EXPANSION OF GREEN'S FUNCTION AND REGULARITY OF ROBIN'S FUNCTION FOR ELLIPTIC OPERATORS IN DIVERGENCE FORM

DAOMIN CAO AND JIE WAN

Abstract. We consider Green's function G_K of an elliptic operator in divergence form $\mathcal{L}_K = -\mathrm{div}(K(x)\nabla)$ on a bounded smooth domain $\Omega \subseteq \mathbb{R}^n (n \geq 2)$ with zero Dirichlet boundary condition, where K is a smooth positively definite matrix-valued function on Ω. We obtain a high-order asymptotic expansion of $G_K(x, y)$, which defines uniquely a regular part $H_K(x, y)$. Moreover, we prove that the associated Robin's function $R_K(x) = H_K(x, x)$ is smooth in Ω, despite the regular part $H_K \notin C^1(\Omega \times \Omega)$ in general.

Keywords: Second-order elliptic operators in divergence form; Green's function; Robin's function.

MSC: Primary 35J08

1. Introduction and main results

Our purpose in this article is to study the expansion of Green's function and regularity of Robin's function for second-order elliptic operators in divergence form in all dimensions. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain with $n \geq 2$. Consider the following elliptic operator:

$$\mathcal{L}_K u = -\mathrm{div}(K(x)\nabla u),$$

where $K = (K_{i,j})_{n \times n}$ is a positively definite matrix satisfying

($\mathcal{K}1$). $K_{i,j} \in C^\infty(\overline{\Omega})$ for $1 \leq i, j \leq n$;
($\mathcal{K}2$). There exist constants $\Lambda_1, \Lambda_2 > 0$ such that

$$\Lambda_1 |\zeta|^2 \leq (K(x)\zeta|\zeta) \leq \Lambda_2 |\zeta|^2, \quad \forall\, x \in \Omega,\ \zeta \in \mathbb{R}^n.$$

Let $G_K(x, y)$ be the Green's function of $\mathcal{L}_K$ associated to zero Dirichlet boundary condition, i.e., for any $y \in \Omega$,

$$\begin{cases} \mathcal{L}_K G_K(x, y) = \delta_y & \text{in } \Omega, \\ G_K(x, y) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here δ_y is the Dirac measure centered at y. Multiplying both sides of the first equation of (1.1) by test functions and integrating on Ω, we get the equivalent characterization of Green's function in integral form, that is, for any $u \in C^2(\Omega) \cap C_0^1(\overline{\Omega})$

$$u(y) = \int_\Omega G_K(x, y)\mathcal{L}_K u(x)dx, \quad \forall y \in \Omega. \tag{1.2}$$

Here $C^2(\Omega)\cap C_0^1(\overline{\Omega})$ is the space of all functions which are twice continuously differentiable and equal 0 on the boundary $\partial\Omega$. The well-posedness of Green's function G_K defined by (1.1) is considered in many articles. When the dimension $n\geq 3$, it is well known that Green's function of $\mathcal{L}_K$ exists and is unique, see [30]. Similar results are obtained in [25] when the dimension $n=2$. More results can be found in [9, 15, 20, 22, 31, 35] for instance.

The asymptotic expansion of Green's function of $\mathcal{L}_K$, meanwhile, has also been widely concerned in recent decades. The understanding of the expansion of Green's function plays an essential role in many fields, especially in the study of concentration phenomena for some fluid mechanics models and semilinear elliptic equations, see, e.g., [3, 4, 5, 12, 18, 33, 36, 38]. For many concentration phenomena, the location of possible singularities of solutions is always determined by the corresponding Green's function, and the solutions are always perturbations of Green's function. A typical example is the construction of concentrated vortex solutions to two dimensional, three dimensional axisymmetric and three dimensional helical symmetric incompressible Euler equations. For planar Euler equations, one always gets concentrated vortex solutions by solving an elliptic problem

$$-\Delta u=\lambda f(u)\ \text{ in }\Omega;\quad u=0\ \text{ on }\partial\Omega,$$

and proving the asymptotic behavior of solutions u_λ as λ tends to infinity, see e.g., [4, 5, 11, 32, 34]. The limiting location is determined by the Kirchhoff-Routh function (see [29]), which is a combination of Green's function of $-\Delta$ and the corresponding Robin's function. For the construction of concentrated vortex solutions to 3D axisymmetric Euler equations (called vortex rings) and 3D Euler equations with helical symmetry, corresponding elliptic operators appeared in elliptic problems are $-\frac{1}{a(x)}\text{div}(a(x)\nabla)$ and $-\text{div}(K_H(x)\nabla)$ for some positive function a and positively definite matrix K_H respectively, see e.g., [6, 7, 12, 14, 18]. When constructing a family of vortex solutions concentrating near several points, one always need to use the asymptotic expansion of the corresponding Green's function for these elliptic operators. Another example is the construction of bubbling solutions to the following anisotropic Emden–Fowler equations in dimension two

$$\text{div}(a(x)\nabla u)+\varepsilon^2 a(x)e^u=0\ \text{ in }\Omega,\quad u=0\ \text{ on }\partial\Omega. \tag{1.3}$$

[38] constructed a family of boundary bubbling solutions by using the asymptotic expansion of Green's function of the operator $-\frac{1}{a(x)}\text{div}(a(x)\nabla)$. See also [37]. Note that both in the construction of vortex solutions of incompressible Euler equations and in the construction of bubbling solutions of the above elliptic equations, the C^1 regularity of the corresponding Robin's function is necessary when constructing concentrated solutions by perturbation arguments, see [5, 8, 37] for example.

The most common case is $K\equiv Id_{n\times n}$, i.e., the identity matrix of order n, and then the operator $\mathcal{L}_K$ is the standard Laplacian. In this case the associated Green's function G_{Id} of $-\Delta$ in a domain Ω with Dirichlet condition is of the decomposition (see [20])

$$G_{Id}(x,y)=\Phi_0(x-y)-H_\Omega(x,y), \tag{1.4}$$

where Φ_0 is the fundamental solution of $-\Delta$ defined by

$$\Phi_0(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, & \text{for } n=2; \\ \frac{1}{n(n-2)w_n}|x|^{2-n}, & \text{for } n \geq 3; \end{cases} \tag{1.5}$$

with w_n the Lebesgue measure of the unit ball in $\mathbb{R}^n$, and H_Ω is the regular part of G_{Id} satisfying for any $y \in \Omega$

$$\begin{cases} -\Delta H_\Omega(x,y) = 0, & x \in \Omega, \\ H_\Omega(x,y) = \Phi_0(x-y), & x \in \partial\Omega. \end{cases} \tag{1.6}$$

Clearly H_Ω is a function determined by Ω. Moreover, H_Ω is smooth and symmetric in $\Omega\times\Omega$, i.e., $H_\Omega(x,y) = H_\Omega(y,x)$. Thus $R_\Omega(x) = H_\Omega(x,x)$, called the Robin's function, is smooth in Ω.

When K is a positively definite constant coefficient matrix K_0, we can also get explicitly the expansion of associated Green's function. Let T_0 be the positively definite matrix satisfying

$$T_0^{-1}T_0^{-t} = K_0,$$

where T_0^{-t} is the transpose of T_0^{-1}. Denote $T_0\Omega = \{T_0 x \mid x \in \Omega\}$. Using the coordinate transformation (see Lemma A.1 in Appendix), we find that Green's function G_{K_0} of $\mathcal{L}_{K_0}$ has the decomposition

$$G_{K_0}(x,y) = \sqrt{\det K_0}^{-1}\Phi_0(T_0(x-y)) - \sqrt{\det K_0}^{-1}H_{T_0\Omega}(T_0x, T_0y) \ \ \forall x,y \in \Omega, \ x \neq y,$$

where $H_{T_0\Omega} : T_0\Omega \times T_0\Omega \to \mathbb{R}$ satisfies for any $y' \in T_0\Omega$

$$\begin{cases} -\Delta H_{T_0\Omega}(x',y') = 0, & x' \in T_0\Omega, \\ H_{T_0\Omega}(x',y') = \Phi_0(x'-y'), & x' \in \partial T_0\Omega. \end{cases}$$

Clearly the corresponding Robin's function $R_{T_0\Omega}(x) = \sqrt{\det K_0}^{-1}H_{T_0\Omega}(T_0x, T_0x)$ is also smooth in Ω.

When K is a matrix-valued function rather than a constant coefficient matrix, the situation turns out to be less clear. If K is a diagonal matrix with the same diagonal elements, i.e., $K(\cdot) = a(\cdot)Id_{n\times n}$ for some positive smooth function a, [26] constructed the expansion of Green's function and proved the smoothness of the corresponding Robin's function. For any $l \in \mathbb{N}$, it was proved in [26] that Green's function G_a of $-\Delta_a = -\frac{1}{a(x)}\mathrm{div}(a(x)\nabla)$ has the high-order expansion

$$G_a(x,y) = \Phi_0(x-y) + \sum_{k=1}^{n+l-2}\Phi_k(x-y) + H_a^l(x,y) \ \text{ in } \overline{\Omega}\times\Omega,$$

where $H_a^l(x,y) \in C^{l,\gamma}(\overline{\Omega}\times\Omega)$, see also [37]. An interesting phenomena is, for any $n \geq 2$ if $\nabla a(y) \neq 0$, then $x \mapsto H_a^0(x,y)$ does not belong to $H^2(\Omega)$ since $-\Delta_a H_a^0(x,y) \notin L^2(\Omega)$. However, $H_a^0(x,x)$ is smooth in Ω, see Proposition 2.7 in [26]. For the case K being a general positively definite matrix-valued function, it follows from [7] that Green's function

G_K of the operator $-\text{div}(K(x)\nabla)$ has the following structure when $n=2$ (see [7], theorem 1.2)

$$G_K(x,y) = \frac{\sqrt{\det K(x)}^{-1} + \sqrt{\det K(y)}^{-1}}{2}\Phi_0\left(\frac{T_x+T_y}{2}(x-y)\right) + S_K(x,y), \tag{1.7}$$

where T_x is a positively definite matrix satisfying $T_x^{-1}(T_x^{-1})^t = K(x)$ for $x\in\Omega$, $S_K \in C^{0,\gamma}(\Omega\times\Omega)$ for all $\gamma\in(0,1)$ and $S_K(x,y)=S_K(y,x)$ for $x,y\in\Omega$. See also [6]. This implies that the regular part S_K of G_K is just $C^{0,\gamma}$, not smooth. Note that (1.7) does not give us a high-order expansion of Green's function $G_K(x,y)$, nor proves whether the corresponding Robin's function $S_K(x,x)$ is smooth. For the dimension $n\geq 3$, such results are also unknown for our knowledge. In this article, we intend to study this aspect. For the cases that dimension $n\geq 2$, we construct a high-order asymptotic expansion of Green's function G_K of (1.1), which permits us to define uniquely a regular part H_K. Then we prove the smoothness of the corresponding Robin's function $R_K(x) = H_K(x,x)$. We find that $R_K\in C^\infty(\Omega)$, although H_K does not belong to $C^1(\Omega\times\Omega)$ in general. Note that the fundamental solution of $-\Delta$ for $n=2$ has a logarithmic term, which is different from that of $n\geq 3$. Therefore, we get results by dealing with the odd and even cases of n separately.

Before stating our main results, let us first introduce some notations. Let $n\geq 2$ and define T_x a C^∞ positively definite matrix-valued function determined by K satisfying

$$T_x^{-1}(T_x^{-1})^t = K(x) \quad \forall x\in\Omega. \tag{1.8}$$

Clearly such T_x exists and is unique (see for example, [21]). For any multi-index $\alpha=(\alpha_i)\in\mathbb{N}^n$ and $x\in\mathbb{R}^n$, we denote $|\alpha| = \sum_{i=1}^n \alpha_i$ and $x^\alpha = \Pi_{i=1}^n x_i^{\alpha_i}$. For any $k\in\mathbb{N}^*, m\in\mathbb{N}$, we define a linear space E_{k+2m}^{n+2m} as follows

Definition 1.1. Given $k\in\mathbb{N}^*, m\in\mathbb{N}$, denote

$$E_{k+2m}^{n+2m} = \text{span}\left\{\frac{x^\alpha}{r^{n+2m}} \mid |\alpha| = k+2m, \alpha\in\mathbb{N}^n\right\},$$

where $r=||x||$ is the classical Euclidean norm of x.

Our first result is on the expansion of Green's function of (1.1) when n is odd.

Theorem 1.2. *Let $n\geq 3$ be odd and $\gamma\in(0,1)$ be an arbitrary constant. Then for any $l\in\mathbb{N}$, there exists a unique $\Phi_i\in E_{i+2+2(2i-1)}^{n+2(2i-1)}$ for $i=1,\cdots,n+l-2$ depending on $y\in\Omega$ and $H^l(x,y) = H_y^l(x)\in C\left(\Omega, C^{l,\gamma}\left(\overline{\Omega}\right)\right)\cap C^l(\Omega\times\Omega)$ such that*

$$G_K(x,y) = \sqrt{\det K(y)}^{-1}\Phi_0\left(T_y(x-y)\right) + \sum_{i=1}^{n+l-2}\Phi_i\left(T_y(x-y)\right) + H^l(x,y) \quad \text{in } \overline{\Omega}\times\Omega.$$

Remark 1.3. Note that from Theorem 1.2, each term in the expansion of Green's function G_K is determined by $\Phi_k\left(T_y(x-y)\right)$, which is different from the term $\Phi_k(x-y)$ in the expansion of Green's function of $-\Delta$ and $-\Delta_a = -\frac{1}{a(x)}\text{div}\left(a(x)\nabla\right)$, see [20, 26]. However, if we choose K to be $Id_{n\times n}$ and $aId_{n\times n}$, the expansion of G_K in Theorem 1.2 coincides with the classical results in [20, 26].

Concerning the cases with even dimension, since the fundamental solution of $-\Delta$ in dimension two is $-\frac{1}{2\pi}\log|x|$, we need to introduce some notations about singular and logarithmic terms. Let $n \in 2\mathbb{N}^*, k \in \mathbb{N}^*, m \in \mathbb{N}$ and E^{n+2m}_{k+2m} be defined as in Definition 1.1. Let $\mathbb{R}[x]$ be the set of real polynomials with variables x_i. When $k \geq n$, we denote by $E^{n+2m,s}_{k+2m}$ the singular set of E^{n+2m}_{k+2m}, i.e., $E^{n+2m,s}_{k+2m} = E^{n+2m}_{k+2m} \backslash \mathbb{R}[x]$. Denote

$$L_m = \operatorname{span}\{x^\alpha \log r \mid |\alpha| = m, \alpha \in \mathbb{N}^n\}.$$

We define F^{n+2m}_{k+2m} as

$$F^{n+2m}_{k+2m} = \begin{cases} E^{n+2m}_{k+2m} & k < n, \\ E^{n+2m,s}_{k+2m} \oplus L_{k-n} & k \geq n. \end{cases}$$

When the dimension n is even we have the following result concerning the expansion of Green's function.

Theorem 1.4. *Let $n \in 2\mathbb{N}^*$ and $\gamma \in (0,1)$ be an arbitrary constant. Then for any $l \in \mathbb{N}$, there exists a unique $\Phi_i \in F^{n+2(2i-1)}_{i+2+2(2i-1)}$ for $i = 1, \cdots, n+l-2$ depending on $y \in \Omega$ and $H^l(x,y) = H^l_y(x) \in C\left(\Omega, C^{l,\gamma}\left(\overline{\Omega}\right)\right) \cap C^l(\Omega \times \Omega)$ such that*

$$G_K(x,y) = \sqrt{\det K(y)}^{-1} \Phi_0\left(T_y(x-y)\right) + \sum_{i=1}^{n+l-2} \Phi_i\left(T_y(x-y)\right) + H^l(x,y) \quad \text{in } \overline{\Omega} \times \Omega.$$

Remark 1.5. Compare with Theorem 1.2, the only difference is $\Phi_i \in F^{n+2(2i-1)}_{i+2+2(2i-1)}$, rather than $E^{n+2(2i-1)}_{i+2+2(2i-1)}$. Note also that when taking $n = 2$ and $l = 0$, Theorem 1.4 yields that

$$G_K(x,y) = \sqrt{\det K(y)}^{-1} \Phi_0(T_y(x-y)) + H^0(x,y),$$

where $H^0(x,y) \in C^{0,\gamma}(\overline{\Omega} \times \Omega)$. This expansion coincides with (1.7) in [7], since in this case $S_K(x,y) = H^0(x,y) + \sqrt{\det K(y)}^{-1} \Phi_0(T_y(x-y)) - \frac{\sqrt{\det K(x)}^{-1} + \sqrt{\det K(y)}^{-1}}{2} \Phi_0\left(\frac{T_x+T_y}{2}(x-y)\right)$ belongs to $C^{0,\gamma}(\Omega \times \Omega)$.

Our last result is on the smoothness of Robin's function, regardless of the parity of n. Let us define the regular part of G_K and the associated Robin's function R_K

$$H_K(x,y) = H^0(x,y) \quad \forall x, y \in \Omega,$$

$$R_K(x) = H_K(x,x) \quad \forall x \in \Omega.$$

It follows from Theorem 1.2 and Theorem 1.4 that $H_K \in C\left(\Omega, C^{0,\gamma}\left(\overline{\Omega}\right)\right) \cap C(\Omega \times \Omega)$. Note that this regularity is optimal since generally $\mathcal{L}_K H_K(\cdot, y) \notin L^2(\Omega)$ and thus $H_K(\cdot, y)$ does not belong to $H^2(\Omega)$ for $n \geq 2$. However, we have

Theorem 1.6. *Let $n \geq 2$ be an integer, Ω be a bounded smooth domain and K be a positively definite smooth matrix-valued function in Ω. Then the Robin's function $R_K(\cdot) \in C^\infty(\Omega)$.*

Remark 1.7. Note that for $n = 2$, it is proved in [7] that Green's function G_K has a structure (1.7), where the regular part S_K belongs to $C^{0,\gamma}$. Since $S_K(x,x) = R_K(x)$ for any $x \in \Omega$, using Theorem 1.6, we actually show that $S_K(x,x)$ is not just $C^{0,\gamma}$, but smooth indeed, which improves the regularity results of the regular part $S_K(x,x)$ in [6, 7].

Remark 1.8. We give a remark about some applications of our results.

Example 1. As mentioned earlier, when $n = 2$ and $K(\cdot) = a(\cdot)Id_{2\times 2}$, [37, 38] constructed bubbling solutions of anisotropic Emden–Fowler equations (1.3) by using Lyapunov-Schmidt reduction method. The key is to use the high-order expansion of Green's function of the operator $-\frac{1}{a(x)}\mathrm{div}(a(x)\nabla)$ and the C^1 regularity of Robin's function. For K in general, it is possible to construct bubbling solutions of the following Emden–Fowler equations in divergence form

$$\mathrm{div}(K(x)\nabla u) + \varepsilon^2 e^u = 0 \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega,$$

by using the expansion of Green's function and the regularity of Robin's function in Theorems 1.4 and 1.6.

Example 2. The study of the Green's functions of the Laplace-Beltrami operators in a complete Riemannian manifold is also a classical problem in geometric analysis and partial differential equations. Let (M_n, g) be a complete Riemannian manifold with metric $g = g_{ij}$ and $(x_1, \cdots, x_n)$ be its local chart. The Laplace-Beltrami operator for a smooth function φ can be written as (using the Einstein notation for repeated indexes)

$$\Delta_g \varphi = \mathrm{div}_g\left(\nabla_g \varphi\right) = \frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}\, g^{ij}\partial_j \varphi\right),$$

which is an elliptic operator in divergence form in local charts. When M is a compact manifold with or without boundary, the existence and lower and upper bounds of Green's function of Δ_g can be found in [1, 23]. When M is a non-compact manifold without boundary, the existence and the number of critical points of Green's function of Δ_g are proved in [16, 28]. However, to our knowledge there are few results showing the expansion of Green's function and the regularity of Robin's function like Theorems 1.2, 1.4 and 1.6. Using our results, it is possible to study the qualitative analysis of bubbling solutions of several kinds of elliptic equations on manifolds. For more results, see, e.g., [2, 10, 13, 17].

Example 3. Another related problem is the expansion of fundamental solutions for a second-order elliptic operator with analytic coefficients, see [19, 24]. For $n \geq 2$, let

$$\mathcal{L} = A_{ij}\partial_{ij} + b_i\partial_i + c,$$

where A_{ij}, b_i, c are analytic in the unit ball $B_1 \subseteq \mathbb{R}^n$ and $A(\cdot)$ is a positive symmetric matrix-valued function in B_1. Given an open set $U \subseteq B_1$ which contains the origin, a function $u \in L^1_{loc}(U)$ is a fundamental solution for the operator $\mathcal{L}$ in U if $\mathcal{L}u = \delta$ in U in the distributional sense, where δ is the Dirac delta distribution. Recently, [19] constructed explicitly an expansion of the fundamental solution of $\mathcal{L}$ in homogeneous terms (homogeneous polynomials divided by a power of $|x|$, plus homogeneous polynomials multiplied by $\log|x|$ if the dimension n is even), which improves the classical results of [24, 27]. Note that the expansion of Green's function in Theorems 1.2 and 1.4 are different from the expansion of fundamental solutions in [19]. It is interesting whether one can use the method in [19] to construct the asymptotic expansion of Green's function of $-\mathrm{div}(K(x)\nabla)$.

The paper is organized as follows. In section 2, we prove the expansion of G_K when n is odd. By similar strategy we prove the even dimensional case in section 3. Based on Theorem 1.2 and 1.4, we finish the proof of Theorem 1.6 at the end of section 3.

2. Odd dimensional case

Now we begin to prove the expansion of Green's function G_K of (1.1) when $n \geq 3$ is odd. For any subset A_j in a vector space V, we denote $A_0 + A_1 + \cdots + A_p = \{\sum_{j=0}^{p} v_j \mid v_j \in A_j\}$. We write $A_0 \oplus A_1 \oplus \cdots \oplus A_p$ when the sum is direct. For any $l \in \mathbb{N}, 2l \leq k + 2m$, we define

$$E_{k+2m,l}^{n+2m} = \mathrm{span}\left\{ \frac{x^\alpha}{r^{n+2m-2l}} \mid |\alpha| = k + 2m - 2l, \alpha \in \mathbb{N}^n \right\}.$$

Note that from Definition 1.1, for any $k \in \mathbb{N}^*$, $E_{k+2m}^{n+2m} \in L^1_{loc}(\mathbb{R}^n)$ and for any $m \in \mathbb{N}$, $i, j \in \mathbb{N}^*, i \neq j$

$$E_{i+2m}^{n+2m} \cap E_{j+2m}^{n+2m} = \{0\}.$$

Moreover, for any $m, i \in \mathbb{N}^*$

$$E_i^n \subsetneqq E_{i+2}^{n+2} \subsetneqq \cdots\cdots \subsetneqq E_{i+2m}^{n+2m}$$

and for any $k \in \mathbb{N}^*, m, l \in \mathbb{N}, 2(l+1) \leq k + 2m$

$$E_{k+2m,l+1}^{n+2m} \subsetneqq E_{k+2m,l}^{n+2m}.$$

Using algebraic construction, we first show that E_{k+2m}^{n+2m} is contained in $\Delta(E_{k+2m+2}^{n+2m})$ for any $k \in \mathbb{N}^*, m \in \mathbb{N}$.

Lemma 2.1. *Let n be odd. Then for any $k \in \mathbb{N}^*, m \in \mathbb{N}$*

$$E_{k+2m}^{n+2m} \subseteq \Delta(E_{k+2m+2}^{n+2m}).$$

That is, for any $f \in E_{k+2m}^{n+2m}$, there exists $g \in E_{k+2m+2}^{n+2m}$ such that $\Delta g = f$.

Proof. Note that $E_{k+2m}^{n+2m} = \sum_{0 \leq 2l \leq k+2m} E_{k+2m,l}^{n+2m}$. Given $f(x) = \frac{x^\alpha}{r^{n+2m-2l}} \in E_{k+2m,l}^{n+2m}$ with $|\alpha| = k + 2m - 2l \geq 0$, let $\bar{g}(x) = r^2 f(x) = \frac{x^\alpha}{r^{n+2m-2(l+1)}} \in E_{k+2m+2,l+1}^{n+2m}$. Then using the fact that $x \cdot \nabla(x^\beta) = |\beta| x^\beta, \forall \beta \in \mathbb{N}^n$, we obtain

$$\Delta \bar{g} = \Delta\left(\frac{x^\alpha}{r^{n+2m-2(l+1)}}\right) = \frac{\Delta x^\alpha}{r^{n+2m-2(l+1)}} + 2(2l + 2 - n - 2m)(|\alpha| + l - m) f.$$

Since n is odd, $2l + 2 - n - 2m \neq 0$. Since $k \geq 1$, $|\alpha| + l - m = \frac{k+|\alpha|}{2} > 0$. Thus $2(2l + 2 - n - 2m)(|\alpha| + l - m) \neq 0$, which implies that

$$\begin{aligned} &\frac{1}{2(2l + 2 - n - 2m)(|\alpha| + l - m)} \Delta \bar{g} - f \\ &= \frac{1}{2(2l + 2 - n - 2m)(|\alpha| + l - m)} \frac{\Delta x^\alpha}{r^{n+2m-2(l+1)}} \in E_{k+2m,l+1}^{n+2m}. \end{aligned} \tag{2.1}$$

From (2.1), we know that if a function $f \in E^{n+2m}_{k+2m,l}$, then $f - \frac{1}{2(2l+2-n-2m)(|\alpha|+l-m)}\Delta\bar{g} \in E^{n+2m}_{k+2m,l+1}$. In the case $|\alpha| \le 1$, we already have $\frac{1}{2(2l+2-n-2m)(|\alpha|+l-m)}\Delta\bar{g} - f = 0$, i.e., $\frac{1}{2(2l+2-n-2m)(|\alpha|+l-m)}\bar{g}$ is a solution. If $|\alpha| > 1$, then we can construct g by decreasing induction on $|\alpha|$, that is, using (2.1) repeatedly to construct the solution g. To be precise, by (2.1) there exist $g_1 \in E^{n+2m}_{k+2m+2,l+1}$ and $f_1 \in E^{n+2m}_{k+2m,l+1}$ such that $f = \Delta g_1 + f_1$. For f_1, using (2.1) again there exist $g_2 \in E^{n+2m}_{k+2m+2,l+2}$ and $f_2 \in E^{n+2m}_{k+2m,l+2}$ such that $f_1 = \Delta g_2 + f_2$. Thus by a recursive method there exist $g_j \in E^{n+2m}_{k+2m+2,l+j}$ and $f_j \in E^{n+2m}_{k+2m,l+j}$ for $j = 2, \cdots, \lfloor \frac{k}{2} \rfloor + m - l$ such that $f_{j-1} = \Delta g_j + f_j$. Note that when $j = \lfloor \frac{k}{2} \rfloor + m - l$, the index of the numerator of $f_{\lfloor \frac{k}{2} \rfloor + m - l}$ is 0 or 1. Thus by (2.1) there exists $g_{\lfloor \frac{k}{2} \rfloor + m - l + 1} \in E^{n+2m}_{k+2m+2, \lfloor \frac{k}{2} \rfloor + m + 1}$ such that $f_{\lfloor \frac{k}{2} \rfloor + m - l} = \Delta g_{\lfloor \frac{k}{2} \rfloor + m - l + 1}$. Let $g = \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor + m - l + 1} g_j$, then g is a solution. Therefore, we prove the existence of $g \in E^{n+2m}_{k+2m+2}$ such that $\Delta g = f$.

□

Remark 2.2. One computes directly that $\Delta(E^{n+2m}_{k+2m+2}) \subseteq E^{n+2m+2}_{k+2m+2}$. Thus the above lemma shows that for any $k \in \mathbb{N}^*, m \in \mathbb{N}$

$$E^{n+2m}_{k+2m} \subseteq \Delta(E^{n+2m}_{k+2m+2}) \subseteq E^{n+2m+2}_{k+2m+2}.$$

Moreover, the function g constructed in Lemma 2.1 is indeed unique. Let $g \in E^{n+2m}_{k+2m+2}$ such that $\Delta g = 0$. Since n is odd, g cannot be in $E^{n+2m}_{l} \backslash \{0\}$ for any $l \in \mathbb{N}^*$. Thus the uniqueness of g holds.

Let $z \in \Omega$ be fixed and T_z be the positively definite matrix defined by (1.8). We define a linear space

$$E^{n+2m}_{k+2m}(T_z) = \{f \circ T_z \mid f \in E^{n+2m}_{k+2m}\}.$$

Clearly, $E^{n+2m}_{i+2m}(T_z) \cap E^{n+2m}_{j+2m}(T_z) = \{0\}$ for any $m \in \mathbb{N}, i, j \in \mathbb{N}^*, i \neq j$. A direct consequence of Lemma 2.1 is

Lemma 2.3. *Let n be odd and $z \in \Omega$ be fixed. Then for any $k \in \mathbb{N}^*, m \in \mathbb{N}$*

$$E^{n+2m}_{k+2m}(T_z) \subseteq -div(K(z)\nabla)(E^{n+2m}_{k+2m+2}(T_z)).$$

That is, for any $f \in E^{n+2m}_{k+2m}(T_z)$, there exists $g \in E^{n+2m}_{k+2m+2}(T_z)$ such that $-div(K(z)\nabla)g = f$.

Proof. For any $f \in E^{n+2m}_{k+2m}(T_z)$, we can find $\hat{f} \in E^{n+2m}_{k+2m}$ such that $f = \hat{f} \circ T_z$. Hence by Lemma 2.1, there exists $\hat{g} \in E^{n+2m}_{k+2m+2}$ such that $-\Delta\hat{g} = \hat{f}$. Let $g = \hat{g} \circ T_z$. One computes directly that $g \in E^{n+2m}_{k+2m+2}(T_z)$ and $-\text{div}(K(z)\nabla)g = f$. □

Using Lemma 2.3, we can solve elliptic equations in divergence form with constant coefficient matrix $K(z)$ when $f \in E^{n+2m}_{k+2m}(T_z)$.

Lemma 2.4. *Let n be odd and $z \in \Omega$ be fixed. Let Ω_0 be a bounded smooth domain in $\mathbb{R}^n$ containing the origin 0 and c be a smooth function defined on $\overline{\Omega}_0$. Then for any $k \in \mathbb{N}^*, m, l \in \mathbb{N}$, $f \in E^{n+2m}_{k+2m}(T_z)$, there exists*

$$g \in E^{n+2m}_{k+2m+2}(T_z) \oplus E^{n+2m}_{k+2m+3}(T_z) \oplus \cdots \oplus E^{n+2m}_{n+2m+l}(T_z) \oplus C^{l,\gamma}(\overline{\Omega}_0) \tag{2.2}$$

such that

$$-div(K(z)\nabla g) = c(x)f \quad in\ \Omega_0; \quad g = 0 \quad on\ \partial\Omega_0.$$

Moreover, the expansion of g is unique.

Remark 2.5. If $k \geq n+l-1$, we just think of $E^{n+2m}_{k+2m+2}(T_z) \oplus E^{n+2m}_{k+2m+3}(T_z) \oplus \cdots \oplus E^{n+2m}_{n+2m+l}(T_z)$ as $\{0\}$.

Proof. For $x \in \mathbb{R}^n$, we denote the operator $[x\cdot\nabla] = \sum_{i=1}^n x_i\partial_i$. It follows from the Taylor's formula that

$$c(x) = \sum_{j=0}^{n+l-k} \frac{1}{j!}[x\cdot\nabla]^j c(0) + \frac{1}{(n+l-k)!}\int_0^1 (1-t)^{n+l-k}[x\cdot\nabla]^{n+l-k+1}c(tx)dt.$$

Thus one has

$$c(x)f \in E^{n+2m}_{k+2m}(T_z) \oplus E^{n+2m}_{k+2m+1}(T_z) \oplus \cdots \oplus E^{n+2m}_{n+2m+l}(T_z) \oplus C^{l,\gamma}(\overline{\Omega}_0).$$

Using Lemma 2.3, we get the existence of $g_1 \in E^{n+2m}_{k+2m+2}(T_z) \oplus E^{n+2m}_{k+2m+3}(T_z) \oplus \cdots \oplus E^{n+2m}_{n+2m+l+2}(T_z) \oplus C^{l+2,\gamma}(\overline{\Omega}_0)$ such that $-\mathrm{div}(K(z)\nabla g_1) = f$. Since the trace of any function in $E^{n+2m}_j(T_z)$ on $\partial\Omega_0$ is smooth, there exists a function $g_2 \in C^\infty(\overline{\Omega}_0)$ such that $-\mathrm{div}(K(z)\nabla g_2) = 0$ in Ω_0 and $g_2 = g_1$ on $\partial\Omega_0$. Note that $E^{n+2m}_{n+2m+l+1}(T_z) \oplus E^{n+2m}_{n+2m+l+2}(T_z) \subseteq C^{l,\gamma}(\overline{\Omega}_0)$, we conclude that $g = g_1 - g_2$ is the desired solution. The uniqueness of the decomposition of g follows directly from the maximum principle and the fact that the sum in (2.2) is direct.

□

Based on Lemma 2.4, we can extend results of Lemma 2.4 to the case of operators $-\mathrm{div}(\tilde{K}(x)\nabla)$, when the coefficient matrix satisfies $\tilde{K}(0) = K(z)$.

Proposition 2.6. *Let n be odd, $z \in \Omega$, Ω_0 and c be as in Lemma 2.4. Let $\tilde{K} \in C^\infty(\overline{\Omega}_0, \mathbb{R}^{n\times n})$ be a positively definite matrix-valued function with $\tilde{K}(0) = K(z)$. Then for any $k \in \mathbb{N}^*, m, l \in \mathbb{N}$, $f \in E^{n+2m}_{k+2m}(T_z)$, there exists*

$$\zeta \in E^{n+2m}_{k+2m+2}(T_z) \oplus E^{n+2m+4}_{k+2m+7}(T_z) \oplus \cdots \oplus E^{n+2m+4(n+l-k-2)}_{n+2m+l+4(n+l-k-2)}(T_z) \oplus C^{l,\gamma}(\overline{\Omega}_0)$$

such that

$$-div(\tilde{K}(x)\nabla\zeta) = c(x)f \quad in\ \Omega_0; \quad \zeta = 0 \quad on\ \partial\Omega_0.$$

Moreover, the expansion of ζ is unique.

Remark 2.7. If $k \geq n+l-1$, we just think of $E^{n+2m}_{k+2m+2}(T_z) \oplus E^{n+2m+4}_{k+2m+7}(T_z) \oplus \cdots \oplus E^{n+2m+4(n+l-k-2)}_{n+2m+l+4(n+l-k-2)}(T_z)$ as $\{0\}$.

Proof. Consider the case $c \equiv 1$. To find a solution ζ, let us first construct a sequence $\{\zeta_j\}_{j=0}^{n+l-k+1}$ by iteration. Let ζ_0 satisfy

$$-\mathrm{div}(\tilde{K}(0)\nabla\zeta_0) = f \quad \text{in } \Omega_0; \quad \zeta_0 = 0 \quad \text{on } \partial\Omega_0. \tag{2.3}$$

It follows from Lemma 2.4 that

$$\zeta_0 \in E^{n+2m}_{k+2m+2}(T_z) \oplus E^{n+2m}_{k+2m+3}(T_z) \oplus \cdots \oplus E^{n+2m}_{n+2m+l+2}(T_z) \oplus C^{l+2,\gamma}(\overline{\Omega}_0).$$

Thus using Taylor's formula, we have

$$\begin{aligned}\operatorname{div}((\tilde{K}(x)-\tilde{K}(0))\nabla\zeta_0) &= \sum_{i,j=1}^{n}(\tilde{K}_{ij}(x)-\tilde{K}_{ij}(0))\partial_{ij}\zeta_0 + \sum_{i,j=1}^{n}\partial_i\tilde{K}_{ij}(x)\partial_j\zeta_0 \\ &\in E^{n+2m+4}_{k+2m+5}(T_z) \oplus E^{n+2m+4}_{k+2m+6}(T_z) \oplus \cdots \oplus E^{n+2m+4}_{n+2m+l+4}(T_z) \oplus C^{l,\gamma}(\overline{\Omega}_0).\end{aligned}$$

Let ζ_1 satisfy

$$-\operatorname{div}(\tilde{K}(0)\nabla\zeta_1) = \operatorname{div}((\tilde{K}(x)-\tilde{K}(0))\nabla\zeta_0) \ \text{ in } \Omega_0; \ \ \zeta_1 = 0 \ \text{ on } \partial\Omega_0.$$

Using Lemma 2.4 again, we obtain

$$\zeta_1 \in E^{n+2m+4}_{k+2m+7}(T_z) \oplus E^{n+2m+4}_{k+2m+8}(T_z) \oplus \cdots \oplus E^{n+2m+4}_{n+2m+l+6}(T_z) \oplus C^{l+2,\gamma}(\overline{\Omega}_0),$$

which implies that

$$\operatorname{div}((\tilde{K}(x)-\tilde{K}(0))\nabla\zeta_1) \in E^{n+2m+8}_{k+2m+10}(T_z) \oplus E^{n+2m+8}_{k+2m+11}(T_z) \oplus \cdots \oplus E^{n+2m+8}_{n+2m+l+8}(T_z) \oplus C^{l,\gamma}(\overline{\Omega}_0)$$

and so on. Thus by a recursive method we conclude that there exists ζ_j for $j = 1, \cdots, n+l-k+1$ with

$$-\operatorname{div}(\tilde{K}(0)\nabla\zeta_j) = \operatorname{div}((\tilde{K}(x)-\tilde{K}(0))\nabla\zeta_{j-1}) \ \text{ in } \Omega_0; \ \ \zeta_j = 0 \ \text{ on } \partial\Omega_0 \tag{2.4}$$

and

$$\zeta_j \in E^{n+2m+4j}_{k+2m+5j+2}(T_z) \oplus E^{n+2m+4j}_{k+2m+5j+3}(T_z) \oplus \cdots \oplus E^{n+2m+4j}_{n+2m+4j+l+2}(T_z) \oplus C^{l+2,\gamma}(\overline{\Omega}_0).$$

In particular, $\zeta_{n+l-k+1} \in C^{l+2,\gamma}(\overline{\Omega}_0)$. Finally, we solve

$$-\operatorname{div}(\tilde{K}(x)\nabla h) = \operatorname{div}((\tilde{K}(x)-\tilde{K}(0))\nabla\zeta_{n+l-k+1}) \ \text{ in } \Omega_0; \ \ h = 0 \ \text{ on } \partial\Omega_0. \tag{2.5}$$

By the classical theory for elliptic equations, $h \in C^{l+2,\gamma}(\overline{\Omega}_0)$.

Denote $\zeta = \sum_{i=0}^{n+l-k+1}\zeta_i + h \in E^{n+2m}_{k+2m+2}(T_z) \oplus E^{n+2m+4}_{k+2m+7}(T_z) \oplus \cdots \oplus E^{n+2m+4(n+l-k)}_{n+2m+4(n+l-k)+l+2}(T_z) \oplus C^{l+2,\gamma}(\overline{\Omega}_0)$. By (2.3), (2.4) and (2.5), using direct computation we get

$$-\operatorname{div}(\tilde{K}(x)\nabla\zeta) = f \ \text{ in } \Omega_0; \ \ \zeta = 0 \ \text{ on } \partial\Omega_0.$$

Note that $E^{n+2m+4(n+l-k-1)}_{n+2m+4(n+l-k-1)+l+1}(T_z) \oplus E^{n+2m+4(n+l-k)}_{n+2m+4(n+l-k)+l+2}(T_z) \subseteq C^{l,\gamma}(\overline{\Omega}_0)$, therefore ζ is the desired solution. For the case $c(x)$ being a smooth function, we can use the Taylor's expansion as that in Lemma 2.4 to get the result. The proof is thus finished.

□

We are now able to prove the expansion of Green's function when n is odd.

Proof of Theorem 1.2: We extend $K(x) : \Omega \to \mathbb{R}^{n\times n}$ to a positively definite smooth matrix-valued function in $\mathbb{R}^n$. This can be done since Ω is smooth. For any $w \in \Omega$, define $\Omega_w = \{x - w \mid x \in \Omega\}$ and $T_w(\Omega) = \{T_w(x) \mid x \in \Omega\}$. By the positively definiteness and smoothness of K, we can choose $M > 0$ sufficiently large such that $T_w(\Omega_w) \subseteq B_M(0)$ for all $w \in \Omega$.

Now we give the expansion of $G_K(x,y)$. Note that $\Phi_0 \in E_2^n$. For fixed $y \in \Omega$, we define $\hat{\Phi}_0 = \sqrt{\det K(y)}^{-1}\Phi_0 \circ T_y \in E_2^n(T_y)$. Then direct computation yields that $-\mathrm{div}(K(y)\nabla\hat{\Phi}_0) = \delta_0$. Consider solutions ζ_y of the following equations

$$\begin{cases} -\mathrm{div}(\tilde{K}_y(x)\nabla\zeta_y) = \mathrm{div}((\tilde{K}_y(x) - K(y))\nabla\hat{\Phi}_0), & x \in B_M(0), \\ \zeta_y = 0, & x \in \partial B_M(0), \end{cases} \tag{2.6}$$

where the coefficient matrix $\tilde{K}_y(\cdot) = K(\cdot + y)$. By assumptions, we have $\tilde{K}_y \in C^\infty(\overline{B_M(0)}, \mathbb{R}^{n\times n})$ and $\tilde{K}_y(0) = K(y)$. Moreover, one computes directly that for any $l \in \mathbb{N}$

$$\mathrm{div}((\tilde{K}_y(x) - K(y))\nabla\hat{\Phi}_0) \in E_3^{n+2}(T_y) \oplus E_4^{n+2}(T_y) \oplus \cdots \oplus E_{n+2+l}^{n+2}(T_y) \oplus C^{l,\gamma}(\overline{B_M(0)}).$$

Therefore using Proposition 2.6 with $z = y$, $\Omega_0 = B_M(0)$ and $\tilde{K} = \tilde{K}_y$, we get the unique solution ζ_y of (2.6) such that

$$\begin{aligned} \zeta_y &= \sum_{i=1}^{n+l-2} \hat{\Phi}_i + h \\ &\in E_5^{n+2}(T_y) \oplus E_{10}^{n+6}(T_y) \oplus \cdots \oplus E_{n+2+4(n+l-3)+l}^{n+2+4(n+l-3)}(T_y) \oplus C^{l,\gamma}(\overline{B_M(0)}), \end{aligned}$$

where $\hat{\Phi}_i \in E_{5i}^{n+2+4(i-1)}(T_y)$ for $i = 1, \cdots, n+l-2$ and $h \in C^{l,\gamma}(\overline{B_M(0)})$.

Define $\eta_y(x) = G_K(x,y) - \hat{\Phi}_0(x-y) - \zeta_y(x-y)$ for any $x \in \Omega$. It follows from the definition of G_K that

$$\mathcal{L}_K G_K(x,y) = \delta_y,\ x \in \Omega; \quad G_K(x,y) = 0,\ x \in \partial\Omega,$$

which implies that for any $x \in \Omega$

$$\mathcal{L}_K\eta_y(x) = \mathrm{div}((\tilde{K}_y(x) - K(y))\nabla\hat{\Phi}_0)(x-y) + \mathrm{div}(\tilde{K}_y(x)\nabla\zeta_y)(x-y) = 0. \tag{2.7}$$

Here we have used (2.6). Note that (2.7) also holds in sense of distribution. Thus by Weyl's lemma, $\eta_y \in C^\infty(\overline{\Omega})$ since it is smooth on $\partial\Omega$. To conclude, we have

$$G_K(x,y) = \hat{\Phi}_0(x-y) + \sum_{i=1}^{n+l-2} \hat{\Phi}_i(x-y) + H^l(x,y), \tag{2.8}$$

where $\hat{\Phi}_i \in E_{5i}^{n+2+4(i-1)}(T_y)$ for $i = 1, \cdots, n+l-2$ and $x \mapsto H^l(x,y) \in C^{l,\gamma}(\overline{\Omega})$. Define $\Phi_i = \hat{\Phi}_i \circ T_y^{-1} \in E_{5i}^{n+2+4(i-1)}$. This combined with (2.8) yields the expansion for $G_K(x,y)$

$$G_K(x,y) = \sqrt{\det K(y)}^{-1}\Phi_0(T_y(x-y)) + \sum_{i=1}^{n+l-2} \Phi_i(T_y(x-y)) + H^l(x,y),$$

where $x \mapsto H^l(x,y) \in C^{l,\gamma}(\overline{\Omega})$.

We now prove the regularity of H^l, i.e., $H^l(x,y)=H^l_y(x)\in C\left(\Omega, C^{l,\gamma}\left(\overline{\Omega}\right)\right)\cap C^l(\Omega\times\Omega)$. Indeed, from the above construction of G_K, we know that $H^l(x,y)$ satisfies for all $y\in\Omega$

$$\begin{aligned}\mathcal{L}_K H^l(x,y) =&\mathcal{L}_K G_K(x,y)-\mathcal{L}_K\hat{\Phi}_0(x-y)-\mathcal{L}_K\left(\sum_{i=1}^{n+l-2}\hat{\Phi}_i(x-y)\right)\\ =&\mathrm{div}\left((K(x)-K(y))\nabla\hat{\Phi}_0(x-y)\right)+\mathrm{div}\left(K(x)\nabla\left(\sum_{i=1}^{n+l-2}\hat{\Phi}_i(x-y)\right)\right)\\ \in& E^{n+2+4(n+l-3)}_{n+1+4(n+l-3)+l}(T_y)(x-y)\oplus E^{n+2+4(n+l-3)}_{n+2+4(n+l-3)+l}(T_y)(x-y)\oplus C^{l,\gamma}(\overline{B_M(0)})(x-y).\end{aligned}\tag{2.9}$$

Here we denote $E^{n+2+4(n+l-3)}_{n+1+4(n+l-3)+l}(T_y)(x-y)=\{f(x-y)\mid f\in E^{n+2+4(n+l-3)}_{n+1+4(n+l-3)+l}(T_y)\}$ and $C^{l,\gamma}(\overline{B_M(0)})(x-y)=\{f(x-y)\mid f\in C^{l,\gamma}(\overline{B_M(0)})\}$. Moreover, it follows from $G_K(\cdot,y)=0$ on $\partial\Omega$ that

$$H^l(x,y)=-\hat{\Phi}_0(x-y)-\sum_{i=1}^{n+l-2}\hat{\Phi}_i(x-y)\quad\ x\in\partial\Omega.$$

By the continuity of the right-hand side of (2.9) and the boundary condition with respect to y in $C^{l-2,\gamma}(\Omega)$ and $C^{l,\gamma}(\partial\Omega)$, respectively, we can get $H^l(x,y)\in C(\Omega,C^{l,\gamma}(\overline{\Omega}))$ and thus $\frac{\partial H^l(x,y)}{\partial x},\frac{\partial^2 H^l(x,y)}{\partial x^2},\cdots,\frac{\partial^l H^l(x,y)}{\partial x^l}\in C(\Omega\times\Omega)$.

Similarly, taking ∇_y to both sides of (2.9), we can check that $\nabla_y H^l(x,y)\in C(\Omega,C^{l-1,\gamma}(\overline{\Omega}))$, which implies that $\frac{\partial H^l(x,y)}{\partial y},\frac{\partial^2 H^l(x,y)}{\partial y\partial x},\cdots,\frac{\partial^l H^l(x,y)}{\partial y\partial x^{l-1}}\in C(\Omega\times\Omega)$. Moreover, we can get that for any integer k,m with $0\le k\le m\le l$,

$$\frac{\partial^m H^l(x,y)}{\partial y^k\partial x^{m-k}}\in C(\Omega\times\Omega).$$

Thus we conclude that $H^l(x,y)$ is a C^l function over $\Omega\times\Omega$.

□

Note that the regular part of G_K and the associated Robin's function R_K are defined by $H_K(x,y)=H^0(x,y)$ and $R_K(x)=H_K(x,x)$ for any $x,y\in\Omega$. Therefore by Theorem 1.2, for any $l\in\mathbb{N}^*$

$$H_K(x,y)=H^0(x,y)=\sum_{i=n-1}^{n+l-2}\Phi_i(T_y(x-y))+H^l(x,y),\quad\forall x,y\in\Omega.\tag{2.10}$$

Now we can prove Theorem 1.6 when n is odd.

Proposition 2.8. *Let n be odd, Ω and K be as in Theorem 1.6. Then the Robin's function $R_K(\cdot)\in C^\infty(\Omega)$.*

Proof. For any $l\in\mathbb{N}^*$, it follows from (2.10) that $R_K(x)=H^l(x,x)$ since $\Phi_i(0)=0$ for any $i\ge n-1$. So $R_K\in C^l(\Omega)$. By the arbitrariness of l, we get the result. □

3. Even dimensional case: proof of Theorems 1.4 and 1.6

Now we prove the even dimensional case. Let $n \in 2\mathbb{N}^*$. Clearly from the definition of $E^{n+2m,s}_{k+2m}$, we have $E^{n+2m}_{k+2m} = E^{n+2m,s}_{k+2m} \oplus \mathbb{R}[x]$ for any $k \geq n, m \in \mathbb{N}$. For L_m, we first have

Lemma 3.1. *For any $m \in \mathbb{N}$, $L_m \subseteq \Delta(L_{m+2} \oplus \mathbb{R}[x])$.*

Proof. We prove it by a direct algebraic construction. Let $l \in \mathbb{N}$, $2l \leq m$, and $L_{m,l} = \operatorname{span}\{x^\alpha r^{2l} \log r \mid |\alpha| = m - 2l\}$. Then $L_m = \sum_{0 \leq 2l \leq m} L_{m,l}$. For $f = x^\alpha r^{2l} \log r \in L_{m,l}$, let $g_1 = r^2 f \in L_{m+2,l+1}$, then one computes directly that

$$\Delta g_1 = r^{2l+2} \log r \Delta(x^\alpha) + (2|\alpha| + n + 4l + 2)x^\alpha r^{2l} + (2l+2)(2l + |\alpha| + n)f.$$

Let g_2 be a polynomial with order $m+2$ satisfying $\Delta g_2 = (2|\alpha| + n + 4l + 2)x^\alpha r^{2l}$. Since $(2l+2)(2l+|\alpha|+n) > 0$, we find that there exists $g_3 = \frac{1}{(2l+2)(2l+|\alpha|+n)}(g_1 - g_2) \in L_{m+2} \oplus \mathbb{R}[x]$ such that

$$\Delta g_3 - f = \frac{1}{(2l+2)(2l+|\alpha|+n)} r^{2l+2} \log r \Delta(x^\alpha) \in L_{m,l+1}.$$

Thus we can get the result by decreasing induction on $|\alpha|$. $\square$

Let us define

$$\tilde{E}^{n+2m}_{k+2m} = \begin{cases} F^{n+2m}_{k+2m} & k < n, \\ F^{n+2m}_{k+2m} \oplus \mathbb{R}[x] & k \geq n. \end{cases}$$

Similar to Lemma 2.1, using the definition of F^{n+2m}_{k+2m} and $\tilde{E}^{n+2m}_{k+2m}$, we get

Lemma 3.2. *There holds*

$$\tilde{E}^{n+2m}_{k+2m} \subseteq \Delta(\tilde{E}^{n+2m}_{k+2m+2}) \quad \forall k \in \mathbb{N}^*, m \in \mathbb{N}.$$

Proof. If $k < n-2$, then $\tilde{E}^{n+2m}_{k+2m} = E^{n+2m}_{k+2m}$, $\tilde{E}^{n+2m}_{k+2m+2} = E^{n+2m}_{k+2m+2}$. The results follows from Lemma 2.1.

When $k = n-2$ or $n-1$, let $f \in E^{n+2m}_{k+2m,l}$ with $0 \leq l \leq \frac{n-2+2m}{2}$. If $l < \frac{n-2+2m}{2}$, we can repeat the construction in Lemma 2.1 with decreasing induction on $|\alpha|$ to get the results. The only case that we need to pay attention to is $2l = n - 2 + 2m$, i.e., $f = \frac{x^\alpha}{r^2}$ with $|\alpha| = k - n + 2 \leq 1$. Let $g = x^\alpha \log r \in L_{k-n+2} \subseteq \tilde{E}^{n+2m}_{k+2m+2}$. One computes directly that $\Delta g = (2|\alpha| + n - 2)f$. Note that $2|\alpha| + n - 2 > 0$ for $n \geq 4$. For $n = 2$, only the case $|\alpha| = 1$ occurs since $k \geq 1$. Thus $2|\alpha| + n - 2 \neq 0$ and we get the results.

For the case $k \geq n$, if $f \in L_{k-n} \oplus \mathbb{R}[x]$, the result is shown by Lemma 3.1. Let $f \in E^{n+2m}_{k+2m,l}$ for some $0 \leq 2l \leq k + 2m$. If $2l < n - 2 + 2m$, we can get the result by decreasing induction on $|\alpha|$. If $2l \geq n + 2m$, then f is a polynomial with order $k - n$. So there exists a polynomial $g \in \mathbb{R}[x] \subseteq \tilde{E}^{n+2m}_{k+2m+2}$ such that $\Delta g = f$. It remains to solve the case $2l = n - 2 + 2m$, i.e., $f = \frac{x^\alpha}{r^2}$ with $|\alpha| = k - n + 2$. Let $g_0 = x^\alpha \log r \in L_{k+2-n}$. Then

$$\Delta g_0 = (2|\alpha| + n - 2)\frac{x^\alpha}{r^2} + \log r \times \Delta(x^\alpha),$$

which implies that $\Delta(\frac{1}{2|\alpha|+n-2} g_0) - f \in L_{k-n}$. Since $L_{k-n} \subseteq \Delta(L_{k-n+2} \oplus \mathbb{R}[x])$, we get the existence of $g \in \mathbb{R}[x] \subseteq \tilde{E}^{n+2m}_{k+2m+2}$ such that $\Delta g = f$.

□

Let $z \in \Omega$ be fixed and T_z be the positively definite matrix defined by (1.8). We define

$$F^{n+2m}_{k+2m}(T_z) = \{f \circ T_z \mid f \in F^{n+2m}_{k+2m}\}; \quad \tilde{E}^{n+2m}_{k+2m}(T_z) = \{f \circ T_z \mid f \in \tilde{E}^{n+2m}_{k+2m}\}.$$

A direct consequence of Lemma 3.2 is

Lemma 3.3. *Let n be even and $z \in \Omega$ be fixed. Then for any $k \in \mathbb{N}^*, m \in \mathbb{N}$*

$$\tilde{E}^{n+2m}_{k+2m}(T_z) \subseteq -div(K(z)\nabla)(\tilde{E}^{n+2m}_{k+2m+2}(T_z)).$$

Based on Lemma 3.3, we get the existence of solutions to elliptic equations in divergence form with constant coefficient matrix $K(z)$ when $f \in F^{n+2m}_{k+2m}(T_z)$.

Lemma 3.4. *Let n be even and $z \in \Omega$ be fixed. Let Ω_0 be a bounded smooth domain in $\mathbb{R}^n$ containing the origin 0 and c be a smooth function defined on $\overline{\Omega}_0$. Then for any $k \in \mathbb{N}^*, m, l \in \mathbb{N}$, $f \in F^{n+2m}_{k+2m}(T_z)$, there exists*

$$g \in F^{n+2m}_{k+2m+2}(T_z) \oplus F^{n+2m}_{k+2m+3}(T_z) \oplus \cdots \oplus F^{n+2m}_{n+2m+l}(T_z) \oplus C^{l,\gamma}(\overline{\Omega}_0)$$

such that

$$-div(K(z)\nabla g) = c(x)f \quad in\ \Omega_0; \quad g = 0 \quad on\ \partial\Omega_0.$$

Moreover, the expansion of g is unique.

Proof. Using Taylor's formula, we have

$$c(x)f \in F^{n+2m}_{k+2m}(T_z) \oplus F^{n+2m}_{k+2m+1}(T_z) \oplus \cdots \oplus F^{n+2m}_{n+2m+l}(T_z) \oplus C^{l,\gamma}(\overline{\Omega}_0).$$

By Lemma 3.3, we get the existence of $\bar{g}_1 \in \tilde{E}^{n+2m}_{k+2m+2}(T_z) \oplus \tilde{E}^{n+2m}_{k+2m+3}(T_z) \oplus \cdots \oplus \tilde{E}^{n+2m}_{n+2m+l+2}(T_z) \oplus C^{l+2,\gamma}(\overline{\Omega}_0)$ such that $-\mathrm{div}(K(z)\nabla\bar{g}_1) = f$. So $\bar{g}_1 \in F^{n+2m}_{k+2m+2}(T_z) \oplus F^{n+2m}_{k+2m+3}(T_z) \oplus \cdots \oplus F^{n+2m}_{n+2m+l+2}(T_z) \oplus C^{l+2,\gamma}(\overline{\Omega}_0)$. Let $\bar{g}_2 \in C^\infty(\overline{\Omega}_0)$ be a function satisfying $-\mathrm{div}(K(z)\nabla\bar{g}_2) = 0$ in Ω_0 and $\bar{g}_2 = \bar{g}_1$ on $\partial\Omega_0$. Since $F^{n+2m}_{n+2m+l+1}(T_z) \oplus F^{n+2m}_{n+2m+l+2}(T_z) \subseteq C^{l,\gamma}(\overline{\Omega}_0)$, we conclude that $g = \bar{g}_1 - \bar{g}_2$ is the desired solution.

□

Then, we further extend results of Lemma 3.4 to elliptic operators in divergence form, which is close to $-\mathrm{div}(K(z)\nabla)$ near the origin 0 when n is even.

Proposition 3.5. *Let n be even, $z \in \Omega$, Ω_0 and c be as in Lemma 3.4. Let $\tilde{K} \in C^\infty(\overline{\Omega}_0, \mathbb{R}^{n\times n})$ be a positively definite matrix-valued function with $\tilde{K}(0) = K(z)$. Then for any $k \in \mathbb{N}^*, m, l \in \mathbb{N}$, $f \in F^{n+2m}_{k+2m}(T_z)$, there exists*

$$\xi \in F^{n+2m}_{k+2m+2}(T_z) \oplus F^{n+2m+4}_{k+2m+7}(T_z) \oplus \cdots \oplus F^{n+2m+4(n+l-k-2)}_{n+2m+l+4(n+l-k-2)}(T_z) \oplus C^{l,\gamma}(\overline{\Omega}_0)$$

such that

$$-div(\tilde{K}(x)\nabla\xi) = c(x)f \quad in\ \Omega_0; \quad \xi = 0 \quad on\ \partial\Omega_0.$$

Moreover, the expansion of ζ is unique.

Proof. The idea is similar to the proof of Proposition 2.6 and we prove it here just for completeness. Consider the case $c \equiv 1$. To find a solution ξ, we construct a sequence $\{\xi_j\}_{j=0}^{n+l-k+1}$ by iterating as follows. First, let ξ_0 satisfy

$$-\mathrm{div}(\tilde{K}(0)\nabla\xi_0) = f \quad \text{in } \Omega_0; \quad \xi_0 = 0 \quad \text{on } \partial\Omega_0.$$

It follows from Lemma 3.4 that

$$\xi_0 \in F^{n+2m}_{k+2m+2}(T_z) \oplus F^{n+2m}_{k+2m+3}(T_z) \oplus \cdots \oplus F^{n+2m}_{n+2m+l+2} \oplus C^{l+2,\gamma}(\overline{\Omega}_0),$$

from which we deduce,

$$\begin{aligned} \mathrm{div}((\tilde{K}(x) - \tilde{K}(0))\nabla\xi_0) &= \sum_{i,j=1}^{n} (\tilde{K}_{ij}(x) - \tilde{K}_{ij}(0))\partial_{ij}\xi_0 + \sum_{i,j=1}^{n} \partial_i\tilde{K}_{ij}(x)\partial_j\xi_0 \\ &\in F^{n+2m+4}_{k+2m+5}(T_z) \oplus F^{n+2m+4}_{k+2m+6}(T_z) \oplus \cdots \oplus F^{n+2m+4}_{n+2m+l+4}(T_z) \oplus C^{l,\gamma}(\overline{\Omega}_0). \end{aligned}$$

Let ξ_1 satisfy

$$-\mathrm{div}(\tilde{K}(0)\nabla\xi_1) = \mathrm{div}((\tilde{K}(x) - \tilde{K}(0))\nabla\xi_0) \quad \text{in } \Omega_0; \quad \xi_1 = 0 \quad \text{on } \partial\Omega_0.$$

By Lemma 3.4, $\xi_1 \in F^{n+2m+4}_{k+2m+7}(T_z) \oplus F^{n+2m+4}_{k+2m+8}(T_z) \oplus \cdots \oplus F^{n+2m+4}_{n+2m+l+6}(T_z) \oplus C^{l+2,\gamma}(\overline{\Omega}_0)$. This yields that $\mathrm{div}((\tilde{K}(x) - \tilde{K}(0))\nabla\xi_1) \in F^{n+2m+8}_{k+2m+10}(T_z) \oplus F^{n+2m+8}_{k+2m+11}(T_z) \oplus \cdots \oplus F^{n+2m+8}_{n+2m+l+8}(T_z) \oplus C^{l,\gamma}(\overline{\Omega}_0)$ and so on. Hence we can find ξ_j for $j = 1, \cdots, n+l-k+1$ satisfying

$$-\mathrm{div}(\tilde{K}(0)\nabla\xi_j) = \mathrm{div}((\tilde{K}(x) - \tilde{K}(0))\nabla\xi_{j-1}) \quad \text{in } \Omega_0; \quad \xi_j = 0 \quad \text{on } \partial\Omega_0$$

and

$$\xi_j \in F^{n+2m+4j}_{k+2m+5j+2}(T_z) \oplus F^{n+2m+4j}_{k+2m+5j+3}(T_z) \oplus \cdots \oplus F^{n+2m+4j}_{n+2m+4j+l+2}(T_z) \oplus C^{l+2,\gamma}(\overline{\Omega}_0).$$

In particular, $\xi_{n+l-k+1} \in C^{l+2,\gamma}(\overline{\Omega}_0)$. Finally, we solve $h \in C^{l+2,\gamma}(\overline{\Omega}_0)$ satisfying

$$-\mathrm{div}(\tilde{K}(x)\nabla h) = \mathrm{div}((\tilde{K}(x) - \tilde{K}(0))\nabla\xi_{n+l-k+1}) \quad \text{in } \Omega_0; \quad h = 0 \quad \text{on } \partial\Omega_0.$$

Denote $\xi = \sum_{i=0}^{n+l-k+1} \xi_i + h \in F^{n+2m}_{k+2m+2}(T_z) \oplus F^{n+2m+4}_{k+2m+7}(T_z) \oplus \cdots \oplus F^{n+2m+4(n+l-k)}_{n+2m+4(n+l-k)+l+2}(T_z) \oplus C^{l+2,\gamma}(\overline{\Omega}_0)$. One computes directly that

$$-\mathrm{div}(\tilde{K}(x)\nabla\xi) = f \quad \text{in } \Omega_0; \quad \zeta = 0 \quad \text{on } \partial\Omega_0.$$

Thus ξ is the desired solution. For the case c being a smooth function, we can use the same Taylor's expansion as in Lemma 3.4 to get the result.

□

Proof of Theorem 1.4: Based on Proposition 3.5, we can use similar method as that in Theorem 1.2 to get the expansion of Green's function G_K of the operator $\mathcal{L}_K$ in case that n is even, just by replacing E^{n+2m}_{k+2m} with F^{n+2m}_{k+2m}. We omit the proof here. □

Similar to the odd dimensional case, we define the regular part of G_K by $H_K(x,y) = H^0(x,y)$ for any $x, y \in \Omega$ and Robin's function $R_K(x) = H_K(x,x)$ for any $x \in \Omega$. By

Theorem 1.4, for any $l \in \mathbb{N}^*$

$$H_K(x,y) = \sum_{i=n-1}^{n+l-2} \Phi_i(T_y(x-y)) + H^l(x,y), \quad \forall x, y \in \Omega. \tag{3.1}$$

Since $\Phi_i(0) = 0$ for any $i \geq n-1$, we can get that R_K is smooth when n is even, which is summarized as follows.

Proposition 3.6. *Let n be even, Ω and K be as in Theorem 1.6. Then Robin's function $R_K(\cdot) \in C^\infty(\Omega)$.*

Proof of Theorem 1.6: Using Propositions 2.8 and 3.6, we finish the proof of Theorem 1.6. □

Remark 3.7. We give a final remark that the smoothness of K and Ω is indeed used to obtain the smoothness of Robin's function R_K. If our aim is to get $R_K \in C^l$ for some l, the assumptions on K and Ω can be weakened.

Appendix

In this Appendix, we give proof of the expansion of Green's function of $\mathcal{L}_K$ when the coefficient matrix K is a constant coefficient matrix K_0.

Lemma A.1. Let T_0 be the positively definite matrix satisfying

$$T_0^{-1} T_0^{-t} = K_0.$$

Denote $T_0\Omega = \{T_0 x \mid x \in \Omega\}$. Let G_{K_0} be the Green's function of $\mathcal{L}_{K_0}$ in Ω with zero Dirichlet boundary condition. Then

$$G_{K_0}(x,y) = \sqrt{\det K_0}^{-1} \Phi_0(T_0(x-y)) - \sqrt{\det K_0}^{-1} H_{T_0\Omega}(T_0x, T_0y) \ \ \forall x, y \in \Omega, x \neq y, \tag{3.2}$$

where $H_{T_0\Omega} : T_0\Omega \times T_0\Omega \to \mathbb{R}$ is the regular part satisfying for any $y' \in T_0\Omega$

$$\begin{cases} -\Delta H_{T_0\Omega}(x', y') = 0, & x' \in T_0\Omega, \\ H_{T_0\Omega}(x', y') = \Phi_0(x' - y'), & x' \in \partial T_0\Omega. \end{cases} \tag{3.3}$$

Proof. For any $u \in C^2(\Omega) \cap C_0^1(\overline{\Omega})$ with $f = \mathcal{L}_{K_0} u$, let $\bar{u}(x') = u(T_0^{-1}x') = u(x)$ and $\bar{f}(x') = f(T_0^{-1}x') = f(x)$ for any $x' \in T_0\Omega$. Here we set $x' = T_0 x$. Then one computes directly that $-\Delta \bar{u} = \bar{f}$ in $T_0\Omega$ and $\bar{u} \in C^2(T_0\Omega) \cap C_0^1(\overline{T_0\Omega})$. From (1.4), we have

$$\bar{u}(y') = \int_{T_0\Omega} \left(\Phi_0(x' - y') - H_{T_0\Omega}(x', y')\right) \bar{f}(x') dx', \quad \forall y' \in T_0\Omega, \tag{3.4}$$

where $H_{T_0\Omega} : T_0\Omega \times T_0\Omega \to \mathbb{R}$ satisfies (3.3). Using coordinate transformation, (3.4) yields

$$\begin{aligned} u(y) = &\int_\Omega \left(\Phi_0(T_0x - T_0y) - H_{T_0\Omega}(T_0x, T_0y)\right) f(x) \cdot \det T_0 dx \\ = &\sqrt{\det K_0}^{-1} \int_\Omega \left(\Phi_0(T_0x - T_0y) - H_{T_0\Omega}(T_0x, T_0y)\right) f(x) dx, \quad \forall y \in \Omega. \end{aligned}$$

This implies that G_{K_0} has decomposition (3.2). The proof is thus complete. □

Acknowledgments: The authors are grateful to the anonymous referees for providing constructive comments that helped to improve representation of the paper. D. Cao was supported by National Key R&D Program (2022YFA1005602) and NNSF of China (grant No.12371212). J. Wan was supported by NNSF of China (grants No. 12101045 and No. 12471190).

Conflict of interest statement. On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data availability statement. All data generated or analysed during this study are included in this published article and its supplementary information files.

References

[1] T. Aubin, "Nonlinear Analysis on Manifolds. Monge-Ampere Equations", Springer-Verlag, New York, 1982.

[2] D. Bartolucci, C.-S. Lin and G. Tarantello, *Uniqueness and symmetry results for solutions of a mean field equation on S^2 via a new bubbling phenomenon*, Comm. Pure Appl. Math., 64 (2011), no. 12, 1677–1730.

[3] M.S. Berger and L.E. Fraenkel, *Nonlinear desingularization in certain free-boundary problems*, Comm. Math. Phys., 77 (1980), 149–172.

[4] L.A. Caffarelli and A. Friedman, *Asymptotic estimates for the plasma problem*, Duke Math. J., 47 (1980), 705–742.

[5] D. Cao, S. Peng and S. Yan, *Planar vortex patch problem in incompressible steady flow*, Adv. Math., 270 (2015), 263–301.

[6] D. Cao and J. Wan, *Helical vortices with small cross-section for 3D incompressible Euler equation*, J. Funct. Anal., 284 (2023), Paper No. 109836.

[7] D. Cao and J. Wan, *Structure of Green's function of elliptic equations and helical vortex patches for 3D incompressible Euler equations*, Math. Ann., 388 (2024), 2627–2669.

[8] D. Cao and J. Wan, *Clustered helical vortices for 3D incompressible Euler equation in infinite cylinders*, arXiv:2311.02676.

[9] S. Chanillo and Y. Li, *Continuity of solutions of uniformly elliptic equations in $\mathbb{R}^2$*, Manuscripta Math., 77 (1992), 415–433.

[10] C.-C. Chen and C.-S. Lin, *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces*, Comm. Pure Appl. Math., 55 (2002), 728–771.

[11] J. Dávila, M. del Pino, M. Musso and J. Wei, *Gluing methods for vortex dynamics in Euler flows*, Arch. Ration. Mech. Anal., 235 (2020), no. 3, 1467–1530.

[12] J. Dekeyser and J. Van Schaftingen, *Vortex motion for the lake equations*, Comm. Math. Phys., 375 (2020), 1459–1501.

[13] F. De Marchis, R. López-Soriano and D. Ruiz, *Compactness, existence and multiplicity for the singular mean field problem with sign-changing potentials*, J. Math. Pures Appl., 115(9) (2018), 237–267.

[14] S. de Valeriola and J. Van Schaftingen, *Desingularization of vortex rings and shallow water vortices by semilinear elliptic problem*, Arch. Ration. Mech. Anal., 210(2) (2013), 409–450.

[15] H.J. Dong and S. Kim, *Green's function for nondivergence elliptic operators in two dimensions*, SIAM J. Math. Anal., 53 (2021), no. 4, 4637–4656.

[16] A. Enciso and D. Peralta-Salas, *Critical points of Green's functions on complete manifolds*, J. Differ. Geom., 92(1) (2012), 1–29.

[17] P. Figueroa and M. Musso, *Bubbling solutions for Moser-Trudinger type equations on compact Riemann surfaces*, J. Funct. Anal., 275 (2018), 2684–2739.

[18] L.E. Fraenkel and M.S. Berger, *A global theory of steady vortex rings in an ideal fluid*, Acta Math., 132 (1974), 13–51.

[19] F. Franceschini and F. Glaudo, *Expansion of the fundamental solution of a second-order elliptic operator with analytic coefficients*, Proc. Lond. Math. Soc., 127(3) (2023), no. 4, 1028–1056.

[20] D. Gilbarg and N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order, Classics in Mathematics", Springer, Berlin, 2001.

[21] W.H. Greub, "Linear Algebra", Third edition, Die Grundlehren der Mathematischen Wissenschaften, Band 97 Springer-Verlag New York, Inc., New York, 1967.

[22] M. Grüter and K.O. Widman, *The Green function for uniformly elliptic equations*, Manuscripta Math., 37 (1982), 303–342.

[23] E. Hebey, "Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities", Courant Lecture Notes in Mathematics, vol. 5, New York University, Courant Institute of Mathematical Sciences/American Mathematical Society, New York/Providence, RI, 1999.

[24] F. John, *The fundamental solution of linear elliptic differential equations with analytic coefficients*, Comm. Pure Appl. Math., 3 (1950), 273–304.

[25] C.E. Kenig and W.-M. Ni, *On the elliptic equation* $Lu+k+Kexp[2u]=0$, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 12(4) (1985), no. 2, 191–224.

[26] S. Khenissy, Y. Rébäi and D. Ye, *Expansion of the Green's function for divergence form operators*, C. R. Acad. Sci. Paris, Ser. I, 348 (2010), 891–896.

[27] K. Kodaira, *Harmonic fields in Riemannian manifolds (generalized potential theory)*, Ann. of Math., 50 (1949), 587–665.

[28] P. Li and L.-F. Tam, *Symmetric Green's functions on complete manifolds*, Amer. J. Math., 109 (1987), 1129–1154.

[29] C.C. Lin, *On the motion of vortices in two dimension - I. Existence of the Kirchhoff-Routh function,* Proc. Natl. Acad. Sci. USA, 27 (1941), 570–575.

[30] W. Littman, G. Stampacchia and H.F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 17(3) (1963), 43–77.

[31] V. Maz'ya and R. McOwen, "On the fundamental solution of an elliptic equation in nondivergence form", Nonlinear Partial Differential Equations and Related Topics, American Mathematical Society, Providence, RI, 2010.

[32] J. Norbury, *Steady planar vortex pairs in an ideal fluid*, Comm. Pure Appl. Math., 28 (1975), 679–700.

[33] O. Rey, *The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent*, J. Funct. Anal., 89(1) (1990), 1–52.

[34] D. Smets and J. Van Schaftingen, *Desingularization of vortices for the Euler equation*, Arch. Ration. Mech. Anal., 198(3) (2010), 869–925.

[35] J.L. Taylor, S. Kim and R.M. Brown, *The Green function for elliptic systems in two dimensions*, Comm. Partial Differential Equations, 38(9) (2013), 1574–1600.

[36] B. Turkington, *On steady vortex flow in two dimensions. I, II*, Comm. Partial Differential Equations, 8 (1983), 999–1030, 1031–1071.

[37] J. Wei, D. Ye and F. Zhou, *Bubbling solutions for an anisotropic Emden–Fowler equation*, Calc. Var. Partial Differential Equations, 28 (2007), 217–247.

[38] J. Wei, D. Ye and F. Zhou, *Analysis of boundary bubbling solutions for an anisotropic Emden-Fowler equation*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 25 (2008), no. 3, 425–447.

Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing 100190, and University of Chinese Academy of Sciences, Beijing 100049, P.R. China

Email address: `dmcao@amt.ac.cn`

School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, P.R. China
Email address: wanjie@bit.edu.cn