DISCRETE AND CONTINUUM AREA-PRESERVING MEAN-CURVATURE FLOW OF RECTANGLES

M. CICALESE AND A. KUBIN

ABSTRACT. We investigate the area-preserving mean-curvature-type motion of a two-dimensional lattice crystal obtained by coupling constrained minimizing movements scheme introduced by Almgren, Taylor and Wang in [4] with a discrete-to-continuous analysis. We first examine the continuum counterpart of the model and establish the existence and uniqueness of the flat flow, originating from a rectangle. Additionally, we characterize the governing system of ordinary differential equations. Subsequently, in the atomistic setting, we identify geometric properties of the discrete-in-time flow and describe the governing system of finite-difference inclusions. Finally, in the limit where both spatial and time scales vanish at the same rate, we prove that a discrete-to-continuum evolution is expressed through a system of differential inclusions which does never reduce to a system of ODEs.

CONTENTS

INTRODUCTION

Comprehending the interplay between the energetic and kinematic aspects of crystal deformation over time is a formidable task, presenting significant challenges in the quest for simplified models that can capture the essential features of this intricate problem. The complexity is compounded by the emergence of geometrically defected microstructures, microscopic atomic arrangements whose flow deviate from the simple geometric description of the motion observed at mesoscopic or continuum scales. Establishing meaningful connections between microscopic and macroscopic observables in this context proves to be a big challenge.

In this paper we provide a simple geometric answer to the problem starting from a well-known microscopic lattice model and considering at several spatial and temporal scales the motion of a two-dimensional crystal which preserves its area along the flow. We first investigate the problem at a continuum level; i.e., when the crystal is modelled as a continuum. In this setting the evolution of the crystal is obtained implementing a

time-discrete iterative minimization scheme proposed by Almgren, Taylor and Wang in
[4] and by Luckhaus and Sturzenhecker in [19], here modified to include the conservation
of area constraint. In this case, passing to the limit as the time step vanishes, we find a
system of forced mean-curvature equations governing the evolution of the crystal. Drawing
from our experience in addressing the continuum problem in the second part of the paper
we turn our attention to the atomistic setting. In this case the crystal is modelled as a
finite union of point masses occupying the position of a subset of the square lattice and
the minimizing movement scheme above needs to be adapted to this framework (see [8] for
the unconstrained problem and [14] for an introduction to the geometric motion of lattice
systems). In this framework we first prove several geometric properties of the solution
of the iterative scheme when the number of particles is kept fixed in the time-discrete
iteration. Then, we provide existence of the flow and characterize it in terms of a system
of finite difference inclusions when the lattice spacing and the time step vanish at the
same rate. We finally conjecture the discrete-to-continuum evolution of the crystal and
propose a regularized version of the flow in which case the convergence to an explicit
system of ordinary differential inclusions can be proved. We emphasize that while most of
our arguments could be extended to higher dimensions or to more general lattices, albeit
at the cost of significantly increased computational complexity, for the sake of readers'
convenience, we limit our consideration to the two-dimensional case.

The simplest geometric model of a two-dimensional crystal assumes its atoms to occupy
some of the nodes of a simple periodic lattice that we fix to be $\mathbb{Z}^2$. The optimal shape
of the crystal is then obtained by minimizing an energy depending on the position of the
atoms in the lattice. Energies with geometric flavours have been proposed by Cahn and
coauthors in several papers (see for instance [24, 25]). The simplest energy functional one
can think about is a perimeter-like energy obtained by referring to its ground states a
classical nearest-neighbors ferromagnetic Ising energy (see [1, 15] and refer to [2] for a more
comprehensive variational treatment of spin systems). The latter can be written as

$$(0.1) \qquad\qquad P_\varepsilon(u) = \frac{1}{4} \sum_{n.n.} \varepsilon(u^i - u^j)^2,$$

where $u : i \in \varepsilon\mathbb{Z}^2 \mapsto u^i \in \{-1, +1\}$ is an Ising variable and the sum is extended to all nearest
neighboring ($n.n.$) points of the lattice $\varepsilon\mathbb{Z}^2$, that is to those $i, j \in \varepsilon\mathbb{Z}^2$ such that $|i - j| = \varepsilon$.
In this model the atoms of the crystal are those lattice points $i \in \varepsilon\mathbb{Z}^2$ where $u^i = 1$ and the
region occupied by the crystal can be thought of as the union of those elementary cells of the
dual lattice of $\varepsilon\mathbb{Z}^2$ centred at those points, namely $E := \bigcup_{\{i \in \varepsilon\mathbb{Z}^2 : u^i = 1\}}(i + \varepsilon[-1/2, 1/2)^2)$.
In the rest of this introduction we generically refer to E as to a $\varepsilon\mathbb{Z}^2$ crystal. With this
identification between functions and sets the energy in (0.1) (also known as *edge perimeter*
energy) can be regarded as defined on sets of finite perimeter. In this sense the energy
of the $\varepsilon\mathbb{Z}^2$ crystal E is defined by $P_\varepsilon(E) := P_\varepsilon(u)$ and one can carry out the variational
coarse-graining of such energies as $\varepsilon \to 0$ in terms of Γ-convergence with respect to the
L^1 distance between sets. In [1] (see also [3, 6, 12, 13] for generalizations covering also
the case of non-periodic lattices) it has been proved that, as $\varepsilon \to 0$, the Γ-limit of the
functionals P_ε is the anisotropic perimeter functional defined on sets of finite perimeter as

$$\mathrm{P}_{|\cdot|_1}(E) := \int_{\partial^* E} |\nu_E|_1 d\mathcal{H}^1,$$

where for $\nu = (\nu_1, \nu_2)$, $|\nu|_1 := |\nu_1| + |\nu_2|$ is the 1-norm of ν. Note that in the functional
above the symmetry properties of the anisotropic surface tension $|\cdot|_1$ reflects the effect of
geometric symmetries of the lattice $\mathbb{Z}^2$ on the continuum limit.

The continuum setting

According to the preceding discussion, within the continuum framework, the area-preserving evolution by mean curvature of a crystal will be modelled as the solution of the following iterative scheme proposed in [4, 19], here modified to enforce the area constraint at each minimization step (see [5] for a similar result using the MBO scheme in the case of convex sets). Let us assume that the initial set is a rectangle, say R, of unitary area $|R| = 1$ and let us fix a time step $\tau > 0$. We introduce the functional $\mathcal{D}(E, F) := \int_{E \Delta F} d_\infty(z, \partial F) \, dz$, where $d_\infty(z, \partial F) := \inf \{|z - \hat{z}|_\infty \colon \hat{z} \in \partial F\}$ and $|z|_\infty = \max\{|z_1|, |z_2|\}$ for $z = (z_1, z_2)$. The functional $\mathcal{D}(E, F)$ is interpreted as the energy cost dissipated for the crystal to evolve from the shape E to the shape F. According to Definition 2.1, an approximate $P_{|\cdot|_1}$ area-preserving mean-curvature flow consists of a family of sets $\{E_t^{(\tau)}\}_{t \geq 0}$, where $E_t^{(\tau)} := E_{k\tau}^{(\tau)}$ for any $t \in [k\tau, (k+1)\tau)$ and where $\{E_{k\tau}^{(\tau)}\}_{k \in \mathbb{N}}$ are obtained by solving the following iterative scheme

$$(0.2) \qquad E_0^{(\tau)} = R, \qquad E_{k\tau}^{(\tau)} \in \operatorname*{arg\,min}_{E \subset \mathbb{R}^2, |E| = 1} \left\{ P_{|\cdot|_1}(E) + \frac{1}{\tau} \mathcal{D}(E, E_{(k-1)\tau}^{(\tau)}) \right\}, \quad k \geq 1.$$

The flat $P_{|\cdot|_1}$ area-preserving mean-curvature flow is then obtained passing to the limit as $\tau \to 0$ (see Definition 2.9). We emphasize that our iterative scheme requires the area to remain constant at each iteration step. Hence, the discrete evolution is area preserving, a requirement that translates the idea that the time step τ holds a physical meaning. Specifically, it is larger or equal than the timescale required for energy relaxation to occur. If instead the time-discrete evolution were merely regarded as an approximation of the time-continuum one, we could relax the area constraint and make it active only in the limit as $\tau \to 0$ as it is the case in [23] (see also [18, 22]). As it will become evident later, keeping the area constraint at each time step significantly complicates our analysis in the continuum setting and even more in the lattice setting. This complexity is also the primary reason why we confine our investigation to initial sets with rectangular shapes, leaving the case of more general initial shapes for future investigations. In Theorem 2.10 we prove that for every $R \subset \mathbb{R}^2$ rectangle with $|R| = 1$ there exists a unique area-preserving $P_{|\cdot|_1}$ flat flow which is given by a family of rectangles $R(t) = [-\frac{a(t)}{2}, \frac{a(t)}{2}] \times [-\frac{b(t)}{2}, \frac{b(t)}{2}]$ such that $R(0) = R$. Moreover their side lengths a and b solve the following system of ODEs:

$$\begin{cases} \dfrac{d}{dt}a(t) = -\dfrac{4}{b(t)} + \dfrac{8}{a(t) + b(t)}, \\[4mm] \dfrac{d}{dt}b(t) = -\dfrac{4}{a(t)} + \dfrac{8}{a(t) + b(t)}. \end{cases}$$

As a byproduct of that, in Remark 2.11 we show that $R(t)$ converge exponentially fast to the square $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$. The proof of Theorem 2.10 is obtained combining several results. In Proposition 2.4 and in Proposition 2.5 we prove that both perimeter and dissipation decrease under Steiner symmetrization in the coordinate directions. Combining this information with the specific geometry of symmetrized sets, in Theorem 2.7 we prove that in 0.2 one can reduce the class of competitors to rectangles with fixed barycenter. As a consequence, we eventually find the solution of the approximate flat flow. Making use of uniform Lipschitz estimates obtained in the step-by-step minimization on the side lengths of the evolving rectangles, we eventually conclude the proof of Theorem 2.10 taking the limit as the time step vanishes.

The discrete setting

In the second part of the paper, we examine the evolution of $\varepsilon \mathbb{Z}^2$ crystals. In the case when the area is not preserved, Braides, Gelli and Novaga in [8] were the first to combine

a variational discrete iterative scheme as the one in (0.2) with a discrete-to-continuum procedure to obtain a corse-grained version of the flow (see also [7, 9, 11, 14, 16, 21] for similar results in this context). In the present case we are interested to the area-preserving evolution of $\varepsilon\mathbb{Z}^2$ crystals. We denote the class of such crystals with unitary area by $\mathcal{AD}_\varepsilon$ and determine their evolution by solving an iterative scheme similar to the one presented in (0.2). Interestingly, due to potential incompatibilities between the lattice spacing and the area constraint, the shape of the solution of the minimization problem may not be a rectangle in successive iterations. Instead, it typically transforms into a quasi-rectangle (see Definition (3.1)). Therefore, the problem we address in the discrete setting is formulated as follows: given a quasi-rectangle QR_ε, solve the iterative minimization problem:
(0.3)
$$E_0^{(\varepsilon,\tau)} = QR_\varepsilon, \qquad E_{k\tau}^{(\varepsilon,\tau)} \in \arg\min\left\{\mathrm{P}_\varepsilon(E) + \frac{1}{\tau}D_\varepsilon(E, E_{(k-1)\tau}^{(\varepsilon,\tau)}), E \in \mathcal{AD}_\varepsilon\right\}, \quad k \geq 1,.$$

Here, the discrete dissipation D_ε is introduced in (3.6) and it is a slight variation of the one introduced in the previous paragraph, more suited for the computation needed for $\varepsilon\mathbb{Z}^2$ crystals. Following an approach similar to the one used in the continuum setting, in Section 3.1, we introduce a discrete Steiner-like symmetrization technique and use it to prove that if QR_ε satisfies an additional symmetry condition, referred to as pseudo-axial symmetry (see Definition (3.2)), then the solution of the minimization problem in (0.3) also shares the same symmetry property. As a consequence, without loss of generality, one can restrict the class of competitors in (0.3) to that of symmetric quasi rectangles. However, within this class of shapes, the process of sending ε and τ to zero and characterizing a coarse-grained continuum flow proves to be an exceedingly challenging endeavor, primarily due to the too many degrees of freedom involved. To have an idea of the difficulties arising in this situation, it is worth pointing out that at each step of the iterative scheme one needs to optimize a fourth order polynomial equation (in the unconstrained case considered in [8] the polynomial is of second order) whose coefficients depend on the shape of the crystal at the previous step. The optimization necessarily goes through a very long and tedious case-by-case study that we have decided to simplify by reducing the class of competitors as much as possible while still keeping the main features of the general flow. Within this class we consider the following coarse-grained procedure. We take a family QR_ε of quasirectangles with pseudo-axial symmetry and assume that, as $\varepsilon \to 0$ QR_ε converge in the Hausdorff distance to the rectangle $[-\frac{a}{2}, \frac{a}{2}] \times [-\frac{b}{2}, \frac{b}{2}]$. We want to prove existence of the flow and obtain some geometric information, and possibly even the governing equations of the flow $E_t^{\varepsilon,\tau}$ with initial datum QR_ε as both ε and τ vanish. As explained in [14], in this kind of problems the interaction between time and space discretization parameters plays a crucial role, the limit motion depending strongly on their relative rate of convergence. If $\varepsilon \ll \tau$ we expect that, as already observed in [8], the limit flow is identified by first letting $\varepsilon \to 0$ and then taking the limit as $\tau \to 0$. At a heuristic level for fixed τ, P_ε would be substituted by the anisotropic perimeter $\mathrm{P}_{|\cdot|_1}$ and D_ε by $\mathcal{D}$. The flat flow would then correspond to the one studied in the continuum setting. If $\varepsilon \gg \tau$ we expect no motion, namely $E_k^{\varepsilon,\tau} = QR_\varepsilon$ at every step k. Indeed, a heuristic argument shows that for $E \neq QR_\varepsilon$ and for τ sufficiently small we have

$$\frac{1}{\tau}D_\varepsilon(E, QR_\varepsilon) = \frac{1}{\tau}\int_{E\Delta QR_\varepsilon} d_\infty^\varepsilon((x,y), \partial QR_\varepsilon)\,dxdy \geq c\frac{\varepsilon}{\tau} \geq \mathrm{P}(QR_\varepsilon),$$

which implies that the limit motion is the constant set $[-\frac{a}{2}, \frac{a}{2}] \times [-\frac{b}{2}, \frac{b}{2}]$. The previous two arguments suggest the most interesting regime to be $\tau = \alpha\varepsilon$, for some $\alpha > 0$. This is the case we focus on in the last part of the paper. More precisely, we compute the minimizer of the incremental problem in (0.3) for E belongin to the special class $\mathcal{SQR}_\varepsilon(QR)$ defined in (3.10) and describe the asymptotic behavior, as $\varepsilon \to 0$, of the approximate flat solution

of the area-preserving mean-curvature flow in the lattice $\varepsilon\mathbb{Z}^2$ within this subclass. In Theorem 3.13 we prove the existence of the time-discrete flat flow and characterize it in terms of a system of finite difference inclusions. At this stage of our investigation, even if one can prove the existence of a continuous-in-time flat flow (again given by a family of rectangles), the characterization of the limit equations is not clear. We conjecture the limit equations in Remark 3.14. We conclude the paper by introducing a regularized version of the flow which we call rectangular flow and that at each step of the evolution is $O(\varepsilon)$ close to the previous one. For such a flow, which however preserves the volume only in the continuum limit, we can write down explicitly the equation of motion in the form of a system of differential inclusions. It is worth observing (see Remark 3.17) that, unlike the unconstrained case considered in [8], here the system of differential inclusions that characterizes the regularized flow never reduces to a single system of ODEs. As a result, the pinning phenomenon, that is when the flow is given by the family of sets identically equal to the initial datum, can only be one of the possible motions. In Remark 3.17 we explicitly write the condition on α, a and b under which the phenomenon occurs.

1. NOTATION

We use the convention that the set $\mathbb{N}$ contains the zero. Let $\mathbb{R}^2$ be the Euclidean plane and let $\{e_1, e_2\}$ denote the canonical basis of $\mathbb{R}^2$. Given $(x, y) \in \mathbb{R}^2$, by $|(x, y)|_1 = |x| + |y|$ and $|(x, y)|_\infty = \max\{|x|, |y|\}$ we denote its $1-$ and the $\infty-$ norms, respectively. The set $\mathcal{S}^1$ stands for the unit circle in $\mathbb{R}^2$. Given $\eta \in \mathcal{S}^1$, we denote by $\eta^\perp \in \mathcal{S}^1$ the vector orthogonal to η such that $\{\eta, \eta^\perp\}$ has the same orientation as $\{e_1, e_2\}$. For every $\eta \in \mathcal{S}^1$ we denote by $\mathbf{p}_\eta \colon \mathbb{R}^2 \to \mathbb{R}^2$ the projection on η. In the case $\eta = e_1, e_2$ we also use the notation Π_i in place of $\mathbf{p}_{e_i}$. We denote the n-dimensional Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ by $\mathcal{L}^n(E)$. In the case $n = 2$ we also use the notation $|E| := \mathcal{L}^2(E)$. Given a measurable set $E \subset \mathbb{R}^2$ we denote its barycenter as $\mathrm{Bar}(E) := \int_E x \, dx$. The symmetric difference of E and F is denoted $E \triangle F$, their Hausdorff distance by $d_\mathcal{H}(E, F)$. If E is a set of finite perimeter we denote by $\partial^* E$ its reduced boundary. The perimeter of E is denoted $\mathrm{P}(E) = \mathcal{H}^1(\partial^* E)$ where $\mathcal{H}^1$ stands for the one dimensional Hausdorff measure. For all $x \in \partial^* E$ we denote by $\nu_E(x)$ the measure theoretic outer normal vector field to E at the point x. We denote by $\left[\mathcal{M}(\mathbb{R}^2)\right]^m$ the space of the $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^2$. If $\mu \in \left[\mathcal{M}(\mathbb{R}^2)\right]^m$ we set $|\mu|$ the total variation measure associated to μ. We denote by $BV(\mathbb{R}^2)$ the space of functions with bounded variation in $\mathbb{R}^2$. If $F \subset \mathbb{R}^2$ we denote its complement $F^c = \mathbb{R}^2 \setminus F$. If $n, m \in \mathbb{Z}$ we write $n \equiv_2 m$ if n, m have the same remainder with respect to the Euclidean division by 2. Given $x \to g(x)$ a real valued function, we denote by $O(g)$ the class of all functions $x \to f(x)$ such that for all x: $|f(x)| \leq C|g(x)|$ for some constant $C > 0$. Finally we denote by $C(\star, \cdots, \star)$ a constant that depends on $\star, \cdots, \star$; this constant can change from line to line.

2. THE CONTINUUM SETTING

In this section we explain the area-preserving crystalline mean-curvature flow of a rectangle in the Euclidean space $\mathbb{R}^2$. We introduce a notion of global flat solution to the area-preserving crystalline mean-curvature flow which is based on the definition introduced independently by Almgren, Taylor & Wang in [4] and by Luckhaus & Sturzenhecker in [19]. Given $E, F \subset \mathbb{R}^2$ of finite perimeter we define

$$(2.1) \qquad \mathcal{F}_\tau(E, F) := \mathrm{P}_{|\cdot|_1}(E) + \frac{1}{\tau} \int_{E \triangle F} d_\infty(z, \partial F) \, dz.$$

In the previous formula the functional $P_{|\cdot|_1}$ is the anisotropic perimeter defined for any set $E \subset \mathbb{R}^2$ of finite perimeter as

$$P_{|\cdot|_1}(E) := \int_{\partial^* E} |\nu_E(z)|_1 d\mathcal{H}^1(z),$$

or equivalently as

(2.2) $\qquad P_{|\cdot|_1}(E) = \sup \left\{ \int_E \mathrm{div}(T)(z)dz \colon T \in C_c^1(\mathbb{R}^2, \mathbb{R}^2),\, |T|_\infty(z) \leq 1,\, \forall z \in \mathbb{R}^2 \right\}.$

In what follows we will make use of the following notation: given an open set $\Omega \subset \mathbb{R}^2$, for all set of finite perimeter $E \subset \mathbb{R}^2$ we denote the relative perimeter of E in Ω by

$$P_{|\cdot|_1}(E, \Omega) = \int_{\partial^* E \cap \Omega} |\nu_E(z)|_1 d\mathcal{H}^1(z).$$

In (2.1) $\mathcal{D}(E, F)$ denotes the dissipation of the two sets $E, F \subset \mathbb{R}^2$, that we define as

$$\mathcal{D}(E, F) := \int_{E \Delta F} d_\infty(z, \partial F) \, dz$$

where

$$d_\infty(z, \partial F) = \inf \left\{ |z - \hat{z}|_\infty \colon \hat{z} \in \partial F \right\}.$$

2.1. **Approximate flat $P_{|\cdot|_1}$ area-preserving mean-curvature flow.** In this subsection we compute the approximate flat solution of the area-preserving crystalline mean-curvature flow with initial datum R_0 a rectangle of unitary area.

Definition 2.1 (Approximate flat $P_{|\cdot|_1}$ area-preserving mean-curvature flow). Let $R_0 \subset \mathbb{R}^2$ be a rectangle with $|R_0| = 1$ and $\mathrm{Bar}(R_0) = (0, 0)$, and $\tau > 0$. Let $\{E_{k\tau}^{(\tau)}\}_{k \in \mathbb{N}}$ be a family of sets defined iteratively as

$$E_0^{(\tau)} = R_0 \quad \text{and} \quad E_{k\tau}^{(\tau)} \in \underset{E \subset \mathbb{R}^2, |E|=1}{\arg\min} \left\{ \mathcal{F}_\tau(E, E_{(k-1)\tau}^{(\tau)}) \right\} \quad k \geq 1.$$

We define

$$E_t^{(\tau)} := E_{k\tau}^{(\tau)} \quad \text{for any } t \in [k\tau, (k+1)\tau).$$

We call $\{E_t^{(\tau)}\}_{t \geq 0}$ an approximate flat $P_{|\cdot|_1}$ solution of the area-preserving mean-curvature flow with initial datum R_0 and time step τ.

In the next theorems we prove that for every $R \subset \mathbb{R}^2$ rectangle with $|R| = 1$ and $\mathrm{Bar}(R) = 0$ there is a unique minimizer of the problem

$(\mathcal{P}_1) \qquad\qquad \min \left\{ P_{|\cdot|_1}(E) + \frac{1}{\tau} \mathcal{D}(E, R) \colon E \subset \mathbb{R}^2,\, |E| = 1 \right\}$

and it is a rectangle. We start by proving that the problem $(\mathcal{P}_1)$ is equivalent to

$(\mathcal{P}_2) \;\; \min \left\{ P_{|\cdot|_1}(\overline{R}) + \frac{1}{\tau} \mathcal{D}(\overline{R}, R) \colon \overline{R} \subset \mathbb{R}^2 \text{ be a rectangle and } |\overline{R}| = 1,\, \mathrm{Bar}(\overline{R}) = (0, 0) \right\}.$

Definition 2.2. Let $E \subset \mathbb{R}^2$ be a measurable set and let $\eta \in \mathcal{S}^1$. For every $z \in \mathbb{R}^2$ we define the section $E_z^\eta \subset \mathbb{R}$ of E as

$$E_z^\eta := \left\{ t \in \mathbb{R} \colon \mathbf{p}_{\eta^\perp} z + t\eta \in E \right\}.$$

The Steiner symmetrization E^η of E in direction η is then defined as

$$E^\eta := \left\{ z \in \mathbb{R}^2 \colon |\mathbf{p}_\eta z| \leq \frac{\mathcal{L}^1(E_z^\eta)}{2} \right\}.$$

We recall the following classical proposition (for a proof see for instance [20], [17]).

Proposition 2.3. *Let $E \subset \mathbb{R}^2$ be a measurable set. The following statements hold true.*

i. *For all $\eta \in \mathcal{S}^1$ and $\mathcal{H}^1$-a.e. $z \in \ker(\mathbf{p}_\eta)$ the set E_z^η is $\mathcal{H}^1$-measurable (the statement is true for all $z \in \ker(\mathbf{p}_\eta)$ if E is a Borel set).*
ii. *The set $E^\eta \subset \mathbb{R}^2$ is $\mathcal{L}^2$-measurable.*
iii. *For all $\eta \in \mathcal{S}^1$ the function $z \in \ker(\mathbf{p}_\eta) \to \mathcal{L}^1(E_z^\eta)$ is $\mathcal{H}^1$-measurable.*
iv. *We set $\pi_1(E) := \{z \in \mathbb{R} \colon \mathcal{L}^1(E_{(x,y)}^{e_2}) > 0\}$ and $\pi_2(E) := \{y \in \mathbb{R} \colon \mathcal{L}^1(E_{(x,y)}^{e_1}) > 0\}$ are $\mathcal{L}^1$-measurable and $|E \setminus \pi_1(E) \times \pi_2(E)| = 0$.*

In the next theorem we state the classical Steiner inequality. For readers' convenience we provide a proof of the inequality regarding the anisotropic perimeter $P_{|\cdot|_1}$.

Proposition 2.4. *Let $E \subset \mathbb{R}^2$ be a set of finite perimeter. Then for every $\eta \in \mathcal{S}^1$ the set E^η is a set of finite perimeter with $|E| = |E^\eta|$ and*

$$P(E) \geq P(E^\eta).$$

Moreover if $\eta \in \{\pm e_1, \pm e_2\}$ it holds that

$$P_{|\cdot|_1}(E) \geq P_{|\cdot|_1}(E^\eta).$$

Proof. For a proof of the first statement see [20][chapter 14]. We present only the proof of the second statement when $\eta = e_2$, the other case being analogous. Given $u \colon \mathbb{R} \to \mathbb{R}$ and $H \subset \mathbb{R}$ we denote the graph of u over H by

$$\Gamma(u, H) := \left\{(x,y) \in \mathbb{R}^2 \colon x \in H, \ y = u(x)\right\}.$$

We divide the proof in two steps.
Step 1)
Let $E \subset \mathbb{R}^2$ be an open bounded set with polyhedral boundary. By this assumption there exists a finite partition of the set $G := \{x \in \mathbb{R} \colon \mathcal{L}^1(E_{(x,0)}^{e_2}) > 0\}$ into intervals $\{(a_h, b_h)\}_{h=1}^M$ and into points $\{c_l\}_{l=1}^L$ in $\mathbb{R}$ i.e.,

$$G = \bigcup_{h=1}^M (a_h, b_h) \cup \bigcup_{l=1}^L \{c_l\},$$

and there exist affine functions $v_h^k, u_h^k \colon (a_h, b_h) \to \mathbb{R}$, $1 \leq h \leq M$, $1 \leq k \leq N(h)$ with $N(h) \in \mathbb{N}$ and there exist numbers $t_l^j, w_l^j \in \mathbb{R}$, $1 \leq l \leq L$, $1 \leq j \leq S(l)$ with $S(l) \in \mathbb{N}$ such that

$$\partial^* E = \bigcup_{h=1}^M \bigcup_{k=1}^{N(h)} \Gamma(u_h^k, (a_h, b_h)) \cup \Gamma(v_h^k, (a_h, b_h)) \cup \bigcup_{l=1}^L \bigcup_{j=1}^{S(l)} \{c_l\} \times (t_l^j, w_l^j)$$

and

$$(2.3) \qquad E = \bigcup_{h=1}^M \left\{(x,y) \in (a_h, b_h) \times \mathbb{R} \colon y \in \bigcup_{k=1}^{N(h)} \left(v_h^k(x), u_h^k(x)\right)\right\}.$$

By formula (2.3) we have that $\forall h \in \{1, \ldots, M\}$ and $\forall x \in (a_h, b_h)$

$$\mathcal{L}^1(E_{(x,0)}^{e_2}) = \sum_{k=1}^{N(h)} u_h^k(x) - v_h^k(x)$$

and $\forall h \in \{1, \ldots, L\}$ and $x = c_l$

$$\mathcal{L}^1(E_{(x,0)}^{e_2}) = \sum_{j=1}^{S(l)} |w_l^j - t_l^j|.$$

We set $m(x) := \mathcal{L}^1(E_{(x,0)}^{e_2})$ for every $x \in \mathbb{R}$ and we observe that such an m is an affine function on every (a_h, b_h). Since $E^{e_2} = \left\{(x,y) \in G \times \mathbb{R} \colon |y| \leq \frac{m(x)}{2}\right\}$ is a bounded set

with polyhedral boundary, its perimeter is finite. We denote by μ_E, $\mu_{E^{e_2}} \in \left[\mathcal{M}(\mathbb{R}^2)\right]^2$ the distributional derivatives of the functions χ_E and $\chi_{E^{e_2}}$, respectively. By the De Giorgi structure theorem we have that $|\mu_E| = \mathcal{H}^1 \llcorner \partial^* E$ and $|\mu_{E^{e_2}}| = \mathcal{H}^1 \llcorner \partial^* E^{e_2}$. For every $h = 1, \ldots, M$, for every $k = 1, \ldots, N(h)$ and for all $x \in (a_h, b_h)$ we have that the exterior normal vector field to E is

$$\nu_E((x, u_h^k(x))) = \frac{\left(-\frac{du_h^k}{dx}(x), 1\right)}{\sqrt{1 + \left|\frac{du_h^k}{dx}(x)\right|^2}}, \quad \nu_E((x, v_h^k(x))) = \frac{\left(\frac{dv_h^k}{dx}(x), -1\right)}{\sqrt{1 + \left|\frac{dv_h^k}{dx}(x)\right|^2}}$$

and similarly

$$\nu_{E^{e_2}}((x, \frac{m(x)}{2})) = \frac{\left(-\frac{1}{2}\frac{dm}{dx}(x), 1\right)}{\sqrt{1 + \left|\frac{1}{2}\frac{dm}{dx}(x)\right|^2}}, \quad \nu_{E^{e_2}}((x, -\frac{m(x)}{2})) = \frac{\left(\frac{1}{2}\frac{dm}{dx}(x), -1\right)}{\sqrt{1 + \left|\frac{1}{2}\frac{dm}{dx}(x)\right|^2}}.$$

Moreover for every $l = 1, \ldots, L$ and for every $j = 1, \ldots, S(l)$ and for all $(x, y) \in \{c_l\} \times (t_l^j, w_l^j)$ we have that

$$\nu_E((x, y)) \in \{-e_1, e_1\},$$

and for $(x, y) \in \{c_l\} \times (-\frac{m(c_l)}{2}, \frac{m(c_l)}{2})$ we have that

$$\nu_{E^{e_2}}((x, y)) \in \{-e_1, e_1\}.$$

Using the formula for the area of a graph, the formulas above and by the very definition of the function m it holds that

(2.4)
$$\begin{aligned}
\mathrm{P}_{|\cdot|_1}(E) &= \int_{\partial^* E} |\nu_E(z)|_1 d\mathcal{H}^1(z) \\
&= \sum_{h=1}^{M} \mathrm{P}_{|\cdot|_1}(E, (a_h, b_h) \times \mathbb{R}) + \sum_{l=1}^{L}\sum_{j=1}^{S(l)} \int_{\{c_l\}\times(t_l^j, w_l^j)} |\nu_E(z)|_1 d\mathcal{H}^1(z) \\
&= \sum_{h=1}^{M} \int_{a_h}^{b_h} \sum_{k=1}^{N(h)} |\nu_E(x, u_h^k(x))|_1 \sqrt{1 + |\frac{du_h^k}{dx}(x)|^2} + |\nu_E(x, v_h^k(x))|_1 \sqrt{1 + |\frac{dv_h^k}{dx}(x)|^2} dx \\
&\quad + \sum_{l=1}^{L}\sum_{j=1}^{S(l)} |t_l^j - w_l^j| \\
&= \sum_{h=1}^{M} \int_{a_h}^{b_h} \sum_{k=1}^{N(h)} 2 + \left|\frac{du_h^k}{dx}(x)\right| + \left|\frac{dv_h^k}{dx}(x)\right| + \sum_{l=1}^{L} m(c_l)
\end{aligned}$$

and we have

(2.5)

$$P_{|\cdot|_1}(E^{e_2}) = \int_{\partial^* E^{e_2}} |\nu_{E^{e_2}}(z)|_1 d\mathcal{H}^1(z)$$

$$= P_{|\cdot|_1}(E^{e_2}, G \times \mathbb{R}) + \sum_{l=1}^{L} \int_{\{c_l\} \times (-\frac{m(c_l)}{2}, \frac{m(c_l)}{2})} |\nu_{E^{e_2}}(z)|_1 d\mathcal{H}^1(z)$$

$$= \int_G \sqrt{1 + \left|\frac{1}{2}\frac{dm}{dx}(x)\right|^2} \left|\nu_{E^{e_2}}\left(x, \frac{m(x)}{2}\right)\right|_1 + \sqrt{1 + \left|\frac{1}{2}\frac{dm}{dx}(x)\right|^2} \left|\nu_{E^{e_2}}\left(x, -\frac{m(x)}{2}\right)\right|_1 dx$$

$$+ \sum_{l=1}^{L} m(c_l) = \int_G 2 + \left|\frac{dm}{dx}(x)\right| + \sum_{l=1}^{L} m(c_l)$$

$$= \sum_{h=1}^{M} \int_{a_h}^{b_h} 2 + \left|\sum_{k=1}^{N(h)} \frac{du_h^k}{dx}(x) - \frac{dv_h^k}{dx}(x)\right| + \sum_{l=1}^{L} m(c_l).$$

Therefore by the formulas (2.4) and (2.5) and from the triangular inequality it follows that

(2.6) $$P_{|\cdot|_1}(E) \geq P_{|\cdot|_1}(E^{e_2}).$$

Step 2)
Let $E \subset \mathbb{R}^2$ be a set of finite perimeter and finite measure. There exists a sequence $\{E_h\}_{h \in \mathbb{N}}$ of open bounded sets with polyhedral boundary such that

(2.7) $$\lim_{h \to +\infty} |E_h \Delta E| = 0, \quad \lim_{h \to +\infty} P(E_h) = P(E).$$

Hence by the formula (2.7) and the Reshetnyak continuity theorem we have

(2.8) $$\lim_{h \to +\infty} P_{|\cdot|_1}(E_h) = P_{|\cdot|_1}(E).$$

Let us set $m_h(x) := \mathcal{L}^1((E_h)_{(x,0)}^{e_2})$ and $G_h := \{x \in \mathbb{R} \colon m_h(x) > 0\}$, by Fubini's theorem, we have

$$|E_h \Delta E| = \int_{\mathbb{R}} \mathcal{L}^1((E_h)_{(x,0)}^{e_2} \Delta E_x) dx \geq \int_{\mathbb{R}} |m_h(x) - m(x)| dx = |E_h^{e_2} \Delta E^{e_2}|.$$

Hence by the above formula and (2.7) we have

(2.9) $$\lim_{h \to +\infty} |E_h^{e_2} \Delta E^{e_2}| = 0.$$

Therefore by (2.6), (2.8), (2.9) and by the lower semicontinuity of the function $E \to P_{|\cdot|_1}(E)$ with respect to the $L^1(\mathbb{R}^2)$ topology we obtain

$$P_{|\cdot|_1}(E) = \lim_{h \to +\infty} P_{|\cdot|_1}(E_h) \geq \liminf_{h \to +\infty} P_{|\cdot|_1}((E_h)^{e_2}) \geq P_{|\cdot|_1}(E^{e_2})$$

i.e. the thesis. $\qquad \square$

Proposition 2.5. *Let $E \subset \mathbb{R}^2$ be a set of finite perimeter with $|E| = 1$ and let $R \subset \mathbb{R}^2$ be a rectangle with $\mathrm{Bar}(R) = (0,0)$ and $|R| = 1$ with horizontal sidelength equal to $a > 0$ and vertical sidelength equal to $b > 0$. Then for every $\eta \in \{\pm e_1, \pm e_2\}$*

$$\mathcal{D}(E, R) \geq \mathcal{D}(E^{\eta}, R).$$

Proof. We only prove the case $\eta = e_2$. For $\mathcal{H}^1$-a.e. $(x,0) \in \mathbb{R}^2$ let as set $E_x := E_{(x,0)}^{e_2}$ and $\tilde{E}_x := (-\frac{\mathcal{L}^1(E_x)}{2}, \frac{\mathcal{L}^1(E_x)}{2})$. We denote by R_x the interval $(-\frac{b}{2}, \frac{b}{2})$. By the Fubini-Tonelli theorem we can write

(2.10) $$\mathcal{D}(E, R) = \int_{E \Delta R} d_\infty(z, \partial R) dz = \int_{\mathbb{R}} dx \int_{E_x \Delta R_x} d_\infty((x,y), \partial R) dy$$

and

$$(2.11) \qquad \mathcal{D}(E^{e_2}, R) = \int_{E^{e_1} \Delta R} d_\infty(z, \partial R) dz = \int_\mathbb{R} dx \int_{\tilde{E}_x \Delta R_x} d_\infty((x,y), \partial R) dy.$$

We claim that

$$(2.12) \qquad \int_{E_x \Delta R_x} d_\infty((x,y), \partial R) dy \geq \int_{\tilde{E}_x \Delta R_x} d_\infty((x,y), \partial R) dy.$$

We first study the case $\mathcal{L}^1(E_x) < \mathcal{L}^1(R_x)$. If moreover $\mathcal{L}^1(E_x \cap R_x^c) \neq 0$ then we have $\mathcal{L}^1(R_x \setminus E_x) > \mathcal{L}^1(E_x \cap R^c)$. Hence there exists a $\mathcal{L}^1$-measurable set $M \subset R_x \setminus E_x$ such that $\mathcal{L}^1(M) = \mathcal{L}^1(E_x \cap R^c)$. Indeed the function $f(t) := \mathcal{L}^1((-t,t) \cap R_x \setminus E_x)$ is $C^0(\mathbb{R})$ and such that $f(0) = 0$ and for $f(\frac{b}{2}) = \mathcal{L}^1(R_x \setminus E_x)$. Therefore there exist $s \in (-t,t)$ such that $f(s) = \mathcal{L}^1((-s,s) \cap R_x \setminus E_x) = \mathcal{L}^1(E_x \cap R^c)$ and hence we define $M =: (-s,s) \cap R_x \setminus E_x$. Therefore we have

$$\int_{E_x \Delta R_x} d_\infty((x,y), \partial R) dy = \int_{E_x \setminus R_x} d_\infty((x,y), \partial R) dy + \int_{R_x \setminus E_x} d_\infty((x,y), \partial R) dy$$

$$\geq \int_{R_x \setminus E_x} d_\infty((x,y), \partial R) dy \geq \int_{R_x \setminus (E_x \cup M)} d_\infty((x,y), \partial R) dy.$$

By the last formula it suffices to study the case $\mathcal{L}^1(E_x) < \mathcal{L}^1(R_x)$ and $E_x \subset R_x$. We observe that

$$(2.13) \quad R_x \setminus E_x = \tilde{E}_x \setminus E_x \cup R_x \setminus (\tilde{E}_x \cup E_x) \text{ and } R_x \setminus \tilde{E}_x = E_x \setminus \tilde{E}_x \cup R_x \setminus (\tilde{E}_x \cup E_x),$$

hence we obtain that

$$(2.14) \qquad |R_x \setminus E_x| = |R_x \setminus \tilde{E}_x| \implies |\tilde{E}_x \setminus E_x| = |E_x \setminus \tilde{E}_x|.$$

Moreover for all $y' \in E_x \setminus \tilde{E}_x$ and $y \in \tilde{E}_x \setminus E_x$ it holds

$$(2.15) \qquad d_\infty((x,y'), \partial R) \leq \frac{b - \mathcal{L}^1(E_x)}{2} \leq d_\infty((x,y), \partial R).$$

Therefore thanks to (2.13), (2.14) and (2.15) we have

$$(2.16) \qquad \begin{aligned} \int_{E_x \Delta R_x} d_\infty((x,y), \partial R) dy &= \int_{R_x \setminus E_x} d_\infty((x,y), \partial R) dy \\ &= \int_{\tilde{E}_x \setminus E_x} d_\infty((x,y), \partial R) dy + \int_{R_x \setminus (\tilde{E}_x \cup E_x)} d_\infty((x,y), \partial R) dy \\ &\geq \int_{E_x \setminus \tilde{E}_x} d_\infty((x,y), \partial R) dy + \int_{R_x \setminus (\tilde{E}_x \cup E_x)} d_\infty((x,y), \partial R) dy \\ &= \int_{R_x \setminus \tilde{E}_x} d_\infty((x,y), \partial R) dy = \int_{\tilde{E}_x \Delta R_x} d_\infty((x,y), \partial R) dy. \end{aligned}$$

If instead $\mathcal{L}^1(E_x) \geq \mathcal{L}^1(R_x)$ and $\mathcal{L}^1(R_x \setminus E_x) \neq 0$ then we have that there exist $M \subset E_x \setminus R_x$ measurable set such that $|M| = |R_x \setminus E_x|$. Therefore we define

$$E_x^1 := R_x \cup (E_x \setminus (M \cup R_x))$$

and we obtain

$$E_x^1 \Delta R_x = E_x \setminus (M \cup R_x) \subset E_x \setminus R_x \subset E_x \setminus R_x \cup R_x \setminus E_x = E_x \Delta R_x,$$

hence

$$\int_{E_x \Delta R_x} d_\infty((x,y), \partial R) dy \geq \int_{E_x^1 \Delta R_x} d_\infty((x,y), \partial R) dy.$$

The last formula show that it suffices to analyze the case $\mathcal{L}^1(E_x) \geq \mathcal{L}^1(R_x)$ and $R_x \subset E_x$. We observe that

$$(2.17) \qquad E_x \Delta R_x = E_x \setminus R_x = \Big[(E_x \setminus R_x) \cap \tilde{E}_x\Big] \cup \Big[(E_x \setminus R_x) \setminus \tilde{E}_x\Big]$$

and

$$(2.18) \qquad \tilde{E}_x \Delta R_x = \tilde{E}_x \setminus R_x = \Big[(E_x \setminus R_x) \cap \tilde{E}_x\Big] \cup \Big[\tilde{E}_x \setminus E_x\Big].$$

Moreover we have that for all $y \in (E_x \setminus R_x) \setminus \tilde{E}_x$ and for all $y' \in \tilde{E}_x \setminus E_x$

$$(2.19) \qquad d_\infty((x,y), \partial R) \geq \frac{\mathcal{L}^1(E_x) - b}{2} \geq d_\infty((x,y'), \partial R).$$

Thank to (2.17), (2.18) and (2.19), following the same arguments leading to (2.16) we have

$$\int_{E_x \Delta R_x} d_\infty((x,y), \partial R) dy \geq \int_{\tilde{E}_x \Delta R_x} d_\infty((x,y), \partial R) dy.$$

The formula above together with (2.16) gives (2.12). The thesis follow from (2.12), (2.10), (2.11). $\qquad\square$

Proposition 2.6. *For every $E \subset \mathbb{R}^2$ set of finite perimeter it holds that*
$$\mathrm{P}_{|\cdot|_1}(E) \geq 2\mathcal{H}^1(\pi_1(E)) + 2\mathcal{H}^1(\pi_2(E)).$$

Proof. By (2.2) we have that

$$\begin{aligned}
\mathrm{P}_{|\cdot|_1}(E) &= \sup\left\{\int_E \mathrm{div}(T)(x,y)dxdy \colon T \in C_c^1(\mathbb{R}^2, \mathbb{R}^2),\ |T|_\infty(x,y) \leq 1,\ \forall x \in \mathbb{R}^2\right\} \\
&= \sup\left\{\int_E \frac{\partial T_1}{\partial x} + \frac{\partial T_2}{\partial y}dxdy \colon T_i \in C_c^1(\mathbb{R}^2, \mathbb{R}),\ |T_i|(x,y) \leq 1,\ \forall x \in \mathbb{R}^2,\ \forall i = 1,2\right\} \\
&= \sup\left\{\int_E \frac{\partial T_1}{\partial x}dxdy \colon T_1 \in C_c^1(\mathbb{R}^2, \mathbb{R}),\ |T_1|(x,y) \leq 1,\ \forall x \in \mathbb{R}^2\right\} \\
&\quad + \sup\left\{\int_E \frac{\partial T_2}{\partial y}dxdy \colon T_2 \in C_c^1(\mathbb{R}^2, \mathbb{R}),\ |T_2|(x,y) \leq 1,\ \forall x \in \mathbb{R}^2\right\}.
\end{aligned}$$

On other hand we have

$$\begin{aligned}
&\sup\left\{\int_E \frac{\partial T_1}{\partial x}dxdy \colon T_1 \in C_c^1(\mathbb{R}^2, \mathbb{R}),\ |T_1(z)| \leq 1,\ \forall z \in \mathbb{R}^2\right\} \\
&\geq \sup\left\{\int_E \frac{\partial T_1}{\partial x}dxdy \colon T_1(x,y) = f(x)g(y),\ f,g \in C_c^1(\mathbb{R}, \mathbb{R}),\ |f|, |g| \leq 1,\ \text{in } \mathbb{R}\right\} \\
&= \sup\left\{\int_{\pi_2(E)} g(y)dy \int_{E_{(0,y)}^{e_1}} \frac{df}{dx}(x)dx \colon f,g \in C_c^1(\mathbb{R}, \mathbb{R}),\ |f|, |g| \leq 1,\ \text{in } \mathbb{R}\right\} \\
&\geq 2\mathcal{H}^1(\pi_2(E))
\end{aligned}$$

where in the last inequality we have used that the slice $E_{(0,y)}^{e_1}$ is a set of finite perimeter in $\mathbb{R}$ for $\mathcal{H}^1$-a.e. $y \in \mathbb{R}$, and for every $y \in \pi_2(E)$ the zero dimensional perimeter of the set $E_{(0,y)}^{e_1}$ is greater that 2, see [17][Theorem 2.5]. With the same argument we obtain that

$$\sup\left\{\int_E \frac{\partial T_2}{\partial y}dxdy \colon T_2 \in C_c^1(\mathbb{R}^2, \mathbb{R}),\ |T_2(z)| \leq 1,\ \forall z \in \mathbb{R}^2\right\} \geq 2\mathcal{H}^1(\pi_1(E)),$$

Hence the thesis. $\qquad\square$

We are now in a position to prove the equivalence of the problem $(\mathcal{P}_2)$ and $(\mathcal{P}_1)$.

Theorem 2.7. *Let $R \subset \mathbb{R}^2$ be a rectangle such that $|R| = 1$. The problem $(\mathcal{P}_2)$ and $(\mathcal{P}_1)$ are equivalent.*

Proof. Let $E \subset \mathbb{R}^2$ be such that $|E| = 1$. We define the set E^s as the set obtained through the Steiner symmetrization with respect to e_1 and e_2; i.e., $E^s = (E^{e_1})^{e_2}$. By Proposition 2.4 and Proposition 2.5 we obtain

$$P_{|\cdot|_1}(E^s) + \frac{1}{\tau}\mathcal{D}(E^s, R) \leq P_{|\cdot|_1}(E) + \frac{1}{\tau}\mathcal{D}(E, R).$$

We first observe that if $\mathcal{H}^1(\pi_1(E^s)) > a$ and $\mathcal{H}^1(\pi_2(E^s)) > b$ then by Proposition 2.6

$$P_{|\cdot|_1}(E^s) + \frac{1}{\tau}\mathcal{D}(E^s, R) \geq 2\mathcal{H}^1(\pi_1(E^s)) + 2\mathcal{H}^1(\pi_2(E^s)) \geq P_{|\cdot|_1}(R) + \frac{1}{\tau}\mathcal{D}(R, R).$$

We complete the proof of the result only in the case $\mathcal{H}^1(\pi_1(E^s)) < a$, the case $\mathcal{H}^1(\pi_2(E^s)) < b$ being analogous. We define the rectangle

$$\overline{R} := \pi_1(E^s) \times \left[-\frac{L}{2}, \frac{L}{2}\right] \quad \text{with } L \text{ such that} \quad L\mathcal{H}^1(\pi_1(E^s)) = 1.$$

Note that $\mathrm{Bar}(\overline{R}) = \mathrm{Bar}(E_s) = (0, 0)$. We claim that

$$(2.20) \qquad\qquad P_{|\cdot|_1}(E^s) \geq 2\mathcal{H}^1(\pi_1(E^s)) + 2L = P_{|\cdot|_1}(\overline{R}).$$

By the Proposition 2.6 we have

$$(2.21) \qquad\qquad P_{|\cdot|_1}(E^s) \geq 2\mathcal{H}^1(\pi_1(E^s)) + 2\mathcal{H}^1(\pi_2(E^s)).$$

Up to a null set we have that

$$E^s \subset \pi_1(E^s) \times \pi_2(E^s)$$

and then

$$(2.22) \qquad \mathcal{H}^1(\pi_1(E^s))\mathcal{H}^1(\pi_2(E^s)) \geq |E^s| = 1 = \mathcal{H}^1(\pi_1(E^s))L \implies L \leq \mathcal{H}^1(\pi_2(E^s)).$$

Hence (2.20) follows from (2.22) and (2.21). We now claim that

$$(2.23) \qquad\qquad \mathcal{D}(E^s, R) \geq \mathcal{D}(\overline{R}, R).$$

We define the sets

$$\mathcal{A} := E^s \cap \mathbb{R} \times \left(-\infty, -\frac{L}{2}\right) \cup \left(\frac{L}{2}, +\infty\right) \quad \mathcal{B} := E^s \cap \mathbb{R} \times \left(-\frac{L}{2}, -\frac{b}{2}\right) \cup \left(\frac{b}{2}, \frac{L}{2}\right)$$

and

$$\mathcal{C} := \overline{R} \cap R \setminus E^s \quad \mathcal{D} := R \setminus \overline{R} \quad \mathcal{E} := \pi_1(E^s) \times \left(-\frac{L}{2}, -\frac{b}{2}\right) \cup \left(\frac{b}{2}, \frac{L}{2}\right) \setminus E^s$$

and we have

$$R\Delta E^s = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \quad R\Delta\overline{R} = \mathcal{B} \cup \mathcal{D} \cup \mathcal{E}.$$

We observe that

$$|E^s| = |\overline{R}| \implies |\mathcal{A}| = |\mathcal{E} \cup \mathcal{C}|,$$

and then we define $\mathcal{A}'$ and $\mathcal{A}''$ measurable sets such that $\mathcal{A} = \mathcal{A}' \cup \mathcal{A}''$ and $|\mathcal{A}'| = |\mathcal{C}|$, $|\mathcal{A}''| = |\mathcal{E}|$. Note that the construction of $\mathcal{A}'$ is straightforward and can be obtained as follows. We define the continuous function $f(t) := |\mathcal{A} \cap \mathbb{R} \times (-\infty, -t) \cup (t, \infty)|$ and observe that $f(\frac{\mathcal{H}^1(\pi_2(E^s))}{2}) = 0$, $f(\frac{b}{2}) = |\mathcal{A}|$. Hence there exists $s \in \mathbb{R}$ such that $f(s) = |\mathcal{C}|$ and $\mathcal{A}' := \mathcal{A} \cap \mathbb{R} \times (-\infty, -s) \cup (s, \infty)$, $\mathcal{A}'' := \mathcal{A} \setminus \mathcal{A}'$. By the construction of $\mathcal{A}'$ and $\mathcal{A}''$ it holds that

$$\text{for all } (x', y') \in \mathcal{A}' \text{ and for all } (x'', y'') \in \mathcal{A}'' \implies y' \geq y''.$$

We introduce a new set $\hat{E} := \mathcal{C} \cup E^s \setminus \mathcal{A}'$ and observe that

$$(2.24) \qquad\qquad R\Delta\hat{E} = \mathcal{A}'' \cup \mathcal{B} \cup \mathcal{D} \subset \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} = R\Delta E^s,$$

which in turn implies that

$$\mathcal{D}(\hat{E}, R) = \int_{R \Delta \hat{E}} d_\infty((x,y), \partial R) dx dy \leq \int_{R \Delta E^s} d_\infty((x,y), \partial R) = \mathcal{D}(E^s, R).$$

We are left to prove that

$$\mathcal{D}(\overline{R}, R) \leq \mathcal{D}(\hat{E}, R).$$

We observe that

$$\text{for all } z'' \in \mathcal{A}'' \text{ and for all } z \in \mathcal{E} \implies d_\infty(z, \partial R) \leq \frac{L-b}{2} \leq d_\infty(z'', \partial R)$$

and since $|\mathcal{E}| = |\mathcal{A}''|$ we have

(2.25)
$$\int_{\mathcal{E}} d_\infty(z, \partial R) dz \leq \int_{\mathcal{A}''} d_\infty(z, \partial R) dz.$$

Therefore by (2.24) and (2.25) we obtain

$$\mathcal{D}(\overline{R}, R) = \int_{\mathcal{D}} d_\infty(z, \partial R) dz + \int_{\mathcal{B}} d_\infty(z, \partial R) dz + \int_{\mathcal{E}} d_\infty(z, \partial R) dz$$

$$\leq \int_{\mathcal{D}} d_\infty(z, \partial R) dz + \int_{\mathcal{B}} d_\infty(z, \partial R) dz + \int_{\mathcal{A}''} d_\infty(z, \partial R) dz = \mathcal{D}(\hat{E}, R)$$

which prove the claim. Eventually by (2.20), (2.23) we obtain the thesis. □

In the next theorem we want to compute the minimizer of the problem $(\mathcal{P}_2)$, namely

$$\min \left\{ \mathrm{P}_{|\cdot|_1}(\overline{R}) + \frac{1}{\tau} \mathcal{D}(\overline{R}, R) \colon \overline{R} \subset \mathbb{R}^2 \text{ be a rectangle and } |\overline{R}| = 1, \text{ Bar}(\overline{R}) = (0,0) \right\},$$

where we remind the reader that $R \subset \mathbb{R}^2$ is a rectangle with horizontal sidelength a, vertical sidelength b, $\text{Bar}(R) = (0,0)$, $|R| = ab = 1$. Without loss of generality we moreover assume that $b \leq a$.

Theorem 2.8. *Let $a, b \in \mathbb{R}$ be such that $0 < b \leq a$ and $ab = 1$ and let us consider $R = \left[-\frac{a}{2}, \frac{a}{2}\right] \times \left[-\frac{b}{2}, \frac{b}{2}\right]$. For all $\Lambda > a + b$ there exists $\tau_0 := \tau_0(\Lambda)$ such that for every $\tau < \tau_0$ there exists a unique minimizer R' of*

(2.26)
$$\min \left\{ \mathrm{P}_{|\cdot|_1}(\tilde{R}) + \frac{1}{\tau} \mathcal{D}(\tilde{R}, R) \colon \tilde{R} \subset \mathbb{R}^2 \text{ be a rectangle and } |\tilde{R}| = 1, \text{ Bar}(\tilde{R}) = 0 \right\}.$$

Moreover we have $R' = \left[-\frac{a'}{2}, \frac{a'}{2}\right] \times \left[-\frac{b'}{2}, \frac{b'}{2}\right]$ with

$$a' = a - 2x(\tau) \text{ with } x(\tau) = x'\tau + O(\tau^2) \text{ and } x' = \frac{2(a-b)}{b(a+b)},$$

$$b' = b + 2y(\tau) \text{ with } y(\tau) = \frac{bx(\tau)}{a - 2x(\tau)}$$

$$a' + b' \leq a + b, \ |a' - a| \leq \tau C(\Lambda), \ |b' - b| \leq \tau C(\Lambda) \text{ and } |R \Delta R'| \leq \tau C(\Lambda).$$

Proof. We divide the proof in to several steps.
Step 1)
In this step we show that if $R' = \left[-\frac{a'}{2}, \frac{a'}{2}\right] \times \left[-\frac{b'}{2}, \frac{b'}{2}\right]$ is a minimizer then

$$a' < a \text{ and } b' > b.$$

We argue by contradiction. Let us suppose $a' = a + x$ for some $x > 0$ and let b' be such that $ab = a'b'$, that is to say $b' = \frac{ab}{a+x}$. Then, using the assumption $b \leq a$ we have that
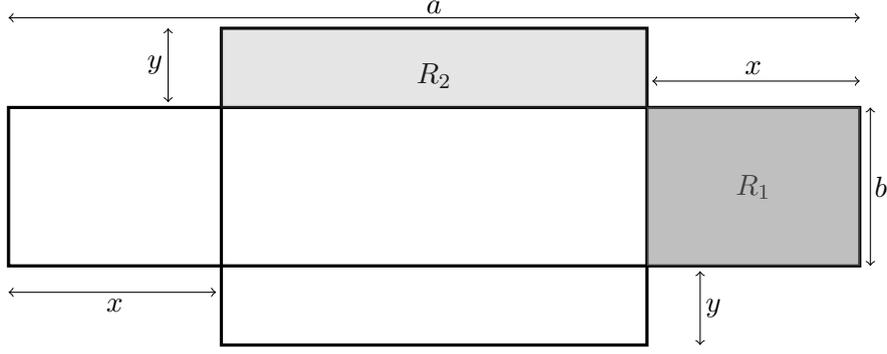
(2.27)
$$a + b \leq a + x + \frac{ab}{a+x} = a' + b'.$$

Figure 1. The rectangle R, $\tilde{R}$, R_1, and R_2.

Therefore by (2.27) we get

$$\mathrm{P}_{|\cdot|_1}(R) + \frac{1}{\tau}\mathcal{D}(R, R) = 2a + 2b \leq 2a' + 2b' < \mathrm{P}_{|\cdot|_1}(R') + \frac{1}{\tau}\mathcal{D}(R', R),$$

which contradicts the minimality of R'.

Step 2)

In this step we show that our minimum problem can by studied by analyzing a function of two real variables. From the previous step we can assume that a competitor of our minimum problem can be written as follows

$$\tilde{R} = \left[-\frac{a}{2} + x, \frac{a}{2} - x\right] \times \left[-\frac{b}{2} - y, \frac{b}{2} + y\right]$$

for some $x \in [0, \frac{a}{2})$ and $y = \frac{xb}{a-2x}$. The perimeter of the set $\tilde{R}$ is

(2.28) $$\mathrm{P}_{|\cdot|_1}(\tilde{R}) = 2(a - 2x + b + 2y).$$

Using the symmetry of the set $\tilde{R}\Delta R$, the dissipation is

(2.29) $$\mathcal{D}(\tilde{R}, R) = \int_{\tilde{R}\Delta R} d_\infty(z, \partial R)dz = 2\int_{R_1} d_\infty(z, \partial R)dz + 2\int_{R_2} d_\infty(z, \partial R)dz$$

where

$$R_1 = \left[-\frac{a}{2} + x, \frac{a}{2} - x\right] \times \left[\frac{b}{2}, \frac{b}{2} + y\right] \text{ and } R_2 = \left[\frac{a}{2} - x, \frac{a}{2}\right] \times \left[-\frac{b}{2}, \frac{b}{2}\right].$$

We now compute explicitly the dissipation as a function of the variables x and y. The first integral in (2.29) is

(2.30) $$\int_{R_1} d_\infty((z_1, z_2), \partial R)dz_1 dz_2 = \int_0^y (a - 2x)z_2 dz_2 = \frac{1}{2}y^2(a - 2x).$$

The formula for the second integral in (2.29) depends on whether $x < \frac{b}{2}$ or $x \geq \frac{b}{2}$.

Case $x < \frac{b}{2}$. It holds that

(2.31)
$$\int_{R_2} d_\infty((z_1, z_2), \partial R)dz_1 dz_2 = \int_{T_1} d_\infty(z, \partial R)dz + \int_{T_2} d_\infty(z, \partial R)dz$$
$$+ \int_{T_3} d_\infty(z, \partial R)dz + \int_{T_4} d_\infty(z, \partial R)dz + \int_H d_\infty(z, \partial R)dz.$$
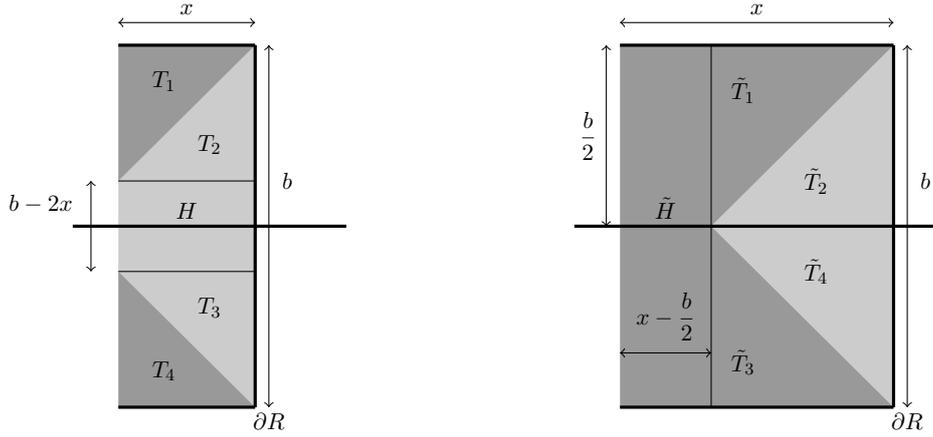
Figure 2. On the left side the case $x \leq \frac{b}{2}$ and on the right side the case $x > \frac{b}{2}$.

Here T_1 denotes the triangle with vertices

$$\left\{ \left(\frac{a}{2} - x, \frac{b}{2} - x \right), \left(\frac{a}{2}, \frac{b}{2} \right), \left(\frac{a}{2} - x, \frac{b}{2} \right) \right\},$$

T_2 denotes the triangle with vertices

$$\left\{ \left(\frac{a}{2} - x, \frac{b}{2} - x \right), \left(\frac{a}{2}, \frac{b}{2} \right), \left(\frac{a}{2}, \frac{b}{2} - x \right) \right\},$$

while T_3 and T_4 are the triangles obtained reflecting T_1 and T_2 with respect to the horizontal axis, respectively. Finally, H denotes the rectangle with vertices

$$\left\{ \left(\frac{a}{2} - x, \frac{b}{2} - x \right), \left(\frac{a}{2}, \frac{b}{2} - x \right), \left(\frac{a}{2} - x, -\frac{b}{2} + x \right), \left(\frac{a}{2}, -\frac{b}{2} + x \right) \right\}.$$

By a change of variables we observe that for every $i \in \{1, 2, 3, 4\}$

$$\int_{T_i} d_\infty(z, \partial R) dz = \int_T d_\infty(z, [0, x] \times \{0\}) dz = \int_T z_2 dz_1 dz_2$$

where T is a triangle with vertices $(0,0), (x,0), (0,x)$. Hence, by (2.31) and the formula above, we obtain that

$$\int_{R_2} d_\infty(z, \partial R) dz = 4 \int_T d_\infty(z, \partial R) dz + \int_H d_\infty(z, \partial R) dz$$

(2.32)
$$= 4 \int_0^x dz_1 \int_0^{x-z_1} z_2 dz_2 + \int_{\frac{a}{2}-x}^{\frac{a}{2}} \int_{-\frac{b}{2}+x}^{\frac{b}{2}-x} (z_1 - \frac{a}{2} + x) dz_1 dz_2$$

$$= \frac{2x^3}{3} + \frac{x^2(b - 2x)}{2}.$$

eventually by (2.29), (2.30), (2.32) we have

(2.33)
$$D(\tilde{R}, R) = \int_{\tilde{R} \triangle R} d_\infty(z, \partial R) dz = y^2(a - 2x) + \frac{4}{3}x^3 + x^2(b - 2x).$$

Case $x \geq \frac{b}{2}$. It holds that

$$\int_{R_2} d_\infty((z_1, z_2), \partial R) dz_1 dz_2 = \int_{\tilde{T}_1} d_\infty(z, \partial R) dz + \int_{\tilde{T}_2} d_\infty(z, \partial R) dz$$

$$+ \int_{\tilde{T}_3} d_\infty(z, \partial R) dz + \int_{\tilde{T}_4} d_\infty(z, \partial R) dz + \int_{\tilde{H}} d_\infty(z, \partial R) dz.$$

Here $\tilde{T}_1$ denotes the triangle with vertices

$$\left\{ \left(\frac{a}{2} - \frac{b}{2}, 0 \right), \left(\frac{a}{2} - \frac{b}{2}, \frac{b}{2} \right), \left(\frac{a}{2}, \frac{b}{2} \right) \right\},$$

$\tilde{T}_2$ denotes the triangle with vertices

$$\left\{ \left(\frac{a}{2} - \frac{b}{2}, 0 \right), \left(\frac{a}{2}, 0 \right), \left(\frac{a}{2}, \frac{b}{2} \right) \right\}$$

and $\tilde{T}_3, \tilde{T}_4$ are the triangles obtained reflecting $\tilde{T}_1, \tilde{T}_2$ with respect to the horizontal axis, respectively. Finally $\tilde{H}$ is the rectangle with vertices

$$\left\{ \left(\frac{a}{2} - x, -\frac{b}{2} \right), \left(\frac{a}{2} - x, \frac{b}{2} \right), \left(\frac{a}{2} - \frac{b}{2}, \frac{b}{2} \right), \left(\frac{a}{2} - \frac{b}{2}, -\frac{b}{2} \right) \right\}.$$

By a change of variables for every $i \in \{1, 2, 3, 4\}$

$$\int_{\tilde{T}_i} d_\infty(z, \partial R) dz = \int_{\tilde{T}} d_\infty(z, [0, \frac{b}{2}] \times \{0\}) dz = \int_{\tilde{T}} z_2 dz_1 dz_2$$

where $\tilde{T}$ is the triangle with vertices $(0,0), (\frac{b}{2}, 0), (0, \frac{b}{2})$. A straightforward computation as in the previous case gives

$$\int_{R_2} d_\infty(z, \partial R) dz = -\frac{b^3}{24} + \frac{xb^2}{4}.$$

Therefore by (2.29), (2.30) and the equality above

$$(2.34) \qquad D(\tilde{R}, R) = \int_{\tilde{R} \Delta R} d_\infty(z, \partial R) dz = y^2(a - 2x) - \frac{b^3}{12} + \frac{xb^2}{2}.$$

Eventually, by (2.28), (2.33), (2.34), we can write the energy

$$\tilde{R} \to P_{|\cdot|_1}(\tilde{R}) + \frac{1}{\tau} \mathcal{D}(\tilde{R}, R)$$

as a function of the variables x and y as follows

$$E(x, y) := \begin{cases} 2(a - 2x + b + 2y) + \dfrac{1}{\tau} \left[y^2(a - 2x) + \dfrac{4}{3}x^3 + x^2(b - 2x) \right], & \text{if } 0 \le x < \dfrac{b}{2}, \\[3mm] 2(a - 2x + b + 2y) + \dfrac{1}{\tau} \left[y^2(a - 2x) - \dfrac{b^3}{12} + \dfrac{b^2}{2}x \right], & \text{if } \dfrac{b}{2} \le x < \dfrac{a}{2}. \end{cases}$$

Step 3)
In this step we want to compute the minimzer of the function E under the area constraint $y = y(x) = \frac{xb}{a - 2x}$. We distinguish two cases.

Case $\frac{b}{2} \le x \le \frac{a}{2}$. In this case the energy can be written as

$$E(x, y(x)) = \frac{1}{\tau} \left[2a\tau + 2b\tau - \frac{b^3}{12} \right] + \frac{1}{\tau} \left[\frac{x(a(b^2 - 8\tau) + 8\tau b + 16\tau x)}{2a - 4x} \right].$$

We observe that

$$(2.35) \qquad \frac{d}{dx} E(x, y(x)) = \frac{1}{\tau} \left[\frac{a^2b^2 - 8\tau a^2 + 8\tau ab + 32\tau ax - 32\tau x^2}{2a^2 - 8ax + 8x^2} \right].$$

We claim the existence of $\tau_1 := \tau_1(\Lambda) > 0$ such that for all $\tau < \tau_1$, $\frac{d}{dx} E(x, y(x)) > 0$ for every x.

To this end, since the denominator is positive, it is enough to prove the positivity of the numerator in (2.35) for τ small enough. The latter follows by the chain of inequalities

$$a^2b^2 - 8\tau a^2 + 8\tau ab + 32\tau ax - 32\tau x^2 > a^2b^2 - (8\tau a^2 + 8\tau ab + 32\tau ax + 32\tau x^2) > 0$$

which holds true for $\tau < \tau_1 := \Lambda^{-k_1}$ with $k_1 \in \mathbb{N}$ such that

$$(8\tau a^2 + 8\tau ab + 32\tau ax + 32\tau x^2) < Ca^2 \Lambda^{-k_1} < \frac{1}{2} < a^2 b^2 = 1,$$

where we have used that $x < a$ and $b < a$. By the same argument we have that there exists $\tau_2(\Lambda)$ such that for all $\tau < \tau_2(\Lambda)$

$$\lim_{x \to \frac{a}{2}^-} E(x, y(x)) = +\infty.$$

By the previous argument the only minimizer of $x \to E(x, y(x))$ for $x \in \left[\frac{b}{2}, \frac{a}{2}\right)$ is $x = \frac{b}{2}$ where the energy takes the value

$$E\left(\frac{b}{2}, y\left(\frac{b}{2}\right)\right) = \frac{1}{\tau}\left[\frac{24\tau a^2 + 2ab^3 - 24\tau ab + b^4 + 24\tau b^3}{12a - 12b}\right].$$

Case $0 \le x \le \frac{b}{2}$. In this case we have

$$E(x, y(x)) = \frac{1}{3\tau(a - 2x)}[6a^2\tau + 6ab\tau + 3abx^2$$
$$- 24a\tau x - 2ax^3 + 3b^2x^2 - 6bx^3 + 24\tau x^2 + 4x^4]$$

and

$$(2.36) \qquad \frac{d}{dx}E(x, y(x)) = \frac{1}{\tau(a - 2x)^2}\Big[-8x^4 + x^3(2a + 8b) + x^2(-8ab - 2b^2 - 16\tau - 2a^2)$$
$$+ x(2a^2b + 2ab^2 + 16a\tau) + 4ab\tau - 4a^2\tau\Big].$$

The minimizer R' of (2.26) satisfies $\mathcal{D}(R', R) \le \tau\Lambda$. Therefore using the volume constraint and (2.33) we obtain

$$(2.37) \qquad x^2\left(b^2 + \frac{4}{3}x(a - 2x) + (b - 2x)(a - 2x)\right) < C(\Lambda)\tau.$$

Hence we have $x = x(\tau) \le C(\Lambda)\tau^{\frac{1}{2}}$. We now look for stationary points of the energy E. To this end we investigate the zeros of the numerator of (2.36) that we denote by
(2.38)
$$f(\tau, x) := -8x^4 + x^3(2a + 8b) + x^2(-8ab - 2b^2 - 16\tau - 2a^2) + x(2a^2b + 2ab^2 + 16a\tau) + 4ab\tau - 4a^2\tau.$$

It holds that
$$\frac{\partial f}{\partial x}(\tau, x) = -32x^3 + x^2(6a + 24b) + x(-16ab - 4b^2 - 32\tau - 4a^2) + (2a^2b + 2ab^2 + 16a\tau)$$

and

$$(2.39) \qquad \frac{\partial^2 f}{\partial^2 x}(\tau, x) = -96x^2 + 6x(2a + 8b) + 2(-8ab - 2b^2 - 2a^2) - 32\tau.$$

We observe that there exists $C_1(\Lambda) > 0$ such that

$$(2.40) \qquad \max_{x \in [0, \frac{b}{2}]} |f(\tau, x) - f(0, x)| \le \tau C_1(\Lambda).$$

By (2.40) there exists $\tau_3 := \tau_3(\Lambda)$ such that for all $\tau < \tau_3$ the points (τ, x) such that $f(\tau, x) = 0$ are in a neighbourhood of the zeros of $f(0, \cdot)$. We have that

$$f(0, 0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial x}(0, 0) = 2a^2b + 2ab^2 > 0 \quad \text{for all } a, b > 0.$$

We first claim that for all $\Lambda > a + b$ there exists $\hat{\delta} := \hat{\delta}(\Lambda)$ such that for all $a, b > 0$ and $a + b < \Lambda$

$$(2.41) \qquad \frac{\partial f}{\partial x}(\tau, x) > 0 \quad \text{for all } (\tau, x) \in (-\hat{\delta}, \hat{\delta}) \times (-\hat{\delta}, \hat{\delta}).$$

The claim follows observing that thanks (2.37) we can assume that $|x| \leq \delta \leq C(\Lambda)\tau^{\frac{1}{2}}$ we have

$$
\begin{aligned}
\frac{\partial f}{\partial x}(\tau, x) =& - 32x^3 + x^2(6a + 24b) + x(-16ab - 4b^2 - 32\tau - 4a^2) + (2a^2b + 2ab^2 + 16a\tau) \\
\geq& - 32x^3 - x^2(6a + 24b) + x(-16ab - 4b^2 - 32\tau - 4a^2) + (2a^2b + 2ab^2) \\
\geq& - 32\delta^3 - \delta^2(6a + 24b + 32) - \delta(+16ab + 4b^2 + 4a^2) + (2a^2b + 2ab^2) \\
=& \; h(\tau) + 2ab(a + b),
\end{aligned}
$$

where $h(\tau) \to 0$ as $\tau \to 0$. As a consequence of this claim and the implicit function Theorem, for every $a, b > 0$ and $a + b \leq \Lambda$ there exists $\bar{x} : (-\frac{\hat{\delta}}{2}, \frac{\hat{\delta}}{2}) \to \mathbb{R}$ smooth function such that $f(\tau, \bar{x}(\tau)) = 0$ and $\bar{x}(0) = 0$. We have proved that there exists τ_0 such that for all $\tau < \tau_0$ such that for all $\tau < \tau_0$ the minimizer of $x \to E(x, y(x))$ is $\bar{x}(\tau)$. We now compute the first coefficient of the Taylor expansion of the function $\tau \to \bar{x}(\tau)$ in the interval $(-\tau_0, \tau_0)$ at the point $\tau = 0$. Substituting $\bar{x}(\tau) = \bar{x}'(0)\tau + o(\tau)$ in (2.36) we have that

$$
\frac{d}{dx}E(\bar{x}(\tau), y(\bar{x}(\tau))) = 0 \iff o(\tau) + \bar{x}'(0)(2a^2b + 2ab^2)\tau + 4ab\tau - 4a^2\tau = 0,
$$

hence

$$
(2.42) \qquad \bar{x}'(0) = \lim_{\tau \to 0^+} \frac{2(-b + a)}{b(a + b)} + \frac{o(\tau)}{\tau(2a^2b + 2ab^2)} = \frac{2(-b + a)}{b(a + b)}.
$$

We observe that $\bar{x}(\tau)$ is the only minimizer of $x \to E(x, y(x))$ in $[0, \frac{a}{2}]$ and such that there exists $\tau_0 := \tau_0(\Lambda)$ such that for all $\tau < \tau_0$

$$
(2.43) \qquad \bar{x}(\tau) = \tau\frac{2(-b + a)}{b(a + b)} + o(\tau).
$$

By (2.38), (2.41) and applying the implicit function Theorem, we have that for all $\tau \in [0, \tau_0]$

$$
\left| \frac{d}{d\tau}\bar{x}(\tau) \right| = \left| \frac{\frac{\partial f}{\partial \tau}(\tau, \bar{x}(\tau))}{\frac{\partial f}{\partial x}(\tau, \bar{x}(\tau))} \right| \leq | - 16(\bar{x}(\tau))^2 + 16a\bar{x}(\tau) + 4ab - 4a^2| \leq C\Lambda^2.
$$

Moreover, using that $\frac{\partial^2 f}{\partial \tau^2} = 0$, the estimate above together with (2.39), (2.41) and the implicit function Theorem gives

$$
(2.44) \qquad \left| \frac{d^2}{d\tau^2}\bar{x}(\tau) \right| = \left| \frac{-\frac{d}{d\tau}\bar{x}(\tau)\frac{\partial^2 f}{\partial x^2}(\tau, \bar{x}(\tau)) + \frac{\partial^2 f}{\partial \tau^2}(\tau, \bar{x}(\tau))}{\left(\frac{\partial f}{\partial x}(\tau, \bar{x}(\tau)) \right)^2} \right| \leq C\Lambda^2.
$$

Thanks to (2.43) and (2.44) we can write that $\bar{x}(\tau) = \bar{x}'(0)\tau + O(\tau^2)$, with $\bar{x}'(0)$ as in (2.42). Recalling the area constraint $y = \frac{bx}{a-2x}$, we now define the function $y(\tau) := \frac{b\bar{x}(\tau)}{a-2\bar{x}(\tau)}$. We eventually have that the minimizer $R' = \left[-\frac{a'}{2}, \frac{a'}{2} \right] \times \left[-\frac{b'}{2}, \frac{b'}{2} \right]$ satisfies the following properties:

$$
a' = a - 2\bar{x}(\tau),
$$
$$
b' = b + 2y(\tau),
$$
$$
|a' - a| = 2|\bar{x}(\tau)| \leq \tau C(\Lambda) \text{ and } |b' - b| = 2|y(\tau)| \leq \frac{b\tau C(\Lambda)}{|a - 2\bar{x}(\tau)|} \leq \tau C(\Lambda),
$$
$$
|R\Delta R'| = 2b\bar{x}(\tau) + 2(a - 2\bar{x}(\tau))y(\tau) = 4b\bar{x}(\tau) \leq \tau C(\Lambda).
$$

By the minimality of R' we have

$$2a' + 2b' = \mathrm{P}_{|\cdot|_1}(R') \leq \mathrm{P}_{|\cdot|_1}(R') + \frac{1}{\tau}\mathcal{D}(R', R)$$

$$\leq \mathrm{P}_{|\cdot|_1}(R) + \frac{1}{\tau}\mathcal{D}(R, R) = 2a + 2b \leq 2\Lambda,$$

hence $a' + b' \leq a + b \leq \Lambda$. $\qquad\square$

2.2. Flat $\mathrm{P}_{|\cdot|_1}$ area-preserving mean-curvature flow.

Definition 2.9. Let $R \subset \mathbb{R}^2$ be a rectangle with unitary area. By flat $\mathrm{P}_{|\cdot|_1}$ area-preserving mean-curvature flow with initial datum R_0, we mean any family of sets $\{E_t\}_{t\geq 0}$ such that $\chi_{E_t} \in BV(\mathbb{R}^2)$ for all $t \geq 0$ and

$$\lim_{k\to+\infty} \|\chi_{E_t^{(\tau_k)}} - \chi_{E_t}\|_{L^1(\mathbb{R}^2)} = 0$$

locally uniformly in t. In the formula above $\{E_t^{(\tau_k)}\}_{t\geq 0}$ denotes an approximate flat $\mathrm{P}_{|\cdot|_1}$ area-preserving mean-curvature flow with initial datum R and $\{\tau_k\}_{k\in\mathbb{N}}$ is a vanishing sequence.

Theorem 2.10. *Let R be a rectangle with horizontal sidelength a_0 and vertical sidelength b_0 such that $a_0 b_0 = 1$, $b_0 < a_0$ and $a_0 + b_0 < \Lambda$, for some $\Lambda > 0$. For any $0 < \tau < \tau_0$ (here τ_0 is as in Theorem 2.8 for $a = a_0$ and $b = b_0$) let $\{R_t^\tau\}_{t\geq 0}$ be an approximate flat $\mathrm{P}_{|\cdot|_1}$ area-preserving mean-curvature flow with initial datum R. Then, for all $k \geq 1$*

$$R_t^\tau = \left[-\frac{a((k+1)\tau)}{2}, \frac{a((k+1)\tau)}{2}\right] \times \left[-\frac{b((k+1)\tau)}{2}, \frac{b((k+1)\tau)}{2}\right] \quad \forall t \in [(k+1)\tau, (k+2)\tau),$$

where $a(k\tau)$ and $b(k\tau)$ are obtained by recurrence setting $a(0) = a_0$, $b(0) = b_0$ and

$$a((k+1)\tau) = a(k\tau) - 2x(k\tau) \text{ with } x(k\tau) = \frac{2(a(k\tau) - b(k\tau))}{b(k\tau)(a(k\tau) + b(k\tau))}\tau + O(\tau^2)$$

$$b((k+1)\tau) = b(k\tau) + 2y(k\tau) \text{ with } y(k\tau) = \frac{b(k\tau)x(k\tau)}{a(k\tau) - 2x(k\tau)},$$

and share the following properties:

$$a((k+1)\tau)b((k+1)\tau) = 1, \ a((k+1)\tau) + b((k+1)\tau) < a(k\tau) + b(k\tau) < \Lambda,$$

$$|a((k+1)\tau) - a(k\tau)| \leq \tau C(\Lambda), \ |b((k+1)\tau) - b(k\tau)| \leq \tau C(\Lambda).$$

Moreover there exists a unique flat $\mathrm{P}_{|\cdot|_1}$ area-preserving mean-curvature flow $\{R_t\}_{t\geq 0}$ such that, for all $t \geq 0$, $R_t = \left[-\frac{\tilde{a}(t)}{2}, \frac{\tilde{a}(t)}{2}\right] \times \left[-\frac{\tilde{b}(t)}{2}, \frac{\tilde{b}(t)}{2}\right]$ with $\tilde{a}(0) = a_0, \tilde{b}(0) = b_0$. The following properties of the flow hold true:

$$\lim_{\tau\to 0} |R_t^\tau \Delta R_t| = 0, \quad \lim_{\tau\to 0} a^\tau(t) = \tilde{a}(t), \quad \lim_{\tau\to 0} b^\tau(t) = \tilde{b}(t),$$

$$\frac{d}{dt}\tilde{a}(t) = \frac{-4}{\tilde{b}(t)} + \frac{8}{\tilde{a}(t) + \tilde{b}(t)}, \quad \frac{d}{dt}\tilde{b}(t) = \frac{-4}{\tilde{a}(t)} + \frac{8}{\tilde{a}(t) + \tilde{b}(t)}$$

where for all $t \geq 0$ we have defined $a^\tau(t) := a((k+1)\tau)$ for all $t \in [(k+1)\tau, (k+2)\tau)$ and $b^\tau(t) := b((k+1)\tau)$ for all $t \in [(k+1)\tau, (k+2)\tau)$.

Remark 2.11. As a consequence of the theorem above, R_t converge to the unit square exponentially fast. These properties can be checked by setting

$$f(t) := \frac{\tilde{a}(t)}{\tilde{b}(t)} - 1.$$

and observing that, by the theorem above, it holds that

$$\frac{d}{dt}f(t) = \frac{\frac{d}{dt}\tilde{a}(t)}{\tilde{b}(t)} - \frac{\tilde{a}(t)\frac{d}{dt}\tilde{b}(t)}{\tilde{b}(t)^2} = \frac{-8}{\tilde{a}(t)\tilde{b}(t) + \tilde{b}(t)^2}\left[\frac{\tilde{a}(t)}{\tilde{b}(t)} - 1\right] < -4f(t).$$

The exponential convergence follows from $f(t) \le f(0)e^{-4t}$ which is a results of the Gronwall Lemma applied to the previous inequality.

Proof of Theorem 2.10. The first part of the statement regarding the properties of the approximate flat flow is a consequence of Theorem (2.8) applied iteratively to the sets $R = R_t^\tau$. By Theorem 2.8 the functions a^τ and b^τ are piecewise constant and for all $s > t$ $|a^\tau(t) - a^\tau(s)| \le C(\Lambda)|s - t + \tau|$. Therefore we have

$$a^\tau(\cdot) \to \tilde{a}(\cdot) \quad b^\tau(\cdot) \to \tilde{b}(\cdot) \text{ as } \tau \to 0^+ \text{ uniformly on the compact sets of } [0, +\infty).$$

We observe that $\tilde{a}, \tilde{b}$ are Lipschitz functions. Hence for almost every $t \ge 0$ there exist the derivative of $\tilde{a}(\cdot)$ and $\tilde{b}(\cdot)$ and

$$(2.45) \qquad \frac{d}{dt}\tilde{a}(t) = \frac{-4}{\tilde{b}(t)} + \frac{8}{\tilde{a}(t) + \tilde{b}(t)}, \quad \frac{d}{dt}\tilde{b}(t) = \frac{-4}{\tilde{a}(t)} + \frac{8}{\tilde{a}(t) + \tilde{b}(t)}.$$

Since the right hand sides of both formulas in (2.45) are Lipschitz continuous functions we have that (2.45) holds true for all $t \ge 0$. $\qquad\square$

3. THE DISCRETE SETTING

For all $\varepsilon > 0$ we consider the square lattice of size ε, namely $\varepsilon\mathbb{Z}^2$. We define its dual lattice as $(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) + \varepsilon\mathbb{Z}^2$. For all $\mathcal{J} \subset \varepsilon\mathbb{Z}^2$ and for all $p, q \in \mathbb{Z}$ we define the sets

$$(3.1) \qquad\qquad \mathcal{J}(\cdot, q) := \{\varepsilon(j_1, q) : \varepsilon(j_1, q) \in \mathcal{J}\}$$

and

$$(3.2) \qquad\qquad \mathcal{J}(p, \cdot) := \{\varepsilon(p, j_2) : \varepsilon(p, j_2) \in \mathcal{J}\}.$$

The set $\mathcal{J}(\cdot, q)$ (respectively $\mathcal{J}(p, \cdot)$) is the subset of $\mathcal{J}$ whose elements have second entry taking the value εq (respectively the first entry taking the value εp). In order to pass from a discrete to a continuous formulation of our problem it is convenient to identify sets $\mathcal{J} \subset \varepsilon\mathbb{Z}^2$ with subsets of $\mathbb{R}^2$. To this end to all $\mathcal{J} \subset \varepsilon\mathbb{Z}^2$ we associate the set

$$E_{\mathcal{J}} = \bigcup_{j \in \mathcal{J}} Q_\varepsilon(j) \subset \mathbb{R}^2$$

where $Q_\varepsilon(j) := j + \varepsilon Q$ and $Q = [-\frac{1}{2}, \frac{1}{2}]^2$. For all $\varepsilon \in (0, 1)$ we define the set

$$(3.3) \qquad\qquad \mathcal{C}_\varepsilon := \{E \subset \mathbb{R}^2 : E = E_{\mathcal{J}} \text{ for some } \mathcal{J} \subset \varepsilon\mathbb{Z}^2\}.$$

In what follows if $E \in \mathcal{C}_\varepsilon$, we write $E = E_{\mathcal{J}}$ to refer to the set $E_{\mathcal{J}}$ in (3.3). For all $\mathcal{J} \subset \varepsilon\mathbb{Z}^2$, we define the discrete l^∞-distance to $\partial\mathcal{J}$ as

$$d_\infty^\varepsilon(i, \partial\mathcal{J}) := \text{dist}(i, \mathcal{J}) + \text{dist}(i, \varepsilon\mathbb{Z}^2 \setminus \mathcal{J}) \quad \text{for all } i \in \varepsilon\mathbb{Z}^2,$$

where

$$\text{dist}(i, \mathcal{J}) := \inf\{\|i - j\|_\infty : j \in \mathcal{J}\} \quad \text{for all } i \in \varepsilon\mathbb{Z}^2.$$

We observe that, if d_∞ denotes the usual l^∞-distance in $\mathbb{R}^2$, then

$$d_\infty^\varepsilon(i, \partial\mathcal{J}) = d_\infty(i, \partial E_{\mathcal{J}}) + \frac{\varepsilon}{2}.$$

The distance defined above can be extended to all $\mathbb{R}^2 \setminus \partial E_{\mathcal{J}}$ by setting

$$d_\infty^\varepsilon(x, \partial\mathcal{J}) := d_\infty^\varepsilon(i, \partial\mathcal{J}) \quad \text{if } x \in Q_\varepsilon(i).$$

The following two polygons will play an important role in what follows.

Definition 3.1 (Quasi-rectangle)**.** We say that a set $E = E_{\mathcal{J}} \in \mathcal{C}_\varepsilon$ is a quasi-rectangle if it satisfies the following properties. The cardinality of $\mathcal{J}$ is $\#\mathcal{J} = nm + r$ for some $n, m, r \in \mathbb{N}$ with $r < n = \max\{n, m\}$ and E can be written as $E = R \cup Q$ with R a rectangle of horizontal sidelength εn and vertical sidelength εm and Q a rectangle with horizontal sidelength εr and vertical sidelength ε and such that for every $j \in Q$ there exists $\hat{j} \in R$ such that $|j - \hat{j}| = \varepsilon$. The set of all quasi rectangles $E \in \mathcal{C}_\varepsilon$ is denoted by $\mathcal{QR}_\varepsilon$.

In what follows we will be using the notation QR for the quasi-rectangle $R \cup Q$.

Definition 3.2 (Pseudo-axial quasi-rectangle)**.** We say that a quasi-rectangle $QR = R \cup Q \in \mathcal{QR}_\varepsilon$ is pseudo axial symmetric if

$$\mathrm{Bar}(R) = \begin{cases} (0, 0) & \text{if } n, m \text{ are odd,} \\ \left(\frac{\varepsilon}{2}, 0\right) & \text{if } n \text{ is even and } m \text{ is odd,} \\ \left(0, \frac{\varepsilon}{2}\right) & \text{if } n \text{ is odd and } m \text{ is even,} \\ \left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) & \text{if } n, m \text{ are even,} \end{cases}$$

and

$$Q = \begin{cases} \left[-\varepsilon\frac{r}{2}, \, \varepsilon\frac{r}{2}\right] \times [L, \, L + \varepsilon] & \text{if } r \text{ is odd,} \\ \left[-\varepsilon\frac{r}{2} + \frac{\varepsilon}{2}, \, \varepsilon\frac{r}{2} + \frac{\varepsilon}{2}\right] \times [L, \, L + \varepsilon] & \text{if } r \text{ is even} \end{cases}$$

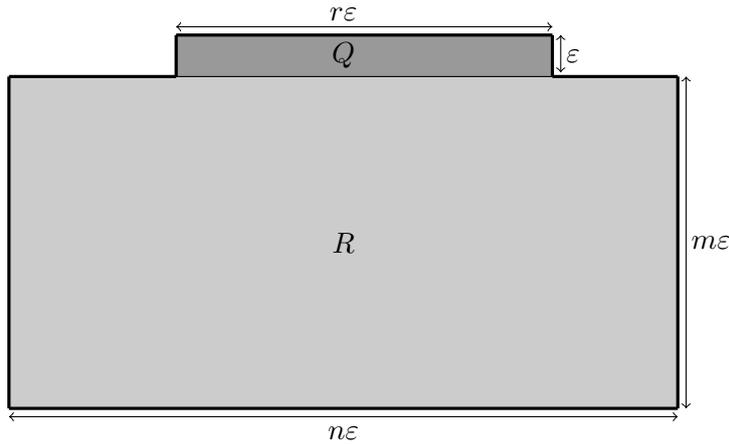where $L := \mathcal{H}^1(\Pi_2(R) \cap (\{0\} \times [0, +\infty)))$.

Figure 3. The quasi-rectangle $QR = R \cup Q \in \mathcal{QR}_\varepsilon$

Definition 3.3 (Rhombus-like Shape)**.** We say that a set $E = E_{\mathcal{J}} \in \mathcal{C}_\varepsilon$ has a rhombus-like shape with respect to $(0, 0) \in \mathbb{R}^2$ if two functions

$$i \in \mathbb{Z} \longrightarrow \#\mathcal{J}(i, \cdot) \text{ and } i \in \mathbb{Z} \longrightarrow \#\mathcal{J}(\cdot, i)$$

are increasing for $i \leq 0$ and non-increasing for $i \geq 0$.

We now define the connectedness by rows or by columns.

Definition 3.4. We say that a set $E = E_{\mathcal{J}} \in \mathcal{C}_\varepsilon$ is connected with respect to rows if for all $j \in \mathbb{Z}$ we have that

$$\varepsilon(i, j) \in \mathcal{J} \quad \forall (i, j) \in \mathbb{Z}^2 \quad \text{such that} \quad i \in \{i_{min}(j), \ldots, i_{max}(j)\}$$

where

$$i_{min}(j) := \min\{i \in \mathbb{Z} \colon \varepsilon(i, j) \in \mathcal{J}\} \text{ and } i_{max}(j) := \max\{i \in \mathbb{Z} \colon \varepsilon(i, j) \in \mathcal{J}\}.$$

The connectedness with respect to columns is defined analogously.

For all $\mathcal{J} \subset \varepsilon\mathbb{Z}^2$ we define the discrete perimeter as

$$\mathrm{P}^\varepsilon(\mathcal{J}) = \varepsilon \, \# \left\{ (i,j) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : \; \varepsilon i \in \mathcal{J}, \; \varepsilon j \in \varepsilon\mathbb{Z}^2 \setminus \mathcal{J}, \; |i-j| = 1 \right\}.$$

We remark that for all $\mathcal{J} \subset \varepsilon\mathbb{Z}^2$

$$\mathrm{P}^\varepsilon(\mathcal{J}) = \mathrm{P}(E_\mathcal{J}) = \mathcal{H}^1(\partial E_\mathcal{J}).$$

In what follows, for $E \in \mathcal{C}_\varepsilon$ we will also drop the dependence on $\mathcal{J}$ and simply write $\mathrm{P}(E) = \mathrm{P}(E_\mathcal{J})$. In the following we also make use of the next two definitions of perimeter of the horizontal and vertical projection of a set $E = E_\mathcal{J}$, namely
(3.4)
$$\mathrm{P}_1(E) := \mathcal{H}^1(\Pi_1(E)) = \max_{q \in \mathbb{Z}} \varepsilon \, \#\mathcal{J}(\cdot, q) \text{ and } \mathrm{P}_2(E) := \mathcal{H}^1(\Pi_2(E)) = \max_{p \in \mathbb{Z}} \varepsilon \, \#\mathcal{J}(p, \cdot)$$

and observe that

(3.5)
$$\mathrm{P}(E) \geq 2\mathrm{P}_1(E) + 2\mathrm{P}_2(E).$$

For every $E, F \in \mathcal{C}_\varepsilon$ we introduce the ε-dissipation of E and F as

(3.6)
$$D_\varepsilon(F, E) := \int_{F \Delta E} d_\infty^\varepsilon(x, \partial E) \, dx.$$

3.1. Discrete Steiner-like rearrangement of a set with respect to a quasi rectangle.
This section is devoted to the definition of a geometric rearrangement of a set with respect to a quasi-rectangle with pseudo axial symmetry. The procedure will turn the set into a rhombus-like shape with the same area and smaller perimeter. The precise construction is given in the proof of the next Proposition 3.5.

Proposition 3.5. *Let $QR \in \mathcal{QR}_\varepsilon$ have pseudo axial symmetry and let us consider $E = E_\mathcal{J} \in \mathcal{C}_\varepsilon$. The discrete Steiner-like rearrangement $\mathcal{R}(E) \in \mathcal{C}_\varepsilon$ of the set E with respect to QR has a rhombus-like shape and is such that*

$$|E| = |\mathcal{R}(E)|, \quad \mathrm{P}(E) \geq \mathrm{P}(\mathcal{R}(E)).$$

Proof. Let $QR = R \cup Q$ be a quasi-rectangle with pseudo axial symmetry. Let $n, m, r \in \mathbb{N}$ be such that εn and εm are the horizontal and vertical sidelengths of R and εr and ε are the horizontal and vertical sidelengths of Q. We define the integer numbers $\underline{p}, \overline{p}, \overline{q}$ as

$$\underline{p} := \min \left\{ p \colon \varepsilon(p, q) \in Q \right\}, \, \overline{p} := \max \left\{ p \colon \varepsilon(p, q) \in Q \right\}, \, \overline{q} := \max \left\{ q \colon \varepsilon(p, q) \in QR \right\}.$$

We divide the proof in two steps.
Step 1)
In this step we construct the set E' obtained via a symmetrization procedure of the rows of the set E. Let $\mathcal{J}(p, \cdot)$, $\mathcal{J}(\cdot, q)$ be the sets defined in (3.2) and (3.1). For all $q \in \mathbb{Z}$ we define the symmetrized set of $\mathcal{J}(\cdot, q)$ with respect to the vertical axis as

$$\mathcal{J}'(q) := \left\{ \varepsilon(j, q) \colon j \in \left\{ -\frac{\#\mathcal{J}(\cdot, q) - 1}{2}, \dots, \frac{\#\mathcal{J}(\cdot, q) - 1}{2} \right\} \right\}$$

if $\#\mathcal{J}(\cdot, q)$ is odd, and we define

$$\mathcal{J}'(q) := \left\{ \varepsilon(j, q) \colon j \in \left\{ -\frac{\#\mathcal{J}(\cdot, q)}{2} + 1, \dots, \frac{\#\mathcal{J}(\cdot, q)}{2} \right\} \right\}$$

if $\#\mathcal{J}(\cdot, q)$ is even.
We define the subsets $\mathcal{J}' \subset \varepsilon\mathbb{Z}^2$ and $E' \subset \mathbb{R}^2$ as

(3.7)
$$\mathcal{J}' := \bigcup_{q \in \mathbb{Z}} \mathcal{J}'(q) \quad \text{and} \quad E' := E_{\mathcal{J}'}.$$

We notice that the set E' can be defined by recurrence as

(3.8)
$$\begin{cases} E'_{q+1} := \left(E'_q \setminus E_{\mathcal{J}(\cdot,\,q)} \right) \cup E_{\mathcal{J}'(q)} & \text{if } q \geq q_{min}, \\ E'_{q_{min}} = E & \text{if } q = q_{min} \end{cases}$$

where

(3.9)
$$q_{min} := \inf \left\{ q \in \mathbb{Z}: \ \mathcal{J}(\cdot,\,q) \neq \emptyset \right\}.$$

We now observe that the function

$$i \in \mathbb{Z} \longrightarrow \#\mathcal{J}'(i,\,\cdot)$$

is increasing for $i \leq 0$ and non-increasing for $i \geq 0$. Indeed, by construction, the set $\mathcal{J}'$ is connected with respect to rows, hence for all $j,\,q \in \mathbb{N}$

$$\text{if} \quad \varepsilon(j+1,\,q) \in \mathcal{J}' \quad \text{then} \quad \varepsilon(j,\,q) \in \mathcal{J}'$$

which proves the monotonicity of the function for positive i. An analogous argument shows the monotonicity for i negative.
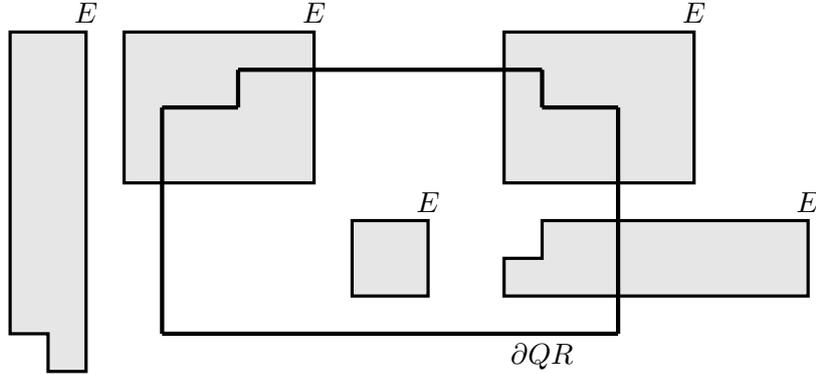

Figure 4. The quasi-rectangle QR and the set E

Step 2)
In this step we construct the set $\mathcal{R}(E)$ obtained from the set E' via a symmetrization of its columns. Also this operation depends on the parity of m. In what follows we keep denoting by $p \in \mathbb{Z}$ the column index. We first consider the case $m \in 2\mathbb{N}$. If $p \in \{\underline{p}, \cdots, \overline{p}\}$ we define the symmetrized column according to the following two cases. If $\#\mathcal{J}'(p,\,\cdot) \in 2\mathbb{N}+1$, then we define

$$\mathcal{J}''(p) := \left\{ \varepsilon(p,\,j): \ j \in \left\{ 1 - \frac{\#\mathcal{J}'(p,\,\cdot)-1}{2}, \ldots, 1 + \frac{\#\mathcal{J}'(p,\,\cdot)-1}{2} \right\} \right\}.$$

If instead $\#\mathcal{J}'(p,\,\cdot) \in 2\mathbb{N}$, then we define

$$\mathcal{J}''(p) := \left\{ \varepsilon(p,\,j): \ j \in \left\{ -\frac{\#\mathcal{J}'(p,\,\cdot)}{2} + 1, \ldots, \frac{\#\mathcal{J}'(p,\,\cdot)}{2} \right\} \right\}.$$

In the case $p \in \mathbb{N} \setminus \{\underline{p}, \cdots, \overline{p}\}$ we define the symmetrized column according to the following two cases. If $\#\mathcal{J}'(p,\,\cdot) \in 2\mathbb{N}+1$, then we define

$$\mathcal{J}''(p) := \left\{ \varepsilon(p,\,j): \ j \in \left\{ 1 - \frac{\#\mathcal{J}'(p,\,\cdot)-1}{2}, \ldots, 1 + \frac{\#\mathcal{J}'(p,\,\cdot)-1}{2} \right\} \right\}.$$

If instead $\#\mathcal{J}'(p,\cdot) \in 2\mathbb{N}$, then we define

$$\mathcal{J}''(p) := \left\{ \varepsilon(p,\,j)\colon\, j \in \left\{ -\frac{\#\mathcal{J}'(p,\,\cdot)}{2} + 1, \ldots, \frac{\#\mathcal{J}'(p,\,\cdot)}{2} \right\} \right\}.$$

Now we consider the case $m \in 2\mathbb{N}+1$. If $p \in \{\underline{p}, \cdots, \overline{p}\}$ we have two possible cases. If $\#\mathcal{J}'(p,\cdot) \in 2\mathbb{N}+1$, then we define

$$\mathcal{J}''(p) := \left\{ \varepsilon(p,\,j)\colon\, j \in \left\{ -\frac{\#\mathcal{J}'(p,\,\cdot)-1}{2}, \ldots, \frac{\#\mathcal{J}'(p,\,\cdot)-1}{2} \right\} \right\}.$$

If instead $\#\mathcal{J}'(p,\cdot) \in 2\mathbb{N}$, then we define

$$\mathcal{J}''(p) := \left\{ \varepsilon(p,\,j)\colon\, j \in \left\{ -\frac{\#\mathcal{J}'(p,\,\cdot)}{2} + 1, \ldots, \frac{\#\mathcal{J}'(p,\,\cdot)}{2} \right\} \right\}.$$

In the alternative case that $p \in \mathbb{N} \setminus \{\underline{p}, \cdots, \overline{p}\}$ we define the new columns as follows. If
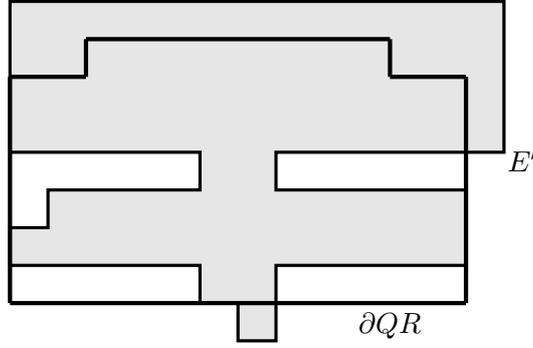

Figure 5. The quasi-rectangle QR and the set E'

$\#\mathcal{J}'(p,\cdot) \in 2\mathbb{N}+1$ then

$$\mathcal{J}''(p) := \left\{ \varepsilon(p,\,j)\colon\, j \in \left\{ -\frac{\#\mathcal{J}'(p,\,\cdot)-1}{2}, \ldots, \frac{\#\mathcal{J}'(p,\,\cdot)-1}{2} \right\} \right\}.$$

If instead $\#\mathcal{J}'(p,\cdot) \in 2\mathbb{N}$, then

$$\mathcal{J}''(p) := \left\{ \varepsilon(p,\,j)\colon\, j \in \left\{ -\frac{\#\mathcal{J}'(p,\,\cdot)}{2} + 1, \ldots, \frac{\#\mathcal{J}'(p,\,\cdot)}{2} \right\} \right\}.$$

We define the subsets $\mathcal{J}'' \subset \varepsilon\mathbb{Z}^2$ and $\mathcal{R}(E) \subset \mathbb{R}^2$ as

$$\mathcal{J}'' := \bigcup_{p\in\mathbb{Z}} \mathcal{J}''(p) \quad \text{and} \quad \mathcal{R}(E) := E_{\mathcal{J}''}.$$

The set $\mathcal{R}(E)$ can be obtained by the following recurrence procedure starting from the initial datum E' in full analogy with formula (3.8):

$$(3.10) \qquad \begin{cases} E''_{p+1} := \left(E''_p \setminus E'_{\mathcal{J}(p,\cdot)} \right) \cup E_{\mathcal{J}''(p)} & \text{if } p \geq p_{min}, \\ E''_{p_{min}} = E' & \text{if } p = p_{min} \end{cases}$$

where

$$(3.11) \qquad p_{min} := \inf \left\{ p \in \mathbb{Z}\colon\, E'_{\mathcal{J}(p,\cdot)} \neq \emptyset \right\}.$$

From the construction of the set $\mathcal{R}(E)$ we have that $|E| = |E^{'}| = |\mathcal{R}(E)|$. We observe that the set $\mathcal{J}^{''}$ is such that the two functions

$$i \in \mathbb{Z} \longrightarrow \#\mathcal{J}^{''}(i, \cdot) \quad i \in \mathbb{Z} \longrightarrow \#\mathcal{J}^{''}(\cdot, i)$$

are increasing for $i \leq 0$ and non-increasing for $i \geq 0$. Note that such a property follows by the same argument detailed above for $\#\mathcal{J}^{'}(i, \cdot)$. According to the definition 3.3 we have that the set $\mathcal{R}(E)$ has a rhombus-like shape and that

$$P(\mathcal{R}(E)) = 2 \max_{q \in \mathbb{Z}} \varepsilon \# \mathcal{J}(\cdot, q) + 2 \max_{p \in \mathbb{Z}} \varepsilon \# \mathcal{J}(p, \cdot) = 2P_1(E) + 2P_2(E).$$

Therefore we observe that

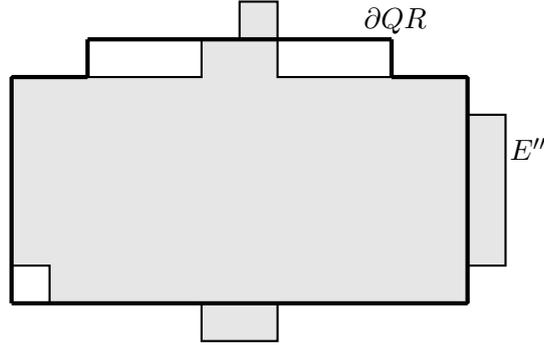$$P(E) \geq 2P_1(E) + 2P_2(E) = P(\mathcal{R}(E)).$$

Figure 6. The quasi-rectangle QR and the set E''

$\square$

Proposition 3.6. *Let $QR \in \mathcal{QR}_\varepsilon$ have pseudo axial symmetry and let us consider $E \in \mathcal{C}_\varepsilon$. Let $\mathcal{R}(E) \in \mathcal{C}_\varepsilon$ be the discrete Steiner-like rearrangement of the set E with respect to QR. Then*

$$D_\varepsilon(E, QR) \geq D_\varepsilon(\mathcal{R}(E), QR).$$

Proof. By the definition of quasi rectangle 3.1, we write $QR = R \cup Q$, with R a rectangle having sidelengths $\varepsilon n, \varepsilon m$ and Q a rectangle with sidelengths $\varepsilon r, \varepsilon$ where $n, m, r \in \mathbb{N}$. By the very definition of quasi-rectangle with pseudo axial symmetry we have that

$$\text{Bar}(R) \in \left\{ (0, 0), \left(\frac{\varepsilon}{2}, 0 \right), \left(0, \frac{\varepsilon}{2} \right), \left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2} \right) \right\}$$

and that

$$Q = \begin{cases} \left[-\frac{\varepsilon}{2}(r-1), \frac{\varepsilon}{2}(r-1) \right] \times [L, L+\varepsilon] & \text{if } r \in 2\mathbb{N}+1, \\ \left[-\frac{\varepsilon}{2}(r+1), \frac{\varepsilon}{2}(r+1) \right] \times [L, L+\varepsilon] & \text{if } r \in 2\mathbb{N}. \end{cases}$$

We define the followings lengths

$L_1(+) := \mathcal{H}^1(\Pi_1(R) \cap ([0, +\infty) \times \{0\})), \quad L_1(-) := \mathcal{H}^1(\Pi_1(R) \cap ((-\infty, 0] \times \{0\})),$

$L_2(+) := \mathcal{H}^1(\Pi_2(R) \cap (\{0\} \times [0, +\infty))), \quad L_2(-) := \mathcal{H}^1(\Pi_2(R) \cap (\{0\} \times (-\infty, 0])),$

$l(+) := \mathcal{H}^1(\Pi_1(Q) \cap ([0, +\infty) \times \{0\})), \quad\quad l(-) := \mathcal{H}^1(\Pi_1(Q) \cap ((-\infty, 0] \times \{0\})),$

$L_1 := L_1(+) + L_1(-) = n\varepsilon,$

$L_2 := L_2(+) + L_2(-) = m\varepsilon.$

By Proposition 3.5 there exists $\mathcal{R}(E)$ such that

$$|E| = |\mathcal{R}(E)|, \quad P(E) \geq P(\mathcal{R}(E)).$$

We recall that the set $\mathcal{R}(E)$ has been obtained by a two steps symmetrization procedure: in the first step we have symmetrized the rows while in the second one we have symmetrized the columns. The proof of the claim is accordingly subdivided in two steps in which we prove that the symmetrization of the rows (respectively the columns) decreases the dissipation. In what follows all the notation are the same as those used in Proposition 3.5.
Step 1)
We start by showing that the dissipation decreases under symmetrization by rows, i.e.,

$$D_\varepsilon(E_q', QR) \geq D_\varepsilon(E_{q+1}', QR) \quad \forall q \geq q_{min}$$

where E', E_q' and q_{min} are defined in (3.7), (3.8) and (3.9), respectively. We start considering the case $q \neq \frac{1}{\varepsilon}\left(L_2(+) + \frac{\varepsilon}{2}\right)$. We need to distinguish two cases, namely $\varepsilon\#\mathcal{J}(\cdot, q) < L_1$ and $\varepsilon\#\mathcal{J}(\cdot, q) \geq L_1$. In the first case if

$$\mathcal{J}(\cdot, q) \cap QR^c \neq \emptyset$$

one can "move" the points of this set to the points of the set

$$\mathcal{J}(\cdot, q)^c \cap ([-L_1(-), L_1(+)] \times \{q\})$$

which is a subset of QR. This procedure decreases the dissipation since the latter is obtained integrating a distance on a subset of $E\Delta QR$. If instead

$$\mathcal{J}(\cdot, q) \cap QR^c = \emptyset,$$

we procede differently. We introduce the sets

$$\mathcal{A} := \left\{\varepsilon(j, q) \colon \varepsilon(j, q) \in \mathcal{J}'(q) \setminus \mathcal{J}(\cdot, q)\right\} \text{ and } \mathcal{B} := \left\{\varepsilon(j, q) \colon \varepsilon(j, q) \in \mathcal{J}'(q)^c \cap \mathcal{J}(\cdot, q)\right\}$$

and define the function $f : \mathcal{A} \to \mathcal{B}$ as follows. If $i = i_{min} := \min\{j \geq 0 : \varepsilon(j, q) \in \mathcal{A}\}$

$$f(\varepsilon(i, q)) \in \arg\min\{|\varepsilon j| : \varepsilon(j, q) \in \mathcal{B}\}$$

with the choice that $f(\varepsilon(i, q)) = \varepsilon(\hat{j}, q)$ in the case that the $\arg\min$ is not unique, namely

$$\arg\min\{|\varepsilon j| : \varepsilon(j, q) \in \mathcal{B}\} = \left\{\varepsilon(-\hat{j}, q), \varepsilon(\hat{j}, q)\right\}.$$

If $i > i_{min}$ we define

$$f(\varepsilon(i, q)) \in \arg\min\{|\varepsilon j| : \varepsilon(j, q) \in \mathcal{B} \setminus \{f(\varepsilon(i_{min}, q)), \ldots, f(\varepsilon(i-1, q))\}\}$$

with the choice $f(\varepsilon(i, q)) = \varepsilon(\hat{j}, q)$ in the case that the $\arg\min$ is not unique, namely

$$\arg\min\{|\varepsilon j| : \varepsilon(j, q) \in \mathcal{B} \setminus \{f(\varepsilon(i_{min}, q)), \ldots, f(\varepsilon(i-1, q))\}\} = \left\{\varepsilon(-\hat{j}, q), \varepsilon(\hat{j}, q)\right\}.$$

If $i < i_{min}$ we define

$$f(\varepsilon(i, q)) \in \arg\min\left\{|\varepsilon j| : \varepsilon(j, q) \in \mathcal{B} \setminus \{f(\varepsilon(\tilde{i}, q)) \colon \tilde{i} \geq i+1\}\right\}$$

with the choice $f(\varepsilon(i, q)) = \varepsilon(\hat{j}, q)$ in the case that the $\arg\min$ is not unique, namely

$$\arg\min\left\{|\varepsilon j| : \varepsilon(j, q) \in \mathcal{B} \setminus \{f(\varepsilon(\tilde{i}, q)) \colon \tilde{i} \geq i+1\}\right\} = \left\{\varepsilon(-\hat{j}, q), \varepsilon(\hat{j}, q)\right\}.$$

For all $\varepsilon(i, q) \in \mathcal{A}$ and for all $(x, y) \in Q_\varepsilon(\varepsilon(i, q))$ we uniquely define $x' := x + \Pi_1(f(\varepsilon(i, q)) - \varepsilon(i, q))$, and observe that $(x', y) \in Q_\varepsilon(f(\varepsilon(i, q)))$. Without loss of generality we consider only the case $\varepsilon(i, q) \in \mathcal{A}$ such that $q \geq 0$, $i \geq 0$ and $\Pi_1(f(\varepsilon(i, q))) > 0$, the argument in the other cases being fully analogous. For all $(x, y) \in Q_\varepsilon(\varepsilon(i, q))$ with $x \geq 0, y \geq 0$ it holds that
(3.12)

$$d_\infty^\varepsilon((x, y), \partial QR) = \min\{d_\infty^\varepsilon((x, y), \partial QR(\pm)), L_1(+) - \varepsilon i\} \geq$$

$$\min\{d_\infty^\varepsilon((x, y), \partial QR(\pm)), L_1(+) - \varepsilon i - \Pi_1(f(\varepsilon(i, q)) - \varepsilon(i, q))\} = d_\infty^\varepsilon((x', y), \partial QR)$$

where

$$\partial QR(\pm) := \partial QR \cap ((\mathbb{R} \times [L_2(+), +\infty)) \cup (\mathbb{R} \times \{L_2(-)\})).$$

Then, by the formula (3.12) we obtain

$$(3.13) \qquad \int_{Q_\varepsilon(i,\,q)} d^\varepsilon_\infty((x,\,y),\,\partial QR)\,dx\,dy \geq \int_{Q_\varepsilon(f(\varepsilon(i,\,q)))} d^\varepsilon_\infty((x',\,y),\,\partial QR)\,dx'\,dy.$$

Moreover, for all $q \in \mathbb{Z}$, we set $S(q) := [L_1(-), L_1(+)] \times \varepsilon\left[q - \frac{1}{2},\,q + \frac{1}{2}\right]$ and, referring to (3.8), we observe that

$$
\begin{aligned}
E'_q \Delta QR \cap S(q) &= \bigcup_{\varepsilon(i,\,q) \in S(q) \cap \mathcal{A}^c \setminus \mathcal{J}(\cdot,\,q)} Q_\varepsilon(\varepsilon(i,\,q)) \cup \bigcup_{\varepsilon(i,\,q) \in \mathcal{A}} Q_\varepsilon(\varepsilon(i,\,q)), \\
(3.14) \\
E'_{q+1} \Delta QR \cap S(q) &= \bigcup_{\varepsilon(i,\,q) \in S(q) \cap \mathcal{A}^c \setminus \mathcal{J}(\cdot,\,q)} Q_\varepsilon(\varepsilon(i,\,q)) \cup \bigcup_{\varepsilon(i,\,q) \in \mathcal{A}} Q_\varepsilon(f(\varepsilon(i,\,q))).
\end{aligned}
$$

As a result, thanks to (3.13) and (3.14), we have that

$$
\begin{aligned}
D_\varepsilon(E'_q,\,QR) &= \int_{E'_q \Delta QR} d^\varepsilon_\infty(z,\,\partial QR)\,dz \\
&= \int_{E'_q \Delta QR \cap S(q)^c} d^\varepsilon_\infty(z,\,\partial QR)\,dz + \int_{E'_q \Delta QR \cap S(q)} d^\varepsilon_\infty(z,\,\partial QR)\,dz \\
&= \int_{E'_q \Delta QR \cap S(q)^c} d^\varepsilon_\infty(z,\,\partial QR)\,dz + \int_{\bigcup_{\varepsilon(i,\,q) \in S(q) \cap \mathcal{A}^c \setminus \mathcal{J}(\cdot,\,q)} Q_\varepsilon(\varepsilon(i,\,q))} d^\varepsilon_\infty(z,\,\partial QR)\,dz \\
&\quad + \int_{\bigcup_{\varepsilon(i,\,q) \in \mathcal{A}} Q_\varepsilon(\varepsilon(i,\,q))} d^\varepsilon_\infty(z,\,\partial QR)\,dz \\
&\geq \int_{E'_q \Delta QR \cap S(q)^c} d^\varepsilon_\infty(z,\,\partial QR)\,dz + \int_{\bigcup_{\varepsilon(i,\,q) \in S(q) \cap \mathcal{A}^c \setminus \mathcal{J}(\cdot,\,q)} Q_\varepsilon(\varepsilon(i,\,q))} d^\varepsilon_\infty(z,\,\partial QR)\,dz \\
&\quad + \int_{\bigcup_{\varepsilon(i,\,q) \in \mathcal{A}} Q_\varepsilon(f(\varepsilon(i,\,q)))} d^\varepsilon_\infty(z,\,\partial QR)\,dz \\
&= \int_{E'_{q+1} \Delta QR} d^\varepsilon_\infty(z,\,\partial QR)\,dz = D_\varepsilon(E'_{q+1},\,QR).
\end{aligned}
$$

Hence we have proved that

$$\text{if} \quad \varepsilon \# \mathcal{J}(\cdot,\,q) < L_1 \quad \text{then} \quad D_\varepsilon(E'_q,\,QR) \geq D_\varepsilon(E'_{q+1},\,QR).$$

We now show the proof in the second case, namely for $\varepsilon \# \mathcal{J}(\cdot,\,q) \geq L_1$. We start by observing that the dissipation $D_\varepsilon(\cdot, QR)$ decreases if we "move" points of the set $QR^c \cap \mathcal{J}(\cdot,\,q)$ to the set

$$QR \cap \mathcal{J}(\cdot,\,q)^c \cap ([-L_1(-),\,L_1(+)] \times \{q\}).$$

Such a procedure stops if

$$QR \cap \mathcal{J}(\cdot,\,q)^c \cap ([-L_1(-),\,L_1(+)] \times \{q\}) = \emptyset.$$

In the latter case we need to proceed differently. We introduce the sets

$$\tilde{\mathcal{C}} := \mathcal{J}(\cdot,\,q) \cap \mathcal{J}'(q) \cap QR^c, \quad \tilde{\mathcal{A}} := \mathcal{J}(\cdot,\,q) \cap QR^c \setminus \tilde{\mathcal{C}}, \quad \tilde{\mathcal{B}} = \mathcal{J}'(q) \cap QR^c \setminus \tilde{\mathcal{C}}$$

and define the function $\tilde{f} : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$ as follows. If $i = i_{min} := \min\{j \geq 0 : \varepsilon(j,\,q) \in \tilde{\mathcal{A}}\}$

$$f(\varepsilon(i,\,q)) \in \arg\min\left\{|\varepsilon j| : \varepsilon(j,\,q) \in \tilde{\mathcal{B}}\right\}$$

with the choice that $\tilde{f}(\varepsilon(i,\,q)) = \varepsilon(\hat{j},\,q)$ in the case

$$\arg\min\left\{|\varepsilon j| : \varepsilon(j,\,q) \in \mathcal{B}\right\} = \left\{\varepsilon(-\hat{j},\,q),\,\varepsilon(\hat{j},\,q)\right\}.$$

If $i > i_{min}$

$$\tilde{f}(\varepsilon(i,\,q)) \in \arg\min\left\{|\varepsilon j| : \varepsilon(j,\,q) \in \tilde{\mathcal{B}} \setminus \left\{\tilde{f}(\varepsilon(i_{min},\,q)),\,\ldots,\,\tilde{f}(\varepsilon(i-1,\,q))\right\}\right\}$$

with the choice $\tilde{f}(\varepsilon(i,\,q)) = \varepsilon(\hat{j},\,q)$ in the case

$$\arg\min\left\{|\varepsilon j| : \varepsilon(j,\,q) \in \tilde{\mathcal{B}} \setminus \left\{\tilde{f}(\varepsilon(i_{min},\,q)),\,\ldots,\,\tilde{f}(\varepsilon(i-1,\,q))\right\}\right\} = \left\{\varepsilon(-\hat{j},\,q),\,\varepsilon(\hat{j},\,q)\right\}.$$

If $i < i_{min}$

$$\tilde{f}(\varepsilon(i,\,q)) \in \arg\min\left\{|\varepsilon j| : \varepsilon(j,\,q) \in \mathcal{B} \setminus \left\{\tilde{f}(\varepsilon(\tilde{i},\,q)) : \tilde{i} \geq i+1\right\}\right\}$$

with the choice $\tilde{f}(\varepsilon(i,\,q)) = \varepsilon(\hat{j},\,q)$ in the case

$$\arg\min\left\{|\varepsilon j| : \varepsilon(j,\,q) \in \tilde{\mathcal{B}} \setminus \left\{\tilde{f}(\varepsilon(\tilde{i},\,q)) : \tilde{i} \geq i+1\right\}\right\} = \left\{\varepsilon(-\hat{j},\,q),\,\varepsilon(\hat{j},\,q)\right\}.$$

We observe that, as long as $\tilde{f}(\varepsilon(i,\,q)) \in \tilde{\mathcal{B}}$ then

$$(3.15) \qquad d_\infty^\varepsilon(\varepsilon(i,\,q),\,\partial QR) \geq d_\infty^\varepsilon(f(\varepsilon(i,\,q)),\,\partial QR).$$

For all $\varepsilon(i,\,q) \in \tilde{\mathcal{A}}$ and for all $(x,\,y) \in Q_\varepsilon(\varepsilon(i,\,q))$ we uniquely define $x' := x + \Pi_1(\tilde{f}(\varepsilon(i,\,q)) - \varepsilon(i,\,q))$, and observe that $(x',\,y) \in Q_\varepsilon(\tilde{f}(\varepsilon(i,\,q)))$ and by (3.15) we have that for all $(x,\,y) \in Q_\varepsilon(\varepsilon(i,\,q))$

(3.16)
$$d_\infty^\varepsilon((x,\,y),\,\partial QR) \geq d_\infty^\varepsilon((x + \Pi_1(\tilde{f}(\varepsilon(i,\,q)) - \varepsilon(i,\,q)),\,y),\,\partial QR) = d_\infty^\varepsilon((x',\,y),\,\partial QR).$$

Moreover we have

$$
\begin{aligned}
E_q' \Delta QR &= \left(\left(E_q' \Delta QR\right) \cap S(q)^c\right) \cup \bigcup_{\varepsilon(i,q) \in \tilde{\mathcal{A}}} Q(\varepsilon(i,\,q)) \cup \bigcup_{\varepsilon(i,q) \in \tilde{\mathcal{C}}} Q(\varepsilon(i,\,q)), \\
(3.17) \\
E_{q+1}' \Delta QR &= \left(\left(E_q' \Delta QR\right) \cap S(q)^c\right) \cup \bigcup_{\varepsilon(i,q) \in \tilde{\mathcal{A}}} Q(\tilde{f}(\varepsilon(i,\,q))) \cup \bigcup_{\varepsilon(i,q) \in \tilde{\mathcal{C}}} Q(\varepsilon(i,\,q)).
\end{aligned}
$$

Hence, thanks to (3.16) and (3.17), we obtain

$$
\begin{aligned}
D_\varepsilon(E_q',\,QR) &= \int_{E_q' \Delta QR} d_\infty^\varepsilon(z,\,\partial QR)\,dz \\
&= \int_{E_q' \Delta QR \cap S(q)^c} d_\infty^\varepsilon(z,\,\partial QR)\,dz + \int_{\bigcup_{\varepsilon(i,q) \in \tilde{\mathcal{A}}} Q_\varepsilon(\varepsilon(i,q))} d_\infty^\varepsilon(z,\,\partial QR)\,dz \\
&\quad + \int_{\bigcup_{\varepsilon(i,q) \in \tilde{\mathcal{C}}} Q_\varepsilon(\varepsilon(i,q))} d_\infty^\varepsilon(z,\,\partial QR)\,dz \\
&\geq \int_{E_q' \Delta QR \cap S(q)^c} d_\infty^\varepsilon(z,\,\partial QR)\,dz + \int_{\bigcup_{\varepsilon(i,q) \in \tilde{\mathcal{A}}} Q_\varepsilon(\tilde{f}(\varepsilon(i,q)))} d_\infty^\varepsilon(z,\,\partial QR)\,dz \\
&\quad + \int_{\bigcup_{\varepsilon(i,q) \in \tilde{\mathcal{C}}} Q_\varepsilon(\varepsilon(i,q))} d_\infty^\varepsilon(z,\,\partial QR)\,dz \\
&= \int_{E_{q+1}' \Delta QR} d_\infty^\varepsilon(z,\,\partial QR)\,dz = D_\varepsilon(E_{q+1}',\,QR)
\end{aligned}
$$

which concludes the second case.

To conclude Step 1 we are left to consider the case $q = \frac{1}{\varepsilon}\left(L_2^\varepsilon(+) + \frac{\varepsilon}{2}\right)$. For such q the proof of $D_\varepsilon(E_q',\,QR) \geq D_\varepsilon(E_{q+1}',\,QR)$ follows the same lines outlined above replacing $L_1(-)$ and $L_1(+)$ by $l(-)$ and $l(+)$, respectively.

Step 2)
In this step we prove that

$$D_\varepsilon(E', QR) \geq D_\varepsilon(\mathcal{R}(E), QR).$$

To this end we need to show that

$$D_\varepsilon(E''_p, QR) \geq D_\varepsilon(E''_{p+1}, QR) \quad \forall p \geq p_{min}$$

where E''_p, p_{min} are defined in (3.10) and (3.11). The proof of this inequality follows by the same arguments of Step 1). This time the geometric construction involves a fully analogous optimization procedure on columns instead of the one used above on rows.

$\square$

3.2. **The incremental problem.** We start by introducing the incremental minimum problem which defines the discrete-in-time area-preserving flow for $\varepsilon\mathbb{Z}^2$ crystals. We begin by introducing, for all $\varepsilon \in (0, 1)$, the space of admissible configurations

$$\mathcal{AD}_\varepsilon := \{ E \in \mathcal{C}_\varepsilon, |E| = 1 \}.$$

Let QR be a quasi rectangle of unitary area, that is $QR \in \mathcal{AD}_\varepsilon$. Fix a real number $\tau > 0$ and consider the problem

(P_1) $$\min \left\{ \mathrm{P}(E) + \frac{1}{\tau} D_\varepsilon(E, QR) : E \in \mathcal{AD}_\varepsilon \right\},$$

where $D_\varepsilon(E, QR)$ is defined in (3.6). In the next theorem, we prove that if QR is pseudo-axially symmetric, then the minimum problem (P_1) is equivalent to

(P_2) $$\min \left\{ \mathrm{P}(E) + \frac{1}{\tau} D_\varepsilon(E, QR) : E \in \mathcal{QR}_\varepsilon, E \text{ pseudo-axially symmetric}, |E| = 1 \right\}.$$

Theorem 3.7. *Let $QR \in \mathcal{QR}_\varepsilon$ be a quasi-rectangle with pseudo-axial symmetry. Then the problem (P_1) is equivalent to the problem (P_2).*

Proof. According to the definition of quasi-rectangle in 3.1 we let $QR = R \cup Q \in \mathcal{QR}_\varepsilon$ with R a rectangle with horizontal sidelength εn and vertical sidelength εm and Q a rectangle with horizontal sidelength εr and vertical sidelength ε, where $n, m, r \in \mathbb{N}$ and moreover $m \leq n$ and $r \leq \max\{n, m\}$. We claim that for all $E \in \mathcal{AD}_\varepsilon$ there exists $\widetilde{QR} \in \mathcal{QR}_\varepsilon$ with pseudo-axial symmetry such that $|\widetilde{QR}| = 1$ and

(3.18) $$\mathrm{P}(E) \geq \mathrm{P}(\widetilde{QR}), \quad D_\varepsilon(E, QR) \geq D_\varepsilon(\widetilde{QR}, QR).$$

Thanks to Propositions 3.5 and 3.6 we have

$$\mathrm{P}(E) \geq \mathrm{P}(\mathcal{R}(E)), \quad D_\varepsilon(E, QR) \geq D_\varepsilon(\mathcal{R}(E), QR).$$

Hence the claim is proved provided we show that

$$\mathrm{P}(\mathcal{R}(E)) \geq \mathrm{P}(\widetilde{QR}), \quad D_\varepsilon(\mathcal{R}(E), QR) \geq D_\varepsilon(\widetilde{QR}, QR).$$

In what follows we need to distinguish three cases.
Case 1
We assume that

(3.19) $$\mathrm{P}_1(\mathcal{R}(E)) = \mathrm{P}_1(E) \geq \mathrm{P}_1(QR) = \text{ and } \mathrm{P}_2(\mathcal{R}(E)) = \mathrm{P}_2(E) \geq \mathrm{P}_2(QR),$$

where P_1 and P_2 are defined in (3.4). In this case we set $\widetilde{QR} := QR$. By formula (3.19) and (3.5) we have that

$$\mathrm{P}(E) \geq 2\mathrm{P}_1(E) + 2\mathrm{P}_2(E) \geq 2\mathrm{P}_1(QR) + 2\mathrm{P}_2(QR) = \mathrm{P}(QR)$$

and

$$D_\varepsilon(E, QR) \geq 0 = D_\varepsilon(QR, QR).$$

Hence we obtain (3.18).

Case 2

In this case we assume

$$P_1(\mathcal{R}(E)) = P_1(E) \geq P_1(QR) \quad \text{and} \quad P_2(\mathcal{R}(E)) = P_2(E) < P_2(QR).$$

We claim that $\widetilde{QR} = QR$. In order to prove the claim we first assume that

$$(3.20) \qquad\qquad\qquad P_2(\mathcal{R}(E)) = P_2(QR) - \varepsilon.$$

We start observing that the following implication, whose proof can be obtained by a contradiction argument, holds true:

$$\text{if } P_2(\mathcal{R}(E)) = P_2(QR) - \varepsilon \quad \text{then} \quad P_1(\mathcal{R}(E)) \geq P_1(QR) + \varepsilon.$$

Therefore by (3.20) we have

$$P(\mathcal{R}(E)) = 2P_1(\mathcal{R}(E)) + 2P_2(\mathcal{R}(E)) \geq 2P_1(QR) + 2\varepsilon + 2P_2(QR) - 2\varepsilon = P(QR)$$

and $D_\varepsilon(\mathcal{R}(E), QR) \geq D_\varepsilon(QR, QR) = 0$. Hence the claim is proved under the assumption (3.20). We are left to consider the case when

$$P_2(\mathcal{R}(E)) \leq P_2(QR) - 2\varepsilon.$$

Let $N \in \mathbb{N}$ be such that $1 = |E| = N\varepsilon^2$ and let $k_1, k_2 \in \mathbb{N}$ be such that $P_1(E) = \varepsilon k_1$ and $P_2(E) = \varepsilon k_2$. By the Euclidean division we have that there exist $q, l \in \mathbb{N}$ such that $N = k_2 q + l$ with $0 \leq l < k_2$. We define the set $QR' := R' \cup Q'$ to be a quasi rectangle with axial symmetry constructed as follows: R' is a rectangle with horizontal sidelength εq and vertical sidelength εk_2 while Q' is a rectangle with horizontal sidelength εl and with vertical sidelength ε. The construction is complete once we fix the position of the barycenter according to the following alternatives:

if $n \equiv_2 q$ and $m \equiv_2 k_2$ then $\mathrm{Bar}(R') = \mathrm{Bar}(R)$,

if $n \equiv_2 q$ and $m \not\equiv_2 k_2$ then $\mathrm{Bar}(R') = (\mathrm{Bar}(R)_x, \mathrm{Bar}(R)_y - \frac{\varepsilon}{2})$ if $k_2 \in 2\mathbb{N} + 1$ and $\mathrm{Bar}(R') = (\mathrm{Bar}(R)_x, \mathrm{Bar}(R)_y + \frac{\varepsilon}{2})$ if $k_2 \in 2\mathbb{N}$,

if $n \not\equiv_2 q$ and $m \equiv_2 k_2$ then $\mathrm{Bar}(R') = (\mathrm{Bar}(R)_x - \frac{\varepsilon}{2}, \mathrm{Bar}(R)_y)$ if $k_2 \in 2\mathbb{N} + 1$ and $\mathrm{Bar}(R') = (\mathrm{Bar}(R)_x + \frac{\varepsilon}{2}, \mathrm{Bar}(R)_y)$ if $k_2 \in 2\mathbb{N}$,

if $n \not\equiv_2 q$ and $m \not\equiv_2 k_2$ then $\mathrm{Bar}(R') = 0$ if $k_2 \in 2\mathbb{N} + 1$ and $\mathrm{Bar}(R') = (\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ if $k_2 \in 2\mathbb{N}$.

We observe that the set QR' is a subset of the rectangle with horizontal sidelength εq and vertical sidelength $P_2(QR) - \varepsilon$, hence the set Q' is a subset of QR. We claim that

$$P(\mathcal{R}(E)) \geq P(QR'),$$

where $P(\mathcal{R}(E)) = 2\varepsilon k_1 + 2\varepsilon k_2$ and $P(QR') = 2\varepsilon q + 2\varepsilon k_2 + 2\varepsilon \chi_{\mathbb{N}\setminus\{0\}}(l)$. We first consider $l = 0$. In this case observing that $\mathcal{R}(E)$ is a subset of a rectangle with horizontal sidelength $P_1(E) = \varepsilon k_1$ and vertical sidelength $P_2(E) = \varepsilon k_2$ and the same barycenter of $\mathcal{R}(E)$, we obtain

$$k_1 k_2 \varepsilon^2 \geq |\mathcal{R}(E)| = 1 = |E| = N\varepsilon^2 = k_2 q \varepsilon^2 \implies k_1 \geq q.$$

Hence $P(\mathcal{R}(E)) \geq P(QR')$. If instead $l \neq 0$, arguing as above, we observe that

$$k_2 k_1 \varepsilon^2 \geq |\mathcal{R}(E)| = 1 = |E| = N\varepsilon^2 = k_2 q \varepsilon^2 + l\varepsilon^2 \quad \text{and} \quad 0 < l < k_2 \implies k_1 > q.$$

Therefore $P(\mathcal{R}(E)) \geq P(QR')$. Now if $Q' \neq \emptyset$ we iterate the argument above replacing the set $\mathcal{R}(E)$ by QR'. In this way we obtain again a set with less perimeter. If instead $Q' = \emptyset$ the procedure above can still be applied, the only difference being in the Euclidean division. In fact in this case we divide N by $\frac{P_2(QR')}{\varepsilon} + 1$. This procedure can be repeated until the vertical component of the perimeter is equal to $P_2(QR) - \varepsilon$. When this happens we fall in the case (3.20) and conclude as before.
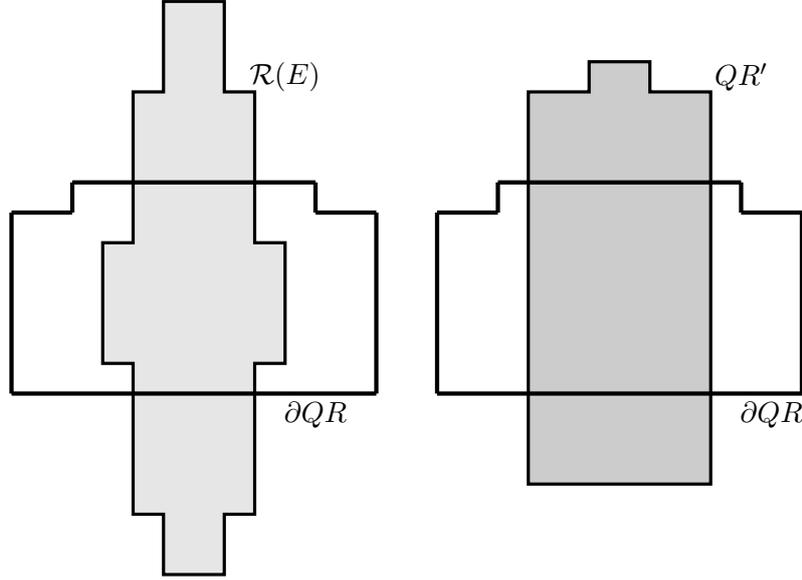
Case 3

In this case we assume

Figure 7. The quasi-rectangle QR and the sets $\mathcal{R}(E)$, QR'

$$\mathrm{P}_1(\mathcal{R}(E)) \leq \mathrm{P}_1(QR) \quad \text{and} \quad \mathrm{P}_2(\mathcal{R}(E)) > \mathrm{P}_2(QR).$$

Let $N \in \mathbb{N}$ be such that $1 = \varepsilon^2 N = |E|$ and let $k_1 \in \mathbb{N}$ be such that $\mathrm{P}_1(E) = \varepsilon k_1$. By the Euclidean division we have that there exist $q, l \in \mathbb{N}$ such that $N = qk_1 + l$ with $l < k_1$. We define the set $QR' := R' \cup Q'$ to be a quasi rectangle with axial symmetry constructed as follows: R' is a rectangle with horizontal sidelength εk_1 and vertical sidelength εq while Q' is a rectangle with horizontal sidelength εl and with vertical sidelength ε. Moreover the coordinate of the barycenter of R' depends on (n, m) as follows:

if $n \equiv_2 k_1$ and $m \equiv_2 q$ then $\mathrm{Bar}(R') = \mathrm{Bar}(R)$,
if $n \equiv_2 k_1$ and $m \not\equiv_2 q$ then $\mathrm{Bar}(R') = (\mathrm{Bar}(R)_x, \mathrm{Bar}(R)_y + \frac{\varepsilon}{2})$ if $q \in 2\mathbb{N} + 1$ and $\mathrm{Bar}(R') = (\mathrm{Bar}(R)_x, \mathrm{Bar}(R)_y + \frac{\varepsilon}{2})$ if $q \in 2\mathbb{N}$,
if $n \not\equiv_2 k_1$ and $m \equiv_2 q$ then $\mathrm{Bar}(R') = (\mathrm{Bar}(R)_x - \frac{\varepsilon}{2}, \mathrm{Bar}(R)_y)$ if $k_1 \in 2\mathbb{N} + 1$ and $\mathrm{Bar}(R') = (\mathrm{Bar}(R)_x + \frac{\varepsilon}{2}, \mathrm{Bar}(R)_y)$ if $k_1 \in 2\mathbb{N}$,
if $n \not\equiv_2 k_1$ and $m \not\equiv_2 q$ then $\mathrm{Bar}(R') = \mathrm{Bar}(R) + (\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ if $k_1 \in 2\mathbb{N}$ and $q \in 2\mathbb{N} + 1$ while $\mathrm{Bar}(R') = \mathrm{Bar}(R) + (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ if $k_1 \in 2\mathbb{N} + 1$ and $q \in 2\mathbb{N}$.

We claim that

$$\mathrm{P}(\mathcal{R}(E)) \geq \mathrm{P}(QR'),$$

where $\mathrm{P}(\mathcal{R}(E)) = 2\varepsilon k_1 + 2\varepsilon k_2$ and $\mathrm{P}(QR') = 2\varepsilon k_1 + 2\varepsilon q + 2\varepsilon \chi_{\mathbb{N} \setminus \{0\}}(l)$. We first assume that $l = 0$. We observe that $\mathcal{R}(E)$ is a subset of a rectangle with horizontal sidelength $\mathrm{P}_1(E) = \varepsilon k_1$, vertical sidelength $\mathrm{P}_2(E) = \varepsilon k_2$ and with the same barycenter of $\mathcal{R}(E)$. We obtain

$$k_1 k_2 \varepsilon^2 \geq |\mathcal{R}(E)| = 1 = |E| = N\varepsilon^2 = k_1 q \varepsilon^2 \implies k_2 \geq q.$$

Therefore $\mathrm{P}(\mathcal{R}(E)) \geq \mathrm{P}(QR')$. If otherwise $l \neq 0$, arguing as above, we observe that

$$k_2 k_1 \varepsilon^2 \geq |\mathcal{R}(E)| = 1 = |E| = N\varepsilon^2 = k_1 q \varepsilon^2 + l\varepsilon^2 \quad \text{and} \quad 0 < l < k_1 \implies k_2 > q.$$

Hence $\mathrm{P}(\mathcal{R}(E)) \geq \mathrm{P}(QR')$. We claim

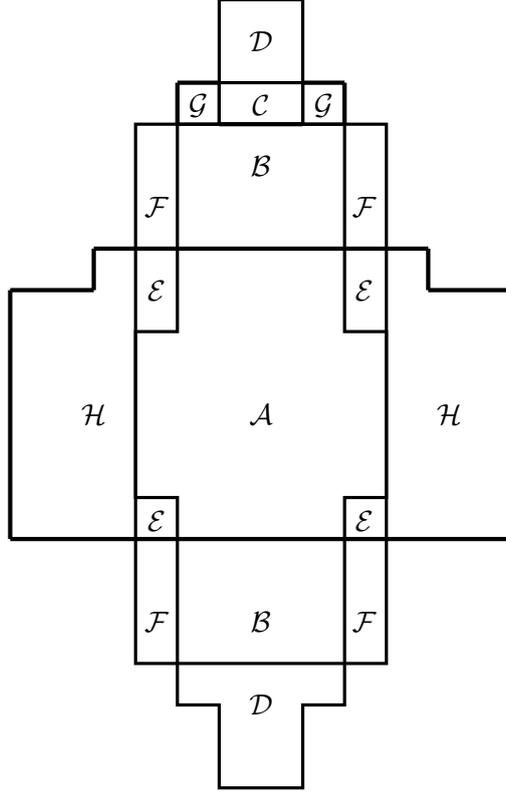$$D_\varepsilon(\mathcal{R}(E), QR) \geq D_\varepsilon(QR', QR).$$

Figure 8. The sets $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$

We define the following sets

$$\mathcal{A} := \varepsilon\mathbb{Z}^2 \cap \mathcal{R}(E) \cap QR \qquad\qquad \mathcal{B} := \varepsilon\mathbb{Z}^2 \cap \mathcal{R}(E) \cap QR^c \cap R'$$

$$\mathcal{C} := \varepsilon\mathbb{Z}^2 \cap \mathcal{R}(E) \cap QR^c \cap Q' \qquad \mathcal{D} := \varepsilon\mathbb{Z}^2 \cap \mathcal{R}(E) \cap QR^c \cap (QR')^c$$

$$\mathcal{E} := \varepsilon\mathbb{Z}^2 \cap R' \cap \mathcal{R}(E)^c \cap QR \qquad \mathcal{F} := \varepsilon\mathbb{Z}^2 \cap R' \cap \mathcal{R}(E)^c \cap QR^c$$

$$\mathcal{G} := \varepsilon\mathbb{Z}^2 \cap Q' \cap QR^c \cap \mathcal{R}(E)^c \qquad \mathcal{H} := \varepsilon\mathbb{Z}^2 \cap QR \cap (QR')^c.$$

We have that $N = \#\varepsilon\mathbb{Z}^2 \cap \mathcal{R}(E) = \#\mathcal{A} + \#\mathcal{B} + \#\mathcal{C} + \#\mathcal{D}$ and that $N = \#\varepsilon\mathbb{Z}^2 \cap QR' = \#\mathcal{A} + \#\mathcal{B} + \#\mathcal{C} + \#\mathcal{E} + \#\mathcal{F} + \#\mathcal{G}$. As a consequence $\#\mathcal{D} = \#\mathcal{E} + \#\mathcal{F} + \#\mathcal{G}$, hence there exists a bijection $f \colon \mathcal{D} \to \mathcal{E} \cup \mathcal{F} \cup \mathcal{G}$. We observe that if $\varepsilon(i, j) \in \mathcal{D}$ then

$$f(\varepsilon(i, j)) \in QR \text{ or}$$
$$f(\varepsilon(i, j)) \in QR^c \text{ and } d_\infty^\varepsilon(\varepsilon(i, j), \partial QR) \geq d_\infty^\varepsilon(f(\varepsilon(i, j)), \partial QR).$$

The last inequality follows from $f(\varepsilon(i, j)) \in QR'$, that is, if $j > 0$ (the case $j \leq 0$ being fully equivalent)

$$d_\infty^\varepsilon(\varepsilon(i, j), \partial QR) \geq \mathcal{H}^1(\Pi_2(QR' \cap \{0\} \times [0, +\infty)))$$
$$- \mathcal{H}^1(\Pi_2(QR \cap \{0\} \times [0, +\infty))) \geq d_\infty^\varepsilon(f(\varepsilon(i, j)), \partial QR).$$

We observe that

$$\mathcal{R}(E)\Delta QR = \bigcup_{\varepsilon(i,j)\in\mathcal{B}} Q_\varepsilon(\varepsilon(i,j)) \cup \bigcup_{\varepsilon(i,j)\in\mathcal{C}} Q_\varepsilon(\varepsilon(i,j)) \cup \bigcup_{\varepsilon(i,j)\in\mathcal{D}} Q_\varepsilon(\varepsilon(i,j))$$

$$\cup \bigcup_{\varepsilon(i,j)\in\mathcal{E}} Q_\varepsilon(\varepsilon(i,j)) \cup \bigcup_{\varepsilon(i,j)\in\mathcal{H}} Q_\varepsilon(\varepsilon(i,j)).$$

(3.21)
$$QR'\Delta QR = \bigcup_{\varepsilon(i,j)\in\mathcal{B}} Q_\varepsilon(\varepsilon(i,j)) \cup \bigcup_{\varepsilon(i,j)\in\mathcal{C}} Q_\varepsilon(\varepsilon(i,j)) \cup \bigcup_{\varepsilon(i,j)\in\mathcal{H}} Q_\varepsilon(\varepsilon(i,j))$$

$$\cup \bigcup_{\varepsilon(i,j)\in\mathcal{F}} Q_\varepsilon(\varepsilon(i,j)) \cup \bigcup_{\varepsilon(i,j)\in\mathcal{G}} Q_\varepsilon(\varepsilon(i,j)).$$

Then by (3.21) we have

$$D_\varepsilon(\mathcal{R}(E), QR) = \int_{\mathcal{R}(E)\Delta QR} d_\infty^\varepsilon(z, \partial QR)\, dz$$

$$= \int_{\bigcup_{\varepsilon(i,j)\in\mathcal{B}} Q_\varepsilon(\varepsilon(i,j))} d_\infty^\varepsilon(z, \partial QR)\, dz + \int_{\bigcup_{\varepsilon(i,j)\in\mathcal{C}} Q_\varepsilon(\varepsilon(i,j))} d_\infty^\varepsilon(z, \partial QR)\, dz$$

$$+ \int_{\bigcup_{\varepsilon(i,j)\in\mathcal{D}} Q_\varepsilon(\varepsilon(i,j))} d_\infty^\varepsilon(z, \partial QR)\, dz + \int_{\bigcup_{\varepsilon(i,j)\in\mathcal{E}} Q_\varepsilon(\varepsilon(i,j))} d_\infty^\varepsilon(z, \partial QR)\, dz$$

$$+ \int_{\bigcup_{\varepsilon(i,j)\in\mathcal{H}} Q_\varepsilon(\varepsilon(i,j))} d_\infty^\varepsilon(z, \partial QR)$$

$$\geq \int_{\bigcup_{\varepsilon(i,j)\in\mathcal{B}} Q_\varepsilon(\varepsilon(i,j))} d_\infty^\varepsilon(z, \partial QR)\, dz + \int_{\bigcup_{\varepsilon(i,j)\in\mathcal{C}} Q_\varepsilon(\varepsilon(i,j))} d_\infty^\varepsilon(z, \partial QR)\, dz$$

$$+ \int_{\bigcup_{\varepsilon(i,j)\in f(\mathcal{D})\setminus\mathcal{E}} Q_\varepsilon(\varepsilon(i,j))} d_\infty^\varepsilon(z, \partial QR)\, dz + \int_{\bigcup_{\varepsilon(i,j)\in\mathcal{H}} Q_\varepsilon(\varepsilon(i,j))} d_\infty^\varepsilon(z, \partial QR)\, dz$$

$$= \int_{\bigcup_{\varepsilon(i,j)\in\mathcal{B}} Q_\varepsilon(\varepsilon(i,j))} d_\infty^\varepsilon(z, \partial QR)\, dz + \int_{\bigcup_{\varepsilon(i,j)\in\mathcal{C}} Q_\varepsilon(\varepsilon(i,j))} d_\infty^\varepsilon(z, \partial QR)\, dz$$

$$+ \int_{\bigcup_{\varepsilon(i,j)\in\mathcal{F}\cup\mathcal{G}} Q_\varepsilon(\varepsilon(i,j))} d_\infty^\varepsilon(z, \partial QR)\, dz + \int_{\bigcup_{\varepsilon(i,j)\in\mathcal{H}} Q_\varepsilon(\varepsilon(i,j))} d_\infty^\varepsilon(z, \partial QR)\, dz$$

$$= \int_{QR'\Delta QR} d_\infty^\varepsilon(z, \partial QR)\, dz = D_\varepsilon(QR', QR).$$

which is the claim. Hence Case 3 is proven for $\widetilde{QR} = QR'$. $\qquad\square$

3.3. Flat $P_{|\cdot|_1}$ area-preserving mean-curvature flow of a quasi rectangle in $\varepsilon\mathbb{Z}^2$.
In this section we introduce the area-preserving mean-curvature flow of a quasi rectangle in the lattice $\varepsilon\mathbb{Z}^2$. We introduce a notion of global flat solution to the area-preserving mean-curvature flow in the lattice $\varepsilon\mathbb{Z}^2$ which is analogous to the one introduced in Section 2. To this end, given $E, F \in \mathcal{AD}_\varepsilon$ we define

$$(3.22) \qquad \mathcal{F}_{\varepsilon,\tau}(E, F) := \mathrm{P}(E) + \frac{1}{\tau} D_\varepsilon(E, F).$$

Definition 3.8. Let $QR \in \mathcal{QR}_\varepsilon$ be a quasi-rectangle with pseudo-axial symmetry such that $|QR| = 1$. Let $\{E_k^{(\varepsilon,\tau)}\}_{k\in\mathbb{N}}$ be a family of sets defined iteratively as

$$E_0^{(\varepsilon,\tau)} = QR \quad \text{and} \quad E_k^{(\varepsilon,\tau)} \in \operatorname*{arg\,min}_{E\in\mathcal{AD}_\varepsilon}\left\{\mathcal{F}_{\varepsilon,\tau}(E, E_{k-1}^{(\varepsilon,\tau)})\right\} \quad k \geq 1.$$

We define

$$E_t^{(\varepsilon,\tau)} := E_k^{(\varepsilon,\tau)} \quad \text{for any } t \in [k\tau, (k+1)\tau)$$

and we call $\{E_t^{(\varepsilon,\tau)}\}_{t \geq 0}$ an approximate flat solution of the area-preserving mean-curvature flow in the lattice $\varepsilon \mathbb{Z}^2$ with initial datum QR.

Remark 3.9. Let QR be a quasi rectangle with pseudo-axial symmetry. Thanks to Theorem 3.7 we have that every minimizer of (3.22) with $E \in \mathcal{AD}_\varepsilon$ and $F = QR$ is a quasi-rectangle with pseudo-axial symmetry. Therefore, we have that an approximate flat solution of the area-preserving mean-curvature flow in the lattice $\varepsilon \mathbb{Z}^2$ with initial datum QR is a family of quasi-rectangles with pseudo-axial symmetry.

Let us consider a family of initial data, say $\{QR_\varepsilon\}_\varepsilon$, such that for every ε the set QR_ε is a quasirectangle with axial symmetry. We assume that, as $\varepsilon \to 0$ QR_ε converge in the Hausdorff distance to the rectangle $[-\frac{a}{2}, \frac{a}{2}] \times [-\frac{b}{2}, \frac{b}{2}]$. In what follows we are interested in proving the existence and give some geometric properties of a limit point, as $\varepsilon, \tau \to 0$, of a subsequence of approximated flat solution of the area-preserving mean-curvature flow $E_t^{\varepsilon,\tau}$ whose initial datum is QR_ε. As already explained in the introduction of this paper, we focus on the regime $\varepsilon \sim \tau$. More precisely we will compute the minimizer of the incremental problem in (3.22) in a subset of $\mathcal{AD}_\varepsilon$ and we will describe the asymptotic behavior, as $\varepsilon \to 0$, of the approximate flat solution of the area-preserving mean-curvature flow in the lattice $\varepsilon \mathbb{Z}^2$ within this subclass. For the sake of simplicity, in the sequel we assume that $\tau = \alpha \varepsilon$ with $\alpha \in (0, +\infty)$.

Definition 3.10. Let $QR \in \mathcal{QR}_\varepsilon$ be a quasi rectangle with pseudo-axial symmetry. We assume that $QR = R \cup Q$ and that the horizontal sidelength of Q is εr, $r \in \mathbb{N}$ (see the definition 3.1). We define $\mathcal{SQR}_\varepsilon(QR)$ as the set of all $QR' \in \mathcal{QR}_\varepsilon$ with $QR' = R' \cup Q'$ such that the horizontal sidelength of R' is εn, $n \in \mathbb{N}$, $|QR'| = |QR|$, $\mathrm{Bar}(R) = \mathrm{Bar}(R')$ and $r \leq n$.

In the next theorem, given $QR \in \mathcal{QR}_\varepsilon$, we want to compute the minimizer of

$$(3.23) \qquad \min \left\{ \mathrm{P}(\widetilde{QR}) + \frac{1}{\alpha\varepsilon} D_\varepsilon(\widetilde{QR}, QR) \colon \widetilde{QR} \in \mathcal{SQR}_\varepsilon(QR) \right\}.$$

Theorem 3.11. *Let* $a, b, c \in \mathbb{R}$ *be such that* $0 < b < a$, $0 \leq c < a$ *and let us consider* $\varepsilon > 0$. *We assume that* $a = \varepsilon A$, $b = \varepsilon B$, $c = \varepsilon C$ *with* $A, B, C \in \mathbb{N}$ *and* $ab + \varepsilon c = 1$. *Let* $QR \in \mathcal{QR}_\varepsilon$ *be the quasi-rectangle with sides* $n = A$, $m = B$ *and* $r = C$ *(see definition 3.1). For all* $\alpha > 0$ *and for all* $\Lambda > a + b$ *there exists* $\varepsilon_0 := \varepsilon_0(\Lambda, \alpha)$ *such that for every* $\varepsilon < \varepsilon_0$ *there exists a minimizer* QR' *of (3.23). Such a minimizer is a quasirectangle with pseudo-axial symmetry* $QR' = R' \cup Q'$ *with* R' *and* Q' *characterized as follows. Setting*

$$x' = \frac{2\alpha(-b+a)}{b(a+b)} - \frac{a}{a+b},$$

$$X_1 = \lfloor x' \rfloor, \ X_2 = \lceil x' \rceil, \ Y_1 = \left\lfloor \frac{c + 2bX_1}{2a - 4\varepsilon X_1} \right\rfloor, Y_2 = \left\lfloor \frac{c + 2bX_2}{2a - 4\varepsilon X_2} \right\rfloor,$$

it holds that

1)

$$R' = \left[-\frac{a'}{2}, \frac{a'}{2} \right] \times \left[-\frac{b'}{2}, \frac{b'}{2} \right] + \mathrm{Bar}(R) \ \text{where}$$

$$a' \in \{a - 2X_1\varepsilon, a - 2X_2\varepsilon\}, \quad b' \in \{b + 2Y_1\varepsilon, b + 2Y_2\varepsilon\},$$

2)

the horizontal sidelength of Q' *is* $d := \varepsilon D \in \{\varepsilon(C + 2X_iB - Y_i(2A - 4X_i)) \text{ for } i = 1, 2\}$,

3)

$$a' + b' \leq a + b, \ |a' - a| \leq \varepsilon C(\Lambda, \alpha),$$

$$|b' - b| \leq \varepsilon C(\Lambda, \alpha) \ \text{and} \ |QR \Delta QR'| \leq \varepsilon C(\Lambda, \alpha).$$

Proof. In what follows all constants depend only on Λ and α. We observe that the problem (3.23) is equivalent to

$$\min\left\{\alpha\varepsilon\mathrm{P}(\widetilde{QR}) + D_\varepsilon(\widetilde{QR}, QR)\colon \widetilde{QR} \in \mathcal{SQR}_\varepsilon(QR)\right\}.$$

To shorten notation we define

(3.24) $$E_\varepsilon(\widetilde{QR}) := \alpha\varepsilon\mathrm{P}(\widetilde{QR}) + D_\varepsilon(\widetilde{QR}, QR).$$

We divide the proof into several steps.

Step 1)

In this step we want to prove that the minimum value and the minimizer of the energy E_ε can be obtained considering the minimum value and the minimizer of a function defined on $\mathbb{N}^3$. According to the definition of $\widetilde{QR} \in \mathcal{SQR}_\varepsilon(QR)$, setting $\widetilde{QR} = \widetilde{R} \cup \widetilde{Q}$, we have that $|QR| = |\widetilde{QR}|$ and that $\mathrm{Bar}(R) = \mathrm{Bar}(\widetilde{R})$. Moreover there exist $X, Y \in \mathbb{N}$ such that the horizontal sidelength of $\widetilde{R}$ is $A - 2X$, the vertical sidelength of $\widetilde{R}$ is $B - 2Y$ and the horizontal sidelength of $\widetilde{Q}$ is $d := \varepsilon D$ with $d < c$. Since $|QR| = |\widetilde{QR}|$, we have that

$$ab + c\varepsilon = (a - 2X\varepsilon)(b + 2Y\varepsilon) + \varepsilon d \implies \varepsilon Y = \frac{c\varepsilon - d\varepsilon + 2\varepsilon b}{2a - 4\varepsilon X}.$$

The perimeter of the set $\widetilde{QR}$ is

(3.25) $$\mathrm{P}(\widetilde{QR}) = 2(a - 2\varepsilon X) + 2(b + 2\varepsilon Y) + 2\varepsilon\chi_{(0,+\infty)}(d).$$

We observe that the function $d \to 2\varepsilon\chi_{(0,+\infty)}(d)$ will not affect the calculation of the energy minimizer, as it only raises or lowers the energy by a constant amount. By the symmetry of the set $\widetilde{QR}\Delta QR$, the dissipation is
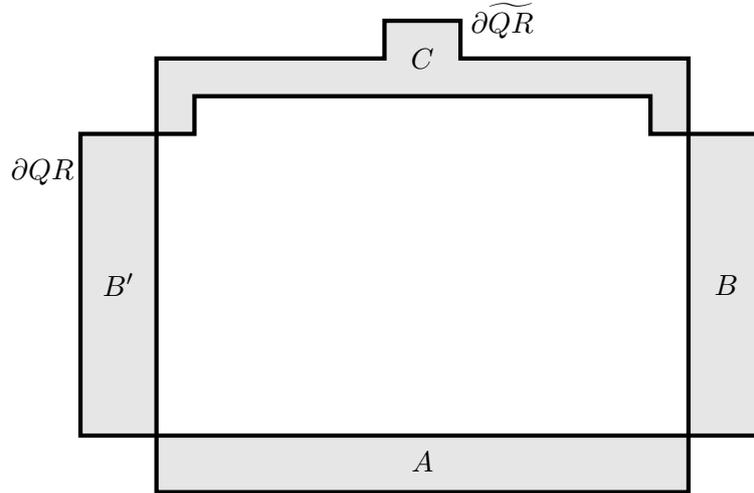

Figure 9. The quasi-rectangles QR and $\widetilde{QR}$

$$\mathcal{D}_\varepsilon(\widetilde{QR}, QR) = \int_{\widetilde{QR}\Delta QR} d_\infty^\varepsilon(z, \partial QR)dz$$

$$= \int_A d_\infty^\varepsilon(z, \partial QR)dz + \int_B d_\infty^\varepsilon(z, \partial QR)dz + \int_{B'} d_\infty^\varepsilon(z, \partial QR)dz + \int_C d_\infty^\varepsilon(z, \partial QR)dz$$

where

$$A = \left[-\frac{a - 2\varepsilon X}{2}, \frac{a - 2\varepsilon X}{2} \right] \times \left[-\frac{\varepsilon Y}{2}, \frac{\varepsilon Y}{2} \right] + \left(\mathrm{Bar}_x(R),\, \mathrm{Bar}_y(R) - \frac{b}{2} - \frac{\varepsilon Y}{2} \right),$$

$$B = \left[-\frac{\varepsilon X}{2}, \frac{\varepsilon X}{2} \right] \times \left[-\frac{b}{2}, \frac{b}{2} \right] + \left(\mathrm{Bar}_x(R) + \frac{a + \varepsilon X}{2},\, \mathrm{Bar}_y(R) \right),$$

$$B' = \left[-\frac{\varepsilon X}{2}, \frac{\varepsilon X}{2} \right] \times \left[-\frac{b}{2}, \frac{b}{2} \right] + \left(\mathrm{Bar}_x(R) - \frac{a + \varepsilon X}{2},\, \mathrm{Bar}_y(R) \right),$$

$$C = \widetilde{QR} \setminus \left(QR \cup A \cup B \cup B' \right).$$

By a change of variables we have that

(3.26)
$$\mathcal{D}_\varepsilon(\widetilde{QR}, QR) = \int_{\widetilde{QR} \triangle QR} d_\infty^\varepsilon(z, \partial QR)\, dz$$
$$= \int_A d_\infty^\varepsilon(z, \partial QR)\, dz + 2 \int_B d_\infty^\varepsilon(z, \partial QR)\, dz + \int_C d_\infty^\varepsilon(z, \partial QR)\, dz.$$

We compute the different terms in the expression above. We have that

(3.27)
$$\mathcal{D}_A := \int_A d_\infty^\varepsilon(z, \partial QR)\, dz = \sum_{j=1}^Y j \varepsilon \varepsilon (a - 2\varepsilon X) = (a - 2\varepsilon X) \frac{\varepsilon Y (\varepsilon Y + \varepsilon)}{2}$$

and

$$\mathcal{D}_C := \int_C d_\infty^\varepsilon(z, \partial QR)\, dz = \int_D d_\infty^\varepsilon(z, \partial QR)\, dz + \int_E d_\infty^\varepsilon(z, \partial QR)\, dz + \int_{E'} d_\infty^\varepsilon(z, \partial QR)\, dz$$
$$+ \int_F d_\infty^\varepsilon(z, \partial QR)\, dz + \int_{F'} d_\infty^\varepsilon(z, \partial QR)\, dz + \int_G d_\infty^\varepsilon(z, \partial QR)\, dz + \int_{G'} d_\infty^\varepsilon(z, \partial QR)\, dz$$
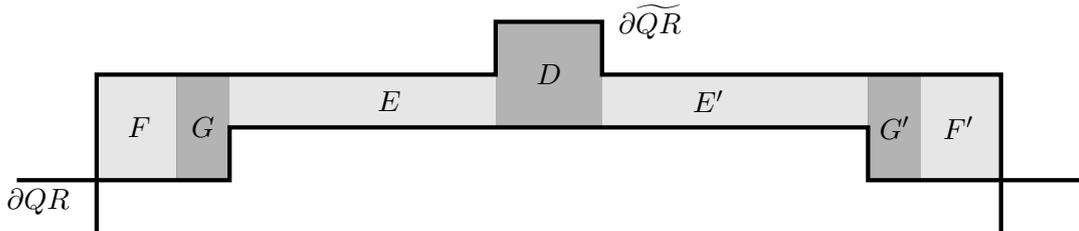

Figure 10. The sets F, F', G, G', E, E', D

where

$$D := \left[-\frac{d}{2}, \frac{d}{2}\right] \times \left[-\frac{\varepsilon Y}{2}, \frac{\varepsilon Y}{2}\right] + \left(\mathrm{Bar}_x(R), \mathrm{Bar}_y(R) + \frac{b}{2} + \varepsilon + \frac{\varepsilon Y}{2}\right),$$

$$E := \left[-\frac{c-d}{2}, \frac{c-d}{2}\right] \times \left[-\frac{\varepsilon Y - \varepsilon}{2}, \frac{\varepsilon Y - \varepsilon}{2}\right] + \left(\mathrm{Bar}_x(R) - \frac{c+d}{4}, \mathrm{Bar}_y + \frac{b}{2} + \frac{\varepsilon Y + \varepsilon}{2}\right),$$

$$E' := \left[-\frac{c-d}{2}, \frac{c-d}{2}\right] \times \left[-\frac{\varepsilon Y - \varepsilon}{2}, \frac{\varepsilon Y - \varepsilon}{2}\right] + \left(\mathrm{Bar}_x(R) + \frac{c+d}{4}, \mathrm{Bar}_y + \frac{b}{2} + \frac{\varepsilon Y + \varepsilon}{2}\right),$$

$$G := \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right] \times \left[-\frac{\varepsilon Y}{2}, \frac{\varepsilon Y}{2}\right] + \left(\mathrm{Bar}_x(R) - \frac{c}{2} - \frac{\varepsilon}{2}, \mathrm{Bar}_y(R) + \frac{b}{2} + \frac{\varepsilon Y}{2}\right),$$

$$G' := \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right] \times \left[-\frac{\varepsilon Y}{2}, \frac{\varepsilon Y}{2}\right] + \left(\mathrm{Bar}_x(R) + \frac{c}{2} + \frac{\varepsilon}{2}, \mathrm{Bar}_y(R) + \frac{b}{2} + \frac{\varepsilon Y}{2}\right),$$

$$F := \left[-\frac{a-c-2\varepsilon X}{2}, \frac{a-c-2\varepsilon X}{2}\right] \times \left[-\frac{\varepsilon Y}{2}, \frac{\varepsilon Y}{2}\right]$$
$$+ \left(\mathrm{Bar}_x(R) - \frac{a - 2\varepsilon X + 2\varepsilon + c}{4}, \mathrm{Bar}_y(R) + \frac{b + \varepsilon Y}{2}\right)$$

and

$$F' := \left[-\frac{a-c-2\varepsilon X}{2}, \frac{a-c-2\varepsilon X}{2}\right] \times \left[-\frac{\varepsilon Y}{2}, \frac{\varepsilon Y}{2}\right]$$
$$+ \left(\mathrm{Bar}_x(R) + \frac{a - 2\varepsilon X + c + 2\varepsilon}{4}, \mathrm{Bar}_y(R) + \frac{b + \varepsilon Y}{2}\right).$$

By a change of variables we obtain that

$$(3.28) \qquad \begin{aligned} \mathcal{D}_C &= \int_C d_\infty^\varepsilon(z, \partial QR)dz = \int_D d_\infty^\varepsilon(z, \partial QR)dz + 2\int_E d_\infty^\varepsilon(z, \partial QR)dz \\ &\quad + 2\int_F d_\infty^\varepsilon(z, \partial QR)dz + 2\int_G d_\infty^\varepsilon(z, \partial QR)dz, \end{aligned}$$

where

$$(3.29) \int_D d_\infty^\varepsilon(z, \partial QR)dz = \sum_{j=1}^{Y} j\varepsilon^2 d = \frac{d\varepsilon^2 Y(Y+1)}{2},$$

$$\int_E d_\infty^\varepsilon(z, \partial QR)dz = \sum_{j=1}^{Y-1} \frac{c-d}{2}\varepsilon^2 j = \frac{(c-d)\varepsilon^2(Y-1)Y}{4},$$

$$\int_F d_\infty^\varepsilon(z, \partial QR)dz = \sum_{j=1}^{Y} \frac{(a-2\varepsilon X - c - 2\varepsilon)}{2}\varepsilon^2 j = \frac{(a - 2\varepsilon X - c - 2\varepsilon)\varepsilon^2(Y+1)Y}{4},$$

$$\int_G d_\infty^\varepsilon(z, \partial QR)dz = \sum_{j=1}^{Y} j\varepsilon^3 = \varepsilon^3 \frac{Y(Y+1)}{2}.$$
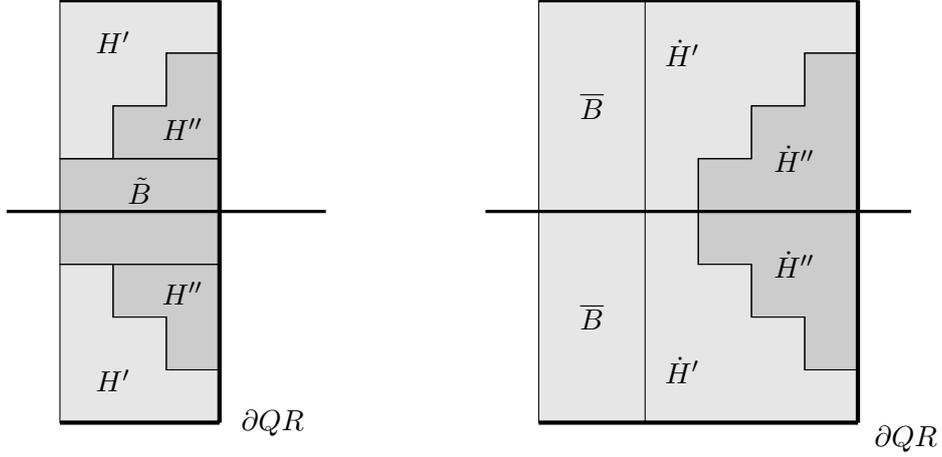
Figure 11. In the left side the case $\varepsilon X \leq \frac{b}{2}$ and in the right side the case $\varepsilon X > \frac{b}{2}$

In order to compute

$$\mathcal{D}_B := 2 \int_B d_\infty^\varepsilon(z, \partial QR) dz$$

we distinguish two cases, namely $\varepsilon X \leq \frac{b}{2}$ or $\varepsilon X > \frac{b}{2}$.

Case $\varepsilon X \leq \frac{b}{2}$ By the symmetry of the quasi rectangle we can write

$$(3.30) \qquad \mathcal{D}_B := 2 \int_B d_\infty^\varepsilon(z, \partial QR) dz = 2 \int_{\tilde{B}} d_\infty^\varepsilon(z, \partial QR) dz + 4 \int_{\hat{B}} d_\infty^\varepsilon(z, \partial QR) dz$$

where

$$\tilde{B} := \left[-\frac{\varepsilon X}{2}, \frac{\varepsilon X}{2} \right] \times \left[-\frac{b - 2\varepsilon X}{2}, \frac{b - 2\varepsilon X}{2} \right] + \left(\mathrm{Bar}_x(R) + \frac{2a - 3\varepsilon X}{4}, \mathrm{Bar}_y(R) \right),$$

$$\hat{B} := \left[-\frac{\varepsilon X}{2}, \frac{\varepsilon X}{2} \right] \times \left[-\frac{\varepsilon X}{2}, \frac{\varepsilon X}{2} \right] + \left(\mathrm{Bar}_x(R) + \frac{a + \varepsilon X}{2}, \mathrm{Bar}_y(R) + \frac{b - \varepsilon X}{2} \right).$$

For the first term we have that

$$(3.31) \qquad 2 \int_{\tilde{B}} d_\infty^\varepsilon(z, \partial QR) dz = 2\varepsilon^2 (b - 2\varepsilon X) \sum_{j=1}^{X} j = (b - 2\varepsilon X)\varepsilon X(\varepsilon X + \varepsilon).$$

For the second term we first perform a change of variables and write that

$$(3.32) \qquad \int_{\hat{B}} d_\infty^\varepsilon(z, \partial QR) dz = \int_H d_\infty^\varepsilon(z, L) dz$$

where

$$H := [0, \varepsilon X] \times [0, \varepsilon X], \quad L := [0, \varepsilon X] \times \{\varepsilon X\} \cup \{\varepsilon X\} \times [0, \varepsilon X].$$

We then simplify the computation by writing

$$(3.33) \qquad \int_H d_\infty^\varepsilon(z, \partial QR) dz = \int_{H'} d_\infty^\varepsilon(z, L) dz + \int_{H''} d_\infty^\varepsilon(z, L) dz$$

where

$$H' := \bigcup_{i=1}^{X} \bigcup_{j=1}^{X+1-j} Q_\varepsilon(\varepsilon(i, j)), \quad H'' := \bigcup_{j=1}^{X} \bigcup_{i=1}^{X-j} Q_\varepsilon(\varepsilon(i, j)).$$

We eventually obtain

$$(3.34) \quad 4\int_H d_\infty^\varepsilon(z, \partial QR)dz = 4\int_{H'} d_\infty^\varepsilon(z, L)dz + 4\int_{H''} d_\infty^\varepsilon(z, L)dz$$

$$= 4\sum_{i=1}^{X}\sum_{j=1}^{X-i+1} j\varepsilon^3 + 4\sum_{j=1}^{X}\sum_{i=1}^{X-j} i\varepsilon^3 = 4\varepsilon^3\sum_{k=1}^{X}\left(2\sum_{h=1}^{X-k} h + X - k + 1\right)$$

$$= 4\varepsilon^3\sum_{k=1}^{X}(X-k+1)^2 = \frac{2}{3}\varepsilon^3 X(2X^2 + 3X + 1).$$

Hence, thanks to (3.30), (3.31), (3.32), (3.33), (3.34) we have

$$(3.35) \quad \mathcal{D}_B = \frac{2}{3}\varepsilon^3 X(2X^2 + 3X + 1) + (b - 2\varepsilon X)\varepsilon X(\varepsilon X + \varepsilon).$$

Case $\varepsilon X > \frac{b}{2}$ We need to consider two sub-cases. The first one is when $b = \varepsilon B$ and $B = 2B'$ with $B, B' \in \mathbb{N}$, while the second one corresponds to $B = 2B' + 1$ with $B, B' \in \mathbb{N}$. In the first case, using the symmetries of the quasi rectangle we can write that

$$(3.36) \quad \mathcal{D}_B := 2\int_B d_\infty^\varepsilon(z, \partial QR)dz = 4\int_{\overline{B}} d_\infty^\varepsilon(z, \partial QR)dz + 4\int_{\dot{B}} d_\infty^\varepsilon(z, \partial QR)dz$$

where

$$\overline{B} := \left[-\frac{(B'-X)\varepsilon}{2}, \frac{(B'-X)\varepsilon}{2}\right] \times \left[-\frac{B'\varepsilon}{2}, \frac{B'\varepsilon}{2}\right]$$

$$+ \left(\text{Bar}_x(R) + \frac{a + (B'-X)\varepsilon}{2}, \text{Bar}_y(R) + \frac{B'\varepsilon}{2}\right)$$

and

$$\dot{B} := \left[-\frac{B'\varepsilon}{2}, \frac{B'\varepsilon}{2}\right] \times \left[-\frac{B'\varepsilon}{2}, \frac{B'\varepsilon}{2}\right] + \left(\text{Bar}_x + \frac{a + 3B'\varepsilon - 2X\varepsilon}{2}, \text{Bar}_y + \frac{B'\varepsilon}{2}\right).$$

For the first term we have that

$$(3.37) \quad 4\int_{\overline{B}} d_\infty^\varepsilon(z, \partial QR)dz = 4\varepsilon(X - B')\sum_{j=1}^{B'}\varepsilon^2 j = 2(X - B')\varepsilon^3 B'(B' + 1).$$

For the second term, by a change of variables, it holds true that

$$(3.38) \quad \int_{\dot{B}} d_\infty^\varepsilon(z, \partial QR)dz = \int_{\dot{H}} d_\infty^\varepsilon(z, \dot{L})dz$$

where

$$\dot{H} := [0, B'\varepsilon] \times [0, B'\varepsilon], \quad \dot{L} := [0, B'\varepsilon] \times \{B'\varepsilon\} \cup \{B'\varepsilon\} \times [0, B'\varepsilon].$$

We can further decompose the integral as follows

$$(3.39) \quad \int_{\dot{H}} d_\infty^\varepsilon(z, \partial QR)dz = \int_{\dot{H}'} d_\infty^\varepsilon(z, L)dz + \int_{\dot{H}''} d_\infty^\varepsilon(z, L)dz$$

where

$$\dot{H}' := \bigcup_{i=1}^{B'}\bigcup_{j=1}^{B'+1-j} Q_\varepsilon(\varepsilon(i,j)), \quad \dot{H}'' := \bigcup_{j=1}^{B'}\bigcup_{i=1}^{B'-j} Q_\varepsilon(\varepsilon(i,j)).$$

We eventually obtain

$$(3.40) \quad 4\int_{\dot{H}'} d_\infty^\varepsilon(z,L)dz + 4\int_{\dot{H}''} d_\infty^\varepsilon(z,L)dz = 4\sum_{i=1}^{B'}\sum_{j=1}^{B'-j+1} j\varepsilon^3 + 4\sum_{j=1}^{B'}\sum_{i=1}^{b'-j} i\varepsilon^3$$

$$= 4\varepsilon^3\sum_{k=1}^{B'}\left(2\sum_{h=1}^{B'-k} h + B' - k + 1\right) = \frac{2}{3}\varepsilon^3\left(2\left(B'\right)^2 + 3B' + 1\right)B'.$$

Gathering together (3.36),(3.37), (3.38), (3.39), (3.40) we have

$$(3.41) \quad \mathcal{D}_B = \frac{b}{3}\left(2\left(\frac{b}{2}\right)^2 + 3\varepsilon\frac{b}{2} + \varepsilon^2\right) + b(\frac{b}{2} + \varepsilon)(\varepsilon X - \frac{b}{2}).$$

We consider the second case, that is when $B = 2B' + 1$ with $B, B' \in \mathbb{N}$. By the symmetry of the quasi rectangle, we can write

$$\mathcal{D}_B := 2\int_B d_\infty^\varepsilon(z,\partial QR)dz = 4\int_{\ddot{B}} d_\infty^\varepsilon(z,\partial QR)dz$$

$$+ 4\int_{\breve{B}} d_\infty^\varepsilon(z,\partial QR)dz + 2\int_V d_\infty^\varepsilon(z,\partial QR)dz + 2\int_W d_\infty^\varepsilon(z,\partial QR)dz$$

where

$$\ddot{B} := \left[-\frac{\varepsilon(X-B')}{2}, \frac{\varepsilon(X-B')}{2}\right] \times \left[-\frac{\varepsilon B'}{2}, \frac{\varepsilon B'}{2}\right]$$

$$+ \left(\mathrm{Bar}_x(R) + \frac{a + X\varepsilon - \varepsilon B'}{2}, \mathrm{Bar}_y(R) + \frac{\varepsilon + \varepsilon B'}{2}\right)$$

and

$$\breve{B} := \left[-\frac{\varepsilon B'}{2}, \frac{\varepsilon B'}{2}\right] \times \left[-\frac{\varepsilon B'}{2}, \frac{\varepsilon B'}{2}\right] + \left(\mathrm{Bar}_x(R) + \frac{a + \varepsilon B' + 2\varepsilon X}{2}, \mathrm{Bar}_y(R) + \frac{\varepsilon + \varepsilon B'}{2}\right),$$

$$V := \left[-\frac{\varepsilon(X-B')}{2}, \frac{\varepsilon(X-B')}{2}\right] \times \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right] + \left(\mathrm{Bar}_x(R) + \frac{a + X\varepsilon - \varepsilon B'}{2}, \mathrm{Bar}_y(R)\right),$$

$$W := \left[-\frac{\varepsilon B'}{2}, \frac{\varepsilon B'}{2}\right] \times \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right] + \left(\mathrm{Bar}_x(R) + \frac{a + \varepsilon B' + 2\varepsilon X}{2}, \mathrm{Bar}_y(R)\right).$$

Arguing as before we show that

$$\mathcal{D}_B = \frac{2}{3}\varepsilon^3\left(2\left(B'\right)^2 + 3B' + 1\right)B'$$

$$+ 2\left(X - B'\right)\varepsilon^3 B'\left(B' + 1\right) + \varepsilon^3 B'\left(B' + 1\right) + \left(X - B'\right)\left(B' + 1\right)\varepsilon^3.$$

Recalling that $\varepsilon B' = \frac{b-\varepsilon}{2}$ we eventually obtain

$$(3.42)$$

$$\mathcal{D}_B = \frac{2}{3}\left(2\left(\frac{b-\varepsilon}{2}\right)^2 + 3\varepsilon\frac{b-\varepsilon}{2} + \varepsilon^2\right)\frac{b-\varepsilon}{2} + 2\left(X\varepsilon - \frac{b-\varepsilon}{2}\right)\frac{b-\varepsilon}{2}\left(\frac{b-\varepsilon}{2} + \varepsilon\right)$$

$$+ \varepsilon\frac{b-\varepsilon}{2}\left(\frac{b-\varepsilon}{2} + \varepsilon\right) + \left(X\varepsilon - \frac{b-\varepsilon}{2}\right)\left(\frac{b-\varepsilon}{2} + \varepsilon\right)\varepsilon.$$

In order to conveniently write the energy functional we introduce the notation

$$x := \varepsilon X, \quad y := \varepsilon Y$$

and we rewrite the area constraint as $y = \frac{c\varepsilon - \varepsilon d + 2xb}{2a - 4x}$. We are now in position to write the energy (3.24) as a function of x and d. We still need to distinguish the two previous cases.

Case $x < \frac{b}{2}$ By (3.25), (3.26), (3.27), (3.28), (3.29), (3.35) the dissipation part of the energy is

$$\mathcal{D}(x, y, d) := \mathcal{D}_\varepsilon(\widetilde{QR}, QR) = \frac{2}{3}x(2x^2 + 3\varepsilon x + \varepsilon^2) + (b - 2x)x(x + \varepsilon) + (a - 2x)\frac{y(y + \varepsilon)}{2}$$
$$+ \frac{d(y + \varepsilon)y}{2} + \frac{y(y - \varepsilon)(c - d)}{2} + \varepsilon y(y + \varepsilon) + \frac{y(y + \varepsilon)}{2}(a - 2x - c - 2\varepsilon).$$

The energy functional becomes

$$E_\varepsilon(x, y, d) = \mathcal{D}(x, y, d) + \varepsilon\alpha(-4x + 4y) + R(d) = a\varepsilon y + ay^2$$
$$- 4\alpha\varepsilon x + 4\varepsilon\alpha y + b\varepsilon x + bx^2 - c\varepsilon y + d\varepsilon y + \frac{2\varepsilon^2 x}{3} - 2\varepsilon xy - \frac{2}{3}x^3 - 2xy^2,$$

where $R(d) := 2a + 2b + 2\varepsilon\chi_{(0,+\infty)}(d)$. Now by the area constraint $y = y(x) = \frac{c\varepsilon - \varepsilon d + 2xb}{2a - 4x}$ we obtain that

$$E_\varepsilon(x, y(x), d) := \frac{1}{12a - 24x}\big(16x^4 + (-24b - 8a)x^3$$
$$+ (-16\varepsilon^2 + (96\alpha - 48b)\varepsilon + 12b^2 + 12ab)x^2$$
$$+ ((12d - 12c + 8a)\varepsilon^2 + ((24a + 48A)b - 48\alpha a)\varepsilon)x$$
$$+ (-3d^2 + (6c - 6a - 24\alpha)d - 3c^2 + (6a + 24\alpha)c)\varepsilon^2\big).$$

In what follows we are going to find the minimizers of our energy above, originally defined in the discrete setting, analyzing its extension in the continuum, that is for x and d real variables. We have that

(3.43)
$$\frac{\partial E_\varepsilon(x, y(x), d)}{\partial x} = \frac{-1}{(24x^2 - 24ax + 6a^2)}\big(48x^4 + (-48b - 48a)x^3$$
$$+ (-16\varepsilon^2 + (96\alpha - 48b)\varepsilon + 12b^2 + 48ab + 12a^2)x^2$$
$$+ (16a\varepsilon^2 + (48ab - 96\alpha a)\varepsilon - 12ab^2 - 12a^2b)x$$
$$+ (3d^2 + (24\alpha - 6c)d + 3c^2 - 2\alpha c - 4a^2)\varepsilon^2 + ((-12a^2 - 24\alpha a)b + 24\alpha a^2)\varepsilon\big).$$

Case $x \geq \frac{b}{2}$ Also in this step we consider the energy as defined on real numbers. We need to distinguish two sub-cases depending wether $\frac{b}{\varepsilon} \equiv_2 0$ or not. If $\frac{b}{\varepsilon} \equiv_2 0$, thanks to (3.25), (3.26), (3.27), (3.28), (3.29), (3.41), we have that

(3.44)
$$E_\varepsilon(x, y(x), d) = \frac{1}{12a - 24x}\big(\varepsilon x^2(48b - 96\alpha)$$
$$+ x((12c - 12d + 8b)\varepsilon^2 + ((-24a - 48\alpha)b + 48\alpha a)\varepsilon - 2b^3 - 6ab^2)$$
$$+ \varepsilon^2(3d^2 + (-6c + 6a + 24\alpha)d + 3c^2 + c(-6a - 24\alpha) - 4ab) + ab^3\big)$$

and

(3.45)
$$\frac{\partial E_\varepsilon(x, y(x), d)}{\partial x} = \frac{1}{8x^2 - 8ax + 2a^2}\big((16b - 32\alpha)\varepsilon x^2 + (32\alpha a - 16ab)\varepsilon x$$
$$+ (-d^2 + (2c - 8\alpha)d - c^2 + 8\alpha c)\varepsilon^2 + ((4a^2 + 8\alpha a)b - 8\alpha a^2)\varepsilon + a^2b^2\big).$$

If instead we are in the case $\frac{b}{\varepsilon} \equiv_2 1$ we can use (3.25), (3.26), (3.27), (3.28), (3.29), (3.42) to write

(3.46)

$$
\begin{aligned}
E_\varepsilon(x, y(x), d) =& \frac{1}{24x - 12a}\Big(x^2(12\varepsilon^2 + (48b - 96\alpha)\varepsilon) \\
& + x((-12d + 12c + 2b - 6a)\varepsilon^2 + ((-24a - 48\alpha)b + 48\alpha a)\varepsilon - 2b^3 + -6ab^2) \\
& + \varepsilon^2(3d^2 + (-6c + 6a + 24\alpha)d + 3c^2 + (-6a - 24\alpha)c - ab) + ab^3\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial E_\varepsilon(x, y(x), d)}{\partial x} =& \frac{1}{8x^2 - 8ax + 2a^2}\Big(x^2(4\varepsilon^2 + (16b - 32\alpha)\varepsilon) \\
& + (-4\varepsilon^2 + (32\alpha a - 16ab)\varepsilon)x + (-d^2 + (2c - 8\alpha)d - c^2 + 8\alpha c + a^2)\varepsilon^2 \\
& + ((4a^2 + 8\alpha a)b - 8\alpha a^2)\varepsilon + a^2 b^2\Big).
\end{aligned}
$$

Step 2)
In this step we compute the minimzers and the minimal values of the function E_ε under the area constraint assuming that x and d are real variables. We distinguish two cases.

Case $x \in [\frac{b}{2}, \frac{a}{2})$ In this case we have to consider the two energies in (3.44) and (3.46). We claim that there exists $\varepsilon_1 := \varepsilon_1(\Lambda, \alpha)$ such that for all $\varepsilon < \varepsilon_1$

$$
\frac{\partial E_\varepsilon(x, y(x), d)}{\partial x} \geq 0
$$

for all $0 \leq d < a$ fixed. We detail the argument only in the case of formula (3.45), the other case being analogous. To this end we observe that the denominator of (3.45) is positive. Hence the claim is proved provided one can show the positivity of the numerator in (3.45). The latter follows by the following chain of inequalities

$$
\begin{aligned}
(16b - 32\alpha)\varepsilon x^2 + (32\alpha a - 16ab)\varepsilon x &+ (-d^2 + (2c - 8\alpha)d - c^2 + 8\alpha c)\varepsilon^2 \\
+ ((4a^2 + 8\alpha a)b - 8\alpha a^2)\varepsilon + a^2 b^2 &> \frac{a^2 b^2}{2} = \frac{1 + \varepsilon^2 c^2 - 2\varepsilon c}{2} > \frac{1}{4} > 0
\end{aligned}
$$

which holds for $\varepsilon < \varepsilon_1 := (\Lambda + \alpha)^{-k_1}$ with $k_1 \in \mathbb{N}$ sufficiently large and uses the assumption $ab + \varepsilon c = 1$. By the previous argument the only minimizer of $x \to E_\varepsilon(x, y(x), d)$ for $x \in [\frac{b}{2}, \frac{a}{2})$ and $a > d \geq 0$ fixed is $x = \frac{b}{2}$, where the energy takes the value

(3.47)
$$
E_\varepsilon\left(\frac{b}{2}, y\left(\frac{b}{2}\right), d\right) = \frac{ab^3 + b^4 + O(\varepsilon)}{12(a - b)} > 0
$$

for all $\varepsilon < \varepsilon_1$. The latter estimate can be extended to the function $(x, d) \to E_\varepsilon(x, y(x), d)$ thanks to the following observation. In formulas (3.44), (3.46) the quantities d and d^2 have ε^2 as a prefactor. Therefore it is possible to write $E_\varepsilon(x, y(x), d) = d\varepsilon^2 g(x) + d^2\varepsilon^2 h(x) + k(x)$ with g, h, k being rational functions of x. Note that the functions g and h have coefficients depending on a, b, c only, while the function k has coefficients depending on a, b, c and ε. As a result there exists $\varepsilon_2 := \varepsilon_2(\Lambda, \alpha) < \varepsilon_1$ such that for all $\varepsilon < \varepsilon_2$ the minimal value of the energy $(x, d) \to E_\varepsilon(x, y(x), d)$ can be written as $E_\varepsilon\left(\frac{b}{2}, y\left(\frac{b}{2}\right), d\right) = \frac{ab^3 + b^4 + O(\varepsilon)}{12(a-b)}$.

Case $x \in [0, \frac{b}{2})$ We again look for stationary points of the energy $x \to E_\varepsilon(x, y(x), d)$ for fixed $0 \leq d < a$. To this end we investigate the zeros of the numerator of (3.43) that we

denote by

$$
\begin{aligned}
f(\varepsilon, x) := - & \left(48x^4 + (-48b - 48a)x^3\right. \\
& + (-16\varepsilon^2 + (96\alpha - 48b)\varepsilon + 12b^2 + 48ab + 12a^2)x^2 \\
& + (16a\varepsilon^2 + (48ab - 96\alpha a)\varepsilon - 12ab^2 - 12a^2 b)x \\
& + (3d^2 + (24\alpha - 6c)d + 3c^2 - 2\alpha c - 4a^2)\varepsilon^2 + ((-12a^2 - 24\alpha a)b + 24\alpha a^2)\varepsilon\big).
\end{aligned}
$$

We observe that there exists $C(\Lambda, \alpha) > 0$ such that

$$
\max_{x \in [0, \frac{b}{2}]} |f(\varepsilon, x) - f(0, x)| \le \varepsilon C(\Lambda, \alpha) \tag{3.48}
$$

where

$$
f(0, x) = -12x \left(-ab^2 - a^2 b + x(a^2 + 4ab + b^2) + (-4b - 4a)x^2 + 4x^3\right).
$$

Thanks to (3.48) we find out that there exists $\varepsilon_3 := \varepsilon_3(\Lambda, \alpha) < \varepsilon_2$ such that for all $\varepsilon < \varepsilon_3$ the points (ε, x) such that $f(\varepsilon, x) = 0$ lie in a neighbourhood of a zero of $f(0, x)$. For this reason we investigate the roots of the polynomial $f(0, x)$. We have that

$$
f(0, x) = -12xp(x), \ p(x) := -ab^2 - a^2 b + x(a^2 + 4ab + b^2) + (-4b - 4a)x^2 + 4x^3.
$$

We observe that for all $x \in [0, \frac{b}{2}]$

$$
\frac{d}{dx} p(x) = 12x^2 + (-8b + 8a)x + a^2 + 4ab + b^2 > 0.
$$

Therefore, since $p(\frac{b}{2})$ is negative we have that $p(x) < 0$ for all $x \in [0, \frac{b}{2}]$. Hence the only root of $f(0, x)$ in $[0, \frac{b}{2}]$ is $x = 0$ and moreover $f(0, x) \ge 0$ for all $x \in [0, \frac{b}{2}]$. In order to obtain information on the zeros of $f(\varepsilon, x)$ from the zeros of $f(0, x)$ we will make use of the implicit function theorem. We first compute

$$
\begin{aligned}
\frac{\partial f}{\partial x}(\varepsilon, x) = - & 192x^3 + (144b + 144a)x^2 \\
& + (32\varepsilon^2 + (96b - 192\alpha) - 24b^2 - 96ab - 24a^2)x \\
& - 16a\varepsilon^2 + (96\alpha a - 48ab)\varepsilon + 12ab^2 + 12a^2 b.
\end{aligned}
$$

From the expression above we obtain that

$$
\frac{\partial f}{\partial x}(0, 0) = 12(a^2 b + ab^2) > 24 > 0 \tag{3.49}
$$

where we have used that $ab = 1$ and $a + b > 2$. Moreover there exists $C(\Lambda, \alpha) > 0$ such that

$$
\max_{x \in [0, \frac{b}{2}]} \left| \frac{\partial f}{\partial x}(\varepsilon, x) - \frac{\partial f}{\partial x}(0, x) \right| \le \varepsilon C(\Lambda, \alpha). \tag{3.50}
$$

As consequence of (3.49), (3.50) there exists $\varepsilon_4 := \varepsilon_4(\Lambda, \alpha) < \varepsilon_3$ such that for all $\varepsilon < \varepsilon_4$

$$
\frac{\partial f}{\partial x}(\varepsilon, x) > 0 \text{ for all } \varepsilon < \varepsilon_4 \text{ and for all } x \in (-\varepsilon_4, \varepsilon_4).
$$

We are now in a position to apply the implicit function theorem to obtain that for every $\alpha, a, b > 0$ and $a + b < \Lambda$ there exists a smooth function $\bar{x} : (-\frac{\varepsilon_4}{2}, \frac{\varepsilon_4}{2}) \to \mathbb{R}$ such that $f(\varepsilon, \bar{x}(\varepsilon)) = 0$ and $\bar{x}(0) = 0$. As a consequence we have that for all $\varepsilon < \varepsilon_4$

$$
f(\varepsilon, x) < 0 \text{ for all } x \in (0, \bar{x}(\varepsilon)) \text{ and } f(\varepsilon, x) > 0 \text{ for all } x \in \left(\bar{x}(\varepsilon), \frac{b}{2}\right).
$$

Hence we obtain that the minimizer of $x \to E_\varepsilon(x, y(x), d)$ is either $\bar{x}(\varepsilon)$ or $\frac{b}{2}$. Now we want to compare the energy E_ε at the points $(\bar{x}(\varepsilon), y(\bar{x}(\varepsilon), d)$ and $(\frac{b}{2}, y(\frac{b}{2}), d)$. To obtain a uniform estimate depending only on Λ and α we first need to investigate the function

$\varepsilon \to \bar{x}(\varepsilon)$ in a neighborhood of the origin. To this end we compute the first coefficient of its Maclaurin expansion. We can write that $\bar{x}(\varepsilon) = x'\varepsilon + o(\varepsilon)$ where x' is obtained observing that, by (3.43),

$$\frac{\partial E_\varepsilon(\bar{x}(\varepsilon), y(\bar{x}(\varepsilon)), d)}{\partial x} = 0 \iff o(\varepsilon) + x'\varepsilon(12ab^2 + 12a^2b) + \varepsilon((12a^2 + 24\alpha a)b - 24\alpha a^2) = 0,$$

hence

$$(3.51) \qquad x' = \frac{2\alpha(a-b)}{b(a+b)} - \frac{a}{a+b} = \lim_{\varepsilon \to 0^+} \frac{2\alpha(a-b)}{b(a+b)} - \frac{a}{a+b} + \frac{o(\varepsilon)}{\varepsilon(12ab^2 + 12a^2b)}.$$

From the formula above we have that $|x'| \le C(\Lambda, \alpha)$. Computing the first and the second derivative of $\varepsilon \to \bar{x}(\varepsilon)$ we have that there exists $\varepsilon_5 := \varepsilon_5(\Lambda, \alpha) < \varepsilon_4$ such that for all $\varepsilon < \varepsilon_5$

$$\left| \frac{d}{d\varepsilon} \bar{x}(\varepsilon) \right| = \left| \frac{\frac{\partial f}{\partial \varepsilon}(\varepsilon, \bar{x}(\varepsilon))}{\frac{\partial f}{\partial x}(\varepsilon, \bar{x}(\varepsilon))} \right| \le C(\Lambda, \alpha)$$

and

$$\left| \frac{d^2}{d\varepsilon^2} \bar{x}(\varepsilon) \right| = \left| \frac{-\frac{d}{d\varepsilon}\bar{x}(\varepsilon)\frac{\partial^2 f}{\partial x^2}(\varepsilon, \bar{x}(\varepsilon)) + \frac{\partial^2 f}{\partial \varepsilon^2}(\varepsilon, \bar{x}(\varepsilon))}{\frac{\partial f}{\partial x}(\varepsilon, \bar{x}(\varepsilon))} \right| \le C(\Lambda, \alpha)$$

where we have used the formula (3.50) to say that $\left| \frac{\partial f}{\partial x}(\varepsilon, \bar{x}(\varepsilon)) \right| > C(\Lambda, \alpha)$. As a result we eventually have that $\bar{x}(\varepsilon) = x'\varepsilon + O(\varepsilon)$. We now check that for ε small enough the only minimizer of $E_\varepsilon(x, y(x), d)$ is $\bar{x}(\varepsilon)$ (and not $\frac{b}{2}$). Indeed, thanks to (3.47), we have that there exists $\varepsilon_6 := \varepsilon_6(\Lambda, \alpha) < \varepsilon_5$ such that for all $\varepsilon < \varepsilon_6$

$$E_\varepsilon(\bar{x}(\varepsilon), y(\bar{x}(\varepsilon)), d) = \frac{O(\varepsilon)}{12a + O(\varepsilon)}$$
$$< \frac{ab^3 + b^4}{12(a+b)} < \frac{ab^3 + b^4 + O(\varepsilon)}{12(a-b)} = E_\varepsilon\left(\frac{b}{2}, y\left(\frac{b}{2}\right), d\right).$$

Arguing as at the end of *Case $x \in [\frac{b}{2}, \frac{a}{2})$*, one can find $\varepsilon_7 := \varepsilon_7(\Lambda, \alpha) < \varepsilon_6$ such that for all $\varepsilon < \varepsilon_7$ the minimal value of the energy $(x, d) \to E_\varepsilon(x, y(x), d)$ is $E_\varepsilon(\bar{x}(\varepsilon), y(\bar{x}(\varepsilon)), d) = \frac{O(\varepsilon)}{12a + O(\varepsilon)}$.

Step 3) This last step is devoted to the computation of the minimizers of the energy (3.24) in $\mathcal{SQR}_\varepsilon(QR)$. Thanks to step 1) we know that this is equivalent to minimize the function

$$(X, Y, D) \to E_\varepsilon(\varepsilon X, \varepsilon Y, \varepsilon D)$$

where $X, Y, D \in \mathbb{N}$. By Step 2) we have that

$$E_\varepsilon(\bar{x}(\varepsilon), y(\bar{x}(\varepsilon)), d) \le E_\varepsilon\left(\varepsilon X, \frac{c\varepsilon - d\varepsilon + 2\varepsilon b}{2a - 4\varepsilon X}, d\right)$$

for all $X \in \mathbb{N}$ such that $\varepsilon X \in (0, a)$, $\varepsilon Y = \frac{c\varepsilon - d\varepsilon + 2\varepsilon b}{2a - 4\varepsilon X}$ with $Y \in \mathbb{N}$ and $d = \varepsilon D \le a$ with $D \in \mathbb{N}$. In step 2) we have proved that the function $x \to E_\varepsilon(x, y(x), d)$ is monotone decreasing in $(0, \bar{x}(\varepsilon))$ and monotone increasing in $(\bar{x}(\varepsilon), a)$. Hence the only possible values of $X \in \mathbb{N}$ that can minimize E_ε are $\lfloor x' \rfloor, \lceil x' \rceil$, where x' is defined in (3.51). For such given X we can exploit the definition of quasi rectangle to determine the values of Y and d which minimize the energy. We recall that, given a quasirectangle $QR = R \cup Q$, the horizontal sidelength of Q is the remainder in the Euclidean division of $|QR|/\varepsilon^2$ (the number of lattice points in QR) by the number of lattice points in the horizontal sidelength of R. In our case this description corresponds to consider as optimal $Y \in \mathbb{N}$ associated to a minimal X,

those values for which the remainder $d = \frac{1}{\varepsilon}\left[c\varepsilon + 2X\varepsilon b - \varepsilon Y(2a - 4\varepsilon X)\right]$ takes the minimal value, namely

$$(3.52) \qquad Y_1 \in \arg\min\left\{\frac{1}{\varepsilon}\left(c\varepsilon + 2\lfloor x'\rfloor\varepsilon b - \varepsilon Y(2a - 4\varepsilon\lfloor x'\rfloor)\right), \varepsilon Y \in (0, +\infty)\right\}$$

and

$$(3.53) \qquad Y_2 \in \arg\min\left\{\frac{1}{\varepsilon}\left(c\varepsilon + 2\lceil x'\rceil\varepsilon b - \varepsilon Y(2a - 4\varepsilon\lceil x'\rceil)\right), \varepsilon Y \in (0, +\infty)\right\}.$$

We denote the candidate minimzers as

$$(\lfloor x'\rfloor, Y_1, D_1), (\lceil x'\rceil, Y_2, D_2) \in \mathbb{N}^3$$

where $d_1 = \varepsilon D_1$ is the minimum value of (3.52) and $d_2 = \varepsilon D_2$ is the minimum value of (3.53). By (3.52) and (3.53) we can also write

$$Y_1 = \left\lfloor \frac{c + 2b\lfloor x'\rfloor}{2a - 4\varepsilon\lfloor x'\rfloor} \right\rfloor, Y_2 = \left\lfloor \frac{c + 2b\lceil x'\rceil}{2a - 4\varepsilon\lceil x'\rceil} \right\rfloor.$$

Exploiting again the area constraint we can also write for $i = 1, 2$

$$D_i = \frac{1}{\varepsilon^2}\left[C\varepsilon^2 + 2X_i\varepsilon^2 B - \varepsilon Y_i(2\varepsilon A - 4\varepsilon X_i)\right] = C + 2X_i B - Y_i(2A - 4X_i)$$

where $X_1 := \lfloor x'\rfloor$ and $X_2 := \lceil x'\rceil$, $c = C\varepsilon$, $a = \varepsilon A$ and $b = \varepsilon B$ with $A, B, C \in \mathbb{N}$ as in the assumptions of the theorem. We recall that in Step 3) we have proved that $|x'| \le C(\Lambda, \alpha)$. As a result we have that $Y_i \le C(\Lambda, \alpha)$ for $i = 1, 2$. Finally we obtain that the minimizer $QR' = R' \cup Q'$ is such that $R' = \left[-\frac{a'}{2}, \frac{a'}{2}\right] \times \left[-\frac{b'}{2}, \frac{b'}{2}\right] + \mathrm{Bar}(R)$ and

$a' = a - 2X_i\varepsilon$ with $X_1 = \lfloor x'\rfloor$ or $X_2 = \lceil x'\rceil$ and $x' = \dfrac{2\alpha(-b + a)}{b(a + b)} - \dfrac{a}{a + b}$,

$b' = b + 2Y_i\varepsilon$ with $Y_1 = \left\lfloor \dfrac{c + 2b\lfloor x'\rfloor}{2a - 4\varepsilon\lfloor x'\rfloor} \right\rfloor$ or $Y_2 = \left\lfloor \dfrac{c + 2b\lceil x'\rceil}{2a - 4\varepsilon\lceil x'\rceil} \right\rfloor$,

the horizontal sidelength of Q' is $\varepsilon D_i = \varepsilon(C + 2X_i B + Y_i(2A - 4X_i))$ for $i = 1, 2$,

$|a' - a| = |2X_i\varepsilon| \le \varepsilon C(\Lambda, \alpha)$, $|b' - b| = |2Y_i\varepsilon| \le \varepsilon C(\Lambda, \alpha)$,

$a' + b' \le a + b$, $|QR\Delta QR'| \le 2bX_i\varepsilon + 2(a - 2X_i\varepsilon)Y_i\varepsilon \le \varepsilon C(\Lambda, \alpha)$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For all $QR, \widetilde{QR} \in \mathcal{QR}_\varepsilon$ we define the energy

$$\mathcal{F}_\varepsilon(\widetilde{QR}, QR) := \mathrm{P}(\widetilde{QR}) + \frac{1}{\alpha\varepsilon}D_\varepsilon(\widetilde{QR}, QR).$$

Definition 3.12. Let $\{QR_\varepsilon\}_{\varepsilon \in (0,1)} \subset \mathcal{QR}_\varepsilon$ have pseudo-axial symmetry, $|QR_\varepsilon| = 1$ and such that $QR_\varepsilon \to \left[-\frac{a}{2}, \frac{a}{2}\right] \times \left[-\frac{b}{2}, \frac{b}{2}\right]$ as $\varepsilon \to 0$ with respect to the Hausdorff distance. For all $\varepsilon \in (0, 1)$ let $\{QR_k^\varepsilon\}_{k \in \mathbb{N}}$ be a family of sets defined iteratively as

$$QR_0^\varepsilon = QR_\varepsilon \quad \text{and} \quad QR_k^\varepsilon \in \arg\min_{QR \in \mathcal{SQR}_\varepsilon(QR_{k-1}^\varepsilon)}\left\{\mathcal{F}_\varepsilon(QR, QR_{k-1}^\varepsilon)\right\} \quad k \ge 1.$$

We define

$$QR_t^\varepsilon := QR_k^\varepsilon \quad \text{for any } t \in [k\alpha\varepsilon, (k+1)\alpha\varepsilon).$$

For all ε we call $\{QR_t^\varepsilon\}_{t \ge 0}$ a symmetric approximate flat solution of the area-preserving mean-curvature flow in the lattice $\varepsilon\mathbb{Z}^2$ with initial datum QR_ε.

Theorem 3.13. *Let R be a rectangle with horizontal sidelength a_0 and vertical sidelength b_0 and such that $a_0 b_0 = 1$ and $b_0 < a_0$ and $a_0 + b_0 < \Lambda$. For any $0 < \varepsilon < \varepsilon_0$ (where ε_0 is the number obtained in the Theorem 3.11 for $QR = QR_\varepsilon$), let $\{QR_t^\varepsilon\}_{t \geq 0}$ be a symmetric approximate flat solution to the area-preserving mean-curvature flow in the lattice $\varepsilon \mathbb{Z}^2$ with initial datum QR_ε defined according to Definition (3.12). Then for all $k \geq 0$ we have $QR_t^\varepsilon = R_t^\varepsilon \cup Q_t^\varepsilon$ where $R_t^\varepsilon = \left[-\frac{a^\varepsilon(k)}{2}, \frac{a^\varepsilon(k)}{2} \right] \times \left[-\frac{b^\varepsilon(k)}{2}, \frac{b^\varepsilon(k)}{2} \right] + \mathrm{Bar}(R^\varepsilon)$ for all $t \in [k\alpha\varepsilon, (k+1)\alpha\varepsilon)$. Here $a^\varepsilon(k)$ and $b^\varepsilon(k)$ and are obtained by recurrence starting from $a^\varepsilon(0), b^\varepsilon(0)$ being the horizontal sidelength and the vertical sidelength of R_ε, respectively. We denote by $c^\varepsilon(k)$ the horizontal sidelength of Q_t^ε for all $t \in [k\alpha\varepsilon, (k+1)\alpha\varepsilon)$. Setting $c^\varepsilon(0)$ to be the horizontal sidelength of Q_ε it holds true that, for all $k \geq 0$*

$$
\begin{aligned}
&X_1^\varepsilon(k) = \lfloor x^\varepsilon(k) \rfloor, \ X_2^\varepsilon(k) = \lceil x^\varepsilon(k) \rceil \\
&x^\varepsilon(k) = \frac{2\alpha(-b^\varepsilon(k) + a^\varepsilon(k))}{b^\varepsilon(k)(a^\varepsilon(k) + b^\varepsilon(k))} - \frac{a^\varepsilon(k)}{a^\varepsilon(k) + b^\varepsilon(k)}, \\
&Y_1^\varepsilon(k) = \left\lfloor \frac{c^\varepsilon(k) + 2b^\varepsilon(k)\lfloor x^\varepsilon(k) \rfloor}{2a^\varepsilon(k) - 4\varepsilon\lfloor x^\varepsilon(k) \rfloor} \right\rfloor, Y_2^\varepsilon(k) = \left\lfloor \frac{c^\varepsilon(k) + 2b^\varepsilon(k)\lceil x^\varepsilon(k) \rceil}{2a^\varepsilon(k) - 4\varepsilon\lceil x^\varepsilon(k) \rceil} \right\rfloor, \\
\text{(3.54)} \quad &a^\varepsilon(k+1) \in \{a^\varepsilon(k) - 2\varepsilon X_1^\varepsilon(k), a^\varepsilon(k) - 2\varepsilon X_2^\varepsilon(k)\}, \\
&b^\varepsilon(k+1) \in \{b^\varepsilon(k) + 2Y_1^\varepsilon(k)\varepsilon, b^\varepsilon(k) + 2Y_2^\varepsilon(k)\varepsilon\}, \\
&a^\varepsilon(k+1) + b^\varepsilon(k+1) \leq a^\varepsilon(k) + b^\varepsilon(k), \ |a^\varepsilon(k+1) - a^\varepsilon(k)| \leq \varepsilon C(\Lambda, \alpha), \\
&|b^\varepsilon(k+1) - b^\varepsilon(k)| \leq \varepsilon C(\Lambda, \alpha), \ |QR_k^\varepsilon \Delta QR_{k+1}^\varepsilon| \leq \varepsilon C(\Lambda, \alpha), \ c^\varepsilon(k) \leq \Lambda \\
&c^\varepsilon(k+1) \in \{c^\varepsilon(k) + 2X_i^\varepsilon(k)b^\varepsilon(k) - Y_i^\varepsilon(k)(2a^\varepsilon(k) - 4X_i^\varepsilon(k)\varepsilon) \quad \text{for } i = 1, 2\}.
\end{aligned}
$$

Setting $a^\varepsilon(t) = a^\varepsilon(\lfloor \frac{t}{\alpha\varepsilon} \rfloor)$, $b^\varepsilon(t) = b^\varepsilon(\lfloor \frac{t}{\alpha\varepsilon} \rfloor)$ and $c^\varepsilon(t) = c^\varepsilon(\lfloor \frac{t}{\alpha\varepsilon} \rfloor)$ we have that QR_t^ε converges locally uniformly in time as $\varepsilon \to 0$ to a rectangle $R(t) = \left[-\frac{a(t)}{2}, \frac{a(t)}{2} \right] \times \left[-\frac{b(t)}{2}, \frac{b(t)}{2} \right]$ with $a(t)b(t) = 1$.

Proof. Formula (3.54) is a consequence of Theorem 3.11 applyed iteratively to the sets $QR = QR_k^\varepsilon$. By Theorem 3.11 we have that the function $a^\varepsilon(t) := a^\varepsilon(\lfloor \frac{t}{\varepsilon} \rfloor)$ and $b^\varepsilon(t) = b^\varepsilon(\lfloor \frac{t}{\varepsilon} \rfloor)$ satisfy

$$
|a^\varepsilon(t) - a^\varepsilon(s)| \leq C(\Lambda, \alpha)|t - s + \varepsilon|, \quad |b^\varepsilon(t) - b^\varepsilon(s)| \leq C(\Lambda, \alpha)|t - s + \varepsilon|
$$

for all $t, s > 0$. Therefore there exist Lipschitz functions a and b obtained as locally uniform limit of a^ε and b^ε. The fact that the limit rectangle has unitary area follows by passing to the limit as $\varepsilon \to 0$ in the area constraint formula

$$
a^\varepsilon(t)b^\varepsilon(t) + \varepsilon c^\varepsilon(t) = 1.
$$

$\square$

Remark 3.14. It is not clear if one can provide a differential inclusion for the motion of the limit rectangle without assuming additional assumptions on the convergence of the remainders $t \mapsto c^\varepsilon(t)$. If one assumes the existence of the limit $c(t) := \lim_{\varepsilon \to 0} c^\varepsilon(t)$ locally uniformly in time (according to the Ascoli-Arzelá theorem this happens if c^ε has a ε-uniform modulo of continuity on all compact subsets of $[0, +\infty)$), passing to the limit in (3.54) as $\varepsilon \to 0$ one can prove that the following system of differential inclusions holds true. We set $x(t) = \frac{2\alpha}{b(t)} - \frac{4\alpha + a(t)}{a(t) + b(t)}$. If $x(t) \notin \mathbb{N}$ then

$$
\begin{cases}
\dfrac{da}{dt} \in -\dfrac{2}{\alpha} \left[\lfloor x(t) \rfloor \lceil x(t) \rceil \right], \\[4mm]
\dfrac{db}{dt} \in \dfrac{c(t)}{\alpha \, a(t)} + \dfrac{2b(t)}{\alpha \, a(t)} \left[\lfloor x(t) \rfloor \lceil x(t) \rceil \right],
\end{cases}
$$

If $x(t) \in \mathbb{N}$ then

$$
\begin{cases}
\dfrac{da}{dt} \in -\dfrac{2}{\alpha} \left[x(t) + 1, x(t) - 1 \right], \\[3mm]
\dfrac{db}{dt} \in \dfrac{c(t)}{\alpha\, a(t)} + \dfrac{2b(t)}{\alpha\, a(t)} \left[x(t) - 1, x(t) + 1 \right],
\end{cases}
$$

In what follows we overcome the existence problem pointed out in the previous remark introducing a different notion of approximate flat solution, that we call rectangle approximate flat solution. The algorithm we propose does not keep the area fixed at each time step, but still produces an area-preserving flow in the limit as $\varepsilon \to 0$.

Definition 3.15. Let $\{R^\varepsilon\}_{\varepsilon \in (0,1)}$ by a family of rectangles with $|R^\varepsilon| = 1$ such that $R^\varepsilon \to \left[-\frac{a}{2}, \frac{a}{2} \right] \times \left[-\frac{b}{2}, \frac{b}{2} \right]$ as $\varepsilon \to 0$ with respect to the Hausdorff distance. For all $\varepsilon \in (0, 1)$ let $\{R_k^\varepsilon\}_{k \in \mathbb{N}}$ be a family of rectangles defined iteratively according to the following scheme. Set $QR_0^\varepsilon = R^\varepsilon$ and define

$$(3.55) \qquad R_k^\varepsilon : \quad Q_k^\varepsilon \cup R_k^\varepsilon = QR_k^\varepsilon \in \mathop{\arg\min}_{QR \in \mathcal{SQR}_\varepsilon(QR_{k-1}^\varepsilon)} \left\{ \mathcal{F}_\varepsilon(QR, QR_{k-1}^\varepsilon) \right\} \quad k \geq 1.$$

We finally set

$$R_t^\varepsilon := R_k^\varepsilon \quad \text{for any } t \in [k\alpha\varepsilon, (k+1)\alpha\varepsilon)$$

and for all ε we call $\{R_t^\varepsilon\}_{t \geq 0}$ a rectangular approximate flat solution of the area-preserving mean-curvature flow in the lattice $\varepsilon\mathbb{Z}^2$ with initial datum R_ε.

Theorem 3.16. *Let R be a rectangle with horizontal sidelenght a_0 and vertical sidelenght b_0 and such that $a_0 b_0 = 1$ and $b_0 < a_0$ and $a_0 + b_0 < \Lambda$. For any $0 < \varepsilon < \varepsilon_0$ (ε_0 as in Theorem 3.11 for $QR = QR_\varepsilon$), let $\{R_t^\varepsilon\}_{t \geq 0}$ be a rectangular approximate flat solution to the area-preserving mean-curvature flow in the lattice $\varepsilon\mathbb{Z}^2$ with initial datum R_ε defined according to Definition (3.15). Then for all $k \geq 0$ we have $R_t^\varepsilon = \left[-\frac{a^\varepsilon(k)}{2}, \frac{a^\varepsilon(k)}{2} \right] \times \left[-\frac{b^\varepsilon(k)}{2}, \frac{b^\varepsilon(k)}{2} \right] + \mathrm{Bar}(R^\varepsilon)$ for all $t \in [k\alpha\varepsilon, (k+1)\alpha\varepsilon)$. Here $a^\varepsilon(k)$ and $b^\varepsilon(k)$ are obtained by recurrence starting from $a^\varepsilon(0), b^\varepsilon(0)$ being the horizontal and the vertical sidelenght of R_ε, respectively. For all $k \geq 0$ it holds*

$$
\begin{aligned}
& X_1^\varepsilon(k) = \lfloor x^\varepsilon(k) \rfloor, \ X_2^\varepsilon(k) = \lceil x^\varepsilon(k) \rceil \\[1mm]
& x^\varepsilon(k) = \frac{2\alpha(-b^\varepsilon(k) + a^\varepsilon(k))}{b^\varepsilon(k)(a^\varepsilon(k) + b^\varepsilon(k))} - \frac{a^\varepsilon(k)}{a^\varepsilon(k) + b^\varepsilon(k)}, \\[1mm]
(3.56) \quad & Y_1^\varepsilon(k) = \left\lfloor \frac{2b^\varepsilon(k)\lfloor x^\varepsilon(k) \rfloor}{2a^\varepsilon(k) - 4\varepsilon\lfloor x^\varepsilon(k) \rfloor} \right\rfloor, Y_2^\varepsilon(k) = \left\lfloor \frac{2b^\varepsilon(k)\lceil x^\varepsilon(k) \rceil}{2a^\varepsilon(k) - 4\varepsilon\lceil x^\varepsilon(k) \rceil} \right\rfloor, \\[1mm]
& a^\varepsilon(k+1) \in \{ a^\varepsilon(k) - 2\varepsilon X_1^\varepsilon(k), a^\varepsilon(k) - 2\varepsilon X_2^\varepsilon(k) \}, \\[1mm]
& b^\varepsilon(k+1) \in \{ b^\varepsilon(k) + 2Y_1^\varepsilon(k)\varepsilon, b^\varepsilon(k) + 2Y_2^\varepsilon(k)\varepsilon \}, \\[1mm]
& a^\varepsilon(k+1) + b^\varepsilon(k+1) \leq a^\varepsilon(k) + b^\varepsilon(k), \ |a^\varepsilon(k+1) - a^\varepsilon(k)| \leq \varepsilon C(\Lambda, \alpha), \\[1mm]
& |b^\varepsilon(k+1) - b^\varepsilon(k)| \leq \varepsilon C(\Lambda, \alpha), \ |QR_k^\varepsilon \Delta QR_{k+1}^\varepsilon| \leq \varepsilon C(\Lambda, \alpha), \ c^\varepsilon(k) \leq \Lambda.
\end{aligned}
$$

Setting $a^\varepsilon(t) = a^\varepsilon(\lfloor \frac{t}{\alpha\varepsilon} \rfloor)$, $b^\varepsilon(t) = b^\varepsilon(\lfloor \frac{t}{\alpha\varepsilon} \rfloor)$ we have that R_t^ε converges locally uniformly in time as $\varepsilon \to 0$ to a rectangle $R(t) = \left[-\frac{a(t)}{2}, \frac{a(t)}{2} \right] \times \left[-\frac{b(t)}{2}, \frac{b(t)}{2} \right]$ with $a(t)b(t) = 1$. Moreover the following system of differential inclusion holds true. Setting $x(t) = \frac{2\alpha}{b(t)} - \frac{4\alpha + a(t)}{a(t) + b(t)}$, if $x(t) \notin \mathbb{N}$ then

$$(3.57) \qquad
\begin{cases}
\dfrac{da}{dt} \in -\dfrac{2}{\alpha} \left[\lfloor x(t) \rfloor \lceil x(t) \rceil \right], \\[3mm]
\dfrac{db}{dt} \in \dfrac{2b(t)}{\alpha\, a(t)} \left[\lfloor x(t) \rfloor \lceil x(t) \rceil \right],
\end{cases}
$$

if $x(t) \in \mathbb{N}$ then

$$(3.58) \qquad \begin{cases} \dfrac{da}{dt} \in -\dfrac{2}{\alpha}\left[x(t)+1, x(t)-1\right], \\[3mm] \dfrac{db}{dt} \in \dfrac{2b(t)}{\alpha\,a(t)}\left[x(t)-1, x(t)+1\right], \end{cases}$$

Proof. By the definition 3.15 we have that the formula (3.56) is a consequence of Theorem 3.11 applied iteratively to the sets $QR = QR_k^\varepsilon$. By Theorem 3.11 we have that the function $a^\varepsilon(t) := a^\varepsilon(\lfloor \frac{t}{\varepsilon} \rfloor)$ and $b^\varepsilon(t) = b^\varepsilon(\lfloor \frac{t}{\varepsilon} \rfloor)$ satisfy

$$|a^\varepsilon(t) - a^\varepsilon(s)| \leq C(\Lambda, \alpha)|t - s + \varepsilon|, \quad |b^\varepsilon(t) - b^\varepsilon(s)| \leq C(\Lambda, \alpha)|t - s + \varepsilon|$$

for all $t, s > 0$. Therefore there exist Lipschitz functions a and b obtained as locally uniform limit of a^ε and b^ε. Let $c^\varepsilon(t) := \mathrm{P}_1(Q_k^\varepsilon)$ for all $t \in [k\alpha\varepsilon, (k+1)\alpha\varepsilon)$ where Q_k^ε is define in (3.55) and P_1 is the function of the definition (3.4). The fact that the limit rectangle has unitary area is a consequence of $a^\varepsilon(t)b^\varepsilon(t) + \varepsilon c^\varepsilon(t) = 1$ and then

$$a^\varepsilon(t)b^\varepsilon(t) = \lim_{\varepsilon \to 0} a^\varepsilon(t)b^\varepsilon(t) = \lim_{\varepsilon \to 0} a^\varepsilon(t)b^\varepsilon(t) + \varepsilon c^\varepsilon(t) = 1$$

where we have used that $c^\varepsilon(t) \leq \Lambda$. In the end passing to the limit in (3.56) as $\varepsilon \to 0$ one can prove (3.57) and (3.58). $\qquad\square$

Remark 3.17. It is worth observing that our system of differential inclusions never reduces to a single system of ODEs. As a result, the pinning phenomenon described in [8], according to which the flow is given by the family of sets identically equal to the initial datum, can only be one of the possible motions. This phenomenon is characterized by the following condition on the initial data a and b

$$0 \leq x(0) < 1 \text{ and } a \cdot b = 1 \iff \frac{b^3}{2(1-b^2)} < \alpha.$$

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(Marco Cicalese) Zentrum Mathematik - M7, Technische Universitat München, Boltzmannstrasse 3, 85748 Garching, Germany
Email address, M. Cicalese: `cicalese@ma.tum.de`

(Andrea Kubin) Jyväskylän Yliopisto, Matematiikan ja Tilastotieteen Laitos, Jyväskylä, Finland
Email address, A. Kubin: `andrea.a.kubin@jyu.fi`