepsilon z)dz$. Assume that the following limit exists:*

$$\lim_{\varepsilon \to 0^+} \int_{S^{N-1}} \int_E \chi_E(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^{rq}} dx d\mathcal{H}^{N-1}(n). \tag{1.20}$$

Then, for every kernel ρ_ε we get

$$\lim_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}\left[u_\varepsilon\right]^q_{W^{r,q}(E,\mathbb{R}^d)}$$
$$=\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q\mathcal{H}^{N-1}\left(S^{N-1}\right)\lim_{\varepsilon\to 0^+}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx$$
$$=\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q\lim_{\varepsilon\to 0^+}\int_{S^{N-1}}\int_E\chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}}dxd\mathcal{H}^{N-1}(n). \quad (1.21)$$

Our next result refers to jumps of functions in Besov spaces $B^{1/p}_{p,\infty}$, which are also functions of bounded variation. This result generalizes (1.9) (The main improvement is that we don't assume that $u\in L^\infty$):

Theorem 1.3. *Let $1<p<\infty$, $u\in BV(\mathbb{R}^N,\mathbb{R}^d)\cap B^{1/p}_{p,\infty}(\mathbb{R}^N,\mathbb{R}^d)$ and $1<q<p$. Then, for every $n\in\mathbb{R}^N$ and every Borel set $B\subset\mathbb{R}^N$ such that $\mathcal{H}^{N-1}(\partial B\cap\mathcal{J}_u)=0$, we have*

$$\lim_{\varepsilon\to 0^+}\int_B\chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx=\int_{B\cap\mathcal{J}_u}\left|u^+(x)-u^-(x)\right|^q|\nu_u(x)\cdot n|d\mathcal{H}^{N-1}(x), \quad (1.22)$$

and for every kernel ρ_ε, we have

$$\lim_{\varepsilon\to 0^+}\int_B\int_B\rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|}dydx$$
$$=\lim_{\varepsilon\to 0^+}\fint_{S^{N-1}}\int_B\chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dxd\mathcal{H}^{N-1}(n)$$
$$=\left(\fint_{S^{N-1}}|z_1|\,d\mathcal{H}^{N-1}(z)\right)\int_{\mathcal{J}_u\cap B}\left|u^+(x)-u^-(x)\right|^q d\mathcal{H}^{N-1}(x). \quad (1.23)$$

Here u^+,u^- are the one-sided approximate limits of u, ν_u is a unit normal and $\mathcal{J}_u$ is the jump set of u (see Definition 10.4).

Corollary 1.1. *Let $1<q<p<\infty$, and $u\in BV(\mathbb{R}^N,\mathbb{R}^d)\cap B^{1/p}_{p,\infty}(\mathbb{R}^N,\mathbb{R}^d)$. Let $\eta\in W^{1,1}\left(\mathbb{R}^N\right)$, and define, for each $\varepsilon\in(0,\infty)$ and $x\in\mathbb{R}^N$, the mollification $u_\varepsilon(x):=\int_{\mathbb{R}^N}\eta(z)u(x-\varepsilon z)dz$. Let $B\subset\mathbb{R}^N$ be a Borel set such that $\mathcal{H}^{N-1}(\partial B\cap\mathcal{J}_u)=0$. Let ρ_ε be a kernel.*

Then we have the following equalities:

$$\frac{1}{\mathcal{H}^{N-1}\left(S^{N-1}\right)}\lim_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}\left[u_\varepsilon\right]^q_{W^{1/q,q}(B,\mathbb{R}^d)}$$
$$=\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q\lim_{\varepsilon\to 0^+}\int_B\int_B\rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|}dydx$$
$$=\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q\lim_{\varepsilon\to 0^+}\fint_{S^{N-1}}\int_B\chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dxd\mathcal{H}^{N-1}(n)$$
$$=\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q\left(\fint_{S^{N-1}}|z_1|\,d\mathcal{H}^{N-1}(z)\right)\int_{\mathcal{J}_u\cap B}\left|u^+(x)-u^-(x)\right|^q d\mathcal{H}^{N-1}(x). \quad (1.24)$$

Notation 1.1. *Throughout the paper, we adopt the following notation: N and d are natural numbers ($N, d \in \mathbb{N}$). We denote S^{N-1} as the $(N-1)$-dimensional sphere in $\mathbb{R}^N$. The N-dimensional Lebesgue measure is denoted as $\mathcal{L}^N$, while $\mathcal{H}^{N-1}$ represents the $(N-1)$-dimensional Hausdorff measure. For an open ball in $\mathbb{R}^N$ centered at x with a radius of r, we use the notation $B_r(x)$. The characteristic function of a set E is denoted as χ_E. Furthermore, we use the notation $A \subset\subset B$ to indicate that $\bar{A}$ is a compact set and $\bar{A} \subset B$, where $\bar{A}$ represents the topological closure of A.*

1.1 Overview of Methodological Framework

In this article, we provide a structured approach to understanding function discontinuities within the framework of Besov spaces, focusing on "Approximations in Besov Spaces and Jump Detection of Besov Functions with Bounded Variation." We outline our proof strategy and main tools as follows:

Approximation Theorems in Besov Spaces: We develop approximation theorems to express functions in Besov spaces as limits of smoother, more regular functions. These approximations are crucial for analysing small-scale variations without relying on potentially discontinuous functions.

Jump Detection Mechanisms: By utilizing specialized norms and metrics for Besov spaces, we develop techniques to detect jumpssignificant discontinuities that highlight essential features in data. These tools are versatile and applicable to various function classes within Besov spaces, aligning with bounded variation constraints.

Combining Approximation and Jump Detection: In the concluding sections, we integrate our approximation strategies with jump-detection methods. This combination is vital as it enables us to quantify the accuracy and reliability of jump detection in practical applications.

Each step builds upon the previous one: approximation manages discontinuities smoothly, while jump detection identifies significant variations within a function.

2 Estimates for Gagliardo Seminorm of Mollified Besov Functions in Terms of Besov Seminorm

In this section we establish estimates for the Gagliardo seminorm of mollified Besov functions in relation to the Besov seminorm of the functions themselves, without mollification (refer to Corollary 2.1). These estimates will enable us to establish a continuity property for the upper and lower G-functionals in the next section (refer to Definition 3.1 and Lemma 3.1).

Definition 2.1. (Besov Seminorm)

Let $1 \leq q < \infty$, $r \in (0,1)$ and $E \subset \mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set. Let $u : E \to \mathbb{R}^d$ be an

$\mathcal{L}^N$-measurable function. The *Besov seminorm* of u with parameters r, q in E is defined by

$$[u]_{B^r_{q,\infty}(E,\mathbb{R}^d)} := \sup_{h\in\mathbb{R}^N\setminus\{0\}} \left(\int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^q}{|h|^{rq}} \chi_E(x+h)\chi_E(x)dx \right)^{1/q}. \tag{2.1}$$

Definition 2.2. (Mollification and Mollifier)

Let $\eta : \mathbb{R}^N \to \mathbb{R}$ be a function. For each $\varepsilon \in (0,\infty)$ we denote $\eta_{(\varepsilon)}(x) := \frac{1}{\varepsilon^N}\eta\left(\frac{x}{\varepsilon}\right), x \in \mathbb{R}^N$. The function $\eta_{(\varepsilon)}$ is called an ε*-mollifier* obtained by η. We call $\{\eta_{(\varepsilon)}\}_{\varepsilon\in(0,\infty)}$ a *family of mollifiers*. For $\eta \in L^1(\mathbb{R}^N)$, $1 \le q \le \infty$, and $u \in L^q(\mathbb{R}^N,\mathbb{R}^d)$, let us define

$$u_\varepsilon(x) := u * \eta_{(\varepsilon)}(x) = \int_{\mathbb{R}^N} \eta_{(\varepsilon)}(x-z)u(z)dz = \int_{\mathbb{R}^N} \eta(z)u(x-\varepsilon z)dz. \tag{2.2}$$

The convolution u_ε is called *mollification* of u by the family of mollifiers $\{\eta_{(\varepsilon)}\}_{\varepsilon\in(0,\infty)}$.

Lemma 2.1. *(Boundedness of Mollified Functions in Besov and Gagliardo Seminorms)*

Let $1 \le q < \infty$, $u \in L^q(\mathbb{R}^N,\mathbb{R}^d)$ and $\eta \in L^1(\mathbb{R}^N)$. Then, for every $z \in \mathbb{R}^N$ and $\varepsilon \in (0,\infty)$

$$\int_{\mathbb{R}^N} |u_\varepsilon(x) - u_\varepsilon(x+z)|^q dx \le \left(\int_{\mathbb{R}^N} |\eta(v)|dv\right)^q \int_{\mathbb{R}^N} |u(x)-u(x+z)|^q dx. \tag{2.3}$$

In particular, for every $r \in (0,1)$

$$\sup_{\varepsilon\in(0,\infty)} \left([u_\varepsilon]_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\right) \le \left(\int_{\mathbb{R}^N} |\eta(v)|dv\right) [u]_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}; \tag{2.4}$$

$$\sup_{\varepsilon\in(0,\infty)} \left([u_\varepsilon]_{W^{r,q}(\mathbb{R}^N,\mathbb{R}^d)}\right) \le \left(\int_{\mathbb{R}^N} |\eta(v)|dv\right) [u]_{W^{r,q}(\mathbb{R}^N,\mathbb{R}^d)}. \tag{2.5}$$

Proof. By (2.2), Hölder's inequality, Fubini's theorem and change of variable formula

$$\begin{aligned}
\int_{\mathbb{R}^N} |u_\varepsilon(x) - u_\varepsilon(x+z)|^q dx &= \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} \eta(v)\left(u(x-\varepsilon v) - u(x+z-\varepsilon v)\right) dv \right|^q dx \\
&\le \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |\eta(v)|\, |u(x-\varepsilon v) - u(x+z-\varepsilon v)|\, dv \right)^q dx \\
&= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |\eta(v)|^{\frac{q-1}{q}} \left(|\eta(v)|^{\frac{1}{q}} |u(x-\varepsilon v) - u(x+z-\varepsilon v)|\right) dv \right)^q dx \\
&\le \left(\int_{\mathbb{R}^N} |\eta(v)|dv\right)^{q-1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\eta(v)|\, |u(x-\varepsilon v) - u(x+z-\varepsilon v)|^q\, dvdx \\
&= \left(\int_{\mathbb{R}^N} |\eta(v)|dv\right)^{q-1} \int_{\mathbb{R}^N} |\eta(v)| \left(\int_{\mathbb{R}^N} |u(x-\varepsilon v) - u(x+z-\varepsilon v)|^q\, dx\right) dv \\
&= \left(\int_{\mathbb{R}^N} |\eta(v)|dv\right)^{q} \int_{\mathbb{R}^N} |u(y) - u(y+z)|^q\, dy.
\end{aligned} \tag{2.6}$$

Let $r \in (0,1)$. Dividing the inequality (2.6) by $|z|^{rq}, z \neq 0$, taking the supremum over $z \in \mathbb{R}^N \setminus \{0\}$ and then the supremum over $\varepsilon \in (0,\infty)$, we obtain (2.4). By Definition 1.4 (Gagliardo seminorm), change of variable formula, Fubini's theorem and (2.3) we get

$$[u_\varepsilon]^q_{W^{r,q}(\mathbb{R}^N,\mathbb{R}^d)} = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{N+rq}}dx\right)dy = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x+y)-u_\varepsilon(y)|^q}{|x|^{N+rq}}dx\right)dy$$
$$= \int_{\mathbb{R}^N}\frac{1}{|x|^{N+rq}}\left(\int_{\mathbb{R}^N}|u_\varepsilon(x+y)-u_\varepsilon(y)|^q dy\right)dx$$
$$\le \|\eta\|^q_{L^1(\mathbb{R}^N)}\int_{\mathbb{R}^N}\frac{1}{|x|^{N+rq}}\left(\int_{\mathbb{R}^N}|u(x+y)-u(y)|^q dy\right)dx = \|\eta\|^q_{L^1(\mathbb{R}^N)}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^q}{|x-y|^{N+rq}}dx\right)dy. \tag{2.7}$$

Inequality (2.5) follows from (2.7). □

Lemma 2.2. *(Estimates for Gagliardo Seminorm of Mollified Besov Functions - part 1)*
Let $1 \le q < \infty$, $r \in (0,1)$, $u \in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$ and $\eta \in W^{1,1}(\mathbb{R}^N)$. For every $\varepsilon \in (0,\infty)$ and $z \in \mathbb{R}^N \setminus \{0\}$ we denote

$$g^\varepsilon(z) := \int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{N+rq}}dx. \tag{2.8}$$

Then, for every $0 < \beta < \gamma < \infty$ it follows that

$$\int_{\mathbb{R}^N\setminus B_\gamma(0)} g^\varepsilon(z)dz \le \|\eta\|^q_{L^1(\mathbb{R}^N)}2^q\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{rq\gamma^{rq}}; \tag{2.9}$$

$$\int_{B_\gamma(0)\setminus B_\beta(0)} g^\varepsilon(z)dz \le \|\eta\|^q_{L^1(\mathbb{R}^N)}[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\mathcal{H}^{N-1}\left(S^{N-1}\right)\left(\ln(\gamma)-\ln(\beta)\right); \tag{2.10}$$

$$\int_{B_\beta(0)} g^\varepsilon(z)dz \le \left(\int_{\mathbb{R}^N}|\nabla\eta(v)|dv\right)^q 2^q\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{q-rq}\frac{\beta^{q-rq}}{\varepsilon^q}. \tag{2.11}$$

If $\varepsilon = \beta$, then we have the following alternative to (2.11) estimate:

$$\int_{B_\varepsilon(0)} g^\varepsilon(z)dz \le \|\nabla\eta\|^{q-1}_{L^1(\mathbb{R}^N,\mathbb{R}^N)}[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\left(\int_{\mathbb{R}^N}|\nabla\eta(v)|(|v|+2)^{rq}dv\right)\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{q-rq}. \tag{2.12}$$

The right hand side of (2.12) can be infinite.

Proof. By Lemma 2.1 and the convexity of the function $r \longmapsto r^q, r \in [0,\infty)$, we have

$$g^\varepsilon(z) \le \frac{\|\eta\|^q_{L^1(\mathbb{R}^N)}}{|z|^{N+rq}}\int_{\mathbb{R}^N}|u(x)-u(x+z)|^q dx \le \|\eta\|^q_{L^1(\mathbb{R}^N)}2^q\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\frac{1}{|z|^{N+rq}}. \tag{2.13}$$

Thus, by polar coordinates (refer to Proposition 10.4)

$$\int_{\mathbb{R}^N\setminus B_\gamma(0)} g^\varepsilon(z)dz \le \|\eta\|^q_{L^1(\mathbb{R}^N)}2^q\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\int_{\mathbb{R}^N\setminus B_\gamma(0)}\frac{1}{|z|^{N+rq}}dz$$
$$= \|\eta\|^q_{L^1(\mathbb{R}^N)}2^q\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{rq\gamma^{rq}}. \tag{2.14}$$

It proves (2.9). By (2.8), (2.4) and polar coordinates

$$\begin{aligned}\int_{B_\gamma(0)\setminus B_\beta(0)} g^\varepsilon(z)dz &= \int_{B_\gamma(0)\setminus B_\beta(0)} \frac{1}{|z|^N}\left(\int_{\mathbb{R}^N} \frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{rq}}dx\right)dz \\ &\le \left(\int_{\mathbb{R}^N}|\eta(v)|dv\right)^q [u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)} \int_{B_\gamma(0)\setminus B_\beta(0)} \frac{1}{|z|^N}dz \\ &= \left(\int_{\mathbb{R}^N}|\eta(v)|dv\right)^q [u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)} \mathcal{H}^{N-1}\left(S^{N-1}\right)(\ln(\gamma)-\ln(\beta)).\end{aligned} \tag{2.15}$$

It proves (2.10). We now prove (2.11). By (2.8) and Fubini's theorem

$$\int_{B_\beta(0)} g^\varepsilon(z)dz = \int_{\mathbb{R}^N}\left(\int_{B_\beta(0)} \frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{N+rq}}dz\right)dx. \tag{2.16}$$

Assume for a moment that $\eta \in C^1(\mathbb{R}^N)\cap W^{1,1}(\mathbb{R}^N)$. By (2.2), change of variable formula, the fundamental theorem of calculus, Fubini's theorem and Jensen's inequality we obtain for every $x\in\mathbb{R}^N$

$$\begin{aligned}&\int_{B_\beta(0)} \frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{N+rq}}dz \\ &= \int_{B_\beta(0)} \frac{1}{|z|^{N+rq}}\left|\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\left(\eta\left(\frac{x-y}{\varepsilon}\right)-\eta\left(\frac{x+z-y}{\varepsilon}\right)\right)u(y)dy\right|^q dz, \qquad [z=\varepsilon w] \\ &= \frac{1}{\varepsilon^{rq}}\int_{B_{\beta/\varepsilon}(0)} \frac{1}{|w|^{N+rq}}\left|\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\left(\eta\left(\frac{x-y}{\varepsilon}\right)-\eta\left(\frac{x-y}{\varepsilon}+w\right)\right)u(y)dy\right|^q dw, \qquad [y=x-\varepsilon v] \\ &= \frac{1}{\varepsilon^{rq}}\int_{B_{\beta/\varepsilon}(0)} \frac{1}{|w|^{N+rq}}\left|\int_{\mathbb{R}^N}(\eta(v)-\eta(v+w))\,u(x-\varepsilon v)dv\right|^q dw \\ &= \frac{1}{\varepsilon^{rq}}\int_{B_{\beta/\varepsilon}(0)} \frac{1}{|w|^{N+rq}}\left|\int_{\mathbb{R}^N}(\eta(v+w)-\eta(v))\,(u(x-\varepsilon v)-u(x))\,dv\right|^q dw \\ &= \frac{1}{\varepsilon^{rq}}\int_{B_{\beta/\varepsilon}(0)} \frac{1}{|w|^{N+rq}}\left|\int_{\mathbb{R}^N}\left(w\cdot\int_0^1\nabla\eta(v+tw)dt\right)(u(x-\varepsilon v)-u(x))\,dv\right|^q dw \\ &\le \frac{1}{\varepsilon^{rq}}\int_{B_{\beta/\varepsilon}(0)} \frac{1}{|w|^{N+rq-q}}\left(\int_0^1\int_{\mathbb{R}^N}|\nabla\eta(v+tw)|\,|u(x-\varepsilon v)-u(x)|\,dvdt\right)^q dw \\ &\le \frac{1}{\varepsilon^{rq}}\int_0^1\int_{B_{\beta/\varepsilon}(0)} \frac{1}{|w|^{N+rq-q}}\left(\int_{\mathbb{R}^N}|\nabla\eta(v+tw)|\,|u(x-\varepsilon v)-u(x)|\,dv\right)^q dwdt \\ &= \frac{1}{\varepsilon^{rq}}\int_0^1\int_{B_{\beta/\varepsilon}(0)} \frac{1}{|w|^{N+rq-q}}\left(\int_{\mathbb{R}^N}|\nabla\eta(v)|\,|u(x-\varepsilon(v-tw))-u(x)|\,dv\right)^q dwdt.\end{aligned} \tag{2.17}$$

In the forth equality we use $\int_{\mathbb{R}^N} (\eta(v+w)-\eta(v))\,dv = 0, w \in \mathbb{R}^N$. By Hölder's inequality

$$\left(\int_{\mathbb{R}^N} |\nabla\eta(v)|\,|u(x-\varepsilon(v-tw))-u(x)|\,dv\right)^q$$
$$= \left(\int_{\mathbb{R}^N} |\nabla\eta(v)|^{\frac{q-1}{q}} |\nabla\eta(v)|^{\frac{1}{q}}\,|u(x-\varepsilon(v-tw))-u(x)|\,dv\right)^q$$
$$\le \left(\int_{\mathbb{R}^N} |\nabla\eta(v)|dv\right)^{q-1} \int_{\mathbb{R}^N} |\nabla\eta(v)|\,|u(x-\varepsilon(v-tw))-u(x)|^q\,dv. \quad (2.18)$$

By (2.18) and Fubini's theorem

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} |\nabla\eta(v)|\,|u(x-\varepsilon(v-tw))-u(x)|\,dv\right)^q dx$$
$$\le \left(\int_{\mathbb{R}^N} |\nabla\eta(v)|dv\right)^{q-1} \int_{\mathbb{R}^N} |\nabla\eta(v)| \left(\int_{\mathbb{R}^N} |u(x-\varepsilon(v-tw))-u(x)|^q\,dx\right) dv$$
$$\le \left(\int_{\mathbb{R}^N} |\nabla\eta(v)|dv\right)^q 2^q \|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}. \quad (2.19)$$

By (2.17) and (2.19) we get

$$\int_{\mathbb{R}^N}\int_{B_\beta(0)} \frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{N+rq}}dzdx$$
$$\le \left(\int_{\mathbb{R}^N} |\nabla\eta(v)|dv\right)^q 2^q \|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)} \frac{1}{\varepsilon^{rq}} \int_{B_{\beta/\varepsilon}(0)} \frac{1}{|w|^{N+rq-q}}dw$$
$$= \left(\int_{\mathbb{R}^N} |\nabla\eta(v)|dv\right)^q 2^q \|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)} \frac{1}{\varepsilon^{rq}} \frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{q-rq}\left(\frac{\beta}{\varepsilon}\right)^{q-rq}$$
$$= \left(\int_{\mathbb{R}^N} |\nabla\eta(v)|dv\right)^q 2^q \|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)} \frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{q-rq}\frac{\beta^{q-rq}}{\varepsilon^q}. \quad (2.20)$$

It proves (2.11) in case $\eta \in C^1(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N)$. We now prove (2.12) in case $\eta \in C^1(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N)$. By (2.18) and Definition 2.1 (definition of Besov seminorm)

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} |\nabla\eta(v)|\,|u(x-\varepsilon(v-tw))-u(x)|\,dv\right)^q dx$$
$$\le \left(\int_{\mathbb{R}^N} |\nabla\eta(v)|dv\right)^{q-1} \int_{\mathbb{R}^N} |\nabla\eta(v)| \left(\int_{\mathbb{R}^N} |u(x-\varepsilon(v-tw))-u(x)|^q\,dx\right) dv$$
$$\le \varepsilon^{rq}\left(\int_{\mathbb{R}^N} |\nabla\eta(v)|dv\right)^{q-1} [u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)} \int_{\mathbb{R}^N} |\nabla\eta(v)||v-tw|^{rq}dv. \quad (2.21)$$

By (2.17) with $\varepsilon = \beta$ and (2.21)

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{B_\varepsilon(0)} \frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{N+rq}}dzdx \\
&\quad\le \|\nabla\eta\|^{q-1}_{L^1(\mathbb{R}^N,\mathbb{R}^N)}[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\int_0^1\int_{B_1(0)}\frac{1}{|w|^{N+rq-q}}\left(\int_{\mathbb{R}^N}|\nabla\eta(v)||v-tw|^{rq}dv\right)dwdt \\
&\quad\le \|\nabla\eta\|^{q-1}_{L^1(\mathbb{R}^N,\mathbb{R}^N)}[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\int_0^1\int_{B_1(0)}\frac{1}{|w|^{N+rq-q}}\left(\int_{\mathbb{R}^N}|\nabla\eta(v)|(|v|+1)^{rq}dv\right)dwdt \\
&\quad= \|\nabla\eta\|^{q-1}_{L^1(\mathbb{R}^N,\mathbb{R}^N)}[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\left(\int_{\mathbb{R}^N}|\nabla\eta(v)|(|v|+1)^{rq}dv\right)\int_{B_1(0)}\frac{1}{|w|^{N+rq-q}}dw \\
&\quad= \|\nabla\eta\|^{q-1}_{L^1(\mathbb{R}^N,\mathbb{R}^N)}[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\left(\int_{\mathbb{R}^N}|\nabla\eta(v)|(|v|+1)^{rq}dv\right)\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{q-rq}. \qquad (2.22)
\end{aligned}$$

It proves (2.12) in case $\eta \in C^1(\mathbb{R}^N)\cap W^{1,1}(\mathbb{R}^N)$. We now generalize (2.11) and (2.12) for $\eta \in W^{1,1}(\mathbb{R}^N)$. For $\eta \in W^{1,1}(\mathbb{R}^N)$, let $\eta_\delta := \eta * \gamma_{(\delta)}, \gamma_{(\delta)}(v) := \frac{1}{\delta^N}\gamma\left(\frac{v}{\delta}\right)$, where $\gamma \in C^1(\mathbb{R}^N)$, $\mathrm{supp}(\gamma)\subset B_1(0)$, $\gamma \ge 0$ and $\|\gamma\|_{L^1(\mathbb{R}^N)} = 1$. Here $\mathrm{supp}(\gamma)$ stands for the support of γ. By (2.22) we get for every $0<\delta<1$

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{B_\varepsilon(0)} \frac{|u*(\eta_\delta)_{(\varepsilon)}(x)-u*(\eta_\delta)_{(\varepsilon)}(x+z)|^q}{|z|^{N+rq}}dzdx \\
&\quad\le \|\nabla\eta_\delta\|^{q-1}_{L^1(\mathbb{R}^N,\mathbb{R}^N)}[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\left(\int_{\mathbb{R}^N}|\nabla\eta_\delta(v)|(|v|+1)^{rq}dv\right)\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{q-rq} \\
&\quad\le \|\nabla\eta\|^{q-1}_{L^1(\mathbb{R}^N,\mathbb{R}^N)}[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\left(\int_{\mathbb{R}^N}|\nabla\eta(v)|(|v|+2)^{rq}dv\right)\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{q-rq}. \qquad (2.23)
\end{aligned}$$

Let us explain the last inequality of (2.23): Since

$$\begin{aligned}
|\nabla\eta_\delta(v)| = |\nabla\eta*\gamma_{(\delta)}(v)| = \left|\int_{\mathbb{R}^N}\nabla\eta(z)\gamma_{(\delta)}(v-z)dz\right| = \left|\int_{\mathbb{R}^N}\nabla\eta(v-\delta y)\gamma(y)dy\right| \\
\le \int_{\mathbb{R}^N}|\nabla\eta(v-\delta y)|\gamma(y)dy, \qquad (2.24)
\end{aligned}$$

then we get by Fubini's theorem, change of variable formula and properties of γ

$$\int_{\mathbb{R}^N}|\nabla\eta_\delta(v)|dv \le \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}|\nabla\eta(v-\delta y)|dv\right)\gamma(y)dy = \int_{\mathbb{R}^N}|\nabla\eta(v)|dv, \qquad (2.25)$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N}|\nabla\eta_\delta(v)|(|v|+1)^{rq}dv \le \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}|\nabla\eta(v-\delta y)|(|v|+1)^{rq}dv\right)\gamma(y)dy \\
&\quad= \int_{B_1(0)}\left(\int_{\mathbb{R}^N}|\nabla\eta(v)|(|v+\delta y|+1)^{rq}dv\right)\gamma(y)dy \le \int_{\mathbb{R}^N}|\nabla\eta(v)|(|v|+2)^{rq}dv. \qquad (2.26)
\end{aligned}$$

The second inequality in (2.23) follows from (2.25) and (2.26). Note that

$$u*(\eta_\delta)_{(\varepsilon)} = u*\left(\eta*\gamma_{(\delta)}\right)_{(\varepsilon)} = u*\left(\eta_{(\varepsilon)}*\left(\gamma_{(\delta)}\right)_{(\varepsilon)}\right) = u*\left(\eta_{(\varepsilon)}*\left(\gamma_{(\varepsilon)}\right)_{(\delta)}\right) = \left(u*\eta_{(\varepsilon)}\right)*\left(\gamma_{(\varepsilon)}\right)_{(\delta)}. \tag{2.27}$$

Since $u*\eta_{(\varepsilon)} \in L^q(\mathbb{R}^N,\mathbb{R}^d)$, $\gamma_{(\varepsilon)} \in C^1_c(\mathbb{R}^N)$, $\gamma_{(\varepsilon)} \geq 0$ and $\|\gamma_{(\varepsilon)}\|_{L^1(\mathbb{R}^N)} = 1$, then the family of functions $\{u*(\eta_\delta)_{(\varepsilon)}\}_{\{0<\delta<1\}}$ converges in $L^q(\mathbb{R}^N,\mathbb{R}^d)$ to the function $u*\eta_{(\varepsilon)}$ as $\delta \to 0^+$, and hence has a subsequence converging almost everywhere. Thus, by (2.23) and Fatou's Lemma we get (2.12) for $\eta \in W^{1,1}(\mathbb{R}^N)$.

Using the same technique we get also (2.11) for $\eta \in W^{1,1}(\mathbb{R}^N)$: Let $\{u*(\eta_{\delta_n})_{(\varepsilon)}\}_{n\in\mathbb{N}}$ be a sequence converging $\mathcal{L}^N$-almost everywhere to the function $u*\eta_{(\varepsilon)}$. By (2.20) and (2.25) we have for every $n \in \mathbb{N}$

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{B_\beta(0)} &\frac{|u*(\eta_{\delta_n})_{(\varepsilon)}(x) - u*(\eta_{\delta_n})_{(\varepsilon)}(x+z)|^q}{|z|^{N+rq}}dzdx \\ &\leq \left(\int_{\mathbb{R}^N}|\nabla\eta_{\delta_n}(v)|dv\right)^q 2^q\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{q-rq}\frac{\beta^{q-rq}}{\varepsilon^q} \\ &\leq \left(\int_{\mathbb{R}^N}|\nabla\eta(v)|dv\right)^q 2^q\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{q-rq}\frac{\beta^{q-rq}}{\varepsilon^q}.\end{aligned} \tag{2.28}$$

Taking the limit as n goes to ∞ and using Fatou's lemma we get (2.11) for $\eta \in W^{1,1}(\mathbb{R}^N)$. □

Corollary 2.1. *(Estimates for Gagliardo Seminorm of Mollified Besov Functions - part 2)* *Let* $1 \leq q < \infty$, $r \in (0,1)$, $u \in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$ *and* $\eta \in W^{1,1}(\mathbb{R}^N)$. *For every* $\varepsilon \in (0,\infty)$ *and* $0 < \beta < \gamma < \infty$ *it follows that*

$$\begin{aligned}[u_\varepsilon]^q_{W^{r,q}(\mathbb{R}^N,\mathbb{R}^d)} \leq{}& \|\eta\|^q_{L^1(\mathbb{R}^N)}2^q\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{rq\gamma^{rq}} \\ &+ \|\eta\|^q_{L^1(\mathbb{R}^N)}[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\mathcal{H}^{N-1}\left(S^{N-1}\right)(\ln(\gamma)-\ln(\beta)) \\ &+ \left(\int_{\mathbb{R}^N}|\nabla\eta(v)|dv\right)^q 2^q\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{q-rq}\frac{\beta^{q-rq}}{\varepsilon^q}.\end{aligned} \tag{2.29}$$

In particular,

$$\begin{aligned}\sup_{\varepsilon\in(0,1/e)}\frac{1}{|\ln\varepsilon|}[u_\varepsilon]^q_{W^{r,q}(\mathbb{R}^N,\mathbb{R}^d)} \leq{}& \|\eta\|^q_{L^1(\mathbb{R}^N)}2^q\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{rq} \\ &+ \|\eta\|^q_{L^1(\mathbb{R}^N)}[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\mathcal{H}^{N-1}\left(S^{N-1}\right)\frac{q}{q-rq} \\ &+ \left(\int_{\mathbb{R}^N}|\nabla\eta(v)|dv\right)^q 2^q\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{q-rq}.\end{aligned} \tag{2.30}$$

Proof. By definition of Gagliardo seminorm (Definition 1.4), change of variable formula, Fubini's theorem and additivity of integral we get

$$[u_\varepsilon]^q_{W^{r,q}(\mathbb{R}^N,\mathbb{R}^d)} = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{N+rq}}dy\right)dx = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{N+rq}}dz\right)dx$$
$$= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{N+rq}}dx\right)dz = \int_{\mathbb{R}^N\setminus B_\gamma(0)} g^\varepsilon(z)dz + \int_{B_\gamma(0)\setminus B_\beta(0)} g^\varepsilon(z)dz + \int_{B_\beta(0)} g^\varepsilon(z)dz, \tag{2.31}$$

where we denote

$$g^\varepsilon(z) := \int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{N+rq}}dx, \quad z\in\mathbb{R}^N\setminus\{0\}. \tag{2.32}$$

Therefore, we get (2.29) by (2.9),(2.10),(2.11) and (2.31). Inequality (2.30) follows from (2.29) choosing $\gamma = 1$, $\beta = \varepsilon^{\frac{q}{q-rq}}$, and using that $\frac{1}{|\ln\varepsilon|} < 1$ for every $\varepsilon \in (0, 1/e)$. □

3 Continuity of G-Functionals

In this section, we define the upper and lower G-functionals (see Definition 3.1). We prove continuity properties for these functionals (see Lemma 3.1). These continuity properties, in particular, allow us to generalize results involving $\eta \in C^1_c(\mathbb{R}^N)$ to cases where $\eta \in W^{1,1}(\mathbb{R}^N)$ (refer to the proof of Corollary 7.2). Additionally, we introduce the Gagliardo constants, which are specific instances of the G-functionals where the function η is fixed (see Definition 3.2).

Definition 3.1. (The Upper and Lower G-Functionals)

Let us define for $q \in [1,\infty)$, $r \in (0,1)$ and an $\mathcal{L}^N$-measurable set $E \subset \mathbb{R}^N$ the *upper G-functional* and the *lower G-functional*, respectively, to be

$$\overline{G}_E, \underline{G}_E : B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)\times W^{1,1}(\mathbb{R}^N) \to [0,\infty),$$
$$\overline{G}_E(u,\eta) := \limsup_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}\left[u*\eta_{(\varepsilon)}\right]^q_{W^{r,q}(E,\mathbb{R}^d)}, \quad \underline{G}_E(u,\eta) := \liminf_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}\left[u*\eta_{(\varepsilon)}\right]^q_{W^{r,q}(E,\mathbb{R}^d)}. \tag{3.1}$$

Remark 3.1. (Well-definedness of the Upper and Lower G-Functionals)

The well-definedness of the upper and lower G-functionals follows immediately from (2.30). Note that $\left[u*\eta_{(\varepsilon)}\right]_{W^{r,q}(\mathbb{R}^N,\mathbb{R}^d)} \in [0,\infty)$ for every $\varepsilon\in(0,\infty)$ assuming only that $u \in L^q(\mathbb{R}^N,\mathbb{R}^d)$ and $\eta \in W^{1,1}(\mathbb{R}^N)$: One can show by Hölder's inequality that the convolution $u*\eta$ lies in $L^q(\mathbb{R}^N,\mathbb{R}^d)$, $1 \le q \le \infty$, whenever $u \in L^q(\mathbb{R}^N,\mathbb{R}^d)$ and $\eta \in L^1(\mathbb{R}^N)$. Therefore, if $1 \le q < \infty$, $u \in L^q(\mathbb{R}^N,\mathbb{R}^d)$ and $\eta \in W^{1,1}(\mathbb{R}^N)$, then $u_\varepsilon \in L^q(\mathbb{R}^N,\mathbb{R}^d)$, and it has weak derivatives $\frac{\partial}{\partial x_i}u_\varepsilon = u*\frac{\partial}{\partial x_i}\eta_{(\varepsilon)} \in L^q(\mathbb{R}^N,\mathbb{R}^d)$ for each $1\le i\le N$. Therefore, $u_\varepsilon \in W^{1,q}(\mathbb{R}^N,\mathbb{R}^d) \subset W^{r,q}(\mathbb{R}^N,\mathbb{R}^d)$, for every $r\in(0,1)$. Thus, $\left[u*\eta_{(\varepsilon)}\right]_{W^{r,q}(\mathbb{R}^N,\mathbb{R}^d)} \in [0,\infty)$ for every $\varepsilon\in(0,\infty)$.

Lemma 3.1. *(Continuity of the Upper and Lower G-Functionals)*

Let $q \in [1,\infty)$, $r\in(0,1)$ and $E\subset\mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set.

1. If $u \in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$ and $\{\eta_n\}_{n=1}^\infty \subset W^{1,1}(\mathbb{R}^N)$ is a sequence such that η_n converges to η in $W^{1,1}(\mathbb{R}^N)$, then

$$\lim_{n\to\infty}\overline{G}_E(u,\eta_n) = \overline{G}_E(u,\eta), \quad \lim_{n\to\infty}\underline{G}_E(u,\eta_n) = \underline{G}_E(u,\eta). \tag{3.2}$$

2. If $\eta \in W^{1,1}(\mathbb{R}^N)$ and $\{u_n\}_{n=1}^\infty \subset B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$ is a sequence such that u_n converges to u in $B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$, which means that $\lim_{n\to\infty}\Big(\|u-u_n\|_{L^q(\mathbb{R}^N,\mathbb{R}^d)}+[u-u_n]_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\Big)=0$, then

$$\lim_{n\to\infty}\overline{G}_E(u_n,\eta)=\overline{G}_E(u,\eta),\quad \lim_{n\to\infty}\underline{G}_E(u_n,\eta)=\underline{G}_E(u,\eta). \tag{3.3}$$

Proof. 1. For every $n\in\mathbb{N}$ we get by (2.30)

$$\sup_{\varepsilon\in(0,1/e)}\frac{1}{|\ln\varepsilon|}\Big[u*(\eta_n)_{(\varepsilon)}\Big]^q_{W^{r,q}(E,\mathbb{R}^d)}<\infty,\ \sup_{\varepsilon\in(0,1/e)}\frac{1}{|\ln\varepsilon|}\big[u*\eta_{(\varepsilon)}\big]^q_{W^{r,q}(E,\mathbb{R}^d)}<\infty. \tag{3.4}$$

Therefore, by Lemma 10.6 we get

$$\begin{aligned}&\left|\liminf_{\varepsilon\to0^+}\frac{1}{|\ln\varepsilon|^{1/q}}\Big[u*(\eta_n)_{(\varepsilon)}\Big]_{W^{r,q}(E,\mathbb{R}^d)}-\liminf_{\varepsilon\to0^+}\frac{1}{|\ln\varepsilon|^{1/q}}\big[u*\eta_{(\varepsilon)}\big]_{W^{r,q}(E,\mathbb{R}^d)}\right|\\&\qquad\le\limsup_{\varepsilon\to0^+}\frac{1}{|\ln\varepsilon|^{1/q}}\left|\Big[u*(\eta_n)_{(\varepsilon)}\Big]_{W^{r,q}(E,\mathbb{R}^d)}-\big[u*\eta_{(\varepsilon)}\big]_{W^{r,q}(E,\mathbb{R}^d)}\right|,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\left|\limsup_{\varepsilon\to0^+}\frac{1}{|\ln\varepsilon|^{1/q}}\Big[u*(\eta_n)_{(\varepsilon)}\Big]_{W^{r,q}(E,\mathbb{R}^d)}-\limsup_{\varepsilon\to0^+}\frac{1}{|\ln\varepsilon|^{1/q}}\big[u*\eta_{(\varepsilon)}\big]_{W^{r,q}(E,\mathbb{R}^d)}\right|\\&\qquad\le\limsup_{\varepsilon\to0^+}\frac{1}{|\ln\varepsilon|^{1/q}}\left|\Big[u*(\eta_n)_{(\varepsilon)}\Big]_{W^{r,q}(E,\mathbb{R}^d)}-\big[u*\eta_{(\varepsilon)}\big]_{W^{r,q}(E,\mathbb{R}^d)}\right|.\end{aligned} \tag{3.6}$$

By the triangle inequality for Gagliardo seminorm we get

$$\begin{aligned}&\limsup_{\varepsilon\to0^+}\frac{1}{|\ln\varepsilon|^{1/q}}\left|\Big[u*(\eta_n)_{(\varepsilon)}\Big]_{W^{r,q}(E,\mathbb{R}^d)}-\big[u*\eta_{(\varepsilon)}\big]_{W^{r,q}(E,\mathbb{R}^d)}\right|\\&\le\limsup_{\varepsilon\to0^+}\frac{1}{|\ln\varepsilon|^{1/q}}\Big[u*(\eta_n)_{(\varepsilon)}-u*\eta_{(\varepsilon)}\Big]_{W^{r,q}(E,\mathbb{R}^d)}=\limsup_{\varepsilon\to0^+}\frac{1}{|\ln\varepsilon|^{1/q}}\Big[u*\Big((\eta_n)_{(\varepsilon)}-\eta_{(\varepsilon)}\Big)\Big]_{W^{r,q}(E,\mathbb{R}^d)}\\&=\limsup_{\varepsilon\to0^+}\frac{1}{|\ln\varepsilon|^{1/q}}\Big[u*(\eta_n-\eta)_{(\varepsilon)}\Big]_{W^{r,q}(E,\mathbb{R}^d)}\le\limsup_{\varepsilon\to0^+}\frac{1}{|\ln\varepsilon|^{1/q}}\Big[u*(\eta_n-\eta)_{(\varepsilon)}\Big]_{W^{r,q}(\mathbb{R}^N,\mathbb{R}^d)}.\end{aligned} \tag{3.7}$$

Therefore, by (2.30)

$$\begin{aligned}&\limsup_{\varepsilon\to0^+}\frac{1}{|\ln\varepsilon|}\left|\Big[u*(\eta_n)_{(\varepsilon)}\Big]_{W^{r,q}(E,\mathbb{R}^d)}-\big[u*\eta_{(\varepsilon)}\big]_{W^{r,q}(E,\mathbb{R}^d)}\right|^q\le\limsup_{\varepsilon\to0^+}\frac{1}{|\ln\varepsilon|}\Big[u*(\eta_n-\eta)_{(\varepsilon)}\Big]^q_{W^{r,q}(\mathbb{R}^N,\mathbb{R}^d)}\\&\quad\le\sup_{\varepsilon\in(0,1/e)}\frac{1}{|\ln\varepsilon|}\Big[u*(\eta_n-\eta)_{(\varepsilon)}\Big]^q_{W^{r,q}(\mathbb{R}^N,\mathbb{R}^d)}\le\|\eta_n-\eta\|^q_{L^1(\mathbb{R}^N)}2^q\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{rq}\\&\qquad+\|\eta_n-\eta\|^q_{L^1(\mathbb{R}^N)}[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\mathcal{H}^{N-1}\left(S^{N-1}\right)\frac{q}{q-rq}\\&\qquad+\left(\int_{\mathbb{R}^N}|\nabla(\eta_n-\eta)(v)|dv\right)^q2^q\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{q-rq}.\end{aligned} \tag{3.8}$$

Taking the limit as $n \to \infty$ in (3.8) we get (3.2) from (3.5) and (3.6).
2. Replacing η_n with η and u with u_n, we get in the same way

$$\limsup_{\varepsilon\to 0^+} \frac{1}{|\ln\varepsilon|}\left|\left[u_n*\eta_{(\varepsilon)}\right]_{W^{r,q}(E,\mathbb{R}^d)}-\left[u*\eta_{(\varepsilon)}\right]_{W^{r,q}(E,\mathbb{R}^d)}\right|^q \le \|\eta\|^q_{L^1(\mathbb{R}^N)}2^q\|u-u_n\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{rq}$$
$$+\|\eta\|^q_{L^1(\mathbb{R}^N)}[u-u_n]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\mathcal{H}^{N-1}\left(S^{N-1}\right)\frac{q}{q-rq}$$
$$+\left(\int_{\mathbb{R}^N}|\nabla\eta(v)|dv\right)^q 2^q\|u-u_n\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{q-rq}. \quad (3.9)$$

Taking the limit as $n \to \infty$ we get (3.3). □

Definition 3.2. (Gagliardo constants)

Let $q \in [1,\infty)$, $r \in (0,1)$, let $E \subset \mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set, and $\eta \in W^{1,1}(\mathbb{R}^N)$. We define the (r,q) *upper Gagliardo constant of* u *in* E *with respect to* η as the quantity:

$$\limsup_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}\left[u*\eta_{(\varepsilon)}\right]^q_{W^{r,q}(E,\mathbb{R}^d)}. \quad (3.10)$$

Similarly, replacing the lim sup by the lim inf, we define the (r,q) *lower Gagliardo constant of* u *in* E *with respect to* η. If the limit exists, we refer to it as the (r,q) *Gagliardo constant of* u *in* E *with respect to* η.

4 $B^{r,q}$-Functions

In this section, we introduce the space of functions $B^{r,q}(E,\mathbb{R}^d)$ (see Definition 4.2). We establish several properties of these functions, as detailed in Propositions 4.1 and 4.2, as well as Corollary 4.1. Additionally, we prove the equivalence between the space $B^{r,q}(\mathbb{R}^N,\mathbb{R}^d)$ and the Besov space $B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$ (refer to Theorem 4.1).

Definition 4.1. ($B^{r,q}$-Seminorms)

Let us define for $r \in (0,1)$, $q \in [1,\infty)$, an $\mathcal{L}^N$-measurable set $E \subset \mathbb{R}^N$ and $\mathcal{L}^N$-measurable function $u : E \to \mathbb{R}^d$ the following two quantities:
The $B^{r,q}$*-seminorm* is defined by

$$|u|_{B^{r,q}(E,\mathbb{R}^d)} := \sup_{\varepsilon\in(0,1)}\left(\int_E \frac{1}{\varepsilon^N}\int_{E\cap B_\varepsilon(x)}\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx\right)^{\frac{1}{q}}; \quad (4.1)$$

the *upper infinitesimal* $B^{r,q}$*-seminorm* is defined by

$$[u]_{B^{r,q}(E,\mathbb{R}^d)} := \limsup_{\varepsilon\to 0^+}\left(\int_E \frac{1}{\varepsilon^N}\int_{E\cap B_\varepsilon(x)}\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx\right)^{\frac{1}{q}}. \quad (4.2)$$

Definition 4.2. (The Space $B^{r,q}$)

Let $r \in (0,1)$, $q \in [1,\infty)$ and an $\mathcal{L}^N$-measurable set $E \subset \mathbb{R}^N$. We define a set

$$B^{r,q}(E,\mathbb{R}^d) := \Big\{ u \in L^q(E,\mathbb{R}^d) : |u|_{B^{r,q}(E,\mathbb{R}^d)} < \infty \Big\}. \tag{4.3}$$

We define the local space $B^{r,q}_{\mathrm{loc}}(E,\mathbb{R}^d)$ as follows: $u \in B^{r,q}_{\mathrm{loc}}(E,\mathbb{R}^d)$ if and only if $u \in L^q_{\mathrm{loc}}(E,\mathbb{R}^d)$ and $u \in B^{r,q}(K,\mathbb{R}^d)$ for every compact set $K \subset E$.

Proposition 4.1. *(Properties of $B^{r,q}$-Seminorms)*

Let $r \in (0,1)$, $q \in [1,\infty)$ and $E \subset \mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set. Then,

1. The $B^{r,q}$-seminorm and the upper infinitesimal $B^{r,q}$-seminorm are seminorms on $B^{r,q}(E,\mathbb{R}^d)$;

2. For $u \in L^q(E,\mathbb{R}^d)$, $|u|_{B^{r,q}(E,\mathbb{R}^d)} < \infty$ if and only if $[u]_{B^{r,q}(E,\mathbb{R}^d)} < \infty$;

3. For an open set $\Omega \subset \mathbb{R}^N$, $u \in B^{r,q}_{loc}(\Omega,\mathbb{R}^d)$ if and only if for every compact set $K \subset \Omega$ we have

$$\limsup_{\varepsilon\to 0^+} \int_K \frac{1}{\varepsilon^N} \int_{B_\varepsilon(x)} \frac{|u(x)-u(y)|^q}{|x-y|^{rq}} dydx < \infty. \tag{4.4}$$

4. Let us denote:

$$\|u\|_1 := [u]_{B^{r,q}(E,\mathbb{R}^d)} + \|u\|_{L^q(E,\mathbb{R}^d)}, \quad \|u\|_2 := |u|_{B^{r,q}(E,\mathbb{R}^d)} + \|u\|_{L^q(E,\mathbb{R}^d)}. \tag{4.5}$$

Then, $\|\cdot\|_1, \|\cdot\|_2$ are norms on the space $B^{r,q}(E,\mathbb{R}^d)$[1] and $\big(B^{r,q}(E,\mathbb{R}^d), \|\cdot\|_2\big)$ is a Banach space.

Proof. 1. Let $u,v \in B^{r,q}(E,\mathbb{R}^d)$ and $a \in \mathbb{R}$. It follows immediately from definitions that $|u|_{B^{r,q}(E,\mathbb{R}^d)}, [u]_{B^{r,q}(E,\mathbb{R}^d)}$ are non-negative and homogeneous, which means that $|au|_{B^{r,q}(E,\mathbb{R}^d)} = |a||u|_{B^{r,q}(E,\mathbb{R}^d)}$ and $[au]_{B^{r,q}(E,\mathbb{R}^d)} = |a|[u]_{B^{r,q}(E,\mathbb{R}^d)}$. We have by Minkowski's inequality

$$\begin{aligned}
&\left(\int_E \int_{E\cap B_\varepsilon(x)} \frac{|(u+v)(x)-(u+v)(y)|^q}{|x-y|^{rq}} dydx\right)^{\frac{1}{q}} \\
&\qquad \le \left(\int_E \int_E \left[\chi_{B_\varepsilon(x)}(y)\frac{|u(x)-u(y)|}{|x-y|^r} + \chi_{B_\varepsilon(x)}(y)\frac{|v(x)-v(y)|}{|x-y|^r}\right]^q dydx\right)^{\frac{1}{q}} \\
&\le \left(\int_E \int_E \left[\chi_{B_\varepsilon(x)}(y)\frac{|u(x)-u(y)|}{|x-y|^r}\right]^q dydx\right)^{\frac{1}{q}} + \left(\int_E \int_E \left[\chi_{B_\varepsilon(x)}(y)\frac{|v(x)-v(y)|}{|x-y|^r}\right]^q dydx\right)^{\frac{1}{q}}.
\end{aligned} \tag{4.6}$$

The triangle inequality for $[\cdot]_{B^{r,q}(E,\mathbb{R}^d)}, |\cdot|_{B^{r,q}(E,\mathbb{R}^d)}$ follows from (4.6).

2. Since for every $\mathcal{L}^N$-measurable function $u : E \to \mathbb{R}^d$ we have $[u]_{B^{r,q}(E,\mathbb{R}^d)} \le |u|_{B^{r,q}(E,\mathbb{R}^d)}$, then the finiteness of $|u|_{B^{r,q}(E,\mathbb{R}^d)}$ implies the finiteness of $[u]_{B^{r,q}(E,\mathbb{R}^d)}$. Assume $[u]_{B^{r,q}(E,\mathbb{R}^d)} < \infty$. Then, there exists a number $0 < \varepsilon_0 < 1$ such that

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \int_E \left(\int_{E\cap B_\varepsilon(x)} \frac{1}{\varepsilon^N} \frac{|u(x)-u(y)|^q}{|x-y|^{rq}} dy\right) dx < \infty. \tag{4.7}$$

[1] As usual, on the space of equivalent classes obtained by equality $\mathcal{L}^N$-almost everywhere.

We have

$$\sup_{\varepsilon\in[\varepsilon_0,1)}\int_E\left(\int_{E\cap B_\varepsilon(x)}\frac{1}{\varepsilon^N}\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dy\right)dx\le\int_E\left(\int_{E\cap B_{\varepsilon_0}(x)}\frac{1}{\varepsilon_0^N}\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dy\right)dx$$
$$+\sup_{\varepsilon\in[\varepsilon_0,1)}\int_E\left(\int_{E\cap\left(B_\varepsilon(x)\setminus B_{\varepsilon_0}(x)\right)}\frac{1}{\varepsilon^N}\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dy\right)dx,\quad(4.8)$$

and

$$\sup_{\varepsilon\in[\varepsilon_0,1)}\int_E\left(\int_{E\cap\left(B_\varepsilon(x)\setminus B_{\varepsilon_0}(x)\right)}\frac{1}{\varepsilon^N}\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dy\right)dx$$
$$\le 2^{q-1}\frac{1}{\varepsilon_0^{N+rq}}\sup_{\varepsilon\in[\varepsilon_0,1)}\int_E\left(\int_{E\cap\left(B_\varepsilon(x)\setminus B_{\varepsilon_0}(x)\right)}|u(x)|^qdy\right)dx$$
$$+2^{q-1}\frac{1}{\varepsilon_0^{N+rq}}\sup_{\varepsilon\in[\varepsilon_0,1)}\int_E\left(\int_{E\cap\left(B_\varepsilon(x)\setminus B_{\varepsilon_0}(x)\right)}|u(y)|^qdy\right)dx\le 2^q\frac{1}{\varepsilon_0^{N+rq}}\mathcal{L}^N(B_1(0))\|u\|^q_{L^q(E,\mathbb{R}^d)}<\infty.\quad(4.9)$$

Thus, $|u|_{B^{r,q}(E,\mathbb{R}^d)}<\infty$.

3. If for a compact set $K\subset\mathbb{R}^N$ we have (4.4), then $[u]_{B^{r,q}(K,\mathbb{R}^d)}<\infty$ and by item 2 we have also $|u|_{B^{r,q}(K,\mathbb{R}^d)}<\infty$, hence $u\in B^{r,q}(K,\mathbb{R}^d)$. For the opposite implication, let $u\in B^{r,q}_{\text{loc}}(\Omega,\mathbb{R}^d)$ and $K\subset\Omega$ be a compact set. Let $\Omega_0\subset\subset\Omega$ be an open set containing K. Since Ω_0 is open, then we have for every small enough $\varepsilon\in(0,\infty)$ that $K+B_\varepsilon(0)\subset\Omega_0$. Hence, by item 2 we have

$$\limsup_{\varepsilon\to0^+}\int_K\frac{1}{\varepsilon^N}\int_{B_\varepsilon(x)}\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx\le\limsup_{\varepsilon\to0^+}\int_{\Omega_0}\frac{1}{\varepsilon^N}\int_{B_\varepsilon(x)\cap\Omega_0}\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx<\infty.\quad(4.10)$$

4. Since by assertion 1, $[\cdot]_{B^{r,q}(E,\mathbb{R}^d)},|\cdot|_{B^{r,q}(E,\mathbb{R}^d)}$ are seminorms and $\|\cdot\|_{L^q(E,\mathbb{R}^d)}$ is a norm, then $\|\cdot\|_1,\|\cdot\|_2$ are norms on the space $B^{r,q}(E,\mathbb{R}^d)$. The space $\left(B^{r,q}(E,\mathbb{R}^d),\|\cdot\|_2\right)$ is complete: let $\{u_n\}_{n=1}^\infty\subset B^{r,q}(E,\mathbb{R}^d)$ be a Cauchy sequence. Then, it is also a Cauchy sequence in $L^q(E,\mathbb{R}^d)$, so, since $L^q(E,\mathbb{R}^d)$ is complete, there exists a function $u\in L^q(E,\mathbb{R}^d)$ such that $\{u_n\}_{n=1}^\infty$ converges to u in $L^q(E,\mathbb{R}^d)$. Let $\{u_{n_k}\}_{n=1}^\infty$ be a subsequence that converges to u also $\mathcal{L}^N$-almost everywhere. Since $\{u_{n_k}\}_{n=1}^\infty$ is a Cauchy sequence, then it is bounded, so there exists a number M such that $|u_{n_k}|_{B^{r,q}(E,\mathbb{R}^d)}\le M$ for every $k\in\mathbb{N}$. By Fatou's lemma we get

$$|u|_{B^{r,q}(E,\mathbb{R}^d)}\le\liminf_{k\to\infty}\sup_{\varepsilon\in(0,1)}\left(\int_E\frac{1}{\varepsilon^N}\int_{E\cap B_\varepsilon(x)}\frac{|u_{n_k}(x)-u_{n_k}(y)|^q}{|x-y|^{rq}}dydx\right)^{\frac{1}{q}}\le M.\quad(4.11)$$

Thus, $u\in B^{r,q}(E,\mathbb{R}^d)$. Let us prove that u_n converges to u in $B^{r,q}(E,\mathbb{R}^d)$. Let $\xi>0$. Since $\{u_n\}_{n=1}^\infty$ is a Cauchy sequence in $B^{r,q}(E,\mathbb{R}^d)$, there exists $N_0\in\mathbb{N}$ such that for every $n,k>N_0$, we get

$$\xi\ \ge\ |u_n-u_k|_{B^{r,q}(E,\mathbb{R}^d)}\ =\ \sup_{\varepsilon\in(0,1)}\int_E\frac{1}{\varepsilon^N}\int_{E\cap B_\varepsilon(x)}\frac{|(u_n-u_k)(x)-(u_n-u_k)(y)|^q}{|x-y|^{rq}}dydx.\quad(4.12)$$

Therefore, for every $n > N_0$, we obtain

$$
\begin{aligned}
\xi &\geq \liminf_{k\to\infty}\left(\sup_{\varepsilon\in(0,1)}\int_E \frac{1}{\varepsilon^N}\int_{E\cap B_\varepsilon(x)} \frac{|(u_n-u_{n_k})(x)-(u_n-u_{n_k})(y)|^q}{|x-y|^{rq}}dydx\right)\\
&\geq \sup_{\varepsilon\in(0,1)}\left(\liminf_{k\to\infty}\int_E \frac{1}{\varepsilon^N}\int_{E\cap B_\varepsilon(x)} \frac{|(u_n-u_{n_k})(x)-(u_n-u_{n_k})(y)|^q}{|x-y|^{rq}}dydx\right)\\
&\geq \sup_{\varepsilon\in(0,1)}\left(\int_E \frac{1}{\varepsilon^N}\int_{E\cap B_\varepsilon(x)} \frac{|(u_n-u)(x)-(u_n-u)(y)|^q}{|x-y|^{rq}}dydx\right) = |u_n-u|_{B^{r,q}(E,\mathbb{R}^d)}. \quad (4.13)
\end{aligned}
$$

□

Proposition 4.2. *(Continuous Embedding of $W^{r,q}$ into $B^{r,q}$, and Negligibility of the Upper Infinitesimal $B^{r,q}$-Seminorm for Sobolev Functions)*

Let $r \in (0,1)$, $q \in [1,\infty)$ and $E \subset \mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set. Then, the space $W^{r,q}(E,\mathbb{R}^d)$ with the norm $[\cdot]_{W^{r,q}(E,\mathbb{R}^d)} + \|\cdot\|_{L^q(E,\mathbb{R}^d)}$ is continuously embedded in the space $B^{r,q}(E,\mathbb{R}^d)$ with the norm $\|\cdot\|_2$ defined in (4.5). Moreover, $[u]_{B^{r,q}(E,\mathbb{R}^d)} = 0$ for every $u \in W^{r,q}(E,\mathbb{R}^d)$.

Proof. We have for every $\varepsilon \in (0,\infty)$ and $u \in W^{r,q}(E,\mathbb{R}^d)$

$$
\begin{aligned}
\infty > \int_E\int_E \frac{|u(x)-u(y)|^q}{|x-y|^{N+rq}}dydx &\geq \int_E\left(\int_{E\cap B_\varepsilon(x)} \frac{|u(x)-u(y)|^q}{|x-y|^{N+rq}}dy\right)dx\\
&\geq \int_E \frac{1}{\varepsilon^N}\int_{E\cap B_\varepsilon(x)} \frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx. \quad (4.14)
\end{aligned}
$$

By (4.14) we conclude that $W^{r,q}(E,\mathbb{R}^d)$ is continuously embedded in $B^{r,q}(E,\mathbb{R}^d)$. Notice that

$$
\begin{aligned}
&\int_E\left(\sup_{\varepsilon\in(0,\infty)}\int_{E\cap B_\varepsilon(x)} \frac{|u(x)-u(y)|^q}{|x-y|^{N+rq}}dy\right)dx \leq \int_E\int_E \frac{|u(x)-u(y)|^q}{|x-y|^{N+rq}}dydx < \infty;\\
&\lim_{\varepsilon\to0^+}\int_{E\cap B_\varepsilon(x)} \frac{|u(x)-u(y)|^q}{|x-y|^{N+rq}}dy = 0, \quad \text{for } \mathcal{L}^N\text{-almost every } x \in E. \quad (4.15)
\end{aligned}
$$

Therefore, by Dominated Convergence Theorem

$$
\begin{aligned}
[u]_{B^{r,q}(E,\mathbb{R}^d)} &\leq \limsup_{\varepsilon\to0^+}\int_E\int_{E\cap B_\varepsilon(x)} \frac{|u(x)-u(y)|^q}{|x-y|^{N+rq}}dydx\\
&= \int_E \lim_{\varepsilon\to0^+}\left(\int_{E\cap B_\varepsilon(x)} \frac{|u(x)-u(y)|^q}{|x-y|^{N+rq}}dy\right)dx = 0. \quad (4.16)
\end{aligned}
$$

□

Corollary 4.1. *(Non-equivalence of the Seminorms $|\cdot|_{B^{r,q}}, [\cdot]_{B^{r,q}}$)*

Let $r \in (0,1)$, $q \in [1,\infty)$ and $\Omega \subset \mathbb{R}^N$ be an open set which is not empty. Let $\|\cdot\|_1, \|\cdot\|_2$ be the norms defined in (4.5). Then, the space $\big(B^{r,q}(\Omega,\mathbb{R}^d), \|\cdot\|_1\big)$ is not a Banach space. In particular, the seminorms $|\cdot|_{B^{r,q}}, [\cdot]_{B^{r,q}}$ are not equivalent.

Proof. Let $u \in L^q(\Omega,\mathbb{R}^d)$ such that $[u]_{B^{r,q}(\Omega,\mathbb{R}^d)} = \infty$. Let $\{u_n\}_{n=1}^\infty \subset C_c^1(\Omega,\mathbb{R}^d)$ be a sequence which converges to u in $L^q(\Omega,\mathbb{R}^d)$, so it is also a Cauchy sequence in $L^q(\Omega,\mathbb{R}^d)$. Therefore, by Proposition 4.2 we have that $\{u_n\}_{n=1}^\infty \subset W^{r,q}(\Omega,\mathbb{R}^d) \subset B^{r,q}(\Omega,\mathbb{R}^d)$ and this sequence is also a Cauchy sequence with respect to the norm $\|\cdot\|_1$ because $\|u_n-u_k\|_1 = \|u_n-u_k\|_{L^q(\Omega,\mathbb{R}^d)}$ for every $k,n\in\mathbb{N}$. Thus, $\{u_n\}_{n=1}^\infty$ is a Cauchy sequence in the space $\left(B^{r,q}(\Omega,\mathbb{R}^d),\|\cdot\|_1\right)$ which does not have a limit in the space. Since by item 4 of Proposition 4.1 $\left(B^{r,q}(\Omega,\mathbb{R}^d),\|\cdot\|_2\right)$ is a Banach space, then the norms $\|\cdot\|_1,\|\cdot\|_2$ are not equivalent and so as the seminorms $|\cdot|_{B^{r,q}},[\cdot]_{B^{r,q}}$. □

Theorem 4.1. *(Equivalence Between $B^{r,q}$-Spaces and Besov Spaces $B^r_{q,\infty}$)*

Let $r\in(0,1)$, $q\in[1,\infty)$. Then,

$$B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d) = B^{r,q}(\mathbb{R}^N,\mathbb{R}^d), \tag{4.17}$$

and for every open set $\Omega\subset\mathbb{R}^N$

$$\left(B^r_{q,\infty}\right)_{loc}(\Omega,\mathbb{R}^d) = B^{r,q}_{loc}(\Omega,\mathbb{R}^d). \tag{4.18}$$

Proof. Assume that $u\in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$. Then, for every $\varepsilon\in(0,\infty)$

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{1}{\varepsilon^N}\int_{B_\varepsilon(x)}\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx &= \int_{\mathbb{R}^N}\int_{B_1(0)}\frac{|u(x)-u(x+\varepsilon z)|^q}{|\varepsilon z|^{rq}}dzdx\\ &= \int_{B_1(0)}\left(\int_{\mathbb{R}^N}\frac{|u(x)-u(x+\varepsilon z)|^q}{|\varepsilon z|^{rq}}dx\right)dz \le [u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\mathcal{L}^N\left(B_1(0)\right)<\infty. \end{aligned}\tag{4.19}$$

Thus, we get

$$\sup_{\varepsilon\in(0,\infty)}\int_{\mathbb{R}^N}\frac{1}{\varepsilon^N}\int_{B_\varepsilon(x)}\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx \le [u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\mathcal{L}^N\left(B_1(0)\right)<\infty. \tag{4.20}$$

Thus, $u\in B^{r,q}(\mathbb{R}^N,\mathbb{R}^d)$. Assume that $u\in B^{r,q}(\mathbb{R}^N,\mathbb{R}^d)$.
Step 1: for every $h_1,h_2\in\mathbb{R}^N$ such that $0\notin\{h_1,h_2,h_1+h_2\}$ we have

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{|u(x+(h_1+h_2))-u(x)|^q}{|h_1+h_2|^{rq}}dx &= \int_{\mathbb{R}^N}\frac{|\left(u(x+(h_1+h_2))-u(x+h_1)\right)+\left(u(x+h_1)-u(x)\right)|^q}{|h_1+h_2|^{rq}}dx\\ &\le 2^{q-1}\int_{\mathbb{R}^N}\frac{|u(x+(h_1+h_2))-u(x+h_1)|^q}{|h_1+h_2|^{rq}}dx + 2^{q-1}\int_{\mathbb{R}^N}\frac{|u(x+h_1)-u(x)|^q}{|h_1+h_2|^{rq}}dx\\ &= \frac{2^{q-1}|h_2|^{rq}}{|h_1+h_2|^{rq}}\int_{\mathbb{R}^N}\frac{|u(x+h_2)-u(x)|^q}{|h_2|^{rq}}dx + \frac{2^{q-1}|h_1|^{rq}}{|h_1+h_2|^{rq}}\int_{\mathbb{R}^N}\frac{|u(x+h_1)-u(x)|^q}{|h_1|^{rq}}dx. \end{aligned}\tag{4.21}$$

Step 2: let $\nu\in S^{N-1}$, $\varepsilon\in(0,\infty)$ and $z\in\mathbb{R}^N$. Denote $h_1:=\varepsilon z$ and $h_2:=\varepsilon(\nu-z)$. Note that $B_{1/2}\left(\frac{1}{2}\nu\right)\subset B_1(0)\cap B_1(\nu)$. For $z\in B_{1/2}\left(\frac{1}{2}\nu\right)$, we get by (4.21)

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon\nu)-u(x)|^q}{\varepsilon^{rq}}dx\\ &\le 2^{q-1}|\nu-z|^{rq}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon(\nu-z))-u(x)|^q}{|\varepsilon(\nu-z)|^{rq}}dx + 2^{q-1}|z|^{rq}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon z)-u(x)|^q}{|\varepsilon z|^{rq}}dx\\ &\le 2^{q-1}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon(\nu-z))-u(x)|^q}{|\varepsilon(\nu-z)|^{rq}}dx + 2^{q-1}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon z)-u(x)|^q}{|\varepsilon z|^{rq}}dx. \end{aligned}\tag{4.22}$$

Taking the average with respect to dz on the ball $B_{1/2}\left(\frac{1}{2}\nu\right)$ of both sides of the inequality (4.22), we get

$$\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon\nu)-u(x)|^q}{\varepsilon^{rq}}dx \le \frac{2^{q-1}}{\mathcal{L}^N\left(B_{1/2}\left(\frac{1}{2}\nu\right)\right)}\int_{B_{1/2}\left(\frac{1}{2}\nu\right)}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon(\nu-z))-u(x)|^q}{|\varepsilon(\nu-z)|^{rq}}dxdz$$
$$+\frac{2^{q-1}}{\mathcal{L}^N\left(B_{1/2}\left(\frac{1}{2}\nu\right)\right)}\int_{B_{1/2}\left(\frac{1}{2}\nu\right)}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon z)-u(x)|^q}{|\varepsilon z|^{rq}}dxdz$$
$$\le \frac{2^{N+q-1}}{\mathcal{L}^N\left(B_1(0)\right)}\int_{B_1(\nu)}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon(\nu-z))-u(x)|^q}{|\varepsilon(\nu-z)|^{rq}}dxdz$$
$$+\frac{2^{N+q-1}}{\mathcal{L}^N\left(B_1(0)\right)}\int_{B_1(0)}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon z)-u(x)|^q}{|\varepsilon z|^{rq}}dxdz = \frac{2^{N+q}}{\mathcal{L}^N\left(B_1(0)\right)}\int_{B_1(0)}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon z)-u(x)|^q}{|\varepsilon z|^{rq}}dxdz. \tag{4.23}$$

Therefore, since $u\in B^{r,q}(\mathbb{R}^N,\mathbb{R}^d)$, then

$$\limsup_{\varepsilon\to 0^+}\left(\sup_{\nu\in S^{N-1}}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon\nu)-u(x)|^q}{\varepsilon^{rq}}dx\right)\le\frac{2^{N+q}}{\mathcal{L}^N\left(B_1(0)\right)}\limsup_{\varepsilon\to 0^+}\int_{B_1(0)}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon z)-u(x)|^q}{|\varepsilon z|^{rq}}dxdz$$
$$=\frac{2^{N+q}}{\mathcal{L}^N\left(B_1(0)\right)}\limsup_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\frac{1}{\varepsilon^N}\int_{B_\varepsilon(x)}\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx<\infty. \tag{4.24}$$

Step 3: notice that

$$[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}=\sup_{h\in\mathbb{R}^N\setminus\{0\}}\int_{\mathbb{R}^N}\frac{|u(x+h)-u(x)|^q}{|h|^{rq}}dx$$
$$=\sup_{\varepsilon\in(0,\infty)}\left(\sup_{|h|=\varepsilon}\int_{\mathbb{R}^N}\frac{|u(x+h)-u(x)|^q}{\varepsilon^{rq}}dx\right)=\sup_{\varepsilon\in(0,\infty)}\left(\sup_{h\in S^{N-1}}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon h)-u(x)|^q}{\varepsilon^{rq}}dx\right). \tag{4.25}$$

By (4.24) there exists $\delta\in(0,\infty)$ such that

$$\sup_{\varepsilon\in(0,\delta)}\left(\sup_{h\in S^{N-1}}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon h)-u(x)|^q}{\varepsilon^{rq}}dx\right)<\infty. \tag{4.26}$$

Therefore, by (4.25) and (4.26) we get

$$[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\le\sup_{\varepsilon\in(0,\delta)}\left(\sup_{h\in S^{N-1}}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon h)-u(x)|^q}{\varepsilon^{rq}}dx\right)+\sup_{\varepsilon\in[\delta,\infty)}\left(\sup_{h\in S^{N-1}}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon h)-u(x)|^q}{\varepsilon^{rq}}dx\right)$$
$$\le\sup_{\varepsilon\in(0,\delta)}\left(\sup_{h\in S^{N-1}}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon h)-u(x)|^q}{\varepsilon^{rq}}dx\right)+\frac{2^q}{\delta^{rq}}\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}<\infty. \tag{4.27}$$

Thus, $u\in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$. It completes the proof of (4.17). We will derive the local case (4.18) from the global one (4.17).
Assume now that $u\in\left(B^r_{q,\infty}\right)_{\text{loc}}(\Omega,\mathbb{R}^d)$. Let $K\subset\Omega$ be a compact set and let $\Omega_0\subset\subset\Omega$ be an open

set such that $K \subset \Omega_0$. Let $g \in B^r_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d)$ be such that $u = g$ $\mathcal{L}^N$-almost everywhere in Ω_0. We have for $\varepsilon \in (0, \infty)$ such that $K + B_\varepsilon(0) \subset \Omega_0$

$$\int_K \frac{1}{\varepsilon^N} \int_{B_\varepsilon(x)} \frac{|u(x)-u(y)|^q}{|x-y|^{rq}} dydx = \int_K \frac{1}{\varepsilon^N} \int_{B_\varepsilon(x)} \frac{|g(x)-g(y)|^q}{|x-y|^{rq}} dydx \\ \leq \int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} \int_{B_\varepsilon(x)} \frac{|g(x)-g(y)|^q}{|x-y|^{rq}} dydx. \quad (4.28)$$

By (4.17), we get

$$\limsup_{\varepsilon\to 0^+} \int_K \frac{1}{\varepsilon^N} \int_{B_\varepsilon(x)} \frac{|u(x)-u(y)|^q}{|x-y|^{rq}} dydx \leq \limsup_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} \int_{B_\varepsilon(x)} \frac{|g(x)-g(y)|^q}{|x-y|^{rq}} dydx < \infty. \quad (4.29)$$

By item 3 of Proposition 4.1 we conclude that $u \in B^{r,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^d)$. Assume that $u \in B^{r,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^d)$. Let $K \subset \Omega$ be a compact set and let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ be open sets such that $K \subset \Omega_0$. Let $f \in C^{0,r}_c(\mathbb{R}^N)$[2] which is constant 1 on K and constant 0 outside Ω_0. We have for $g := uf$ and $\varepsilon \in (0, \infty)$ such that $\mathbb{R}^N \setminus \Omega_1 + B_\varepsilon(0) \subset \mathbb{R}^N \setminus \Omega_0$

$$\int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} \int_{B_\varepsilon(x)} \frac{|g(x)-g(y)|^q}{|x-y|^{rq}} dydx \\ = \int_{\Omega_1} \frac{1}{\varepsilon^N} \int_{B_\varepsilon(x)} \frac{|u(x)f(x)-u(x)f(y)+u(x)f(y)-u(y)f(y)|^q}{|x-y|^{rq}} dydx \\ \leq 2^{q-1} \int_{\Omega_1} \frac{|u(x)|^q}{\varepsilon^N} \int_{B_\varepsilon(x)} \frac{|f(x)-f(y)|^q}{|x-y|^{rq}} dydx + 2^{q-1} \|f\|^q_{L^\infty(\mathbb{R}^N)} \int_{\Omega_1} \frac{1}{\varepsilon^N} \int_{B_\varepsilon(x)} \frac{|u(x)-u(y)|^q}{|x-y|^{rq}} dydx \\ \leq 2^{q-1} C^q \mathcal{L}^N(B_1(0)) \int_{\Omega_1} |u(x)|^q dx + 2^{q-1} \|f\|^q_{L^\infty(\mathbb{R}^N)} \int_{\Omega_1} \frac{1}{\varepsilon^N} \int_{B_\varepsilon(x)} \frac{|u(x)-u(y)|^q}{|x-y|^{rq}} dydx, \quad (4.30)$$

where C is a number such that $|f(x) - f(y)| \leq C|x-y|^r$ for $x, y \in \mathbb{R}^N$. Therefore,

$$\limsup_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} \int_{B_\varepsilon(x)} \frac{|g(x)-g(y)|^q}{|x-y|^{rq}} dydx < \infty, \quad (4.31)$$

and by (4.17) we conclude that $g \in B^r_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d)$. Thus, $u \in \left(B^r_{q,\infty}\right)_{\mathrm{loc}}(\Omega, \mathbb{R}^d)$. □

5 Kernels

In this section, we analyse the concept of a kernel (see Definition 1.2). Additionally, we discuss specific kernels, namely the logarithmic and trivial kernels (see Definitions 5.2 and 5.3), and establish their properties.

Definition 5.1. (Compact Support Property)

Let $a \in (0, \infty]$ and $\rho_\varepsilon : (0, \infty) \to [0, \infty), \varepsilon \in (0, a)$, be a family of functions. We say that the family $\{\rho_\varepsilon\}_{\varepsilon\in(0,a)}$ has the *compact support property* if for every $r > 0$ there exists $\delta_r > 0$ such that $\mathrm{supp}(\rho_\varepsilon) \subset B_r(0)$ for every $\varepsilon \in (0, \delta_r)$.

[2]The space of Hlder continuous functions with exponent r and compact support.

Note that, if the functions $\{\rho_\varepsilon\}_{\varepsilon\in(0,a)}$ are $\mathcal{L}^1$-measurable, the compact support property implies the decreasing support property (see Definition 1.1).

Definition 5.2. (Logarithmic Kernel)

For every $\varepsilon \in (0, 1/e)$ and $\omega \in (0,1)$ let us define a function

$$\rho_{\varepsilon,\omega}(r) := \frac{1}{\mathcal{H}^{N-1}(S^{N-1})\left(|\ln\varepsilon| - |\ln R_{\varepsilon,\omega}|\right)}\frac{1}{r^N}\chi_{[\varepsilon,R_{\varepsilon,\omega})}(r), \quad \rho_{\varepsilon,\omega} : (0,\infty)\to[0,\infty), \tag{5.1}$$

where $R_{\varepsilon,\omega} := \frac{1}{|\ln\varepsilon|^\omega}$, and $\chi_{[\varepsilon,R_{\varepsilon,\omega})}$ is the characteristic function of the interval $[\varepsilon, R_{\varepsilon,\omega})$. We call the family of functions $\{\rho_{\varepsilon,\omega}\}_{\varepsilon\in(0,1/e)}$ the *N-dimensional logarithmic kernel*, or just *logarithmic kernel*.

Remark 5.1. (Comments about the Logarithmic Kernel)

1. Note that for every $\varepsilon, \omega \in (0,1)$ we have $\varepsilon < R_{\varepsilon,\omega}$: $\varepsilon < R_{\varepsilon,\omega}$ if and only if $\varepsilon < \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)^\omega}$ if and only if $\varepsilon^{1/\omega}\ln\left(\frac{1}{\varepsilon}\right) < 1$. The last inequality holds since $\ln(z) < z$ for every $z\in(0,\infty)$.
2. Note that for $\varepsilon\in(0,1)$, $\ln R_{\varepsilon,\omega} = -\omega\ln\left(\ln\left(\frac{1}{\varepsilon}\right)\right)$, and for $\varepsilon\in(0,1/e)$, $|\ln R_{\varepsilon,\omega}| = \omega\ln\left(\ln\left(\frac{1}{\varepsilon}\right)\right)$, so $|\ln\varepsilon| - |\ln R_{\varepsilon,\omega}| = \ln\left(\frac{1}{\varepsilon}\right) - \omega\ln\left(\ln\left(\frac{1}{\varepsilon}\right)\right) = \ln\left(\frac{1}{\varepsilon}\right) + \ln\left(\frac{1}{\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^\omega}\right) = \ln\left(\frac{1}{\varepsilon\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^\omega}\right) > 0$. The last inequality holds since $\varepsilon\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^\omega < 1$.
3. By L'hopital's rule we have $\lim_{x\to\infty}\frac{\ln(\ln(x))}{\ln(x)} = 0$, so we get by definition of $R_{\varepsilon,\omega}$

$$\lim_{\varepsilon\to0^+}\frac{|\ln R_{\varepsilon,\omega}|}{|\ln\varepsilon|} = \lim_{\varepsilon\to0^+}\frac{\omega\ln\left(\ln\left(\frac{1}{\varepsilon}\right)\right)}{\ln\left(\frac{1}{\varepsilon}\right)} = 0. \tag{5.2}$$

Proposition 5.1. *(Properties of the Logarithmic Kernel)*

For $\omega\in(0,1)$, the logarithmic kernel $\{\rho_{\varepsilon,\omega}\}_{\varepsilon\in(0,1/e)}$ has the following properties:
1. The logarithmic kernel is a kernel that also possesses the compact support property;
2. $\lim_{\varepsilon\to0^+}\varepsilon^\alpha\int_{\mathbb{R}^N}\frac{\rho_{\varepsilon,\omega}(|z|)}{|z|^\alpha}dz = 0, \ \forall\alpha\in(0,\infty)$.

Proof. 1. It is easy to see that for every $\varepsilon\in(0,1/e)$, $\omega\in(0,1)$, the function $\rho_{\varepsilon,\omega}$ is $\mathcal{L}^1$-measurable. By polar coordinates

$$\int_{\mathbb{R}^N}\frac{1}{|z|^N}\chi_{[\varepsilon,R_{\varepsilon,\omega})}(|z|)dz = \int_{B_{R_{\varepsilon,\omega}}(0)\setminus B_\varepsilon(0)}\frac{1}{|z|^N}dz = \int_\varepsilon^{R_{\varepsilon,\omega}}\left(\int_{\partial B_r(0)}\frac{1}{|z|^N}d\mathcal{H}^{N-1}(z)\right)dr$$
$$= \int_\varepsilon^{R_{\varepsilon,\omega}}\frac{1}{r^N}r^{N-1}\mathcal{H}^{N-1}\left(S^{N-1}\right)dr = \mathcal{H}^{N-1}\left(S^{N-1}\right)(\ln R_{\varepsilon,\omega} - \ln\varepsilon) = \mathcal{H}^{N-1}\left(S^{N-1}\right)(|\ln\varepsilon| - |\ln R_{\varepsilon,\omega}|). \tag{5.3}$$

Note that, since $\varepsilon\in(0,1/e)$, then $\varepsilon, R_{\varepsilon,\omega} < 1$, and therefore $-\ln\varepsilon = |\ln\varepsilon|$ and $\ln R_{\varepsilon,\omega} = -|\ln R_{\varepsilon,\omega}|$, so $\ln R_{\varepsilon,\omega} - \ln\varepsilon = |\ln\varepsilon| - |\ln R_{\varepsilon,\omega}|$. Thus, $\int_{\mathbb{R}^N}\rho_{\varepsilon,\omega}(|z|)dz = 1$. The logarithmic kernel satisfies the compact support property: for every $r\in(0,\infty)$ let $\delta_r := e^{-\frac{1}{r^{1/\omega}}}$. Note that if $\varepsilon\in(0,\delta_r)$, then $R_{\varepsilon,\omega} < r$, so $\operatorname{supp}(\rho_{\varepsilon,\omega}) \subset B_{R_{\varepsilon,\omega}}(0) \subset B_r(0)$, where $R_{\varepsilon,\omega} := \frac{1}{|\ln\varepsilon|^\omega}$.

2. By polar coordinates

$$\int_{\mathbb{R}^N} \frac{\rho_{\varepsilon,\omega}(|z|)}{|z|^\alpha}dz = \frac{1}{\mathcal{H}^{N-1}\left(S^{N-1}\right)\left(|\ln\varepsilon| - |\ln R_{\varepsilon,\omega}|\right)} \int_{B_{R_{\varepsilon,\omega}}(0)\setminus B_\varepsilon(0)} \frac{1}{|z|^{N+\alpha}}dz$$

$$= \frac{1}{\mathcal{H}^{N-1}\left(S^{N-1}\right)\left(|\ln\varepsilon| - |\ln R_{\varepsilon,\omega}|\right)} \int_\varepsilon^{R_{\varepsilon,\omega}} \frac{1}{r^{N+\alpha}} r^{N-1}\mathcal{H}^{N-1}\left(S^{N-1}\right)dr$$

$$= \frac{1}{\left(|\ln\varepsilon| - |\ln R_{\varepsilon,\omega}|\right)}\frac{1}{\alpha}\left(\frac{1}{\varepsilon^\alpha} - \frac{1}{R^\alpha_{\varepsilon,\omega}}\right) = \frac{1}{\alpha\varepsilon^\alpha}\left(\frac{1-\frac{\varepsilon^\alpha}{R^\alpha_{\varepsilon,\omega}}}{|\ln\varepsilon| - |\ln R_{\varepsilon,\omega}|}\right) = \frac{1}{\alpha\varepsilon^\alpha}\left(\frac{1-\varepsilon^\alpha|\ln\varepsilon|^{\alpha\omega}}{|\ln\varepsilon| - \omega\ln|\ln\varepsilon|}\right). \tag{5.4}$$

Hence,

$$\lim_{\varepsilon\to 0^+}\varepsilon^\alpha\int_{\mathbb{R}^N}\frac{\rho_{\varepsilon,\omega}(|z|)}{|z|^\alpha}dz = \frac{1}{\alpha}\lim_{\varepsilon\to 0^+}\frac{1-\varepsilon^\alpha|\ln\varepsilon|^{\alpha\omega}}{|\ln\varepsilon| - \omega\ln|\ln\varepsilon|} = \frac{1}{\alpha}\lim_{\varepsilon\to 0^+}\frac{\frac{1}{|\ln\varepsilon|} - \left(\varepsilon|\ln\varepsilon|^\omega\right)^\alpha\frac{1}{|\ln\varepsilon|}}{1-\omega\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}} = \frac{0}{1} = 0. \tag{5.5}$$

□

Definition 5.3. (The Trivial Kernel)

Let us define the *N-dimensional trivial kernel*, or just *trivial kernel*, to be

$$\tilde{\rho}_\varepsilon(r) := \begin{cases} \frac{1}{\varepsilon^N\mathcal{L}^N(B_1(0))} & if \quad 0 < r < \varepsilon \\ 0 & if \quad r \geq \varepsilon \end{cases}, \quad \varepsilon \in (0,\infty). \tag{5.6}$$

Remark 5.2. Notice that the trivial kernel is a kernel. Moreover, it satisfies the compact support property: for every $r \in (0,\infty)$, let $\delta_r := r$. Thus, if $\varepsilon \in (0,\delta_r)$, then $\operatorname{supp}(\tilde{\rho}_\varepsilon) \subset B_\varepsilon(0) \subset B_r(0)$.

Definition 5.4. (σ-Approximating Kernels)

For every number $\sigma \in (0,\infty)$, the *N-dimensional σ-approximating kernel* is defined to be

$$\rho^\sigma_\varepsilon(r) := \frac{1}{2\sigma\mathcal{H}^{N-1}(S^{N-1})r^{N-1}}\chi_{[\varepsilon-\sigma,\varepsilon+\sigma]}(r), \quad \rho^\sigma_\varepsilon : (0,\infty) \to [0,\infty). \tag{5.7}$$

Remark 5.3. (σ-Approximating Kernels Give us Kernels)

Note that σ-approximating kernels are not kernels because they lack the decreasing support property (see Definition 1.1). However, if we select a number $\sigma_\varepsilon \in (0,\varepsilon)$ for every $\varepsilon \in (0,\infty)$, then the family $\{\rho^{\sigma_\varepsilon}_\varepsilon\}_{\varepsilon\in(0,\infty)}$ possesses the compact support property, and in particular, it satisfies the decreasing support property. By employing polar coordinates, we find that $\int_{\mathbb{R}^N}\rho^\sigma_\varepsilon(|z|)dz = 1$ for every choice of ε and σ in $(0,\infty)$ with $\sigma < \varepsilon$. Therefore, $\{\rho^{\sigma_\varepsilon}_\varepsilon\}_{\varepsilon\in(0,\infty)}$ is a kernel, as defined in Definition 1.2.

6 Variations and Besov constants

In this section, we introduce the notion of (r,q)-Variation (see Definition 6.1). We prove that (r,q)-variations control Besov Constants (see Lemma 6.1). Furthermore, we demonstrate that (r,q)-variation can be represented as a Besov constant (see Corollary 6.1). Additionally, we establish the continuity of Variations and Besov constants with respect to convergence in Besov Space (see Lemma 6.3).

Definition 6.1. ((r,q)-Variation and Directional (r,q)-Variation)

Let $r, q \in (0,\infty)$, and $u : \mathbb{R}^N \to \mathbb{R}^d$ be an $\mathcal{L}^N$-measurable function. Suppose $E \subset \mathbb{R}^N$ is an $\mathcal{L}^N$-measurable set, and let $n \in S^{N-1}$ be a direction. Then, the (r,q) *upper variation of* u *in* E *in the direction* n is defined by

$$(r,q) - \overline{V}(u,E,n) := \limsup_{\varepsilon\to 0^+} \int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}}dx. \tag{6.1}$$

Similarly, replacing the lim sup by the lim inf, we define the (r,q) *lower variation of* u *in* E *in the direction* n and denote it by $(r,q)-\underline{V}(u,E,n)$. If the limit exists, we denote it by $(r,q)-V(u,E,n)$, and we call it the (r,q) *variation of* u *in* E *in the direction* n.

The (r,q) *upper variation of* u *in* E is defined by

$$(r,q) - \overline{V}(u,E) := \limsup_{\varepsilon\to 0^+} \int_{S^{N-1}}\int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}}dx d\mathcal{H}^{N-1}(n). \tag{6.2}$$

Similarly, replacing the lim sup by the lim inf, we define the (r,q) *lower variation of* u *in* E and denote it by $(r,q)-\underline{V}(u,E)$. If the limit exists, we denote it by $(r,q)-V(u,E)$, and we call it the (r,q) *variation of* u *in* E. We also define the notions of (r,q) lower (upper) essential variation of u in E, replacing the lower (upper) limit by the essential lower (upper) limit.

Definition 6.2. (Besov Constants)

Let $r, q \in (0,\infty)$, and $u : \mathbb{R}^N \to \mathbb{R}^d$ be an $\mathcal{L}^N$-measurable function. Suppose $E \subset \mathbb{R}^N$ is an $\mathcal{L}^N$-measurable set, and let $\{\rho_\varepsilon\}_{\varepsilon\in(0,a)}$ be a kernel for some $a \in (0,\infty]$. The *upper infinitesimal* (r,q) *Besov constant of* u *in* E *with respect to the kernel* ρ_ε is defined as the quantity:

$$\limsup_{\varepsilon\to 0^+} \int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx. \tag{6.3}$$

Similarly, replacing the lim sup by the lim inf, we define the *lower infinitesimal* (r,q) *Besov constant of* u *in* E *with respect to the kernel* ρ_ε. If the limit exists, we refer to it as the *infinitesimal* (r,q) *Besov constant of* u *in* E *with respect to the kernel* ρ_ε.

Remark 6.1. (The Upper Infinitesimal $B^{r,q}$-seminorm is a Besov Constant)

Note that if we select the trivial kernel as defined in Definition 5.3 in (6.3), multiply the result by $\mathcal{L}^N(B_1(0))$, and then take the result to the power of $\frac{1}{q}$, we obtain the upper infinitesimal $B^{r,q}$-seminorm as defined in 4.1.

Remark 6.2. (Variations of $W^{1,q}$, BV and $B^r_{q,\infty}$)

From the BBM formula, for an open and bounded set $\Omega \subset \mathbb{R}^N$ with a Lipschitz boundary, where $1 < q < \infty$ and $u \in W^{1,q}(\Omega)$, we have

$$(1,q) - V(u,\Omega) = C_{q,N}\|\nabla u\|^q_{L^q(\Omega)}; \tag{6.4}$$

for $u \in BV(\Omega)$, we have

$$(1,1) - V(u,\Omega) = C_{1,N}\|Du\|(\Omega), \tag{6.5}$$

where $C_{q,N} := \int_{S^{N-1}} |z_1|^q d\mathcal{H}^{N-1}(n)$ for every $q \geq 1$. For proof of this result see [17].

For $r \in (0,1)$ and $q \in [1,\infty)$, we observe from Sandwich Lemma 6.1 and Theorem 4.1 that the finiteness of the upper variation $(r,q) - \overline{V}(u,\mathbb{R}^N)$ of u together with $u \in L^q(\mathbb{R}^N,\mathbb{R}^d)$ is equivalent to $u \in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$.

Lemma 6.1. *(The Sandwich Lemma with Variations and Besov Constants Included)*

Let $E \subset \mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set and $u : E \to \mathbb{R}^d$ be an $\mathcal{L}^N$-measurable function. Let $a \in (0,\infty]$ and let $\rho_\varepsilon : (0,\infty) \to [0,\infty), \varepsilon \in (0,a)$, be a kernel, and $\alpha, q \in (0,\infty)$. Assume that at least one of the following three assumptions holds:

1. $\operatorname{ess\,sup}_{\varepsilon\in(0,\infty)} \int_{S^{N-1}} \int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^\alpha} dx d\mathcal{H}^{N-1}(n) < \infty$;
2. $u \in L^q(\mathbb{R}^N,\mathbb{R}^d)$;
3. *The kernel $\{\rho_\varepsilon\}_{\varepsilon\in(0,a)}$ has the compact support property as defined in Definition 5.1.*

Then,

$$\begin{aligned}
&\liminf_{\varepsilon\to0^+} \fint_{S^{N-1}} \int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^\alpha} dx d\mathcal{H}^{N-1}(n)\\
&\qquad\leq \operatorname{ess\,liminf}_{\varepsilon\to0^+} \fint_{S^{N-1}} \int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^\alpha} dx d\mathcal{H}^{N-1}(n)\\
&\leq \liminf_{\varepsilon\to0^+} \int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^\alpha} dy dx \leq \limsup_{\varepsilon\to0^+} \int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^\alpha} dy dx\\
&\qquad\leq \operatorname{ess\,limsup}_{\varepsilon\to0^+} \fint_{S^{N-1}} \int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^\alpha} dx d\mathcal{H}^{N-1}(n)\\
&\qquad\qquad\leq \limsup_{\varepsilon\to0^+} \fint_{S^{N-1}} \int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^\alpha} dx d\mathcal{H}^{N-1}(n).
\end{aligned} \tag{6.6}$$

In particular, for $r \in (0,\infty)$ and $\alpha = rq$, we get (6.6) *for (r,q) variations and Besov constants.*

Proof. By using polar coordinates, we get for every $\delta \in (0, \infty)$

$$\begin{aligned}
\int_E \int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^\alpha}dydx &= \int_E \left(\int_{\mathbb{R}^N} \chi_E(y)\rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^\alpha}dy\right)dx \\
&= \int_E \left(\int_{\mathbb{R}^N} \chi_E(x+z)\rho_\varepsilon(|z|)\frac{|u(x)-u(x+z)|^q}{|z|^\alpha}dz\right)dx \\
&= \int_{\mathbb{R}^N} \left(\int_E \chi_E(x+z)\rho_\varepsilon(|z|)\frac{|u(x)-u(x+z)|^q}{|z|^\alpha}dx\right)dz \\
&= \int_0^\infty \int_{\partial B_t(0)} \left(\int_E \chi_E(x+z)\rho_\varepsilon(|z|)\frac{|u(x)-u(x+z)|^q}{|z|^\alpha}dx\right)d\mathcal{H}^{N-1}(z)dt \\
&= \int_0^\infty \int_{S^{N-1}} t^{N-1}\left(\int_E \chi_E(x+tn)\rho_\varepsilon(t)\frac{|u(x)-u(x+tn)|^q}{t^\alpha}dx\right)d\mathcal{H}^{N-1}(n)dt \\
&= \int_0^\infty t^{N-1}\rho_\varepsilon(t)\left(\int_{S^{N-1}}\int_E \chi_E(x+tn)\frac{|u(x)-u(x+tn)|^q}{t^\alpha}dxd\mathcal{H}^{N-1}(n)\right)dt \\
&= \int_0^\delta t^{N-1}\rho_\varepsilon(t)V(t)dt + \int_\delta^\infty t^{N-1}\rho_\varepsilon(t)V(t)dt. \qquad (6.7)
\end{aligned}$$

In formula (6.7) we denote

$$V(t) := \int_{S^{N-1}}\int_E \chi_E(x+tn)\frac{|u(x)-u(x+tn)|^q}{t^\alpha}dxd\mathcal{H}^{N-1}(n). \qquad (6.8)$$

By polar coordinates we see that

$$\int_{\mathbb{R}^N} \rho_\varepsilon(|z|)dz = 1 \quad \Longrightarrow \quad \frac{1}{\mathcal{H}^{N-1}(S^{N-1})} = \int_0^\infty t^{N-1}\rho_\varepsilon(t)dt. \qquad (6.9)$$

Since $\{\rho_\varepsilon\}_{\varepsilon\in(0,a)}$ is a kernel, then it has the decreasing support property (see Definition 1.1). Therefore, for every $\delta > 0$, we get $\lim_{\varepsilon\to 0^+}\int_\delta^\infty t^{N-1}\rho_\varepsilon(t)dt = 0$, and by (6.9) we obtain

$$\frac{1}{\mathcal{H}^{N-1}(S^{N-1})} = \lim_{\varepsilon\to 0^+}\left\{\int_0^\delta t^{N-1}\rho_\varepsilon(t)dt + \int_\delta^\infty t^{N-1}\rho_\varepsilon(t)dt\right\} = \lim_{\varepsilon\to 0^+}\int_0^\delta t^{N-1}\rho_\varepsilon(t)dt. \qquad (6.10)$$

By equation (6.7) we obtain

$$\begin{aligned}
\operatorname*{ess\,sup}_{t\in(0,\delta)} V(t)\int_0^\delta t^{N-1}\rho_\varepsilon(t)dt &+ \int_\delta^\infty t^{N-1}\rho_\varepsilon(t)V(t)dt \\
&\geq \int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^\alpha}dydx \\
&\geq \operatorname*{ess\,inf}_{t\in(0,\delta)} V(t)\int_0^\delta t^{N-1}\rho_\varepsilon(t)dt + \int_\delta^\infty t^{N-1}\rho_\varepsilon(t)V(t)dt. \qquad (6.11)
\end{aligned}$$

If $\operatorname{ess\,sup}_{t\in(0,\infty)} V(t) < \infty$, then we get $\lim_{\varepsilon\to 0^+}\int_\delta^\infty t^{N-1}\rho_\varepsilon(t)V(t)dt = 0$. If $u \in L^q(\mathbb{R}^N, \mathbb{R}^d)$, then

$$\sup_{t\in(\delta,\infty)} V(t) \leq \frac{2^q\mathcal{H}^{N-1}(S^{N-1})}{\delta^\alpha}\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)} < \infty. \qquad (6.12)$$

So we get again that $\lim_{\varepsilon\to 0^+}\int_\delta^\infty t^{N-1}\rho_\varepsilon(t)V(t)dt = 0$. Therefore, in both cases, we obtain (6.6) by first taking the liminf(limsup) as $\varepsilon \to 0^+$ and then the limit as $\delta \to 0^+$ in inequality (6.11).

In case $\{\rho_\varepsilon\}_{\varepsilon\in(0,a)}$ has the compact support property, for $r > 0$ there exists δ_r such that for every $\varepsilon \in (0, \delta_r)$ we obtain $\operatorname{supp}(\rho_\varepsilon) \subset (0, r)$, and by (6.7) we get

$$\int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^\alpha}dydx = \int_0^r t^{N-1}\rho_\varepsilon(t)V(t)dt \\ \leq \left(\int_0^r t^{N-1}\rho_\varepsilon(t)dt\right)\operatorname*{ess\,sup}_{t\in(0,r)} V(t) = \frac{1}{\mathcal{H}^{N-1}(S^{N-1})}\operatorname*{ess\,sup}_{t\in(0,r)} V(t), \quad (6.13)$$

$$\int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^\alpha}dydx = \int_0^r t^{N-1}\rho_\varepsilon(t)V(t)dt \\ \geq \left(\int_0^r t^{N-1}\rho_\varepsilon(t)dt\right)\operatorname*{ess\,inf}_{t\in(0,r)} V(t) = \frac{1}{\mathcal{H}^{N-1}(S^{N-1})}\operatorname*{ess\,inf}_{t\in(0,r)} V(t). \quad (6.14)$$

Taking the upper limit as $\varepsilon \to 0^+$ and then the limit as $r \to 0^+$ in (6.13), we obtain the forth inequality in (6.6). Similarly, taking the lower limit as $\varepsilon \to 0^+$ and then the limit as $r \to 0^+$ in (6.14), we obtain the second inequality of (6.6). □

Lemma 6.2. *(Variations and Essential Variations)*

Let $q, \alpha \in (0,\infty)$, and let $u \in L^q(\mathbb{R}^N, \mathbb{R}^d)$. Assume that $E \subset \mathbb{R}^N$ is a Lebesgue measurable set such that for every $v \in \mathbb{R}^N$ we have $\mathcal{L}^N(E \cap (\partial E + v)) = 0$. Then,

$$\liminf_{\varepsilon\to 0^+} \fint_{S^{N-1}}\int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^\alpha}dxd\mathcal{H}^{N-1}(n) \\ = \operatorname*{ess\,liminf}_{\varepsilon\to 0^+} \fint_{S^{N-1}}\int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^\alpha}dxd\mathcal{H}^{N-1}(n) \quad (6.15)$$

and

$$\limsup_{\varepsilon\to 0^+} \fint_{S^{N-1}}\int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^\alpha}dxd\mathcal{H}^{N-1}(n) \\ = \operatorname*{ess\,limsup}_{\varepsilon\to 0^+} \fint_{S^{N-1}}\int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^\alpha}dxd\mathcal{H}^{N-1}(n). \quad (6.16)$$

In particular, for $r \in (0,\infty)$ and $\alpha = rq$, we get the result for (r,q) essential variations and Besov constants.

Proof. Let us denote

$$V(t) := \int_{S^{N-1}}\int_E \chi_E(x+tn)\frac{|u(x)-u(x+tn)|^q}{t^\alpha}dxd\mathcal{H}^{N-1}(n), \quad (6.17)$$

and

$$F(t) := \int_{S^{N-1}}\int_E \chi_E(x+tn)|u(x+tn)-u(x)|^q dxd\mathcal{H}^{N-1}(n), \quad F:\mathbb{R}\to\mathbb{R}. \quad (6.18)$$

Note that $F(t) = t^\alpha V(t)$. We prove the continuity of F in $\mathbb{R}$, and consequently establish the continuity of V in $(0,\infty)$. Thus, every point in $(0,\infty)$ is a Lebesgue point of V. Therefore, by Proposition 10.1 and Corollary 10.1, we conclude that

$$\liminf_{\varepsilon\to 0^+} V(\varepsilon) = \operatorname{ess}\liminf_{\varepsilon\to 0^+} V(\varepsilon), \quad \limsup_{\varepsilon\to 0^+} V(\varepsilon) = \operatorname{ess}\limsup_{\varepsilon\to 0^+} V(\varepsilon). \tag{6.19}$$

Let $t_0 \in \mathbb{R}$ be any number, and let us show that F is continuous at t_0. Note first that

$$\begin{aligned}
&\Big|\chi_E(x+tn)|u(x+tn)-u(x)|^q - \chi_E(x+t_0n)|u(x+t_0n)-u(x)|^q\Big| \\
&\qquad \le \chi_E(x+tn)\Big||u(x+tn)-u(x)|^q - |u(x+t_0n)-u(x)|^q\Big| \\
&\qquad\quad + |\chi_E(x+tn)-\chi_E(x+t_0n)|\,|u(x+t_0n)-u(x)|^q \\
&\qquad \le \Big||u(x+tn)-u(x)|^q - |u(x+t_0n)-u(x)|^q\Big| \\
&\qquad\quad + \chi_{(E-tn)\Delta(E-t_0n)}(x)|u(x+t_0n)-u(x)|^q.
\end{aligned} \tag{6.20}$$

Therefore, by (6.20)

$$\begin{aligned}
|F(t)-F(t_0)| &= \Big|\int_{S^{N-1}}\int_E \chi_E(x+tn)|u(x+tn)-u(x)|^q dx d\mathcal{H}^{N-1}(n) \\
&\qquad - \int_{S^{N-1}}\int_E \chi_E(x+t_0n)|u(x+t_0n)-u(x)|^q dx d\mathcal{H}^{N-1}(n)\Big| \\
&\le \int_{S^{N-1}}\int_E \Big|\chi_E(x+tn)|u(x+tn)-u(x)|^q - \chi_E(x+t_0n)|u(x+t_0n)-u(x)|^q\Big| dx d\mathcal{H}^{N-1}(n) \\
&\le \int_{S^{N-1}}\int_E \Big||u(x+tn)-u(x)|^q - |u(x+t_0n)-u(x)|^q\Big| dx d\mathcal{H}^{N-1}(n) \\
&\qquad + \int_{S^{N-1}}\int_E \chi_{(E-tn)\Delta(E-t_0n)}(x)|u(x+t_0n)-u(x)|^q dx d\mathcal{H}^{N-1}(n).
\end{aligned} \tag{6.21}$$

By Dominated Convergence Theorem and continuity of translations in L^q we obtain

$$\begin{aligned}
&\lim_{t\to t_0}\int_{S^{N-1}}\int_E \Big||u(x+tn)-u(x)|^q - |u(x+t_0n)-u(x)|^q\Big| dx d\mathcal{H}^{N-1}(n) \\
&\qquad = \int_{S^{N-1}}\left(\lim_{t\to t_0}\int_E \Big||u(x+tn)-u(x)|^q - |u(x+t_0n)-u(x)|^q\Big| dx\right) d\mathcal{H}^{N-1}(n) = 0.
\end{aligned} \tag{6.22}$$

We utilized the continuity of translations in L^q as follows: since $u(\cdot+tn)$ converges to $u(\cdot+t_0n)$ in L^q as t tends to t_0, then $u(\cdot+tn)-u$ converges to $u(\cdot+t_0n)-u$ in L^q as t tends to t_0. Consequently, $|u(\cdot+tn)-u|$ converges to $|u(\cdot+t_0n)-u|$ in L^q as t tends to t_0, and thus $|u(\cdot+tn)-u|^q$ converges to $|u(\cdot+t_0n)-u|^q$ in L^1 as t tends to t_0.

Let us define for every $\varepsilon \in (0,\infty)$ the ε-neighbourhood of $\partial E - t_0n$ by

$$E_\varepsilon := \left\{x \in \mathbb{R}^N \,\middle|\, \operatorname{dist}(x, \partial E - t_0n) \le \varepsilon\right\}. \tag{6.23}$$

Note that $\cap_{\varepsilon\in(0,\infty)}E_\varepsilon = \partial E - t_0 n$. Therefore, for every $\varepsilon \in (0,\infty)$, there exists $R(\varepsilon) \in (0,\infty)$ such that for every $t \in \mathbb{R}$ with $|t - t_0| < R(\varepsilon)$, we have $(E - tn)\Delta(E - t_0 n) \subset E_\varepsilon$ and so $\chi_{(E-tn)\Delta(E-t_0 n)}(x) \le \chi_{E_\varepsilon}(x)$ for every $x \in \mathbb{R}^N$. Therefore,

$$\limsup_{t\to t_0} \int_{S^{N-1}} \int_E \chi_{(E-tn)\Delta(E-t_0n)}(x)|u(x+t_0n) - u(x)|^q dx d\mathcal{H}^{N-1}(n) \\ \le \int_{S^{N-1}} \int_E \chi_{E_\varepsilon}(x)|u(x+t_0n) - u(x)|^q dx d\mathcal{H}^{N-1}(n). \quad (6.24)$$

Therefore, taking the limit as $\varepsilon \to 0^+$ in (6.24), we get by Dominated Convergence Theorem and the assumption about E

$$\limsup_{t\to t_0} \int_{S^{N-1}} \int_E \chi_{(E-tn)\Delta(E-t_0n)}(x)|u(x+t_0n) - u(x)|^q dx d\mathcal{H}^{N-1}(n) \\ \le \int_{S^{N-1}} \int_{E\cap(\cap_{\varepsilon>0}E_\varepsilon)} |u(x+t_0n) - u(x)|^q dx d\mathcal{H}^{N-1}(n) \\ = \int_{S^{N-1}} \int_{E\cap(\partial E - t_0 n)} |u(x+t_0n) - u(x)|^q dx d\mathcal{H}^{N-1}(n) = 0. \quad (6.25)$$

Using (6.21), (6.22), and (6.25), we conclude the continuity of F at $t_0 \in \mathbb{R}$. It completes the proof. □

Proposition 6.1. *(Besov Constants and Essential Variations)*

Let $q, \alpha \in (0,\infty)$, and let $u \in L^q(\mathbb{R}^N, \mathbb{R}^d)$. Assume that $E \subset \mathbb{R}^N$ is a Lebesgue measurable set. Then, there exists a kernel $\{\rho_\varepsilon\}_{\varepsilon\in(0,\infty)}$ such that

$$\liminf_{\varepsilon\to 0^+} \int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^\alpha} dy dx = \operatorname*{ess\,liminf}_{\varepsilon\to 0^+} \int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^\alpha} dy dx \\ = \operatorname*{ess\,liminf}_{\varepsilon\to 0^+} ⨍_{S^{N-1}} \int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n) - u(x)|^q}{\varepsilon^\alpha} dx d\mathcal{H}^{N-1}(n), \quad (6.26)$$

and

$$\limsup_{\varepsilon\to 0^+} \int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^\alpha} dy dx = \operatorname*{ess\,limsup}_{\varepsilon\to 0^+} \int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^\alpha} dy dx \\ = \operatorname*{ess\,limsup}_{\varepsilon\to 0^+} ⨍_{S^{N-1}} \int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n) - u(x)|^q}{\varepsilon^\alpha} dx d\mathcal{H}^{N-1}(n). \quad (6.27)$$

In particular, for $r \in (0,\infty)$ and $\alpha = rq$, we get the result for (r,q) variations and Besov constants.

Proof. For every $\varepsilon \in (0,\infty)$ and $\sigma \in (0,\varepsilon)$ let ρ_ε^σ as in (5.7). By (6.7), we get

$$\int_E\int_E \rho_\varepsilon^\sigma(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^\alpha} dy dx \\ = \int_0^\infty t^{N-1}\rho_\varepsilon^\sigma(t)\left(\int_{S^{N-1}}\int_E \chi_E(x+tn)\frac{|u(x)-u(x+tn)|^q}{t^\alpha} dx d\mathcal{H}^{N-1}(n)\right) dt \\ = \frac{1}{2\sigma\mathcal{H}^{N-1}(S^{N-1})}\int_{\varepsilon-\sigma}^{\varepsilon+\sigma} V(t) dt, \quad (6.28)$$

where

$$V(t) := \int_{S^{N-1}} \int_E \chi_E(x+tn) \frac{|u(x)-u(x+tn)|^q}{t^\alpha} dx d\mathcal{H}^{N-1}(n). \tag{6.29}$$

Since $u \in L^q(\mathbb{R}^N, \mathbb{R}^d)$, the function V is locally integrable in $(0,\infty)$, so almost every point in $(0,\infty)$ is a Lebesgue point of V. Let $\varepsilon \in (0,\infty)$ be a Lebesgue point of V. There exists $0 < \sigma_\varepsilon < \varepsilon$ such that $\left| \frac{1}{2\sigma_\varepsilon} \int_{\varepsilon-\sigma_\varepsilon}^{\varepsilon+\sigma_\varepsilon} V(t)dt - V(\varepsilon) \right| < \varepsilon$. Therefore,

$$\begin{aligned} \int_E \int_E \rho_\varepsilon^{\sigma_\varepsilon}(|x-y|) \frac{|u(x)-u(y)|^q}{|x-y|^\alpha} dy dx &= \frac{1}{2\sigma_\varepsilon \mathcal{H}^{N-1}(S^{N-1})} \int_{\varepsilon-\sigma_\varepsilon}^{\varepsilon+\sigma_\varepsilon} V(t)dt \\ &= \frac{1}{\mathcal{H}^{N-1}(S^{N-1})} V(\varepsilon) + \frac{1}{\mathcal{H}^{N-1}(S^{N-1})} \left(\frac{1}{2\sigma_\varepsilon} \int_{\varepsilon-\sigma_\varepsilon}^{\varepsilon+\sigma_\varepsilon} V(t)dt - V(\varepsilon) \right). \end{aligned} \tag{6.30}$$

By taking the lower limit in (6.30) as $\varepsilon \to 0^+$, with ε being a Lebesgue point of V, we derive the second equation in (6.26) using Proposition 10.1 and Corollary 10.1. Similarly, by taking the upper limit in (6.30) as $\varepsilon \to 0^+$, with ε being a Lebesgue point of V, we obtain the second equation in (6.27). Note that $\{\rho_\varepsilon^{\sigma_\varepsilon}\}_{\varepsilon \in (0,\infty)}$ is a kernel as was explained in Remark 5.3.

By the definition of ess liminf and the second equation of (6.26) we obtain

$$\liminf_{\varepsilon \to 0^+} \int_E \int_E \rho_\varepsilon^{\sigma_\varepsilon}(|x-y|) \frac{|u(x)-u(y)|^q}{|x-y|^\alpha} dy dx \leq \operatorname*{ess\,liminf}_{\varepsilon \to 0^+} \frac{1}{\mathcal{H}^{N-1}(S^{N-1})} V(\varepsilon). \tag{6.31}$$

By Lemma 6.1, we get

$$\liminf_{\varepsilon \to 0^+} \int_E \int_E \rho_\varepsilon^{\sigma_\varepsilon}(|x-y|) \frac{|u(x)-u(y)|^q}{|x-y|^\alpha} dy dx \geq \operatorname*{ess\,liminf}_{\varepsilon \to 0^+} \frac{1}{\mathcal{H}^{N-1}(S^{N-1})} V(\varepsilon). \tag{6.32}$$

We get the first equation of (6.26) by (6.31) and (6.32). We get the first equation of (6.27) in a similar way. □

Corollary 6.1. *(Representability of Variations as Besov Constants)*

Let $q, \alpha \in (0,\infty)$, and let $u \in L^q(\mathbb{R}^N, \mathbb{R}^d)$. Assume that $E \subset \mathbb{R}^N$ is a Lebesgue measurable set such that for every $v \in \mathbb{R}^N$ we have $\mathcal{L}^N(E \cap (\partial E + v)) = 0$. Then, there exists a kernel $\{\rho_\varepsilon\}_{\varepsilon \in (0,\infty)}$ such that

$$\begin{aligned} &\liminf_{\varepsilon \to 0^+} \int_E \int_E \rho_\varepsilon(|x-y|) \frac{|u(x)-u(y)|^q}{|x-y|^\alpha} dy dx \\ &\qquad = \liminf_{\varepsilon \to 0^+} \fint_{S^{N-1}} \int_E \chi_E(x+\varepsilon n) \frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^\alpha} dx d\mathcal{H}^{N-1}(n), \end{aligned} \tag{6.33}$$

and

$$\begin{aligned} &\limsup_{\varepsilon \to 0^+} \int_E \int_E \rho_\varepsilon(|x-y|) \frac{|u(x)-u(y)|^q}{|x-y|^\alpha} dy dx \\ &\qquad = \limsup_{\varepsilon \to 0^+} \fint_{S^{N-1}} \int_E \chi_E(x+\varepsilon n) \frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^\alpha} dx d\mathcal{H}^{N-1}(n). \end{aligned} \tag{6.34}$$

In particular, for $r \in (0,\infty)$ and $\alpha = rq$, we get the result for (r,q) variations and Besov constants.

Proof. Formulas (6.33) and (6.34) follow immediately from Lemma 6.2 and Proposition 6.1. □

Lemma 6.3. *(Continuity of Variations and Besov Constants in Besov Spaces $B^r_{q,\infty}$)*

Let $q \in [1,\infty)$, $r \in (0,1)$, and $E \subset \mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set. Consider a sequence $\{u_k\}_{k=1}^\infty \subset B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$ such that u_k converges to u in $B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$. Then,

1. *for every $n \in \mathbb{R}^N$, we have*

$$\lim_{k\to\infty}\left(\limsup_{\varepsilon\to 0^+}\int_E \chi_E(x+\varepsilon n)\frac{|u_k(x+\varepsilon n)-u_k(x)|^q}{\varepsilon^{rq}}dx\right) = \limsup_{\varepsilon\to 0^+}\int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}}dx, \quad (6.35)$$

   and a similar result also holds when replacing the lim sup *with the* lim inf*.*

2. *It follows that*

$$\lim_{k\to\infty}\left(\limsup_{\varepsilon\to 0^+}\int_{S^{N-1}}\int_E \chi_E(x+\varepsilon n)\frac{|u_k(x+\varepsilon n)-u_k(x)|^q}{\varepsilon^{rq}}dx d\mathcal{H}^{N-1}(n)\right) = \limsup_{\varepsilon\to 0^+}\int_{S^{N-1}}\int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}}dx d\mathcal{H}^{N-1}(n), \quad (6.36)$$

   and a similar result also holds when replacing the lim sup *with the* lim inf*.*

3. *Let $a \in (0,\infty]$. For every kernel $\rho_\varepsilon : (0,\infty) \to [0,\infty), \varepsilon \in (0,a)$, we get*

$$\lim_{k\to\infty}\left(\limsup_{\varepsilon\to 0^+}\int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u_k(x)-u_k(y)|^q}{|x-y|^{rq}}dydx\right) = \limsup_{\varepsilon\to 0^+}\int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx, \quad (6.37)$$

   and a similar result also holds when replacing the lim sup *with the* lim inf*.*

Proof. Let us prove assertion 1. Note that if $n = 0$, then equation (6.35) in both lim inf and lim sup cases trivially holds. Assume $n \neq 0$. Let us denote

$$I_\varepsilon(u_k,n)(x) := \chi_E(x+\varepsilon n)\frac{|u_k(x+\varepsilon n)-u_k(x)|^q}{\varepsilon^{rq}}, \quad (6.38)$$

and

$$I_\varepsilon(u,n)(x) := \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}}. \quad (6.39)$$

By Lemma 10.6, Minkowski's inequality, and the definition of the Besov seminorm $[\cdot]_{B^r_{q,\infty}}$, we obtain

$$
\begin{aligned}
&\left|\limsup_{\varepsilon\to0^+}\left(\int_E I_\varepsilon(u_k,n)(x)dx\right)^{\frac1q}-\limsup_{\varepsilon\to0^+}\left(\int_E I_\varepsilon(u,n)(x)dx\right)^{\frac1q}\right|\\
&\qquad\le\limsup_{\varepsilon\to0^+}\left|\left(\int_E I_\varepsilon(u_k,n)(x)dx\right)^{\frac1q}-\left(\int_E I_\varepsilon(u,n)(x)dx\right)^{\frac1q}\right|\\
&\qquad=\limsup_{\varepsilon\to0^+}\left|\left(\int_E\left[(I_\varepsilon(u_k,n)(x))^{1/q}\right]^q dx\right)^{\frac1q}-\left(\int_E\left[(I_\varepsilon(u,n)(x))^{1/q}\right]^q dx\right)^{\frac1q}\right|\\
&\qquad\le\limsup_{\varepsilon\to0^+}\left(\int_E\left|(I_\varepsilon(u_k,n)(x))^{1/q}-(I_\varepsilon(u,n)(x))^{1/q}\right|^q dx\right)^{\frac1q}\\
&\qquad=\limsup_{\varepsilon\to0^+}\left(\int_E\left|\chi_E(x+\varepsilon n)\frac{|u_k(x+\varepsilon n)-u_k(x)|}{\varepsilon^r}-\chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|}{\varepsilon^r}\right|^q dx\right)^{\frac1q}\\
&\qquad=\limsup_{\varepsilon\to0^+}\left(\int_E\chi_E(x+\varepsilon n)\frac{||u_k(x+\varepsilon n)-u_k(x)|-|u(x+\varepsilon n)-u(x)||^q}{\varepsilon^{rq}}dx\right)^{\frac1q}\\
&\qquad\le\limsup_{\varepsilon\to0^+}\left(\int_E\chi_E(x+\varepsilon n)\frac{|(u_k-u)(x+\varepsilon n)-(u_k-u)(x)|^q}{\varepsilon^{rq}}dx\right)^{\frac1q}\le|n|^r[u_k-u]_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}.
\end{aligned}
\tag{6.40}
$$

We take the limit as $k\to\infty$ on both sides of (6.40) to obtain (6.35). Similarly, we get

$$
\left|\liminf_{\varepsilon\to0^+}\left(\int_E I_\varepsilon(u_k,n)(x)dx\right)^{\frac1q}-\liminf_{\varepsilon\to0^+}\left(\int_E I_\varepsilon(u,n)(x)dx\right)^{\frac1q}\right|\le|n|^r[u_k-u]_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}. \tag{6.41}
$$

Assertion 2 of the Lemma is proven in the same way. By replacing the integral $\int_E(\cdot)dx$ with the integral $\int_{S^{N-1}}\int_E(\cdot)dxd\mathcal{H}^{N-1}(n)$ in (6.40) throughout, we obtain

$$
\begin{aligned}
&\left|\limsup_{\varepsilon\to0^+}\left(\int_{S^{N-1}}\int_E I_\varepsilon(u_k,n)(x)dxd\mathcal{H}^{N-1}(n)\right)^{\frac1q}-\limsup_{\varepsilon\to0^+}\left(\int_{S^{N-1}}\int_E I_\varepsilon(u,n)(x)dxd\mathcal{H}^{N-1}(n)\right)^{\frac1q}\right|\\
&\qquad\le\left(\mathcal{H}^{N-1}(S^{N-1})\right)^{1/q}|n|^r[u_k-u]_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)},
\end{aligned}
\tag{6.42}
$$

and

$$
\begin{aligned}
&\left|\liminf_{\varepsilon\to0^+}\left(\int_{S^{N-1}}\int_E I_\varepsilon(u_k,n)(x)dxd\mathcal{H}^{N-1}(n)\right)^{\frac1q}-\liminf_{\varepsilon\to0^+}\left(\int_{S^{N-1}}\int_E I_\varepsilon(u,n)(x)dxd\mathcal{H}^{N-1}(n)\right)^{\frac1q}\right|\\
&\qquad\le\left(\mathcal{H}^{N-1}(S^{N-1})\right)^{1/q}|n|^r[u_k-u]_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}.
\end{aligned}
\tag{6.43}
$$

Taking the limit as k tends to infinity in inequalities (6.42) and (6.43), we get formula (6.36) in both cases $\liminf$ and $\limsup$.

We prove assertion 3. Let us denote

$$
B_{\varepsilon,u_k}(x,y):=\rho_\varepsilon(|x-y|)\frac{|u_k(x)-u_k(y)|^q}{|x-y|^{rq}},\quad B_{\varepsilon,u}(x,y):=\rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}. \tag{6.44}
$$

As in (6.40), for $n \neq 0$, by Lemma 10.6, Minkowski's inequality, Sandwich Lemma with $\alpha = rq$ (Lemma 6.1) and the definition of the Besov seminorm $[\cdot]_{B^r_{q,\infty}}$, we obtain

$$\begin{aligned}
&\left|\limsup_{\varepsilon\to 0^+}\left(\int_E\int_E B_{\varepsilon,u_k}(x,y)dydx\right)^{\frac{1}{q}} - \limsup_{\varepsilon\to 0^+}\left(\int_E\int_E B_{\varepsilon,u}(x,y)dydx\right)^{\frac{1}{q}}\right| \\
&\leq \limsup_{\varepsilon\to 0^+}\left|\left(\int_E\int_E\left[(B_{\varepsilon,u_k}(x,y))^{\frac{1}{q}}\right]^q dydx\right)^{\frac{1}{q}} - \left(\int_E\int_E\left[(B_{\varepsilon,u}(x,y))^{\frac{1}{q}}\right]^q dydx\right)^{\frac{1}{q}}\right| \\
&\leq \limsup_{\varepsilon\to 0^+}\left(\int_E\int_E\left|(B_{\varepsilon,u_k}(x,y))^{\frac{1}{q}} - (B_{\varepsilon,u}(x,y))^{\frac{1}{q}}\right|^q dydx\right)^{\frac{1}{q}} \\
&= \limsup_{\varepsilon\to 0^+}\left(\int_E\int_E\left|(\rho_\varepsilon(|x-y|))^{\frac{1}{q}}\,\frac{|u_k(x)-u_k(y)|}{|x-y|^r} - (\rho_\varepsilon(|x-y|))^{\frac{1}{q}}\,\frac{|u(x)-u(y)|}{|x-y|^r}\right|^q dydx\right)^{\frac{1}{q}} \\
&\leq \limsup_{\varepsilon\to 0^+}\left(\int_E\int_E \rho_\varepsilon(|x-y|)\frac{|(u_k-u)(x)-(u_k-u)(y)|^q}{|x-y|^{rq}}dydx\right)^{\frac{1}{q}} \\
&\leq \limsup_{\varepsilon\to 0^+}\left(\fint_{S^{N-1}}\int_E \chi_E(x+\varepsilon n)\frac{|(u_k-u)(x+\varepsilon n)-(u_k-u)(x)|^q}{\varepsilon^{rq}}dxd\mathcal{H}^{N-1}(n)\right)^{\frac{1}{q}}. \\
&\leq |n|^r[u_k-u]_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}. \qquad (6.45)
\end{aligned}$$

Taking the limit as $k\to\infty$ in the inequality (6.45), we obtain (6.37). We get this result for lim inf in a similar way. □

Remark 6.3. In Lemma 6.3, we can utilize Corollary 6.1 to derive assertion 2 from assertion 3 in Lemma 6.3, provided that we limit ourselves to sets E satisfying the conditions of Corollary 6.1.

7 Equivalence Between Gagliardo Constants and Besov Constants

In this section, we demonstrate that the upper and lower variations control Gagliardo constants (refer to Theorem 7.1). Furthermore, we establish that, under certain conditions, Gagliardo constants and Besov constants are equivalent (see Theorem 7.2). As a special case, we derive the equivalence between Gagliardo constants and infinitesimal $B^{r,q}$-seminorms (refer to Corollary 7.4).

Corollary 7.1. *(Besov Constants Bounded by Besov Seminorms)*

Let $q\in[1,\infty)$, $r\in(0,1)$ and $u\in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$. Let $\rho_\varepsilon:(0,\infty)\to[0,\infty), \varepsilon\in(0,a)$, be a kernel for some $a\in(0,\infty)$. Then

$$\limsup_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx \leq [u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)} < \infty. \qquad (7.1)$$

Proof. By Lemma 6.1 with $\alpha = rq$ and $E = \mathbb{R}^N$, Definition 2.1 (definition of Besov seminorm) and the assumption $u \in B^r_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d)$ we get

$$\begin{aligned} &\limsup_{\varepsilon\to 0^+} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx \\ &\le \frac{1}{\mathcal{H}^{N-1}(S^{N-1})}\limsup_{\varepsilon\to 0^+}\int_{S^{N-1}}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}}dxd\mathcal{H}^{N-1}(n) \le [u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)} < \infty. \quad (7.2) \end{aligned}$$

□

Lemma 7.1. *(Approximation of Gagliardo Constants by Besov Constants through the Logarithmic Kernel)*

Let $1 \le q < \infty$, $r \in (0,1)$. Let $\omega \in (0,1)$ be such that $rq < 1/\omega$. Let $u \in B^r_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d)$, $E \subset \mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set, and let η be such that

$$\eta \in W^{1,1}\left(\mathbb{R}^N\right), \quad \int_{\mathbb{R}^N} |\nabla\eta(v)||v|^{rq}dv < \infty. \tag{7.3}$$

Then for every $\varepsilon \in (0, 1/e)$ it follows that

$$\frac{1}{|\ln\varepsilon|}[u_\varepsilon]^q_{W^{r,q}(E,\mathbb{R}^d)} = \mathcal{H}^{N-1}\left(S^{N-1}\right)\int_E\int_E \rho_{\varepsilon,\omega}(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx + o_\varepsilon(1), \tag{7.4}$$

where $u_\varepsilon(x) := \int_{\mathbb{R}^N}\eta(z)u(x-\varepsilon z)dz$, $\lim_{\varepsilon\to 0^+} o_\varepsilon(1) = 0$ and $\rho_{\varepsilon,\omega}$ is the logarithmic kernel defined in Definition 5.2.

Proof. Let $\varepsilon \in (0, 1/e)$ be fixed. By definition of Gagliardo seminorm $[\cdot]_{W^{r,q}}$ (Definition 1.4), change of variable formula, Fubini's theorem and additivity of integral

$$\begin{aligned} \frac{1}{|\ln\varepsilon|}[u_\varepsilon]^q_{W^{r,q}(E,\mathbb{R}^d)} &= \frac{1}{|\ln\varepsilon|}\int_E\left(\int_E\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{N+rq}}dy\right)dx \\ &= \frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{N+rq}}\chi_E(x+z)\chi_E(x)dz\right)dx \\ &= \frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{N+rq}}\chi_E(x+z)\chi_E(x)dx\right)dz \\ &= \frac{1}{|\ln\varepsilon|}\left\{\int_{\mathbb{R}^N\setminus B_{R_{\varepsilon,\omega}}(0)} g^\varepsilon(z)dz + \int_{B_{R_{\varepsilon,\omega}}(0)\setminus B_\varepsilon(0)} g^\varepsilon(z)dz + \int_{B_\varepsilon(0)} g^\varepsilon(z)dz\right\}, \quad (7.5) \end{aligned}$$

where we set

$$g^\varepsilon(z) := \int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{N+rq}}\chi_E(x+z)\chi_E(x)dx. \tag{7.6}$$

Using (2.9) with $\gamma = R_{\varepsilon,\omega} := |\ln\varepsilon|^{-\omega}$ we get

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}^N\setminus B_{R_{\varepsilon,\omega}}(0)} g^\varepsilon(z)dz \le \|\eta\|^q_{L^1(\mathbb{R}^N)}2^q\|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{|\ln\varepsilon|^{1-rq\omega}rq} = o_\varepsilon(1). \tag{7.7}$$

Using (2.12) we get

$$\frac{1}{|\ln\varepsilon|}\int_{B_\varepsilon(0)} g^\varepsilon(z)dz$$
$$\leq \|\nabla\eta\|^{q-1}_{L^1(\mathbb{R}^N,\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}|\nabla\eta(v)|(|v|+2)^{rq}dv\right)[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{q-rq}\frac{1}{|\ln\varepsilon|}=o_\varepsilon(1). \quad (7.8)$$

Therefore, we obtain by (7.5),(7.7),(7.8) and the definition of the logarithmic kernel $\rho_{\varepsilon,\omega}$

$$\frac{1}{|\ln\varepsilon|}[u_\varepsilon]^q_{W^{r,q}(E,\mathbb{R}^d)}=\frac{1}{|\ln\varepsilon|}\int_{B_{R_{\varepsilon,\omega}}(0)\setminus B_\varepsilon(0)} g^\varepsilon(z)dz+o_\varepsilon(1)$$
$$=\frac{1}{|\ln\varepsilon|}\int_{B_{R_{\varepsilon,\omega}}(0)\setminus B_\varepsilon(0)}\left(\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{N+rq}}\chi_E(x+z)\chi_E(x)dx\right)dz+o_\varepsilon(1)$$
$$=\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}^N}\frac{\chi_{[\varepsilon,R_{\varepsilon,\omega})}(|z|)}{|z|^N}\left(\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{rq}}\chi_E(x+z)\chi_E(x)dx\right)dz+o_\varepsilon(1)$$
$$=\left(1-\frac{|\ln R_{\varepsilon,\omega}|}{|\ln\varepsilon|}\right)\mathcal{H}^{N-1}\left(S^{N-1}\right)\int_{\mathbb{R}^N}\rho_{\varepsilon,\omega}(|z|)\left(\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{rq}}\chi_E(x+z)\chi_E(x)dx\right)dz+o_\varepsilon(1)$$
$$=\mathcal{H}^{N-1}\left(S^{N-1}\right)\int_E\int_E\rho_{\varepsilon,\omega}(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dxdy$$
$$-\frac{|\ln R_{\varepsilon,\omega}|}{|\ln\varepsilon|}\mathcal{H}^{N-1}\left(S^{N-1}\right)\int_{\mathbb{R}^N}\rho_{\varepsilon,\omega}(|z|)\left(\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{rq}}\chi_E(x+z)\chi_E(x)dx\right)dz+o_\varepsilon(1)$$
$$=\mathcal{H}^{N-1}\left(S^{N-1}\right)\int_E\int_E\rho_{\varepsilon,\omega}(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dxdy+o_\varepsilon(1). \quad (7.9)$$

In the last equality we used (5.2), item 1 of Proposition 5.1, (2.4) and $u\in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$ in order to get

$$\frac{|\ln R_{\varepsilon,\omega}|}{|\ln\varepsilon|}\mathcal{H}^{N-1}\left(S^{N-1}\right)\int_{\mathbb{R}^N}\rho_{\varepsilon,\omega}(|z|)\left(\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{rq}}\chi_E(x+z)\chi_E(x)dx\right)dz$$
$$\leq\frac{|\ln R_{\varepsilon,\omega}|}{|\ln\varepsilon|}\mathcal{H}^{N-1}\left(S^{N-1}\right)\left(\int_{\mathbb{R}^N}|\eta(v)|dv\right)^q[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}=o_\varepsilon(1). \quad (7.10)$$

It completes the proof. □

Lemma 7.2. *(The η-Separating Lemma)*

Assume $q\in[1,\infty)$, $r\in(0,1)$ and $u\in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$. Let $\eta\in L^1(\mathbb{R}^N)$ be such that

$$\int_{\mathbb{R}^N}|\eta(z)||z|^{rq}dz<\infty. \quad (7.11)$$

Let $\{\rho_\varepsilon\}_{\varepsilon\in(0,a)}, a\in(0,\infty]$, $\rho_\varepsilon:(0,\infty)\to[0,\infty)$, be a kernel such that

$$\lim_{\varepsilon\to0^+}\varepsilon^{rq}\int_{\mathbb{R}^N}\frac{\rho_\varepsilon(|z|)}{|z|^{rq}}dz=0. \quad (7.12)$$

Then for every $\mathcal{L}^N$-measurable set $E \subset \mathbb{R}^N$ we have

$$\liminf_{\varepsilon\to 0^+} \int_E \int_E \rho_\varepsilon(|x-y|) \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}} dydx \\ = \left| \int_{\mathbb{R}^N} \eta(z)dz \right|^q \liminf_{\varepsilon\to 0^+} \int_E \int_E \rho_\varepsilon(|x-y|) \frac{|u(x)-u(y)|^q}{|x-y|^{rq}} dydx, \quad (7.13)$$

and

$$\limsup_{\varepsilon\to 0^+} \int_E \int_E \rho_\varepsilon(|x-y|) \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}} dydx \\ = \left| \int_{\mathbb{R}^N} \eta(z)dz \right|^q \limsup_{\varepsilon\to 0^+} \int_E \int_E \rho_\varepsilon(|x-y|) \frac{|u(x)-u(y)|^q}{|x-y|^{rq}} dydx. \quad (7.14)$$

Proof. Let $0 < \alpha < 1$. It follows for $\mathcal{L}^N$-almost every $x, z \in \mathbb{R}^N$ that

$$\begin{aligned} |u(x)-u(x+z)|^q &= |(u(x)-u_\varepsilon(x)) + (u_\varepsilon(x)-u_\varepsilon(x+z)) + (u_\varepsilon(x+z)-u(x+z))|^q \\ &\le (|u(x)-u_\varepsilon(x)| + |u_\varepsilon(x)-u_\varepsilon(x+z)| + |u_\varepsilon(x+z)-u(x+z)|)^q \\ &\le \frac{1}{\alpha^{q-1}} |u_\varepsilon(x)-u_\varepsilon(x+z)|^q + \frac{1}{(1-\alpha)^{q-1}} (|u(x)-u_\varepsilon(x)| + |u_\varepsilon(x+z)-u(x+z)|)^q. \quad (7.15) \end{aligned}$$

In the last inequality we use the following convex inequality: for numbers $A, B \ge 0$ and convex function $\Psi : [0,\infty) \to \mathbb{R}$ it follows that

$$\Psi(A+B) = \Psi\left(\alpha\frac{A}{\alpha} + (1-\alpha)\frac{B}{1-\alpha}\right) \le \alpha\Psi\left(\frac{A}{\alpha}\right) + (1-\alpha)\Psi\left(\frac{B}{1-\alpha}\right). \quad (7.16)$$

In the inequality (7.15) we choose

$$A = |u_\varepsilon(x)-u_\varepsilon(x+z)|, \quad B = |u(x)-u_\varepsilon(x)| + |u_\varepsilon(x+z)-u(x+z)|, \quad \Psi(r) = r^q. \quad (7.17)$$

Therefore, by (7.15)

$$\begin{aligned} \int_E \int_E \rho_\varepsilon(|x-y|) \frac{|u(x)-u(y)|^q}{|x-y|^{rq}} dydx &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho_\varepsilon(|x-y|) \frac{|u(x)-u(y)|^q}{|x-y|^{rq}} \chi_E(y)\chi_E(x) dydx \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho_\varepsilon(|z|) \frac{|u(x)-u(x+z)|^q}{|z|^{rq}} \chi_E(x+z)\chi_E(x) dzdx \\ &\le \frac{1}{\alpha^{q-1}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho_\varepsilon(|z|) \frac{|u_\varepsilon(x)-u_\varepsilon(x+z)|^q}{|z|^{rq}} \chi_E(x+z)\chi_E(x) dzdx \\ &+ \frac{1}{(1-\alpha)^{q-1}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho_\varepsilon(|z|) \frac{(|u(x)-u_\varepsilon(x)| + |u_\varepsilon(x+z)-u(x+z)|)^q}{|z|^{rq}} dzdx \\ &= \frac{1}{\alpha^{q-1}} \int_E \int_E \rho_\varepsilon(|x-y|) \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}} dydx \\ &+ \frac{1}{(1-\alpha)^{q-1}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho_\varepsilon(|z|) \frac{(|u(x)-u_\varepsilon(x)| + |u_\varepsilon(x+z)-u(x+z)|)^q}{|z|^{rq}} dzdx. \quad (7.18) \end{aligned}$$

Notice that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho_\varepsilon(|z|)\frac{(|u(x)-u_\varepsilon(x)|+|u(x+z)-u_\varepsilon(x+z)|)^q}{|z|^{rq}}dzdx$$
$$\le 2^{q-1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho_\varepsilon(|z|)\frac{|u(x)-u_\varepsilon(x)|^q+|u(x+z)-u_\varepsilon(x+z)|^q}{|z|^{rq}}dzdx$$
$$=2^q\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho_\varepsilon(|z|)\frac{|u(x)-u_\varepsilon(x)|^q}{|z|^{rq}}dzdx=2^q\left(\int_{\mathbb{R}^N}\frac{\rho_\varepsilon(|z|)}{|z|^{rq}}dz\right)\|u-u_\varepsilon\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}. \quad (7.19)$$

Assume for a moment that $\int_{\mathbb{R}^N}\eta(z)dz=1$. Then, by Hlder's inequality

$$\|u-u_\varepsilon\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}=\int_{\mathbb{R}^N}|u(x)-u_\varepsilon(x)|^qdx=\int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N}\eta(v)\left(u(x)-u(x-\varepsilon v)\right)dv\right|^qdx$$
$$\le\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}|\eta(v)|^{\frac{q-1}{q}}\left(|\eta(v)|^{\frac{1}{q}}|u(x)-u(x-\varepsilon v)|\right)dv\right)^qdx$$
$$\le\|\eta\|^{q-1}_{L^1(\mathbb{R}^N)}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}|\eta(v)|\,|u(x)-u(x-\varepsilon v)|^q\,dv\right)dx$$
$$=\|\eta\|^{q-1}_{L^1(\mathbb{R}^N)}\int_{\mathbb{R}^N}|\eta(v)|\left(\int_{\mathbb{R}^N}|u(x)-u(x-\varepsilon v)|^q\,dx\right)dv$$
$$\le\varepsilon^{rq}\|\eta\|^{q-1}_{L^1(\mathbb{R}^N)}[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\int_{\mathbb{R}^N}|\eta(v)||v|^{rq}dv. \quad (7.20)$$

Hence, by (7.18), (7.19) and (7.20)

$$\int_E\int_E\rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx\le\frac{1}{\alpha^{q-1}}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx$$
$$+\frac{2^q}{(1-\alpha)^{q-1}}\left(\varepsilon^{rq}\int_{\mathbb{R}^N}\frac{\rho_\varepsilon(|z|)}{|z|^{rq}}dz\right)\frac{\|u-u_\varepsilon\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}}{\varepsilon^{rq}}\le\frac{1}{\alpha^{q-1}}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx$$
$$+\frac{2^q}{(1-\alpha)^{q-1}}\left(\varepsilon^{rq}\int_{\mathbb{R}^N}\frac{\rho_\varepsilon(|z|)}{|z|^{rq}}dz\right)\|\eta\|^{q-1}_{L^1(\mathbb{R}^N)}[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\int_{\mathbb{R}^N}|\eta(v)||v|^{rq}dv. \quad (7.21)$$

By (7.11), (7.12), $u\in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$ and (7.21) we obtain

$$\liminf_{\varepsilon\to0^+}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx\le\frac{1}{\alpha^{q-1}}\liminf_{\varepsilon\to0^+}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx, \quad (7.22)$$

and

$$\limsup_{\varepsilon\to0^+}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx\le\frac{1}{\alpha^{q-1}}\limsup_{\varepsilon\to0^+}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx. \quad (7.23)$$

Taking the limit as $\alpha\to1^-$ we get

$$\liminf_{\varepsilon\to0^+}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx\le\liminf_{\varepsilon\to0^+}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx, \quad (7.24)$$

and

$$\limsup_{\varepsilon\to 0^+}\int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx \le \limsup_{\varepsilon\to 0^+}\int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx. \tag{7.25}$$

Replacing the roles of $u_\varepsilon(x), u_\varepsilon(y)$ with $u(x), u(y)$, respectively, one can prove similarly (to the inequality (7.21)) the inequality

$$\begin{aligned}\int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx &\le \frac{1}{\alpha^{q-1}}\int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx\\ &+\frac{2^q}{(1-\alpha)^{q-1}}\left(\varepsilon^{rq}\int_{\mathbb{R}^N}\frac{\rho_\varepsilon(|z|)}{|z|^{rq}}dz\right)\|\eta\|^{q-1}_{L^1(\mathbb{R}^N)}[u]^q_{B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\int_{\mathbb{R}^N}|\eta(v)||v|^{rq}dv\end{aligned} \tag{7.26}$$

in order to obtain

$$\liminf_{\varepsilon\to 0^+}\int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx \le \liminf_{\varepsilon\to 0^+}\int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx, \tag{7.27}$$

and

$$\limsup_{\varepsilon\to 0^+}\int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx \le \limsup_{\varepsilon\to 0^+}\int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx. \tag{7.28}$$

Assume now that $\int_{\mathbb{R}^N}\eta(z)dz \neq 0$. Replacing η with $c\eta$, where $c := \frac{1}{\int_{\mathbb{R}^N}\eta(z)dz}$, and using the homogeneity of the convolution $u*(c\eta) = c\,(u*\eta)$, one can get (7.13) and (7.14). In case $\int_{\mathbb{R}^N}\eta(z)dz = 0$, let us choose any $\eta_0 \in C_c(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N}\eta_0(z)dz = 1$, and for each $n \in \mathbb{N}$ define $\eta_n := \eta - \frac{1}{n}\eta_0$. It follows that

$$\begin{aligned}\left|u*\eta_{(\varepsilon)}(x) - u*\eta_{(\varepsilon)}(y)\right| &= \left|u*\left(\eta_n+\frac{1}{n}\eta_0\right)_{(\varepsilon)}(x) - u*\left(\eta_n+\frac{1}{n}\eta_0\right)_{(\varepsilon)}(y)\right|\\ &= \left|u*\left((\eta_n)_{(\varepsilon)}+\left(\frac{1}{n}\eta_0\right)_{(\varepsilon)}\right)(x) - u*\left((\eta_n)_{(\varepsilon)}+\left(\frac{1}{n}\eta_0\right)_{(\varepsilon)}\right)(y)\right|\\ &= \left|\left(u*(\eta_n)_{(\varepsilon)}(x) - u*(\eta_n)_{(\varepsilon)}(y)\right) + \left(u*\left(\frac{1}{n}\eta_0\right)_{(\varepsilon)}(x) - u*\left(\frac{1}{n}\eta_0\right)_{(\varepsilon)}(y)\right)\right|.\end{aligned} \tag{7.29}$$

Therefore,

$$\begin{aligned}&\left|u*\eta_{(\varepsilon)}(x) - u*\eta_{(\varepsilon)}(y)\right|^q\\ &\quad\le 2^{q-1}\left(\left|u*(\eta_n)_{(\varepsilon)}(x) - u*(\eta_n)_{(\varepsilon)}(y)\right|^q + \frac{1}{n^q}\left|u*(\eta_0)_{(\varepsilon)}(x) - u*(\eta_0)_{(\varepsilon)}(y)\right|^q\right).\end{aligned} \tag{7.30}$$

Thus, since $\int_{\mathbb{R}^N}\eta_n(v)dv=-\frac{1}{n}\neq 0$, $\int_{\mathbb{R}^N}\eta_0(v)dv=1\neq 0$, then

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx\\
&\qquad\le 2^{q-1}\limsup_{\varepsilon\to 0^+}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{\left|u*(\eta_n)_{(\varepsilon)}(x)-u*(\eta_n)_{(\varepsilon)}(y)\right|^q}{|x-y|^{rq}}dydx\\
&\qquad+\frac{2^{q-1}}{n^q}\limsup_{\varepsilon\to 0^+}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{\left|u*(\eta_0)_{(\varepsilon)}(x)-u*(\eta_0)_{(\varepsilon)}(y)\right|^q}{|x-y|^{rq}}dydx\\
&=\frac{2^{q-1}}{n^q}\limsup_{\varepsilon\to 0^+}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx+\frac{2^{q-1}}{n^q}\limsup_{\varepsilon\to 0^+}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx\\
&\qquad=\frac{2^q}{n^q}\limsup_{\varepsilon\to 0^+}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx. \quad (7.31)
\end{aligned}$$

Taking the limit as $n\to\infty$ and using Corollary 7.1 we get

$$\lim_{\varepsilon\to 0^+}\int_E\int_E\rho_\varepsilon(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx=0. \quad (7.32)$$

□

Corollary 7.2. *(Equivalence Between Gagliardo and Besov Constants Including the Logarithmic Kernel)*

Let $q\in[1,\infty)$, $r\in(0,1)$. Let $\omega\in(0,1)$ be such that $rq<1/\omega$. Let $u\in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$, $E\subset\mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set and $\eta\in W^{1,1}\left(\mathbb{R}^N\right)$. Then,

$$\begin{aligned}
&\liminf_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}\left[u*\eta_{(\varepsilon)}\right]^q_{W^{r,q}(E,\mathbb{R}^d)}\\
&\qquad=\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q\mathcal{H}^{N-1}\left(S^{N-1}\right)\liminf_{\varepsilon\to 0^+}\int_E\int_E\rho_{\varepsilon,\omega}(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx, \quad (7.33)
\end{aligned}$$

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}\left[u*\eta_{(\varepsilon)}\right]^q_{W^{r,q}(E,\mathbb{R}^d)}\\
&\qquad=\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q\mathcal{H}^{N-1}\left(S^{N-1}\right)\limsup_{\varepsilon\to 0^+}\int_E\int_E\rho_{\varepsilon,\omega}(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx, \quad (7.34)
\end{aligned}$$

where $\rho_{\varepsilon,\omega}$ is the logarithmic kernel.

Proof. Assume for a moment that $\eta\in C^1_c(\mathbb{R}^N)$. By Lemma 7.1 we have

$$\liminf_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}\left[u_\varepsilon\right]^q_{W^{r,q}(E,\mathbb{R}^d)}=\mathcal{H}^{N-1}\left(S^{N-1}\right)\liminf_{\varepsilon\to 0^+}\int_E\int_E\rho_{\varepsilon,\omega}(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx, \quad (7.35)$$

$$\limsup_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}\left[u_\varepsilon\right]^q_{W^{r,q}(E,\mathbb{R}^d)}=\mathcal{H}^{N-1}\left(S^{N-1}\right)\limsup_{\varepsilon\to 0^+}\int_E\int_E\rho_{\varepsilon,\omega}(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx. \quad (7.36)$$

Note that by item 2 of Proposition 5.1 with $\alpha = rq$, the logarithmic kernel satisfies condition (7.12) of Lemma 7.2. Therefore, by Lemma 7.2

$$\liminf_{\varepsilon\to 0^+}\int_E\int_E \rho_{\varepsilon,\omega}(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx = \left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \liminf_{\varepsilon\to 0^+}\int_E\int_E \rho_{\varepsilon,\omega}(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx, \quad (7.37)$$

and

$$\limsup_{\varepsilon\to 0^+}\int_E\int_E \rho_{\varepsilon,\omega}(|x-y|)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^q}{|x-y|^{rq}}dydx = \left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \limsup_{\varepsilon\to 0^+}\int_E\int_E \rho_{\varepsilon,\omega}(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx. \quad (7.38)$$

Now, (7.33) and (7.34) follow from (7.35),(7.36),(7.37),(7.38). For $\eta \in W^{1,1}(\mathbb{R}^N)$ choose any sequence $\{\eta_n\}_{n=1}^{\infty} \subset C_c^1(\mathbb{R}^N)$ which converges to η in $W^{1,1}(\mathbb{R}^N)$. So we have (7.33) and (7.34) for η_n, for every $n \in \mathbb{N}$. Taking the limit as n goes to ∞ and using item 1 of Lemma 3.1, we obtain (7.33) and (7.34) for $\eta \in W^{1,1}(\mathbb{R}^N)$. □

Corollary 7.3. *(Gagliardo Constants are Controlled by Besov Seminorms)*

Let $1 \le q < \infty$, $r \in (0,1)$, $u \in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$ and $E \subset \mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set. Let $\eta \in W^{1,1}(\mathbb{R}^N)$. Then,

$$\limsup_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}[u_\varepsilon]^q_{W^{r,q}(E,\mathbb{R}^d)} \le \left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \mathcal{H}^{N-1}(S^{N-1})[u]^q_{B^r_{q,\infty}(E,\mathbb{R}^d)} < \infty. \quad (7.39)$$

Proof. Let $\omega \in (0,1)$ be such that $rq < 1/\omega$ and $\rho_{\varepsilon,\omega}$ be the logarithmic kernel as defined in Definition 5.2. By (7.34), (6.6) and Definition 2.1 (Definition of Besov seminorm) we get

$$\begin{aligned}\limsup_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}[u_\varepsilon]^q_{W^{r,q}(E,\mathbb{R}^d)} &= \left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \mathcal{H}^{N-1}\left(S^{N-1}\right)\limsup_{\varepsilon\to 0^+}\int_E\int_E \rho_{\varepsilon,\omega}(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx \\ &\le \left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \limsup_{\varepsilon\to 0^+}\int_{S^{N-1}}\int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}}dxd\mathcal{H}^{N-1}(n) \\ &\le \left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \mathcal{H}^{N-1}(S^{N-1})[u]^q_{B^r_{q,\infty}(E,\mathbb{R}^d)} < \infty. \quad (7.40)\end{aligned}$$

□

Theorem 7.1. *(Variations Control Gagliardo Constants)*

Let $q \in [1,\infty)$ and $r \in (0,1)$. Suppose $u \in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$, $E \subset \mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set and $\eta \in W^{1,1}(\mathbb{R}^N)$. Then,

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \liminf_{\varepsilon\to 0^+}\int_{S^{N-1}}\int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}}dxd\mathcal{H}^{N-1}(n) \\ &\le \liminf_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}[u_\varepsilon]^q_{W^{r,q}(E,\mathbb{R}^d)} \le \limsup_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}[u_\varepsilon]^q_{W^{r,q}(E,\mathbb{R}^d)} \\ &\le \left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \limsup_{\varepsilon\to 0^+}\int_{S^{N-1}}\int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}}dxd\mathcal{H}^{N-1}(n). \quad (7.41)\end{aligned}$$

Proof. Let $\omega \in (0,1)$ be such that $rq < 1/\omega$ and $\rho_{\varepsilon,\omega}$ be the logarithmic kernel as defined in Definition 5.2. By Sandwich Lemma (Lemma 6.1) we get for $\alpha = rq$ and $\rho_\varepsilon = \rho_{\varepsilon,\omega}$

$$\begin{aligned}
&\liminf_{\varepsilon\to 0^+} \fint_{S^{N-1}} \int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}} dx d\mathcal{H}^{N-1}(n) \\
&\le \liminf_{\varepsilon\to 0^+} \int_E\int_E \rho_{\varepsilon,\omega}(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}} dydx \le \limsup_{\varepsilon\to 0^+} \int_E\int_E \rho_{\varepsilon,\omega}(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}} dydx \\
&\qquad\qquad \le \limsup_{\varepsilon\to 0^+} \fint_{S^{N-1}} \int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}} dx d\mathcal{H}^{N-1}(n). \quad (7.42)
\end{aligned}$$

If $\int_{\mathbb{R}^N} \eta(z)dz = 0$, then (7.41) follows from Corollary 7.3. Assume that $\int_{\mathbb{R}^N} \eta(z)dz \neq 0$. By Corollary 7.2 and (7.42) we get

$$\begin{aligned}
&\liminf_{\varepsilon\to 0^+} \fint_{S^{N-1}} \int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}} dx d\mathcal{H}^{N-1}(n) \\
&\qquad \le \frac{\liminf_{\varepsilon\to 0^+} \frac{1}{|\ln\varepsilon|}\left[u*\eta_{(\varepsilon)}\right]^q_{W^{r,q}(E,\mathbb{R}^d)}}{\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \mathcal{H}^{N-1}\left(S^{N-1}\right)} \le \frac{\limsup_{\varepsilon\to 0^+} \frac{1}{|\ln\varepsilon|}\left[u*\eta_{(\varepsilon)}\right]^q_{W^{r,q}(E,\mathbb{R}^d)}}{\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \mathcal{H}^{N-1}\left(S^{N-1}\right)} \\
&\qquad\qquad \le \limsup_{\varepsilon\to 0^+} \fint_{S^{N-1}} \int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}} dx d\mathcal{H}^{N-1}(n). \quad (7.43)
\end{aligned}$$

Multiplying both sides of inequality (7.43) by $\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \mathcal{H}^{N-1}\left(S^{N-1}\right)$ we obtain (7.41). □

Theorem 7.2. *(Equivalence Between Gagliardo and Beosv Constants)*

Let $q \in [1,\infty)$, $r \in (0,1)$. Let $u \in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$, $E \subset \mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set and $\eta \in W^{1,1}\left(\mathbb{R}^N\right)$. If the following limit exists:

$$\lim_{\varepsilon\to 0^+} \int_{S^{N-1}} \int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}} dx d\mathcal{H}^{N-1}(n), \quad (7.44)$$

then, for every kernel ρ_ε we get

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{1}{|\ln\varepsilon|}\left[u_\varepsilon\right]^q_{W^{r,q}(E,\mathbb{R}^d)} &= \left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \lim_{\varepsilon\to 0^+} \int_{S^{N-1}} \int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}} dx d\mathcal{H}^{N-1}(n) \\
&= \left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \mathcal{H}^{N-1}\left(S^{N-1}\right) \lim_{\varepsilon\to 0^+} \int_E\int_E \rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}} dydx. \quad (7.45)
\end{aligned}$$

Proof. Formulas (7.45) follow from assumption (7.44), Theorem 7.1 and Sandwich Lemma with $\alpha = rq$ (Lemma 6.1). □

Corollary 7.4. *(Equivalence Between Gagliardo Constants and $B^{r,q}$-Seminorms) Let $1 \le q < \infty$, $r \in (0,1)$, $u \in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$, $E \subset \mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set and $\eta \in W^{1,1}\left(\mathbb{R}^N\right)$. If the following limit exists:*

$$\lim_{\varepsilon\to 0^+} \int_{S^{N-1}} \int_E \chi_E(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon^{rq}} dx d\mathcal{H}^{N-1}(n), \quad (7.46)$$

then

$$\lim_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}[u_\varepsilon]^q_{W^{r,q}(E,\mathbb{R}^d)} = N\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \lim_{\varepsilon\to 0^+}\int_E\frac{1}{\varepsilon^N}\int_{E\cap B_\varepsilon(x)}\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx$$
$$= N\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q [u]^q_{B^{r,q}(E,\mathbb{R}^d)}, \quad (7.47)$$

where $[u]_{B^{r,q}(E,\mathbb{R}^d)}$ *is the upper infinitesimal* $B^{r,q}$*-seminorm defined in 4.1.*

Proof. By Remark 5.2 the trivial kernel, $\tilde{\rho}_\varepsilon$, is a kernel. Therefore, by assumption (7.46) and Theorem 7.2 we obtain

$$\lim_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}[u_\varepsilon]^q_{W^{r,q}(E,\mathbb{R}^d)} = \left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \mathcal{H}^{N-1}\left(S^{N-1}\right)\lim_{\varepsilon\to 0^+}\int_E\int_E\tilde{\rho}_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx$$
$$= \left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{\mathcal{L}^N\left(B_1(0)\right)}\lim_{\varepsilon\to 0^+}\int_E\frac{1}{\varepsilon^N}\int_{E\cap B_\varepsilon(x)}\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx$$
$$= N\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \lim_{\varepsilon\to 0^+}\int_E\frac{1}{\varepsilon^N}\int_{E\cap B_\varepsilon(x)}\frac{|u(x)-u(y)|^q}{|x-y|^{rq}}dydx. \quad (7.48)$$

The equation $\mathcal{L}^N\left(B_1(0)\right) = \frac{\mathcal{H}^{N-1}\left(S^{N-1}\right)}{N}$ follows from polar coordinates.

□

Remark 7.1. (Consistency with Previous Results)

Equation (7.47) can be derived for functions $u \in BV(\mathbb{R}^N,\mathbb{R}^d)\cap L^\infty(\mathbb{R}^N,\mathbb{R}^d)$, $1 < q < \infty$, $r = \frac{1}{q}$, $\eta \in W^{1,1}(\mathbb{R}^N)$, and an open set $\Omega \subset \mathbb{R}^N$ with a bounded Lipschitz boundary such that $\|Du\|(\partial\Omega) = 0$. By combining Theorem 1.2 in [15] and Theorem 1.1 in [16], we get

$$\lim_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}[u_\varepsilon]^q_{W^{\frac{1}{q},q}(\Omega,\mathbb{R}^d)}$$
$$= \frac{\int_{\mathbb{R}^{N-1}}2\left(1+|v|^2\right)^{-\frac{N+1}{2}}dv}{\frac{1}{N}\int_{S^{N-1}}|z_1|d\mathcal{H}^{N-1}(z)}\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q \lim_{\varepsilon\to 0^+}\int_\Omega\frac{1}{\varepsilon^N}\int_{\Omega\cap B_\varepsilon(x)}\frac{|u(x)-u(y)|^q}{|x-y|}dydx. \quad (7.49)$$

According to Proposition 10.3, we obtain $\int_{\mathbb{R}^{N-1}}2\left(1+|v|^2\right)^{-\frac{N+1}{2}}dv = \int_{S^{N-1}}|z_1|d\mathcal{H}^{N-1}(z)$.

8 Jump Detection in $BV \cap B^{1/p,p}$

In this section we prove formulas for $\lim_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}[u_\varepsilon]^q_{W^{1/q,q}(B,\mathbb{R}^d)}$, where $u \in BV \cap B^{1/p,p}$ and $B \subset \mathbb{R}^N$ is a Borel set (refer to Corollary 8.2).

Remark 8.1. ($BV \cap L^\infty$ is a Subset of $B^r_{q,\infty}, rq \leq 1$)

Let $u \in BV\left(\mathbb{R}^N, \mathbb{R}^d\right) \cap L^\infty\left(\mathbb{R}^N, \mathbb{R}^d\right)$. Let $1 \leq q < \infty$ and $r \in (0,1)$ be such that $rq \leq 1$. By Lemma 10.9 we get

$$\begin{aligned}
&\sup_{h\in\mathbb{R}^N\setminus\{0\}} \int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^q}{|h|^{rq}} dx \\
&\quad\leq \sup_{h\in\mathbb{R}^N\setminus B_1(0)} \int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^q}{|h|^{rq}} dx + \sup_{h\in B_1(0)\setminus\{0\}} \int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^q}{|h|^{rq}} dx \\
&\quad\leq 2^q \|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)} + 2^{q-1}\|u\|^{q-1}_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)} \sup_{h\in B_1(0)\setminus\{0\}} \int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|}{|h|} dx \\
&\quad\leq 2^q \|u\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)} + 2^{q-1}\|u\|^{q-1}_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)} \|Du\|(\mathbb{R}^N) < \infty. \quad (8.1)
\end{aligned}$$

Note that since $u \in L^1\left(\mathbb{R}^N, \mathbb{R}^d\right) \cap L^\infty\left(\mathbb{R}^N, \mathbb{R}^d\right)$, then $u \in L^q\left(\mathbb{R}^N, \mathbb{R}^d\right)$. Thus, by Definition 1.3 (definition of Besov space) we get $u \in B^r_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d)$.

Lemma 8.1. *(Interpolation for Besov Seminorms)*
Let $p \in (1,\infty)$ and $u \in BV(\mathbb{R}^N, \mathbb{R}^d) \cap B^{1/p,p}(\mathbb{R}^N, \mathbb{R}^d)$. Then for every $q \in (1,p)$ we have $u \in B^{1/q,q}(\mathbb{R}^N, \mathbb{R}^d)$ and

$$[u]^q_{B^{1/q}_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)} \leq \left(\|Du\|(\mathbb{R}^N)\right)^\alpha \left([u]^p_{B^{1/p}_{p,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\right)^{1-\alpha}, \quad (8.2)$$

where $\alpha := \frac{p-q}{p-1}$.

Proof. Since $\alpha = \frac{p-q}{p-1}$, then $q = \alpha + (1-\alpha)p$. By Hölder's inequality and Lemma 10.9 we get

$$\begin{aligned}
[u]^q_{B^{1/q}_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)} &= \sup_{h\in\mathbb{R}^N\setminus\{0\}} \int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^q}{|h|} dx \\
&= \sup_{h\in\mathbb{R}^N\setminus\{0\}} \int_{\mathbb{R}^N} \left(\frac{|u(x+h)-u(x)|}{|h|}\right)^\alpha \left(\frac{|u(x+h)-u(x)|^p}{|h|}\right)^{1-\alpha} dx \\
&\leq \left(\sup_{h\in\mathbb{R}^N\setminus\{0\}} \int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|}{|h|} dx\right)^\alpha \left(\sup_{h\in\mathbb{R}^N\setminus\{0\}} \int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^p}{|h|} dx\right)^{1-\alpha} \\
&\leq \left(\|Du\|(\mathbb{R}^N)\right)^\alpha \left([u]^p_{B^{1/p}_{p,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\right)^{1-\alpha}. \quad (8.3)
\end{aligned}$$

□

Corollary 8.1. *(Convergence of the Truncated Family in Besov Seminorm)*
Let $p \in (1,\infty)$ and $u \in BV(\mathbb{R}^N, \mathbb{R}^d) \cap B^{1/p}_{p,\infty}(\mathbb{R}^N, \mathbb{R}^d)$. Then, for every $q \in (1,p)$ we have

$$\lim_{l\to\infty} [u-u_l]_{B^{1/q}_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)} = 0, \quad (8.4)$$

where $\{u_l\}_{l\in[0,\infty)}$ is the truncated family obtained by u as defined in Definition 10.2. In particular, the truncated family u_l converges to u in the norm of the space $B^{1/q}_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d)$, which means that

$$\lim_{l\to\infty} \left([u-u_l]_{B^{1/q}_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)} + \|u-u_l\|_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\right) = 0. \quad (8.5)$$

Proof. Let $q \in (1,p)$ and denote $\alpha := \frac{p-q}{p-1}$. From Lemma 8.1 we get

$$[u-u_l]^q_{B^{1/q}_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)} \le \left(\|D(u-u_l)\|(\mathbb{R}^N)\right)^\alpha \left([u-u_l]^p_{B^{1/p}_{p,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\right)^{1-\alpha}. \tag{8.6}$$

Note that

$$\begin{aligned}[u-u_l]^p_{B^{1/p}_{p,\infty}(\mathbb{R}^N,\mathbb{R}^d)} &= \sup_{h\in\mathbb{R}^N\setminus\{0\}} \int_{\mathbb{R}^N} \frac{|(u-u_l)(x+h)-(u-u_l)(x)|^p}{|h|}dx \\ &\le 2^{p-1}\sup_{h\in\mathbb{R}^N\setminus\{0\}} \int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^p}{|h|}dx + 2^{p-1}\sup_{h\in\mathbb{R}^N\setminus\{0\}} \int_{\mathbb{R}^N} \frac{|u_l(x+h)-u_l(x)|^p}{|h|}dx \\ &\le 2^p[u]^p_{B^{1/p}_{p,\infty}(\mathbb{R}^N,\mathbb{R}^d)} < \infty. \end{aligned} \tag{8.7}$$

Since by Lemma 10.11 we have $\lim_{l\to\infty}\|D(u-u_l)\|(\mathbb{R}^N)=0$, then (8.4) follows. The convergence of u_l to u as $l\to\infty$ in the norm of the space $B^{1/q}_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$ follows from (8.4) and Lemma 10.10. □

Recall Definition 10.4 for $u^+, u^-, \mathcal{J}_u, \nu_u$.

Theorem 8.1. *(Proposition 2.4 in [16])*

Let $\Omega \subset \mathbb{R}^N$ be an open set, $1 < q < \infty$, $u \in BV_{loc}(\Omega,\mathbb{R}^d) \cap L^\infty_{loc}(\Omega,\mathbb{R}^d)$. Then, for every $h \in \mathbb{R}^N$ and every compact set $K \subset \Omega$ such that $\|Du\|(\partial K) = 0$ we have

$$\lim_{\varepsilon\to0^+}\int_K \frac{|u(x+\varepsilon h)-u(x)|^q}{\varepsilon}dx = \int_{K\cap\mathcal{J}_u} \left|u^+(x)-u^-(x)\right|^q |\nu_u(x)\cdot h| d\mathcal{H}^{N-1}(x). \tag{8.8}$$

Remark 8.2. (Variation Negligibility of Boundaries of Sets)

The purpose of this remark is to explain the condition $\|Du\|(\partial K) = 0$ in Theorem 8.1. For an open set $\Omega \subset \mathbb{R}^N$ and a function $u \in BV(\Omega,\mathbb{R}^d)$, it follows that $\|Du\| \ge |u^+ - u^-|\mathcal{H}^{N-1}\llcorner\mathcal{J}_u$ (refer to Lemma 3.76 in [1]). It is important to note that according to Definition 10.4, $|u^+(x) - u^-(x)| > 0$ for $x \in \mathcal{J}_u$. Therefore, for a set $E \subset \Omega$, the assumption $\|Du\|(\partial E) = 0$ indicates that $\mathcal{H}^{N-1}(\partial E \cap \mathcal{J}_u) = 0$, implying that the portion of the jump set $\mathcal{J}_u$ within the topological boundary of E is negligible with respect to $\mathcal{H}^{N-1}$.

Lemma 8.2. *(Equivalence Between Variation and Jump Variation in the BV Case)*

Let $1 < p < \infty$, $u \in BV(\mathbb{R}^N,\mathbb{R}^d) \cap B^{\frac{1}{p},p}(\mathbb{R}^N,\mathbb{R}^d)$ and $1 < q < p$. Then, for every $n \in \mathbb{R}^N$ and every Borel set $B \subset \mathbb{R}^N$ such that $\mathcal{H}^{N-1}(\partial B \cap \mathcal{J}_u) = 0$ we have

$$\lim_{\varepsilon\to0^+}\int_B \chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx = \int_{B\cap\mathcal{J}_u} \left|u^+(x)-u^-(x)\right|^q |\nu_u(x)\cdot n| d\mathcal{H}^{N-1}(x). \tag{8.9}$$

In particular, the following limit exists:

$$\begin{aligned}&\lim_{\varepsilon\to0^+}\int_{S^{N-1}}\int_B \chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx d\mathcal{H}^{N-1}(n) \\ &\qquad = \left(\int_{S^{N-1}} |z_1|\, d\mathcal{H}^{N-1}(z)\right)\int_{B\cap\mathcal{J}_u} \left|u^+(x)-u^-(x)\right|^q d\mathcal{H}^{N-1}(x). \end{aligned} \tag{8.10}$$

Remark 8.3. (The Assumption of Bounded Variation in Lemma 8.2)

In Lemma 8.2, the assumption that u has bounded variation cannot be dropped in general to obtain inequality (8.10). There are examples of functions in $B^{\frac{1}{p},p}(\mathbb{R}^N,\mathbb{R}^d)$ for which equation (8.10) does not hold. Examples can be found in [12]. Here, we will mention that bi-Hölder functions can be used to demonstrate that the jump variation of such a function is zero (the right-hand side of (8.10)), but the variation of u (the left-hand side of (8.10)) is positive.

Proof. **Step 1**: $u \in L^\infty(\mathbb{R}^N,\mathbb{R}^d)$. Let $n \in \mathbb{R}^N$, and let $B \subset \mathbb{R}^N$ be a Borel set. Let $K \subset B^o$ be a compact set such that $\|Du\|(\partial K) = 0$, where B^o is the topological interior of B. By Theorem 8.1 we obtain

$$\begin{aligned}\liminf_{\varepsilon\to 0^+}\int_B \chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx &\geq \liminf_{\varepsilon\to 0^+}\int_{B^o} \chi_{B^o}(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx\\ \geq \liminf_{\varepsilon\to 0^+}\int_K \frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx &= \int_{K\cap\mathcal{J}_u}\left|u^+(x)-u^-(x)\right|^q|\nu_u(x)\cdot n|d\mathcal{H}^{N-1}(x). \quad (8.11)\end{aligned}$$

Taking the supremum over compact sets $K \subset B^o$ such that $\|Du\|(\partial K) = 0$ we get by Lemma 10.2

$$\liminf_{\varepsilon\to 0^+}\int_B \chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx \geq \int_{B^o\cap\mathcal{J}_u}\left|u^+(x)-u^-(x)\right|^q|\nu_u(x)\cdot n|d\mathcal{H}^{N-1}(x). \quad (8.12)$$

Let Ω be an open set such that $\overline{B} \subset \Omega$ and $\|Du\|(\partial\Omega) = 0$. By Lemma 10.1, there exists a sequence of numbers $\{R_k\}_{k=1}^\infty$ such that for every $k \in \mathbb{N}$: $R_k > 0$, $R_k < R_{k+1}$, $\|Du\|(\partial B_{R_k}(0)) = 0$, and $\lim_{k\to\infty} R_k = \infty$. Note that since $\partial\left(\overline{\Omega}\cap\overline{B}_{R_k}(0)\right) \subset \partial\overline{\Omega}\cup\partial\overline{B}_{R_k}(0)$, then $\|Du\|\left(\partial\left(\overline{\Omega}\cap\overline{B}_{R_k}(0)\right)\right) = 0$. Note that if $n = 0$, then equation (8.9) holds trivially. Assume $n \neq 0$. It follows from Theorem 8.1, Lemma 10.9 and Remark 8.2

$$\begin{aligned}\limsup_{\varepsilon\to 0^+}\int_B \chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx &\leq \limsup_{\varepsilon\to 0^+}\int_\Omega \frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx\\ &\leq \limsup_{\varepsilon\to 0^+}\int_{\overline{\Omega}} \frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx\\ \leq \limsup_{\varepsilon\to 0^+}\int_{\overline{\Omega}\cap\overline{B}_{R_{k+1}}(0)} \frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx &+ \limsup_{\varepsilon\to 0^+}\int_{\overline{\Omega}\setminus\overline{B}_{R_{k+1}}(0)} \frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx\\ \leq \int_{\left(\overline{\Omega}\cap\overline{B}_{R_{k+1}}(0)\right)\cap\mathcal{J}_u}&\left|u^+(x)-u^-(x)\right|^q|\nu_u(x)\cdot n|d\mathcal{H}^{N-1}(x)\\ + 2^{q-1}\|u\|^{q-1}_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}&\limsup_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus\overline{B}_{R_{k+1}}(0)}\frac{|u(x+\varepsilon n)-u(x)|}{\varepsilon}dx\\ \leq \int_{\Omega\cap\mathcal{J}_u}\left|u^+(x)-u^-(x)\right|^q|\nu_u(x)\cdot n|d\mathcal{H}^{N-1}(x) &+ 2^{q-1}\|u\|^{q-1}_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}|n|\|Du\|\left(\mathbb{R}^N\setminus\overline{B}_{R_k}(0)\right).\end{aligned} \quad (8.13)$$

Taking the limit as $k \to \infty$ in (8.13), we get

$$\limsup_{\varepsilon\to 0^+}\int_B \chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx \le \int_{\Omega\cap\mathcal{J}_u}\left|u^+(x)-u^-(x)\right|^q|\nu_u(x)\cdot n|d\mathcal{H}^{N-1}(x). \tag{8.14}$$

Therefore, by the Lemma 10.3 we get

$$\limsup_{\varepsilon\to 0^+}\int_B \chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx \le \int_{\overline{B}\cap\mathcal{J}_u}\left|u^+(x)-u^-(x)\right|^q|\nu_u(x)\cdot n|d\mathcal{H}^{N-1}(x). \tag{8.15}$$

By (8.12) and (8.15) we get (8.9) for every Borel set $B\subset\mathbb{R}^N$ such that $\mathcal{H}^{N-1}(\partial B\cap\mathcal{J}_u)=0$.

Since by Lemma 10.9

$$\begin{aligned}\sup_{\varepsilon\in(0,\infty)}\left(\int_B \chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx\right) &\le \sup_{\varepsilon\in(0,\infty)}\left(\int_B \frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx\right)\\ &\le 2^{q-1}\|u\|^{q-1}_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\sup_{\varepsilon\in(0,\infty)}\left(\int_B \frac{|u(x+\varepsilon n)-u(x)|}{\varepsilon}dx\right)\\ &\le 2^{q-1}\|u\|^{q-1}_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}|n|\|Du\|(\mathbb{R}^N)<\infty,\end{aligned} \tag{8.16}$$

then we get by Dominated Convergence Theorem, equation (8.9), Fubini's Theorem and Proposition 10.2

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\int_{S^{N-1}}\int_B \chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dxd\mathcal{H}^{N-1}(n)\\ &= \int_{S^{N-1}}\left(\lim_{\varepsilon\to 0^+}\int_B \chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx\right)d\mathcal{H}^{N-1}(n)\\ &= \int_{S^{N-1}}\left(\int_{B\cap\mathcal{J}_u}\left|u^+(x)-u^-(x)\right|^q|\nu_u(x)\cdot n|d\mathcal{H}^{N-1}(x)\right)d\mathcal{H}^{N-1}(n)\\ &= \int_{B\cap\mathcal{J}_u}\left|u^+(x)-u^-(x)\right|^q\left(\int_{S^{N-1}}|\nu_u(x)\cdot n|d\mathcal{H}^{N-1}(n)\right)d\mathcal{H}^{N-1}(x)\\ &= \left(\int_{S^{N-1}}|e_1\cdot n|d\mathcal{H}^{N-1}(n)\right)\int_{B\cap\mathcal{J}_u}\left|u^+(x)-u^-(x)\right|^q d\mathcal{H}^{N-1}(x).\end{aligned} \tag{8.17}$$

In particular, the limit in (8.10) exists.

Step 2: u is not necessarily bounded. For every $l\in[0,\infty)$ we have $u_l\in L^\infty(\mathbb{R}^N,\mathbb{R}^d)$, where $\{u_l\}_{l\in[0,\infty)}$ is the truncated family defined in Definition 10.2. So we get for every $l\in[0,\infty)$ by the previous step the formulas

$$\lim_{\varepsilon\to 0^+}\int_B \chi_B(x+\varepsilon n)\frac{|u_l(x+\varepsilon n)-u_l(x)|^q}{\varepsilon}dx = \int_{B\cap\mathcal{J}_{u_l}}\left|(u_l)^+(x)-(u_l)^-(x)\right|^q|\nu_{u_l}(x)\cdot n|d\mathcal{H}^{N-1}(x), \tag{8.18}$$

and

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\int_{S^{N-1}}\int_B \chi_B(x+\varepsilon n)\frac{|u_l(x+\varepsilon n)-u_l(x)|^q}{\varepsilon}dxd\mathcal{H}^{N-1}(n)\\ &= \left(\int_{S^{N-1}}|z_1|\,d\mathcal{H}^{N-1}(z)\right)\int_{B\cap\mathcal{J}_{u_l}}\left|(u_l)^+(x)-(u_l)^-(x)\right|^q d\mathcal{H}^{N-1}(x).\end{aligned} \tag{8.19}$$

By Lemma 10.8 we obtain

$$\lim_{l\to\infty}\int_{B\cap\mathcal{J}_{u_l}}\left|(u_l)^+(x)-(u_l)^-(x)\right|^q|\nu_{u_l}(x)\cdot n|d\mathcal{H}^{N-1}(x)=\int_{B\cap\mathcal{J}_u}\left|u^+(x)-u^-(x)\right|^q|\nu_u(x)\cdot n|d\mathcal{H}^{N-1}(x), \tag{8.20}$$

and

$$\lim_{l\to\infty}\int_{B\cap\mathcal{J}_{u_l}}\left|(u_l)^+(x)-(u_l)^-(x)\right|^q d\mathcal{H}^{N-1}(x)=\int_{B\cap\mathcal{J}_u}\left|u^+(x)-u^-(x)\right|^q d\mathcal{H}^{N-1}(x). \tag{8.21}$$

By Corollary 8.1, we know that the truncated family u_l converges to u in Besov space $B^{1/q}_{q,\infty}$. Let us denote

$$F_\varepsilon(u):=\int_B\chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx. \tag{8.22}$$

By Lemma 6.3, we get

$$\lim_{l\to\infty}\left(\limsup_{\varepsilon\to0^+}F_\varepsilon(u_l)\right)=\limsup_{\varepsilon\to0^+}F_\varepsilon(u),\quad \lim_{l\to\infty}\left(\liminf_{\varepsilon\to0^+}F_\varepsilon(u_l)\right)=\liminf_{\varepsilon\to0^+}F_\varepsilon(u). \tag{8.23}$$

By (8.18) the limit $\lim_{\varepsilon\to0^+}F_\varepsilon(u_l)$ exists for every $l\in[0,\infty)$. Thus, by (8.23), we conclude the existence of the limit $\lim_{\varepsilon\to0^+}F_\varepsilon(u)$, and

$$\lim_{l\to\infty}\left(\lim_{\varepsilon\to0^+}F_\varepsilon(u_l)\right)=\lim_{\varepsilon\to0^+}F_\varepsilon(u). \tag{8.24}$$

Taking the limit in (8.18) as $l\to\infty$, and using (8.20) and (8.24), we obtain (8.9). By the Dominated Convergence Theorem, we deduce (8.10) from (8.9), as shown in calculation (8.17). □

Definition 8.1. (q-Jump Variation)

Let $\Omega\subset\mathbb{R}^N$ be an open set, $u\in L^1_{\text{loc}}(\Omega,\mathbb{R}^d)$, $q\in\mathbb{R}$, and $S\subset\Omega$ is an $\mathcal{H}^{N-1}$-measurable set. We define the *q-jump variation of u in S* by

$$JV_{u,q}(S):=\int_{S\cap\mathcal{J}_u}|u^+(x)-u^-(x)|^q d\mathcal{H}^{N-1}(x). \tag{8.25}$$

Let $n\in S^{N-1}$. We define the *q-jump variation of u in S in direction n* by

$$JV_{u,q,n}(S):=\int_{S\cap\mathcal{J}_u}|u^+(x)-u^-(x)|^q|\nu_u(x)\cdot n|d\mathcal{H}^{N-1}(x). \tag{8.26}$$

Corollary 8.2. *(Equivalence Between Gagliardo Constants and the q-Jump Variations)*

Let $p\in(1,\infty)$, $q\in(1,p)$, $u\in BV(\mathbb{R}^N,\mathbb{R}^d)\cap B^{\frac{1}{p},p}(\mathbb{R}^N,\mathbb{R}^d)$, $\eta\in W^{1,1}\left(\mathbb{R}^N\right)$ and $B\subset\mathbb{R}^N$ be a Borel set such that $\mathcal{H}^{N-1}(\partial B\cap\mathcal{J}_u)=0$. Then, for every kernel ρ_ε, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to0^+}\frac{1}{|\ln\varepsilon|}[u_\varepsilon]^q_{W^{1/q,q}(B,\mathbb{R}^d)}\\
&\quad=\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q\mathcal{H}^{N-1}\left(S^{N-1}\right)\lim_{\varepsilon\to0^+}\int_B\int_B\rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|}dydx\\
&\quad=\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q\lim_{\varepsilon\to0^+}\int_{S^{N-1}}\int_B\chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dxd\mathcal{H}^{N-1}(n)\\
&\quad=\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q\left(\int_{S^{N-1}}|z_1|\,d\mathcal{H}^{N-1}(z)\right)\int_{\mathcal{J}_u\cap B}\left|u^+(x)-u^-(x)\right|^q d\mathcal{H}^{N-1}(x).
\end{aligned} \tag{8.27}$$

Proof. By Lemma 8.2 the limit

$$\lim_{\varepsilon\to 0^+}\int_{S^{N-1}}\int_B \chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dxd\mathcal{H}^{N-1}(n) \tag{8.28}$$

exists, and

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_{S^{N-1}}\int_B \chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dxd\mathcal{H}^{N-1}(n)&\\ =\left(\int_{S^{N-1}}|z_1|\,d\mathcal{H}^{N-1}(z)\right)\int_{B\cap\mathcal{J}_u}\left|u^+(x)-u^-(x)\right|^q d\mathcal{H}^{N-1}(x).&\end{aligned} \tag{8.29}$$

Since the limit in (8.28) exists, we get by Theorem 7.2 with $r=\frac{1}{q}$ and $E=B$ that

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}[u_\varepsilon]^q_{W^{1/q,q}(B,\mathbb{R}^d)}&=\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q\mathcal{H}^{N-1}\left(S^{N-1}\right)\lim_{\varepsilon\to 0^+}\int_B\int_B\rho_\varepsilon(|x-y|)\frac{|u(x)-u(y)|^q}{|x-y|}dydx\\ &=\left|\int_{\mathbb{R}^N}\eta(z)dz\right|^q\lim_{\varepsilon\to 0^+}\int_{S^{N-1}}\int_B\chi_B(x+\varepsilon n)\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dxd\mathcal{H}^{N-1}(n).\end{aligned} \tag{8.30}$$

We get (8.27) by equations (8.29) and (8.30). □

8.1 Some observations about jumps of functions in $B^r_{q,\infty}=B^{r,q}$

Lemma 8.3. *(Besov Spaces Embed in Fractional Sobolev Spaces)*
Let $0<r<s<1$, $q\in[1,\infty)$. *Then,*

$$B^s_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)\subset W^{r,q}_{loc}(\mathbb{R}^N,\mathbb{R}^d). \tag{8.31}$$

Proof. Let $u\in B^s_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$ and $K\subset\mathbb{R}^N$ be a compact set. We have by additivity of integral

$$\begin{aligned}&\int_K\int_K\frac{|u(x)-u(y)|^q}{|x-y|^{rq+N}}dydx\\ &=\int_K\left(\int_{K\cap B_1(x)}\frac{|u(x)-u(y)|^q}{|x-y|^{rq+N}}dy\right)dx+\int_K\left(\int_{K\setminus B_1(x)}\frac{|u(x)-u(y)|^q}{|x-y|^{rq+N}}dy\right)dx.\end{aligned} \tag{8.32}$$

By Change of variable formula, Fubini's theorem, definition of the Besov seminorm, polar coordinates and the assumption $u\in B^s_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$, we have that

$$\begin{aligned}&\int_K\left(\int_{K\cap B_1(x)}\frac{|u(x)-u(y)|^q}{|x-y|^{rq+N}}dy\right)dx=\int_K\left(\int_{B_1(x)}\chi_K(y)\frac{|u(x)-u(y)|^q}{|x-y|^{rq+N}}dy\right)dx\\ &=\int_K\left(\int_{B_1(0)}\chi_K(x+z)\frac{|u(x)-u(x+z)|^q}{|z|^{rq+N}}dz\right)dx=\int_{B_1(0)}\left(\int_K\chi_K(x+z)\frac{|u(x)-u(x+z)|^q}{|z|^{rq+N}}dx\right)dz\\ &=\int_{B_1(0)}|z|^{sq-rq-N}\left(\int_K\chi_K(x+z)\frac{|u(x)-u(x+z)|^q}{|z|^{sq}}dx\right)dz\\ &\le[u]^q_{B^s_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\int_{B_1(0)}|z|^{sq-rq-N}dz=[u]^q_{B^s_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)}\frac{\mathcal{H}^{N-1}(S^{N-1})}{(s-r)q}<\infty.\end{aligned} \tag{8.33}$$

By monotonicity of integral, the convexity of the function $r \longmapsto r^q, r \in [0,\infty)$, and $u \in L^q(K,\mathbb{R}^d)$ we obtain that

$$\begin{aligned}\int_K \left(\int_{K\setminus B_1(x)} \frac{|u(x)-u(y)|^q}{|x-y|^{rq+N}}dy\right)dx &\le \int_K \left(\int_{K\setminus B_1(x)} |u(x)-u(y)|^q dy\right)dx \\ &\le \int_K \left(\int_K |u(x)-u(y)|^q dy\right)dx \le \int_K \left(2^{q-1}\int_K |u(x)|^q + |u(y)|^q dy\right)dx \\ &= \int_K \left(2^{q-1}|u(x)|^q \mathcal{L}^N(K) + 2^{q-1}\|u\|^q_{L^q(K,\mathbb{R}^d)}\right)dx = 2^q \|u\|^q_{L^q(K,\mathbb{R}^d)}\mathcal{L}^N(K) < \infty. \quad (8.34)\end{aligned}$$

Thus, we derive from (8.32), (8.33) and (8.34) that $u \in W^{r;q}_{\text{loc}}(\mathbb{R}^N,\mathbb{R}^d)$. □

Theorem 8.2. *($\mathcal{H}^{N-1}$-Negligibility of the Jump Set of Fractional Sobolev Functions, Theorem 1.7 in [12])*

Let $\Omega \subset \mathbb{R}^N$ be an open set, $q \in (1,\infty)$ and $u \in W^{1/q,q}_{loc}(\Omega,\mathbb{R}^d)$. Then $\mathcal{H}^{N-1}(\mathcal{J}_u) = 0$.

Corollary 8.3. *($\mathcal{H}^{N-1}$-Negligibility of the Jump Set of $u \in B^r_{q,\infty}, rq > 1$)*

Let $r \in (0,1)$ and $q \in [1,\infty)$ be such that $rq > 1$ and $u \in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$. Then $\mathcal{H}^{N-1}(\mathcal{J}_u) = 0$.

Proof. By Lemma 8.3 we have $B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d) \subset W^{1/q,q}_{\text{loc}}(\mathbb{R}^N,\mathbb{R}^d)$, so by Theorem 8.2 we get $\mathcal{H}^{N-1}(\mathcal{J}_u) = 0$. □

Remark 8.4. (Functions in $B^r_{q,\infty}, rq \le 1$, Have Jumps) If $r \in (0,1)$, $q \in [1,\infty)$ are such that $rq \le 1$, then, as was proved in Remark 8.1, $BV(\mathbb{R}^N,\mathbb{R}^d) \cap L^\infty(\mathbb{R}^N,\mathbb{R}^d) \subset B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$. Therefore, for functions $u \in B^r_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$, the measure of the jump set with respect to Hausdorff measure, $\mathcal{H}^{N-1}(\mathcal{J}_u)$, can be any value in the interval $[0,\infty]$.

9 Open questions

Question 9.1. Let $1 < q < \infty$ and $u \in B^{1/q}_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$. Does the following limit exist?

$$\lim_{\varepsilon\to 0^+}\int_{S^{N-1}}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx d\mathcal{H}^{N-1}(n). \quad (9.1)$$

Note that if the limit

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx \quad (9.2)$$

exists for $\mathcal{H}^{N-1}$-almost every $n \in S^{N-1}$, then the limit in (9.1) exists by Dominated Convergence Theorem: Since $u \in B^{1/q}_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)$, then we get by Definition 2.1 that

$$\sup_{n\in S^{N-1}}\sup_{\varepsilon\in(0,\infty)}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx \le [u]^q_{B^{1/q}_{q,\infty}(\mathbb{R}^N,\mathbb{R}^d)} < \infty, \quad (9.3)$$

so by Dominated Convergence Theorem we have the existence of the limit in (9.1) and

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_{S^{N-1}}\int_{\mathbb{R}^N}&\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx d\mathcal{H}^{N-1}(n) \\ &= \int_{S^{N-1}}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\frac{|u(x+\varepsilon n)-u(x)|^q}{\varepsilon}dx d\mathcal{H}^{N-1}(n). \quad (9.4)\end{aligned}$$

Question 9.2. Let $1 < q < \infty$, $u \in B^{1/q}_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d)$, $\eta \in W^{1,1}(\mathbb{R}^N)$. Does the following inequality hold?

$$\liminf_{\varepsilon \to 0^+} \frac{1}{|\ln \varepsilon|} [u_\varepsilon]^q_{W^{1/q,q}(\mathbb{R}^N,\mathbb{R}^d)} \\ \geq \left| \int_{\mathbb{R}^N} \eta(z) dz \right|^q \left(\int_{S^{N-1}} |z_1| \, d\mathcal{H}^{N-1}(z) \right) \int_{\mathcal{J}_u} \left| u^+(x) - u^-(x) \right|^q d\mathcal{H}^{N-1}(x). \quad (9.5)$$

Question 9.3. Let $1 < q < \infty$, $u \in L^q(\mathbb{R}^N, \mathbb{R}^d)$. Does the following implication hold?

$$\forall \eta \in W^{1,1}(\mathbb{R}^N), \quad \limsup_{\varepsilon \to 0^+} \frac{1}{|\ln \varepsilon|} [u_\varepsilon]^q_{W^{1/q,q}(\mathbb{R}^N,\mathbb{R}^d)} < \infty \quad \Longrightarrow \quad u \in B^{1/q}_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d). \quad (9.6)$$

Theorem 9.1. *(Theorem 1.3 in [12]) Let $1 \leq q < \infty$, $\Omega \subset \mathbb{R}^N$ be an open set and $u \in L^1_{loc}(\Omega, \mathbb{R}^d)$. Then,*

$$\left(\frac{1}{N} \int_{S^{N-1}} |z_1| \, d\mathcal{H}^{N-1}(z) \right) \int_{\mathcal{J}_u} |u^+(x) - u^-(x)|^q d\mathcal{H}^{N-1}(x) \\ \leq \liminf_{\varepsilon \to 0^+} \int_\Omega \left(\int_{\Omega \cap B_\varepsilon(x)} \frac{1}{\varepsilon^N} \frac{|u(x) - u(y)|^q}{|x-y|} dy \right) dx. \quad (9.7)$$

Remark 9.1. If the limit in (9.1) exists, then the answer on the other questions is yes: If the limit in (9.1) exists, then we get (9.5) from equation (7.47), Theorem 9.1 and Proposition 10.3; and we get (9.6) from equation (7.47) and Theorem 4.1.

Question 9.4. Assume $r \in (0,1)$, $q \in [1, \infty)$ and $u \in B^r_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d)$. Does the following limit hold?

$$\lim_{l \to \infty} \left(\sup_{h \in \mathbb{R}^N \setminus \{0\}} \int_{\{x \in \mathbb{R}^N \,|\, |u(x)| > l\}} \frac{|u(x+h) - u(x)|^q}{|h|^{rq}} dx \right) = 0. \quad (9.8)$$

10 Appendix

10.1 Aspects of Measure Theory

Lemma 10.1. *(Countability of Measurable Sets with Finite Measure)*

Let $(X, \mathcal{E}, \sigma)$ be a measure space, which means that X is a set, $\mathcal{E}$ is a sigma-algebra on X and $\sigma : \mathcal{E} \to [0, \infty]$ is a measure. Assume that $E \in \mathcal{E}$ is such that $\sigma(E) < \infty$. Assume $\{E_\alpha\}_{\alpha \in I}$ is a family of sets, where I is a set of indexes, such that for every $\alpha \in I$, $E_\alpha \subset E, E_\alpha \in \mathcal{E}$, and $E_\alpha \cap E_{\alpha'} = \emptyset$ for every different $\alpha, \alpha' \in I$. Define the set

$$F := \left\{ \alpha \in I : \sigma(E_\alpha) > 0 \right\}. \quad (10.1)$$

Then, F is at most countable.

Proof. Let us decompose $F = \cup_{k\in\mathbb{N}} F_k, F_k := \big\{\alpha \in I : \sigma(E_\alpha) > \frac{1}{k}\big\}$. For each $k \in \mathbb{N}$ the set F_k is finite. Otherwise, there exists a sequence $\{\alpha_j\}_{j\in\mathbb{N}} \subset F_k$ of different elements such that

$$\infty > \sigma(E) \geq \sigma\left(\bigcup_{j\in\mathbb{N}} E_{\alpha_j}\right) = \sum_{j\in\mathbb{N}} \sigma(E_{\alpha_j}) \geq \sum_{j\in\mathbb{N}} \frac{1}{k} = \infty. \tag{10.2}$$

This contradiction shows that each F_k is a finite set and hence F is at most countable set as a countable union of finite sets. □

Lemma 10.2. *(The Compact Negligible Boundary Property)*

Let (X, d) be a locally compact metric space and let μ be a positive Borel measure on X which is finite on compact sets. Then for every compact set $K \subset X$ there exists a compact set $E \subset X$ such that $K \subset E$ and $\mu(\partial E) = 0$.

Proof. Since K is compact and X is locally compact, then there exists an open set W such that $K \subset W$ and $\overline{W}$ is compact, where $\overline{W}$ is the topological closure of W. Note that since $\partial W \subset X \setminus W$, then $d(\partial W, K) \geq d(X \setminus W, K)$. Since K is compact and $X \setminus W$ is closed and $K \cap (X \setminus W) = \emptyset$, then $d(X \setminus W, K) > 0$. Therefore, $D := d(\partial W, K) > 0$. For each $\varepsilon \in (0, \infty)$ we define a set

$$W_\varepsilon := \Big\{x \in W : d(x, \partial W) \geq \varepsilon\Big\}. \tag{10.3}$$

Note that $K \subset W_\varepsilon$ for every $\varepsilon \in (0, D)$. If $\partial W = \emptyset$, then we can choose $E = \overline{W}$, because E is compact and since $\partial E \subset \partial W = \emptyset$, then $\mu(\partial E) = 0$. So we can assume that $\partial W \neq \emptyset$. Notice that for a general non-empty set $S \subset X$, the map $f(x) := d(x, S), f : X \to [0, \infty)$ is Lipschitz and so continuous. Thus, the set W_ε is a closed set. Since W_ε is a subset of the compact set $\overline{W}$, then it is compact. For every different $\varepsilon, \varepsilon' \in (0, D)$ we have $\partial W_\varepsilon \cap \partial W_{\varepsilon'} = \emptyset$: since W is open and the distance function f is continuous, then $W \cap \big\{x \in X : d(x, \partial W) > \varepsilon\big\}$ is an open set, and it is a subset of W_ε. Therefore, $\big\{x \in W : d(x, \partial W) > \varepsilon\big\} \subset W_\varepsilon^{\mathrm{o}}$, where $W_\varepsilon^{\mathrm{o}}$ is the topological interior of W_ε. Hence,

$$\partial W_\varepsilon = W_\varepsilon \setminus W_\varepsilon^{\mathrm{o}} \subset W_\varepsilon \setminus \big\{x \in W : d(x, \partial W) > \varepsilon\big\} = \big\{x \in W : d(x, \partial W) = \varepsilon\big\}, \tag{10.4}$$

and the sets $\big\{x \in W : d(x, \partial W) = \varepsilon\big\}$ are disjoint for different numbers ε. Using Lemma 10.1 with the family of sets $\{\partial W_\varepsilon\}_{\varepsilon\in(0,D)} \subset W$, $\mu(W) < \infty$, we derive the existence of $\varepsilon \in (0, D)$ such that $\mu(\partial W_\varepsilon) = 0$. We choose $E := W_\varepsilon$. □

Lemma 10.3. *(The Open Negligible Boundary Property)*

Let (X, d) be a metric space and let μ be a finite positive Borel measure on X. Let $C \subset X$ be a closed set. Then, there exists a monotone decreasing sequence of open sets $\Omega_k \subset X$ such that for every $k \in \mathbb{N}$ $\mu(\partial \Omega_k) = 0$, and $C = \bigcap_{k\in\mathbb{N}} \Omega_k$.

Proof. Define for every $\varepsilon \in (0, \infty)$

$$\Omega_\varepsilon := \big\{x \in X : d(x, C) < \varepsilon\big\}. \tag{10.5}$$

Assume that $C \neq \emptyset$; if $C = \emptyset$, then we can choose $\Omega_k = \emptyset$. Since the function $x \longmapsto d(x, C)$ is continuous, then Ω_ε is an open set. We have

$$\partial\Omega_\varepsilon = \overline{\Omega}_\varepsilon \setminus \Omega_\varepsilon \subset \big\{x \in X : d(x, C) \leq \varepsilon\big\} \setminus \Omega_\varepsilon = \big\{x \in X : d(x, C) = \varepsilon\big\}. \tag{10.6}$$

Therefore, for every different $\varepsilon_1, \varepsilon_2 \in (0, \infty)$ we get $\partial\Omega_{\varepsilon_1} \cap \partial\Omega_{\varepsilon_2} = \emptyset$. Thus, we get by Lemma 10.1 for the family $\{\partial\Omega_\varepsilon\}_{\varepsilon \in (0,\infty)}$ the existence of an infinitesimal sequence $\varepsilon_k \in (0, \infty)$ such that $\mu(\partial\Omega_{\varepsilon_k}) = 0$. Since C is closed we have $C = \bigcap_{k \in \mathbb{N}} \Omega_{\varepsilon_k}$. □

Proposition 10.1. *(Extremal Sets for Essential Infimum and Supremum)*

Let X be a set and μ be a positive measure on X. Let $f : X \to \mathbb{R}$ be a μ-measurable function. Assume that $K \subset X$ is a set with the following two properties:

1. *$\mu(X \setminus K) = 0$;*
2. *For every $\sigma \in (0, \infty)$ and $x_0 \in K$, $\mu\left(\{x \in X \,|\, |f(x) - f(x_0)| < \sigma\}\right) > 0$.*

Then,

$$\operatorname*{ess\,inf}_{x \in X} f(x) = \inf_{x \in K} f(x), \quad \operatorname*{ess\,sup}_{x \in X} f(x) = \sup_{x \in K} f(x). \tag{10.7}$$

We call K an extremal set for the function f.

Proof. Recall that

$$\operatorname*{ess\,inf}_{x \in X} f(x) := \sup_{\Theta \subset X, \mu(\Theta) = 0} \left(\inf_{x \in X \setminus \Theta} f(x)\right), \quad \operatorname*{ess\,sup}_{x \in X} f(x) := \inf_{\Theta \subset X, \mu(\Theta) = 0} \left(\sup_{x \in X \setminus \Theta} f(x)\right). \tag{10.8}$$

By property 1 of K, we obtain

$$\operatorname*{ess\,inf}_{x \in X} f(x) = \operatorname*{ess\,inf}_{x \in K} f(x), \quad \operatorname*{ess\,sup}_{x \in X} f(x) = \operatorname*{ess\,sup}_{x \in K} f(x). \tag{10.9}$$

Let us consider a set $\Theta \subset K$ such that $\mu(\Theta) = 0$. We aim to show that $\inf_{x \in K \setminus \Theta} f(x) = \inf_{x \in K} f(x)$. By taking the supremum over all such Θ, we obtain $\operatorname{ess\,inf}_{x \in K} f(x) = \inf_{x \in K} f(x)$, and hence $\operatorname{ess\,inf}_{x \in X} f(x) = \inf_{x \in K} f(x)$.

It follows from the definition of infimum that $\inf_{x \in K \setminus \Theta} f(x) \geq \inf_{x \in K} f(x)$. Suppose, by contradiction, that $\inf_{x \in K \setminus \Theta} f(x) > \inf_{x \in K} f(x)$. This implies that $\inf_{x \in K} f(x) = \inf_{x \in \Theta} f(x)$. Otherwise, if $\inf_{x \in K} f(x) < \inf_{x \in \Theta} f(x)$, then $\inf_{x \in K} f(x) = \min\big\{\inf_{x \in \Theta} f(x), \inf_{x \in K \setminus \Theta} f(x)\big\} > \inf_{x \in K} f(x)$, which leads to a contradiction.

Therefore, for any $\varepsilon \in (0, \infty)$, there exists $x_0 \in \Theta$ such that $f(x_0) - \inf_{x \in K} f(x) < \frac{\varepsilon}{2}$. By properties 1,2 of K, there exists $y \in K \setminus \Theta$ such that $|f(y) - f(x_0)| < \frac{\varepsilon}{2}$. Hence,

$$0 < \inf_{x \in K \setminus \Theta} f(x) - \inf_{x \in K} f(x) = \left(f(x_0) - \inf_{x \in K} f(x)\right) + (f(y) - f(x_0)) + \left(\inf_{x \in K \setminus \Theta} f(x) - f(y)\right) < \varepsilon. \tag{10.10}$$

Since ε is arbitrarily small, we arrive at a contradiction, which proves that $\inf_{x \in K \setminus \Theta} f(x) = \inf_{x \in K} f(x)$. The proof of of formula $\operatorname{ess\,sup}_{x \in X} f(x) = \sup_{x \in K} f(x)$ is similar. □

Corollary 10.1. *(Existence of Extremal Sets for Lebesgue Functions)*

Let X be a metric space, and let μ be a Borel measure on X such that $0 < \mu(B_r(x)) < \infty$ for every $r \in (0,\infty)$ and every $x \in X$. Suppose $p \in [1,\infty)$ and $f \in L^p(X)$. Then, there exists a set $K \subset X$ with properties 1 and 2 as outlined in Proposition 10.1. More precisely, the set of Lebesgue points of f possesses these properties.

Proof. Since $f \in L^p(X)$, by the Lebesgue Differentiation Theorem, we know that almost every point in X is a Lebesgue point of f with respect to μ. Let us denote this set by K. Therefore, we have property 1: $\mu(X \setminus K) = 0$. To establish property 2, let $x_0 \in K$ and $\alpha \in (0,1)$. Note that for an arbitrary positive number σ, there exists R such that

$$\int_{B_R(x_0)} |f(x) - f(x_0)|^p d\mu(x) < \alpha\sigma^p \mu\left(B_R(x_0)\right). \tag{10.11}$$

By Chebyshev's inequality

$$\frac{\mu\left(\{x \in B_R(x_0) \,|\, |f(x) - f(x_0)| > \sigma\}\right)}{\mu(B_R(x_0))} \leq \frac{1}{\sigma^p} \fint_{B_R(x_0)} |f(x) - f(x_0)|^p d\mu(x) < \alpha. \tag{10.12}$$

Since f is μ-measurable, we obtain

$$\frac{\mu\left(\{x \in B_R(x_0) \,|\, |f(x) - f(x_0)| > \sigma\}\right)}{\mu(B_R(x_0))} + \frac{\mu\left(\{x \in B_R(x_0) \,|\, |f(x) - f(x_0)| \leq \sigma\}\right)}{\mu(B_R(x_0))} = 1. \tag{10.13}$$

Therefore,

$$\frac{\mu\left(\{x \in B_R(x_0) \,|\, |f(x) - f(x_0)| \leq \sigma\}\right)}{\mu(B_R(x_0))} \geq 1 - \alpha, \tag{10.14}$$

and hence,

$$\mu\left(\{x \in X \,|\, |f(x) - f(x_0)| \leq \sigma\}\right) \geq (1-\alpha)\mu(B_R(x_0)) > 0. \tag{10.15}$$

□

10.2 Vector Valued Measures and Variation

Definition 10.1. (Vector Valued Measures and Variation)

Let X be a set and $\mathcal{E}$ be a σ-algebra on X. Let $\mu : \mathcal{E} \to \mathbb{R}^d$ be a *measure*, which means that $\mu(\emptyset) = 0$ and for any sequence $\{E_j\}_{j\in\mathbb{N}} \subset \mathcal{E}$ of pairwise disjoint sets we have $\mu\left(\bigcup_{j\in\mathbb{N}} E_j\right) = \sum_{j\in\mathbb{N}} \mu\left(E_j\right)$. The *variation* of μ is defined to be

$$\|\mu\|(E) := \sup\left\{\sum_{j\in\mathbb{N}} |\mu(E_j)| : E_j \in \mathcal{E} \text{ pairwise disjoint}, E = \bigcup_{j\in\mathbb{N}} E_j\right\}, \quad E \in \mathcal{E}. \tag{10.16}$$

Lemma 10.4. *(Variation of Multiplication of a Vector Valued Function with Positive Measure, Proposition 1.23 in [1])*

Let μ be a positive measure on the measurable space $(X, \mathcal{E})$, X is a set and $\mathcal{E}$ is a σ-algebra on X. Let $f \in L^1(X, \mathbb{R}^N)$. Then, the variation of the $\mathbb{R}^N$-valued measure

$$f\mu(B) := \int_B f d\mu, \quad B \in \mathcal{E} \tag{10.17}$$

satisfies

$$\|f\mu\|(B) = \int_B |f| d\mu, \quad B \in \mathcal{E}. \tag{10.18}$$

Lemma 10.5. *(Variation of Multiplication of Scalar Function with Vector Valued Measure)*

Let X be a set, $\mathcal{E}$ be a σ-algebra on X and $\mu : \mathcal{E} \to \mathbb{R}^N$ be a measure. Let $f : X \to \mathbb{R}$ be such that $f \in L^1(X, \|\mu\|)$. Then,

$$\|f\mu\|(E) \leq N^{1/2}|f|\|\mu\|(E), \quad E \in \mathcal{E}. \tag{10.19}$$

Proof. Let us denote $\mu := (\mu_1, ..., \mu_N)$. For every $E \in \mathcal{E}$

$$\begin{aligned} |f\mu(E)| = |(f\mu_1(E), ..., f\mu_N(E))| &= \left(\sum_{i=1}^N (f\mu_i(E))^2\right)^{1/2} \\ &\leq \left(\sum_{i=1}^N (\|f\mu_i\|(E))^2\right)^{1/2} \leq \left(\sum_{i=1}^N (|f|\|\mu_i\|(E))^2\right)^{1/2} \leq N^{1/2}|f|\|\mu\|(E). \end{aligned} \tag{10.20}$$

Therefore,

$$\begin{aligned} &\|f\mu\|(E) = \sup\left\{\sum_{j\in\mathbb{N}} |f\mu(E_j)| : E_j \in \mathcal{E} \text{ pairwise disjoint}, E = \cup_{j\in\mathbb{N}} E_j\right\} \\ &\leq N^{1/2} \sup\left\{\sum_{j\in\mathbb{N}} |f|\|\mu\|(E_j) : E_j \in \mathcal{E} \text{ pairwise disjoint}, E = \cup_{j\in\mathbb{N}} E_j\right\} = N^{1/2}|f|\|\mu\|(E). \end{aligned} \tag{10.21}$$

□

10.3 Aspects of Integration on S^{N-1} with respect to $\mathcal{H}^{N-1}$

Proposition 10.2. *For every $v_1, v_2 \in S^{N-1}$ we have*

$$\int_{S^{N-1}} |v_1 \cdot n| \, d\mathcal{H}^{N-1}(n) = \int_{S^{N-1}} |v_2 \cdot n| \, d\mathcal{H}^{N-1}(n). \tag{10.22}$$

Proof. Take an isometry $A : \mathbb{R}^N \to \mathbb{R}^N$ such that $A(v_2) = v_1$. Then,

$$\int_{S^{N-1}} |v_1 \cdot n| \, d\mathcal{H}^{N-1}(n) = \int_{A^{-1}(S^{N-1})} |A(v_2) \cdot A(w)| \, d\mathcal{H}^{N-1}(w) = \int_{S^{N-1}} |v_2 \cdot w| \, d\mathcal{H}^{N-1}(w). \tag{10.23}$$

□

Proposition 10.3. *It follows that*

$$\int_{\mathbb{R}^{N-1}} \frac{2dv}{\left(\sqrt{1+|v|^2}\right)^{N+1}} = \int_{S^{N-1}} |z_1| d\mathcal{H}^{N-1}(z), \quad z = (z_1, ..., z_N). \tag{10.24}$$

Proof. Note that (10.24) holds for $N = 1$. So we can assume that $N > 1$. Let $B_1^{N-1}(0)$ be the ball of radius 1 around the origin in $\mathbb{R}^{N-1}$. Define

$$g : B_1^{N-1}(0) \to \mathbb{R}^N, \quad g(z_2, ..., z_N) := \left(f(z_2, ..., z_N), z_2, ..., z_N\right), \quad f(z_2, ..., z_N) := \sqrt{1 - \sum_{j=2}^{N} z_j^2}. \tag{10.25}$$

The image of g is $S^+ := \left\{z = (z_1, ..., z_N) \in S^{N-1} : z_1 > 0\right\}$. Denote $z = (z_1, z')$, $z' := (z_2, ..., z_N)$. By the area formula

$$\begin{aligned}\int_{S^{N-1}} |z_1| d\mathcal{H}^{N-1}(z) &= 2\int_{S^+} |z_1| d\mathcal{H}^{N-1}(z) = 2\int_{B_1^{N-1}(0)} \sqrt{1 - \sum_{j=2}^{N} z_j^2}\, \sqrt{1 + |\nabla f(z')|^2}\, d\mathcal{L}^{N-1}(z') \\ &= 2\int_{B_1^{N-1}(0)} \sqrt{1 - \sum_{j=2}^{N} z_j^2}\, \sqrt{1 + \frac{1}{1 - \sum_{j=2}^{N} z_j^2} \sum_{j=2}^{N} z_j^2}\, d\mathcal{L}^{N-1}(z') = 2\Big(\mathcal{L}^{N-1}\left(B_1^{N-1}(0)\right)\Big).\end{aligned} \tag{10.26}$$

In addition, by polar coordinates we obtain for $N \geq 2$

$$\int_{\mathbb{R}^{N-1}} \frac{dv}{\left(\sqrt{1+|v|^2}\right)^{N+1}} = \Big(\mathcal{H}^{N-2}(S^{N-2})\Big) \int_0^\infty \frac{r^{N-2} dr}{\left(\sqrt{1+r^2}\right)^{N+1}}. \tag{10.27}$$

Let us denote

$$A_N := \int_0^\infty \frac{r^{N-2} dr}{\left(\sqrt{1+r^2}\right)^{N+1}}. \tag{10.28}$$

Assume for the moment that $N > 3$. Integration by parts gives

$$\begin{aligned}A_N &= \int_0^\infty r^{N-3} \frac{2r}{2\left(1+r^2\right)^{\frac{N+1}{2}}} dr \\ &= \left(r^{N-3} \frac{1}{(1-N)\left(1+r^2\right)^{\frac{N-1}{2}}}\right)\Bigg|_{r=0}^{r=\infty} - \int_0^\infty (N-3) r^{N-4} \frac{1}{(1-N)\left(1+r^2\right)^{\frac{N-1}{2}}} dr \\ &= \frac{N-3}{N-1} \int_0^\infty r^{N-4} \frac{1}{\left(1+r^2\right)^{\frac{N-1}{2}}} dr = \frac{N-3}{N-1} A_{N-2}.\end{aligned} \tag{10.29}$$

We got a recursive sequence. Note for example

$$A_N = \frac{N-3}{N-1} A_{N-2} = \frac{N-3}{N-1}\frac{N-5}{N-3} A_{N-4} = \frac{N-5}{N-1} A_{N-4} = \frac{N-5}{N-1}\frac{N-7}{N-5} A_{N-6} = \frac{N-7}{N-1} A_{N-6}. \tag{10.30}$$

Therefore, we get from (10.29) for every natural $m > 1$

$$A_{2m} = \frac{1}{2m-1} A_2 \quad \text{and} \quad A_{2m+1} = \frac{2}{2m} A_3 \,. \tag{10.31}$$

Let us calculate A_2, A_3 separately. Note that

$$A_3 := \int_0^\infty \frac{2r dr}{2\big(1+r^2\big)^2} = -\frac{1}{2\big(1+r^2\big)}\bigg|_{r=0}^{r=\infty} = \frac{1}{2}. \tag{10.32}$$

Let us prove that

$$A_2 := \int_0^\infty \frac{1}{\big(\sqrt{1+r^2}\big)^3} dr = 1. \tag{10.33}$$

Changing variables $r = \frac{z}{2} - \frac{1}{2z}$ in the last integral gives:

$$\begin{aligned} A_2 &= \int_1^\infty \frac{1}{\left(\sqrt{1+\left(\frac{z}{2}-\frac{1}{2z}\right)^2}\right)^3}\left(\frac{1}{2}+\frac{1}{2z^2}\right)dz = \int_1^\infty \frac{1}{\left(\sqrt{1+\frac{1}{4z^2}\left(z^2-1\right)^2}\right)^3}\frac{1}{2z^2}\left(z^2+1\right)dz \\ &= \int_1^\infty \frac{1}{\left(\sqrt{4z^2+\left(z^2-1\right)^2}\right)^3} 4z\left(z^2+1\right)dz = \int_1^\infty \frac{1}{\left(\sqrt{\left(z^2+1\right)^2}\right)^3} 4z\left(z^2+1\right)dz \\ &= \int_1^\infty \frac{4z}{(z^2+1)^2} dz = -\frac{2}{z^2+1}\bigg|_{z=1}^{z=\infty} = 1. \end{aligned} \tag{10.34}$$

Therefore, by (10.31), (10.32) and (10.33)

$$A_{2m} = \frac{1}{2m-1} \quad \text{and} \quad A_{2m+1} = \frac{1}{2m}. \tag{10.35}$$

Thus, for every natural $N > 1$

$$A_N = \frac{1}{N-1} \,. \tag{10.36}$$

Therefore, by (10.27), (10.28), (10.36) and polar coordinates we get for every $N > 1$

$$\int_{\mathbb{R}^{N-1}} \frac{dv}{\big(\sqrt{1+|v|^2}\big)^{N+1}} = \Big(\mathcal{H}^{N-2}(S^{N-2})\Big)\frac{1}{N-1} = \mathcal{L}^{N-1}\big(B_1^{N-1}(0)\big). \tag{10.37}$$

Thus, by (10.26) and (10.37) we get (10.24). □

Proposition 10.4. *(Polar coordinates, see 3.4.4 in [9])*
Let $g \in L^1\left(\mathbb{R}^N, \mathbb{R}^d\right)$. Then

$$\int_{\mathbb{R}^N} g(x)dx = \int_0^\infty \left(\int_{\partial B_r(0)} g(z) d\mathcal{H}^{N-1}(z)\right) dr = \int_0^\infty r^{N-1}\left(\int_{S^{N-1}} g(rz) d\mathcal{H}^{N-1}(z)\right) dr. \tag{10.38}$$

10.4 Sequences of Real Numbers

Lemma 10.6. *(Liminfsup Lemma)*

Let $\{a_k\}_{k=1}^{\infty}, \{b_k\}_{k=1}^{\infty} \subset \mathbb{R}$ be bounded sequences. Then,

$$\max\left\{|\liminf_{k\to\infty} a_k - \liminf_{k\to\infty} b_k|, |\limsup_{k\to\infty} a_k - \limsup_{k\to\infty} b_k|\right\} \le \limsup_{k\to\infty} |a_k - b_k|. \tag{10.39}$$

Proof. Recall the general inequalities:

$$\limsup_{k\to\infty}(a_k + b_k) \le \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k, \tag{10.40}$$

$$\liminf_{k\to\infty}(a_k + b_k) \le \limsup_{k\to\infty} a_k + \liminf_{k\to\infty} b_k. \tag{10.41}$$

By (10.40) we get

$$\limsup_{k\to\infty} a_k = \limsup_{k\to\infty}(a_k - b_k + b_k) \le \limsup_{k\to\infty}(a_k - b_k) + \limsup_{k\to\infty} b_k. \tag{10.42}$$

Changing the roles of a_k and b_k, we get

$$\left|\limsup_{k\to\infty} a_k - \limsup_{k\to\infty} b_k\right| \le \limsup_{k\to\infty} |a_k - b_k|. \tag{10.43}$$

By (10.41) we get

$$\liminf_{k\to\infty} a_k = \liminf_{k\to\infty}(a_k - b_k + b_k) \le \limsup_{k\to\infty}(a_k - b_k) + \liminf_{k\to\infty} b_k. \tag{10.44}$$

Changing the roles of a_k and b_k, we get

$$\left|\liminf_{k\to\infty} a_k - \liminf_{k\to\infty} b_k\right| \le \limsup_{k\to\infty} |a_k - b_k|. \tag{10.45}$$

□

10.5 The Truncated Family

Definition 10.2. (Truncated Family)

Let $E \subset \mathbb{R}^N$ be a set and let $u : E \to \mathbb{R}^d, u = (u^1, ..., u^d)$ be a function. For every $1 \le i \le d, i \in \mathbb{N}$, $l \in [0, \infty)$ and $x \in E$ we define $u_l^i(x) := l \wedge (-l \vee u^i(x))$, where $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, for $a, b \in \mathbb{R}$; and we define $u_l(x) := (u_l^1(x), ..., u_l^d(x))$. We call the family of functions $\{u_l\}_{l\in[0,\infty)}$ the *truncated family obtained by u*.

Proposition 10.5. *(Properties of the Truncated Family)*

Let $E \subset \mathbb{R}^N$ be a set and let $u : E \to \mathbb{R}^d, u = (u^1, ..., u^d)$ be a function. Let $\{u_l\}_{l\in[0,\infty)}$ be the truncated family obtained by u. Then,

1. $\lim_{l\to\infty} u_l(x) = u(x), \quad \forall x \in E$;

2. For every $x, y \in E$ and $l, m \in [0, \infty), l \le m$, we have $|u_l(x) - u_l(y)| \le |u_m(x) - u_m(y)| \le |u(x) - u(y)|$;

3. For every $x, y \in E$, the family $\{|u_l(x) - u_l(y)|\}_{l\in[0,\infty)}$ is monotone increasing to $|u(x) - u(y)|$.

Proof. For $x \in \mathbb{R}, l \in [0,\infty)$ we define $x_l := l \wedge (-l \vee x)$. Notice that for every $x, y \in \mathbb{R}$ and $l, m \in [0,\infty), l < m$, we have $|x_l - y_l| \leq |x_m - y_m|$. For a point $x = (x^1, ..., x^d) \in \mathbb{R}^d$, and $l \in [0,\infty)$ we define $x_l := (x_l^1, ..., x_l^d)$. Similarly, we have for every $x, y \in \mathbb{R}^d$ and $l, m \in [0,\infty), l < m$, the inequality $|x_l - y_l| \leq |x_m - y_m|$. Notice also that for $x \in \mathbb{R}^d$, the family $\{x_l\}_{l \in [0,\infty)}$ has the property $\lim_{l\to\infty} x_l = x$. In particular, for every $x, y \in \mathbb{R}^d$, the family $\{|x_l - y_l|\}_{l\in[0,\infty)}$ is monotone increasing to $|x-y|$. Therefore, we get items 1,2 and 3 by choosing the points $u(x), u(y)$ in place of the points x, y. □

10.6 Approximate Continuity and Differentiability of L^1_{loc}-functions

Definition 10.3. (Approximate Limit)

Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^d)$. We say that u has approximate limit at $x \in \Omega$ if and only if there exists $z \in \mathbb{R}^d$ such that

$$\lim_{\rho\to 0^+} \fint_{B_\rho(x)} |u(y) - z| dy = 0. \tag{10.46}$$

The set $\mathcal{S}_u$ of points where this property does not hold is called the *approximate discontinuity set.* For any $x \in \Omega$ the point z, uniquely determined by (10.46), is called the *approximate limit of* u *at* x and denoted by $\tilde{u}(x)$.

Definition 10.4. (Approximate Jump Points)

Let $\Omega \subset \mathbb{R}^N$ be an open set, $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^d)$ and $x \in \Omega$. We say that x is an *approximate jump point* of u if and only if there exist different $a, b \in \mathbb{R}^d$ and $\nu \in S^{N-1}$ such that

$$\lim_{\rho\to 0^+} \frac{1}{\rho^N} \left(\int_{B^+_\rho(x,\nu)} |u(z) - a| dz + \int_{B^-_\rho(x,\nu)} |u(z) - b| dz \right) = 0, \tag{10.47}$$

where

$$B^+_\rho(x,\nu) := \big\{ y \in B_\rho(x) : (y - x)\cdot \nu > 0 \big\}, \quad B^-_\rho(x,\nu) := \big\{ y \in B_\rho(x) : (y - x)\cdot \nu < 0 \big\}. \tag{10.48}$$

The triple (a, b, ν), uniquely determined by (10.47) up to a permutation of (a, b) and the change of sign of ν, is denoted by $(u^+(x), u^-(x), \nu_u(x))$. The set of approximate jump points is denoted by $\mathcal{J}_u$. Note that $\mathcal{J}_u \subset \mathcal{S}_u$.

Definition 10.5. (Approximate Differentiability, definition 3.70 in [1])

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^d)$. Let $x \in \Omega \setminus \mathcal{S}_u$. We say that u is approximately differentiable at x if there exists a $d \times N$ matrix L such that

$$\lim_{\rho\to 0^+} \fint_{B_\rho(x)} \frac{|u(y) - \tilde{u}(x) - L(y-x)|}{\rho} dy = 0. \tag{10.49}$$

If u is approximately differentiable at x, the matrix L, uniquely determined by (10.49), is called the *approximate differential* of u at x and denoted by $\nabla u(x)$. The set of approximate differentiability points of u is denoted by $\mathcal{D}_u$.

Proposition 10.6. *(Properties of Approximate Differential, Proposition 3.71 in [1])*

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in L^1_{loc}(\Omega, \mathbb{R}^d)$. Then, $\mathcal{D}_u$ is a Borel set and the map $\nabla u : \mathcal{D}_u \to \mathbb{R}^{dN}$ is a Borel map.

Proposition 10.7. *(Locality Properties of Approximate Differential, Proposition 3.73 in [1])*

Let $\Omega \subset \mathbb{R}^N$ be an open set, $u, v \in L^1_{loc}(\Omega, \mathbb{R}^d)$. If $x \in \mathcal{D}_u \cap \mathcal{D}_v$ and the set $\{u = v\}$ has density 1 at x, then $\nabla u(x) = \nabla v(x)$. In particular, $\nabla u(x) = \nabla v(x)$ for $\mathcal{L}^N$-almost every $x \in \{u = v\} \cap \mathcal{D}_u \cap \mathcal{D}_v$.

Proposition 10.8. *(Properties of Approximate Limits, Proposition 3.64 in [1])*

Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in L^1_{loc}(\Omega, \mathbb{R}^d)$.
(a) $\mathcal{S}_u$ is a Borel set, $\mathcal{L}^N(\mathcal{S}_u) = 0$ and $\tilde{u} : \Omega \setminus \mathcal{S}_u \to \mathbb{R}^d$ is a Borel function, coinciding $\mathcal{L}^N-$almost everywhere in $\Omega \setminus \mathcal{S}_u$ with u;
(b) if $f : \mathbb{R}^d \to \mathbb{R}^p$ is a Lipschitz map and $v = f \circ u$, then $\mathcal{S}_v \subset \mathcal{S}_u$ and $\tilde{v}(x) = f(\tilde{u}(x))$ for any $x \in \Omega \setminus \mathcal{S}_u$.

Proposition 10.9. *(Properties of One-Sided Approximate Limits, Proposition 3.69 in [1])*

Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in L^1_{loc}(\Omega, \mathbb{R}^d)$.
(a) The set $\mathcal{J}_u$ is a Borel subset of $\mathcal{S}_u$ and there exist Borel functions

$$\left(u^+, u^-, \nu_u\right) : \mathcal{J}_u \to \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1} \tag{10.50}$$

such that for every $x \in \mathcal{J}_u$ we have

$$\lim_{\rho \to 0^+} \fint_{B^+_\rho(x, \nu_u(x))} |u(y) - u^+(x)| dy = 0, \quad \lim_{\rho \to 0^+} \fint_{B^-_\rho(x, \nu_u(x))} |u(y) - u^-(x)| dy = 0. \tag{10.51}$$

(b) if $f : \mathbb{R}^d \to \mathbb{R}^p$ is a Lipschitz map, $v = f \circ u$ and $x \in \mathcal{J}_u$, then $x \in \mathcal{J}_v$ if and only if $f(u^+(x)) \neq f(u^-(x))$, and in this case

$$\left(v^+(x), v^-(x), \nu_v(x)\right) = \left(f(u^+(x)), f(u^-(x)), \nu_u(x)\right). \tag{10.52}$$

Otherwise, $x \notin \mathcal{S}_v$ and $\tilde{v}(x) = f(u^+(x)) = f(u^-(x))$.

Proposition 10.10. *(Truncation and Jumps)*

Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $u \in L^1_{loc}(\Omega, \mathbb{R}^d)$. For each $l \in [0, \infty)$, let us define the l-truncated function by $T_l : \mathbb{R}^d \to \mathbb{R}^d, T_l(x) := x_l$, where x_l is defined as in the proof of Proposition 10.5. Then we have the following assertions:
1. T_l is a Lipschitz map;
2. The jumps set of u can be decomposed in terms of the jump sets of $T_l \circ u$ through the formula:

$$\mathcal{J}_u = \bigcup_{l \in [0, \infty)} \mathcal{J}_{T_l \circ u} \cap \mathcal{J}_u; \tag{10.53}$$

3. For every $l, m \in [0, \infty)$ such that $l \leq m$ we have the following monotonicity property:

$$\mathcal{J}_{T_l \circ u} \cap \mathcal{J}_u \subset \mathcal{J}_{T_m \circ u} \cap \mathcal{J}_u. \tag{10.54}$$

Proof. 1. For each $l \in [0, \infty)$, by Proposition 10.5 we get that the map $T_l : \mathbb{R}^d \to \mathbb{R}^d, T_l(x) := x_l$, is Lipschitz.
2. For $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^d)$, where $\Omega \subset \mathbb{R}^N$ is an open set, and $x \in \mathcal{J}_u$, we know by Proposition 10.9 that $x \in \mathcal{J}_{T_l \circ u}$ if and only if $T_l(u^+(x)) \neq T_l(u^-(x))$, and in this case

$$\left((T_l \circ u)^+(x), (T_l \circ u)^-(x), \nu_{T_l \circ u}(x)\right) = \left(T_l(u^+(x)), T_l(u^-(x)), \nu_u(x)\right); \tag{10.55}$$

and if $T_l(u^+(x)) = T_l(u^-(x))$, then $x \notin \mathcal{S}_{T_l \circ u}$. Thus, since for every $x \in \mathcal{J}_u$ there exists a big enough $l \in [0, \infty)$ such that $T_l(u^+(x)) = u^+(x) \neq u^-(x) = T_l(u^-(x))$, we have

$$\mathcal{J}_u = \bigcup_{l \in [0,\infty)} \mathcal{J}_{T_l \circ u} \cap \mathcal{J}_u. \tag{10.56}$$

3. we have for every $l, m \in [0, \infty), l \leq m$, that $\mathcal{J}_{T_l \circ u} \cap \mathcal{J}_u \subset \mathcal{J}_{T_m \circ u} \cap \mathcal{J}_u$: If $x \in \mathcal{J}_{T_l \circ u} \cap \mathcal{J}_u$, then $T_l(u^+(x)) \neq T_l(u^-(x))$ and so $T_m(u^+(x)) \neq T_m(u^-(x))$. If not, then $T_m(u^+(x)) = T_m(u^-(x))$ and then $x \notin \mathcal{S}_{T_m \circ u}$, and since $T_l \circ (T_m \circ u) = T_l \circ u$, then, by part (b) of Proposition 10.8 with T_l in place of f and $T_m \circ u$ in place of u, we obtain $\mathcal{S}_{T_l \circ u} \subset \mathcal{S}_{T_m \circ u}$ and so $x \notin \mathcal{S}_{T_l \circ u}$. It is a contradiction since $x \in \mathcal{J}_{T_l \circ u} \subset \mathcal{S}_{T_l \circ u}$. From $T_m(u^+(x)) \neq T_m(u^-(x))$ and $x \in \mathcal{J}_u$ we get $x \in \mathcal{J}_{T_m \circ u} \cap \mathcal{J}_u$. □

Lemma 10.7. *(Lower Semi-Continuity for Jump-Integral with respect to the Truncated Family) Let $\Omega \subset \mathbb{R}^N$ be an open set, $u \in L^1_{loc}(\Omega, \mathbb{R}^d)$, $h : \mathbb{R}^N \to \mathbb{R}$ be a non-negative, $\mathcal{H}^{N-1}$-measurable function and $F : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. Then,*

$$\liminf_{l \to \infty} \int_{\mathcal{J}_{u_l}} F\left(|(u_l)^+(x) - (u_l)^-(x)|\right) h(x) d\mathcal{H}^{N-1}(x) \geq \int_{\mathcal{J}_u} F\left(|u^+(x) - u^-(x)|\right) h(x) d\mathcal{H}^{N-1}(x), \tag{10.57}$$

where $\{u_l\}_{l \in [0,\infty)}$ is the truncated family obtained by u.

Proof. By Proposition 10.10 we obtain

$$\begin{aligned}\int_{\mathcal{J}_{u_l}} F\left(|(u_l)^+(x) - (u_l)^-(x)|\right) h(x) d\mathcal{H}^{N-1}(x) &\geq \int_{\mathcal{J}_{u_l} \cap \mathcal{J}_u} F\left(|(u_l)^+(x) - (u_l)^-(x)|\right) h(x) d\mathcal{H}^{N-1}(x) \\ &= \int_{\mathcal{J}_{u_l} \cap \mathcal{J}_u} F\left(|(u^+(x))_l - (u^-(x))_l|\right) h(x) d\mathcal{H}^{N-1}(x) \\ &= \int_{\mathcal{J}_u} \chi_{\mathcal{J}_{u_l} \cap \mathcal{J}_u}(x) F\left(|(u^+(x))_l - (u^-(x))_l|\right) h(x) d\mathcal{H}^{N-1}(x). \end{aligned} \tag{10.58}$$

By Proposition 10.10 we have $\lim_{l \to \infty} \chi_{\mathcal{J}_{u_l} \cap \mathcal{J}_u}(x) = \chi_{\mathcal{J}_u}(x), \forall x \in \mathcal{J}_u$, and by Proposition 10.5 we have $\lim_{l \to \infty} |(u^+(x))_l - (u^-(x))_l| = |u^+(x) - u^-(x)|, \forall x \in \mathcal{J}_u$. Taking the lower limit as $l \to \infty$ on both sides of (10.58) and using Fatou's lemma we obtain (10.57). □

10.7 Aspects of BV-Functions

Definition 10.6. (Definition of BV Functions) Let $\Omega \subset \mathbb{R}^N$ be an open set. We say that $u \in BV(\Omega, \mathbb{R}^d)$ if and only if $u \in L^1(\Omega, \mathbb{R}^d)$ and there exists an $d \times N$ matrix valued measure μ :

$\mathcal{B}(\Omega)^3 \to \mathbb{R}^{d\times N}$ such that for every $\varphi \in C_c^\infty(\Omega)$ it follows that

$$\int_\Omega u(x)\nabla\varphi(x)dx = -\int_\Omega \varphi(x)d\mu(x). \tag{10.59}$$

In this case we denote $\mu := Du$. In formula (10.59) we think about u as a column vector $u = (u_1, ..., u_d)^T$ and $\nabla\varphi = (\partial_1\varphi, ..., \partial_N\varphi)$.

Lemma 10.8. *(Continuity for Jump-Integral with respect to the Truncated Family)*

Let $\Omega \subset \mathbb{R}^N$ be an open set, $u \in BV_{loc}(\Omega, \mathbb{R}^d)$. Let $h : \mathbb{R}^N \to \mathbb{R}$ be a non-negative $\mathcal{H}^{N-1}$-measurable function, and $F : \mathbb{R} \to \mathbb{R}$ be a non-negative, monotone increasing function. Let $\{u_l\}_{l\in[0,\infty)}$ be the truncated family obtained by u. Then,

1.

$$\lim_{l\to\infty}\int_{\mathcal{J}_{u_l}} F\left(|(u_l)^+(x) - (u_l)^-(x)|\right) h(x)d\mathcal{H}^{N-1}(x) = \int_{\mathcal{J}_u} F\left(|u^+(x) - u^-(x)|\right) h(x)d\mathcal{H}^{N-1}(x). \tag{10.60}$$

2. *For every $n \in \mathbb{R}^N$*

$$\lim_{l\to\infty}\int_{\mathcal{J}_{u_l}} F\left(|(u_l)^+(x) - (u_l)^-(x)|\right) |\nu_{u_l}(x)\cdot n| h(x)d\mathcal{H}^{N-1}(x)$$
$$= \int_{\mathcal{J}_u} F\left(|u^+(x) - u^-(x)|\right) |\nu_u(x)\cdot n| h(x)d\mathcal{H}^{N-1}(x). \tag{10.61}$$

Proof. Let us prove assertion 1. Since $u \in BV_{\mathrm{loc}}(\Omega, \mathbb{R}^d)$, then for every $l \in [0, \infty)$ we have by chain rule for BV-functions (refer to Theorem 10.5) that $u_l \in BV_{\mathrm{loc}}(\Omega, \mathbb{R}^d)$, and by Federer-Vol'pert theorem (refer to Theorem 10.2) we have $\mathcal{H}^{N-1}(\mathcal{S}_u \setminus \mathcal{J}_u) = \mathcal{H}^{N-1}(\mathcal{S}_{u_l} \setminus \mathcal{J}_{u_l}) = 0$. Therefore, $\mathcal{H}^{N-1}(\mathcal{J}_{u_l} \setminus \mathcal{J}_u) = \mathcal{H}^{N-1}(\mathcal{S}_{u_l} \setminus \mathcal{S}_u) = 0$, because $\mathcal{S}_{u_l} \subset \mathcal{S}_u$. Therefore, by item (b) of Proposition 10.9 we get

$$\int_{\mathcal{J}_{u_l}} F\left(|(u_l)^+(x) - (u_l)^-(x)|\right) h(x)d\mathcal{H}^{N-1}(x) = \int_{\mathcal{J}_{u_l}\cap\mathcal{J}_u} F\left(|(u_l)^+(x) - (u_l)^-(x)|\right) h(x)d\mathcal{H}^{N-1}(x)$$
$$+\int_{\mathcal{J}_{u_l}\setminus\mathcal{J}_u} F\left(|(u_l)^+(x) - (u_l)^-(x)|\right) h(x)d\mathcal{H}^{N-1}(x) = \int_{\mathcal{J}_{u_l}\cap\mathcal{J}_u} F\left(|(u^+(x))_l - (u^-(x))_l|\right) h(x)d\mathcal{H}^{N-1}(x)$$
$$= \int_{\mathcal{J}_u} \chi_{\mathcal{J}_{u_l}\cap\mathcal{J}_u}(x) F\left(|(u^+(x))_l - (u^-(x))_l|\right) h(x)d\mathcal{H}^{N-1}(x). \tag{10.62}$$

By Proposition 10.5, Proposition 10.10 and monotone convergence theorem we get (10.60) by taking the limit as $l \to \infty$ on both sides of (10.62).

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For assertion 2, note that, by item (b) of Proposition 10.9 we get

$$\begin{aligned}
&\int_{\mathcal{J}_{u_l}} F\left(|(u_l)^+(x)-(u_l)^-(x)|\right)|\nu_{u_l}(x)\cdot n|h(x)d\mathcal{H}^{N-1}(x)\\
&\qquad=\int_{\mathcal{J}_{u_l}\cap\mathcal{J}_u} F\left(|(u_l)^+(x)-(u_l)^-(x)|\right)|\nu_{u_l}(x)\cdot n|h(x)d\mathcal{H}^{N-1}(x)\\
&\qquad+\int_{\mathcal{J}_{u_l}\setminus\mathcal{J}_u} F\left(|(u_l)^+(x)-(u_l)^-(x)|\right)|\nu_{u_l}(x)\cdot n|h(x)d\mathcal{H}^{N-1}(x)\\
&\qquad=\int_{\mathcal{J}_{u_l}} F\left(|(u_l)^+(x)-(u_l)^-(x)|\right)|\nu_{u}(x)\cdot n|\chi_{\mathcal{J}_u}(x)h(x)d\mathcal{H}^{N-1}(x). \quad (10.63)
\end{aligned}$$

Using item 1 with $|\nu_u(x)\cdot n|\chi_{\mathcal{J}_u}(x)h(x)$ in place of $h(x)$, we conclude (10.61). □

Theorem 10.1. *(Calderón-Zygmund, Theorem 3.83 in [1])*

Let $\Omega\subset\mathbb{R}^N$ be an open set. Any function $u\in BV(\Omega,\mathbb{R}^d)$ is approximately differentiable at $\mathcal{L}^N$-almost every point of Ω. Moreover, the approximate differential ∇u is the density of the absolutely continuous part of Du with respect to $\mathcal{L}^N$, in particular $\nabla u\in L^1(\Omega,\mathbb{R}^{d\times N})$.

Theorem 10.2. *(Federer-Vol'pert Theorem, Theorem 3.78 in [1])*

Let $\Omega\subset\mathbb{R}^N$ be an open set, and $u\in BV_{loc}(\Omega,\mathbb{R}^d)$. Then, the jump set $\mathcal{J}_u$ is countably $(N-1)-$rectifiable set, oriented with the jump vector $\nu_u(x)$, and moreover, we have $\mathcal{H}^{N-1}\big(\mathcal{S}_u\setminus\mathcal{J}_u\big)=0$. In particular, $\mathcal{S}_u$ is σ-finite with respect to $\mathcal{H}^{N-1}$.

Lemma 10.9. *(Variation Inequality)*

Let $\Omega\subset\mathbb{R}^N$ be an open set and $u\in BV(\Omega,\mathbb{R}^d)$. Let $E\subset\Omega$ be an $\mathcal{L}^N$-measurable set and let $h\in\mathbb{R}^N\setminus\{0\}$. Assume that $\operatorname{dist}(E,\partial\Omega)>|h|$. Then,

$$\int_E \frac{|u(x+h)-u(x)|}{|h|}dx\le\|Du\|(\Omega). \quad (10.64)$$

In particular, if $\Omega=\mathbb{R}^N$, then

$$\sup_{h\in\mathbb{R}^N\setminus\{0\}}\int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|}{|h|}dx\le\|Du\|(\mathbb{R}^N). \quad (10.65)$$

Proof. Let $\{u_k\}_{k=1}^{\infty}\subset C^1(\Omega,\mathbb{R}^d)$ be a sequence of functions which converges to u $\mathcal{L}^N$-almost everywhere and $\lim_{k\to\infty}\|Du_k\|(\Omega)=\|Du\|(\Omega)$. Then, for every $k\in\mathbb{N}$, by the fundamental theorem of calculus and Fubini's theorem we get

$$\begin{aligned}
\int_E \frac{|u_k(x+h)-u_k(x)|}{|h|}dx&=\int_E\frac{|\int_0^1\nabla u_k(x+th)\cdot hdt|}{|h|}dx\le\int_0^1\int_E|\nabla u_k(x+th)|dxdt\\
&=\int_0^1\int_{E+th}|\nabla u_k(y)|dydt\le\|Du_k\|(\Omega). \quad (10.66)
\end{aligned}$$

Taking the lower limit as $k\to\infty$ and using Fatou's Lemma we get (10.64). To get (10.65) note that for every $h\in\mathbb{R}^N$, $\operatorname{dist}(\mathbb{R}^N,\emptyset)=\infty>|h|$. □

10.8 Negligibility of Sets with respect to $\|Du\|$

Definition 10.7. (Measure-theoretic Boundary)

Let $E \subset \mathbb{R}^N$ be a set. We write $x \in \partial^* E$ if and only if the following two inequalities hold:

$$\limsup_{\varepsilon\to 0^+} \frac{\mathcal{L}^N\left(B_\varepsilon(x)\cap E\right)}{\mathcal{L}^N\left(B_\varepsilon(x)\right)} > 0, \quad \limsup_{\varepsilon\to 0^+} \frac{\mathcal{L}^N\left(B_\varepsilon(x)\cap \left(\mathbb{R}^N\setminus E\right)\right)}{\mathcal{L}^N\left(B_\varepsilon(x)\right)} > 0. \tag{10.67}$$

Equivalently, $x \in \partial^* E$ if and only if E and its complement $\mathbb{R}^N \setminus E$ do not have density 0 at x; if and only if the set E does not have density neither 0 nor 1. In other words, if we denote by E^0 the set of points at which E has density 0 and by E^1 the set of points at which E has density 1, namely

$$E^0 := \left\{x\in\mathbb{R}^N : \lim_{\varepsilon\to 0^+} \frac{\mathcal{L}^N\left(B_\varepsilon(x)\cap E\right)}{\mathcal{L}^N\left(B_\varepsilon(x)\right)} = 0\right\}, \quad E^1 := \left\{x\in\mathbb{R}^N : \lim_{\varepsilon\to 0^+} \frac{\mathcal{L}^N\left(B_\varepsilon(x)\cap E\right)}{\mathcal{L}^N\left(B_\varepsilon(x)\right)} = 1\right\}, \tag{10.68}$$

then $x \in \partial^* E$ if and only if $x \notin E^0 \cup E^1$. We call $\partial^* E$ the *measure-theoretic boundary of the set* E.

Theorem 10.3. *(The Co-Area Formula for BV-Functions, see equation (3.63) in [1])*

Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $u \in BV(\Omega)$. Then, for every Borel set $B \subset \Omega$

$$\|Du\|(B) = \int_{\mathbb{R}} \mathcal{H}^{N-1}\left(B\cap \partial^*\{u>t\}\right) d\mathcal{H}^1(t). \tag{10.69}$$

Proposition 10.11. *(Variation-Negligibility of Sets with $\mathcal{H}^1$-Negligible Images) Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in BV(\Omega, \mathbb{R}^d)$. Let $B \subset \Omega \setminus \mathcal{S}_u$ be a Borel set such that $\mathcal{H}^1\left(\tilde{u}(B)\right) = 0$. Then, $\|Du\|(B) = 0$.*

Proof. Assume first that $d = 1$. Let us first prove that for every $t \in \mathbb{R}$ we have

$$\tilde{u}\big((\Omega\setminus\mathcal{S}_u)\cap\partial^*\big\{z\in\Omega : u(z)>t\big\}\big) \subset \{t\}. \tag{10.70}$$

It means that the approximate limit $\tilde{u}$ takes the measure-theoretic boundaries of super-level sets $\partial^*\big\{z \in \Omega : u(z) > t\big\}$, which are outside $\mathcal{S}_u$, to the corresponding points t. We use the short notation $\{u > t\} := \big\{z \in \Omega : u(z) > t\big\}$, as well as for similar sets. Assume that $z_0 \in (\Omega \setminus \mathcal{S}_u) \cap \partial^*\{u > t\}$. Therefore, if $\tilde{u}(z_0) < t$, then for every $\varepsilon \in (0, \infty)$ we have by Chebyshev's inequality

$$\frac{\mathcal{L}^N\left(B_\varepsilon(z_0)\cap\{u>t\}\right)}{\mathcal{L}^N\left(B_\varepsilon(z_0)\right)} = \frac{\mathcal{L}^N\left(B_\varepsilon(z_0)\cap\{u-\tilde{u}(z_0)>t-\tilde{u}(z_0)\}\right)}{\mathcal{L}^N\left(B_\varepsilon(z_0)\right)} \le \frac{1}{t-\tilde{u}(z_0)} \fint_{B_\varepsilon(z_0)} |u(x)-\tilde{u}(z_0)|dx. \tag{10.71}$$

Since $z_0 \in \Omega \setminus \mathcal{S}_u$, then we get from (10.71) that the density of $\{u > t\}$ at z_0 is zero, which contradicts the assumption that $z_0 \in \partial^*\{u > t\}$. Similarly, if $\tilde{u}(z_0) > t$, then for every $\varepsilon \in (0, \infty)$

we have by Chebyshev's inequality

$$\frac{\mathcal{L}^N\left(B_\varepsilon(z_0)\cap\{u\le t\}\right)}{\mathcal{L}^N\left(B_\varepsilon(z_0)\right)}=\frac{\mathcal{L}^N\left(B_\varepsilon(z_0)\cap\{\tilde u(z_0)-u\ge\tilde u(z_0)-t\}\right)}{\mathcal{L}^N\left(B_\varepsilon(z_0)\right)}$$
$$\le\frac{1}{\tilde u(z_0)-t}\fint_{B_\varepsilon(z_0)}|u(x)-\tilde u(z_0)|dx. \quad (10.72)$$

Since $z_0\in\Omega\setminus\mathcal{S}_u$, then we get from (10.72) that the density of $\{u\le t\}$ at z_0 is zero, which contradicts the assumption that $z_0\in\partial^*\{u>t\}$. We conclude that $\tilde u(z_0)=t$, which proves (10.70).

By (10.70) we get that, if $t\notin\tilde u(B)$, then $B\cap\partial^*\{u>t\}=\emptyset$. We get from the co-area formula (Theorem 10.3) and the assumption $\mathcal{H}^1\left(\tilde u(B)\right)=0$ that

$$\|Du\|(B)=\int_{\mathbb{R}}\mathcal{H}^{N-1}\left(B\cap\partial^*\{u>t\}\right)d\mathcal{H}^1(t)=\int_{\tilde u(B)}\mathcal{H}^{N-1}\left(B\cap\partial^*\{u>t\}\right)d\mathcal{H}^1(t)=0. \quad (10.73)$$

In the general case, $d\in\mathbb{N}$, let us denote $u=(u_1,...,u_d)$. Notice that for every natural $1\le j\le d$ we have $\mathcal{S}_{u_j}\subset\mathcal{S}_u$, and for $x\in\mathcal{S}_u$ we have by uniqueness of approximate limit $(\tilde{u_j})(x)=(\tilde u)_j\,(x)$. Therefore,

$$B\subset\Omega\setminus\mathcal{S}_u\subset\Omega\setminus\mathcal{S}_{u_j},\quad \mathcal{H}^1((\tilde{u_j})(B))=\mathcal{H}^1((\tilde u)_j\,(B))=\mathcal{H}^1(P_j(\tilde u(B)))\le\mathcal{H}^1(\tilde u(B))=0. \quad (10.74)$$

Here $P_j:\mathbb{R}^d\to\mathbb{R}$ is the projection on the j-th coordinate which is a Lipschitz function. Therefore,

$$\|Du\|(B)\le\sum_{j=1}^d\|Du_j\|(B)=0. \quad (10.75)$$

□

Proposition 10.12. *(Properties of Cantor Part D^cu, Proposition 3.92 in [1])*

Let $\Omega\subset\mathbb{R}^N$ be an open set, and let $u\in BV(\Omega,\mathbb{R}^d)$. Then, the Cantor part D^cu (see Definition 10.8) of the distributional derivative Du vanishes on sets which are σ-finite with respect to $\mathcal{H}^{N-1}$ and on sets of the form $\tilde u^{-1}(E)$ with $E\subset\mathbb{R}^d$, $\mathcal{H}^1(E)=0$.

Remark 10.1. (Variation of Cantor Part Vanishes on $\mathcal{H}^{N-1}$ σ-Finite Sets)

Since D^cu vanishes on sets which are σ-finite with respect to $\mathcal{H}^{N-1}$, and any subset of such a set is also σ-finite with respect to $\mathcal{H}^{N-1}$, then the variation $\|D^cu\|$ vanishes on sets which are σ-finite with respect to $\mathcal{H}^{N-1}$ (recall that a variation of a vector valued measure μ vanishes on a set if and only if μ vanishes on every subset of the set).

10.9 Decomposition of Du and the Chain Rule for BV-Functions

Definition 10.8. (Jump and Cantor Parts)

Let $\Omega\subset\mathbb{R}^N$ be an open set, and let $u\in BV(\Omega,\mathbb{R}^d)$. Let $Du=D^au+D^su$ be the decomposition of the distributional derivative Du of u into the absolutely continuous and singular parts with respect to $\mathcal{L}^N$. We define the jump part and the Cantor part of Du, respectively, to be the following measures:

$$D^j u := D^s u \llcorner \mathcal{J}_u, \quad D^c u := D^s u \llcorner (\Omega \setminus \mathcal{S}_u)\,. \tag{10.76}$$

Theorem 10.4. *(Decomposition of Du into the Absolutely Continuous, Jump and Cantor Parts)*

Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $u \in BV(\Omega, \mathbb{R}^d)$. Then,

$$Du = D^a u + D^j u + D^c u, \tag{10.77}$$

where $D^a u, D^j u, D^c u$ are defined in Definition 10.8. They have the following properties:
1. $D^a u, D^j u, D^c u$ are finite Radon measures in Ω (it means that they are measures from $\mathcal{B}(\Omega)$, the Borel σ-algebra, into $\mathbb{R}^{d\times N}$, the set of all matrices of size $d \times N$ with entries from $\mathbb{R}$);
2. They are orthogonal to each other;
3. It follows that:

$$D^a u = \nabla u \mathcal{L}^N, \quad D^j u = \left(u^+ - u^-\right) \otimes \nu_u \mathcal{H}^{N-1} \llcorner \mathcal{J}_u, \tag{10.78}$$

where for points $a = (a_1, ..., a_d) \in \mathbb{R}^d, b = (b_1, ..., b_N) \in \mathbb{R}^N$ we define $a \otimes b$ to be the $d \times N$ matrix given by $(a \otimes b)_{ij} := a_i b_j$.
4. We have

$$\|Du\| = |\nabla u| \mathcal{L}^N + \left|u^+ - u^-\right| \mathcal{H}^{N-1} \llcorner \mathcal{J}_u + \|D^c u\|. \tag{10.79}$$

One can find proofs for the assertions of Theorem 10.4 in section 3.9 in [1].

Theorem 10.5. *(Chain Rule in BV, Theorem 3.99 in [1])*

Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $u \in BV(\Omega, \mathbb{R})$ and let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function satisfying $f(0) = 0$ if $\mathcal{L}^N(\Omega) = \infty$. Then, $v := f \circ u$ belongs to $BV(\Omega, \mathbb{R})$ and

$$Dv = f'(u) \nabla u \mathcal{L}^N + \left(f(u^+) - f(u^-)\right) \nu_u \mathcal{H}^{N-1} \llcorner \mathcal{J}_u + f'(\tilde{u}) D^c u. \tag{10.80}$$

Remark 10.2. (Well-Definedness of Compositions in Chain Rule for BV-Functions) In this remark we would like to explain why $f' \circ u \in L^1(\Omega, |\nabla u| \mathcal{L}^N)$ and $f' \circ \tilde{u} \in L^1(\Omega, \|D^c u\|)$. Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $u \in BV(\Omega, \mathbb{R})$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function.
1. By Rademacher's theorem there exists a Borel set $\Theta \subset \mathbb{R}$ such that f is differentiable at every $x \in \mathbb{R} \setminus \Theta$ and $\mathcal{H}^1(\Theta) = 0$.
2. Since the approximate limit $\tilde{u} : \Omega \setminus \mathcal{S}_u \to \mathbb{R}$ is a Borel function, then $\tilde{u}^{-1}(\Theta) \subset \Omega \setminus \mathcal{S}_u$ is a Borel set.
3. Therefore, we get by Proposition 10.11 that $\|Du\|(\tilde{u}^{-1}(\Theta)) = 0$.
4. Since f is Lipschitz, then its $\mathcal{L}^1$-almost everywhere derivative $f' : \mathbb{R} \setminus \Theta \to \mathbb{R}$ is a Borel function. Therefore, the composition $f' \circ \tilde{u} : \Omega \setminus (\mathcal{S}_u \cup \tilde{u}^{-1}(\Theta)) \to \mathbb{R}$ is a Borel function.
5. Since by Remark 10.1 we have $\|D^c u\|(\mathcal{S}_u) = 0$, then $f' \circ \tilde{u}$ is defined almost everywhere in Ω with respect to the measure $\|D^c u\|$. Since $\|D^c u\|$ is a Borel measure, then $f' \circ \tilde{u}$ is a measurable function with respect to the measure $\|D^c u\|$. Since f is Lipschitz and $u \in BV(\Omega, \mathbb{R})$, then

$$\int_\Omega |f'(\tilde{u}(x))| d\|D^c u\|(x) \le \|f'\|_{L^\infty(\mathbb{R})} \|D^c u\|(\Omega) \le \|f'\|_{L^\infty(\mathbb{R})} \|Du\|(\Omega) < \infty. \tag{10.81}$$

Therefore, $f' \circ \tilde{u} \in L^1(\Omega, \|D^c u\|)$.

6. Without loss of generality assume that the $\mathcal{L}^N$-almost everywhere defined function u is defined on all of Ω and it is a Borel function. Let us denote by

$$E := \mathcal{S}_u \cup \{x \in \Omega \setminus \mathcal{S}_u : u(x) \neq \tilde{u}(x)\} \cup \tilde{u}^{-1}(\Theta) \cup (\Omega \setminus \mathcal{D}_u)\,. \tag{10.82}$$

The function $f' \circ u : \Omega \setminus E \to \mathbb{R}$ is a Borel function because it is a restriction of the Borel function $f' \circ \tilde{u}$ to the Borel set $\Omega \setminus E$. Hence, it is a measurable function with respect to the Borel measure $|\nabla u|\mathcal{L}^N$. The function $f' \circ u$ is defined almost everywhere in Ω with respect to the measure $|\nabla u|\mathcal{L}^N$: by item (a) of Proposition 10.8 and Theorem 10.1 we get that $|\nabla u|\mathcal{L}^N(E) = 0$. Since f is Lipschitz and $u \in BV(\Omega, \mathbb{R})$, then

$$\int_\Omega |f'(u(x))| d|\nabla u|\mathcal{L}^N(x) \leq \|f'\|_{L^\infty(\mathbb{R})} |\nabla u|\mathcal{L}^N(\Omega) \leq \|f'\|_{L^\infty(\mathbb{R})} \|Du\|(\Omega) < \infty. \tag{10.83}$$

Therefore, $f' \circ u \in L^1(\Omega, |\nabla u|\mathcal{L}^N)$.

10.10 Convergence of the Truncated Family in the Space BV

Lemma 10.10. *(Convergence of the Truncated Family in Lebesgue Spaces)*
Let $p \in (0, \infty)$, $E \subset \mathbb{R}^N$ be an $\mathcal{L}^N$-measurable set and $u \in L^p(E, \mathbb{R}^d)$. Then,

$$\lim_{l \to \infty} \int_E |u(z) - u_l(z)|^p d\mathcal{L}^N(z) = 0, \tag{10.84}$$

where $\{u_l\}_{l \in [0,\infty)}$ is the truncated family obtained by u.

Proof. Since for $\mathcal{L}^N$-almost every $z \in E$ $\lim_{l \to \infty} |u(z) - u_l(z)| = 0$, $|u(z) - u_l(z)| \leq 2|u(z)|$, and $u \in L^p(E, \mathbb{R}^d)$, then we get (10.84) from Dominated Convergence Theorem. □

Lemma 10.11. *(Convergence of the Truncated Family in BV)*
Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in BV(\Omega, \mathbb{R}^d)$. Let $\{u_l\}_{l \in [0,\infty)}$ be the truncated family obtained by u. Then, for every $l \in [0, \infty)$ we have $u_l \in BV(\Omega, \mathbb{R}^d)$, and

$$\lim_{l \to \infty} \|D(u - u_l)\|(\Omega) = 0. \tag{10.85}$$

In particular, u_l converges to u as $l \to \infty$ in the norm of the space $BV(\Omega, \mathbb{R}^d)$, which means that $\lim_{l \to \infty} \big(\|D(u - u_l)\|(\Omega) + \|u - u_l\|_{L^1(\Omega, \mathbb{R}^d)}\big) = 0$.

Proof. Assume first that $u \in BV(\Omega, \mathbb{R})$. By Theorem 10.4, we can decompose the distributional derivative Du into the sum of the absolutely continuous part, the jump part and the Cantor part:

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner \mathcal{J}_u + D^c u. \tag{10.86}$$

For each $l \in [0, \infty)$ let us define a function $f_l : \mathbb{R} \to \mathbb{R}$, $f_l(z) := l \wedge (-l \vee z)$. By Proposition 10.5 we have that f_l is a Lipschitz function and by the definition of the truncated family (Definition

10.2) we have $u_l = f_l \circ u$. By the chain rule for BV-functions (refer to Theorem 10.5) we have $u_l \in BV(\Omega, \mathbb{R})$ and

$$Du_l = f_l'(u)\nabla u\mathcal{L}^N + (f_l(u^+) - f_l(u^-))\nu_u\mathcal{H}^{N-1}\llcorner\mathcal{J}_u + f_l'(\tilde{u})D^c u. \tag{10.87}$$

By (10.86), (10.87) and Remark 10.2 we have

$$\begin{aligned} D(u - u_l) = (1 - f_l'(u))\,\nabla u\mathcal{L}^N \\ + \big((u^+ - u^-) - (f_l(u^+) - f_l(u^-))\big)\,\nu_u\mathcal{H}^{N-1}\llcorner\mathcal{J}_u + (1 - f_l'(\tilde{u}))D^c u. \end{aligned} \tag{10.88}$$

By Lemma 10.4 we get

$$\|\,(1 - f_l'(u))\,\nabla u\mathcal{L}^N\|(\Omega) = \int_\Omega |1 - f_l'(u(x))|\,|\nabla u(x)|\,d\mathcal{L}^N(x) \tag{10.89}$$

and

$$\begin{aligned} \|\,\big((u^+ - u^-) - (f_l(u^+) - f_l(u^-))\big)\,\nu_u\mathcal{H}^{N-1}\llcorner\mathcal{J}_u\|(\Omega) \\ = \int_{\mathcal{J}_u} \big|(u^+(x) - u^-(x)) - (f_l(u^+(x)) - f_l(u^-(x)))\big|\,d\mathcal{H}^{N-1}(x). \end{aligned} \tag{10.90}$$

Note that for getting (10.90) we use that $|\nu_u| = 1$ (refer to Definition 10.4). By Lemma 10.5 we get

$$\|(1 - f_l'(\tilde{u}))D^c u\|(\Omega) \le N^{1/2}\int_\Omega |1 - f_l'(\tilde{u}(x))|\,d\|D^c u\|(x), \tag{10.91}$$

where $\|\cdot\|$ stands for the variation (refer to Definition 10.1). Therefore, we get by (10.88), the triangle inequality of the variation, (10.89),(10.90) and (10.91) that

$$\begin{aligned} \|D(u - u_l)\|(\Omega) \le \int_\Omega |1 - f_l'(u(x))|\,|\nabla u(x)|\,d\mathcal{L}^N(x) \\ + \int_{\mathcal{J}_u} \big|(u^+(x) - u^-(x)) - (f_l(u^+(x)) - f_l(u^-(x)))\big|\,d\mathcal{H}^{N-1}(x) \\ + N^{1/2}\int_\Omega |1 - f_l'(\tilde{u}(x))|\,d\|D^c u\|(x). \end{aligned} \tag{10.92}$$

For every $l \in [0, \infty)$, from item (a) of Proposition 10.8 we get

$$|\nabla u|\mathcal{L}^N\left(\big\{x \in \Omega : |u(x)| = l\big\}\right) = |\nabla u|\mathcal{L}^N\left(\big\{x \in \Omega \setminus \mathcal{S}_u : |\tilde{u}(x)| = l\big\}\right), \tag{10.93}$$

and from Proposition 10.11 we have

$$\|D^c u\|\left(\big\{x \in \Omega \setminus \mathcal{S}_u : |\tilde{u}(x)| = l\big\}\right) = |\nabla u|\mathcal{L}^N\left(\big\{x \in \Omega \setminus \mathcal{S}_u : |\tilde{u}(x)| = l\big\}\right) = 0. \tag{10.94}$$

Note that for getting (10.94) we use the assumption that u is a scalar function in order to get that

$$\mathcal{H}^1\left(\tilde{u}\left(E_l\right)\right) \le \mathcal{H}^1(\{l, -l\}) = 0, \quad E_l := \big\{x \in \Omega \setminus \mathcal{S}_u : |\tilde{u}(x)| = l\big\}. \tag{10.95}$$

For every $l \in (0,\infty)$ we have

$$f_l'(z) := \begin{cases} 1, & if \quad |z| < l \\ 0, & if \quad |z| > l \end{cases}. \tag{10.96}$$

By (10.93) and (10.96) we get for every $l \in (0,\infty)$ that

$$\begin{aligned}
&\int_\Omega |1 - f_l'(u(x))|\, |\nabla u(x)|\, d\mathcal{L}^N(x) = \int_\Omega |1 - f_l'(u(x))|\, d\, |\nabla u|\, \mathcal{L}^N(x) \\
&\quad = \int_{\left\{x\in\Omega : |u(x)|>l\right\}} |1 - f_l'(u(x))|\, d\, |\nabla u|\, \mathcal{L}^N(x) + \int_{\left\{x\in\Omega : |u(x)|=l\right\}} |1 - f_l'(u(x))|\, d\, |\nabla u|\, \mathcal{L}^N(x) \\
&\qquad + \int_{\left\{x\in\Omega : |u(x)|<l\right\}} |1 - f_l'(u(x))|\, d\, |\nabla u|\, \mathcal{L}^N(x) = \int_{\left\{x\in\Omega : |u(x)|>l\right\}} d\, |\nabla u|\, \mathcal{L}^N(x).
\end{aligned} \tag{10.97}$$

By Calderón-Zygmund theorem (refer to Theorem 10.1), we have $\nabla u \in L^1(\Omega, \mathbb{R}^N)$. Therefore, we get by (10.97) and the decreasing monotonicity of the measure $|\nabla u|\, \mathcal{L}^N$ that

$$\lim_{l\to\infty} \int_\Omega |1 - f_l'(u(x))|\, |\nabla u(x)|\, d\mathcal{L}^N(x) = 0. \tag{10.98}$$

Since $u \in BV(\Omega, \mathbb{R})$, then we get from Federer-Vol'pert Theorem (refer to Theorem 10.2) that $\mathcal{S}_u$ is σ-finite with respect to $\mathcal{H}^{N-1}$. Thus, by Proposition 10.12 and Remark 10.1 we have $\|D^c u\|(\mathcal{S}_u) = 0$. Therefore, by (10.94) and (10.96) we obtain

$$\begin{aligned}
&\int_\Omega |1 - f_l'(\tilde{u}(x))|\, d\|D^c u\|(x) = \int_{\Omega\setminus\mathcal{S}_u} |1 - f_l'(\tilde{u}(x))|\, d\|D^c u\|(x) \\
&\quad = \int_{\left\{x\in\Omega\setminus\mathcal{S}_u : |\tilde{u}(x)|>l\right\}} |1 - f_l'(\tilde{u}(x))|\, d\|D^c u\|(x) + \int_{\left\{x\in\Omega\setminus\mathcal{S}_u : |\tilde{u}(x)|=l\right\}} |1 - f_l'(\tilde{u}(x))|\, d\|D^c u\|(x) \\
&\qquad + \int_{\left\{x\in\Omega\setminus\mathcal{S}_u : |\tilde{u}(x)|<l\right\}} |1 - f_l'(\tilde{u}(x))|\, d\|D^c u\|(x) = \int_{\left\{x\in\Omega\setminus\mathcal{S}_u : |\tilde{u}(x)|>l\right\}} 1 d\|D^c u\|(x).
\end{aligned} \tag{10.99}$$

Note that since $\mathcal{S}_u$ is a Borel set in Ω and $\tilde{u} : \Omega \setminus \mathcal{S}_u \to \mathbb{R}$ is a Borel function (refer to Proposition 10.8), then the sets $\left\{x \in \Omega\setminus\mathcal{S}_u : |\tilde{u}(x)| > l\right\}$, $\left\{x \in \Omega\setminus\mathcal{S}_u : |\tilde{u}(x)| < l\right\}$ and $\left\{x \in \Omega\setminus\mathcal{S}_u : |\tilde{u}(x)| = l\right\}$ are Borel sets in Ω, so they are measurable with respect to the measure $\|D^c u\|$, because $\|D^c u\|$ is a Borel measure (refer to Theorem 10.4). Since $\|D^c u\|$ is a finite Borel measure in Ω, then we get by (10.99) and the decreasing monotonicity of the measure $\|D^c u\|$ that

$$\begin{aligned}
&\lim_{l\to\infty} \int_\Omega |1 - f_l'(\tilde{u}(x))|\, d\|D^c u\|(x) = \lim_{l\to\infty} \|D^c u\| \left(\left\{x \in \Omega \setminus \mathcal{S}_u : |\tilde{u}(x)| > l\right\}\right) \\
&\qquad = \|D^c u\| \left(\bigcap_{l\in\mathbb{N}} \left\{x \in \Omega \setminus \mathcal{S}_u : |\tilde{u}(x)| > l\right\}\right) = \|D^c u\|(\emptyset) = 0.
\end{aligned} \tag{10.100}$$

At last, by Proposition 10.5 we get

$$\lim_{l\to\infty} \left|(u^+(x) - u^-(x)) - (f_l(u^+(x)) - f_l(u^-(x)))\right| = 0, \quad x \in \mathcal{J}_u; \tag{10.101}$$

$$\left|(u^+(x) - u^-(x)) - (f_l(u^+(x)) - f_l(u^-(x)))\right| \le 2\left|u^+(x) - u^-(x)\right|, \quad x \in \mathcal{J}_u. \tag{10.102}$$

By Theorem 10.4 we get

$$\left|u^+ - u^-\right| \in L^1(\mathcal{J}_u, \mathcal{H}^{N-1}). \tag{10.103}$$

Therefore, Dominated Convergence Theorem gives

$$\lim_{l\to\infty} \int_{\mathcal{J}_u} \left|(u^+(x) - u^-(x)) - (f_l(u^+(x)) - f_l(u^-(x)))\right| d\mathcal{H}^{N-1}(x) = 0. \tag{10.104}$$

Equation (10.85) follows from (10.92), (10.98), (10.100) and (10.104) in case $u \in BV(\Omega, \mathbb{R})$. The general case, $u \in BV(\Omega, \mathbb{R}^d)$, follows from the inequality

$$\max_{1\leq i\leq d, i\in\mathbb{N}} \|Du^i\|(\Omega) \leq \|Du\|(\Omega) \leq \sum_{i=1}^{d} \|Du^i\|(\Omega), \tag{10.105}$$

where $u = (u^1, ..., u^d)$. Indeed, note that by the definition of the truncated family, Definition 10.2, it follows that $(u_l)^i = (u^i)_l$ for every natural $1 \leq i \leq d$ and $l \in [0, \infty)$. Therefore, we get

$$\|D(u - u_l)\|(\Omega) \leq \sum_{i=1}^{d} \|D(u - u_l)^i\|(\Omega) = \sum_{i=1}^{d} \|D(u^i - (u^i)_l)\|(\Omega). \tag{10.106}$$

Therefore, since for every natural $1 \leq i \leq d$ we have that $u^i \in L^1(\Omega, \mathbb{R})$, then we get by (10.105) that $u^i \in BV(\Omega, \mathbb{R})$. Therefore, we obtain (10.85) from (10.106) taking the limit as l goes to infinity. The convergence of u_l to u as $l \to \infty$ in the norm of the space $BV(\Omega, \mathbb{R}^d)$ follows from Lemma 10.10 with $p = 1$ and (10.85). □

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