

MARKED NODAL CURVES WITH VECTOR FIELDS

ADRIAN ZAHARIUC

ABSTRACT. We discuss two operations on nodal curves with (logarithmic) vector fields, which resemble the ‘stabilization’ construction in Knudsen’s proof that $\overline{\mathcal{M}}_{g,n+1}$ is the universal curve over $\overline{\mathcal{M}}_{g,n}$. We prove that both operations work in families (commute with base change). We also construct inverse operations under suitable assumptions, which allow us to prove a technical result somewhat similar to Knudsen’s.

As an application, we prove that the Losev–Manin compactification of the space of configurations of n points on $\mathbb{P}^1 \setminus \{0, \infty\}$ modulo scaling degenerates isotrivially to a compactification of the space of configurations of n points on \mathbb{A}^1 modulo translation, and the natural group actions fit together globally.

1. INTRODUCTION

A *logarithmic vector field* on a (connected but possibly reducible, reduced, projective) curve C with only nodal singularities over an algebraically closed field \mathbb{K} is a global section of ω_C^\vee , the dual of the dualizing sheaf. Concretely, using the well-known description of ω_C in terms of meromorphic differentials and residues, we see that a logarithmic vector field on C corresponds to a global section $v \in \Gamma(T_{\tilde{C}})$, where $\nu : \tilde{C} \rightarrow C$ is the normalization of C , such that for any node $p \in C$,

- v vanishes at q_1 and q_2 , where $\nu^{-1}(p) = \{q_1, q_2\}$, and
- $c_1 + c_2 = 0$, where $c_i \in \mathbb{K}$ is the image of v under the natural map

$$\Gamma(T_{\tilde{C}}(-q_i)) \rightarrow T_{\tilde{C}}(-q_i) \otimes \mathcal{O}_{\tilde{C}}/\mathcal{O}_{\tilde{C}}(-q_i) = T_{q_i, \tilde{C}} \otimes T_{q_i, \tilde{C}}^\vee = \mathbb{K}.$$

If the arithmetic genus of C is at least 2, the vector field must be identically 0 on many irreducible components of C (though not all in general), and there are strong restrictions on what situations can occur. The higher genus case is thus somewhat artificial, though we have decided to include it since many of the constructions below still go through. The adjective ‘logarithmic’ will typically be omitted, and we will write just ‘field’ or ‘vector field’.

1.1. Bubbling up in the presence of vector fields. One direction in Knudsen’s proof that $\overline{\mathcal{M}}_{g,n+1}$ is the universal curve over $\overline{\mathcal{M}}_{g,n}$ amounts the procedure called stabilization in [18]. In the case of a single curve, stabilization works as follows. (We are discussing this procedure out of context; perhaps ‘pre-stabilization’ would be a better name in our setup.) Given a projective nodal curve C over an algebraically closed field \mathbb{K} , with several nonsingular distinct marked points $w_1, \dots, w_m \in C(\mathbb{K})$, imagine inserting a new marked point at $x \in C(\mathbb{K})$. If x is singular or $x = w_i$ for some i , then we deem this configuration ‘degenerate’. In this situation, stabilization inserts a new component $\Sigma \simeq \mathbb{P}^1$ at x , and produces non-degenerate data $C', w'_1, \dots, w'_m, x'$, by placing x' (and w'_i , if $x = w_i$) on Σ . (In the main body of

2020 *Mathematics Subject Classification.* 14H10, 14M27.

the paper, we will instead use different notation, but for now the prime notation is adequate.) Here are two related reasons why stabilization is important:

- it *works in families*: it can be generalized to families of curves in a manner which commutes with base change (and this commutativity satisfies the suitable ‘cocycle conditions’); and
- iterating the stabilization construction in families suitably intertwined with base changes produces important moduli spaces. For instance, in this way it is possible to obtain $\overline{M}_{0,n}$ (or even $\overline{M}_{g,n}$ given \overline{M}_g), or the Fulton-MacPherson compactifications of configuration spaces of curves.

In this paper, we will discuss a related (now two-step) procedure when a vector field on the nodal curve is given.

Knudsen stabilization with vector fields. The first step of the procedure operates in the same setup as Knudsen’s stabilization, with the extra data of a vector field ϕ on C , which vanishes at w_1, \dots, w_m . The meaning of degenerate is the same as in the usual Knudsen stabilization, and fixing it entails inserting a \mathbb{P}^1 component under the same circumstances. It can be shown that ϕ lifts uniquely to a vector field ϕ' on the curve C' obtained by Knudsen stabilization, if we require $\phi'(w'_i) = 0$ for $i = 1, \dots, m$. (This boils down to the fact that, given $r \in \mathbb{K}^\times$, there exists a unique meromorphic 1-form on \mathbb{P}^1 with poles with residues r at 0 and $-r$ at ∞ .)

Inflating at zero vector. The second operation is applied after the first, so it presumes milder degeneracy to begin with: the point x is nonsingular and distinct from w_1, \dots, w_m . We consider such a situation ‘degenerate’ if $\phi(x) = 0$. Then we insert a \mathbb{P}^1 component at x , and fix the degeneracy by placing x' on \mathbb{P}^1 and lifting ϕ to a field ϕ' on C' such that $\phi'(w'_i) = 0$, but $\phi'(x') \neq 0$.

Remark 1.1. Here are a few remarks regarding this construction.

- (1) Indeed, x behaves differently from w_1, \dots, w_m . In applications, we will get around this by thinking of x as part of a different set of markings x_1, \dots, x_n .
- (2) The properties above don’t determine ϕ' uniquely, although they do determine it up to automorphisms of C' that fix all the rest of the data and the map $C' \rightarrow C$ which contracts the new component, if there is a new component. Still, a canonical choice exists in a certain sense. A good analogy is with the following question: in Knudsen stabilization, where do we place x' on the new \mathbb{P}^1 component? It seems that all points on \mathbb{P}^1 except two are equally good, yet, in [18, §2], there is a definite formula for x' .
- (3) The new field ϕ' depends nonlinearly on the old field ϕ . For instance, $\phi = 0$ everywhere is possible (or, more frequently in applications, $\phi = 0$ on the component containing x), so $\phi'(x') \neq 0$ rules out linearity.

The main result concerning these operations, which forms the technical core of the paper and will be stated more precisely and proved in §4, is the following.

Theorem 1.2 (Theorems 4.7 and 4.20). *‘Knudsen stabilization with vector fields’ and ‘Inflating at zero vector’ work in families: they can be generalized to families in a manner which commutes with base change.*

Knudsen’s original stabilization admits an inverse operation called contraction. Similarly, we will show that under suitable circumstances, our two operations admit inverse operations (Theorem 2.7), which will allow us to prove a technical result

(Theorem 2.4) reminiscent of Knudsen’s theorem that $\overline{\mathcal{M}}_{g,n+1}$ is the universal curve over $\overline{\mathcal{M}}_{g,n}$. Very roughly, instead of n -marked curves, we will consider curves with a vector field and $m+n$ markings, the first m of which are vanishing points of the vector field and the remaining n give stability; then, the ‘universal curve’ in this situation is isomorphic to the analogous space with m markings of the first kind, and $n+1$ markings of the second kind, by Theorem 2.4.

Due to obvious numerical constraints, the range of meaningful geometric applications of these technicalities is surely limited, though it does include a few very interesting examples. In this paper, we will focus on the applications given in Theorems 1.3 and 1.4 below, and to a much lesser extent the application mentioned in §8.1.2, though a few other interesting setups are possible. These applications require and illustrate the technicalities above essentially in full generality.

1.2. Application: configurations on a line modulo scaling or translation.

Consider the following open ended (and seemingly unrelated) problems:

- (1) compactifying the space of configurations of n not necessarily distinct points on a punctured line modulo scaling; and
- (2) compactifying the space of configurations of n not necessarily distinct points on a line modulo translation.

Below are two arguably optimal answers to these problems. We first discuss the constructions concretely over an algebraically closed field \mathbb{K} , and then over \mathbb{Z} . In what follows, \mathbb{G}_m and \mathbb{G}_a will denote the multiplicative, respectively additive group scheme over a ground ring $R \in \{\mathbb{K}, \mathbb{Z}\}$ that will be clear from context. Recall that their formal definitions using functors of points are $\mathbb{G}_m(T) = (\Gamma(T, \mathcal{O}_T^\times), \times)$ and $\mathbb{G}_a(T) = (\Gamma(T, \mathcal{O}_T), +)$ respectively, and thus their respective underlying schemes are $\text{Spec } R[x, y]/(xy - 1)$ and \mathbb{A}_R^1 .

The Losev–Manin space. An n -marked Losev–Manin string is a genus 0 nodal curve C over \mathbb{K} whose dual graph is a chain, with $n+2$ smooth points $p_0, p_\infty, x_1, \dots, x_n \in C(\mathbb{K})$, such that p_0 and p_∞ live on components at opposite ends of the chain, $p_0 \neq p_\infty$ (if there is just one component), $x_i \neq p_0, p_\infty$ for all i , and all components contain at least one of the points x_1, \dots, x_n . (However, $x_i = x_j$ is allowed. The concentric circles in Figure 1 represent overlapping markings.) Note that there is

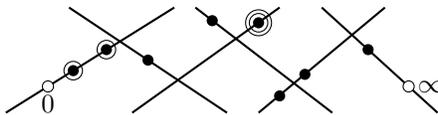


FIGURE 1. A 12-marked Losev–Manin string.

a unique \mathbb{G}_m -action on C which fixes p_0 and p_∞ , and acts with weight 1 on each irreducible component.

The Losev–Manin space $\overline{L}_{n,\mathbb{K}}$ is a variety which parametrizes the n -marked Losev–Manin strings. It is shown in [20] that it is a smooth irreducible toric projective variety of dimension $n-1$. There is an action of \mathbb{G}_m^n on $\overline{L}_{n,\mathbb{K}}$ such that, on \mathbb{K} -points,

$$(c_1, \dots, c_n) \cdot (C, p_0, p_\infty, x_1, \dots, x_n) = (C, p_0, p_\infty, c_1 \cdot x_1, \dots, c_n \cdot x_n),$$

where \cdot on the right hand side is the \mathbb{G}_m -action on C above. The action restricts to a trivial one on the diagonal $\mathbb{G}_m \hookrightarrow \mathbb{G}_m^n$, and thus restricts to the toric \mathbb{G}_m^{n-1} -action on $\{1\} \times \mathbb{G}_m^{n-1} \subset \mathbb{G}_m^n$. Moreover, the open stratum $L_{n,\mathbb{K}} \simeq \mathbb{G}_m^n/\mathbb{G}_m \hookrightarrow \overline{L}_{n,\mathbb{K}}$ where the string has a single component can be thought of as the space of configurations of n points (not necessarily distinct) on $\mathbb{P}^1 \setminus \{0, \infty\}$ modulo scaling.

The construction of the Losev-Manin space in [20, §1.3 and §2.1] clearly goes through over $\text{Spec } \mathbb{Z}$. From now on, we will typically denote this finite type scheme over \mathbb{Z} by \overline{L}_n , and the projective variety above is then $\overline{L}_{n,\mathbb{K}} = \text{Spec } \mathbb{K} \times_{\text{Spec } \mathbb{Z}} \overline{L}_n$.

A modular equivariant compactification of $\mathbb{G}_a^n/\mathbb{G}_a$. A remarkable compactification of the space of configurations of n *distinct* points on \mathbb{A}^1 modulo translation is the moduli space \overline{Q}_n of ‘stable scaled marked curves’ constructed as a projective variety by Ma'u and Woodward in [21], after Ziltener introduced it in a symplectic setting [29, 30]. The moduli space \overline{Q}_n plays an important role in the context of gauged stable maps [27, 5, 6].

We will construct a related space \overline{P}_n which compactifies the space of configurations of n not necessarily distinct points on \mathbb{A}^1 modulo translation. This space will turn out to be an equivariant compactification of \mathbb{G}_a^{n-1} in the sense of [15]. Besides this additional feature and simplicity, Problem 8.3 (discussed at the end of the paper) suggests that the version in which the points may coincide is the suitable object to consider in certain applications.

An n -marked \mathbb{G}_a -rational tree is a connected projective curve C over \mathbb{K} of arithmetic genus 0 with at worst nodal singularities, with a \mathbb{G}_a -action which operates trivially or ‘by translation’ on each irreducible component of C (i.e. $a \cdot [X : Y] = [X + aY : Y]$ in suitable coordinates), and $n+1$ nonsingular points $p_\infty, x_1, \dots, x_n \in C(\mathbb{K})$, such that p_∞ is fixed by the \mathbb{G}_a -action on C , but x_1, \dots, x_n are not. (The condition that \mathbb{G}_a acts trivially or by translation on each component is actually automatic in characteristic 0.) The n -marked \mathbb{G}_a -rational tree is *stable* if any irreducible component of C which doesn't contain any of the points x_1, \dots, x_n , either intersects at least 3 other irreducible components of C , or contains p_∞ and intersects at least 2 other irreducible components of C . Note that all marked points

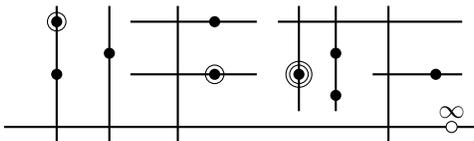


FIGURE 2. A stable 13-marked \mathbb{G}_a -rational tree.

x_1, \dots, x_n live on tail components of C (leaves of the dual tree), and that the action is trivial on all irreducible components of C except the tails if C is reducible.

There exists an (irreducible) projective \mathbb{G}_a^{n-1} -variety $\overline{P}_{n,\mathbb{K}}$ which parametrizes stable n -marked \mathbb{G}_a -rational trees, cf. Remark 7.11. We use G -variety in the sense of [15, Definition 2.1]. The \mathbb{G}_a^n -action on $\overline{P}_{n,\mathbb{K}}$ comes from the \mathbb{G}_a -action on the stable n -marked \mathbb{G}_a -rational trees in precisely the same way the \mathbb{G}_m^n -action on $\overline{L}_{n,\mathbb{K}}$ comes from the \mathbb{G}_m -action on n -marked Losev-Manin strings. Again, we may restrict to $\{0\} \times \mathbb{G}_a^{n-1} \subset \mathbb{G}_a^n$ to mimic modding out the trivial diagonal action. Similarly to \overline{Q}_n , $\overline{P}_{n,\mathbb{K}}$ is mildly singular for $n \geq 4$. The open stratum $P_{n,\mathbb{K}} \simeq \mathbb{G}_a^n/\mathbb{G}_a \hookrightarrow \overline{P}_{n,\mathbb{K}}$

where the rational trees have a single component is the space of configurations of n points on \mathbb{A}^1 modulo translation.

A similar space can be defined over $\text{Spec } \mathbb{Z}$, and one of our main goals is to show that this space is a fine moduli space (as opposed to accomplishing just some naive parametrization). In this context, the additional structure on the curves will be given by a vector field rather than a \mathbb{G}_a -action (though please see Remark 7.11).

Theorem 1.3. *Let F be the functor which associates to each noetherian scheme S the set of all collections of data as follows, modulo isomorphism:*

- a genus 0 prestable curve $\pi : C \rightarrow S$;
- smooth sections $x_1, \dots, x_n, p_\infty : S \rightarrow C$ of π (possibly not disjoint); and
- an \mathcal{O}_C -module homomorphism $\phi : \omega_{C/S} \rightarrow \mathcal{O}_C$,

such that:

- (1) ϕ factors through the inclusion $\mathcal{O}_C(-2p_\infty(S)) \rightarrow \mathcal{O}_C$;
- (2) $x_i^* \phi : x_i^* \omega_{C/S} \rightarrow \mathcal{O}_S$ is an isomorphism for $i = 1, \dots, n$; and
- (3) the natural stability condition holds: for any geometric point $\bar{s} \rightarrow S$,
 - (a) with the possible exception of the component which contains $p_{\infty, \bar{s}}$, no irreducible component of $C_{\bar{s}}$ intersects exactly two other components;
 - (b) any irreducible component of $C_{\bar{s}}$ which intersects exactly one other irreducible component contains at least one of the points $x_{1, \bar{s}}, \dots, x_{n, \bar{s}}$ but not the point $p_{\infty, \bar{s}}$.

Then F is represented by a projective local complete intersection flat geometrically integral scheme (over $\text{Spec } \mathbb{Z}$).

By geometrically integral, we simply mean that all geometric fibers are integral (equivalently, all fibers are geometrically integral in the sense of [26, Tag 020H]).

From now on, \bar{P}_n will denote the moduli space in Theorem 1.3, and the projective variety above will turn out to be $\bar{P}_{n, \mathbb{K}} = \text{Spec } \mathbb{K} \times_{\text{Spec } \mathbb{Z}} \bar{P}_n$.

Relation between \bar{L}_n and \bar{P}_n . Our main result is that the two compactification problems discussed above are in fact related (or at least, the proposed answers are): we will show that \bar{L}_n degenerates to \bar{P}_n , and the actions fit together globally.

First, let's review the fact that \mathbb{G}_m degenerates isotrivially to \mathbb{G}_a , e.g. [17, 3.1]. Consider the commutative cocommutative Hopf $\mathbb{Z}[t]$ -algebra $H = \mathbb{Z}[t, x]_{1+tx}$ with the structure described below, where the maps are always uniquely determined by the property on the right and the requirement that they are $\mathbb{Z}[t]$ -algebra homomorphisms, and x is always the element $x \in H$.

$$\begin{array}{ll}
 \text{(multiplication)} & H \otimes_{\mathbb{Z}[t]} H \rightarrow H \quad x \otimes 1 \mapsto x \text{ and } 1 \otimes x \mapsto x \\
 \text{(comultiplication)} & H \rightarrow H \otimes_{\mathbb{Z}[t]} H \quad x \mapsto x \otimes 1 + 1 \otimes x + tx \otimes x \\
 \text{(unit)} & \mathbb{Z}[t] \rightarrow H \\
 \text{(counit)} & H \rightarrow \mathbb{Z}[t] \quad x \mapsto 0 \\
 \text{(antipode)} & H \rightarrow H \quad x \mapsto -\frac{x}{1+tx}
 \end{array}$$

The verification of the axioms is a tedious exercise left to the patient reader.

Recall that for any ring R , $R[y, y^{-1}]$ has a natural R -Hopf algebra structure with comultiplication characterized by $y \mapsto y \otimes y$. Note that

$$(1) \quad \mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}[t]} H = \mathbb{Z}[t, t^{-1}, x]_{1+tx} \simeq \mathbb{Z}[t, t^{-1}, y, y^{-1}]$$

as $\mathbb{Z}[t, t^{-1}]$ -Hopf algebras. Indeed, the isomorphism is given by $y \mapsto 1 + tx$.

Then $G = \text{Spec } H$ with projection $\gamma : G \rightarrow \text{Spec } \mathbb{Z}[t]$ via the unit in H is a flat group scheme over $\text{Spec } \mathbb{Z}[t]$. Note that $G_{\mathbb{Z}[t,t^{-1}]} \simeq \mathbb{G}_{m,\mathbb{Z}[t,t^{-1}]}$ by (1), and that

$$G_{\bar{z}} \simeq \begin{cases} \mathbb{G}_{m,\bar{z}} & \text{if } t \notin z, \\ \mathbb{G}_{a,\bar{z}} & \text{if } t \in z, \end{cases}$$

for any geometric point $\bar{z} \rightarrow \text{Spec } \mathbb{Z}[t]$ with the corresponding usual point denoted by $z \in \text{Spec } \mathbb{Z}[t]$. We write $G_{\mathbb{Z}[t]}^k = \underbrace{G \times_{\mathbb{Z}[t]} \cdots \times_{\mathbb{Z}[t]} G}_{k \text{ copies of } G}$ for any integer $k \geq 0$.

Theorem 1.4. *For any positive integer n , there exists a flat projective geometrically integral local complete intersection morphism $\xi : X \rightarrow \text{Spec } \mathbb{Z}[t]$, and an action of $G_{\mathbb{Z}[t]}^n$ on X over $\text{Spec } \mathbb{Z}[t]$ such that for any geometric point $\bar{z} \rightarrow \text{Spec } \mathbb{Z}[t]$ (with the corresponding point denoted by $z \in \text{Spec } \mathbb{Z}[t]$),*

- if $t \notin z$, then $X_{\bar{z}}$ is isomorphic to $\bar{L}_{n,\bar{z}}$, and the induced action of $G_{\bar{z}}^n$ on $X_{\bar{z}}$ is isomorphic to the canonical action of $\mathbb{G}_{m,\bar{z}}^n$ on $\bar{L}_{n,\bar{z}}$;
- if $t \in z$, then $X_{\bar{z}}$ is isomorphic to $\bar{P}_{n,\bar{z}}$, and the induced action of $G_{\bar{z}}^n$ on $X_{\bar{z}}$ is isomorphic to the canonical action of $\mathbb{G}_{a,\bar{z}}^n$ on $\bar{P}_{n,\bar{z}}$.

Example 1.5. Recall that $\bar{L}_{3,\mathbb{C}}$ is the blowup of \mathbb{P}^2 at 3 non-collinear points. It can be checked that $\bar{P}_{3,\mathbb{C}}$ is the blowup of \mathbb{P}^2 at 3 collinear points. There exist

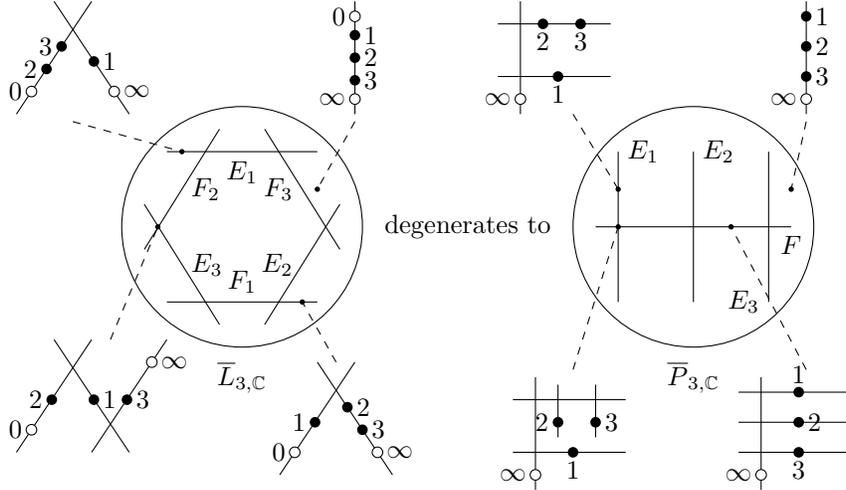


FIGURE 3. The degeneration of $\bar{L}_{3,\mathbb{C}}$ to $\bar{P}_{3,\mathbb{C}}$.

$p_1(t), p_2(t), p_3(t) \in \mathbb{P}^2$ ($t \in \mathbb{C}$), which are collinear if and only if $t = 0$, and such that the blowup of $\mathbb{A}^1 \times \mathbb{P}^2$ at the union of the (images of the) 3 sections $t \mapsto (t, p_i(t))$ can be identified with $X_{\mathbb{C}}$, with X as in Theorem 1.4 for $n = 3$. Let E_1, E_2, E_3 be the exceptional curves when \mathbb{P}^2 is blown up at $p_1(t), p_2(t), p_3(t)$, and let F_i be the proper transform of the line through $p_j(t)$ and $p_k(t)$, for any permutation (i, j, k) of $(1, 2, 3)$. (It would probably be more correct to write $E_{i,t}$ and $F_{i,t}$ instead of E_i and F_i , but we hope this abuse of notation will not lead to confusion.) When $t = 0$, $F_1 = F_2 = F_3$, and we denote this (-2) -curve by F . The table below specifies the flat limits (in $\bar{P}_{3,\mathbb{C}}$) of the strata of the blowups of \mathbb{P}^2 isomorphic to $\bar{L}_{3,\mathbb{C}}$, as $t \rightarrow 0$.

stratum for $t \neq 0$ ($\overline{L}_{3,\mathbb{C}}$)	E_i	F_i	$E_i \cap F_j$ ($i \neq j$)
union of strata for $t = 0$ ($\overline{P}_{3,\mathbb{C}}$)	E_i	$F \cup E_i$	$E_i \cap F$

On the other hand, Figure 3 specifies the modular behaviour on strata (we may ‘consistently’ permute the indices of the markings and strata), so, together with the table above, we obtain a complete picture of the degeneration when $n = 3$.

The generalization of this example to any n , and some other combinatorial issues regarding \overline{P}_n and its deformation to \overline{L}_n will be discussed in a future note.

Remark 1.6. Here are a few further remarks regarding Theorem 1.4.

- (1) In fact, $X \setminus X_{(t)} \simeq \text{Spec } \mathbb{Z}[t, t^{-1}] \times \overline{L}_n$ over $\text{Spec } \mathbb{Z}[t, t^{-1}]$, and the restriction of the $G_{\mathbb{Z}[t]}^n$ -action to $X \setminus X_{(t)}$ is the pullback of the action on \overline{L}_n along $X \setminus X_{(t)} \simeq \text{Spec } \mathbb{Z}[t, t^{-1}] \times \overline{L}_n \rightarrow \overline{L}_n$.
- (2) Although $\overline{L}_{n,\mathbb{C}}$ and $\overline{P}_{n,\mathbb{C}}$ are homeomorphic only for $n \leq 3$ since $\overline{P}_{n,\mathbb{C}}$ is singular for $n \geq 4$, Theorem 1.4 still shows that \overline{L}_n and \overline{P}_n are related topologically.
- (3) The Losev-Manin space \overline{L}_n is the moduli space of weighted pointed stable curves [14] for weight $(1, 1, \epsilon, \dots, \epsilon)$. However, \overline{P}_n doesn’t resemble any of Hassett’s moduli spaces. I would like to thank Valery Alexeev for raising this issue by email, which prompted me to include it here.
- (4) It is very tempting to imagine the p_0 and p_∞ markings for \overline{L}_n coalescing into the p_∞ marking for \overline{P}_n , and then hope to understand the picture with no reference to vector fields. However, this doesn’t work. In fact, the p_0 marking doesn’t even extend to a section over the total space X if $n \geq 2$.

We conclude with some general comments on the proofs of Theorems 1.3 and 1.4. From a ‘modular’ perspective, the key point is that the degeneration X from Theorem 1.4 admits a modular interpretation quite similar to that of \overline{P}_n in Theorem 1.3 if the double vanishing of the vector field at p_∞ is relaxed to a simple vanishing requirement (Theorem 6.3 and Corollary 6.10), and the map to \mathbb{A}^1 comes from the natural trivialization of $\omega_{C/S}^\vee(-p_\infty)/\omega_{C/S}^\vee(-2p_\infty)$ (§6.1), up to an unimportant sign. From a ‘constructive’ perspective, the constructions of \overline{P}_n and X follow the same inductive pattern reminiscent of the inductive construction of $\overline{M}_{0,n}$, with the exception that eliminating degeneracy is done in two steps in our setup. Starting with the trivial case $n = 1$, we repeatedly (1) make a base change to the universal curve at the last step; then (2) apply ‘Knudsen stabilization with vector fields’ (Construction 4.4) and (3) ‘inflate at zero vector’ (Construction 4.14) to eliminate degeneracy. To connect the explicit constructions with the ‘modular’ point of view, we require the inverse operations alluded to in §1.1 (§5). Finally, to deal with the group actions in Theorem 1.4, we first construct a relative action on the family of curves (which requires some careful geometry), then use the modular interpretation of X to ‘transfer’ the group action from the curves to the base by moving around the markings on the curves, as explained above for \overline{L}_n and \overline{P}_n individually.

Another degeneration in the style of that in Theorem 1.4 is sketched in §8.1.2, without a completely formal statement or proof.

Conventions. All schemes in this paper, including those in the definitions of moduli functors or fibered categories, are assumed noetherian. In fact, this assumption can probably be removed with more work, but we will not attempt to do so. ‘ \mathbb{K} ’ is always an algebraically closed field. We write $\mathbb{P}\mathcal{F} = \text{Proj}_X \text{Sym } \mathcal{F}$, for any coherent

\mathcal{O}_X -module \mathcal{F} . A prestable curve is a proper flat morphism (of schemes), whose geometric fibers are connected curves with at worst nodal singularities, e.g. [1, Definition 2.1]. We say that it is of genus g if all geometric fibers have arithmetic genus g . If $\pi : C \rightarrow S$ is a prestable curve and $x : S \rightarrow C$ is a section, we will often abuse notation by writing x instead of $x(S)$ for the scheme-theoretic image of x (sections of separated morphisms are closed immersions [7, 5.4.6]).

Acknowledgments. This project was initiated a few years ago, during my visit at the MSRI. I would like to thank Professor David Eisenbud for interesting discussions and encouragement. I am also grateful to the anonymous referee for many excellent suggestions, which greatly improved the quality of the paper.

This material is based upon work supported by the National Science Foundation under Grant No. 1440140, while the author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2020 semester.



We acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), RGPIN-2020-05497. Cette recherche a été financée par le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG), RGPIN-2020-05497.

2. UNIVERSAL CURVES

We start by introducing some notation.

Definition 2.1. Let $g, m, n \geq 0$ integers. An object of $\mathfrak{V}_{g,m,n}^+$ consists of

- a (noetherian, according to our conventions) scheme S ;
- a prestable curve $\pi : C \rightarrow S$ of genus g ;
- sections $w_1, \dots, w_m : S \rightarrow C$ and $x_1, \dots, x_n : S \rightarrow C$ of π ; and
- an \mathcal{O}_C -module homomorphism $\phi : \omega_{C/S} \rightarrow \mathcal{O}_C$,

satisfying the following conditions

- (1) π is smooth at $w_i(s)$ and $x_j(s)$, for all $s \in S$;
- (2) $w_i(s) \neq w_j(s)$ if $i \neq j$, and $w_i(s) \neq x_j(s)$, for all $s \in S$;
- (3) $w_i^* \phi = 0$ as homomorphism $w_i^* \omega_{C/S} \rightarrow \mathcal{O}_S$, for all i ;
- (4) $x_j^* \phi : x_j^* \omega_{C/S} \rightarrow \mathcal{O}_S$ is an isomorphism for all j .

Note that we allow $x_i \cap x_j \neq \emptyset$. We will sometimes write \bar{x} and \bar{w} instead of (x_1, \dots, x_n) and (w_1, \dots, w_m) . Arrows in $\mathfrak{V}_{g,m,n}^+$ are pullbacks: an arrow

$$(S', C', \pi', \bar{w}', \bar{x}', \phi') \rightarrow (S, C, \pi, \bar{w}, \bar{x}, \phi)$$

is a pair of morphisms $(h : S' \rightarrow S, r : C' \rightarrow C)$ such that the following diagrams

$$\begin{array}{ccc} C' & \xrightarrow{r} & C \\ \pi' \downarrow & & \downarrow \pi \\ S' & \xrightarrow{h} & S \end{array} \quad \begin{array}{ccc} C' & \xrightarrow{r} & C \\ x_j' \uparrow & & \uparrow x_j \\ S' & \xrightarrow{h} & S \end{array} \quad \begin{array}{ccc} C' & \xrightarrow{r} & C \\ w_i' \uparrow & & \uparrow w_i \\ S' & \xrightarrow{h} & S \end{array}$$

are commutative and the first one is cartesian, and ϕ' corresponds to the pullback of ϕ under the isomorphism $C' \rightarrow C_{S'}$ induced by the first diagram.

An object of $\mathfrak{C}_{g,m,n}^+$ is defined verbatim the same as an object of $\mathfrak{V}_{g,m,n}^+$, with the sole exception that, as part of the data, we consider an additional section $x : S \rightarrow C$ of π . No conditions involve this section in any way. Arrows are pullbacks again (as above, and another commutative diagram saying $rx' = xh$).

Let $\mathfrak{V}_{g,m,n}$ (respectively $\mathfrak{C}_{g,m,n}$) be the full subcategory of $\mathfrak{V}_{g,m,n}^+$ (respectively $\mathfrak{C}_{g,m,n}^+$) consisting of objects for which $\omega_{C/S}(w_1 + \cdots + w_m + 2x_1 + \cdots + 2x_n)$ is π -ample (by condition 1, x_j and w_i are Cartier divisors on C).

Instead of imposing condition 3, we could have equally well defined ϕ directly as a global section of $\omega_C^\vee(-w_1 - \cdots - w_m)$. Depending on context, we will sometimes regard ϕ as a section of ω_C^\vee , and sometimes as a section of $\omega_C^\vee(-w_1 - \cdots - w_m)$.

Unless $2g + m \leq 2$, a curve in $\mathfrak{V}_{g,m,n}(\mathbb{K})$ with a vector field that is not everywhere 0 must be reducible, which is quite artificial, though shouldn't be discarded altogether, as illustrated by §8.1.1. Examples 2.2 and 2.3 below aim to clarify what types of curves we can expect to see in $\mathfrak{V}_{g,m,n}(\mathbb{K})$. They are slightly informal, since they will not be used in any proofs.

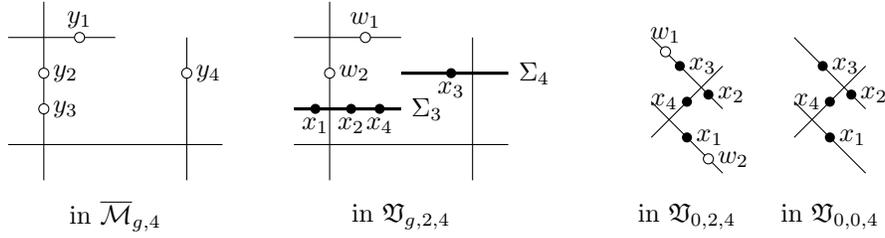


FIGURE 4. The two pictures on the left show how a curve in $\mathfrak{V}_{g,2,4}$ can be obtained from a curve in $\overline{\mathcal{M}}_{g,4}$. The two pictures on the right show two ‘exceptional’ situations which can’t be obtained by a construction of this type. (The examples are very similar, though the two vanishing points of the field are ‘ w ’-markings in one example but not in the other.)

Example 2.2. Given g, m, n, N such that $m < N \leq m + n$, there is a simple way to systematically produce curves in $\mathfrak{V}_{g,m,n}(\mathbb{K})$ out of curves (Y, y_1, \dots, y_N) in $\overline{\mathcal{M}}_{g,N}(\mathbb{K})$. Let $\lambda : \{1, \dots, n\} \rightarrow \{m+1, \dots, N\}$ be a surjective function. Let

$$C = Y \cup \Sigma_{m+1} \cup \cdots \cup \Sigma_N, \quad \text{where } \Sigma_{m+1} \simeq \cdots \simeq \Sigma_N \simeq \mathbb{P}^1,$$

and each Σ_i is attached transversally to Y at y_i , for $i = m+1, \dots, N$, $w_j = y_j$ for $j = 1, \dots, m$, and $\phi \in \Gamma(\omega_C^\vee)$ whose restriction to Y is everywhere 0, and whose restriction to $\Sigma_i \setminus \{y_i\} \simeq \mathbb{A}^1$ is $\frac{d}{dx}$ in suitable coordinates. Finally, let us choose $x_\alpha \in \Sigma_{\lambda(\alpha)} \setminus \{y_{\lambda(\alpha)}\}$ arbitrary, for $\alpha = 1, \dots, n$ (with automorphisms suitably taken into account, there are $n + m - N$ ‘moduli’ for x_1, \dots, x_n). Then,

$$(C, w_1, \dots, w_m, x_1, \dots, x_n, \phi) \in \mathfrak{V}_{g,m,n}(\mathbb{K}),$$

by $\omega_Y(y_1 + \cdots + y_N)$ ample, the classical description of dualizing sheaves in terms of meromorphic forms, and Definition 2.1. An example is shown in Figure 4, on the left. (As a trivial extension of this example to the case $m = N$, $n = 0$, we have a functor $\overline{\mathcal{M}}_{g,m} \rightarrow \mathfrak{V}_{g,m,0}$ by taking the vector field to be equal to 0.)

Example 2.3. Can all curves in $\mathfrak{V}_{g,m,n}(\mathbb{K})$ be obtained using the construction in Example 2.2? The answer for most values of g, m, n is positive if N is allowed variable (the proof of Proposition 5.2 implicitly contains the justification of this assertion), although there are a few exceptions, such as

- (1) smooth genus 0 curves with a vector field of the form $\frac{d}{dx}$ in suitable coordinates, in $\mathfrak{V}_{0,0,n}$ and $\mathfrak{V}_{0,1,n}$;
- (2) chains similar to those shown on the right side of Figure 4 (and also of the ‘mixed’ type, with one vanishing point a ‘ w ’-marking and one not, which is important in the proof of Theorem 1.4), in $\mathfrak{V}_{0,0,n}$, $\mathfrak{V}_{0,1,n}$, $\mathfrak{V}_{0,2,n}$; or
- (3) genus 1 curves (either smooth, or cycles of projective lines) with a nowhere vanishing vector field, in $\mathfrak{V}_{1,0,n}$.

The main technical result is the following.

Theorem 2.4. *If $2g + 2n + m \geq 3$, then the (fibered) categories $\mathfrak{V}_{g,m,n+1}$ and $\mathfrak{C}_{g,m,n}$ are equivalent.*

As we will see in a moment, the equivalence between the two categories is constructed explicitly, and, in fact, it is given by the composition of Constructions 4.4 and 4.14. Indeed, unlike in [18], the passage from the universal curve to the family of curves with an extra marking requires not one, but two steps. To prove Theorem 2.4, we introduce a class of curves $\mathfrak{C}_{g,m,n}^2$ which will serve as an intermediate step between $\mathfrak{V}_{g,m,n+1}$ and $\mathfrak{C}_{g,m,n}$.

Definition 2.5. For $c \in \{1, 2, 3\}$, let $\mathfrak{C}_{g,m,n}^{c,+}$ be the full subcategory of $\mathfrak{C}_{g,m,n}^+$ (cf. Definition 2.1) which consists of objects $(S, C, \pi, \bar{w}, \bar{x}, x, \phi)$ such that:

- (1) if $c \geq 2$, π is smooth at $x(s)$ and $w_i(s) \neq x(s)$, for all $s \in S$ and $i = 1, \dots, n$;
- (2) if $c = 3$, $x^* \phi : x^* \omega_{C/S} \rightarrow \mathcal{O}_S$ is an isomorphism.

Let $\mathfrak{C}_{g,m,n}^c$ be the full subcategory of $\mathfrak{C}_{g,m,n}^{c,+}$ whose objects also satisfy:

- (3) $\omega_{C/S}(w_1 + \dots + w_m + 2x_1 + \dots + 2x_n + (c-1)x)$ is π -ample.

Remark 2.6. Note that $\mathfrak{V}_{g,m,n+1}$ is isomorphic to $\mathfrak{C}_{g,m,n}^3$ and $\mathfrak{C}_{g,m,n}$ to $\mathfrak{C}_{g,m,n}^1$.

As we will see in Proposition 5.1, the constructions in §4 induce functors

$$(2) \quad \mathfrak{C}_{g,m,n}^1 \longrightarrow \mathfrak{C}_{g,m,n}^2 \longrightarrow \mathfrak{C}_{g,m,n}^3.$$

Theorem 2.7. *The functors in (2) give equivalences $\mathfrak{C}_{g,m,n}^1 \simeq \mathfrak{C}_{g,m,n}^2$ if $2g + 2n + m \geq 3$, respectively $\mathfrak{C}_{g,m,n}^2 \simeq \mathfrak{C}_{g,m,n}^3$ if $2g + 2n + m \geq 2$.*

In light of Remark 2.6, Theorem 2.4 follows from Theorem 2.7.

Although the results are very similar and the techniques overlap to a significant extent, there does not seem to be a clear-cut mathematical relation between Theorem 2.4 and Knudsen’s theorem that $\overline{\mathcal{M}}_{g,N+1}$ is the universal curve over $\overline{\mathcal{M}}_{g,N}$.

Theorem 2.4 should be taken as a purely formal result. However, especially in the range $2g + m \leq 2$, there is reasonable amount of ‘elbow room’ for interesting geometry, and it is usually the ‘exceptional’ situations (Example 2.3) that lead to more surprising phenomena. This will be illustrated using Theorems 1.3 and 1.4 (with $(g, m) = (0, 1)$) and §8.1.2 (with $(g, m) = (1, 0)$), though there are other setups which are at least nontrivial, even if arguably less aesthetic.

3. CONTRACTING COMPONENTS OF PRESTABLE CURVES

3.1. Morphisms between prestable curves. We start by collecting some basic technicalities needed to deal with contracting (rational) irreducible components of prestable curves in families. These technicalities are quite well-known (cf. [26, Tag 0E7B] for some deeper aspects than the ones discussed here).

Definition 3.1. Let $\pi : X \rightarrow S$ and $\rho : Y \rightarrow S$ be prestable curves, and $f : X \rightarrow Y$ an S -morphism. We say that f has property R if $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is an isomorphism and $R^1f_*\mathcal{O}_X = 0$, and these hold universally, that is, $f_{S'}^\# : \mathcal{O}_{Y_{S'}} \rightarrow f_{S',*}\mathcal{O}_{X_{S'}}$ is an isomorphism and $R^1f_{S',*}\mathcal{O}_{X_{S'}} = 0$ for all $S' \rightarrow S$.

Remark 3.2. In the situation of Definition 3.1, the map $\mathcal{L} \rightarrow f_*f^*\mathcal{L}$ is an isomorphism, for any $\mathcal{L} \in \text{Pic}(Y)$. This is actually true for any morphism f for which $f^\#$ is an isomorphism.

Lemma 3.3. *If $\pi : X \rightarrow S$ and $\rho : Y \rightarrow S$ are prestable curves, and $f : X \rightarrow Y$ is an S -morphism, then f has property R if and only if the property in Definition 3.1 holds on the geometric fibers, i.e. $f_{\bar{s}}^\# : \mathcal{O}_{Y_{\bar{s}}} \rightarrow f_{\bar{s},*}\mathcal{O}_{X_{\bar{s}}}$ is an isomorphism and $R^1f_{\bar{s},*}\mathcal{O}_{X_{\bar{s}}} = 0$ for all geometric points $\bar{s} \rightarrow S$.*

Proof. The ‘only if’ direction is trivial. [26, Tag 0E88] reduces the ‘if’ direction to the special case when S is the spectrum of a field. For the lack of a reference, we explain this elementary case in some detail. Let $S = \text{Spec } K$, and $\alpha : X_{\bar{K}} \rightarrow X$ and $\beta : Y_{\bar{K}} \rightarrow Y$ the natural projections. The base change map $\beta^*R^i f_*\mathcal{O}_X \rightarrow R^i f_{\bar{K},*}(\alpha^*\mathcal{O}_X) = R^i f_{\bar{K},*}\mathcal{O}_{X_{\bar{K}}}$ is an isomorphism for all i by the cohomology and flat base change theorem. Then $\beta^*R^1 f_*\mathcal{O}_X = 0$, so $R^1 f_*\mathcal{O}_X = 0$. Moreover, $\beta^*f^\# : \beta^*\mathcal{O}_Y \rightarrow \beta^*f_*\mathcal{O}_X$ is an isomorphism because it fits in a commutative diagram with $\beta^*\mathcal{O}_Y = \mathcal{O}_{Y_{\bar{K}}}$, $f_{\bar{K}}^\#$, and $\beta^*f_*\mathcal{O}_X \rightarrow f_{\bar{K},*}\mathcal{O}_{X_{\bar{K}}}$, which are all isomorphisms. It follows that $f^\#$ is an isomorphism. \square

The following remark will not be used in any proof (so its proof will be omitted), but it is important for context and surely well-known.

Remark 3.4. If $f : X \rightarrow Y$ has property R and $S = \text{Spec } \mathbb{K}$, then, it can be shown that f is obtained by repeatedly contracting *rational tails* and/or *rational bridges*. (Summary of an inductive argument: (1) unless it is an isomorphism, f must contract some tail or bridge $\Sigma \simeq \mathbb{P}^1 \subset X$ by some standard H^1 calculations; (2) f factors as $X \xrightarrow{h} X' \xrightarrow{f'} Y$, where $h : X \rightarrow X'$ is the contraction of Σ , by [1, Lemma 2.2]; and (3) f' has property R by some purely formal/cohomological arguments, such as the five-term sequence of the Grothendieck spectral sequence.) Another way to look at this fact amounts to the following slightly imprecisely phrased ‘picture’ of morphisms with property R (when $S = \text{Spec } \mathbb{K}$): f is obtained by contracting several disjoint rational trees in X which intersect the union of the other components of X at either 1 (tail-like) or 2 (bridge-like) points (Figure 5).

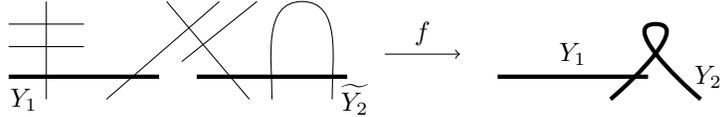


FIGURE 5. A map f with property R.

Lemma 3.5. *In the situation of Definition 3.1, let $Y^\circ \subseteq Y$ consist of all $y \in Y$ such that $f^{-1}(y)$ is a singleton set-theoretically, and $X^\circ = f^{-1}(Y^\circ)$. Then:*

- (1) Y° is open in Y , and X° is open in X .

- (2) $Y \setminus Y^\circ$ is finite over S .
- (3) The restriction of f to X° induces an isomorphism $X^\circ \simeq Y^\circ$. Moreover, if $U \subseteq Y$ is open, then the restriction of f to $f^{-1}(U)$ induces an isomorphism $U \simeq f^{-1}(U)$ if and only if $U \subseteq Y^\circ$.
- (4) The formation of Y° and X° commutes with base change.

Proof. The fibers of f are connected by Zariski's connectedness theorem [9, Corollaire (4.3.2)] and hence they are either positive dimensional or singletons set-theoretically, since they are of finite type over a field. Item 1 then follows from Chevalley's theorem [10, Corollaire (13.1.5)]. Item 4 is set-theoretic and clear, since the set underlying the preimage of an open subscheme is the set-theoretic preimage of the underlying set. Item 4 reduces item 2 to the case when S is the spectrum of a field, when it is clear – quasi-finiteness suffices, since $Y \setminus Y^\circ$ is proper over S . For item 3, the general observation is that, if $f : X \rightarrow Y$ is a proper morphism such that $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is an isomorphism, then f is an isomorphism if and only if it is a bijection. Indeed, f is a continuous closed bijection, hence a homeomorphism, and we are assuming that $f^\#$ is an isomorphism. \square

The practical use of Lemma 3.5 is that it often allows us to easily check on X° or Y° nontrivial statements about f on X or Y , and then ‘bootstrap’ to X or Y , using tricks such as the following.

Lemma 3.6. *If $\pi : X \rightarrow S$ is a prestable curve, and $U \subseteq X$ is an open whose complement $X \setminus U$ is finite over S , then the restriction map $\Gamma(V, \mathcal{L}) \rightarrow \Gamma(U \cap V, \mathcal{L})$ is injective for any invertible \mathcal{L} , and any open $V \subseteq X$.*

Proof. It suffices to prove the lemma for $\mathcal{L} = \mathcal{O}_X$. By [26, Tag 0B3L], it suffices to prove that all associated points of V are in $U \cap V$. If S is the spectrum of a field, then X is reduced and hence V is reduced, so all associated points of V are generic points of irreducible components of X by [26, Tag 05AL] and the well-known fact that noetherian reduced schemes have no embedded components, and the claim follows in this special case. In the general case, if x is an associated point of V , then it is also an associated point of $(\pi|_V)^{-1}(\pi(x))$ by [26, Tag 05DB], which boils down the claim to the special case in the previous sentence, completing the proof. \square

3.2. Differentials. Next, we discuss the differentials of morphisms with property R. These will allow us to push forward vector fields, and turn out to be an essential operation later in the paper. The differentials will be constructed using a simple case of coherent duality. (Logarithmic structures seem a natural approach, but, with hindsight, they seem a worse option overall.) For the basic properties of the relative dualizing sheaf of a prestable curve, please see [18, §1] and [26, Tag 0E6N].

Lemma 3.7. *In the situation of Definition 3.1, if \mathcal{L} is an invertible \mathcal{O}_Y -module, then $f^!\mathcal{L} \simeq \omega_{X/S} \otimes f^*\omega_{Y/S}^\vee \otimes f^*\mathcal{L}$ regarded as a complex in degree 0.*

Proof. We have $\pi^!\mathcal{O}_S = \omega_{X/S}[1]$ and $\rho^!\mathcal{O}_S = \omega_{Y/S}[1]$. Moreover, $f^!$ is the right adjoint of Rf_* because f is proper. By [26, Tag 0A9T],

$$(3) \quad f^!(\omega_{Y/S}[1]) = f^!(\omega_{Y/S}[1] \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \mathcal{O}_Y) = Lf^*\omega_{Y/S}[1] \otimes_{\mathcal{O}_X}^{\mathbf{L}} f^!\mathcal{O}_Y,$$

since noetherian ensures quasi-compact and quasi-separated. Therefore,

$$\begin{aligned}\omega_{X/S}[1] &= \pi^! \mathcal{O}_S = f^!(\rho^! \mathcal{O}_S) \quad \text{by [26, Tag 0ATX]} \\ &= f^!(\omega_{Y/S}[1]) = Lf^* \omega_{Y/S}[1] \otimes_{\mathcal{O}_X}^{\mathbf{L}} f^! \mathcal{O}_Y \quad \text{by (3)}\end{aligned}$$

so $f^! \mathcal{O}_Y \simeq \omega_{X/S} \otimes f^* \omega_{Y/S}^\vee$ as a complex in degree 0. However, $f^! \mathcal{L} = Lf^* \mathcal{L} \otimes_{\mathcal{O}_X}^{\mathbf{L}} f^! \mathcal{O}_Y$ by another application of [26, Tag 0A9T], and the lemma follows. \square

By Lemma 3.7, we may think of $f^! \mathcal{L}$ as an invertible sheaf rather than an object of the derived category. Lemma 3.7 also shows that the formation of $f^! \mathcal{L}$ commutes with base changes $S' \rightarrow S$, since the formation of $\omega_{X/S}$ and $\omega_{Y/S}$ commutes with base change [26, Tag 0E6R].

Example 3.8. If $f : X \rightarrow Y$ over $S = \text{Spec } \mathbb{K}$ has property R, and \mathcal{L} is a line bundle on Y , $f^! \mathcal{L}$ can be understood concretely using Lemma 3.7. In particular, if $\Sigma \simeq \mathbb{P}^1$ is a contracted irreducible component of X , then $\deg(f^! \mathcal{L})|_\Sigma = \deg(\omega_X)|_\Sigma = n_\Sigma - 2$, where n_Σ is the number of nodes of X on Σ . For instance, Figure 6 illustrates this on the map in Figure 5, for an arbitrary \mathcal{L} on Y . On non-contracted components, in this example, we have $(f^! \mathcal{L})|_{Y_1} = \mathcal{L}|_{Y_1} \otimes \mathcal{O}_{Y_1}(p)$ and $(f^! \mathcal{L})|_{\widetilde{Y}_2} = \nu^*(\mathcal{L}|_{Y_2})$ by Lemma 3.7 again, where ν is the normalization of Y_2 , and p is the point indicated in Figure 6.

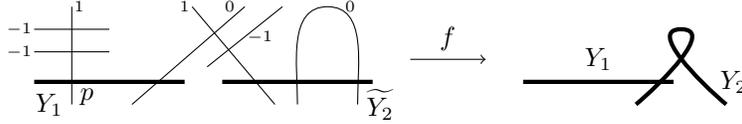


FIGURE 6. The integers on the contracted components of the source specify the degrees of $f^! \mathcal{L}$ on the respective components.

Another concrete way to understand $f^! \mathcal{L}$ when $S = \text{Spec } \mathbb{K}$ combines:

- (1) if $f : X \rightarrow Y$ is the contraction of a rational bridge, then $f^! \mathcal{L} = f^* \mathcal{L}$;
- (2) if f is the contraction of a rational tail $T \simeq \mathbb{P}^1 \subset X$, then $f^! \mathcal{L} \simeq \mathcal{J} \otimes f^* \mathcal{L}$, where $\mathcal{J}|_T \simeq \mathcal{O}_T(-1)$, and $\mathcal{J}|_W = \mathcal{O}_W(p)$, where W is the closure of $X \setminus T$, and $\{p\} = T \cap W$ (this follows from Lemma 3.7 again);

and the functoriality $(f^! g^! = (gf)^!)$ of ‘!’. Note also that $\deg f^! \mathcal{L} = \deg \mathcal{L}$.

Situation 3.9. Let $\pi : X \rightarrow S$ and $\rho : Y \rightarrow S$ be prestable curves over S , and $f : X \rightarrow Y$ with property R. Let $x_1, \dots, x_n : S \rightarrow X$ be sections of π , and $y_i = f x_i : S \rightarrow Y$ the corresponding sections of ρ , $i = 1, \dots, n$. Assume that:

- (1) π is smooth at $x_i(s)$ and ρ is smooth at $y_i(s)$, for all $s \in S$, and $i = 1, \dots, n$;
- (2) the sections x_1, \dots, x_n respectively y_1, \dots, y_n are pairwise disjoint.

In Situation 3.9, $x_i \subset X$ and $y_i \subset Y$ (cf. Conventions subsection) are effective Cartier divisors.

Proposition 3.10. *In Situation 3.9, the following hold.*

- (1) The isomorphism $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ restricts to an isomorphism
- (4)
$$\mathcal{O}_Y(-y_1 - \dots - y_n) \simeq f_* \mathcal{O}_X(-x_1 - \dots - x_n).$$

Moreover, $R^1 f_* \mathcal{O}_X(-x_1 - \dots - x_n) = 0$.

(2) For any invertible \mathcal{O}_Y -module \mathcal{L} , there exists an \mathcal{O}_X -module homomorphism

$$(5) \quad \phi : f^*(\mathcal{L}(y_1 + \cdots + y_n)) \rightarrow (f^!\mathcal{L})(x_1 + \cdots + x_n)$$

such that for any open $U \subseteq Y^\circ$, $\phi|_{f^{-1}(U)}$ is the isomorphism $f^*(\mathcal{L}(y_1 + \cdots + y_n))|_{f^{-1}(U)} \simeq (f^!\mathcal{L})(x_1 + \cdots + x_n)|_{f^{-1}(U)}$ induced by $(f|_U)^* = (f|_U)^!$.

Proof. Let $D = \sum_{i=1}^n x_i$ and $E = \sum_{i=1}^n y_i$. If

$$0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \bigoplus_{i=1}^n x_{i,*}\mathcal{O}_S \rightarrow 0$$

is pushed forward along f , we obtain a (solid arrow) commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & f_*\mathcal{O}_X(-D) & \rightarrow & f_*\mathcal{O}_X & \rightarrow & f_*\bigoplus_{i=1}^n x_{i,*}\mathcal{O}_S \rightarrow R^1f_*\mathcal{O}_X(-D) \rightarrow \cdots \\ & & \uparrow \widehat{} & & \uparrow f^\# & & \parallel \\ 0 & \rightarrow & \mathcal{O}_Y(-E) & \rightarrow & \mathcal{O}_Y & \rightarrow & \bigoplus_{i=1}^n y_{i,*}\mathcal{O}_S \rightarrow 0. \end{array}$$

Then $f_*\mathcal{O}_X \rightarrow f_*\bigoplus_{i=1}^n x_{i,*}\mathcal{O}_S$ is surjective as $\mathcal{O}_Y \rightarrow \bigoplus_{i=1}^n y_{i,*}\mathcal{O}_S$ is surjective. We claim that

$$(6) \quad R^j f_*\mathcal{O}_X(-D) = 0$$

for all $j > 0$. First, $R^1 f_* x_{i,*}\mathcal{O}_S = 0$ because $R^1 y_{i,*}\mathcal{O}_S = 0$ (as y_i is finite) and $0 \rightarrow R^1 f_* x_{i,*}\mathcal{O}_S \rightarrow R^1 y_{i,*}\mathcal{O}_S$ is the beginning of the five-term sequence of the Grothendieck spectral sequence. Second, in the piece

$$R^{j-1} f_*\mathcal{O}_X \rightarrow R^{j-1} f_*\bigoplus_{i=1}^n x_{i,*}\mathcal{O}_S \rightarrow R^j f_*\mathcal{O}_X(-D) \rightarrow R^j f_*\mathcal{O}_X$$

of the top row, the first map is surjective (if $j = 1$, we've shown above that it is surjective; if $j = 2$, we've shown above that the second term is 0; if $j \geq 3$, the second term is trivially 0 [9, Corollaire (4.2.2)]), and the last term is 0 by [9, Corollaire (4.2.2)] again or Definition 3.1, completing the proof of (6).

In particular, there exists a unique \mathcal{O}_Y -module homomorphism $\gamma : \mathcal{O}_Y(-E) \rightarrow f_*\mathcal{O}_X(-D)$ that can play the role of the dashed arrow and make the diagram commute, and this γ is an isomorphism, completing the proof of part 1. Combining with (6), we obtain a quasi-isomorphism

$$(7) \quad Rf_*\mathcal{O}_X(-D) \simeq \mathcal{O}_Y(-E).$$

To explain the last step in more detail, let $0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{J}^0 \rightarrow \mathcal{J}^1 \rightarrow \cdots$ be an injective resolution of $\mathcal{O}_X(-D)$ in $\mathbf{Qcoh}(X)$. We have $H^0(f_*\mathcal{J}^\bullet) = \mathcal{O}_Y(-E)$ and $H^j(f_*\mathcal{J}^\bullet) = 0$ for $j \neq 0$, and hence the map of complexes $\mathcal{O}_Y(-E) \rightarrow f_*\mathcal{J}^\bullet = [\cdots \rightarrow 0 \rightarrow f_*\mathcal{J}^0 \rightarrow f_*\mathcal{J}^1 \rightarrow \cdots]$, with $\mathcal{O}_Y(-E)$ in degree 0, which maps $\mathcal{O}_Y(-E) \cong \text{Ker}(f_*\mathcal{J}^0 \rightarrow f_*\mathcal{J}^1)$ to $f_*\mathcal{J}^0$ in the natural way, is a quasi-isomorphism. Since f is proper, Rf_* and $f^!$ are adjoint, so there is a bijection

$$(8) \quad \begin{aligned} \text{Hom}_{D+(\mathbf{Qcoh}(Y))}(Rf_*\mathcal{O}_X(-D), \mathcal{O}_Y(-E)) &\simeq \\ \text{Hom}_{D+(\mathbf{Qcoh}(X))}(\mathcal{O}_X(-D), f^!\mathcal{O}_Y(-E)), \end{aligned}$$

and let $\mathcal{O}_X(-D) \rightarrow f^!\mathcal{O}_Y(-E)$ correspond to (7) under this bijection. Taking H^0 , we may think of this as a map of \mathcal{O}_X -modules, cf. Lemma 3.7. Twisting by $\mathcal{O}_X(D) \otimes f^*(\mathcal{L}(E))$ on both sides and keeping Lemma 3.7 in mind, we obtain the desired homomorphism $\phi : f^*(\mathcal{L}(E)) \rightarrow (f^!\mathcal{L})(D)$. The fact that ϕ restricts to the ‘obvious’ isomorphism on each open subset $U \subseteq Y$ on which f induces an isomorphism $f^{-1}(U) \simeq U$ follows from the fact that duality behaves naturally relative to restricting to open subschemes. \square

Lemma 3.11. *In Situation 3.9, assume that*

$$(9) \quad f_{\bar{s}}^* \omega_{Y_{\bar{s}}}(y_{1,\bar{s}} + \cdots + y_{n,\bar{s}}) \simeq \omega_{X_{\bar{s}}}(x_{1,\bar{s}} + \cdots + x_{n,\bar{s}})$$

for all geometric points $\bar{s} \rightarrow S$. Then there exists a unique homomorphism ϕ as in part 2 of Proposition 3.10, and this ϕ is an isomorphism. Moreover, its formation commutes with base change.

Proof. Let $D = \sum_{i=1}^n x_i$ and $E = \sum_{i=1}^n y_i$. By Lemma 3.7, we have an isomorphism

$$(10) \quad \text{Hom}(f^*(\mathcal{L}(E)), (f^!\mathcal{L})(D)) \simeq \text{Hom}(f^*(\omega_{Y/S}(E)), \omega_{X/S}(D)).$$

Let \mathcal{H} be the left hand side of (10), so that $\phi \in \Gamma(X, \mathcal{H})$. We have $\mathcal{H}_{\bar{s}} \simeq \mathcal{O}_{X_{\bar{s}}}$ for all geometric points $\bar{s} \rightarrow S$, by (10) and the assumption, hence $\phi_{\bar{s}} \in \Gamma(X_{\bar{s}}, \mathcal{H}_{\bar{s}})$ is nowhere vanishing since it isn’t identically 0 (for instance, by Lemma 3.5), and the fact that $X_{\bar{s}}$ is connected and proper. Then ϕ is a nowhere vanishing section of \mathcal{H} , thus an isomorphism, proving one of the claims. Uniqueness follows from Lemmas 3.5 and 3.6. Commutativity with respect to base change follows from uniqueness, combined with items 3 and 4 in Lemma 3.5. \square

Definition 3.12. Let $f : X \rightarrow Y$ be a map with property R, and let \mathcal{L} be an invertible \mathcal{O}_Y -module. Then define $f^i\mathcal{L} = (f^!\mathcal{L}^\vee)^\vee$.

We’ve seen earlier that the formation of $f^!(\mathcal{L}^\vee)$ commutes with base change, and hence so does the formation of $f^i\mathcal{L}$, since $f^!\mathcal{L}^\vee$ is invertible by Lemma 3.7.

Example 3.13. This is the analogue of Example 3.8 for f^i . Note that the degrees on contracted components switch sign from $f^!\mathcal{L}$ to $f^i\mathcal{L}$, cf. Figures 7 and 6. In the same example, we also have $(f^i\mathcal{L})|_{Y_1} = \mathcal{L}|_{Y_1} \otimes \mathcal{O}_{Y_1}(-p)$ and $(f^i\mathcal{L})|_{\widetilde{Y}_2} = \nu^*(\mathcal{L}|_{Y_2})$, with notation similar to that in Example 3.8.

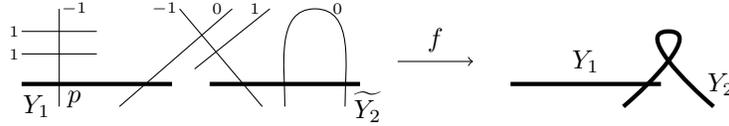


FIGURE 7. The integers on the contracted components of the source indicate the degrees of $f^i\mathcal{L}$ on the respective components.

Proposition 3.14. *In Situation 3.9, there exists a unique homomorphism*

$$(11) \quad \psi : f_*((f^i\mathcal{L})(-x_1 - \cdots - x_n)) \rightarrow \mathcal{L}(-y_1 - \cdots - y_n).$$

such that, for any open $U \subseteq Y$ above which f restricts to an isomorphism $f^{-1}(U) \simeq U$, $\psi|_U$ is the natural isomorphism induced by $f^i\mathcal{L}|_U \cong f^*\mathcal{L}|_U$.

If condition (9) holds for all geometric points $\bar{s} \rightarrow S$, then ψ is an isomorphism.

Moreover, the formation of ψ is compatible with base change in the following sense: if $h : S' \rightarrow S$ is a morphism, X', Y', \dots is the pullback along h of the data in Situation 3.9, $m : Y' = S' \times_S Y \rightarrow Y$ is the projection map, and ψ' is the analogue of (11), then the diagram

$$\begin{array}{ccc} m^* f_*((f^i \mathcal{L})(-x_1 - \dots - x_n)) & \xrightarrow{m^* \psi} & m^* \mathcal{L}(-y_1 - \dots - y_n) \\ \downarrow & & \parallel \\ f'_*((f^i \mathcal{L}')(-x'_1 - \dots - x'_n)) & \xrightarrow{\psi'} & \mathcal{L}'(-y'_1 - \dots - y'_n) \end{array}$$

in which the left vertical map comes from the standard base change map $m^* f_*(\cdot) \rightarrow f'_* p^*(\cdot)$ (where $p : X' = S' \times_S X \rightarrow X$ is the projection map) and the remark before Definition 3.12, is commutative.

Proof. Let $D = \sum_{i=1}^n x_i$ and $E = \sum_{i=1}^n y_i$, and let $f^*(\mathcal{L}^\vee(E)) \rightarrow (f^i \mathcal{L}^\vee)(D)$ obtained by replacing \mathcal{L} with \mathcal{L}^\vee in (5). Dualizing, we obtain $(f^i \mathcal{L})(-D) \rightarrow f^*(\mathcal{L}(-E))$. Pushing the last homomorphism forward along f with Remark 3.2 in mind, we obtain $f_*((f^i \mathcal{L})(-D)) \rightarrow \mathcal{L}(-E)$, which is (11). This proves existence. The isomorphism criterion follows easily from Lemma 3.11 and the construction of ψ above. Both uniqueness and compatibility with base change follow from Lemmas 3.5 and 3.6. Since uniqueness can be regarded as a sort of special case of compatibility with base change, we only explain the latter. Let $Y^\circ \subseteq Y$ as in Lemma 3.5. It is clear that the restriction of the square diagram to $m^{-1}(Y^\circ)$ is commutative. Then the square diagram commutes a fortiori. Indeed, the two images in $\mathcal{L}'(-y'_1 - \dots - y'_n)$ of any local section of $f'_*((f^i \mathcal{L}')(-x'_1 - \dots - x'_n))$ above some open $V \subseteq Y$ via the two possible routes restrict to the same local section of $\mathcal{L}'(-y'_1 - \dots - y'_n)$ on $V \cap m^{-1}(Y^\circ)$, so they must coincide on V by Lemma 3.6, which applies in our situation in light of items 2 and 4 in Lemma 3.5. \square

4. BUBBLING UP

4.1. Knudsen stabilization with sections of line bundles. In §4.1, we generalize to families the operation called ‘Knudsen stabilization with vector fields’ in §1.1. (We also allow for sections of arbitrary line bundles instead of vector fields, though this won’t be a major improvement in generality.) There is very little to add to Knudsen’s well-known work; we will merely check that, if we have a vector field, it lifts uniquely to one with the desired properties.

Situation 4.1. Assume that $\pi : C \rightarrow S$ is a prestable curve, x is a section of π , \mathcal{L} is an invertible \mathcal{O}_C -module, $w_1, \dots, w_m : S \rightarrow C$ are disjoint smooth sections of π , and σ is a global section of $\mathcal{L}(-w_1 - \dots - w_m)$.

In §4.1, we consider such data *degenerate* if π is singular at $x(s)$ for some $s \in S$, or $x(s) = w_i(s)$, for some $s \in S$ and $i \in \{1, \dots, m\}$.

In Situation 4.1, we construct $S_{\ddagger} = S$, C_{\ddagger} , π_{\ddagger} , x_{\ddagger} , \mathcal{L}_{\ddagger} , σ_{\ddagger} , $w_{1,\ddagger}, \dots, w_{m,\ddagger}$ such that the new data satisfies the requirements of Situation 4.1 and is nondegenerate, and the construction commutes with base change (in a manner which satisfies the natural ‘cocycle conditions’, at least implicitly). Moreover, the construction will also provide a morphism $f : C_{\ddagger} \rightarrow C$ with property R, which satisfies some natural compatibilities: $f x_{\ddagger} = x$, $f w_{i,\ddagger} = w_i$, and a few others explained below. If \mathcal{L} and

σ weren't in discussion, this would be nothing else but the famous stabilization procedure of Knudsen [18, §2].

Example 4.2. We discuss the desired effect of the construction when $S = \text{Spec } \mathbb{K}$ and the input data is degenerate, in advance of the general statement. Observation (12) in this example will be referred to in Construction 4.4, though it is easy to see that there is no logical circularity, and this order facilitates exposition.

The local picture of Knudsen stabilization is reviewed in Figure 8. Specifically, if x is a node of C (or $x = w_i$ for some i), we insert a component $\Sigma \simeq \mathbb{P}^1$ at x , and place x_{\dagger} (respectively x_{\dagger} and $w_{i,\dagger}$) on Σ . The line bundle will be $\mathcal{L}_{\dagger} = f^i \mathcal{L}$, and the

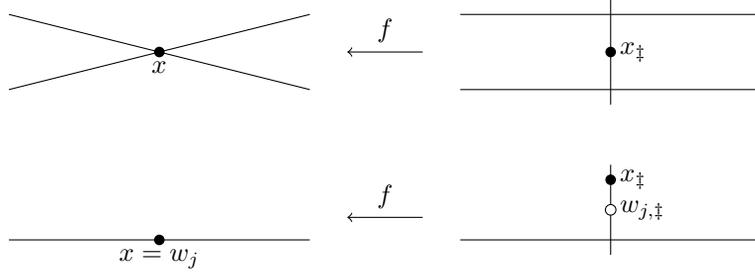


FIGURE 8. Knudsen stabilization (local picture).

choice of the section σ_{\dagger} is forced by the agreement of σ and σ_{\dagger} on the open where f is an isomorphism. Inspecting the two cases in Figure 8, we can also check that

$$(12) \quad f^* \omega_C(w_1 + \cdots + w_m) \simeq \omega_{C_{\dagger}}(w_{1,\dagger} + \cdots + w_{m,\dagger}),$$

that is, condition (9) is satisfied in the current setup.

Finally, note that if $\mathcal{L} = \omega_C^{\vee}$, then $\mathcal{L}_{\dagger} = \omega_{C_{\dagger}}^{\vee}$, so this construction indeed generalizes the one called ‘Knudsen stabilization with vector fields’ in §1.1.

Remark 4.3. We will frequently use the description of the functor of points of projectivizations of coherent modules, as written at the beginning of [18, §2]. This remark serves as an indirect link to the respective property, so, when we say ‘by Remark 4.3’, we actually mean ‘by the remark at the beginning of [18, §2]’.

Construction 4.4. In Situation 4.1, let $\delta : \mathcal{O}_C \rightarrow \mathcal{O}_C(w_1 + \cdots + w_m) \oplus \mathcal{I}_{x,C}^{\vee}$ be the homomorphism of \mathcal{O}_C -modules such that $\delta(1) = 1 \oplus \iota$, where $\iota : \mathcal{I}_{x,C} \hookrightarrow \mathcal{O}_C$ is the inclusion. Note that δ is injective, and let \mathcal{K} be its cokernel. Then

$$(13) \quad 0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C(w_1 + \cdots + w_m) \oplus \mathcal{I}_{x,C}^{\vee} \rightarrow \mathcal{K} \rightarrow 0$$

is exact. Define

$$(14) \quad C_{\dagger} = \mathbb{P}(\mathcal{K}),$$

and let $f : C_{\dagger} \rightarrow C$ be the natural projection, and $\pi_{\dagger} = \pi f$. For the definition of the lifts $x_{\dagger}, w_{1,\dagger}, \dots, w_{m,\dagger}$ of x, w_1, \dots, w_m , please see [18, §2]. As in [18, §2],

- (1) the morphism $\pi_{\dagger} : C_{\dagger} \rightarrow S$ is a prestable curve;
- (2) for each $s \in S$, $x_{\dagger}(s) \neq w_{i,\dagger}(s)$ for all i , and π_{\dagger} is smooth at $x_{\dagger}(s)$;
- (3) f has property R (by Lemma 3.3 and item 5 below);
- (4) f induces an isomorphism $f^{-1}(U) \simeq U$ if $U = \pi^{\text{sm}} \setminus \bigcup_{i=1}^m (x \cap w_i)$; and

(5) the formation of the output data commutes with base change.

(Although [18, §2] operates under the assumption that (C, w_1, \dots, w_m) is a *stable* m -marked curve over S , while Situation 4.1 ensures *prestability* only, this weaker assumption suffices to establish the claims above using precisely the same arguments as in [18, §2]. A similar situation occurs in [20, §1.3].) Let

$$(15) \quad \mathcal{L}_{\ddagger} = f^i \mathcal{L},$$

cf. Definition 3.12. The homomorphism (11) from Proposition 3.14 reads

$$(16) \quad f_*(\mathcal{L}_{\ddagger}(-w_{1,\ddagger} - \dots - w_{m,\ddagger})) \rightarrow \mathcal{L}(-w_1 - \dots - w_m)$$

in our case, and in fact it is an isomorphism by Proposition 3.14, since the condition on geometric fibers (9) holds, by (12). (Indeed, we already know at least that the behaviour of the curve and markings agree with that described in Example 4.2, so we may invoke (12).) Define σ_{\ddagger} to be the preimage of σ under (16).

Remark 4.5. Here are a few remarks on Construction 4.4.

- (1) The map f is projective in the sense of [8, §5.5]. Indeed, $\mathcal{I}_{x,C}^{\vee}$ is a finitely generated \mathcal{O}_S -module since the dual of a finitely generated module over a noetherian ring is finitely generated.
- (2) The scheme-theoretic vanishing locus of $x_{\ddagger}^* \sigma_{\ddagger}$ coincides with the scheme-theoretic vanishing locus of $x^* \sigma$. We may argue as follows. Applying Lemma 3.11 with \mathcal{L}^{\vee} in the role of \mathcal{L} and dualizing, we obtain an isomorphism $\mathcal{L}_{\ddagger}^{\vee}(-w_{1,\ddagger} - \dots - w_{m,\ddagger}) \simeq f^* \mathcal{L}^{\vee}(-w_1 - \dots - w_m)$ whose adjoint maps σ_{\ddagger} to σ , by Construction 4.4. Pulling back along x_{\ddagger} , we obtain an isomorphism $x_{\ddagger}^* \mathcal{L}_{\ddagger}^{\vee}(-w_{1,\ddagger} - \dots - w_{m,\ddagger}) \simeq x^* \mathcal{L}^{\vee}(-w_1 - \dots - w_m)$ under which $x_{\ddagger}^* \sigma_{\ddagger}$ corresponds to $x^* \sigma$.
- (3) If $(S, C, \pi, x, w_1, \dots, w_m, \mathcal{L}, \sigma)$ satisfies Situation 4.1, and $\alpha \in \Gamma(S, \mathcal{O}_S)$, then $(S, C, \pi, x, w_1, \dots, w_m, \mathcal{L}, \pi^* \alpha \cdot \sigma)$ also satisfies Situation 4.1, and, moreover, the output of Construction 4.4 on the second set of input data will be (isomorphic to) $(S, C_{\ddagger}, \pi_{\ddagger}, x_{\ddagger}, w_{1,\ddagger}, \dots, w_{m,\ddagger}, \mathcal{L}_{\ddagger}, \pi_{\ddagger}^* \alpha \cdot \sigma_{\ddagger})$. Indeed, if ψ denotes (16), then $\psi(\pi_{\ddagger}^* \alpha \cdot \sigma_{\ddagger}) = \psi(f^* \pi^* \alpha \cdot \sigma_{\ddagger}) = \pi^* \alpha \cdot \psi(\sigma_{\ddagger}) = \pi^* \alpha \cdot \sigma$, where the second equality is simply by \mathcal{O}_C -linearity.

Proposition 4.6. *Construction 4.4 commutes with base change.*

Proof. For $C_{\ddagger}, \pi_{\ddagger}, f, x_{\ddagger}, w_{i,\ddagger}$ this is completely analogous to [18, p. 176] (the key is that \mathcal{K} is stably reflexive, everything else is clear). For \mathcal{L}_{\ddagger} , this is an easy consequence of Lemma 3.7, as noted after Definition 3.12 too. For σ_{\ddagger} , this follows from Lemma 3.11. \square

To summarize, we have the following.

Theorem 4.7. *Construction 4.4 has the following features:*

- (1) *the output data is nondegenerate;*
- (2) *if the input data is nondegenerate, then the output data is isomorphic to the input data;*
- (3) *if $S = \text{Spec } \mathbb{K}$, and the input data is degenerate, then the output data is (isomorphic to) that described in Example 4.2; and*
- (4) *it commutes with base change.*

Proof. The first two items are stated in Construction 4.4, the third item is elementary to check from Construction 4.4 and left to the reader (and well-known, being completely analogous to [18]), while the last item is Proposition 4.6. \square

Finally, we state a version of [18, Lemma 2.5], which will be used later to construct an inverse of this bubbling up operation, under suitable circumstances.

Proposition 4.8. *Let $(S, C, \pi, x, \mathcal{L}, w_1, \dots, w_m, \sigma)$ and $(S, Y, \varpi, y, \mathcal{J}, u_1, \dots, u_m, \rho)$ satisfy the requirements of Situation 4.1, and assume moreover that the second set of data is nondegenerate, i.e. $y(s) \neq u_i(s)$ for $i = 1, \dots, m$ and ϖ is smooth at $y(s)$, for all $s \in S$. Let $q : Y \rightarrow C$ be a morphism with property R such that:*

- (1) $x = qy$, $w_i = qu_i$, $\mathcal{J} = q^*\mathcal{L}$, and the homomorphism

$$q_*\mathcal{J}(-u_1 - \dots - u_m) \rightarrow \mathcal{L}(-w_1 - \dots - w_m)$$

obtained as a special case of (11) from $\mathcal{J} = q^*\mathcal{L}$ maps ρ to σ ;

- (2) for any geometric point $\bar{s} \rightarrow S$, there exists an isomorphism $\alpha : Y_{\bar{s}} \rightarrow (C_{\dagger})_{\bar{s}}$ over \bar{s} , such that $q_{\bar{s}} = f_{\bar{s}}\alpha$, and α is compatible in the natural sense with all the rest of the data (restrictions of line bundles, sections, etc.).

Then, there exists a canonical isomorphism $q_{\dagger} : Y \rightarrow C_{\dagger}$ over S (compatible with all given sections of ϖ and π_{\dagger} , and with σ_{\dagger} and ρ), such that $q = fq_{\dagger}$.

To emphasize the nuance of the proposition, the isomorphisms $Y_{\bar{s}} \simeq (C_{\dagger})_{\bar{s}}$ in item 2 are not required a priori to ‘fit together’ in any nice way, but the conclusion is essentially the statement that they do.

Proof. The proof is analogous to the proof of [18, Lemma 2.5]. By Definition 3.1, $\mathcal{O}_C \simeq q_*\mathcal{O}_Y$, and, by part 1 of Proposition 3.10, $\mathcal{O}_C(-W) \simeq q_*\mathcal{O}_Y(-U)$, where $W = \sum_{i=1}^m w_i$ and $U = \sum_{i=1}^m u_i$. Following the analogous arguments in the proof of [18, Lemma 2.5], the natural pairing

$$\mathcal{I}_{x,C} \otimes q_*\mathcal{O}_Y(y - U) \rightarrow q_*\mathcal{O}_Y(-y) \otimes q_*\mathcal{O}_Y(y - U) \rightarrow q_*\mathcal{O}_Y(-U) \simeq \mathcal{O}_C(-W)$$

induces a homomorphism $q_*\mathcal{O}_Y(y - U) \rightarrow \mathcal{I}_{x,C}^{\vee}(-W)$, which is an isomorphism on geometric fibers by assumption 2, and hence an isomorphism. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \rightarrow & q_*\mathcal{O}_Y(-U) & \rightarrow & q_*\mathcal{O}_Y(y - U) \oplus q_*\mathcal{O}_Y & \rightarrow & q_*\mathcal{O}_Y(y) \rightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \rightarrow & \mathcal{O}_C(-W) & \longrightarrow & \mathcal{I}_{x,C}^{\vee}(-W) \oplus \mathcal{O}_C & \longrightarrow & \mathcal{K}(-W) \rightarrow 0. \end{array}$$

Exactness of the top row follows from part 1 of Proposition 3.10, while the bottom row is simply (13) twisted by $\mathcal{O}_C(-W)$.

As in [18], the next step is to apply [18, Corollary 1.5]; for the reader’s convenience, we include the details of verifying the hypotheses of the corollary. We claim that for any closed point $z \in C$ (with residue field denoted by K), either $q^{-1}(z) \simeq \text{Spec } K$, or $q^{-1}(z) \simeq \mathbb{P}_K^1$ and $\mathcal{O}_Y(y)|_{q^{-1}(z)} \simeq \mathcal{O}(1)$. First, let’s assume that z is contained in the maximal open of C where f is an isomorphism (Lemma 3.5). Then, in fact, z is in the maximal open of C where q is an isomorphism, by assumption 2, the last item in Lemma 3.5, and the fact that two open subsets $U_1, U_2 \subseteq C$ which agree on ‘geometric fibers’ (that is, $U_{1,\bar{s}} = U_{2,\bar{s}}$ as subsets of $C_{\bar{s}}$, for any geometric point $\bar{s} \rightarrow S$) must coincide. The last fact follows from the fact

that $C_{\bar{s}} \rightarrow C_s$ is surjective (on the underlying topological spaces), which in turn follows from $\bar{s} \rightarrow s$ faithfully flat, and the fact that ‘faithfully flat’ is preserved by base change. Hence, in this case, $q^{-1}(z) \simeq \text{Spec } K$. It remains to deal with the case when z is not contained in the maximal open of C where f is an isomorphism (Lemma 3.5). Then, $z \in x(S)$ and the fiber of (S, C, π, \dots) over $s \in S$ is degenerate. Moreover, if $\pi(z) = s \in S$, then $z = x(s)$ necessarily and, in particular, $s \in S$ has the same residue field K as $z \in C$. Since $(q^{-1}(z))_{\bar{s}}$ and $(f^{-1}(z))_{\bar{s}}$ are the fibers of $z \in C$ by the morphisms $Y_{\bar{s}} \rightarrow C$ and $(C_{\dagger})_{\bar{s}} \rightarrow C$ respectively, assumption 2 implies that they are isomorphic, so $(q^{-1}(z))_{\bar{s}} \simeq (f^{-1}(z))_{\bar{s}} \simeq \mathbb{P}_{\bar{K}}^1$. However, $q^{-1}(z)$ contains the K -point (i.e. rational point) $y(s) \in q^{-1}(z)$ by the degeneracy assumption and assumption 2, so $q^{-1}(z) \simeq \mathbb{P}_K^1$ since genus 0 curves with rational points are projective lines. Moreover, $\mathcal{O}_Y(y)|_{q^{-1}(z)} \simeq \mathcal{O}(1)$ since $y(s) \in q^{-1}(z)$, completing the proof of the claims made earlier in this paragraph.

Therefore, $\mathcal{O}_Y(y)$ satisfies conditions (1) and (2) in [18, Corollary 1.5] relative to $q : Y \rightarrow C$. By [18, Corollary 1.5], $q^* \mathcal{K}(-W) = q^* q_* \mathcal{O}_Y(y) \rightarrow \mathcal{O}_Y(y)$ is surjective. Therefore, by Remark 4.3, we obtain a morphism

$$q_{\dagger} : Y \rightarrow C_{\dagger}$$

such that $q = f q_{\dagger}$. It can be checked from assumption 2 that q_{\dagger} is an isomorphism on geometric fibers (for the purposes of this claim, we may assume that $S = \text{Spec } \mathbb{K}$, when everything is straightforward), and therefore an isomorphism everywhere. All compatibilities are straightforward to check. \square

4.2. Inflating at nonsingular zero of a section. In §4.2, we generalize the operation ‘Inflating at zero vector’ from §1.1 to families (and arbitrary line bundles).

Situation 4.9. Assume that $\pi : C \rightarrow S$ is a prestable curve, x is a section of π , \mathcal{L} is an invertible \mathcal{O}_C -module, and σ is a global section of \mathcal{L} , such that π is smooth at $x(s)$, for all $s \in S$.

We deem such data *degenerate* if σ vanishes at $x(s)$ for some $s \in S$ (vanishes at $x(s)$ means $\sigma_{x(s)} \in \mathfrak{m} \mathcal{L}_{x(s)}$, where $\mathfrak{m} \subset \mathcal{O}_{C, x(s)}$ is the maximal ideal).

In Situation 4.9, we will construct $S_{\dagger} = S$, C_{\dagger} , π_{\dagger} , x_{\dagger} , \mathcal{L}_{\dagger} , σ_{\dagger} such that the new (output) data satisfies the respective requirements of Situation 4.9, $x_{\dagger}^* \sigma_{\dagger}$ is nowhere vanishing (the output data is nondegenerate), and the construction commutes with base change. Once again, the construction will also provide a map $f : C_{\dagger} \rightarrow C$ with property R, which satisfies various compatibilities, similarly to §4.1.

Example 4.10. Similarly to Example 4.2, we state the effect of the construction when $S = \text{Spec } \mathbb{K}$ and the data is degenerate in advance of its general statement. The local picture on the level of curves and sections is very simple, by Figure 9 below. Specifically, we insert $\Sigma \simeq \mathbb{P}^1$ at x , and place x_{\dagger} on Σ . The line bundle is

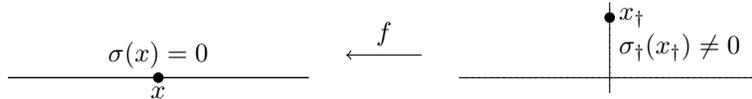


FIGURE 9. Inflating at nonsingular zero of a section (local picture).

again $\mathcal{L}_{\dagger} = f^! \mathcal{L}$. The section σ_{\dagger} is required to not vanish at x_{\dagger} and to agree with

σ at all generic points of C ; please see Remark 4.11 for further clarifications. In contrast to Example 4.2, $f^*\omega_C \not\cong \omega_{C_\dagger}$, i.e. condition (9) fails. Finally, if $\mathcal{L} = \omega_C^\vee$, then $\mathcal{L}_\dagger = \omega_{C_\dagger}^\vee$, so this construction indeed generalizes the construction called ‘inflating at zero vector’ in §1.1.

Remark 4.11. The behaviour of σ_\dagger in Example 4.10 is rather subtle. There are infinitely many sections satisfying the requirements, but they are ‘indistinguishable’ up to automorphisms compatible with all structure (including with f). To sketch the argument, let us identify (non-canonically) Σ with \mathbb{P}^1 such that x_\dagger maps to 0 and the node of C_\dagger on Σ maps to ∞ . Then, the restriction of σ_\dagger to $\Sigma \simeq \mathbb{P}^1$ should correspond to a vector field v on \mathbb{P}^1 such that $v(0) \neq 0$, $v(\infty) = 0$, and

$$v = (\alpha y + by^2) \frac{d}{dy} = -(\alpha x + b) \frac{d}{dx} \quad \text{where } y = \frac{1}{x} = \frac{Y}{X} \in K(\mathbb{P}^1),$$

for a *specified* $\alpha \in \mathbb{K}$, by Example 3.13 and the considerations at the beginning of §1. (For instance, if $\mathcal{L}|_U \simeq \omega_C^\vee|_U$ is an isomorphism in a neighbourhood U of x ; then $\mathcal{L}_\dagger|_{f^{-1}(U)} \simeq \omega_{C_\dagger}^\vee|_{f^{-1}(U)}$, and the ‘logarithmic vector field’ condition at the node of C_\dagger on Σ shows that α is fixed. Please see Example 6.2 also.) Hence, the claimed indistinguishability amounts to the fact that the standard \mathbb{K}^\times -action on \mathbb{P}^1 , $a \cdot [X : Y] = [aX : Y]$, induces a (free and) transitive action on

$$\left\{ (c - \alpha x) \frac{d}{dx} : c \in \mathbb{K}^\times \right\} \subset \Gamma(\mathcal{T}_{\mathbb{P}^1}),$$

which is true. Thus, the fact that a *canonical* choice of σ_\dagger exists (as Construction 4.14 shows) is quite remarkable. In Remark 1.1, we proposed the analogy with similar subtleties in [18] as a way to come to terms with these issues.

We will first carry out the required calculations in Proposition 4.12 below, and only then return to Situation 4.9 and state the construction.

Proposition 4.12. *Let X be a (noetherian) scheme and σ a section of a rank 2 locally free \mathcal{O}_X -module \mathcal{E} . Assume that the homomorphism $\mathcal{O}_X \rightarrow \mathcal{E}$ which maps 1 to σ is injective. Let*

$$Y = \mathbb{P}\mathcal{F} \quad \text{where } \mathcal{F} = \text{Coker}(\mathcal{O}_X \xrightarrow{\times\sigma} \mathcal{E}),$$

$f : Y \rightarrow X$ the projection, and $\mathcal{O}_Y(1)$ the twisting sheaf.

- (1) *The following hold.*
 - (a) f is a projective (in the sense of [8, §5.5]) global lci morphism.
 - (b) The adjoint $\mathcal{F} \rightarrow f_*\mathcal{O}_Y(1)$ of $f^*\mathcal{F} \rightarrow \mathcal{O}_Y(1)$ is an isomorphism. Moreover, this isomorphism is compatible with base changes $h : X' \rightarrow X$ such that the multiplication by $h^*\sigma$ map $\mathcal{O}_{X'} \rightarrow h^*\mathcal{E}$ is injective.
 - (c) There exists an isomorphism $f^!\mathcal{O}_X \simeq (f^*\det \mathcal{F})(-1)$; in particular, we think of $f^!\mathcal{O}_X$ as a sheaf. Moreover, if U is an open subset of X such that σ vanishes at no point of U , then f restricts to an isomorphism $V := f^{-1}(U) \simeq U$, and there exists a commutative diagram of isomorphisms

$$(f^!\mathcal{O}_X)|_V \begin{array}{c} \xrightarrow{\cong} \\ \xrightarrow{\cong} \end{array} \mathcal{O}_V \begin{array}{c} \xleftarrow{\cong} \\ \xleftarrow{\cong} \end{array} ((f^*\det \mathcal{F})(-1))|_V$$

in which the curved arrow is the restriction of the isomorphism above, the left arrow comes from $f^!|_V = f^*|_V$, since $f|_V$ is an isomorphism, and the one on the right commutes with any base change $X' \rightarrow X$.

- (2) Assume that $g : X \rightarrow T$ is a flat morphism, and that for all morphisms $T' \rightarrow T$, the multiplication by $h^*\sigma$ map $\mathcal{O}_{X'} \rightarrow h^*\mathcal{E}$ is injective (where $h : X' = T' \times_T X \rightarrow X$ is the projection). Then fg is flat.
- (3) If X is a prestable curve over \mathbb{K} and the vanishing locus of σ is a reduced closed subscheme of dimension 0 contained in the nonsingular locus of X , then Y is also a prestable curve over \mathbb{K} , and f has property R.

Proof. The assumption that the map $\mathcal{O}_X \rightarrow \mathcal{E}$ such that $1 \mapsto \sigma$ (i.e. the multiplication by σ map) is injective in fact implies a seemingly stronger hypothesis which will be used in this proof, that the germ σ_x of σ at x is nonzero, and not a zero-divisor in the ring $\text{Sym } \mathcal{E}_x$, for all $x \in X$. We will sometimes refer to this fact as the ‘condition on germs’. Indeed, $1 \mapsto \sigma_x$ is an injective homomorphism $\mathcal{O}_{X,x} \rightarrow \mathcal{E}_x$, and then the claim follows from a theorem of McCoy [22, Theorem 3] that zero-divisors in polynomial rings are in fact annihilated by non-zero scalars (in our case, applied to a linear homogeneous polynomial in two variables).

We have a short exact sequence

$$(17) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0.$$

Let $P = \mathbb{P}\mathcal{E}$, and $p : P \rightarrow X$ the projection which exhibits P as a \mathbb{P}^1 -bundle over X . Since $\text{Sym}^k \mathcal{E} \rightarrow \text{Sym}^k \mathcal{F}$ is surjective for all k , it follows from [26, Tag 07ZK] that the graded \mathcal{O}_X -algebras homomorphism $\text{Sym } \mathcal{E} \rightarrow \text{Sym } \mathcal{F}$ induces a closed immersion $j : Y \rightarrow P$.

Claim 4.13. If ψ is the image of $p^*\sigma$ under the map $p^*\mathcal{E} \rightarrow \mathcal{O}_P(1)$, then ψ is a regular section of $\mathcal{O}_P(1)$, and $j(Y)$ is the Cartier divisor on P cut out by ψ .

Proof. We may assume that $X = \text{Spec } R$ and that \mathcal{E} is trivial. Then $P = \text{Proj } R[T, S]$, and let σ and ψ correspond to the linear polynomial $aT + bS \in R[T, S]$. First, we claim that $aT + bS$ is a nonzero non-zero-divisor in $R[T, S]$. Indeed, the ‘condition on germs’ implies that this is the case in $R_{\mathfrak{p}}[T, S]$ for all prime ideals $\mathfrak{p} \in \text{Spec } R$, so any $p \in R[T, S]$ such that $(aT + bS)p(T, S) = 0$ is killed by all homomorphisms $R[T, S] \rightarrow R_{\mathfrak{p}}[T, S]$, and thus must be equal to 0. Then $at + b$ and $a + bs$ are nonzero non-zero-divisors in $R[t]$ and $R[s]$, so ψ indeed cuts out a Cartier divisor on P . Moreover, by construction, this Cartier divisor is $\text{Proj } R[T, S]/(aT + bS) \rightarrow \text{Proj } R[T, S]$, which is nothing but j . \square

1a: The $f = pj$ factorization settles both issues.

1b: Note that $\mathcal{L}_{j(Y), P} = \mathcal{O}_P(-1)$ by Claim 4.13 and $j_*\mathcal{O}_Y \otimes \mathcal{O}_P(1) = j_*(\mathcal{O}_Y(1))$ by construction. Twisting $0 \rightarrow \mathcal{L}_{j(Y), P} \rightarrow \mathcal{O}_P \rightarrow j_*\mathcal{O}_Y \rightarrow 0$ by $\mathcal{O}_P(1)$, we obtain the exact sequence

$$(18) \quad 0 \rightarrow \mathcal{O}_P \rightarrow \mathcal{O}_P(1) \rightarrow j_*\mathcal{O}_Y(1) \rightarrow 0.$$

We have $R^1p_*\mathcal{O}_P = 0$, and $\mathcal{O}_X \rightarrow p_*\mathcal{O}_P$ and $\mathcal{E} \rightarrow p_*\mathcal{O}_P(1)$ are isomorphisms (the claims are local on X , so they reduce to X affine and \mathcal{E} trivial), so we have a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} \longrightarrow 0 \\
 & & \simeq \downarrow & & \simeq \downarrow & & \downarrow \\
 0 & \longrightarrow & p_*\mathcal{O}_P & \longrightarrow & p_*\mathcal{O}_P(1) & \longrightarrow & f_*\mathcal{O}_Y(1) \longrightarrow 0
 \end{array}$$

in which the top row is (17), the bottom row is the pushforward of (18) along p , and the left and central vertical arrows are isomorphisms. It follows that the right vertical arrow is also an isomorphism, as desired. For compatibility with base change, we first clarify that the map $h^*f_*\mathcal{O}_Y(1) \rightarrow f'_*\mathcal{O}_{Y'}(1)$ is the usual map discussed in context of ‘cohomology and base change’. It is clear that $h^*\mathcal{F} \cong \mathcal{F}'$, since (17) remains exact after applying h^* . The commutativity of the square diagram which expresses that $\mathcal{F} \rightarrow f_*\mathcal{O}_Y(1)$ commutes with base change is standard; in particular, $h^*f_*\mathcal{O}_Y(1) \rightarrow f'_*\mathcal{O}_{Y'}$ is a fortiori an isomorphism, since the other three maps in the square are isomorphisms.

1c: Taking determinants in the Euler sequence of P , we obtain

$$(19) \quad \omega_{P/X} = p^* \det \mathcal{E} \otimes \mathcal{O}_P(-2).$$

Keeping in mind that j is the immersion of an effective Cartier divisor, we have

$$\begin{aligned}
 f^!\mathcal{O}_X &= j^!p^!\mathcal{O}_X = j^!(\omega_{P/X}[1]) = j^*\omega_{P/X}[1] \otimes j^!\mathcal{O}_P \\
 &= j^*\omega_{P/X}[1] \otimes j^*\mathcal{O}_P(j(Y))[-1] \quad \text{by [26, Tag 0B4B]} \\
 (20) \quad &= f^*\det \mathcal{E} \otimes \mathcal{O}_Y(-1) \quad \text{by (19) and } \mathcal{O}_P(j(Y)) \cong \mathcal{O}_P(1), \\
 &= f^*\det \mathcal{F} \otimes \mathcal{O}_Y(-1) \quad \text{since (17) implies } \det \mathcal{F} = \det \mathcal{E},
 \end{aligned}$$

as desired.

For the claim regarding the restriction to U , it is alright to assume that $U = X$ since both (19) and (20) obviously commute with restricting to open subschemes of X . With this assumption, \mathcal{F} is invertible and isomorphic to $\det \mathcal{E}$, and $Y = \mathbb{P}\mathcal{F} = X$ with $\mathcal{O}_Y(1) = \mathcal{F}$. This provides the isomorphism on the right, and the rest is clear.

2: Clearly, $P \rightarrow T$ is flat, because it is a composition of flat morphisms $P \rightarrow X \rightarrow T$. Recall from Claim 4.13 that the section ψ of $\mathcal{O}_P(1)$ cuts out the effective Cartier divisor Y on P . However, the assumption in the statement of 2 implies that the latter remains true after any base change $T' \rightarrow T$, so we may conclude by [26, Tag 056Y].

3: Indeed, $j(Y) = \Sigma + p^{-1}(\{\sigma = 0\})$ as divisors on P for some section $\Sigma \subset P$ of the \mathbb{P}^1 -bundle $P \rightarrow X$, and the claim follows easily. (Lemma 3.3 is implicitly relied on.) \square

We can now state the main construction.

Construction 4.14. In Situation 4.9, consider the section $(-\sigma, 1)$ of $\mathcal{L} \oplus \mathcal{O}_C(x)$. (Note the sign!) We are in a situation covered by Proposition 4.12, with C in the role of X , $\mathcal{L} \oplus \mathcal{O}_C(x)$ in the role of \mathcal{E} , and $(-\sigma, 1)$ in the role of σ .

Let $\gamma : \mathcal{O}_C \rightarrow \mathcal{L} \oplus \mathcal{O}_C(x)$ be the homomorphism such that $\gamma(1) = (-\sigma, 1)$. Then

$$(21) \quad 0 \rightarrow \mathcal{O}_C \xrightarrow{\gamma} \mathcal{L} \oplus \mathcal{O}_C(x) \xrightarrow{\kappa} \mathcal{K} \rightarrow 0$$

is exact, where \mathcal{K} is the cokernel of γ . Define

$$(22) \quad C_{\dagger} = \mathbb{P}(\mathcal{K}),$$

and let $f : C_{\dagger} \rightarrow C$ be the natural projection, and $\pi_{\dagger} = \pi f$. The complex $\mathcal{O}_S \xrightarrow{x^*\gamma} x^*\mathcal{L} \oplus x^*\mathcal{O}_C(x) \rightarrow x^*\mathcal{O}_C(x) \rightarrow 0$ is exact at $x^*\mathcal{O}_C(x)$, and we get a surjective

homomorphism $x^*\mathcal{K} \rightarrow x^*\mathcal{O}_C(x)$ by the universal property of cokernels, since x^* is right exact. This map induces a lift

$$(23) \quad x_{\dagger} : S \rightarrow C_{\dagger}$$

of $x : S \rightarrow C$ by Remark 4.3. It is clear that all constructions so far commute with base change.

In particular, part 3 of Proposition 4.12 implies that the geometric fibers of π_{\dagger} are curves with at worst nodal singularities. On the other hand, part 2 of Proposition 4.12 implies that π_{\dagger} is flat, since, once more, our constructions are functorial and the non-zero-divisor condition holds universally just as well as it holds in the given case – simply repeat the first paragraph in this construction after the base change. Thus π_{\dagger} is a prestable curve. Part 3 of Proposition 4.12 again, the functoriality of the construction so far, and Lemma 3.3 imply that f has property R. Let

$$(24) \quad \mathcal{L}_{\dagger} = f^i\mathcal{L},$$

cf. Definition 3.12 and the remark thereafter that $f^i\mathcal{L}$ is an invertible sheaf. Some preliminary calculations are needed before we can define σ_{\dagger} . We have

$$(25) \quad \begin{aligned} f^i\mathcal{L} &= f^*\mathcal{L} \otimes f^i\mathcal{O}_C \quad \text{by the analogous property for } f^! \\ &= f^*\mathcal{L} \otimes f^*\mathcal{L}^{\vee}(-x) \otimes \mathcal{O}_{C_{\dagger}}(1) \quad \text{by part 1c of Proposition 4.12} \\ &= f^*\mathcal{O}_C(-x) \otimes \mathcal{O}_{C_{\dagger}}(1) \\ &= \mathcal{O}_{\mathbb{P}\mathcal{K}(-x)}(1) \quad \text{by a well-known fact,} \end{aligned}$$

and it follows that

$$(26) \quad \begin{aligned} f_*f^i\mathcal{L} &= f_*(f^*\mathcal{O}_C(-x) \otimes \mathcal{O}_{C_{\dagger}}(1)) \quad \text{by (25)} \\ &= \mathcal{O}_C(-x) \otimes f_*\mathcal{O}_{C_{\dagger}}(1) \quad \text{by the projection formula} \\ &= \mathcal{K}(-x) \quad \text{by part 1b of Proposition 4.12.} \end{aligned}$$

If we combine the composition $\mathcal{O}_C \xrightarrow{0 \oplus \text{id}} \mathcal{L}(-x) \oplus \mathcal{O}_C \xrightarrow{\kappa \otimes \text{id}} \mathcal{K}(-x)$ with (26), we obtain an \mathcal{O}_C -module homomorphism $\mathcal{O}_C \rightarrow f_*f^i\mathcal{L} = f_*\mathcal{L}_{\dagger}$. Let

$$(27) \quad \sigma_{\dagger} \in \Gamma(C_{\dagger}, \mathcal{L}_{\dagger})$$

be the image of $1 \in \Gamma(C, \mathcal{O}_C)$ under the last homomorphism.

Remark 4.15. Here are some technical remarks on Construction 4.14.

- (1) The composition $\mathcal{L}|_{C \setminus x} \xrightarrow{\text{id} \oplus 0} (\mathcal{L} \oplus \mathcal{O}_C(x))|_{C \setminus x} \xrightarrow{\kappa} \mathcal{K}|_{C \setminus x}$ is an isomorphism $\mathcal{L}|_{C \setminus x} \simeq \mathcal{K}|_{C \setminus x}$. The section $1 \in \Gamma(C \setminus x, \mathcal{O}_C)$ is mapped to $\sigma|_{C \setminus x}$ under the composition

$$\mathcal{O}_C|_{C \setminus x} = \mathcal{O}_C(x)|_{C \setminus x} \xrightarrow{0 \oplus \text{id}} (\mathcal{L} \oplus \mathcal{O}_C(x))|_{C \setminus x} \rightarrow \mathcal{K}|_{C \setminus x} \simeq \mathcal{L}|_{C \setminus x},$$

which begins to explain the sign convention in Construction 4.14.

- (2) The map f restricts to an isomorphism $f^{-1}(C \setminus x) \simeq C \setminus x$ since $\mathcal{L}|_{C \setminus x} \simeq \mathcal{K}|_{C \setminus x}$, cf. item 1.
- (3) The adjoint of (26) is the homomorphism $f^*\mathcal{K}(-x) \rightarrow f^i\mathcal{L}$ which induces (via Remark 4.3) the isomorphism $C_{\dagger} \simeq \mathbb{P}\mathcal{K}(-x) = \mathbb{P}\mathcal{K}$.
- (4) If the data (S, C, \dots) from Situation 4.9 is nondegenerate, then the output data $(S_{\dagger} = S, C_{\dagger}, \dots)$ is canonically isomorphic to the input data (S, C, \dots) . We also note that (21) takes the form $0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{L} \oplus \mathcal{O}_C(x) \rightarrow \mathcal{L}(x) \rightarrow 0$ in this situation.

First, let us check that Construction 4.14 commutes with base change.

Proposition 4.16. *The formation of C_{\dagger} , π_{\dagger} , x_{\dagger} , \mathcal{L}_{\dagger} , σ_{\dagger} , and f commutes with base change.*

Proof. The upshot is that the relevant constructions visibly commute with base changes $S' \rightarrow S$, with the exception of those involving duality (as customary, we will not check the ‘cocycle conditions’ for compositions $S'' \rightarrow S' \rightarrow S$ for the isomorphisms which express commutativity with respect to base change). Thus functoriality of $C_{\dagger}, \pi_{\dagger}, f, x_{\dagger}$ is straightforward. The fact that \mathcal{L}_{\dagger} commutes with base change comes from $\mathcal{L}_{\dagger} = f^* \mathcal{L} \otimes f^* \omega_{C/S} \otimes \omega_{C_{\dagger}/S}^{\vee}$ (by (24), Definition 3.12, and Lemma 3.7) and the fact that the formation of the relative dualizing sheaf of prestable curves commutes with base change, e.g. [26, Tag 0E6R].

The functoriality of most ingredients involved in the functoriality of σ_{\dagger} is quite straightforward and left to the reader, with one exception: the application of part 1c of Proposition 4.12. Specifically, given a base change $h : S' \rightarrow S$, let $C', \pi', x', \mathcal{L}', \sigma'_{\dagger}$ be the pullback of the data in Situation 4.9 along h , if $q : C'_{\dagger} \rightarrow C_{\dagger}$ is the induced morphism, then we need the diagram

$$\begin{array}{ccc} q^* f^! \mathcal{O}_C & \longrightarrow & q^*(f^* \mathcal{L}(x) \otimes \mathcal{O}_{C_{\dagger}}(1)) \\ \downarrow & & \downarrow \\ f^! \mathcal{O}_{C'} & \longrightarrow & f'^* \mathcal{L}'(x') \otimes \mathcal{O}_{C'_{\dagger}}(1) \end{array}$$

in which all sheaves are invertible $\mathcal{O}_{C_{\dagger}}$ -modules, and all homomorphisms are isomorphisms (the horizontal ones come from the application of part 1c of Proposition 4.12, the left vertical one comes from $f^! \mathcal{O}_C = \omega_{C_{\dagger}/S} \otimes f^* \omega_{C/S}^{\vee}$ and the fact that the formation of relative dualizing sheaves commutes with base change) to commute. Luckily, we can circumvent the issue of base change in Grothendieck duality [3] using the following trick. The commutativity of the square boils down to a statement of the form that an automorphism (say, α) of $q^* f^! \mathcal{O}_C$ is the identity (compose all arrows around the square, two reversed). We have

$$\text{Aut}(q^* f^! \mathcal{O}_C) \cong \mathcal{O}_{C'_{\dagger}}^{\times} \quad (\text{the sheaf of nowhere vanishing functions on } C'_{\dagger})$$

since $q^* f^! \mathcal{O}_C$ is invertible. However, by the second half of part 1c of Proposition 4.12, it is clear that the restriction of the square diagram above to $(f')^{-1}(C' \setminus x')$ commutes, so the invertible section corresponding to α restricts to 1 on $(f')^{-1}(C' \setminus x')$. It then suffices to check that the restriction map on sections of the structure sheaf of C'_{\dagger} from global sections to sections over $C'_{\dagger} \setminus (f')^{-1}(x')$ is injective. Since $f'^{\#}$ is an isomorphism, this boils down to the statement that the restriction map on sections of $\mathcal{O}_{C'}$ from global sections to sections over $C' \setminus x'$ is injective, which follows from Lemma 3.6. \square

Second, let us check that σ_{\dagger} is indeed a lift of σ .

Lemma 4.17. *In the situation of Construction 4.14, the homomorphism (11) reads*

$$(28) \quad f_* \mathcal{L}_{\dagger} = f_* f^! \mathcal{L} \rightarrow \mathcal{L}.$$

Then the map $\Gamma(C_{\dagger}, \mathcal{L}_{\dagger}) \rightarrow \Gamma(C, \mathcal{L})$ on global sections maps $\sigma_{\dagger} \mapsto \sigma$.

Proof. It suffices to check that $f_*\mathcal{L}_\dagger \rightarrow \mathcal{L}$ maps the restriction of σ_\dagger to $f^{-1}(C \setminus x)$ to the restriction of σ to $C \setminus x$. Indeed, $\Gamma(C, \mathcal{L}) \rightarrow \Gamma(C \setminus x, \mathcal{L})$ is injective by Lemma 3.6, so this suffices. We have $\mathcal{K}(-x)|_{C \setminus x} \simeq \mathcal{L}|_{C \setminus x}$ by item 1 in Remark 4.15 and $f_*f^i\mathcal{L}|_{C \setminus x} \simeq \mathcal{L}|_{C \setminus x}$ by item 2 in Remark 4.15 compatibly with the restriction of (26), that is, the resulting triangle is commutative. To justify this compatibility, it is necessary to revisit (26) and (25), but everything is clear. Then the diagram

$$\begin{array}{ccccc} \mathcal{O}_C|_{C \setminus x} & \longrightarrow & \mathcal{K}(-x)|_{C \setminus x} & \xlongequal{\quad} & \mathcal{L}|_{C \setminus x} \\ & \searrow & \text{(26)}|_{C \setminus x} & \swarrow & \\ & & f_*f^i\mathcal{L}|_{C \setminus x} & & \end{array}$$

in which the curved arrow is the restriction of the homomorphism $\mathcal{O}_C \rightarrow f_*f^i\mathcal{L}$ used in Construction 4.14 is obviously commutative. It remains to see where $1 \in \Gamma(C \setminus x, \mathcal{O}_C)$ ends up in the diagram above. On one hand, its image in $\mathcal{L}|_{C \setminus x}$ is $\sigma|_{C \setminus x}$ by item 1 in Remark 4.15. On the other hand, its image in $f_*f^i\mathcal{L}|_{C \setminus x}$ is the restriction of σ_\dagger to $f^{-1}(C \setminus x)$ by Construction 4.14. Then the claim in the beginning of this proof follows, since the diagram is commutative, and $f_*f^i\mathcal{L} \rightarrow \mathcal{L}$ restricts on $C \setminus x$ to the isomorphism in the diagram, cf. part 2 of Proposition 3.10. \square

Remark 4.18. Here are some remarks related to Lemma 4.17.

- (1) It may or may not be the case that there exists a unique global section of \mathcal{L}_\dagger which maps to σ by (28).
 - (a) For instance, when $S = \text{Spec } \mathbb{K}$ and the data is degenerate, this is not the case, as explained at length in Remark 4.11.
 - (b) However, if, for instance, we assume that C and C_\dagger are integral, and f is a birational morphism, then $\Gamma(C_\dagger, \mathcal{L}_\dagger) \rightarrow \Gamma(C, \mathcal{L})$ is injective, so σ can have only one preimage. Note also that, in this case, if ζ and ζ_\dagger are the generic points of C and C_\dagger and $K = K(C) = K(C_\dagger)$ is the field of fractions, then there is a natural identification of the fibers at the generic points $K \otimes \mathcal{L} \simeq K \otimes \mathcal{L}_\dagger$ as (one-dimensional) K -vector spaces, under which $\sigma(\zeta) = \sigma_\dagger(\zeta_\dagger)$.
- (2) Note that if $(S, C, \pi, x, \mathcal{L}, \sigma)$ satisfies Situation 4.9, and $\alpha \in \Gamma(S, \mathcal{O}_S^\times)$, then $(S, C, \pi, x, \mathcal{L}, \pi^*\alpha \cdot \sigma)$ also satisfies Situation 4.9. Let $(S, C'_\dagger, \pi'_\dagger, x'_\dagger, \mathcal{L}'_\dagger, \sigma'_\dagger)$ be the output of Construction 4.14 on this data. The commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_C & \xrightarrow{1 \mapsto (-\sigma, 1)} & \mathcal{L} \oplus \mathcal{O}_C(x) & \xrightarrow{\kappa} & \mathcal{K} \longrightarrow 0 \\ & & \parallel & & \begin{bmatrix} \pi^*\alpha & 0 \\ 0 & 1 \end{bmatrix} \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_C & \xrightarrow{1 \mapsto (-\pi^*\alpha \cdot \sigma, 1)} & \mathcal{L} \oplus \mathcal{O}_C(x) & \xrightarrow{\kappa \begin{bmatrix} \pi^*\alpha & 0 \\ 0 & 1 \end{bmatrix}^{-1}} & \mathcal{K} \longrightarrow 0 \end{array}$$

in which the top row is (21) and the bottom row is the version of (21) for the new set of data, shows that the cokernels \mathcal{K} for the two constructions are isomorphic, so we have a natural identification $C_\dagger = C'_\dagger$ compatible with the projections to S . By (24), we can also identify $\mathcal{L}'_\dagger = \mathcal{L}_\dagger$.

If we also assume furthermore for simplicity that the hypotheses from item 1b (that C_\dagger and C are integral and f is birational) hold, then, by 1b,

$\sigma'_\dagger(\zeta_\dagger) = \pi^*\alpha \cdot \sigma(\zeta) = \pi^*\alpha \cdot \sigma_\dagger(\zeta_\dagger)$, and hence $\sigma'_\dagger = \pi^*\alpha \cdot \sigma_\dagger$ everywhere. It is also clear that $x_\dagger(\zeta) = x'_\dagger(\zeta)$, so $x_\dagger = x'_\dagger$.

Finally, let's check that Construction 4.14 eliminated the degeneracy.

Lemma 4.19. *In Construction 4.14, $x_\dagger^*\sigma_\dagger^*$ is a nowhere vanishing global section of $x_\dagger^*\mathcal{L}_\dagger^*$.*

Proof. Let's first check that $\mathcal{L}_\dagger = f^i\mathcal{L}$ is trivial along x_\dagger . We have

$$\begin{aligned} (29) \quad x_\dagger^*f^i\mathcal{L} &= x_\dagger^*\mathcal{O}_{\mathbb{P}\mathcal{K}(-x)}(1) \quad \text{by (25)} \\ &= x^*\mathcal{O}_C(-x) \otimes x_\dagger^*\mathcal{O}_{\mathbb{P}\mathcal{K}}(1) \quad \text{taking a step back in (25)} \\ &= x^*\mathcal{O}_C(-x) \otimes x^*\mathcal{O}_C(x) = \mathcal{O}_S \quad \text{by the definition of } x_\dagger \end{aligned}$$

as desired. Consider the following diagram of \mathcal{O}_S -modules

$$\begin{array}{ccccc} \mathcal{O}_S & \longrightarrow & x^*\mathcal{K}(-x) & \equiv & x_\dagger^*f^*f_*\mathcal{O}_{\mathbb{P}\mathcal{K}(-x)}(1) \\ & \searrow \text{id}_{\mathcal{O}_S} & \downarrow & \searrow & \downarrow \\ & & \mathcal{O}_S & \xrightarrow{(29)} & x_\dagger^*\mathcal{O}_{\mathbb{P}\mathcal{K}(-x)}(1) & \xrightarrow{(29)} & x_\dagger^*f^i\mathcal{L} \end{array}$$

in which the map $\mathcal{O}_S \rightarrow x^*\mathcal{K}(-x)$ is the composition $\mathcal{O}_S \rightarrow x^*(\mathcal{L}(-x) \oplus \mathcal{O}_C) \rightarrow x^*\mathcal{K}(-x)$, the map $x^*\mathcal{K}(-x) \rightarrow \mathcal{O}_S$ is a twist of the map $x^*\mathcal{K} \rightarrow x^*\mathcal{O}_C(x)$ used in Construction 4.14 to define x_\dagger , the map $x^*\mathcal{K}(-x) \rightarrow x_\dagger^*\mathcal{O}_{\mathbb{P}\mathcal{K}(-x)}(1)$ also comes from the definition of x_\dagger , and the isomorphism $x^*\mathcal{K}(-x) = x_\dagger^*f^*f_*\mathcal{O}_{\mathbb{P}\mathcal{K}(-x)}(1)$ comes from $x_\dagger^*f^* = x^*$ and

$$\begin{aligned} (30) \quad f_*\mathcal{O}_{\mathbb{P}\mathcal{K}(-x)}(1) &= f_*(\mathcal{O}_{\mathbb{P}\mathcal{K}}(1) \otimes f^*\mathcal{O}_C(-x)) \\ &= f_*\mathcal{O}_{\mathbb{P}\mathcal{K}}(1) \otimes \mathcal{O}_C(-x) \quad \text{by the projection formula} \\ &= \mathcal{K}(-x) \quad \text{by part 1b of Proposition 4.12,} \end{aligned}$$

similarly to some steps in (26). It is not hard to check that the diagram is commutative.

Consider the image of $1 \in \Gamma(S, \mathcal{O}_S)$ in $\Gamma(S, x_\dagger^*f^i\mathcal{L})$. On one hand, if we take the upper route in the diagram, we see that this image is $x_\dagger^*\sigma_\dagger$ by revisiting Construction 4.14. Indeed, the composition of (30) with the pushforward of (25) along f is (26), hence the image of 1 in $\Gamma(S, x_\dagger^*f^*f_*f^i\mathcal{L})$ is $x_\dagger^*f^*\sigma_\dagger = x^*\sigma_\dagger$, and the claim follows. On the other hand, the lower route is clearly an isomorphism, so the upper route is also an isomorphism. Hence $x_\dagger^*\sigma_\dagger$ is nowhere vanishing. \square

To summarize, we have the following.

Theorem 4.20. *Construction 4.14 has the following features:*

- (1) *the output data is nondegenerate;*
- (2) *if the input data is nondegenerate, then the output data is isomorphic to the input data;*
- (3) *if $S = \text{Spec } \mathbb{K}$, and the input data is degenerate, then the output data is (isomorphic to) that described in Example 4.10; and*
- (4) *it commutes with base change.*

Proof. The first claim is Lemma 4.19, the second claim is item 4 in Remark 4.15, the third claim is elementary and left to the reader, and the last claim is Proposition 4.16. \square

At this point, we consider Theorem 1.2 proved.

Finally, we prove a technical proposition which contains most of the work needed later to prove that the bubbling down operation constructed in §5 is inverse to bubbling up.

Proposition 4.21. *Let $(S, C, \pi, x, \mathcal{L}, \sigma)$ and $(S, Y, \varpi, y, \mathcal{J}, \rho)$ both satisfy the requirements of Situation 4.9, and assume in addition that $y^*\rho$ is a nowhere vanishing section of $y^*\mathcal{J}$. Let $q : Y \rightarrow C$ with property R, such that:*

- (1) $x = qy$, $\mathcal{J} = q^i\mathcal{L}$ and $q_*\mathcal{J} = q_*q^i\mathcal{L} \rightarrow \mathcal{L}$, cf. (11), maps $\rho \mapsto \sigma$;
- (2) for any geometric point $\bar{s} \rightarrow S$, there exists an isomorphism $\alpha : Y_{\bar{s}} \rightarrow (C_{\dagger})_{\bar{s}}$ over \bar{s} , such that $q_{\bar{s}} = f_{\bar{s}}\alpha$, and α is compatible in the natural sense with all the rest of the data (restrictions of line bundles, sections, etc.).

Then there exists a canonical isomorphism $q_{\dagger} : Y \rightarrow C_{\dagger}$ such that $q = fq_{\dagger}$, $x_{\dagger} = q_{\dagger}y$, and $q_{\dagger}^*\sigma_{\dagger} \mapsto \rho$ under the isomorphism $q_{\dagger}^*f^i\mathcal{L} \simeq q^i\mathcal{L}$.

Proof. By item 1 in Proposition 3.10, we have an isomorphism $\mathcal{O}_C(-x) \simeq q_*\mathcal{O}_Y(-y)$. Proposition 3.10 gives a homomorphism $q^*(\mathcal{L}^\vee(x)) \rightarrow (q^i\mathcal{L}^\vee)(y)$. By assumption 2 and Example 3.8, this homomorphism is an isomorphism on geometric fibers, and hence an isomorphism. Dualizing, we obtain

$$(31) \quad q^*(\mathcal{L}(-x)) \simeq (q^i\mathcal{L})(-y).$$

By (31) and Remark 3.2,

$$(32) \quad q_*[(q^i\mathcal{L})(-y)] = q_*q^*\mathcal{L}(-x) = \mathcal{L}(-x).$$

We claim that the following diagram in which the top row is induced by σ and the bottom one by ρ is commutative.

$$\begin{array}{ccc} \mathcal{O}_C(-x) & \longrightarrow & \mathcal{L}(-x) \\ \simeq \downarrow & & \parallel (32) \\ q_*\mathcal{O}_Y(-y) & \longrightarrow & q_*(q^i\mathcal{L})(-y) \end{array}$$

Indeed, the claim amounts to a statement of the form that two elements of the group $\text{Hom}(\mathcal{O}_C(-x), \mathcal{L}(-x)) = \Gamma(C, \mathcal{L})$ coincide. However, it is clear that they coincide over $C \setminus x$ by assumption 1, the fact that q induces an isomorphism $q^{-1}(C \setminus x) \simeq C \setminus x$, and part 2 of Remark 4.15, so they must coincide on C by Lemma 3.6.

Since (S, Y, \dots) is nondegenerate in the sense of the current section, part 4 of Remark 4.15 gives a short exact sequence $0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{J} \oplus \mathcal{O}_Y(y) \rightarrow \mathcal{J}(y) \rightarrow 0$. If we twist this sequence by $\mathcal{O}_Y(-y)$ and push forward along q , we obtain a short exact sequence

$$(33) \quad 0 \rightarrow q_*\mathcal{O}_Y(-y) \rightarrow q_*[q^i(\mathcal{L}(-y))] \oplus q_*\mathcal{O}_Y \rightarrow q_*q^i\mathcal{L} \rightarrow 0,$$

since $R^1q_*\mathcal{O}_Y(-y) = 0$ by item 1 in Proposition 3.10 once more, since $qy = x$ is a smooth section. We have a (only solid arrow, for now) diagram with exact rows

$$(34) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_C(-x) & \longrightarrow & \mathcal{L}(-x) \oplus \mathcal{O}_C & \longrightarrow & \mathcal{K}(-x) \longrightarrow 0 \\ & & \parallel & & \left[\begin{array}{cc} (32) & 0 \\ 0 & q^\# \end{array} \right] \parallel & & \downarrow \simeq \\ 0 & \longrightarrow & q_*\mathcal{O}_Y(-y) & \longrightarrow & q_*(q^i\mathcal{L})(-y) \oplus q_*\mathcal{O}_Y & \xrightarrow{[-1 \ \rho]} & q_*q^i\mathcal{L} \longrightarrow 0 \end{array}$$

The top row is (21) twisted by $\mathcal{O}_C(-x)$, the bottom row is (33). This diagram is commutative, by the commutativity of the first diagram in this proof, and part 1 of Proposition 3.10. A trivial diagram chase shows that there exists a (unique) homomorphism $\mathcal{K}(-x) \rightarrow q_*q^i\mathcal{L}$ that makes the diagram above commute if assigned as the dashed arrow and moreover, it is an isomorphism $\mathcal{K}(-x) \simeq q_*q^i\mathcal{L}$. We claim that its adjoint

$$(35) \quad q^*\mathcal{K}(-x) \rightarrow q^i\mathcal{L}$$

is surjective. Consider the ‘adjoint’ diagram of (34).

$$(36) \quad \begin{array}{ccccccc} q^*\mathcal{O}_C(-x) & \longrightarrow & q^*\mathcal{L}(-x) \oplus \mathcal{O}_Y & \longrightarrow & q^*\mathcal{K}(-x) & \longrightarrow & 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_Y(-y) & \longrightarrow & (q^i\mathcal{L})(-y) \oplus \mathcal{O}_Y & \longrightarrow & q^i\mathcal{L} \longrightarrow 0 \end{array}$$

The central vertical map is still an isomorphism by (31). The map $q^*\mathcal{K}(-x) \rightarrow q^i\mathcal{L}$ we are interested in is the right vertical map in the diagram, and it is surjective by the snake lemma. Twisting, we obtain a surjection $q^*\mathcal{K} \rightarrow q^*\mathcal{O}_C(x) \otimes q^i\mathcal{L}$. By Remark 4.3, this induces an S -morphism $q_{\dagger} : Y \rightarrow C_{\dagger}$ such that $f_{q_{\dagger}} = q$.

Let’s first check that $q_{\dagger}y = x_{\dagger}$. In light of Remark 4.3, $q_{\dagger}y$ corresponds to the surjection $x^*\mathcal{K} \rightarrow x^*\mathcal{O}_C(x) \otimes y^*q^i\mathcal{L}$. Earlier, x_{\dagger} was defined to correspond to the natural map $x^*\mathcal{K} \rightarrow x^*\mathcal{O}_C(x)$. Since $y^*\rho$ is nowhere vanishing, the homomorphism $\mathcal{O}_S \rightarrow y^*\mathcal{J}$ is an isomorphism, and hence so is its twist $x^*\mathcal{O}_C(x) \rightarrow x^*\mathcal{O}_C(x) \otimes y^*q^i\mathcal{L}$. It remains to check that the triangle with the three homomorphisms among $x^*\mathcal{K}$, $x^*\mathcal{O}_C(x)$, $x^*\mathcal{O}_C(x) \otimes y^*q^i\mathcal{L}$ is commutative. To this end, consider the diagram

$$\begin{array}{ccc} & & \mathcal{O}_S \\ & \nearrow & \downarrow \simeq \\ x^*\mathcal{L}(-x) \oplus \mathcal{O}_S & \longrightarrow & x^*\mathcal{K}(-x) \\ \simeq \uparrow & & \searrow \\ y^*[(q^i\mathcal{L})(-y)] \oplus \mathcal{O}_S & \xrightarrow{[0 \quad y^*\rho]} & y^*(q^i\mathcal{L}) \end{array}$$

in which all arrows in the bottom (trapezoidal) face are obtained as pullbacks along y of homomorphisms in (36), the $x^*\mathcal{K}(-x) \rightarrow \mathcal{O}_S$ arrow is a twist of the map $x^*\mathcal{K} \rightarrow x^*\mathcal{O}_C(x)$ discussed earlier, the curved arrow is projection to the second factor, and the right vertical arrow is the map discussed earlier which maps $1 \mapsto y^*\rho$. We argue commutativity as follows. First, the top (curved, triangular) face commutes, essentially by construction. Second, the bottom (trapezoidal) face commutes, thanks to (36). Third, the convex hull commutes. Indeed, we may check this separately on the submodule $y^*[(q^i\mathcal{L})(-y)] \oplus 0$ and the section $(0, 1)$ of $y^*[(q^i\mathcal{L})(-y)] \oplus \mathcal{O}_S$; via either route to $y^*(q^i\mathcal{L})$, the former is mapped to 0, and the latter to $y^*\rho$. Since $x^*\mathcal{L}(-x) \oplus \mathcal{O}_S \rightarrow x^*\mathcal{K}(-x)$ is surjective, the three points above imply that the right triangular face commutes as well. After twisting by $x^*\mathcal{O}_C(x)$, this is precisely what had to be checked.

Now it is easy to check that the morphism $q_{\dagger} : Y \rightarrow C_{\dagger}$ over S constructed earlier is an isomorphism on geometric fibers, and hence an isomorphism.

It remains to check that $q_{\dagger}^*\sigma_{\dagger} \mapsto \rho$ under the isomorphism $q_{\dagger}^*f^i\mathcal{L} \simeq q^i\mathcal{L}$. This isomorphism can be obtained, for instance, from $q_{\dagger}^*f^i = q_{\dagger}^!f^i = q^i$. Moreover, we have maps $q^*\mathcal{K}(x) = q_{\dagger}^*f^*\mathcal{K}(-x) \rightarrow q_{\dagger}^*f^i\mathcal{L}$ coming from (26) and the adjoint

property, and $q^*\mathcal{K}(-x) \rightarrow q^i\mathcal{L}$, cf. (35). We claim that these 3 homomorphisms fit into a commutative triangle. We will give an indirect argument. The starting point is part 3 in Remark 4.15. Pulling back along q_{\dagger} , we see that

$$q^*\mathcal{K}(-x) = q_{\dagger}^* f^* \mathcal{K}(-x) \rightarrow q_{\dagger}^* f^i \mathcal{L}$$

corresponds to $q_{\dagger} : Y \rightarrow C_{\dagger} = \mathbb{P}\mathcal{K}(-x)$ in the sense of Remark 4.3. On the other hand, by the same remark, $q_{\dagger} : Y \rightarrow \mathbb{P}\mathcal{K}(-x)$ also corresponds to (35), since (35) is a twist of the morphism defining q_{\dagger} as a morphism to $\mathbb{P}\mathcal{K}$. It follows that there exists an isomorphism $q_{\dagger}^* f^i \mathcal{L} \simeq q^i \mathcal{L}$, such that the triangle diagram with this isomorphism and the maps $q^*\mathcal{K}(-x) \rightarrow q_{\dagger}^* f^i \mathcal{L}$ and $q^*\mathcal{K}(-x) \rightarrow q^i \mathcal{L}$ commutes. Although there is no obvious formal reason why this isomorphism $q_{\dagger}^* f^i \mathcal{L} \simeq q^i \mathcal{L}$ coincides with the earlier isomorphism $q_{\dagger}^* f^i \mathcal{L} \simeq q^i \mathcal{L}$, this is true, and the reason is the following. Since all geometry becomes trivial over $C \setminus x$, it is straightforward to check that the two isomorphisms above coincide over $q^{-1}(C \setminus x) \simeq C \setminus x$. However, $\mathcal{A}ut(q^i \mathcal{L}) \simeq \mathcal{O}_Y^{\times}$ and $\Gamma(Y, \mathcal{O}_Y^{\times}) \rightarrow \Gamma(Y \setminus y, \mathcal{O}_Y^{\times})$ is injective since $\Gamma(Y, \mathcal{O}_Y) \rightarrow \Gamma(Y \setminus y, \mathcal{O}_Y)$ is injective by Lemma 3.6, so the equality of the isomorphisms a fortiori extends to all of Y . This concludes the proof of the desired commutativity. Let $\tau \in \Gamma(Y, q^*\mathcal{K}(-x))$ be the image of $1 \in \Gamma(Y, \mathcal{O}_Y)$ under the composition

$$\mathcal{O}_Y \rightarrow q^*(\mathcal{L}(-x) \oplus \mathcal{O}_C) \rightarrow q^*\mathcal{K}(-x).$$

Then the image of τ in $q_{\dagger}^* f^i \mathcal{L}$ is $q_{\dagger}^* \sigma_{\dagger}^*$ by Construction 4.14, while the image of τ in $q^i \mathcal{L}$ is ρ by the commutativity of the right square in (36). Then the commutativity of the triangle above concludes the proof. \square

5. BUBBLING DOWN

Recall the categories of curves $\mathfrak{C}_{g,m,n}^{c,+}$ and $\mathfrak{C}_{g,m,n}^c$ from Definition 2.1.

Proposition 5.1. *Constructions 4.4 and 4.14 induce functors*

$$(37) \quad \mathfrak{C}_{g,m,n}^{1,+} \xrightarrow{\ddagger} \mathfrak{C}_{g,m,n}^{2,+} \xrightarrow{\dagger} \mathfrak{C}_{g,m,n}^{3,+}$$

which, in turn, restrict to functors

$$(38) \quad \mathfrak{C}_{g,m,n}^1 \xrightarrow{\ddagger} \mathfrak{C}_{g,m,n}^2 \xrightarrow{\dagger} \mathfrak{C}_{g,m,n}^3.$$

Proof. We take $\mathcal{L} = \omega_{C/S}^{\vee}$ in Construction 4.4 (resp. Construction 4.14); then $(\omega_{C/S}^{\vee})_{\ddagger} = \omega_{C_{\ddagger}/S_{\ddagger}}^{\vee}$ (resp. $(\omega_{C/S}^{\vee})_{\dagger} = \omega_{C_{\dagger}/S_{\dagger}}^{\vee}$), by (15), (24), Definition 3.12, and Lemma 3.7. To be very accurate, in Construction 4.4, σ is the restriction $\omega_{C/S} \rightarrow \mathcal{O}_C(-w_1 - \dots - w_m)$ of ϕ rather than ϕ itself. In Construction 4.14, σ is simply ϕ . The apparent problem is that \bar{x} is not specified in Construction 4.4, and neither \bar{x} nor \bar{w} is specified in Construction 4.14. However, the conditions in Definitions 2.1 and 2.5 force these sections to be contained in the maximal open subset U such that $f^{-1}(U) \simeq U$ (this is the open in Lemma 3.5), and then lifting them is a trivial matter. Propositions 4.6 and 4.16 state that \ddagger and \dagger commute with base change. Finally, the functors in (37) restrict to the functors in (38). Indeed, condition 3 in Definition 2.5 may be checked on geometric fibers, and is then elementary. \square

The two operations above suffice to construct the degeneration in Theorem 1.4. However, they do not suffice to interpret the central fiber in any meaningful way, and, in particular, to prove Theorem 1.3.

Therefore, in this section, we construct inverse operations to the two operations above, and in particular prove Theorem 2.4. Artificial as it may be, it is often feasible to treat the two inverse operations simultaneously.

We start with some geometric preliminaries. Recall that a coherent \mathcal{O}_X -module \mathcal{F} is *normally generated* if the canonical map $\Gamma(X, \mathcal{F})^{\otimes k} \rightarrow \Gamma(X, \mathcal{F}^{\otimes k})$ is surjective for all $k \geq 1$ (e.g. [18, Definition 1.7]).

Proposition 5.2. *Let $g, m, n, c \geq 0$ be integers such that $2g + 2n + m + c \geq 4$ and $c \in \{1, 2\}$. Let $(S = \text{Spec } \mathbb{K}, C, \dots)$ be an object of $\mathfrak{C}_{g,m,n}^c(\mathbb{K})$ with notation as in Definition 2.5, and*

$$\mathcal{L} = \omega_C(w_1 + \dots + w_m + 2x_1 + \dots + 2x_n + (c-1)x).$$

Then:

- (1) $\mathcal{L}^{\otimes k}$ is generated by global sections if $k \geq 2$;
- (2) $H^1(C, \mathcal{L}^{\otimes k}) = 0$ if $k \geq 2$;
- (3) $\mathcal{L}^{\otimes k}$ is normally generated if $k \geq 3$.

Proof. First, let us prove the proposition in the case when all components of C on which ϕ is not identically zero are rational tails. Let T_1, \dots, T_a ($a \geq 0$) be these components. Let Y be the nodal curve obtained by contracting T_1, \dots, T_a ; then we have both a contraction $C \rightarrow Y$, and an immersion $j: Y \hookrightarrow C$. Note that

$$(39) \quad w_1, \dots, w_m \notin T_1 \cup \dots \cup T_a$$

unless $a = 1$ and $C = T_1$ because ϕ must vanish at the point where T_α is attached to the rest of the curve, and hence cannot vanish anywhere else on T_α , as $\deg \omega_C^\vee|_{T_\alpha} = 1$. We assume (39) holds since the alternative is trivial. We have $x_1, \dots, x_n \in T_1 \cup \dots \cup T_a$ since ϕ cannot vanish at x_1, \dots, x_n . Let y_1, \dots, y_r be the following points on Y :

- w_1, \dots, w_m (or $j^{-1}(w_1), \dots, j^{-1}(w_m)$ to be exceedingly accurate);
- the images of the tails T_1, \dots, T_a under the contraction $C \rightarrow Y$; and
- x , but only in case $c = 2$ and $x \notin T_1 \cup \dots \cup T_a$.

Thus, $r = m + a + 1$ if $c = 2$ and $x \notin T_1 \cup \dots \cup T_a$, and $r = m + a$ otherwise. Note that y_1, \dots, y_r are smooth points of Y , and that $\omega_Y(y_1 + \dots + y_r) \simeq j^* \mathcal{L}$. Then $(Y, y_1, \dots, y_r) \in \overline{\mathcal{M}}_{g,r}(\mathbb{K})$ since \mathcal{L} is ample by condition 3 in Definition 2.5 and the fact that ample restricts to ample. The analogues of items 1 – 3 above for Y, y_1, \dots, y_r hold by [18, Theorem 1.8]. All items follow inductively, by reattaching the tails T_1, \dots, T_a back to Y one by one. The inductive step is essentially the argument in the last paragraph in the proof of [18, Theorem 1.8].

It remains to consider the case when ϕ is not identically 0 on some component Σ of C which is not a rational tail. Such cases occur extremely rarely. Note that, if $\Sigma_1 \neq \Sigma_2$ are irreducible components of C such that $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, $\phi|_{\Sigma_1} \equiv 0$, and $\phi|_{\Sigma_2} \not\equiv 0$, then Σ_2 is a rational tail intersecting Σ_1 . (By the elementary description of fields in §1, $\phi|_{\Sigma_2}$ regarded as a global section of T_{Σ_2} vanishes *doubly* (by the assumption $\phi|_{\Sigma_1} \equiv 0$) at any point of $\Sigma_1 \cap \Sigma_2$ and also vanishes at all other nodes of C on Σ_2 , so the claim follows by simple degree considerations.) However, this scenario with $\Sigma_2 = \Sigma$ is ruled out, so the only possible conclusion is that ϕ doesn't vanish identically on any irreducible component of C . The only nodal curves C which admit (logarithmic) vector fields which don't vanish identically on any irreducible component are smooth genus 1 curves, rational chains, and rational cycles, so it remains to check all items in all these cases. If C is irreducible, then

all claims amount to well-known facts about elliptic curves. In the remaining cases (rational cycles or chains), it is straightforward to check that there exist smooth, *distinct* points $p_1, \dots, p_N \in C$ such that $\mathcal{O}_C(w_1 + \dots + w_m + 2x_1 + \dots + 2x_n + (c-1)x) \simeq \mathcal{O}_C(p_1 + \dots + p_N)$, or equivalently, $\mathcal{L} \simeq \omega_C(p_1 + \dots + p_N)$. Since \mathcal{L} is ample, $(C, p_1, \dots, p_N) \in \overline{\mathcal{M}}_{g,N}(\mathbb{K})$, and the claims follow from [18, Theorem 1.8]. \square

Definition 5.3. Let $g, m, n, c \geq 0$ be integers such that $2g + 2n + m + c \geq 5$ and $c \in \{2, 3\}$. Let $\mathbf{o} = (S, C, \pi, \overline{w}, \overline{x}, x, \phi)$ be an object of $\mathfrak{C}_{g,m,n}^c$. A *bubbling down* of \mathbf{o} is an object $\mathbf{o}_b = (S, C_b, \pi_b, \overline{w}_b, \overline{x}_b, x_b, \phi_b)$ of $\mathfrak{C}_{g,m,n}^{c-1}$, together with a morphism $h : C \rightarrow C_b$ with property R, such that:

- (1) $\pi_b h = \pi$, $hx_j = x_{j,b}$ for all j , $hx = x_b$, $hw_i = w_{i,b}$ for all i ; and
- (2) ϕ_b is the image of ϕ under $h_* \omega_{C/S}^\vee \rightarrow \omega_{C_b/S}^\vee$, which is a special case of (11), in light of Definition 3.12 and Lemma 3.7.

Remark 5.4. If ϕ and ϕ_b are regarded as sections of $\omega_{C/S}^\vee(-w_1 - \dots - w_n)$ and $\omega_{C_b/S}^\vee(-w_{1,b} - \dots - w_{n,b})$, condition 2 in Definition 5.3 is equivalent to asking that the homomorphism $h_* \omega_{C/S}^\vee(-\sum_{i=1}^n w_i) \rightarrow \omega_{C_b/S}^\vee(-\sum_{i=1}^n w_{i,b})$, which is also an example of (11), maps ϕ to ϕ_b .

A base change of a bubbling down is a bubbling down by Proposition 3.14.

Proposition 5.5. *Let g, m, n, c as in Definition 5.3. The bubbling down of any object of $\mathfrak{C}_{g,m,n}^c$ exists and is unique up to unique isomorphism in $\mathfrak{C}_{g,m,n}^{c-1}$ compatible with the maps with property R.*

Proof. To prove existence, we follow the proof of [1, Proposition 3.10] extremely closely (which in turn relates to [18, §1]). Let

$$\mathcal{L} = \omega_{C/S}(w_1 + \dots + w_m + 2x_1 + \dots + 2x_n + (c-2)x),$$

and define

$$(40) \quad C_b = \text{Proj}_S \mathcal{S} \quad \text{where} \quad \mathcal{S} = \bigoplus_{k \geq 0} \mathcal{S}_k = \bigoplus_{k \geq 0} \pi_* \mathcal{L}^{\otimes k},$$

with structure map $\pi_b : C_b \rightarrow S$.

First, we analyse in depth the case $S = \text{Spec } \mathbb{K}$. An irreducible component Σ of C is a *component to be contracted* if $\Sigma \simeq \mathbb{P}^1$, $x \in \Sigma$, $x_1, \dots, x_n \notin \Sigma$, and the following conditions dependent on c are also satisfied:

- if $c = 3$, then Σ contains precisely one node of C , and $w_1, \dots, w_m \notin \Sigma$;
- if $c = 2$, then *either*
 - (1) Σ contains precisely one node of C , and one of w_1, \dots, w_m ; *or*
 - (2) Σ contains precisely two nodes of C , and $w_1, \dots, w_m \notin \Sigma$.

Clearly, there is at most one component to be contracted. We construct $\chi : C \rightarrow Y$ which contracts the component to be contracted (if there is one) and nothing else, such that Y is nodal. The construction is by elementary means, similar to [1]. Then, Y with the markings $\chi(\overline{w})$, $\chi(\overline{x})$, $\chi(x)$, and the image of ϕ under the homomorphism $\chi_* \omega_C^\vee \rightarrow \omega_Y^\vee$ (an example of (11)) is a bubbling down of the given data (note also that χ has property R). Let

$$\mathcal{J} = \omega_Y(\chi(w_1) + \dots + \chi(w_m) + 2\chi(x_1) + \dots + 2\chi(x_n) + (c-2)\chi(x)).$$

Claim 5.6. For all $k \geq 0$, $\chi^* \mathcal{J}^{\otimes k} \simeq \mathcal{L}^{\otimes k}$, $\mathcal{J}^{\otimes k} \simeq \chi_* \mathcal{L}^{\otimes k}$, and $R^1 \chi_* \mathcal{L}^{\otimes k} = 0$.

Proof. The proof is straightforward. We will only mention that the second claim follows from the first claim and Remark 3.2. \square

Claim 5.7. The following hold:

- (1) $\mathcal{L}^{\otimes k}$ is generated by global sections if $k \geq 2$;
- (2) $H^1(C, \mathcal{L}^{\otimes k}) = 0$ if $k \geq 2$;
- (3) $\mathcal{L}^{\otimes k}$ is normally generated if $k \geq 3$.

Proof. Apply Proposition 5.2 with $c - 1$ in the role of c , Y in the role of C , and \mathcal{J} in the role of \mathcal{L} from the statement of the proposition. The claim follows from this and Claim 5.6. (Please compare also with Lemma 1.6, Theorem 1.8, and Corollary 1.10 in [18] on one hand, and with Proposition 3.9 and Lemmas 3.11 and 3.12 in [1], on the other hand.) \square

Claim 5.8. With C_b defined as in (40), $Y \simeq C_b$. Moreover, in the language of [8, (3.7.1)], the composition $C \xrightarrow{\chi} Y \simeq C_b$ coincides with the morphism associated to \mathcal{L} and $\mathcal{O}_C \otimes \bigoplus_{k \geq 0} \Gamma(C, \mathcal{L}^{\otimes k}) \rightarrow \bigoplus_{k \geq 0} \mathcal{L}^{\otimes k}$.

Proof. By (40), Claim 5.6, [8, Proposition (4.6.3)], and the ampleness of \mathcal{J} ,

$$C_b = \text{Proj} \bigoplus_{k \geq 0} \Gamma(C, \mathcal{L}^{\otimes k}) = \text{Proj} \bigoplus_{k \geq 0} \Gamma(Y, \chi_* \mathcal{L}^{\otimes k}) = \text{Proj} \bigoplus_{k \geq 0} \Gamma(Y, \mathcal{J}^{\otimes k}) = Y.$$

The second claim is trivial if χ is an isomorphism, so let us assume this is not the case, and let $\Sigma \subset C$ be the component to be contracted. By Claim 5.6, we have maps $\mathcal{O}_C \otimes \Gamma(Y, \mathcal{J}^{\otimes k}) = \mathcal{O}_C \otimes \Gamma(C, \mathcal{L}^{\otimes k}) \rightarrow \mathcal{L}^{\otimes k}$ for all $k \geq 0$. Then, by [8, (3.7.1)], the claim amounts to the statement that the composition

$$(41) \quad C = \text{Proj}_C \bigoplus_{k \geq 0} \mathcal{L}^{\otimes k} \xrightarrow{j} \text{Proj}_C \bigoplus_{k \geq 0} \mathcal{O}_C \otimes \Gamma(Y, \mathcal{J}^{\otimes k}) = C \times Y \rightarrow Y$$

equals χ . Note that j is a closed immersion (in particular, defined everywhere) by [26, Tag 07ZK], the general fact that $\text{Proj}_X \bigoplus_{k \geq 0} \mathcal{A}_k \cong \text{Proj}_X \bigoplus_{k \geq 0} \mathcal{A}_{k_0+k}$ (X is a scheme, $\bigoplus_{k \geq 0} \mathcal{A}_k$ is a graded \mathcal{O}_X -algebra, and $k_0 > 0$ is an integer), the graded ring isomorphism $\bigoplus_{k \geq 0} \Gamma(C, \mathcal{L}^{\otimes 3k}) \cong \bigoplus_{k \geq 0} \Gamma(Y, \mathcal{J}^{\otimes 3k})$, and item 3 in Claim 5.7. The fact that (41) coincides with χ follows from the easily checked fact that the restriction of (41) to $C \setminus \Sigma$ coincides with $C \setminus \Sigma \simeq Y \setminus \chi(\Sigma) \hookrightarrow Y$. Indeed, the explicit description of Y and χ clarifies that this a priori weaker statement suffices. \square

Let us now consider the general case, when the base S is arbitrary.

Claim 5.9. The following hold:

- (1) the formation of C_b commutes with base change;
- (2) π_b is flat and projective;
- (3) the morphism associated to \mathcal{L} and $\pi^* \mathcal{S} \rightarrow \bigoplus_{k \geq 0} \mathcal{L}^{\otimes k}$ (in the language of [8, (3.7.1)]) is defined everywhere; and
- (4) the formation of the morphism $h : C \rightarrow C_b$ from item 3 commutes with base change.

Proof. The proof of the first three items is entirely analogous to the proofs of Claims 1–3 inside the proof of [1, Proposition 3.10], but we will briefly review the arguments for the reader's convenience. (Claim 5.7 ensures that the required hypotheses are satisfied, so the arguments in [1] can be repeated almost verbatim – indeed, the

analogue of item (4) in Lemma 3.12 in loc. cit. is not used in Claims 1–3.) As in [1], the idea is to take $d \gg 0$ (in fact, $d = 3$ suffices), and to use the fact that

$$(42) \quad C_b = \text{Proj}_S \mathcal{S}^{(d)}, \quad \text{where} \quad \mathcal{S}^{(d)} = \bigoplus_{d|k} \mathcal{S}_k.$$

First, note that whenever $d|k$ (or more generally, either $k = 0$ or $k \gg 0$), \mathcal{S}_k is locally free and its formation commutes with base change. Indeed, this is trivial for $k = 0$ since $\pi_* \mathcal{O}_C = \mathcal{O}_S$, and follows from Claim 5.7 and the ‘cohomology and base change’ theorem otherwise. Hence, the formation of C_b commutes with base change, and, as in [1], flatness of π_b follows from local freeness.

Moreover, if $d|k$, the homomorphism

$$(43) \quad \pi^* \mathcal{S}_k = \pi^* \pi_* \mathcal{L}^{\otimes k} \rightarrow \mathcal{L}^{\otimes k}$$

is surjective. Indeed, we have established that $\pi_* \mathcal{L}^{\otimes k}$ commutes with base change, so the surjectivity may be checked on (geometric) fibers, when it follows from item 1 in Claim 5.7.

We claim that $\mathcal{S}^{(d)}$ is generated by $\mathcal{S}_1^{(d)} = \mathcal{S}_d$ over $\mathcal{S}_0^{(d)} = \mathcal{O}_S$. Since all summands are locally free and commute with base change, in view of Nakayama’s lemma, the claim can be checked on fibers and hence on geometric fibers, when it follows from item 3 in Claim 5.7. It also follows that the graded \mathcal{O}_C -algebra $\pi^* \mathcal{S}^{(d)}$ is generated by $\pi^* \mathcal{S}_1^{(d)} = \pi^* \mathcal{S}_d$ over \mathcal{O}_C . Projectivity of π_b follows from the claim at the beginning of this paragraph and [8, Proposition 5.5.1], as in [1].

Unwinding the meaning of the definition of h (using [8, (3.7.1)]), we see that h is equal to the composition

$$(44) \quad C = \text{Proj}_C \bigoplus_{d|k} \mathcal{L}^{\otimes k} \xrightarrow{j} \text{Proj}_C \pi^* \mathcal{S}^{(d)} = C \times_S C_b \rightarrow C_b.$$

The fact that the map denoted j is defined everywhere (in fact, a closed immersion) follows from the surjectivity of (43), the fact that the graded \mathcal{O}_C -algebra $\pi^* \mathcal{S}^{(d)}$ is generated by $\pi^* \mathcal{S}_1^{(d)}$ over \mathcal{O}_C , and [26, Tag 07ZK]. Then, it is clear from (44) that the formation of h commutes with base change. \square

By Claims 5.8 and 5.9, $\pi_b : C_b \rightarrow S$ is a prestable curve (recall that Y is nodal by construction), and, using Lemma 3.3 as well, h has property R.

It is straightforward to construct the rest of the data in the bubbling down. For the sections of π_b , we compose with h , i.e. $x_b = hx$, etc., then we simply define ϕ_b to be the image required by Definition 5.3. It remains to check that the data defines indeed an object of $\mathfrak{C}_{g,m,n}^{c-1}$, i.e. that $\omega_{C_b}(w_{1,b} + \cdots + w_{m,b} + 2x_{1,b} + \cdots + 2x_{n,b} + (c-2)x_b)$ is π_b -ample, and, if $c = 3$, that π_b is smooth at x_b and $w_{i,b} \cap x_b = \emptyset$ for all i . Both can be checked on geometric fibers, completing the proof of existence.

It remains to prove uniqueness. First, we claim that $U := h^{-1}(C_b \setminus x_b) = C \setminus h^{-1}(h(x))$ is independent of the choice of bubbling down. This claim can obviously be checked on fibers. In fact, it can be checked even on geometric fibers, when it becomes elementary. Indeed, if $s \in S$ and $\bar{s} \rightarrow S$ is the corresponding geometric point, then the surjectivity of $C_{\bar{s}} \rightarrow C_s$ can be used to show that, set-theoretically, $h_s^{-1}(h_s(x_s))$ is the image of $h_{\bar{s}}^{-1}(h_{\bar{s}}(x_{\bar{s}}))$ under $C_{\bar{s}} \rightarrow C_s$. Second, by Lemma 3.5 part 3, h restricts to an isomorphism $h|_U : U \rightarrow C_b \setminus x_b$. Let

$$\mathfrak{o}'_b = (S, C'_b, \pi'_b, \bar{w}'_b, \bar{x}'_b, x'_b, \phi'_b) \quad \text{with} \quad h' : C \rightarrow C'_b$$

be another bubbling down of \mathbf{o} . By [1, Lemma 2.2], there exists an S -morphism $r : C_b \rightarrow C'_b$ such that $rh = h'$. It is straightforward to check that r must be an isomorphism on the geometric fibers, and hence simply an isomorphism by flatness (this trick is used frequently in [18]). The compatibility of r with all sections is elementary. In particular, if U is as above, we have a commutative diagram

$$\begin{array}{ccccc}
 & & U & \xrightarrow{\quad} & C \\
 & \swarrow & \downarrow & \xrightarrow{\quad} & \downarrow \\
 C_b \setminus x_b & \xleftarrow{\quad} & & \xrightarrow{\quad} & C_b \xleftarrow{h} \\
 & \searrow & & \searrow & \downarrow \\
 & & C'_b \setminus x'_b & \xrightarrow{\quad} & C'_b \xleftarrow{r} \\
 & & & & \downarrow \\
 & & & & h'
 \end{array}$$

in which the left triangular face consists of isomorphisms only. It remains to check that the isomorphism r is compatible with ϕ_b and ϕ'_b , that is, that ϕ_b is mapped to ϕ'_b by the isomorphism $\Gamma(C_b \setminus x_b, \omega_{C_b/S}^\vee) \simeq \Gamma(C'_b \setminus x'_b, \omega_{C'_b/S'}^\vee)$ induced by r . The commutative diagram above clearly induces a commutative diagram on the level of vector fields. The map $\Gamma(C'_b, \omega_{C'_b/S'}^\vee) \rightarrow \Gamma(C'_b \setminus x'_b, \omega_{C'_b/S'}^\vee)$ is injective by Lemma 3.6.

$$\begin{array}{ccccc}
 & & \Gamma(U, \omega_{C/S}^\vee) & \xleftarrow{\quad} & \Gamma(C, \omega_{C/S}^\vee) \\
 & \swarrow & \downarrow & \xrightarrow{\quad} & \downarrow \\
 \Gamma(C_b \setminus x_b, \omega_{C_b/S}^\vee) & \xleftarrow{\quad} & & \xrightarrow{\quad} & \Gamma(C_b, \omega_{C_b/S}^\vee) \xleftarrow{(11)} \\
 & \searrow & & \searrow & \downarrow \\
 & & \Gamma(C'_b \setminus x'_b, \omega_{C'_b/S'}^\vee) & \xleftarrow{\quad} & \Gamma(C'_b, \omega_{C'_b/S'}^\vee) \xleftarrow{(11)}
 \end{array}$$

The claim then follows by a simple diagram chase. Uniqueness of the isomorphism is equivalent to triviality of all ‘automorphisms’ of any given bubbling down, and, in fact, there are even no nontrivial automorphisms of C_b compatible with h . \square

If \mathbf{o} is an object of $\mathfrak{C}_{g,m,n}^1$ (resp. $\mathfrak{C}_{g,m,n}^2$), then \mathbf{o} together with the morphism $f : C_\dagger \rightarrow C$ (resp. $f : C_\dagger \rightarrow C$) provided by Construction 4.4 (resp. Construction 4.14), satisfies all the requirements of Definition 5.3. Note that besides minor verifications, this also relies on the quite nontrivial Lemma 4.17 (keeping Remark 5.4 in mind). Hence Proposition 5.5 has the following important consequence.

Corollary 5.10. *Let $g, m, n, c \geq 0$ be integers such that $2g + 2n + m + c \geq 4$ and $c \in \{1, 2\}$, and let $\mathfrak{C}_{g,m,n}^c \xrightarrow{\#} \mathfrak{C}_{g,m,n}^{c+1}$, such that, with the notation in (38),*

$$\# = \begin{cases} \ddagger & \text{if } c = 1 \\ \dagger & \text{if } c = 2. \end{cases}$$

Then, the composition $\flat \circ \#$ is naturally isomorphic to the identity on $\mathfrak{C}_{g,m,n}^c$.

It remains to prove the other direction.

Proposition 5.11. *Let $g, m, n, c \geq 0$ be integers such that $2g + 2n + m + c \geq 5$ and $c \in \{2, 3\}$, and let $\mathfrak{C}_{g,m,n}^{c-1} \xrightarrow{\#} \mathfrak{C}_{g,m,n}^c$, such that, with the notation in (38),*

$$\# = \begin{cases} \ddagger & \text{if } c = 2 \\ \dagger & \text{if } c = 3. \end{cases}$$

Then, the composition $\# \circ \flat$ is naturally isomorphic to the identity on $\mathfrak{C}_{g,m,n}^c$.

Proof. We apply Proposition 4.8 if $c = 2$, respectively Proposition 4.21 if $c = 3$, with the current (C_b, \dots) , (C, \dots) , and h in the roles of (C, \dots) , (Y, \dots) , and q from the respective propositions. In either case (Proposition 4.8 or Proposition 4.21), the first condition is automatic by construction, while the second condition follows from the fact that \sharp and \flat are inverses when $S = \text{Spec } \mathbb{K}$, which is elementary (by Examples 4.2 and 4.10, Definition 5.3, and the basic considerations on components to be contracted in the proof of Proposition 5.5). \square

Corollary 5.10 and Proposition 5.11 complete the proof of Theorem 2.7, and hence also of Theorem 2.4.

6. CONSTRUCTING MODULI SPACES

We now turn our attention towards Theorems 1.3 and 1.4.

6.1. Coresidues. The map to \mathbb{A}^1 that exhibits the space X in Theorem 1.4 as a degeneration is obtained from the following construction, which is essentially the ‘dual’ of taking residues at poles of 1-forms (please see also Example 6.2 below). Recall that, if $\pi : C \rightarrow S$ is a prestable curve and $x : S \rightarrow C$ is any smooth section, then $x^*\Omega_{C/S} \cong x^*\omega_{C/S}$ since $\Omega_{C/S}$ and $\omega_{C/S}$ agree on the open where π is smooth [18, p. 163], and $x^*\mathcal{O}_C(-x(S)) \cong x^*\Omega_{C/S}$, since $x(S)$ is a relative effective Cartier divisor on C over S , and both sides are canonically the conormal sheaf of x .

Definition 6.1. Let $\pi : C \rightarrow S$ be a prestable curve, $x : S \rightarrow C$ a smooth section, and ϕ a homomorphism $\omega_{C/S} \rightarrow \mathcal{O}_S(-x)$. The element

$$x^*\mathcal{O}_C(-x) \cong x^*\Omega_{C/S} \cong x^*\omega_{C/S} \xrightarrow{-x^*\phi} x^*\mathcal{O}_C(-x)$$

(note the sign) of $\text{End}_{\mathcal{O}_S}(x^*\mathcal{O}_C(-x)) = \Gamma(\mathcal{O}_S)$ induces a map $S \rightarrow \text{Spec } \mathbb{Z}[t]$, called the *negative coresidue (NCR) morphism* of the datum.

In particular, if $\mathfrak{o} = (S, C, \dots)$ is an object of $\mathfrak{V}_{0,1,n}^+$ or $\mathfrak{C}_{0,1,n}^+$ (with notation as in Definition 2.1), then the NCR morphism of \mathfrak{o} is the NCR morphism relative to the section $w_1 : S \rightarrow C$, and ϕ regarded as a homomorphism $\omega_{C/S} \rightarrow \mathcal{O}_S(-w_1)$.

Note that the formation of the NCR map commutes with base change.

Example 6.2. If p is a smooth point on a curve C over \mathbb{K} , and v is a vector field which vanishes at p , then,

$$(45) \quad v = (c_1x + c_2x^2 + c_3x^3 + \dots) \frac{d}{dx} \quad \text{for some } c_1, c_2, c_3, \dots \in \mathbb{K}$$

in local coordinates x near p . Then, the ‘coresidue’ of v at p is c_1 . Indeed, if we change coordinates to $u = g(x)$, where $g \in \mathbb{K}[[x]]$, $g(0) = 0$, $g'(0) \neq 0$, then a calculation shows that a formula similar to (45) with the same c_1 (different c_2, c_3, \dots) holds in the u coordinate. For instance, if $v = (7x + 3x^3) \frac{d}{dx}$ and $u = 2x$, then $v = (7u + \frac{3}{4}u^3) \frac{d}{du}$ since $\frac{d}{dx} = 2 \frac{d}{du}$ (assuming $\text{char}(\mathbb{K}) \neq 2$). In this example, the coresidue at p ($x = 0$) is 7 and the negative coresidue is -7 .

Theorem 6.3. *The category $\mathfrak{V}_{0,1,n}$ admits a terminal object. The NCR morphism of the terminal object is a flat, projective, local complete intersection (lci), geometrically integral morphism of relative dimension $n - 1$ to $\text{Spec } \mathbb{Z}[t]$.*

The base of the terminal object in Theorem 6.3 will turn out to be precisely the space of the degeneration of \bar{L}_n to \bar{P}_n from Theorem 1.4. Theorem 6.3 will be proved towards the end of §6.3.

6.2. Iterating bubbling up. To obtain interesting spaces from Constructions 4.4 and 4.14, we usually proceed as follows: start with a simple setup, then do:

- (1) a base change to the universal curve at the last step;
- (2) one application of Knudsen stabilization with vector fields;
- (3) one application of Inflating at zero vector;

and repeat. This is stated formally in Construction 6.4 below.

Construction 6.4. Given an object \mathbf{v}_{n_0} of \mathfrak{V}_{g,m,n_0}^+ (referred to as the *initial data*), we construct inductively a sequence of objects as follows:

- (1) $\mathbf{c}_{n,1} = \varpi^* \mathbf{v}_n$ in $\mathfrak{C}_{g,m,n}^{1,+}$ where ϖ is the projection (denoted by π in Definition 2.1) in \mathbf{v}_n ;
- (2) $\mathbf{c}_{n,2} = \dagger \mathbf{c}_{n,1}$ in $\mathfrak{C}_{g,m,n}^{2,+}$;
- (3) $\mathbf{c}_{n,3} = \dagger \mathbf{c}_{n,2}$ in $\mathfrak{C}_{g,m,n}^{3,+}$; and
- (4) \mathbf{v}_{n+1} corresponds to $\mathbf{c}_{n,3}$ under $\mathfrak{V}_{g,m,n+1}^+ \cong \mathfrak{C}_{g,m,n}^{3,+}$.

We refer to \mathbf{v}_n as the n th object, to its base as the n th space, etc.

Thus, the only choice is the choice of the initial data. Note also that, if any of the objects in the sequence belongs to $\mathfrak{C}_{g,m,n}^c$ or $\mathfrak{V}_{g,m,n}$ (that is, we can drop the ‘+’), then we can always drop the ‘+’ thereafter.

Below, we discuss four examples of what can be obtained by applying Construction 6.4, all four of which are relevant to Theorems 1.3 or 1.4. Two other examples not relevant to our main results will be sketched in §8.1. As agreed in §1.2, \bar{L}_n denotes the ‘Losev–Manin space over $\text{Spec } \mathbb{Z}$ ’, that is, the space obtained by repeating the construction in [20, 1.3 and 2.1] over $\text{Spec } \mathbb{Z}$. (Alternatively, \bar{L}_n can be obtained as a blowup, similarly to $\bar{L}_{n,\mathbb{C}}$, though for our purposes, it is quicker and more convenient to take the inductive construction as the definition.) The first two examples below are artificial, but they are meant to prepare the more interesting examples that follow.

6.2.1. The Losev–Manin space: first artificial construction. Run Construction 6.4 with the following initial data in $\mathfrak{V}_{0,2,1}$:

$$S = \text{Spec } \mathbb{Z}, \quad C = \mathbb{P}_{\mathbb{Z}}^1, \quad w_1 = [1 : 0], \quad w_2 = [0 : 1], \quad x_1 = [1 : 1], \quad \phi = x \frac{d}{dx},$$

in the chart $Y \neq 0$, where $x = X/Y$ and $[X : Y]$ are the projective coordinates on $\mathbb{P}_{\mathbb{Z}}^1$. The resulting n th space is \bar{L}_n . This is clear because, at all steps, the vector field will only vanish at w_1 and w_2 , so the application of Construction 4.14 (that is, of step 3 in Construction 6.4) will never have any effect by (4) in Remark 4.15. Thus we recover the inductive construction of \bar{L}_n .

6.2.2. The Losev–Manin space: second artificial construction. Run Construction 6.4 with the following initial data in $\mathfrak{V}_{0,1,1}$:

$$S = \text{Spec } \mathbb{Z}, \quad C = \mathbb{P}_{\mathbb{Z}}^1, \quad w_1 = [1 : 0], \quad x_1 = [1 : 1], \quad \phi = x \frac{d}{dx},$$

with notation and conventions as in 6.2.1. We claim that by applying Construction 6.4, the resulting n th space will once again be \bar{L}_n . We briefly argue this inductively, comparing with 6.2.1. Assume inductively that this is the case for a given n , and that the object denoted by $\mathbf{c}_{n,1}$ in Construction 6.4 has base \bar{L}_{n+1} and prestable curve $\bar{L}_{n+1} \times_{\bar{L}_n} \bar{L}_{n+1} \rightarrow \bar{L}_{n+1}$, just like in the situation of 6.2.1. The only difference

between these objects is that, in 6.2.2, the marking w_2 is not given, and that point is just a nameless isolated vanishing point of the vector field, the only one besides the marking w_1 . We claim that after applying Constructions 4.4 and 4.14 we will obtain essentially the same objects, again with the sole difference that w_2 is not considered marked in 6.2.2 (though after applying just Construction 4.4, the curves are truly different). Indeed, this may be checked on the open subset of the base

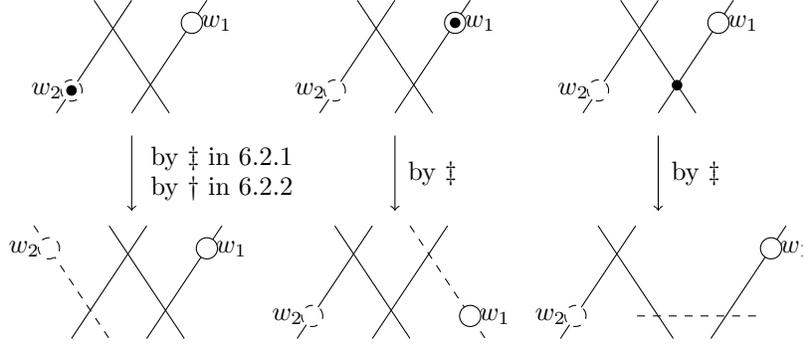


FIGURE 10. Comparison between 6.2.1 and 6.2.2 (the 3 nontrivial cases). The newly added marking is drawn as ‘•’; the other x -markings are omitted in the picture. The point w_2 is a marking in 6.2.1, but only an unmarked vanishing point of the vector field in 6.2.2. In the situation on the left, although a \mathbb{P}^1 component is inserted at w_2 both in 6.2.1 and 6.2.2, this happens in different ways: in the former case, by Construction 4.4 (the degeneracy is inserting a new marking over a w marking); in the latter case, by Construction 4.14 (the degeneracy is the vanishing of the field at the newly inserted marking).

where $x \neq w_1$ and x is smooth, since both Construction 4.4 and Construction 4.14 agree in 6.2.1 and 6.2.2 over a neighbourhood of the complement, such as the open subset of the base where $x \neq w_2$. However, on the former open subset, Construction 4.14 is trivial in 6.2.1 and Construction 4.4 is trivial in 6.2.2, so we just need to compare Construction 4.4 in 6.2.1 with Construction 4.14 in 6.2.2. It is easy to check that the respective cokernels to be projectivized, cf. (13) and (21), agree up to a twist by an invertible sheaf, hence their projectivizations coincide. The other details are straightforward and skipped. In Figure 10, 2 and 3 refer to the items in Construction 6.4, i.e. Constructions 4.4 and 4.14.

6.2.3. *Compactifying the space of configurations modulo translation.* Run Construction 6.4 with the following initial data in $\mathfrak{A}_{0,1,1}$:

$$S = \text{Spec } \mathbb{Z}, \quad C = \mathbb{P}_{\mathbb{Z}}^1, \quad w_1 = [1 : 0], \quad x_1 = [0 : 1], \quad \phi = \frac{d}{dx},$$

with the usual notation (so the vector field vanishes doubly at w_1 , but nowhere else). We will prove in §6.3 (see the proof of Theorem 1.3 towards the end of §6.3) that the resulting n th space is the moduli space \bar{P}_n from §1.2. For now, we take this explicit construction to be the definition of \bar{P}_n , and we will prove in §6.3 that it represents the functor in Theorem 1.3.

6.2.4. *The first glimpse of the degeneration in Theorem 1.4.* Run Construction 6.4 with the following initial data in $\mathfrak{V}_{0,1,1}$:

$$(46) \quad S = \text{Spec } \mathbb{Z}[t], \quad C = \mathbb{P}_{\mathbb{Z}[t]}^1, \quad w_1 = [1 : 0], \quad x_1 = [0 : 1], \quad \phi = (1 + tx) \frac{\partial}{\partial x},$$

with the usual notation, and π is the projection $\mathbb{P}_{\mathbb{Z}[t]}^1 \rightarrow \text{Spec } \mathbb{Z}[t]$. Throughout the application of Construction 6.4, the bases will retain the morphisms to $\text{Spec } \mathbb{Z}[t]$.

Remark 6.5. All objects that occur throughout the application of Construction 6.4 in this situation have the property that their base (what is usually denoted by S) and their curve (what is usually denoted by C) are integral. This is straightforward to check inductively. Flatness of the curve over the base guarantees that all generic points of the curve map to generic points of the base. On the other, there exists a nonempty open $U \subset S$ such that $\pi^{-1}(U) \simeq \mathbb{P}_U^1$ over U , and Cohen-Macaulay-ness rules out embedded components.

Construction 6.4 commutes with base change, because Constructions 4.4 and 4.14 do, cf. Propositions 4.6 and 4.16. Moreover, in our situation, it also commutes with multiplying the vector field with the pullback of an invertible regular function on the base by item 3 in Remark 4.5, item 2 in Remark 4.18, and Remark 6.5. Then we can conclude the following regarding our current construction:

- (1) The fibers over $(t) \in \text{Spec } \mathbb{Z}[t]$ mimic 6.2.3.
- (2) After the change of coordinates $X' = tX + Y$, $Y' = Y$ on $\mathbb{P}_{\mathbb{Z}[t, t^{-1}]}^1$, the preimages of $\text{Spec } \mathbb{Z}[t, t^{-1}] \subset \text{Spec } \mathbb{Z}[t]$ mimic the direct product of 6.2.2 with $\text{Spec } \mathbb{Z}[t, t^{-1}]$, with the sole exception that ϕ is multiplied by t . (In the new coordinates, $\phi = tx' \frac{\partial}{\partial x'}$ where $x' = X'/Y'$.) In particular, if X_n (temporary notation) is the n -th space obtained this way, then

$$(47) \quad \text{Spec } \mathbb{Z}[t, t^{-1}] \times_{\text{Spec } \mathbb{Z}[t]} X_n \simeq \text{Spec } \mathbb{Z}[t, t^{-1}] \times \bar{L}_n$$

by §6.2.2 and the considerations above.

This hints at Theorem 1.4. Indeed, we will see (essentially in Corollary 5.10) that the space X_n above is X from Theorem 1.4.

6.3. The modular interpretation of the degeneration. We continue the analysis of the construction in 6.2.4.

Definition 6.6. Let \mathbf{t}_n (respectively $\mathbf{t}_{n,c}$) be the objects of $\mathfrak{V}_{0,1,n}$ (respectively $\mathfrak{C}_{0,1,n}^c$) constructed in 6.2.4. Let η_n (respectively $\eta_{n,c}$) be the morphism from the base of \mathbf{t}_n (respectively $\mathbf{t}_{n,c}$) to $\text{Spec } \mathbb{Z}[t]$ constructed in 6.2.4.

We will prove that $\mathbf{t}_n, \mathbf{t}_{n,c}$ are terminal objects.

For now, we state some geometric properties that follow from our construction. In the statement and proof of Proposition 6.7 below, we agree to write $S[\mathbf{o}]$, $C[\mathbf{o}]$, etc. for the base, curve, etc. of an object \mathbf{o} when there is possibility of confusion, respecting the letters used in Definitions 2.1 and 2.5.

Proposition 6.7. *The following hold.*

- (1) Let \mathbf{t}_* be either \mathbf{t}_n or $\mathbf{t}_{n,c}$. Then the schemes $S[\mathbf{t}_*]$ and $C[\mathbf{t}_*]$ are normal and integral, as well as geometrically integral, separated, flat, of finite type, and lci over both $\text{Spec } \mathbb{Z}[t]$ and $\text{Spec } \mathbb{Z}$, and projective over $\text{Spec } \mathbb{Z}[t]$. The geometric generic fiber of $\pi[\mathbf{t}_*]$ is integral.

- (2) (a) $x[\mathbf{t}_{n,1}] \neq w_1[\mathbf{t}_{n,1}]$ and $x[\mathbf{t}_{n,1}] \cap w_1[\mathbf{t}_{n,1}]$ is integral.
 (b) The scheme-theoretic vanishing locus of $x[\mathbf{t}_{n,2}]^* \phi[\mathbf{t}_{n,2}]$ (viewed as a global section of $x[\mathbf{t}_{n,2}]^* \omega_{C[\mathbf{t}_{n,2}]/S[\mathbf{t}_{n,2}]}^\vee$) is an integral (prime) Cartier divisor on $S[\mathbf{t}_{n,2}]$.
 (3) The morphism $\eta_{n,c}$ from Definition 6.6 is the NCR morphism of $\mathbf{t}_{n,c}$, cf. Definition 6.1, and a similar statement holds for η_n .

Proof. 1: The claims that the bases and curves are integral has already been established in Remark 6.5. The other claims concerning (geometric) integrality follow using the same techniques. The flatness, projectivity, lci-ness, and normality claims can be proved comfortably by induction, following through the construction in 6.2.4, though we quote the relevant results for the reader's convenience.

Projective: Since projective in the sense of [8] is the same as projective in the sense of [12] if the base is itself quasi-projective over an affine scheme [12, p. 103], and composition of Hartshorne-projective morphisms is Hartshorne-projective, it follows inductively from these observations and Remark 4.5, Constructions 4.4 and 4.14, and item 1a in Proposition 4.12 that 'everything in sight' is Hartshorne-projective over $\text{Spec } \mathbb{Z}[t]$.

Local complete intersection: Prestable curves are lci over their base [25, Corollary 13.2.7], compositions of lci morphisms are lci [26, Tag 069J], and flat base changes of lci morphisms are lci [26, Tag 069I].

Normal: We need to check the R_1 and S_2 conditions. First, S_2 is automatic: both the bases and the curves are flat and lci over $\text{Spec } \mathbb{Z}[t]$, hence Gorenstein over $\text{Spec } \mathbb{Z}[t]$ [26, Tag 0C15], hence Cohen-Macaulay over $\text{Spec } \mathbb{Z}[t]$ [26, Tag 0C06], hence absolutely (over $\text{Spec } \mathbb{Z}$) Cohen-Macaulay [26, Tag 0C0W], hence satisfy all S_k conditions [26, Tag 0342]. For R_1 , if $\pi : C \rightarrow S$ denotes the prestable curve in question, consider the open subset $U \subset C$ defined as the intersection of π^{sm} with the preimage under π of the open subset of S where $S \rightarrow \text{Spec } \mathbb{Z}[t]$ is smooth. It is clear that C is regular at all points of U , and moreover, $C \setminus U$ has codimension at least 2, and the R_1 condition follows too.

2a: Recall that $C[\mathbf{t}_{n,1}] = C[\mathbf{t}_n] \times_{S[\mathbf{t}_n]} C[\mathbf{t}_n]$, $x[\mathbf{t}_{n,1}]$ the diagonal of this fiber square, and $w_1[\mathbf{t}_{n,1}]$ is the pullback of $w_1[\mathbf{t}_n]$. Thus $x[\mathbf{t}_{n,1}] \cap w_1[\mathbf{t}_{n,1}] \simeq S[\mathbf{t}_n]$, which we know to be integral.

2b: We write ϕ_0 for ϕ regarded as a section of $\omega_{C/S}^\vee(-w_1)$ (rather than $\omega_{C/S}^\vee$) for clarity. Then, if we write $\{\dots = 0\}$ for the vanishing loci, we have

$$\begin{aligned} \{x[\mathbf{t}_{n,2}]^* \phi[\mathbf{t}_{n,2}] = 0\} &= \{x[\mathbf{t}_{n,2}]^* \phi_0[\mathbf{t}_{n,2}] = 0\} \quad \text{since } x[\mathbf{t}_{n,2}] \cap w_1[\mathbf{t}_{n,2}] = \emptyset \\ &= \{x[\mathbf{t}_{n,1}]^* \phi_0[\mathbf{t}_{n,1}] = 0\} \quad \text{by item 2 in Remark 4.5} \\ &\cong \{\phi_0[\mathbf{t}_n] = 0\} \quad \text{by construction,} \end{aligned}$$

so it suffices to check that the last one is integral. The restriction to the complement of the central fiber, $\{\phi_0[\mathbf{t}_n] = 0\} \setminus C[\mathbf{t}_n]_{(t)}$, is a smooth section of the respective restriction of $\pi[\mathbf{t}_n]$, hence integral. Moreover, $\{\phi_0[\mathbf{t}_n] = 0\}$ doesn't contain $C[\mathbf{t}_n]_{(t)}$ (which is irreducible), thus $\{\phi_0[\mathbf{t}_n] = 0\}$ is irreducible and also reduced at all generic points, hence it is reduced everywhere since embedded components are ruled out by Cohen-Macaulay-ness.

3: We proceed inductively. If $y = Y/X = 1/x$, then

$$\phi[\mathbf{t}_1] = (1 + tx) \frac{\partial}{\partial x} = (1 + tx) \frac{dy}{dx} \frac{\partial}{\partial y} = -\frac{1 + tx}{x^2} \frac{\partial}{\partial y} = -y(t + y) \frac{\partial}{\partial y}.$$

Therefore,

$$w_1[\mathbf{t}_1]^* \left(\frac{\phi[\mathbf{t}_1]}{y} \right) = -t \frac{\partial}{\partial y},$$

so the NCR morphism of \mathbf{t}_1 is the identity. Inductively, we have the following. If $c = 1$, then the NCR morphism doesn't change at the next step: the claim is clear over the dense open subset of the base where Construction 4.4 is trivial (density is left to the reader), so it's true everywhere by reducedness. If $c = 2$, then the NCR morphism doesn't change at the next step. This is actually obvious because, in Construction 4.14, the sections w_1, \dots, w_m are always contained in the open subset of the source where the operation is trivial. If $c = 3$, then the NCR morphism at the next step simply pulls back together with the curve and the rest of the data, completing the proof. \square

Finally, we prove Theorem 6.3 inductively using Theorem 2.7. The only missing ingredient is the base case of the induction.

Lemma 6.8. *The object \mathbf{t}_1 (cf. Definition 6.6) is a terminal object of $\mathfrak{V}_{0,1,1}$. For any object $\mathbf{o} = (S, C, \dots)$ of $\mathfrak{V}_{0,1,1}$, the morphism $S \rightarrow \text{Spec } \mathbb{Z}[t]$ that exhibits \mathbf{o} as a pullback of \mathbf{t}_1 is the NCR morphism of \mathbf{o} , cf. Definition 6.1.*

Proof. In this proof, let $\mathbf{t}_1 = (S_0, C_0, \dots)$, and let $\mathbf{o} = (S, C, \dots)$ be an arbitrary object of $\mathfrak{V}_{0,1,1}$. We claim that $C \simeq \mathbb{P}_S^1$ over S . This requires a few steps.

- All geometric fibers of π are irreducible by a simple combinatorial argument using the ampleness of $\omega_{C/S}(w_1 + 2x_1)$. Otherwise, a leaf of the dual graph will correspond to a component which doesn't contain x_1 , and this is destabilizing.
- The morphism $\pi : C \rightarrow S$ is actually a projective \mathbb{P}^1 -bundle since it is a geometrically integral conic bundle with a section. This is well-known, we leave as hint that $C \cong \mathbb{P}(\pi_* \mathcal{O}_C(x_1))$ in fact.
- The projective bundle $\pi : C \rightarrow S$ is the projectivization of the direct sum of two line bundles because w_1 and x_1 are disjoint sections.
- We have $C \simeq \mathbb{P}_S^1$ by the previous step and $x_1^* \omega_{C/S}^\vee$ trivial.

If \mathbf{o} is isomorphic to the pullback of \mathbf{t}_1 along a morphism $S \xrightarrow{\alpha} \text{Spec } \mathbb{Z}[t]$, then α must be the NCR morphism of \mathbf{o} by the functoriality of NCR maps and item 3 in Proposition 6.7. Conversely, to see that \mathbf{o} is indeed isomorphic to the pullback of \mathbf{t}_1 along η , form the fiber product $S \times_{\mathbb{Z}[t]} C_0 \simeq \mathbb{P}_S^1$ relative to the NCR morphism $\eta : S \rightarrow \text{Spec } \mathbb{Z}[t]$ of \mathbf{o} and $\pi_0 : C_0 \rightarrow \text{Spec } \mathbb{Z}[t]$, choose an isomorphism $C \simeq S \times_{\mathbb{Z}[t]} C_0$ over S compatible with $x_1, w_1, x_{1,0}, w_{1,0}$, and note that the remaining \mathcal{O}_S^\times -ambiguity is precisely what is needed to make the vector fields ϕ and $\eta^* \phi_0$ match, since they already both vanish on w_1 with equal 'coresidues', and both are nowhere vanishing on x_1 . Indeed, after composing with a suitable automorphism which fixes both x_1 and w_1 , we may arrange so that $x_1^* \phi = x_1^* \eta^* \phi_0$, and then $\phi - \eta^* \phi_0$ will be a global section of $\omega_{C/S}^\vee(-2w_1 - x_1)$, but $\pi_* \omega_{C/S}^\vee(-2w_1 - x_1) = 0$ by the cohomology and base change theorem, completing the proof. \square

Theorem 6.9. *The object \mathbf{t}_n (respectively $\mathbf{t}_{n,c}$) from Definition 6.6 is a terminal object of $\mathfrak{V}_{0,1,n}$ (respectively $\mathfrak{C}_{0,1,n}$).*

Proof. Follows from Lemma 6.8, Theorem 2.7 and the obvious fact that the pullback of a terminal object of $\mathfrak{C}_{0,1,n-1}^3 \simeq \mathfrak{V}_{0,1,n}$ along the projection from the curve to the base is a terminal object of $\mathfrak{C}_{0,1,n} \simeq \mathfrak{C}_{0,1,n}^1$. \square

Theorem 6.3 follows from Theorem 6.9 and Proposition 6.7.

Proof of Theorem 1.3. We will prove that F is represented by the space \overline{P}_n constructed in 6.2.3. Indeed, F corresponds to the subcategory of $\mathfrak{V}_{0,1,n}$ whose objects have identically 0 NCR morphisms. Since NCR morphisms commute with base change, it follows from Theorems 6.3 and 6.9 that F is represented by $\eta_n^{-1}(\text{Spec } \mathbb{Z})$, where $\eta_n : S[\mathfrak{t}_n] \rightarrow \text{Spec } \mathbb{Z}[t]$ is the NCR morphism of \mathfrak{t}_n , and $\text{Spec } \mathbb{Z} \hookrightarrow \text{Spec } \mathbb{Z}[t]$ by $\mathbb{Z}[t] \rightarrow \mathbb{Z}[t]/(t) = \mathbb{Z}$. By the remarks in 6.2.4 and Proposition 6.7, this is nothing but \overline{P}_n . The fact that \overline{P}_n is projective, local complete intersection, flat, and geometrically integral over $\text{Spec } \mathbb{Z}$ follows from Proposition 6.7. \square

Corollary 6.10. *Let $X = S[\mathfrak{t}_n]$ be the base of the terminal object \mathfrak{t}_n of $\mathfrak{V}_{0,1,n}$ cf. Theorem 6.9, and $\eta_n : X \rightarrow \text{Spec } \mathbb{Z}[t]$ its NCR morphism. Then,*

$$\eta_n^{-1}(\text{Spec } \mathbb{Z}) \simeq \overline{P}_n \quad \text{and} \quad \eta_n^{-1}(\text{Spec } \mathbb{Z}[t, t^{-1}]) \simeq \text{Spec } \mathbb{Z}[t, t^{-1}] \times \overline{L}_n,$$

the latter over $\text{Spec } \mathbb{Z}[t, t^{-1}]$, where $\text{Spec } \mathbb{Z} \hookrightarrow \text{Spec } \mathbb{Z}[t]$ by $\mathbb{Z}[t] \rightarrow \mathbb{Z}[t]/(t) = \mathbb{Z}$.

Proof. To review, (1) \mathfrak{t}_n is both the terminal object of $\mathfrak{V}_{0,1,n}$ and the object constructed in §6.2.4 (Theorem 6.9), (2) \overline{P}_n is both the scheme which represents the functor F and the space constructed in §6.2.3 (proof of Theorem 1.3 above), and (3) η_n is both the NCR morphism of \mathfrak{t}_n , and the morphism to $\text{Spec } \mathbb{Z}[t]$ which comes from the inductive construction in §6.2.4 (item 3 in Proposition 6.7). Then, the fact that $\eta_n^{-1}(\text{Spec } \mathbb{Z}) \simeq \overline{P}_n$ comes directly from the proof of Theorem 1.3 above (the third and fourth sentences in the respective proof). The fact that $\eta_n^{-1}(\text{Spec } \mathbb{Z}[t, t^{-1}]) \simeq \text{Spec } \mathbb{Z}[t, t^{-1}] \times \overline{L}_n$ is simply (47). \square

By Corollary 6.10, we know that \overline{L}_n degenerates to \overline{P}_n . To complete the proof of Theorem 1.4, it only remains to deal with the group actions.

7. ACTIONS OF \mathbb{G}_a AND \mathbb{G}_m ON CURVES AND THEIR MODULI

7.1. Preliminaries. In this subsection, we review some generalities. These may be skipped and referred back to as necessary.

7.1.1. Review of Weil divisors and associated reflexive sheaves. Let X be an integral normal separated excellent scheme. For any Weil divisor D on X , let $\mathcal{O}_X(D)$ be the sheaf associated to the divisor D , that is, $\Gamma(U, \mathcal{O}_X(D)) = \{f \in K(X) : (\text{div}(f) + D)|_U \geq 0\}$. However, in other sections we will reserve the notation $\mathcal{O}_X(D)$ for the Cartier case, unless specified otherwise. We collect here some basic facts regarding associated sheaves. I have been unable to find a suitable published reference, but the excellent notes [24] contain what is needed.

- (1) $\mathcal{O}_X(D)$ is a rank 1 reflexive coherent \mathcal{O}_X -module [24, Proposition 3.4];
- (2) conversely, if \mathcal{F} is a rank 1 reflexive coherent \mathcal{O}_X -module, then there exists a Weil divisor D such that $\mathcal{F} \simeq \mathcal{O}_X(D)$ [24, Propositions 3.6 and 3.7];
- (3) if D is prime, then $\mathcal{I}_{D,X} = \mathcal{O}_X(-D)$ [24, Proposition 3.4];
- (4) if D is prime, then $\mathcal{I}_{D,X}^\vee = \mathcal{O}_X(D)$, by 1, 3, and [24, Proposition 3.13.(b)];
- (5) if \mathcal{F} is a rank 1 reflexive coherent \mathcal{O}_X -module, then $\mathcal{E}nd(\mathcal{F}) = \mathcal{O}_X$ by item 2 and [24, Proposition 3.13.(c)]. In particular, $\mathcal{A}ut(\mathcal{F}) = \mathcal{O}_X^\times$.

Lemma 7.1. *Let X and X' be integral normal separated excellent schemes, and $f : X' \rightarrow X$ a flat morphism. Let $D \subset X$ be a prime Weil divisor such that $D' = f^{-1}(D)$ is a prime Weil divisor on X' . Then $f^*\mathcal{O}_X(D) \simeq \mathcal{O}_{X'}(D')$.*

Proof. First, we claim that $f^*\mathcal{I}_{D,X} \simeq \mathcal{I}_{D',X'}$. This is a consequence of flatness, as follows. We have the following solid arrow commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & f^*\mathcal{I}_{D,X} & \longrightarrow & f^*\mathcal{O}_X & \longrightarrow & f^*l_*\mathcal{O}_D \longrightarrow 0 \\ & & \downarrow \simeq & & \parallel & & \downarrow \simeq \\ 0 & \longrightarrow & \mathcal{I}_{D',X'} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & l'_*\mathcal{O}_{D'} \longrightarrow 0 \end{array}$$

in which the top row is exact because f is flat. The right vertical isomorphism comes from the flat cohomology and base change theorem [12, Proposition 9.3]. Then there must exist a dashed arrow which makes the diagram commute, proving $f^*\mathcal{I}_{D,X} \simeq \mathcal{I}_{D',X'}$. However, as in the proof of [13, Proposition 1.8], f^* commutes with dualizing, so $f^*\mathcal{I}_{D,X}^\vee \simeq \mathcal{I}_{D',X'}^\vee$. Then item 4 completes the proof. \square

7.1.2. *Some calculations on \bar{L}_n .* The Losev-Manin space has already been reviewed in §1.2 and discussed in §6.2.1 and §6.2.2. As agreed in §1.2, \bar{L}_n is the Losev-Manin space over $\text{Spec } \mathbb{Z}$. [20, Theorem 2.2] still holds over $\text{Spec } \mathbb{Z}$, and \bar{L}_{n+1} is still the universal curve over \bar{L}_n . In particular, \bar{L}_n is smooth over $\text{Spec } \mathbb{Z}$. Indeed, it is clear inductively that it is flat and of finite type, since $\bar{L}_{n+1} \rightarrow \bar{L}_n$ is flat and of finite type, and the fact that the geometric fibers are regular is stated on page 446 of [20] – nothing changes in positive characteristic. (Since no details on smoothness are provided in [20], we mention an alternative argument: use the moduli space interpretation and deformation theory. The usual deformation theory of marked curves still applies.)

Moreover, there is a natural action of $\bar{L}_n \times \mathbb{G}_m$ on \bar{L}_{n+1} over \bar{L}_n – on the open subset corresponding to Losev-Manin chains of length 1, this is just the \mathbb{G}_{m,L_n} -action on $\mathbb{P}_{L_n}^1$ fixing 0 and ∞ . We denote it by $\nu : \mathbb{G}_m \times \bar{L}_{n+1} \rightarrow \bar{L}_{n+1}$. This also follows from [20], although it is worth noting that we will actually recover this as a byproduct of an inductive argument below (it will take one moment of thought to convince ourselves that there is no logical circularity here).

Let $Y_n = \bar{L}_{n+1} \times_{\bar{L}_n} \bar{L}_{n+1}$. The action $\nu : \mathbb{G}_m \times \bar{L}_{n+1} \rightarrow \bar{L}_{n+1}$ pulls back along $Y_n \rightarrow \bar{L}_{n+1}$ to an action $\mu : \mathbb{G}_m \times Y_n \rightarrow Y_n$. Let $\Delta \subset Y_n$ be the diagonal of the fiber product, and $p_2 : \mathbb{G}_m \times Y_n \rightarrow Y_n$ the projection to the second factor.

First, we show that Y_n admits a small resolution (over $\text{Spec } \mathbb{Z}$).

Lemma 7.2. *Let $R_n = \mathbb{P}\mathcal{I}_{\Delta,Y_n}^\vee$, and $\lambda : R_n \rightarrow Y_n$ the natural projection. Then R_n is smooth over $\text{Spec } \mathbb{Z}$, and there exists a closed subscheme $N \subset Y_n$ such that:*

- (1) N has relative (over $\text{Spec } \mathbb{Z}$) codimension 3 in Y_n ;
- (2) $\lambda^{-1}(N)$ has relative codimension 2 in R_n ; and
- (3) λ restricts to an isomorphism $R_n \setminus \lambda^{-1}(N) \simeq Y_n \setminus N$.

Proof. Roughly, the point is that R_n resolves the singularities of Y_n in the same way \bar{L}_{n+2} does. Let $y_0, y_\infty : \bar{L}_{n+1} \rightarrow Y_n$ be 0 and ∞ sections, i.e. the pullbacks of the 0 and ∞ sections of \bar{L}_{n+1} over \bar{L}_n . As reviewed earlier, the Losev-Manin spaces are constructed inductively in [20, 1.3 and 2.1] by $\bar{L}_{n+2} = \mathbb{P}\mathcal{K}$, where

$$(48) \quad \mathcal{K} = \text{Coker} \left(\mathcal{O}_{Y_n} \xrightarrow{b \rightarrow (b,b)} \mathcal{I}_{\Delta,Y_n}^\vee \oplus \mathcal{O}_{Y_n}(y_0 + y_\infty) \right).$$

Then \bar{L}_{n+2} and R_n are isomorphic above $Y_n \setminus (y_0 \cup y_\infty)$, and hence $R_n \rightarrow \text{Spec } \mathbb{Z}$ is smooth everywhere in $\lambda^{-1}(Y_n \setminus (y_0 \cup y_\infty))$. Let $N \subset \Delta \simeq \bar{L}_{n+1}$ correspond to the locus where $\bar{L}_{n+1} \rightarrow \bar{L}_n$ fails to be smooth, taken with, say, the reduced closed subscheme structure. It is clear that λ is an isomorphism above $Y_n \setminus N$. Since $\bar{L}_{n+1} \rightarrow \text{Spec } \mathbb{Z}$ is smooth, $Y_n \rightarrow \text{Spec } \mathbb{Z}$ is clearly smooth at all points where at least one of the two projection maps $Y_n \rightarrow \bar{L}_{n+1}$ is smooth, and in particular, it is smooth in a neighbourhood of $y_0 \cup y_\infty$. Therefore, $R_n \rightarrow \text{Spec } \mathbb{Z}$ is also smooth in a neighborhood of $\lambda^{-1}(y_0) \cup \lambda^{-1}(y_\infty)$ as $N \cap (y_0 \cup y_\infty) = \emptyset$, completing the proof of the smoothness claim. The dimension claims are straightforward. \square

Proposition 7.3. $\mu^{-1}(\Delta) \sim p_2^{-1}(\Delta)$ on $\mathbb{G}_m \times Y_n$.

Proof. Let $D = p_2^{-1}(\Delta)$ and $D' = \mu^{-1}(\Delta)$, and E and E' the ‘proper transforms’ of D and D' on $\mathbb{G}_m \times R_n$, that is, with notation as in Lemma 7.2, E is the closure of $(\text{id}_{\mathbb{G}_m} \times \lambda)^{-1}(D \setminus (\mathbb{G}_m \times N))$ with the reduced structure, and E' is the closure of $(\text{id}_{\mathbb{G}_m} \times \lambda)^{-1}(D' \setminus (\mathbb{G}_m \times N))$ with the reduced structure. Let $p_1 : \mathbb{G}_m \times Y_n \rightarrow \mathbb{G}_m$ and $q_1 : \mathbb{G}_m \times R_n \rightarrow \mathbb{G}_m$ be the projections to the first factors. Since $\mathbb{G}_m \times R_n$ is regular, E and E' are effective Cartier divisors.

Let $z = (f, p) \in \mathbb{G}_m = \text{Spec } \mathbb{Z}[x, x^{-1}]$ be a closed point, where p is a prime number, and $f \in \mathbb{Z}[x]$ is irreducible mod p , $f \not\equiv_p x$. First, we claim that

$$(49) \quad D_z \sim D'_z$$

on $p_1^{-1}(z)$. Let $K = \mathbb{F}_p[x]/(f)$, the residue field of \mathbb{G}_m at z . As usual, write $\square_K = \text{Spec } K \times_{\text{Spec } \mathbb{Z}} \square$. Let $w \in \mathbb{G}_{m,K}(K)$ be a preimage of $z \in \mathbb{G}_m(K)$, and let 1 denote the point $(t-1) \in \mathbb{G}_{m,K}(K)$. We have $D'_z = D'_{K,w} \sim D'_{K,1} = D_{K,1} = D_{K,w} = D_z$ in $p_1^{-1}(z) \cong Y_{n,K}$, where the linear equivalence comes from [4, Proposition 1.6] (the family over $\mathbb{G}_{m,K}$ may be extended to a family over \mathbb{P}_K^1 simply by taking the closure of D_K in $\mathbb{P}_K^1 \times Y_{n,K}$), which proves (49). Second, we claim that

$$(50) \quad E_z \sim E'_z$$

on $q_1^{-1}(z)$. Note that E_z and E'_z are the proper transforms of D_z and D'_z under the small resolution $\lambda_K : q_1^{-1}(z) = R_{n,K} \rightarrow Y_{n,K} = p_1^{-1}(z)$. (Since $((\text{id}_{\mathbb{G}_m} \times \lambda)^{-1}(D'))_z$ contains E'_z and has a unique component of codimension 1 in $(\mathbb{G}_m \times R_n)_z$ as D'_z is irreducible and $\text{id}_{\mathbb{G}_{m,K}} \times \lambda_K$ is an isomorphism in codimension 2, and E'_z is cut out by a single equation (line bundle section) on $(\mathbb{G}_m \times R_n)_z$ being the pullback of the Cartier divisor E , it follows that E'_z is a fortiori the codimension 1 irreducible component above, and also that it is the proper transform of D'_z . Similarly for E_z .) By Lemma 7.2 and [12, Proposition 6.5, part b], we have $\text{Cl}(R_{n,K}) \cong \text{Cl}(Y_{n,K})$ compatible with taking proper transforms, so (50) follows from (49).

Let $\mathcal{J} = \mathcal{O}_{\mathbb{G}_m \times R_n}(E' - E)$. We will show that $\mathcal{J} \simeq \mathcal{O}_{\mathbb{G}_m \times R_n}$ using a standard argument. First, we claim that

$$(51) \quad q_* \mathcal{J} \simeq \mathcal{O}_{\mathbb{G}_m}.$$

By (50), $\dim_{\kappa(z)} \Gamma(\mathcal{J}_z) = 1$, for all closed points $z \in \mathbb{G}_m$. Since all closed and all open subsets of $|\mathbb{G}_m|$ contain at least one closed point, the semicontinuity theorem [12, III, Theorem 12.8] implies that $\dim_{\kappa(z)} \Gamma(\mathcal{J}_z) = 1$ holds for all $z \in \mathbb{G}_m$, not just the closed points. By Grauert’s Theorem [12, III, Corollary 12.9], $q_* \mathcal{J}$ is invertible. Then (51) follows as $\text{Pic}(\mathbb{G}_m)$ is trivial. However, $\dim_{\kappa(z)} \Gamma(\mathcal{J}_z^\vee) = 1$ for all closed

points $z \in \mathbb{G}_m$ follows equally well from (50), so a completely analogous argument shows that

$$(52) \quad q_*(\mathcal{J}^\vee) \simeq \mathcal{O}_{\mathbb{G}_m}.$$

Let s_1 and s_2 be the global sections of \mathcal{J} and \mathcal{J}^\vee which correspond to the section 1 of $\mathcal{O}_{\mathbb{G}_m}$ under the isomorphisms (51) and (52). Clearly, $s_1 \otimes s_2 \neq 0$, so $s_1 \otimes s_2$ is nowhere vanishing as $\mathcal{J} \otimes \mathcal{J}^\vee \simeq \mathcal{O}_{\mathbb{G}_m \times R_n}$. Then s_1 is nowhere vanishing, so $\mathcal{J} \simeq \mathcal{O}_{\mathbb{G}_m \times R_n}$. Hence $E \sim E'$ on $\mathbb{G}_m \times R_n$. By Lemma 7.2 and [12, Proposition 6.5, part b)], $\text{Cl}(\mathbb{G}_m \times R_n) \cong \text{Cl}(\mathbb{G}_m \times Y_n)$ compatible with taking proper transforms, and $D \sim D'$ follows. \square

7.1.3. *Projectivization of equivariant sheaves.* Finally, we review the fact that group actions lift to projectivizations of equivariant sheaves.

Lemma 7.4. *Let G be an S -group scheme, Y an S -scheme, and α an action of G on Y relative to S . Let \mathcal{F} be a G -equivariant coherent \mathcal{O}_Y -module, $X = \mathbb{P}\mathcal{F}$, and $f : X \rightarrow Y$ the natural projection map. Then there exists a G -action on X over S , relative to which f is G -equivariant.*

Proof. Given that \mathbb{P} commutes with base change, this is purely formal and surely well-known. In essence, the discussion in [23, p. 31] still applies – it doesn't truly matter that we have $\mathbb{P} = \text{Proj Sym}$ instead of Spec Sym , and that [23] operates in a more restrictive setup (invertible sheaf, etc.). \square

7.2. **The G -action on the universal curve.** We return to the main logical thread of the paper, and deal with the last remaining aspect, the group actions. The main goal of §7.2 is to ‘integrate’ the vector field on the universal curve in the terminal object of $\mathfrak{V}_{0,1,n}$ to obtain a group action.

Let $\gamma : G \rightarrow \text{Spec } \mathbb{Z}[t]$ as in §1.2, $e : \text{Spec } \mathbb{Z}[t] \rightarrow G$ the identity section, and $\mathfrak{g} = e^* \mathcal{T}_{G/\mathbb{Z}[t]}$ its Lie algebra, where $\mathcal{T}_{G/\mathbb{Z}[t]}$ is the relative tangent bundle of γ . For simplicity, let $\ell = \text{Spec } \mathbb{Z}[t]$ and $\ell^* = \text{Spec } \mathbb{Z}[t, t^{-1}]$. We will use the notation $X[\varepsilon] = \text{Spec } \mathbb{Z}[\varepsilon]/(\varepsilon^2) \times_{\text{Spec } \mathbb{Z}} X$. Since \mathfrak{g} is the normal sheaf of $\ell \xrightarrow{e} G$, $\frac{\partial}{\partial x} \in \Gamma(\mathfrak{g})$ gives a first order thickening $j : \ell[\varepsilon] \rightarrow G$ of e by standard deformation theory.

Let $\mathfrak{o} = (S, C, \dots, \phi)$ be an object of $\mathfrak{V}_{0,1,n}^+$ or $\mathfrak{C}_{0,1,n}^+$, with notation as in Definition 2.1. Consider the pullback $G \times_\ell S$ of G along the NCR morphism $S \rightarrow \ell$. An action α of $G \times_\ell S$ on C over S induces an automorphism of $G \times_\ell C$ as follows: $\alpha : (G \times_\ell S) \times_S C = G \times_\ell C \rightarrow C$ and the projection to the first factor $G \times_\ell C \rightarrow G$ induce an endomorphism of $G \times_\ell C$, which is easily checked to be an automorphism (over $G \times_\ell S$). This restricts to an automorphism of $C[\varepsilon]$ over $S[\varepsilon]$ (equal to id_C on C), by $C[\varepsilon] = \ell[\varepsilon] \times_\ell C \subset G \times_\ell C$ via $j : \ell[\varepsilon] \rightarrow G$ and similarly for $S[\varepsilon]$. In turn, by standard deformation theory, this automorphism gives an element of $\text{Hom}(\Omega_{C/S}, \mathcal{O}_C)$. The discussion can be summarized in the following diagram.

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{actions of } G \times_\ell S \\ \text{on } C \text{ over } S \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{l} \text{automorphisms of } C[\varepsilon] \text{ over } S[\varepsilon] \\ \text{which restrict to } \text{id}_C \text{ on } C \end{array} \right\} \\ & & \downarrow \\ \phi \in \text{Hom}(\omega_{C/S}, \mathcal{O}_C) & \longrightarrow & \text{Hom}(\Omega_{C/S}, \mathcal{O}_C) \end{array}$$

The lower horizontal map is obtained by dualizing the map $\Omega_{C/S} \rightarrow \omega_{C/S}$ from e.g. [18, §1]. Henceforth, $G \times_\ell S$, $G \times_\ell C$, etc. are relative to the NCR morphism $S \rightarrow \ell$.

Definition 7.5. In the situation above, we say that an action α of $G \times_\ell S$ on C over S is *compatible with ϕ* if the images of ϕ and α in $\text{Hom}(\Omega_{C/S}, \mathcal{O}_C)$ in the diagram above coincide. If only the restrictions of these images to an open $U \subset C$ coincide, we say the compatibility holds on U .

Theorem 7.6. *An action compatible with the respective field exists for the terminal object \mathfrak{t}_n of $\mathfrak{A}_{0,1,n}$, cf. Theorem 6.9, and fixes the respective section w_1 .*

We recall the following elementary fact which will be used soon.

Remark 7.7. Let $D, E \subset X$ be effective Cartier divisors such that $D|_E = D \cap E$ is Cartier on E , that is, the restriction $\mathcal{O}_E \rightarrow (\mathcal{O}_X(D))|_E$ of $\mathcal{O}_X \rightarrow \mathcal{O}_X(D)$ is injective. Then we have a short exact sequence $0 \rightarrow \mathcal{I}_{D+E, X} \rightarrow \mathcal{I}_{D, X} \oplus \mathcal{I}_{E, X} \rightarrow \mathcal{I}_{D \cap E, X} \rightarrow 0$. Indeed, the elements cutting out D and E locally form a regular sequence, and then the exactness of the sequence is essentially the exactness of the Koszul complex.

Lemma 7.8. *Let $\mathfrak{o} = (S, C, \pi, w_1, \bar{x}, x, \phi)$ be an object of $\mathfrak{C}_{0,1,n}^1$ such that*

- (1) *C and S are integral and separated;*
- (2) *$x \neq w_1$, that is, there exists a point $s \in S$ such that $x(s) \neq w_1(s)$; and*
- (3) *the scheme theoretic intersection $x \cap w_1$ is integral.*

For simplicity, we write $w = w_1$. Let α be a $G \times_\ell S$ -action compatible with ϕ which fixes w . Assume that $\mathcal{I}_{x, C}^\vee$ is $G \times_\ell S$ -equivariant. Let $\dagger \mathfrak{o} = (S, C_\dagger, \dots)$ cf. (38). Then there exists an action α_\dagger of $G \times_\ell S$ on C_\dagger compatible with ϕ_\dagger , which fixes $w_{1, \dagger}$.

Formally, the requirement that α fixes w means that the restriction of $\alpha : G \times_S C \rightarrow C$ to $G \times_S w$ is the composition of the projection $G \times_S w \rightarrow w$ with the closed immersion $w \hookrightarrow C$.

Proof. We will first show that if some twist $\mathcal{K} \otimes \mathcal{J}$ of \mathcal{K} (cf. Construction 4.4) by a line bundle admits an equivariant structure relative to α , then the conclusion holds. Indeed, Lemma 7.4 then produces an action α_\dagger of $G \times_\ell S$ on $C_\dagger = \mathbb{P}(\mathcal{K} \otimes \mathcal{J})$ such that $f : C_\dagger \rightarrow C$ (Construction 4.4) is $G \times_\ell S$ -equivariant. The compatibility of α_\dagger with ϕ_\dagger holds at least over a dense open subset above which f is an isomorphism, and then it holds everywhere a fortiori, since $\Omega_{C_\dagger/S}^\vee$ is torsion free. Moreover, α_\dagger fixes $w_\dagger \setminus x_\dagger$, and hence it must fix w_\dagger .

Since the claim is local on the base, it suffices to analyze separately two situations: x is contained in the open subset of C where π is smooth, and $x \cap w = \emptyset$. In the first case, x is an effective Cartier divisor on C and $\mathcal{I}_{x, C}^\vee = \mathcal{O}_C(x)$. We have a (solid arrow) commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_C(-x-w) & \longrightarrow & \mathcal{O}_C(-x) \oplus \mathcal{O}_C(-w) & \longrightarrow & \mathcal{I}_{x \cap w, C} \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \simeq \\ 0 & \longrightarrow & \mathcal{O}_C(-x-w) & \longrightarrow & \mathcal{O}_C(-x) \oplus \mathcal{O}_C(-w) & \longrightarrow & \mathcal{K}(-x-w) \longrightarrow 0 \end{array}$$

in which the top row comes from Remark 7.7, and the bottom row is (13) twisted by $\mathcal{O}_C(-x-w)$. Then there exists a unique dashed isomorphism $\mathcal{I}_{x \cap w, C} \simeq \mathcal{K}(-x-w)$ which makes the diagram commute. However, $\mathcal{I}_{x \cap w, C}$ is $G \times_\ell S$ -equivariant because α fixes w and hence $x \cap w$ too, and then $\mathcal{I}_{x \cap w, C}$ is the kernel of the homomorphism

$\mathcal{O}_C \rightarrow (x \cap w \hookrightarrow C)_* \mathcal{O}_{x \cap w}$ in the category of $G \times_\ell S$ -equivariant coherent \mathcal{O}_C -modules. In the second case, $\mathcal{K}(-w) = \mathcal{I}_{x,C}^\vee$, and we are done since we are assuming that $\mathcal{I}_{x,C}^\vee$ is equivariant. \square

Lemma 7.9. *Let $\mathfrak{o} = (S, C, \pi, w_1, \bar{x}, x, \phi)$ be an object of $\mathfrak{C}_{0,1,n}^2$ such that*

- (1) *C and S are integral and separated; and*
- (2) *if $Z = \{x^* \phi = 0\} \subseteq S$ is the scheme-theoretic vanishing locus of $x^* \phi$, then Z is an integral (prime) effective Cartier divisor on S .*

Let $w = w_1$. Let α be a $G \times_\ell S$ -action on C compatible with ϕ which fixes w . Let $\dagger \mathfrak{o} = (S, C_\dagger, \dots)$, cf. (38). Then there exists a $G \times_\ell S$ -action α_\dagger on C_\dagger compatible with ϕ_\dagger , which fixes $w_{1,\dagger}$.

Proof. As in the proof of Lemma 7.8, if some twist $\mathcal{K} \otimes \mathcal{J}$ of \mathcal{K} (cf. Construction 4.14) by a line bundle admits an equivariant structure relative to α , then we are done. We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \omega_{C/S}(-x) & \longrightarrow & \mathcal{O}_C(-x) \oplus \omega_{C/S} & \longrightarrow & \mathcal{I}_{x(Z),C} \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \simeq \\ 0 & \longrightarrow & \omega_{C/S}(-x) & \longrightarrow & \mathcal{O}_C(-x) \oplus \omega_{C/S} & \longrightarrow & \mathcal{K} \otimes \omega_{C/S}(-x) \longrightarrow 0 \end{array}$$

in which the top row comes from Remark 7.7, and the bottom row is (21) twisted by $\omega_{C/S}(-x)$. It follows that $\mathcal{K} \otimes \omega_{C/S}(-x) \simeq \mathcal{I}_{x(Z),C}$.

We claim that $x(Z) \subset C$ is fixed by α . Formally, this means that the restriction of $\alpha : G \times_\ell C \rightarrow C$ to $G \times_\ell x(Z)$ is the composition of the projection $G \times_\ell x(Z) \rightarrow x(Z)$ with the closed immersion $x(Z) \hookrightarrow C$. We are thus claiming that two S -morphisms $G \times_\ell x(Z) \rightarrow C$ coincide, or equivalently, that a morphism $G \times_\ell x(Z) \rightarrow C \times_S C$ factors through the diagonal immersion $C \hookrightarrow C \times_S C$. It is elementary to check on (geometric) fibers over S that the image of the restriction of α to $G \times_\ell x(Z)$ is contained in $\pi^{\text{sm}} \subseteq C$, the open subset where π is smooth, and it follows that the image of the morphism $G \times_\ell x(Z) \rightarrow C \times_S C$ is contained in the open subset $\pi^{\text{sm}} \times_S \pi^{\text{sm}}$. However, the diagonal Δ of $\pi^{\text{sm}} \times_S \pi^{\text{sm}}$ is a Cartier divisor, and our claim boils down to the statement that the section 1 of $\mathcal{O}_{\pi^{\text{sm}} \times_S \pi^{\text{sm}}}(\Delta)$ pulls back to 0 on $G \times_\ell x(Z)$. However, the claim, and in particular the vanishing of the section above, are elementary to check on geometric fibers over S ; then they hold on the fibers of S , and hence everywhere since $G \times_\ell x(Z)$ is integral as a consequence of the assumption that $x(Z)$ is integral. Then $\mathcal{I}_{x(Z),C} = \text{Ker}(\mathcal{O}_C \rightarrow x_* \mathcal{O}_Z)$ is equivariant (as in the proof of Lemma 7.8), which completes the proof. \square

To prove Theorem 7.6, we need to show that the assumption that the dual ideal sheaf in Lemma 7.8 is equivariant holds for the terminal object of $\mathfrak{C}_{0,1,n}^1$.

Proposition 7.10. *Let $\mathfrak{t}_{n,1} = (S, C, \pi, w_1, \bar{x}, x, \phi)$ be the terminal object of $\mathfrak{C}_{0,1,n}^1$ cf. Definition 6.6 and Theorem 6.9, and α an action of $G \times_\ell S$ on C over S compatible with ϕ , cf. Definition 7.5. Then $\mathcal{I}_{x,C}^\vee$ is α -equivariant.*

Proof. Throughout this proof, we rely (sometimes implicitly) on Proposition 6.7. All of C , $G \times_\ell C$, and $G \times_\ell G \times_\ell C$ are dense open in some $\mathbb{A}^k \times C$, hence normal and integral by Proposition 6.7. Clearly, we may think of α as an action of G on C

over ℓ , in view of the identification $G \times_{\ell} C = (G \times_{\ell} S) \times_S C$. Let $\varpi_2 : G \times_{\ell} C \rightarrow C$ be the projection to the second factor. First, we will show that

$$(53) \quad \alpha^{-1}(x) \sim \varpi_2^{-1}(x).$$

Let $G^* = \ell^* \times_{\ell} G \cong \ell^* \times \mathbb{G}_m$. With notation as in §7.1.2, $\ell^* \times_{\ell} C \cong \ell^* \times Y_n$ and this isomorphism restricts on $\ell^* \times_{\ell} x$ to an isomorphism $\ell^* \times_{\ell} x \cong \ell^* \times \Delta$. With μ also as in §7.1.2, we have the following commutative diagram.

$$\begin{array}{ccccc} G \times_{\ell} C & \hookrightarrow & G^* \times_{\ell} C & \xrightarrow{\cong} & \ell^* \times \mathbb{G}_m \times Y_n \\ \alpha \downarrow & & \downarrow & & \downarrow (\text{id}_{\ell^*}, \mu) \\ C & \hookrightarrow & \ell^* \times_{\ell} C & \xrightarrow{\cong} & \ell^* \times Y_n \end{array}$$

(We briefly explain this claim. A priori, α restricts to some action of $\mathbb{G}_m \times (S \setminus S_{(t)})$ on $C \setminus C_{(t)} \simeq \ell^* \times Y_n$ over $S \setminus S_{(t)}$. If we restrict to, say, the fiber of $(t-1) \in \ell^*$, we recover the $\mathbb{G}_m \times \bar{L}_{n+1}$ action μ on Y_n in §7.1.2. All the \mathbb{G}_m actions in discussion satisfy various elementary properties (such as having weight 1 on generic fibers and fixing the two suitable sections) that determine them uniquely, so there is no concern that we have obtained a different action. The same type of uniqueness argument establishes commutativity of the right half of the diagram.) Note that $\text{Cl}(G^* \times_{\ell} C) \cong \text{Cl}(G \times_{\ell} C)$ by [12, Proposition 6.5, part c] and $C_{(t)} \sim 0$ (as $C_{(t)}$ is integral by Proposition 6.7), so (53) may be checked on the restriction to $G^* \times_{\ell} C$, i.e. the complement of the fiber over $(t) \in \ell$. Then it becomes $\ell^* \times \mu^{-1}(\Delta) \sim \ell^* \times p_2^{-1}(\Delta)$, and it follows from Proposition 7.3 and the elementary fact that, for any (integral, normal, separated, noetherian) X , $\text{Cl}(X) \cong \text{Cl}(\ell^* \times X)$ via $[D] \mapsto [\ell^* \times D]$ by e.g. [12, Propositions 6.5.c and 6.6] and their proofs.

Let's temporarily reinstate the \mathcal{O} notation for associated sheaves from §7.1.1. We have $\mathcal{O}_{G \times_{\ell} C}(\varpi_2^{-1}(x)) \simeq \varpi_2^* \mathcal{O}_C(x)$ and $\mathcal{O}_{G^* \times_{\ell} C}(\alpha^{-1}(x)) \simeq \alpha^* \mathcal{O}_C(x)$ by Lemma 7.1. On the other hand, $\mathcal{O}_{G^* \times_{\ell} C}(\alpha^{-1}(x)) \simeq \mathcal{O}_{G \times_{\ell} C}(\varpi_2^{-1}(x))$ by (53), so $\alpha^* \mathcal{O}_C(x) \simeq \varpi_2^* \mathcal{O}_C(x)$, or $\alpha^* \mathcal{I}_{x,C}^{\vee} \simeq \varpi_2^* \mathcal{I}_{x,C}^{\vee}$ by item 4. Let $\mathcal{F} = \mathcal{I}_{x,C}^{\vee}$ for simplicity, so $\alpha^* \mathcal{F} \simeq \varpi_2^* \mathcal{F}$. We will see that what we've done so far suffices for proving Proposition 7.10.

First, we claim that there exists an isomorphism $\psi : \alpha^* \mathcal{F} \simeq \varpi_2^* \mathcal{F}$ which is 'unitary', that is, $(e_G \times_{\ell} \text{id}_C)^* \psi = \text{id}_{\mathcal{F}}$. Let $\psi_0 : \alpha^* \mathcal{F} \simeq \varpi_2^* \mathcal{F}$ be an isomorphism. Note that $(e_G \times_{\ell} \text{id}_C)^* \psi_0$ is an automorphism of \mathcal{F} , and then it is easy to check that if ψ is the composition

$$\alpha^* \mathcal{F} \xrightarrow{\psi_0} \varpi_2^* \mathcal{F} \xrightarrow{(\varpi_2^*(e_G \times_{\ell} \text{id}_C)^* \psi_0)^{-1}} \varpi_2^* \mathcal{F},$$

then ψ is unitary.

We will show that ψ actually satisfies the 'cocycle condition' automatically. The cocycle condition is a statement that two isomorphisms between two sheaves isomorphic to $q_3^* \mathcal{F}$ coincide, where $q_3 : G \times_{\ell} G \times_{\ell} C \rightarrow C$ is the projection to the third factor. However, since ψ is unitary, these isomorphisms at least coincide over $B \times_{\ell} C$, where $B = G \times_{\ell} \{e_G\} \cup \{e_G\} \times_{\ell} G \subset G \times_{\ell} G$. Then it suffices to prove that the only automorphism of $q_3^* \mathcal{F}$ which restricts to the identity on $B \times_{\ell} C$ is the identity. Note that $q_3^* \mathcal{F}$ is reflexive by [13, Proposition 1.8]. It follows from item 5 in the review of Weil divisors (§7.1.1) that $\text{Aut}(q_3^* \mathcal{F}) \cong \mathcal{O}_{G \times_{\ell} G \times_{\ell} C}^{\times}$. Let $\text{Spec } \mathbb{K} \rightarrow C$ be the geometric generic point of X . Because C is integral and $(G \times_{\ell} G \times_{\ell} C)_{\mathbb{K}} \simeq \mathbb{G}_{m,\mathbb{K}}^2$, to conclude the proof, it suffices to check the following: the only (multiplicatively) invertible regular function on $\mathbb{G}_{m,\mathbb{K}}^2$ which restricts to

the constant 1 on $\mathbb{G}_{m,\mathbb{K}} \times \{1\} \cup \{1\} \times \mathbb{G}_{m,\mathbb{K}}$ is the constant 1. Indeed, the invertible regular functions are all monomials, so the claim is clear. \square

Proof of Theorem 7.6. With the notation in Definition 6.6, we will prove by induction that \mathfrak{t}_n and $\mathfrak{t}_{n,c}$ admit group actions compatible with their vector fields in the sense of Definition 7.5. For \mathfrak{t}_1 , it can be checked that the standard action of G on itself extends to an action on the relative compactification $G \hookrightarrow \ell \times \mathbb{P}^1$, and that the extended G -action is compatible with the vector field. Going from \mathfrak{t}_n (or equivalently, $\mathfrak{t}_{n-1,3}$) to $\mathfrak{t}_{n,1}$ is elementary, just pull back the action accordingly. Going from $\mathfrak{t}_{n,1}$ to $\mathfrak{t}_{n,2}$ is Lemma 7.8 backed up by Propositions 7.10 and 6.7, while going from $\mathfrak{t}_{n,2}$ to $\mathfrak{t}_{n,3}$ is Lemma 7.9 backed up by Proposition 6.7. \square

Remark 7.11. It is elementary to see that there is a bijection between the set of stable n -marked \mathbb{G}_a -rational trees over \mathbb{K} (§1.2) up to isomorphism and the set of objects of $\mathfrak{V}_{0,1,n}(\mathbb{K})$ with 0 NCR morphisms up to isomorphism. It is then obvious from Theorem 1.3 that the fibers over the closed points of $\overline{P}_{n,\mathbb{K}}$ are the set of stable n -marked \mathbb{G}_a -rational trees over \mathbb{K} .

7.3. The G_ℓ^n -action on the moduli space. Finally, we make use of the modular interpretation to ‘transfer’ the group actions on the curves in Theorem 7.6 to group actions on the moduli spaces, thereby completing the proof of Theorem 1.4. Let $\mathfrak{t}_n = (S, C, \pi, \bar{x}, w_1, \phi)$ be the terminal object of $\mathfrak{V}_{0,1,n}$, cf. Theorem 6.9 and Definition 6.6.

Let’s construct the object $\mathfrak{t}_n^\circ = (S^\circ, C^\circ, \dots)$ of $\mathfrak{V}_{0,1,n}$ which corresponds to the open stratum of the terminal object where the parametrized curves are integral:

- $S^\circ = G_\ell^{n-1}$, $C^\circ = G_\ell^{n-1} \times \mathbb{P}^1$, π° is the projection to the first factor;
- w_1° is the constant $[1 : 0]$ section;
- x_i° is the constant $[0 : 1]$ section, for $i \geq 2$, x_i° is the graph of the map

$$G_\ell^{n-1} \xrightarrow{\text{projection to } (i-1)\text{-st factor}} G \rightarrow \ell \times \mathbb{P}^1 \rightarrow \mathbb{P}^1;$$

- $\phi^\circ = (1 + tx) \frac{\partial}{\partial x}$, where $[X : Y]$ are the coordinates on \mathbb{P}^1 and $x = X/Y$.

The details of the verification that this is indeed the open stratum are skipped. There is a tautological G_ℓ^{n-1} -action on S° , but if we are to think of \mathfrak{t}_n° more symmetrically, we should extend this to a G_ℓ^n -action as follows: the composition

$$G \xrightarrow{\text{inverse}} G \xrightarrow{\text{diagonal}} G_\ell^{n-1}$$

and the tautological action produce a G -action on S° , and we may combine the latter action and the tautological action into a $G_\ell^n = G \times_\ell G_\ell^{n-1}$ -action on S° , called the natural action on S° .

Proposition 7.12. *With notation as above, there exists a G_ℓ^n -action on S over ℓ which extends the natural G_ℓ^n -action on S° .*

Proof. Consider the object \mathfrak{y}_n of $\mathfrak{V}_{0,1,n}$ which coincides with the pullback $p_2^* \mathfrak{t}_n$, where $p_2 : G_\ell^n \times_\ell S \rightarrow S$ is the projection to the second factor, with the sole exception that the i -th section (denoted by x_i in Definition 2.1) is the composition

$$G_\ell^n \times_\ell S \xrightarrow{(p_1, x_i)} G_\ell^n \times_\ell C \xrightarrow{(q_1, \alpha_i)} G_\ell^n \times_\ell C,$$

where p_1, q_1 are the projections to the first factors respectively, and α_i uses the i -th factor of G_ℓ^n to act on C via the action provided by Theorem 7.6. Note

that \mathbf{y}_n is indeed an object in $\mathfrak{Y}_{0,1,n}$. By Theorem 6.9, there exists a morphism $\beta : G_\ell^n \times_\ell S \rightarrow S$ such that $\mathbf{y}_n = \beta^* \mathbf{t}_n$.

We only sketch the verification that β is a G_ℓ^n -action on S over ℓ . Let pr_i be the projection to the i th factor of $G_\ell^n \times_\ell G_\ell^n \times_\ell S$, and let $\mu : G \times_\ell G \rightarrow G$ and $\mu_\ell^n : G_\ell^n \times_\ell G_\ell^n \rightarrow G_\ell^n$ be the group laws on G and G_ℓ^n . We need to check that

$$(54) \quad \beta \circ (\mu_\ell^n, \text{pr}_3) = \beta \circ (\text{pr}_1, \text{pr}_1^* \beta)$$

as morphisms $G_\ell^n \times_\ell G_\ell^n \times_\ell S \rightarrow S$. It is straightforward to check that β restricts to the natural action on S° . Then the restriction of (54) to $G_\ell^n \times_\ell G_\ell^n \times_\ell S^\circ$ must hold, so (54) must hold a fortiori since everything in sight is reduced and separated. \square

Proposition 7.12 also shows that $\overline{P}_{n,\mathbb{K}}$ is a $\mathbb{G}_{a,\mathbb{K}}^{n-1}$ -variety, as claimed in §1.2.

Theorem 1.4 follows from the examples in §6.2.2, §6.2.3, and §6.2.4, Theorem 6.9, Theorem 1.3 (serving as the definition of \overline{P}_n), and Proposition 7.12.

8. CONCLUDING REMARKS AND QUESTIONS

8.1. Further examples of iterative constructions. We briefly discuss two more applications of Construction 6.4, this time in strictly positive genus. Their details will be left to the interested reader: for clarity, *anything not proved or not stated precisely in §8.1 is included in Figure 11 and Remarks 8.1 and 8.2.*

8.1.1. An example in arbitrary genus. This example is very similar to [27, Example 4.2.(e)]. Let Y be a smooth, projective, connected curve over \mathbb{K} of genus $g \geq 1$. Run Construction 6.4 with the following initial data in $\mathfrak{Y}_{g,0,0}^+$:

$$S = \text{Spec } \mathbb{K}, \quad C = Y, \quad \phi = 0,$$

and denote the resulting n -th space by $Y\{n\}$.

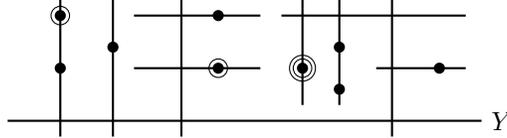


FIGURE 11. An object parametrized by $Y\{13\}$. One can think of it as 4 rational trees, in $\overline{P}_{3,\mathbb{K}}$, $\overline{P}_{1,\mathbb{K}}$, $\overline{P}_{3,\mathbb{K}}$, and $\overline{P}_{6,\mathbb{K}}$, attached to a copy of Y , on which the vector field is 0.

Remark 8.1. It is possible to establish a modular interpretation of $Y\{n\}$, similar to that of \overline{P}_n in Theorem 1.3, using essentially the same methods, though we also need to prescribe a map to Y satisfying suitable properties as part of the data, and suitably update the stability condition. (In the stability/ampleness condition in item 3 in Definition 2.5, we need to twist our line bundle by the pullback of an ample line bundle on Y .) As Figure 11 suggests, the relation between \overline{P}_n and $Y\{n\}$ is reminiscent of the relation between $\overline{M}_{0,n+1}$ and the Fulton–MacPherson space $Y[n]$. However, $Y\{n\}$ only parametrizes reducible curves for $n \geq 1$, and $Y\{n\}$ is reducible of pure dimension n for $n \geq 2$.

8.1.2. *A feature of 8.1.1 in genus 1.* Let E be a smooth genus 1 curve over \mathbb{K} with a nonzero vector field denoted rather abusively by $\frac{\partial}{\partial z}$. Run Construction 6.4 with the following initial data in $\mathfrak{R}_{1,0,0}^+$:

$$S = \text{Spec } \mathbb{K}[t] = \mathbb{A}^1, \quad C = \mathbb{A}^1 \times E, \quad \phi = t \frac{\partial}{\partial z},$$

and denote the resulting n -th space by W . As in 6.2.4, the bases retain projective flat morphisms to \mathbb{A}^1 throughout the application of Construction 6.4. Then:

- $W_0 \simeq E\{n\}$, cf. §8.1.1.
- $W \setminus W_0 \simeq (\mathbb{A}^1 \setminus \{0\}) \times E^n$ over $\mathbb{A}^1 \setminus \{0\}$, parametrizing simply n -tuples of points on E together with a nonzero vector field on E .

The reason for the second point is that, in the complement of the fiber over 0, the applications of Constructions 4.4 and 4.14 inside Construction 6.4 are always trivial – the bubbling up will simply never get off the ground.

In conclusion, $E\{n\}$ is a degeneration of E^n .

Remark 8.2. Accepting the interpretation of $E\{n\}$ from Remark 8.1, we may form the quotient $E\{n\}/S_n$ by permuting the markings, and we may even restrict to a fixed linear system $|\mathcal{L}|$ for the divisor on E equal to the sum of the markings. Then, the resulting space $E\{\mathcal{L}\}$ is a flat degeneration of \mathbb{P}^{n-1} .

8.2. Primitive linear systems on abelian (and K3) surfaces. The author’s motivation to study the constructions in the current paper originally came from [28, Question 4.14]. Although the techniques developed here do not answer this question, they might hint at some progress, according to the speculations below.

As in [28], consider a family $\{(V_t, \mathcal{L}_t) : t \in \Delta\}$ of $(1, d)$ -polarized abelian surfaces specializing over $0 \in \Delta$ to $V_0 = E \times F$, a product of two smooth genus 1 curves, polarized by $\mathcal{L}_0 = \mathcal{J}_E \boxtimes \mathcal{J}_F$, with $\deg \mathcal{J}_E = d$ and $\deg \mathcal{J}_F = 1$. [28, Question 4.14] proposes to try to construct an alternate degeneration of the space of stable maps, in the style of the well-known work of J. Li [19] or Kim, Kresh, and Oh [16], by allowing V_0 to ‘expand’ as in [28, Figure A].

Let us consider a similar task: to find an alternate degeneration $\mathfrak{L} \neq |\mathcal{L}_0|$ of the primitive linear systems $\{|\mathcal{L}_t| : t \in \Delta, t \neq 0\}$ based on this idea, by taking the limit divisors D to live on expansions W_0 of V_0 (and imposing the ‘correct’ conditions, which will not be discussed here). We expect an isomorphism

$$\mathfrak{L} \simeq E\{\mathcal{J}_E\}$$

given by $D \mapsto (\varpi^{-1}(p), D \cap \varpi^{-1}(p), v)$, where $\varpi : W_0 \rightarrow F$ is the projection map, $p \in F$ is such that $\mathcal{J}_F \not\cong \mathcal{O}_F(p)$ ($D \cap \varpi^{-1}(p)$ are the unordered markings on the curve $\varpi^{-1}(p)$), and the vector field v is related to the \mathbb{G}_a -action on the ruled surface in [28, §3]. Then \mathfrak{L} , and hence $E\{\mathcal{J}_E\}$, needs to be a degeneration of \mathbb{P}^{d-1} , which is a very strong requirement. However, by Remark 8.2, this very strong requirement is actually satisfied! Unfortunately, despite this, it is still not clear (to the author, at least) how to construct the desired degeneration, or if it even exists.

Question 8.3. Does an alternate degeneration of $|\mathcal{L}_t|$ ($t \neq 0$) with the features suggested above exist?

Similar ideas might also apply to K3 surfaces specializing to elliptically fibered K3s with 24 nodal fibers, as in e.g. [2].

8.3. Other toric-to- \mathbb{G}_a^d degenerations. Recall the program in [15] of classifying \mathbb{G}_a^d -varieties, and comparing the resulting picture with toric geometry. Although [15] makes it clear that no direct analogy with toric geometry holds, Theorem 1.4 raises the following question.

Question 8.4. Which projective toric varieties of dimension d degenerate isotrivially to \mathbb{G}_a^d -varieties in a manner compatible with the group actions? Classify all such degenerations.

For instance, the toric variety associated with the permutohedron (the Losev-Manin space) has the property in Problem 8.4, by Theorem 1.4.

REFERENCES

- [1] K. Behrend and Y. Manin, *Stacks of stable maps and Gromov-Witten invariants*, Duke Math. J. **85** (1996), 1–60.
- [2] J. Bryan and C. Leung, *The enumerative geometry of K3 surfaces and modular forms*, J. Amer. Math. Soc. **12** (2000), 371–410.
- [3] B. Conrad, “Grothendieck duality and base change”, Springer Berlin, Heidelberg, 2000.
- [4] W. Fulton, “Intersection Theory”, second edition, Springer New York, NY, 1998.
- [5] E. González, P. Solis, C. Woodward, *Properness for scaled gauged maps*, J. Algebra **490** (2017), 104–157.
- [6] E. González, P. Solis, C. Woodward, *Stable gauged maps*, In: “Algebraic geometry: Salt Lake City 2015”, T. de Fernex et al. (eds.) AMS and Clay Mathematics Institute, Proc. Sympos. Pure Math., 2018, 243–275.
- [7] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique: I. Le langage des schémas*, Publ. Math. Inst. Hautes Études Sci. **4** (1960), 5–228.
- [8] A. Grothendieck, *Éléments de géométrie algébrique: II. Étude globale élémentaire de quelques classes de morphismes*, Publ. Math. Inst. Hautes Études Sci. **8** (1961), 5–222.
- [9] A. Grothendieck, *Éléments de géométrie algébrique: III. Étude cohomologique des faisceaux cohérents, Première partie*, Publ. Math. Inst. Hautes Études Sci. **11** (1961), 5–167.
- [10] A. Grothendieck, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Troisième partie*, Publ. Math. Inst. Hautes Études Sci. **28** (1966), 5–255.
- [11] A. Grothendieck, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie*, Publ. Math. Inst. Hautes Études Sci. **32** (1967), 5–361.
- [12] R. Hartshorne, “Algebraic geometry”, Graduate Texts in Mathematics, Springer New York, NY, 1977.
- [13] R. Hartshorne, *Stable reflexive sheaves*, Math. Ann. **254** (1980), 121–176.
- [14] B. Hassett, *Moduli spaces of weighted pointed stable curves*, Adv. Math. **173** (2003), 316–352.
- [15] B. Hassett and Y. Tschinkel, *Geometry of equivariant compactifications of \mathbb{G}_a^n* , Int. Math. Res. Notices (IMRN) **22** (1999), 1211–1230.
- [16] B. Kim, A. Kresch, and Y.-G. Oh, *A compactification of the space of maps from curves*, Trans. Amer. Math. Soc. **366** (2014), 51–74.
- [17] T. Kambayashi and M. Miyanishi, *On flat fibrations by the affine line*, Illinois J. Math., **4** (1978), 662–671.
- [18] F. Knudsen, *The projectivity of the moduli space of stable curves II: the stacks $\overline{\mathcal{M}}_{g,n}$* , Math. Scan. **52** (1983), 161–199.
- [19] J. Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Diff. Geom. **57** (2001), 509–578.
- [20] A. Losev and Y. Manin, *New moduli spaces of pointed curves and pencils of flat connections*, Michigan Math. J. **48** (2000), 443–472.
- [21] S. Ma’u and C. Woodward, *Geometric realizations of the multiplihedra*, Compos. Math. **146** (2010), 1002–1028.
- [22] N. H. McCoy, *Remarks on divisors of zero*, Amer. Math. Monthly **49** (1942), 286–295.

- [23] D. Mumford, J. Fogarty, and F. Kirwan, “Geometric invariant theory”, Springer Berlin, Heidelberg, 1994.
- [24] K. Schwede, *Generalized divisors and reflexive sheaves*, <https://www.math.utah.edu/~schwede/Notes/GeneralizedDivisors.pdf>, 2010.
- [25] M. Olsson, *Algebraic spaces and stacks*, AMS Colloquium Publications **62**, American Mathematical Society, 2016.
- [26] Stacks project authors, “The Stacks project”, <https://stacks.math.columbia.edu/>, 2024.
- [27] C. Woodward, *Quantum Kirwan morphism and Gromov-Witten invariants of quotients II*, *Transformation Groups* **20** (2015), 881–920.
- [28] A. Zahariuc, *The Severi problem for abelian surfaces in the primitive case*, *J. Math. Pures Appl.* **158** (2022), 320–349.
- [29] F. Ziltener, “Symplectic vortices on the complex plane and quantum cohomology”, PhD thesis, ETH Zürich, 2006.
- [30] F. Ziltener, *A Quantum Kirwan map: bubbling and Fredholm theory for symplectic vortices over the plane*, *Mem. Amer. Math. Soc.* **1082**, vol. 230, 129 pages, 2014.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF WINDSOR, 401 SUNSET AVE,
WINDSOR, ON, N9B 3P4, CANADA

Email address: `adrian.zahariuc@uwindsor.ca`