ON THE VISCOSITY LINEARIZATION METHOD WITHOUT COMPACTNESS

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Abstract. Savin's small perturbation approach has had far reaching applications in the theory of non-linear elliptic and parabolic PDE. In this short note, we revisit his seminal proof of De-Giorgi's improvement of flatness theorem for minimal surfaces and provide an approach based on the Harnack inequality that avoids the use of compactness arguments.

1. Introduction

In this paper we consider minimizers of the perimeter functional in B_1. Precisely, given any measurable set $E \subset \mathbb{R}^n$ the perimeter of ∂E in B_1 is defined as

$$P(E) = \int_{B_1} |D\chi_E| := \sup_{\substack{g \in C_0^1(B_1;\mathbb{R}^n) \\ \|g\|_{L^\infty} \le 1}} \int_E \operatorname{div}(g).$$

If a set $E \subset \mathbb{R}^n$ is a minimizer of the perimeter functional then ∂E is called a minimal surface (see Definition 2.1 for a precise statement).

By viewing ∂E as a multi-valued graph in $\mathbb{R}^{n-1}$, it is shown in [3] that minimal surfaces are viscosity solutions (see Section 2 for a precise statement) of the minimal surface equation

$$(1 + |Du|^2)\Delta u - Du^T D^2 u Du = 0. \tag{1.1}$$

In the seminal work [14], Savin showed that minimal surfaces in this setting satisfy the following Harnack inequality.

Theorem 1.1 (Harnack Inequality). *There exists universal constants $\varepsilon_1(n)$ and $\eta > 0$ such that if ∂E is a minimal surface and*

$$\partial E \cap B_1 \subset \{|x_n| \le \varepsilon\}$$

for some $\varepsilon \le \varepsilon_1(n)$, then

$$\partial E \cap B_{1/2} \subset \{|x_n| \le \varepsilon(1 - \eta)\}.$$

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Using Theorem 1.1 and a compactness argument, Savin in [14, Theorem 5.2] then showed that there exists universal constants $\varepsilon_\circ > 0$ and $r_\circ > 0$ such that if

$$\partial E \cap B_1 \subset \{|x_n| \le \varepsilon\}$$

for some $\varepsilon \le \varepsilon_\circ$, then there exists $\nu \in \mathbb{S}^{n-1}$ such that

$$\partial E \cap B_{r_\circ} \subset \left\{|x \cdot \nu| \le \frac{\varepsilon}{2} r_\circ\right\}.$$

Geometrically, this implication states that if a minimal surface is flat enough in B_1, then up to a rotation, it is flatter in a smaller ball. This is the celebrated improvement of flatness theorem for minimal surfaces and is a key step in the corresponding regularity theory. Indeed, by iterating this result one readily obtains that a minimal surface is a $C^{1,\alpha}$ graph in $B_{3/4}$ (see for instance [14, Section 5] for the proof). Once the minimal surface is $C^{1,\alpha}$, Schauder estimates for (1.1) along with a bootstrapping argument then imply that it is C^∞ (see for instance [11]). In analogy then with the regularity theory for uniformly elliptic PDE, we have that the Harnack inequality implies improvement of flatness, which implies $C^{1,\alpha}$ regularity, which in turn implies C^∞ regularity (see Section 1.1 below for more on this point).

This improvement of flatness acts as a prototype for a large number of "ε-regularity results" - if a solution u of some PDE is close enough to a special solution, then u is smooth. In the case of minimal surfaces, the special solution considered is just the hyperplane.

The key idea in establishing such an ε-regularity result is to consider solutions to the linearized equation around the special solution. For instance, if the PDE is of the form $F(D^2u, Du) = 0$ (as is (1.1)), then a function v satisfies the linearized equation around a solution w if $L_w(v) = 0$, where

$$L_w(v) = \lim_{\varepsilon \to 0} \frac{F(D^2(w + \varepsilon v), D(w + \varepsilon v)) - F(D^2w, Dw)}{\varepsilon}. \tag{1.2}$$

The next step is to then show that if u is close enough to the special solution, then certain properties of solutions of the linearized equation are inherited by u.

This idea has become ubiquitous in the regularity theory of nonlinear elliptic PDE and free boundary problems [1, 5, 7–9, 12, 13]. For instance, in the same seminal paper [14], Savin uses the linearization method to obtain an improvement of flatness theorem for certain solutions of the Allen-Cahn equation (which he then uses to conclude the 1D-symmetry of such solutions). For the Bernoulli free boundary problem, an analogous improvement of flatness theorem around free boundary points is obtained using these same ideas in both the elliptic [7] and parabolic [8] setting.

In the case of the minimal surface equation (1.1), the linearized PDE is simply $\Delta u = 0$ and so one expects that minimal surfaces are well approximated by harmonic functions. Although Savin used this linearization technique with a viscosity approach in [14], the improvement of flatness theorem was first shown by De-Giorgi in the variational setting [4]. In both [14] and [4] a compactness argument is used to obtain a harmonic function that appropriately approximates the minimal surface and as a result, the value $\varepsilon_\circ$ appearing in their results is not quantified. Schoen and Simons in [15] provided a constructive proof in the variational setting, that is, a proof in which all the constants can be explicitly computed. Moreover, Caffarelli and Córdoba in [3] give another constructive proof using viscosity techniques.

While also based on viscosity methods, the approach given in [14] is distinct to that in [3] as it is based on a Harnack inequality for minimal surfaces (Theorem 1.1). Broadly speaking, the framework established by [14] is that a Harnack inequality implies the improvement of flatness result. The objective of this note is to provide a constructive proof of the improvement of flatness theorem within the framework of [14]. Similar to the proof in [14], the previously mentioned applications of the viscosity linearization method utilise a compactness argument to obtain solutions to the linearized PDE. We believe the methods developed in this note can be applied to these related problems in order to obtain quantitative statements analogous to our main result, Theorem 1.2 below.

Recalling the values ε_1 and η from Theorem 1.1, our main result can be stated as follows.

Theorem 1.2 (Improvement of Flatness). *Let* $\alpha = -\log(1-\eta)$ *and* $\gamma = \frac{1}{1-\alpha}$ *and define the constants*

$$r_\circ = \frac{1}{2^{16}n^2}, \quad \varepsilon_\circ = \left(\frac{\varepsilon_1^{\gamma\alpha}}{2^{33+5\alpha}n^3}\right)^{\frac{8}{\gamma\alpha^2}}.$$

If ∂E *is a minimal surface in* B_1, $0 \in \partial E$ *and*

$$\partial E \cap B_1 \subset \{|x_n| \leq \varepsilon\}$$

for some $\varepsilon \leq \varepsilon_\circ$, *then there exists* $\nu \in \mathbb{S}^{n-1}$ *such that*

$$\partial E \cap B_{r_\circ} \subset \left\{|x \cdot \nu| \leq \frac{\varepsilon}{2} r_\circ\right\}.$$

We note that Theorem 1.1 was already proven in [14] using constructive methods and so the constants ε_1 and η appearing in the statement of

Theorem 1.2 are known. Moreover, we refer to [6, Proposition 2.6] for a short proof of Theorem 1.1 in which the constants can be more easily quantified. Therefore, in this note we will only be concerned with obtaining estimates for $r_\circ$ and $\varepsilon_\circ$ in terms of the constants ε_1 and η.

1.1. **Structure of the proof.** Beginning with the Harnack inequality, Theorem 1.1, we will show that the minimal surface ∂E satisfying the flatness assumption is *almost* Hölder continuous, see (3.1). In contrast with uniformly elliptic PDE we do not obtain Hölder continuity of the minimal surface directly from the Harnack inequality, rather we obtain the following two-scale behaviour. For points that are further apart than ε^γ we can show that the oscillation of ∂E *decays* and from this we obtain the desired estimate on the Hölder semi-norm of ∂E. On the other hand, if the points are closer than ε^γ we can only show that the bound on the oscillation of ∂E between these points is of order $\varepsilon^{1+\gamma\alpha}$. (Notice that this follows from (3.1) which is stated for the mulivalued function A by the relationship $\partial E = \varepsilon A$, see Lemma 3.1).

This improved control on the oscillation of ∂E is in fact enough to construct a Hölder continuous function u which is suitably close to ∂E (see (3.5)). This function is contructed in Lemma 3.3 via inf-convolution while in Lemma 3.4 we show that it inherits some of the viscosity properties of ∂E. This then allows us to show in Proposition 3.6 that u is close to its harmonic replacement in L^∞. Recalling the closeness of u to ∂E in (3.5), this shows that ∂E is in fact very close to a harmonic function.

In this way we are able to explicitly construct the desired harmonic approximation. Once this is obtained, we can conclude similarly to [14].

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2. Notation and Preliminaries

2.1. **Notation.** Throughout this work $\mathbb{R}^n$ will be endowed with the Euclidean inner product $x \cdot y$ and its induced norm $|x|$. For any matrix

$Q \in \mathbb{R}^{n\times n}$ we will denote by $|Q|$ the induced norm

$$|Q| = \max_{|x|\le 1} |Qx|.$$

$B_r(x)$ represents the ball of radius of r with centre x. We will also represent points $x \in \mathbb{R}^n$ as $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and use $B_r'(x')$ to represent a ball in $\mathbb{R}^{n-1}$ with centre x' and radius r. Moreover, given any measurable set $E \subset \mathbb{R}^n$ we will denote by E^c the set $\mathbb{R}^n \backslash E$.

We will denote by $2^{\mathbb{R}}$ all possible subsets of $\mathbb{R}$. For a multi-valued graph $A : \mathbb{R}^{n-1} \to 2^{\mathbb{R}}$, we define

$$A(x') - A(y') = \sup\{t - s; t \in A(x'), s \in A(y')\},$$

and

$$|A| = \sup_{x' \in \mathbb{R}^{n-1}} \sup\{s; s \in A(x')\}.$$

Note that the oscillation of A in any $\Omega \subset \mathbb{R}^{n-1}$ is well defined as

$$\operatorname*{osc}_{\Omega} A = \sup_{\Omega} A - \inf_{\Omega} A.$$

Finally, given any $x \in \mathbb{R}$, $\lfloor x \rfloor$ represents the number $n \in \mathbb{Z}$ such that $n \le x \le n + 1$.

2.2. **Preliminaries.** We will follow the notation and definitions given in [14]. We begin by giving the definition of a minimal surface.

Definition 2.1. *A measurable set $E \subset B_1$ is said to be a set with minimal perimeter in B_1 if for any set $F \subset B_1$ such that $(E\backslash F) \cup (F\backslash E) \subset\subset B_1$ it holds that $P(E) \le P(F)$. Moreover, ∂E is called a minimal surface.*

We also recall the following.

Definition 2.2. *Given a measurable set $E \subset \mathbb{R}^n$ we say that $x \in \partial E$ if for any $r > 0$ there holds that $|B_r(x) \cap E| > 0$ and $|B_r(x) \cap E^c| > 0$.*

Recalling the density estimates for minimal surfaces ([14, Theorem 4.1]) it is obvious that ∂E is closed in the sense of Definition 2.2.

We will now recall the definition of viscosity solutions for (1.1) (see [14, Section 5]). Given any smooth function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, we define its sub-graph to be the set $\Gamma_\varphi = \{x' \in \mathbb{R}^{n-1} : x_n < \varphi(x')\}$. Similarly its super-graph is the set $\Gamma^\varphi := \{x' : x_n > \varphi(x')\}$.

Definition 2.3 (Viscosity solution)**.** *A set $E \subset B_1$ is a viscosity solution of (1.1) if for any smooth function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\Gamma_\varphi \cap B_r(y)$ is contained in either E or E^c for some $y \in \partial E \cap \partial\Gamma_\varphi$ and $r > 0$, then*

$$(1 + |D\varphi|^2)\Delta\varphi - D\varphi^T D^2\varphi D\varphi \le 0.$$

If $\Gamma^\varphi \cap B_r(y)$ is contained in either E or E^c for some $y \in \partial E \cap \partial\Gamma^\varphi$ and $r > 0$, then

$$(1 + |D\varphi|^2)\Delta\varphi - D\varphi^T D^2\varphi D\varphi \geq 0.$$

We have the following (see [14, Lemma 5.4]).

Theorem 2.4. *Let ∂E be a minimal surface in B_1. Then ∂E is a viscosity solution of (1.1) in the sense of Definition 2.3.*

A remark on the constants appearing in the Harnack inequality, Theorem 1.1, is in order.

Remark 2.5. *A careful computation of the constants $\varepsilon_1(n)$ and η appearing in Theorem 1.1 reveals that these constants are very small. We will therefore use the (crude) upper bounds $\varepsilon_1 \leq \frac{1}{4}$ and $\eta \leq \frac{1}{5}$ to simplify some our estimates. In particular, these bounds imply that $-\log_2(1-\eta) \leq \frac{1}{4}$.*

Finally, we will make use of the following estimate which is found in [2, Proposition 4.13] for viscosity solutions of fully non-linear elliptic PDE.

Proposition 2.6. *Suppose that $u \in C^0(B_1)$ satisfies*

$$\begin{cases} \Delta u = 0 & \text{in } B_1 \\ u = g & \text{on } \partial B_1, \end{cases}$$

for some $g \in C^\sigma(\partial B_1)$. Then $u \in C^{\frac{\sigma}{2}}(\overline{B}_1)$ with

$$\|u\|_{C^{0,\frac{\sigma}{2}}(\overline{B}_1)} \leq 2n5^\sigma \|g\|_{C^\sigma(\partial B_1)}. \tag{2.1}$$

Proof. We begin by recalling [2, Proposition 4.12] which states that for any $x_0 \in \partial B_1$ the function u is $C^{\sigma/2}$-Hölder continuous at x_0 with

$$\sup_{x\in\overline{B_1}} \frac{|u(x)-u(y)|}{|x-y|^{\frac{\sigma}{2}}} \leq 2^{\sigma/2} \sup_{x\in\partial B_1} \frac{|g(x)-g(x_0)|}{|x-x_0|^\sigma} \tag{2.2}$$

Now suppose that $x_0 \in B_1$ and let $0 < r \leq \operatorname{dist}(x_0, \partial B_1)$. Applying [10, Section 2.2, Equation (20)] to each point $x \in B_{\frac{r}{2}}(x_0)$ we find that $\|Du\|_{L^\infty(B_{\frac{r}{2}}(x))} \leq 2nr^{-1}\|u\|_{L^\infty(\partial B_{r/2}(x))}$ and since $B_{\frac{r}{2}}(x) \subset B_r(x_0)$ we have by the maximum principle that $\|Du\|_{L^\infty(B_{\frac{r}{2}}(x_0))} \leq 2nr^{-1}\|u\|_{L^\infty(\partial B_r(x_0))}$. Hence for each $x, y \in B_{r/2}(x_0)$ we have that

$$r^{\frac{\sigma}{2}} \frac{|u(x)-u(y)|}{|x-y|^{\frac{\sigma}{2}}} \leq 2n\|u\|_{L^\infty(\partial B_r(x_0))}. \tag{2.3}$$

Now following the proof of [2, Proposition 4.13] we take any two points $x, y \in B_1$ and denote by $d_x = \operatorname{dist}(x, \partial B_1)$ and $d_y = \operatorname{dist}(y, \partial B_1)$ and let

x_0, y_0 be the points on ∂B_1 such that $|x - x_0| = d_x$ and $|y - y_0| = d_y$. We can also assume with no loss of generality that $d_x \le d_y$. Therefore, if $|x - y| \le \frac{d_x}{2}$ then we can apply (2.3) in $B_{d_x/2}(x)$ to $u(x) - u(x_0)$ to find that

$$d_x^{\sigma/2}\frac{|u(x) - u(y)|}{|x - y|^{\frac{\sigma}{2}}} \le 2n\|u - u(x_0)\|_{L^\infty(B_{d_x}(x))},$$

and so using (2.2) we conclude that

$$\frac{|u(x) - u(y)|}{|x - y|^{\frac{\sigma}{2}}} \le 2n2^{\sigma/2}\|g\|_{C^\sigma(\partial B_1)}. \tag{2.4}$$

We now suppose $d_y \le d_x \le 2|x - y|$ and we notice that by (2.2) we have

$$\begin{aligned}|u(x) - u(y)| &\le |u(x) - u(x_0)| + |u(x_0) - u(y_0)| + |u(y) - u(y_0)|\\ &\le \|g\|_{C^\sigma}\left(2^{\sigma/2}d_x^{\sigma/2} + |x_0 - y_0|^\sigma + 2^{\sigma/2}d_y^{\sigma/2}\right).\end{aligned}$$

Since $|x_0 - y_0| \le 2$ we have that

$$|x_0 - y_0|^\sigma \le 2^{\sigma/2}(|x - y| + d_x + d_y)^{\sigma/2} \le 2^{\sigma/2}5^{\sigma/2}|x - y|^{\sigma/2},$$

and so we obtain

$$\begin{aligned}|u(x) - u(y)| &\le 2^{\sigma/2}\|g\|_{C^\sigma}(d_x^{\sigma/2} + 5^{\sigma/2}|x - y|^{\sigma/2} + d_y^{\sigma/2})\\ &\le 2^{\sigma/2}\|g\|_{C^\sigma}|x - y|^{\sigma/2}(2^{\sigma/2} + 5^{\sigma/2} + 2^{\sigma/2})\\ &\le 2n5^\sigma\|g\|_{C^\sigma}|x - y|^{\frac{\sigma}{2}},\end{aligned}$$

where in the last line we have used the fact that $3 \le 2n$. This and (2.4) establishes the bound on $[u]_{C^{\frac{\sigma}{2}}(\overline{B_1})}$. Since by the maximum principle we have $\|u\|_{L^\infty(B_1)} \le \|g\|_{L^\infty(\partial B_1)}$, we have established (3.7) and this completes the proof. ☐

3. Proof of Theorem 1.2

Lemma 3.1. *Suppose ∂E is a minimal surface with $\partial E \cap B_1 \subset \{|x_n| \le \varepsilon\}$ for some $0 < \varepsilon \le \frac{1}{8}\varepsilon_1(n)$ and that $0 \in \partial E$. There exists a multivalued graph $A : B_1' \to 2^{\mathbb{R}}$ such that $|A| \le 1$, $0 \in A(0)$ and $\partial E = \varepsilon A$. Moreover there holds for any $x', y' \in B_{3/4}'$ that*

$$|A(x') - A(y')| \le C_1(|x' - y'| + C_2\varepsilon^\gamma)^\alpha \tag{3.1}$$

where $\alpha = -\log_2(1 - \eta)$, $\gamma = \frac{1}{1-\alpha}$, $C_1 = 2^{4+4\alpha}$ and $C_2 = 2^{\frac{3\alpha}{1-\alpha}-1}\varepsilon_1^{-\gamma}$ are all universal constants.

Proof. The first part of the lemma is easily seen by setting

$$A(x') = \left\{\frac{x_n}{\varepsilon};\ (x', x_n) \in \partial E\right\}.$$

To deduce (3.1) we iteratively apply Theorem 1.1 to each point in $\partial E \cap B_{3/4}$. To this end, we fix some $x_0 \in \partial E \cap B_{3/4}$ and note that by assumption

$$|(x - x_0) \cdot e_n| \le 2\varepsilon, \text{ for all } x \in \partial E \cap B_{1/4}(x_0).$$

Since $8\varepsilon \le \varepsilon_1$ we can apply Theorem 1.1 in $B_{1/4}(x_0)$ and obtain

$$\partial E \cap B_{1/8}(x_0) \subset \{|(x - x_0) \cdot e_n| \le 2\varepsilon(1 - \eta)\}.$$

Iterating this procedure, we obtain for all $m \ge 3$ satisfying

$$2^m \varepsilon (1 - \eta)^{m-3} \le \varepsilon_1 \tag{3.2}$$

that

$$\partial E \cap B_{2^{-m}}(x_0) \subset \left\{|(x - x_0) \cdot e_n| \le 2\varepsilon(1 - \eta)^{m-2}\right\}. \tag{3.3}$$

Using now (3.2) and Remark 2.5 we have that

$$2\varepsilon(1 - \eta)^{m-2} = \varepsilon_1^{-1} \varepsilon (1 - \eta)^{m-3}(2\varepsilon_1(1 - \eta)) \le 2^{-m-1},$$

and so

$$\partial E \cap \left\{|x' - x_0'| \le 2^{-1-m}\right\} \subset B_{2^{-m}}(x_0).$$

Hence as a consequence of (3.3) we have that

$$A \cap \left\{|x' - x_0'| \le 2^{-1-m}\right\} \subset \left\{|(x - x_0) \cdot e_n| \le 2(1 - \eta)^{m-2}\right\},$$

which implies

$$\operatorname*{osc}_{B'_{2^{-1-m}}(x_0')} A \le 2(1 - \eta)^{m-2}. \tag{3.4}$$

Moreover using (3.2) we deduce that the oscillation decay of (3.4) holds for $m \le \lfloor M(\varepsilon, n) \rfloor =: \tilde{M}$ where $M(\varepsilon, n)$ achieves the equality in (3.2). We note that with this maximal value of m, we have that $2^{-\tilde{M}-1} \le 2^{\frac{3\alpha}{1-\alpha}-1} \varepsilon_1^{-\gamma} \varepsilon^\gamma = C_2 \varepsilon^\gamma$.

We now show that this implies that A is Hölder continuous at x_0' outside of $B_{C_2\varepsilon^\gamma}(x_0')$. Fix any $y' \in B_{1/4}(x_0') \backslash B_{C_2\varepsilon^\gamma}(x_0')$ and choose $m \in \mathbb{N}$ so that

$$2^{-m-1} \le |x_0' - y'| \le 2^{-m}.$$

Since $m \le \tilde{M}$ we have by (3.4) that

$$\begin{aligned} |A(x_0') - A(y')| &\le \operatorname*{osc}_{B'_{2^{-m}}(x_0')} A \\ &\le 2(1 - \eta)^{m-3} \\ &\le 2^{-\alpha(m-3)+1} \\ &\le \frac{C_1}{6} |x_0' - y'|^\alpha. \end{aligned}$$

If on the other hand $y' \in B_{C_2\varepsilon^\gamma}(x_0')$ we note that $2^{-\tilde{M}-1} \le |x'-y'| + C_2\varepsilon^\gamma$ and so using the oscillation decay in $B_{2^{-\tilde{M}}}(x_0')$ we obtain

$$|A(x_0') - A(y')| \le \frac{C_1}{6}\left(|x'-y'| + C_2\varepsilon^\gamma\right)^\alpha.$$

In particular we have shown (3.1) for all $y' \in B'_{1/4}(x_0')$. Taking any $y' \in B'_{3/4}\backslash B'_{1/4}(x_0')$ we can find a sequence of $I \le 5$ points $\{x_i'\}_{1\le i\le I} \subset B'_{3/4}$ such that $x_I' = y'$ and $|x_i' - x_{i+1}'| \le \frac{1}{4}$ for $0 \le i \le 5$. Applying the triangle inequality then establishes (3.1). □

Remark 3.2. *By Remark 2.5, the constants γ and C_2 are well defined. Moreover, using the bound $\alpha \le \frac{1}{4}$ we find that $C_2 \le \varepsilon_1^{-\gamma}$.*

Recalling that ∂E is closed, we now define the lower semi-continuous function

$$u^-(x') = \varepsilon \inf A(x'),$$

as well as the upper semi-continuous function

$$u^+(x') = \varepsilon \sup A(x').$$

Geometrically, these correspond to the lower and upper parts of the minimal surface. Since $0 \in \partial E$ we can assume that $u^-(0) = 0$ (otherwise we could translate the minimal surface and complete the proof under the assumption that $\partial E \cap B_1 \subset \{|x_n| \le 2\varepsilon\}$).

Lemma 3.3. *Under the assumptions of Lemma 3.1 there exists some $u \in C^{0,\alpha}(B'_{3/4})$ with $[u]_{C^{0,\alpha}(B'_{3/4})} \le 2^{4+5\alpha}$ and $u(0) = 0$, such that*

$$\varepsilon u(x') \le u^-(x') \le u^+(x') \le \varepsilon u(x') + C_3\varepsilon^{1+\gamma\alpha} \tag{3.5}$$

where $C_3 = 2^{4+5\alpha}\varepsilon_1^{-\gamma\alpha}$. In particular we have that

$$\|A - u\|_{L^\infty(B'_{3/4})} \le C_3\varepsilon^{\gamma\alpha}, \tag{3.6}$$

and

$$\|u\|_{C^{0,\alpha}(\overline{B'}_{1/2})} \le 2^{6+5\alpha}. \tag{3.7}$$

Proof. We define the function

$$u(x_0') = \inf_{x' \in B'_{3/4}} \left\{ \frac{u^-(x')}{\varepsilon} + 2^\alpha C_1 |x' - x_0'|^\alpha \right\}, \;\; x_0' \in B'_{3/4}.$$

To see that $u \in C^{0,\alpha}$ take $x_0', x_1' \in B_{3/4}'$ and let $x_0'^* \in \overline{B'}_{3/4}$ such that $u(x_0') = \frac{u^-(x_0'^*)}{\varepsilon} + 2^\alpha C_1 |x_0'^* - x_0'|^\alpha$. Then we observe that

$$\begin{aligned} u(x_1') - u(x_0') &\le \left(\frac{u^-(x_0'^*)}{\varepsilon} + 2^\alpha C_1 |x_0'^* - x_1'|^\alpha \right) - \left(\frac{u^-(x_0'^*)}{\varepsilon} + 2^\alpha C_1 |x_0'^* - x_0'|^\alpha \right) \\ &\le 2^\alpha C_1 \left(|x_0'^* - x_1'|^\alpha - |x_0'^* - x_0'|^\alpha \right) \\ &\le 2^\alpha C_1 |x_0' - x_1'|^\alpha. \end{aligned}$$

A symmetrical argument yields $u(x_0') - u(x_1') \le 2^\alpha C_1 |x_0' - x_1'|^\alpha$ which establishes $[u]_{C^{0,\alpha}(B_{3/4}')} \le 2^{4+5\alpha}$. The fact that $u(0) = 0$ follows immediately from $u^-(0) = 0$ and the definition of u.

The lower bound in (3.5) follows directly from the definition of u and multiplying by ε. For the upper bound we first observe that (3.1) yields

$$\frac{u^-(x')}{\varepsilon} \ge \frac{u^+(x_0')}{\varepsilon} - C_1 (|x' - x_0'| + C_2 \varepsilon^\gamma)^\alpha$$

for any $x_0', x' \in B_{3/4}'$. Consequently we have that

$$\begin{aligned} \frac{u^-(x')}{\varepsilon} + 2^\alpha C_1 |x' - x_0'|^\alpha &\ge \frac{u^+(x_0')}{\varepsilon} - C_1 (|x' - x_0'| + C_2 \varepsilon^\gamma)^\alpha + 2^\alpha C_1 |x' - x_0'|^\alpha \\ &\ge \frac{u^+(x_0')}{\varepsilon} - 2^\alpha C_1 C_2^\alpha \varepsilon^{\gamma\alpha}. \end{aligned}$$

Taking now the infimum and multiplying by ε yields the upper bound in (3.5). By subtracting εu in (3.5) and then dividing by ε we obtain (3.6).

The estimate (3.7) follows from (3.6), the fact that $|A| \le 1$ and the estimate $[u]_{C^{0,\alpha}(B_{3/4}')} \le 2^{4+5\alpha}$. Indeed, since $\varepsilon \le \varepsilon_1$, we have that

$$\begin{aligned} \|u\|_{C^{0,\alpha}(\overline{B'}_{1/2})} &\le \|u\|_{L^\infty(B_{3/4}')} + [u]_{C^{0,\alpha}(B_{3/4}')} \\ &\le (1 + C_3 \varepsilon^{\gamma\alpha}) + 2^{4+5\alpha} \\ &\le 2^{4+5\alpha} (2^{-4-5\alpha} + \varepsilon_1^{-\gamma\alpha} \varepsilon^{\gamma\alpha}) + 2^{4+5\alpha} \\ &\le 2^{6+5\alpha}. \end{aligned}$$

□

We must now understand what properties of u we can deduce from the minimal surface equation that ∂E satisfies. To this end, we define the universal constant $r = \varepsilon^{\frac{\gamma\alpha}{4}}$ and we note that $\frac{\gamma\alpha}{4} \le \frac{1}{12}$.

Lemma 3.4. *Suppose $\varphi \in C^\infty(B_{1/2}')$ touches $u + C_3 \varepsilon^{\gamma\alpha}$ from above at some $x_0' \in B_{1/2-2r}'$. If $\max\{\sup_{i,j} |D_{ij}\varphi|, |D\varphi|, |D^2\varphi|\} \le \varepsilon^{-\frac{1}{2}}$ in $B_{1/2-r}'$ for* *some* $\varepsilon \le (2^{-6-5\alpha}\varepsilon_1^{\gamma\alpha})^{\frac{2}{1+\gamma\alpha}}$, *then there exits* $x_1' \in B_r'(x_0')$ *so that*

$$\Delta\varphi(x_1') \ge -C_4\varepsilon^{\frac{\gamma\alpha}{2}}, \tag{3.8}$$

where $C_4 = 2^{8+5\alpha}n\varepsilon_1^{-\gamma\alpha}$.

Proof. We add a sufficiently large paraboloid P to $\varepsilon\varphi$ so that a translation of $\varepsilon\varphi + P$ touches u^+ from above in $B_r'(x_0')$. That is, for some $\delta > 0$ to be determined there exists some $x_1' \in B_r'(x_0')$ such that

$$\varepsilon\varphi(x_1') + \frac{\delta}{2}|x_1' - x_0'|^2 - u^+(x_1') = \min_{B_r'(x_0')}\left\{\varepsilon\varphi + \frac{\delta}{2}|x' - x_0'|^2 - u^+(x')\right\}. \tag{3.9}$$

Now if $x_1' \in \partial B_r'(x_0')$ we have by (3.9) that

$$\varepsilon\varphi(x_1') + \frac{\delta}{2}r^2 - u^+(x_1') \le \varepsilon\varphi(x_0') - u^+(x_0'),$$

and since $\varepsilon\varphi(x_1') - u^+(x_1') \ge 0$ we obtain

$$\varepsilon\varphi(x_0') \ge \frac{\delta}{2}r^2 + u^+(x_0').$$

We then find

$$\begin{aligned} 0 = \varepsilon u(x_0') + C_3\varepsilon^{1+\gamma\alpha} - \varepsilon\varphi(x_0') &\le \varepsilon u(x_0') + C_3\varepsilon^{1+\gamma\alpha} - \frac{\delta}{2}r^2 - u^+(x_0') \\ &\le C_3\varepsilon^{1+\gamma\alpha} - \frac{\delta}{2}r^2, \end{aligned}$$

which is a contradiction if

$$\delta = 4C_3\varepsilon^{1+\frac{\gamma\alpha}{2}}.$$

Now with this δ we have that $\varepsilon\varphi(x') + \frac{\delta}{2}|x' - x_0'|^2$ is tangent from above to u^+ at x_1' in $B_r'(x_0')$. Since ∂E is a viscosity solution of (1.1) we have that

$$\begin{aligned} &\left(1 + |\varepsilon D\varphi(x_1') + \delta(x_1' - x_0')|^2\right)(\varepsilon\Delta\varphi(x_1') + (n-1)\delta) \\ &+ (\varepsilon D\varphi(x_1') + \delta(x_1' - x_0'))^T\left(\varepsilon D^2\varphi(x_1') + \delta I\right)(\varepsilon D\varphi(x_1') + \delta(x_1' - x_0')) \ge 0. \end{aligned}$$

Using the bounds on $|D\varphi|$ and $|D^2\varphi|$, along with the fact that $|x_0' - x_1'| \le r$, we find

$$\begin{aligned} |\varepsilon D\varphi(x_1') + \delta(x_1' - x_0')| &\le \varepsilon^{\frac{1}{2}} + 4C_3\varepsilon^{1+\frac{3\gamma\alpha}{4}} \\ &\le \varepsilon^{\frac{1}{2}}\left(1 + 2^{6+5\alpha}\varepsilon_1^{-\gamma\alpha}\varepsilon^{\frac{1}{2}+\frac{3}{4}\gamma\alpha}\right) \\ &\le 2\varepsilon^{\frac{1}{2}} \end{aligned}$$

for $\varepsilon \le (2^{-6-5\alpha}\varepsilon_1^{\gamma\alpha})^{\frac{4}{2+3\gamma\alpha}}$. A similar computation yields the bound

$$\left|\varepsilon D^2\varphi(x_1') + \delta I\right| \le 2\varepsilon^{\frac{1}{2}}$$

for $\varepsilon \le (2^{-6-5\alpha}\varepsilon_1^{\gamma\alpha})^{\frac{2}{1+\gamma\alpha}} \le (2^{-6-5\alpha}\varepsilon_1^{\gamma\alpha})^{\frac{4}{2+3\gamma\alpha}}$. Using these bounds we see that $(1+|\varepsilon D\varphi(x_1')+\delta(x_1'-x_0')|^2)(n-1)\delta \le 2n\delta$ and so we find that

$$\begin{aligned}\varepsilon\Delta\varphi(x_1') &\ge -\varepsilon|\Delta\varphi(x_1')||\varepsilon D\varphi(x_1')+\delta(x_1'-x_0')|^2\\ &\quad -(1+|\varepsilon D\varphi(x_1')+\delta(x_1'-x_0')|^2)(n-1)\delta\\ &\quad -|\varepsilon D\varphi(x_1')+\delta(x_1'-x_0')|^2\left|\varepsilon D^2\varphi(x_1')+\delta I\right|\\ &\ge -12n\varepsilon^{3/2}-2n\delta\\ &\ge -C_4\varepsilon^{1+\frac{\gamma\alpha}{2}},\end{aligned}$$

where in the last inequality we have used that $\frac{\gamma\alpha}{2}\le\frac{1}{2}$. □

We will now construct a function that is strictly above u for ε small enough. Recalling that $u\in C^{0,\alpha}(\overline{B}'_{1/2})$, there exists $v\in C^{0,\alpha/2}(\overline{B}'_{1/2})$ satisfying

$$\begin{cases}\Delta v = -2C_4\varepsilon^{\frac{\gamma\alpha}{2}} & \text{in } B'_{1/2}\\ v=u & \text{on } \partial B'_{1/2}\end{cases}.$$

Moreover, splitting $v=v_1+v_2$ where

$$\begin{cases}\Delta v_1 = 0 & \text{in } B'_{1/2}\\ v_1=u & \text{on } \partial B'_{1/2},\end{cases}$$

and $v_2=-C_4\varepsilon^{\frac{\gamma\alpha}{2}}\frac{1}{n-1}(|x'|^2-\frac{1}{4})$, we have using Proposition 2.6, (3.7) and $\varepsilon\le\varepsilon_1$ that

$$\begin{aligned}\|v\|_{C^{0,\alpha/2}(\overline{B}'_{1/2})} &\le 2n\cdot 5^\alpha\|u\|_{C^{0,\alpha}(\overline{B}'_{1/2})}+2C_4\varepsilon^{\frac{\gamma\alpha}{2}}\\ &\le 2n\cdot(2^3)^\alpha(2^{6+5\alpha})+2C_4\varepsilon^{\frac{\gamma\alpha}{2}}\\ &\le 2^{10+5\alpha}n\varepsilon_1^{-\frac{\gamma\alpha}{2}} =: C_5.\end{aligned}$$

Lemma 3.5. *Let $w_+\in C^{0,\frac{\alpha}{2}}(\overline{B}'_{1/2})\cap C^\infty(B'_{1/2})$ satisfy*

$$\begin{cases}\Delta w_+ = -2C_4\varepsilon^{\frac{\gamma\alpha}{2}} & \text{in } B'_{1/2}\\ w_+=u+4C_5r^{\frac{\alpha}{2}} & \text{on } \partial B'_{1/2}\end{cases}. \tag{3.10}$$

For all $x_0'\in B'_{\frac{1}{2}-r}$ we have that

$$\max\left\{\sup_{i,j}|D_{ij}w_+(x_0')|,|Dw_+(x_0')|,\left|D^2w_+(x_0')\right|\right\}\le\varepsilon^{-\frac{1}{2}} \tag{3.11}$$

as long as $\varepsilon\le(2^{-18-5\alpha}n^{-5}\varepsilon_1^{\gamma\alpha})^{\frac{2}{1-\gamma\alpha}}$. Consequently, $w_+>u$ in $B'_{1/2}$.

Proof. We split $w_+ = (w_+)_1 + (w_+)_2$ where

$$\begin{cases} \Delta(w_+)_1 = 0 & \text{in } B'_{1/2} \\ (w_+)_1 = u + 4C_5 r^{\frac{\alpha}{2}} & \text{on } \partial B'_{1/2} \end{cases}$$

and $(w_+)_2 = -C_4 \varepsilon^{\frac{\gamma\alpha}{2}} \frac{1}{n-1}(|x'|^2 - \frac{1}{4})$.

We first obtain estimates on the derivatives of $(w_+)_1$. For any $x'_0 \in B'_{1/2-r}$ we can apply the derivative estimates for harmonic functions in $B'_{\frac{r}{2}}(x'_0) \subset B'_{1/2}$ and obtain for any $1 \le i \le n$ that

$$|D_i(w_+)_1(x'_0)| \le \frac{2n}{r}\|(w_+)_1\|_{L^\infty(B'_{1/2})} \le 2n\varepsilon^{-\frac{\gamma\alpha}{4}}(\|u\|_{L^\infty(B'_{1/2})} + 4C_5 r^{\frac{\alpha}{2}}),$$

where in the last inequality we have used the maximum principle. Similarly for any $1 \le i, j \le n$ we find that

$$|D_{i,j}(w_+)_1(x'_0)| \le \left(\frac{4n}{r}\right)^2 \|(w_+)_1\|_{L^\infty(B'_{1/2})} \le 16n^2\varepsilon^{-\frac{\gamma\alpha}{2}}(\|u\|_{L^\infty(B'_{1/2})} + 4C_5 r^{\frac{\alpha}{2}}).$$

Choosing $\varepsilon \le (2^{-18-5\alpha} n^{-5} \varepsilon_1^{\gamma\alpha})^{\frac{2}{1-\gamma\alpha}}$ we see that

$$\max\left\{\sup_{1\le i\le n}|D_i(w_+)_1(x'_0)|, \sup_{1\le i,j\le n}|D_{ij}(w_+)_1(x'_0)|\right\} \le \frac{1}{2n^2}\varepsilon^{-\frac{1}{2}}.$$

Consequently we obtain that

$$\sup_{1\le k\le 2}\left|D^k(w_+)_1\right| \le \frac{1}{2}\varepsilon^{-\frac{1}{2}} \text{ in } B'_{1/2-r}. \tag{3.12}$$

Moreover a direct computation shows that

$$\sup_{1\le i\le n}|D_i(w_+)_2| \le \sup_{1\le i,j\le n}|D_{ij}(w_+)_2| \le \frac{2}{n-1}C_4\varepsilon^{\frac{\gamma\alpha}{2}}$$

and so if $\varepsilon \le (2^{-10-5\alpha} n^{-2} \varepsilon_1^{\gamma\alpha})^{\frac{2}{1+\gamma\alpha}}$ we have that

$$\sup_{1\le k\le 2}\left|D^k(w_+)_2\right| \le \frac{1}{2}\varepsilon^{-\frac{1}{2}} \text{ in } B'_{1/2}. \tag{3.13}$$

Now since $(2^{-18-5\alpha} n^{-5} \varepsilon_1^{\gamma\alpha})^{\frac{2}{1-\gamma\alpha}} \le (2^{-10-5\alpha} n^{-1} \varepsilon_1^{\gamma\alpha})^{\frac{2}{1+\gamma\alpha}}$ we have that (3.11) follows from (3.12) and (3.13) for $\varepsilon \le (2^{-18-5\alpha} n^{-5} \varepsilon_1^{\gamma\alpha})^{\frac{2}{1-\gamma\alpha}}$.

Noticing that $(2^{-18-5\alpha} n^{-5} \varepsilon_1^{\gamma\alpha})^{\frac{2}{1-\gamma\alpha}} \le (2^{-6-5\alpha} \varepsilon_1^{\gamma\alpha})^{\frac{2}{1+\gamma\alpha}}$, we have that w_+ will not touch u in $B'_{1/2-2r}$ or else this would contradict Lemma 3.4. Moreover w_+ will not touch u in the annulus $B'_{1/2} \backslash B'_{1/2-2r}$ or else that would violate the Hölder continuity of both u and w_+. □

Arguing similarly as in Lemma 3.4 and Lemma 3.5, we can construct a function w_- satisfying

$$\begin{cases} \Delta w_- = 2C_4\varepsilon^{\frac{\gamma\alpha}{2}} & \text{in } B'_{1/2} \\ w_- = u - 4C_5 r^{\frac{\alpha}{2}} & \text{on } \partial B'_{1/2}, \end{cases} \tag{3.14}$$

and which lives below the graph of u in $B'_{1/2}$ for $\varepsilon \le (2^{-18-5\alpha}n^{-5}\varepsilon_1^{\gamma\alpha})^{\frac{2}{1-\gamma\alpha}}$. The existence of w_+ and w_- now allows us to show that u is actually very close to its harmonic replacement.

Proposition 3.6. *Let $w \in C^{0,\frac{\alpha}{2}}(\overline{B}'_{1/2}) \cap C^\infty(B'_{1/2})$ be the solution of*

$$\begin{cases} \Delta w = 0 & \text{in } B'_{1/2} \\ w = u & \text{on } \partial B'_{1/2}. \end{cases}$$

If $\varepsilon \le (2^{-18-5\alpha}n^{-5}\varepsilon_1^{\gamma\alpha})^{\frac{2}{1-\gamma\alpha}}$ then $\|u - w\|_{L^\infty(B'_{1/2})} \le C_6\varepsilon^{\frac{\gamma\alpha^2}{8}}$ where $C_6 = 2^{14+5\alpha}n\varepsilon_1^{-\gamma\alpha}$.

Proof. Using (3.10) and (3.14) we have that

$$\begin{cases} \Delta(w_+ - w_-) = -4C_4\varepsilon^{\frac{\gamma\alpha}{2}} & \text{in } B'_{1/2} \\ w_+ - w_- = 8C_5 r^{\frac{\alpha}{2}} & \text{on } \partial B'_{1/2} \end{cases}$$

and so by the maximum principle we have

$$\|w_+ - w_-\|_{L^\infty(B'_{1/2})} \le 8C_5\varepsilon^{\frac{\gamma\alpha^2}{8}} + \frac{1}{2}\frac{C_4}{n-1}\varepsilon^{\frac{\gamma\alpha}{2}} \le C_6\varepsilon^{\frac{\gamma\alpha^2}{8}}.$$

Since for $\varepsilon \le (2^{-18-5\alpha}n^{-5}\varepsilon_1^{\gamma\alpha})^{\frac{2}{1-\gamma\alpha}}$ we have $w_- < u < w_+$, and since $w_- < w < w_+$, the result follows. $\square$

We can now finally give the

Proof of Theorem 1.2. We assume that $\varepsilon \le (2^{-18-5\alpha}n^{-5}\varepsilon_1^{\gamma\alpha})^{\frac{2}{1-\gamma\alpha}}$ so that Proposition 3.6 holds. Moreover, in order to simplify our estimates we will use the bound $\alpha \le \frac{1}{4}$ (c.f. Remark 2.5) so that in particular $\|u\|_{L^\infty(\overline{B}'_{1/2})} \le 2^6$.
Since w is harmonic, we have for any $x' \in B'_{2r_\circ}(0)$ with $r_\circ \le \frac{1}{8}$ that

$$\begin{aligned} |w(x') - w(0) - \nabla w(0)\cdot x'| &\le 2r_\circ^2 \max_{1\le i,j\le n} \|D_{ij}w\|_{L^\infty(B'_{\frac{1}{4}})} \\ &\le 128n^2r_\circ^2\|w\|_{L^\infty(\partial B'_{1/2})} \\ &\le 128n^2r_\circ^2\|u\|_{L^\infty(\overline{B}'_{1/2})} \\ &\le 2^{13}n^2r_\circ^2. \end{aligned}$$

Choosing $r_\circ = 2^{-16} n^{-2}$ we have that

$$|w(x') - w(0) - \nabla w(0) \cdot x'| \leq \frac{r_\circ}{8}. \tag{3.15}$$

Since $u(0) = 0$ we have using Proposition 3.6 for any $x' \in B'_{2r_\circ}(0)$ that

$$\begin{aligned}
|u(x') - \nabla w(0) \cdot x'| &\leq |u(x') - w(x')| + |u(0) - w(0)| + |w(x') - w(0) - \nabla w(0) \cdot x'| \\
&\leq 2C_6 \varepsilon^{\frac{\gamma\alpha^2}{8}} + \frac{r_\circ}{8} \\
&\leq \frac{3r_\circ}{8}
\end{aligned}$$

by choosing $\varepsilon \leq \left(\frac{r_\circ}{8C_6}\right)^{\frac{8}{\gamma\alpha^2}}$. Since with this choice of ε we have that $\varepsilon \leq \left(\frac{r_\circ}{8C_3}\right)^{\frac{1}{\gamma\alpha}}$, we obtain using (3.6) that

$$A \cap \{|x'| \leq 2r_\circ\} \subset \left\{|x_n - \nabla w(0) \cdot x'| \leq \frac{r_\circ}{2}\right\}.$$

It then immediately follows that

$$\partial E \cap B_{r_0} \subset \left\{|x \cdot \nu| \leq \frac{\varepsilon}{2} r_0\right\},$$

where $\nu = \frac{(-\varepsilon \nabla w(0), 1)}{|(-\varepsilon \nabla w(0), 1)|}$. Since $\alpha \leq \frac{1}{4}$ we also readily have that $\left(\frac{r_\circ}{8C_6}\right)^{\frac{8}{\gamma\alpha^2}} \leq \left(2^{-18-5\alpha} n^{-5} \varepsilon_1^{\gamma\alpha}\right)^{\frac{2}{1-\gamma\alpha}}$ and so we can take $\varepsilon_\circ = \left(\frac{r_\circ}{8C_6}\right)^{\frac{8}{\gamma\alpha^2}}$ which concludes the proof. □

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