THE CLASSIFICATION OF REFLEXIVE MODULES OF RANK ONE OVER RATIONAL AND MINIMALLY ELLIPTIC SINGULARITIES

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Abstract. We classify the reflexive modules of rank one over rational and minimally elliptic singularities. Equivalently, we classify full line bundles (in the sense of Esnault) on the resolutions of rational and minimally elliptic singularities. As an application, we determine among such reflexive modules of rank one all the special ones (in the sense of Wunram) and all the flat ones. In this way, we also classify the non-flat reflexive modules as well (as a generalization of a construction of Dan and Romano). In particular, we prove (in the rank one case) a conjecture of Behnke (respectively of Gustavsen and Ile), namely that in the case of a cusp (respectively log-canonical) singularity any reflexive module admits a flat connection. Additionally, we characterize those normal surface singularities for which any rank one reflexive module is flat.

The results generalize the classical Mckay correspondence, and results of Artin, Verdier, Esnault, Kahn and Wunram valid for different particular families of singularities.

1. Introduction

Let (X,x) be normal surface singularity, embedded in $(\mathbb{C}^n,0)$. Denote by Σ its link, i.e., $\Sigma = X\cap \mathbb{S}^{2n-1}_\epsilon$ with $\epsilon>0$ small enough. The first complete classification of the finite dimensional representations of $\pi_1(\Sigma)$ was done by McKay [20] in the case of rational double point singularities. This provides an identification of the McKay graph of the non-trivial irreducible representations with the dual resolution graph of the minimal resolution. This is called the McKay correspondence. Later Artin and Verdier in [2] reformulate the McKay correspondence in a more geometrical setting. For rational double point singularities the McKay correspondence by Artin and Verdier gives a complete classification of the indecomposable reflexive $\mathcal{O}_X$-modules. The classification of reflexive modules has been studied by several people in different cases: Esnault [7] introduced full sheaves and improved the results of Artin and Verdier [2] for rational surface singularities, and classified rank one reflexive modules for quotient singularities. Wunram [30] classified the family of *special* reflexive modules for rational surface singularities, nevertheless the complete classification of reflexive modules on rational singularities remained open. For a non-rational singularity, Kahn [15] classified all the reflexive modules in the simply elliptic case using the Atiyah's classification of vector bundles on elliptic curves. For a general normal Gorenstein singularity, Bobadilla and Romano [8] classified the family of *cohomological special* reflexive modules. For a general surface singularity, the classification problem for reflexive $\mathcal{O}_X$-modules remains open.

In this article, we provide a complete classification of reflexive modules of rank one for rational surfaces singularities and minimally elliptic singularities. Let $\pi\colon \tilde{X}\to X$ be a resolution of singularities. Denote the exceptional divisor by $E:=\pi^{-1}(x)$. Set $L:=H_2(\tilde{X},\mathbb{Z})$ and denote by L' its dual. Set $H:=\mathrm{Tors}(H_1(\partial\tilde{X},\mathbb{Z}))=\mathrm{Tors}(H_1(\Sigma,\mathbb{Z}))\cong L'/L$. We set $[\ell']$ the class of $\ell'\in L'$ in H. Let $\mathcal{S}'$ be

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the Lipman (anti-nef) cone. For any $h \in H$, there exists a unique $s_h \in \mathcal{S}'$ such that $[s_h] = h$ and s_h is minimal (see Section 2 for more details). Note that if (X, x) is a rational singularity, then the first Chern class $c_1 : \mathrm{Pic}(\tilde{X}) \to H^2(\tilde{X}, \mathbb{Z}) \simeq L'$ realizes a bijection between $\mathrm{Pic}(\tilde{X})$ and L'. In particular, for any $s_h \in L'$ there exists a unique line bundle denoted by $\mathcal{O}_{\tilde{X}}(s_h)$ such that $c_1(\mathcal{O}_{\tilde{X}}(s_h)) = s_h$.

Rank one reflexive sheaves on (X, x) can also be described by the local divisor class group $\mathrm{Cl}(X, x)$ of (X, x) (see e.g. [14]). This can be compared with the group $\mathrm{Pic}(\tilde{X})$ of locally free sheaves of rank one (that is, of the group of the line bundles) on $\tilde{X}$. One has the following exact sequence [21]

$$0 \to L \xrightarrow{\lambda} \mathrm{Pic}(\tilde{X}) \xrightarrow{\pi_*} \mathrm{Cl}(X, x) \to 0, \tag{1}$$

where $\lambda(l) = \mathcal{O}_{\tilde{X}}(l)$. This shows that any reflexive sheaf of rank one of (X, x) can be lifted to a line bundle of $\tilde{X}$, however this lift is well-defined only modulo L.

Now, following Esnault [7] and Kahn [15] we say that a locally free sheaf $\mathcal{M}$ over $\tilde{X}$ is *full* if and only if $\mathcal{M} \cong (\pi^* M)^{\vee\vee}$ with M a reflexive $\mathcal{O}_X$-module. In this note we focus on rank one full sheaves: there is a bijection between full sheaves of rank one and reflexive modules of (X, x) of rank one by the correspondences $M \mapsto \mathcal{M} := (\pi^* M)^{\vee\vee}$ and $\mathcal{M} \mapsto M := \pi_*(\mathcal{M})$. Our goal is to determine all the full sheaves, and in this way to identify for any $M \in \mathrm{Cl}(X, x)$ from its lifts $(\pi_*)^{-1}(M)$ the unique full sheave $\mathcal{M} \in \mathrm{Pic}(\tilde{X})$. Our first main result (Theorem 3.1) is the following.

Theorem 1. *Let (X, x) be a rational normal surface singularity and let $\pi \colon \tilde{X} \to X$ be any resolution of (X, x). Then the set of full sheaves of rank one is exactly $\{\mathcal{O}_{\tilde{X}}(-s_h)\}_{h \in H}$.*

We wish to emphasize that in this result we do not impose any speciality condition (compare with Wunram [30] and with the discussion below).

The cycles s_h originally were introduced and used in the study of the topological properties of normal surface singularities, e.g. in the computation of the equivariant Seiberg–Witten invariants of their links, see e.g. [26]. The above statement highlights the role of the cycles s_h in this new situation as well. Their appearance in this context is one of the novelties of the present article.

The above classification generalizes the McKay correspondence (for rank one) for rational surface singularities in the following sense. In Theorem 3.10 and Corollary 3.11 we classify all rank one *special* reflexive modules (full sheaves) supported on rational singularities, as part of reflexive modules (full sheaves). This classification of special reflexive modules is indeed guided by certain irreducible exceptional curves (as in the case of McKay correspondence).

If the singularity is not rational then the first Chern class $c_1 : \mathrm{Pic}(\tilde{X}) \to L'$ (though it is surjective) does not provide an isomorphism anymore, hence the identification of certain line bundles (with special properties) from the class of line bundles with given Chern class is considerably harder.

We set $\mathrm{Pic}^{\ell'}(\tilde{X}) := c_1^{-1}(\ell')$ for any $\ell' \in L'$. It is a torsor of $\mathrm{Pic}^0(\tilde{X}) \simeq \mathbb{C}^{p_g}/H^1(\tilde{X}, \mathbb{Z})$ (where p_g denotes the geometric genus). We also denote by $\mathrm{Full}^1(\tilde{X})$ the set of full sheaves over $\tilde{X}$ of rank one regarded as a subset of $\mathrm{Pic}(\tilde{X})$.

Our second main result (Theorem 4.6) is the following.

Theorem 2. *Let (X, x) be a minimally elliptic singularity. Let $\pi \colon \tilde{X} \to X$ be any resolution such that the support of the elliptic cycle is equal to E. (This happens e.g. in the minimal resolution or in the minimal good resolution.) Then*

$$\mathrm{Full}^1(\tilde{X}) = \left(\bigcup_{h \in H \setminus \{0\}} \mathrm{Pic}^{-s_h}(\tilde{X}) \right) \cup \left(\mathrm{Pic}^{-Z_{min}}(\tilde{X}) \setminus \{\mathcal{O}_{\tilde{X}}(-Z_{min})\} \right) \cup \{\mathcal{O}_{\tilde{X}}\}.$$

Again, note that this classification theorem does not require the 'specialty' property.

Next, we analyze the existence of flat connections in the reflexive modules. Recall that the original McKay correspondence establishes a one-to-one correspondence between finite dimensional non-trivial irreducible representations of $\pi_1(\Sigma)$ and the irreducible components of the exceptional divisor of the minimal resolution. By the Riemann-Hilbert correspondence (see [11]) there is a one-to-one correspondence between finite dimensional representations of $\pi_1(\Sigma)$ and reflexive $\mathcal{O}_X$-modules equipped with a flat connection. Denote by

$$\begin{aligned} \mathrm{Ref}^1_X &:= \text{the category of reflexive } \mathcal{O}_X\text{-modules of rank one},\\ \mathrm{Ref}^{1,\nabla}_X &:= \text{the category of pairs } (M,\nabla) \text{ where } M \text{ is a reflexive } \mathcal{O}_X\text{-module of rank one}\\ &\qquad \text{and } \nabla \text{ is an integrable connection.} \end{aligned}$$

In general, the forgetful functor

$$\mathrm{Ref}^{1,\nabla}_X \to \mathrm{Ref}^1_X$$

is not an equivalence of categories. Moreover, the forgetful functor may not be essentially surjective, i.e., the forgetful functor may not be onto on objects. In the case of quotient singularities, Esnault [7] proved that the forgetful functor is essentially surjective. In the case of simple elliptic singularities, Kahn [15] also proved that the forgetful functor is essentially surjective.

In Theorem 3.6 we prove that in the case of rational singularities *any rank one full sheaf is flat*, i.e. the forgetful functor is essentially surjective.

The first example of a singularity such that the forgetful functor is not essentially surjective was done by Dan and Romano [6]. As our first application (Theorem 5.1), we generalize the results and ideas of Dan and Romano as follows:

Theorem 3. *Let (X,x) be a minimally elliptic singularity such that its link is a rational homology sphere. Let $\pi\colon \tilde{X} \to X$ be the minimal resolution. Corresponding to $h = [c_1(\mathcal{L})] = 0$ there is only one flat full sheaf, namely the trivial one. Let $h \in H$ be different from zero. Then, again, in the family $\mathrm{Pic}^{-s_h}(\tilde{X}) \simeq \mathbb{C}$ of full sheaves only one element admits a flat connection (which is concretely characterized).*

In [3] Behnke conjectured that every reflexive module on a cusp singularity admits a flat connection. Later Gustavsen and Ile [12] extended this conjecture for any log-canonical surface singularity (recall that any cusp singularity is log-canonical). Our next application (Theorem 5.2 and Corollary 5.3) is a partial positive answer to these conjectures (valid for rank one modules).

Theorem 4. *Let (X,x) be a log-canonical singularity. Then, any reflexive module of rank one admits a flat connection.*

In fact, we can derive an even more general statement from the proof of this theorem.

Theorem 5. *Let (X,x) be a normal surface singularity. Then any rank one reflexive sheaf of (X,x) admits a flat connection if and only if the natural map $H^1(\tilde{X},\mathbb{C}) \to H^1(\tilde{X},\mathcal{O}_{\tilde{X}})$ is onto.*

The paper is organized as follows. In Section 2 we review some general properties of normal surface singularities, the divisor lattice structure and the Lipman cone, flat reflexive modules and full sheaves. In Section 3 we classify the rank one full sheaves for rational surfaces singularities using techniques based on the properties of the divisor lattice structure and of the Lipman cone. We also classify and characterize in several different ways the special full sheaves for rational singularities. In Section 4 we generalize the techniques of the previous section to the case of minimally elliptic singularities. In Section 5 we use our classification theorems to study the flat reflexive modules over minimally elliptic singularities, cusps and log-canonical singularities. Here we also characterize all the normal surface singularities for which any rank one reflexive module is flat.

2. Preliminaries

In this section we recall basics on reflexive modules, full sheaves and topological invariants of normal surface singularities. We assume basic familiarity with these objects, see [4, 14, 22, 26] for more details.

2.1. **Setting and notation. Normal singularities.** Throughout this article, we denote by (X,x) the germ of a complex analytic normal surface singularity, i.e., the germ of a complex surface such that its local ring of functions $\mathcal{O}_{X,x}$ is integrally closed in its field of fractions. In this situation X has a *dualizing sheaf* ω_X, and let $\omega_{X,x}$ be its stalk at $x \in X$, which is called the *dualizing module* of the ring $\mathcal{O}_{X,x}$ (see [14, Chapter 5 § 3] for more details). We say that (X,x) has a *Gorenstein* normal singularity if the dualizing module is isomorphic to $\mathcal{O}_{X,x}$.

2.2. **Good resolutions and dual graphs.** Let (X,x) be the germ of a complex analytic normal surface singularity. Let

$$\pi\colon \tilde{X} \to X,$$

be a *resolution of* (X,x), i.e., a proper holomorphic map from a smooth surface $\tilde{X}$ to a given representative of (X,x) such that π restricted to the complement of $\pi^{-1}(x)$ is biholomorphic. Sometimes we will require π to be a *good resolution*, which means that the exceptional divisor $E := \pi^{-1}(x)$ is a normal crossing divisor and each irreducible component of E is smooth. For any normal surface singularity there is always a good resolution, however it is not unique. Given a good resolution, the *dual graph* is a decorated graph Γ constructed as follows: the set of vertices, say V, is in bijection with the set of irreducible components of E, say $\{E_v\}_{v\in V}$, two vertices u and v are connected by k edges in Γ if and only if the corresponding components E_u and E_v have k intersection points. Each vertex v is decorated with two numbers: the self-intersection E_v^2 in $\tilde{X}$ and the genus $g(E_v)$.

The *intersection matrix* $M = (m_{u,v})_{u,v\in V}$ associated to the dual graph Γ is the intersection matrix of the curves $\{E_v\}_{v\in V}$, i.e., $m_{u,v} = (E_u, E_v)$. It is negative definite.

2.3. **The link.** Let us embed (X,x) in a certain $(\mathbb{C}^n, 0)$. Via this embedding, the *link* Σ of (X,x) is defined as the intersection $\Sigma = X \cap \mathbb{S}^{2n-1}_\epsilon$, where $\mathbb{S}^{2n-1}_\epsilon = \{z \in \mathbb{C}^n \,:\, |z| = \epsilon\}$ and $\epsilon > 0$ is small enough. The diffeomorphism type of the link does not depend on the embedding or on $0 < \epsilon \ll 1$.

2.4. **The divisor lattice structure and the Lipman cone.** Let (X,x) be the germ of a complex analytic normal surface singularity. Let $\pi\colon \tilde{X} \to X$ be any resolution. Set $L := H_2(\tilde{X},\mathbb{Z})$, i.e., L is the free abelian group generated by the classes $\{E_v\}_{v\in V}$. Denote also $L' := H_2(\tilde{X}, \partial\tilde{X}, \mathbb{Z})$. Notice that L' is the dual of L, indeed by the Lefschetz–Poincaré duality we have the perfect pairing $H_2(\tilde{X},\mathbb{Z}) \otimes H_2(\tilde{X},\partial\tilde{X},\mathbb{Z}) \to \mathbb{Z}$, hence

$$\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z}) \cong H_2(\tilde{X},\partial\tilde{X},\mathbb{Z}) = L'.$$

Since the intersection matrix M is non-degenerate, the homological map $L \to L'$ is injective. Thus, by the homological long exact sequence of the pair $(\tilde{X},\partial\tilde{X})$ (where $\partial\tilde{X} \simeq \Sigma$) we have

$$0 \to L \to L' \to H_1(\partial\tilde{X},\mathbb{Z}) \to H_1(\tilde{X},\mathbb{Z}) \to 0. \tag{2}$$

Since $H_1(\tilde{X},\mathbb{Z})$ is free, $H_1(\partial\tilde{X},\mathbb{Z}) \cong \mathrm{Tors}(H_1(\partial\tilde{X},\mathbb{Z})) \oplus H_1(\tilde{X},\mathbb{Z})$, and the quotient L'/L is identified with

$$H := \mathrm{Tors}\big(H_1(\partial\tilde{X},\mathbb{Z})\big) = \mathrm{Tors}\left(H_1(\Sigma,\mathbb{Z})\right).$$

We extend the intersection form $(,)$ from L to $L_{\mathbb{Q}} := L \otimes \mathbb{Q}$, which we denote by $(,)_{\mathbb{Q}}$. Via $(,)_{\mathbb{Q}}$ the lattice $L' \cong \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$ can be identified with a lattice of rational cycles

$$\{\ell' \in L_{\mathbb{Q}} \,:\, (\ell',\ell)_{\mathbb{Q}} \in \mathbb{Z} \text{ for any } \ell \in L \,\} \subset L_{\mathbb{Q}}.$$

Hence, we can identify L' with $\bigoplus_{v\in V} \mathbb{Z}\langle E_v^*\rangle$, the lattice generated by the rational cycles $E_v^* \in L_{\mathbb{Q}}$ ($v \in V$) defined via

$$(E_u^*, E_v)_{\mathbb{Q}} = -\delta_{u,v} \text{ (Kronecker delta) for any } u, v \in V.$$

Let $\ell'_1, \ell'_2 \in L_{\mathbb{Q}}$ where $\ell'_j = \sum_v l'_{jv} E_v$ for $j \in \{1,2\}$. We consider the partial order in $L_{\mathbb{Q}}$ given by $\ell'_1 \geq \ell'_2$ if and only if $l'_{1v} \geq l'_{2v}$ for all $v \in V$. We set $\min\{\ell'_1,\ell'_2\} := \sum_v \min\{l'_{1v}, l'_{2v}\} E_v$.

Let $\ell' \in L'$. We denote its class in H by $[\ell']$. The lattice L' has a partition parametrized by H, where for any $h \in H$ we have

$$L'_h := \{\ell' \in L' \,|\, [\ell'] = h\}.$$

Note that $L'_0 = L$. Given any $h \in H$ we define $r_h := \sum_v l'_v E_v$ as the unique element of L'_h such that $0 \leq l'_v < 1$. We define the *rational Lipman cone* by

$$\mathcal{S}_{\mathbb{Q}} := \{\ell' \in L_{\mathbb{Q}} \,|\, (\ell', E_v) \leq 0 \text{ for any } v \in V\}.$$

It is a cone generated over $\mathbb{Q}_{\geq 0}$ by E_v^*. We set

$$\mathcal{S}' := \mathcal{S}_{\mathbb{Q}} \cap L' \quad \text{and} \quad \mathcal{S} := \mathcal{S}_{\mathbb{Q}} \cap L.$$

Note that $\mathcal{S}'$ is the monoid of anti-nef rational cycles of L', it is generated over $\mathbb{Z}_{\geq 0}$ by the cycles E_v^*. The Lipman cone $\mathcal{S}'$ also has a natural equivariant partition indexed by H. We denote $\mathcal{S}'_h = \mathcal{S}' \cap L'_h$. The monoid $\mathcal{S} = \mathcal{S}'_0$ has the following properties:

(1) if $Z = \sum n_v E_v \in \mathcal{S}$ and $Z \neq 0$, then $n_v > 0$ for all $v \in V$,
(2) if $Z_1, Z_2 \in \mathcal{S}$, then $Z_1 + Z_2 \in \mathcal{S}$,
(3) if $Z_1, Z_2 \in \mathcal{S}$, then $\min\{Z_1, Z_2\} \in \mathcal{S}$.

Thus, $\mathcal{S} \setminus \{0\}$ has a unique minimal element Z_{min} called the *Artin's fundamental cycle (minimal cycle or numerical cycle).* Similarly, for any $h \in H$, the set $\mathcal{S}'_h$ has the following properties [22,23]:

(1) If $s_1, s_2 \in \mathcal{S}'_h$, then $s_1 - s_2 \in L$ and $\min\{s_1, s_2\} \in \mathcal{S}'_h$,
(2) for any h there exists a unique *minimal cycle* $s_h := \min\{\mathcal{S}'_h\}$.

Remark 2.1. *Clearly $s_0 = 0$. In fact, $s_h \neq 0$ if and only if $h \neq 0$.*

From now on, we will denote by $\lfloor \cdot \rfloor$ the integral part, that is $\lfloor \sum_v r_v E_v \rfloor := \sum_v \lfloor r_v \rfloor E_v$.

Definition 2.2. *The* geometric genus p_g *of X is the $\mathbb{C}$-vector space dimension of $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$. It is independent of the choice of the resolution. We say that (X,x) has a* rational singularity *if $p_g = 0$.*

We denote by $\Omega^2_{\tilde{X}}$ the sheaf of holomorphic 2-forms. The divisor of any meromorphic section of $\Omega^2_{\tilde{X}}$ is called *the canonical divisor*; it is denoted by $K_{\tilde{X}}$. The rational cycle $Z_K \in L'$ (supported on E) that satisfies $(Z_K, E_v)_{\mathbb{Q}} = -K_{\tilde{X}} \cdot E_v$ for any $v \in V$ is called *the canonical cycle*. (It is independent of the choice of $K_{\tilde{X}}$.) In the case of Gorenstein singularities, one proves that $Z_K \in L$, see [22].

We define the Riemann-Roch expression

$$\chi : L \to \mathbb{Z},\ \chi(\ell) = -(\ell, \ell - Z_K)_{\mathbb{Q}}/2.$$

By Artin's criterion [1] (X,x) is rational if and only if $\chi(\ell) > 0$ for any $\ell > 0$ ($\ell \in L$). For rational graphs (graphs satisfying Artin's criterion) one also has $\chi(\ell) \geq 0$ for any $\ell \in L$. Furthermore, rational graphs are trees of $\mathbb{P}^1$'s.

The next level of complexity of graphs is realized by elliptic graphs.

Definition 2.3. *The germ (X,x) is called elliptic if (X,x) is not rational but $\chi(l)\geq 0$ for any $l\in L$. By an equivalent definition, (X,x) is elliptic if $\chi(Z_{min})=0$. (Cf. [22, 26].) Both topological criteria are independent of the choice of the resolution.*

Definition 2.4. *Let (X,x) be an elliptic normal surface singularity. Let $\pi\colon\tilde{X}\to X$ be a resolution. A non-zero effective cycle C supported on E is called* minimally elliptic cycle *if $\chi(C)=0$ and for any $0<l<C$, one has $\chi(l)>0$.*

Once a resolution is fixed, a minimally elliptic cycle always exists and it is unique.

Definition 2.5. *We say that (X,x) is a* minimally elliptic singularity*, if the geometric genus of (X,x) is one and (X,x) is Gorenstein.*

Laufer proved the following topological characterization: an elliptic singularity is minimally elliptic if and only if in the *minimal resolution* $C=Z_K=Z_{min}$, see [17] or [26, Theorem 7.2.15]. In an arbitrary resolution these identities do not necessarily hold, even the support $|C|$ of C can be smaller than E. (The support $|\ell|$ of $\ell=\sum_v m_vE_v$ is $\cup_{v:m_v\neq 0}E_v$.) However, in the *minimal good resolution* $|C|=E$ still holds, cf. [26, p. 300].

As usual, denote $H^1(\tilde{X},\mathcal{O}^*_{\tilde{X}})$ by $\mathrm{Pic}(\tilde{X})$. Let $c_1\colon\mathrm{Pic}(\tilde{X})\to H^2(\tilde{X},\mathbb{Z})\cong H_2(\tilde{X},\partial\tilde{X},\mathbb{Z})=L'$ denote the surjective 'first Chern class morphism', and set

$$\mathrm{Pic}^{\ell'}(\tilde{X}):=c_1^{-1}(\ell')\qquad\text{for any }\ell'\in L'.$$

It is a $\mathrm{Pic}^0(\tilde{X})\simeq H^1(\tilde{X},\mathcal{O}_{\tilde{X}})/H^1(\tilde{X},\mathbb{Z})\simeq\mathbb{C}^{p_g}/H^1(\tilde{X},\mathbb{Z})$ torsor.

The following vanishing theorems will be used several times.

Theorem 2.6 (Generalized Grauert–Riemenschneider vanishing [26, Theorem 6.4.3])**.** *Let (X,x) be a normal surface singularity. Let $\pi\colon\tilde{X}\to X$ be any resolution. Let $\mathcal{L}\in\mathrm{Pic}(\tilde{X})$ such that $c_1(\mathcal{L}(-K_{\tilde{X}}))\in\Delta-\mathcal{S}_{\mathbb{Q}}$ for some $\Delta\in L\otimes\mathbb{Q}$ such that $\lfloor\Delta\rfloor=0$. Then for any $\ell\in L_{>0}$ one has the vanishing $h^1(\ell,\mathcal{L}|_\ell)=0$. In particular, $h^1(\tilde{X},\mathcal{L})=0$.*

Theorem 2.7 (Lipman's vanishing [19, Theorem 11.1])**.** *Let (X,x) be a rational normal surface singularity. Let $\pi\colon\tilde{X}\to X$ be any resolution. If $\mathcal{L}\in\mathrm{Pic}(\tilde{X})$ with $-c_1(\mathcal{L})\in\mathcal{S}'$, then $h^1(\tilde{X},\mathcal{L})=0$.*

2.5. **Generalized Laufer's algorithm.** [23, Lemma 7.4] For any $\ell'\in L'$, there exists a unique minimal element $s(\ell')\in\mathcal{S}'$ such that $s(\ell')\geq\ell'$ and $s(\ell')-\ell'\in L$. Furthermore, $s(\ell')$ can be constructed by a computation sequence $\{x_i\}_{i=1}^t$ as follows:

(1) set $x_0:=\ell'$,
(2) if x_i is already constructed and $x_i\notin\mathcal{S}'$, then there exists some E_{v_i} such that $(x_i,E_{v_i})>0$. Then define $x_{i+1}:=x_i+E_{v_i}$ and repeat the algorithm.

The procedure stops after finitely many steps, and the last term x_t is $s(\ell')$.

Laufer's algorithm will be used in several different situations. E.g., one might consider a line bundle $\mathcal{L}\in\mathrm{Pic}^{-c}(\tilde{X})$, and then run the algorithm starting with c and ending with $s(c)$: $x_0=c$, $x_{i+1}=x_i+E_{v_i}$, $(x_i,E_{v_i})>0$. Then in the cohomological long exact sequence associated with

$$0\to\mathcal{L}(c-x_{i+1})\to\mathcal{L}(c-x_i)\to\mathcal{L}(c-x_i)|_{E_{v_i}}\to 0,\tag{3}$$

the Chern class of $\mathcal{L}(c-x_i)|_{E_{v_i}}$ is negative, so $H^0(E_{v_i},\mathcal{L}(c-x_i)|_{E_{v_i}})=0$. Hence we have the following statements proved inductively along the computation sequence.

Lemma 2.8. *Let (X,x) be the germ of a normal surface singularity. Let $\pi\colon \tilde{X}\to X$ be a resolution and fix $\mathcal{L}\in \mathrm{Pic}^{-c}(\tilde{X})$. Then*

(a) the natural map $H^0(\tilde{X},\mathcal{L}(c-s(c)))\to H^0(\tilde{X},\mathcal{L})$ is an isomorphism,

(b) $h^1(\tilde{X},\mathcal{L})=h^1(\tilde{X},\mathcal{L}(c-s(c))+\sum_i h^1(E_{v_i},\mathcal{L}(c-x_i))$.

In particular, if (X,x) is rational then $h^1(\tilde{X},\mathcal{L}(c-s(c))=0$ by Lipman's vanishing, hence $h^1(\tilde{X},\mathcal{L})=\sum_i h^1(\mathcal{O}_{\mathbb{P}^1}(-(x_i,E_{v_i})))$.

The last sentence shows that in the rational case, $h^1(\tilde{X},\mathcal{L})=0$ if and only if along the computation sequence connecting c and $s(c)$ at all the steps $(x_i,E_{v_i})=1$ holds.

This can be compared with Laufer's criterion of rationality [16]. For any singularity (X,x) with resolution $\tilde{X}$ (and all E_v rational) consider the computation sequence starting with one of the exceptional curves, say E_{v_0}, and ending with $s(E_{v_0})=Z_{min}$. Then (X,x) is rational if and only if $h^1(\mathcal{O}_{Z_{min}})=0$ if and only if along the sequence $(x_i,E_{v_i})=1$ for every i.

Part *(a)* of the above theorem will be used e.g. when $c=r_h$ and $s(r_h)=s_h$ for some $h\in H$.

2.6. **Reflexive and flat modules.** Let X be a normal variety. Let $\mathscr{H}om_{\mathcal{O}_X}(\cdot,\cdot)$ be the sheaf theoretical Hom functor. The dual of an $\mathcal{O}_X$-module M is denoted by $M^{\vee}:=\mathscr{H}om_{\mathcal{O}_X}(M,\mathcal{O}_X)$. An $\mathcal{O}_X$-module M is called *reflexive* if the natural homomorphism from M to $M^{\vee\vee}$ is an isomorphism (see [14, Definition 5.1.12]). We denote by $\mathrm{Cl}(X,x)$ the *local divisor class group* of (X,x) (see [26, Chapter 6]). The group $\mathrm{Cl}(X,x)$ can also be interpreted as the group of isomorphism classes of reflexive sheaves on (X,x) of rank one [29].

One has the following commutative diagram of exact sequences (see e.g. [21] or [26, 6.1]):

$$
\begin{array}{ccccccccc}
 & & & & & 0 & & 0 & \\
 & & & & & \downarrow & & \downarrow & \\
 & & & & & L & \xrightarrow{=} & L & \\
 & & & & & \downarrow \lambda & & \downarrow & \\
0 & \to & H^1(\tilde{X},\mathbb{Z}) & \to & H^1(\tilde{X},\mathcal{O}_{\tilde{X}}) & \to & \mathrm{Pic}(\tilde{X}) & \xrightarrow{c_1} & L' & \to & 0 \\
 & & \downarrow = & & \downarrow = & & \downarrow \pi_* & & \downarrow & \\
0 & \to & H^1(\tilde{X},\mathbb{Z}) & \to & H^1(\tilde{X},\mathcal{O}_{\tilde{X}}) & \to & \mathrm{Cl}(X,x) & \xrightarrow{\bar{c}_1} & H & \to & 0 \\
 & & & & & \downarrow & & \downarrow & \\
 & & & & & 0 & & 0 &
\end{array}
\qquad (4)
$$

where $\bar{c}_1$ is the class of c_1, or, it assigns to a Weil divisor the homological class of its intersection with the link.

Let M be a coherent $\mathcal{O}_X$-module. Following [11] a *connection* on M is an $\mathcal{O}_X$-linear map

$$\nabla : \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X)\to \mathrm{End}_{\mathbb{C}}(M),$$

which for all $f\in\mathcal{O}_X$, $m\in M$ and $D\in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X)$ satisfies the *Leibniz rule*

$$\nabla(D)(fm)=D(f)m+f\nabla(D)(m).$$

A connection ∇ is called *integrable or flat* if it is a $\mathbb{C}$-Lie-algebra homomorphism. Denote by:

Ref_X := the category of reflexive $\mathcal{O}_X$-modules,

Ref_X^{∇} := the category of pairs (M,∇) where M is a reflexive $\mathcal{O}_X$-module and ∇ is integrable,

$\mathrm{Rep}_{\pi_1(\Sigma)}$:= the category of complex finite dimensional representations of $\pi_1(\Sigma)$.

By [11] there is an equivalence

$$\mathrm{Ref}_X^{\nabla}\cong \mathrm{Rep}_{\pi_1(\Sigma)}.$$

In general, the *forgetful functor*

$$\mathrm{Ref}_X^{\nabla} \to \mathrm{Ref}_X,$$

is not an equivalence of categories. Moreover, the forgetful functor may not be essentially surjective, i.e., the forgetful functor may not be onto on objects. This problem has an easier reformulation in the case of reflexive modules of rank one. Denote by

$$\begin{aligned}
\mathrm{Ref}_X^1 &:= \text{the category of reflexive } \mathcal{O}_X\text{-modules of rank one},\\
\mathrm{Ref}_X^{1,\nabla} &:= \text{the category of pairs } (M,\nabla) \text{ where } M \text{ is a reflexive } \mathcal{O}_X\text{-module of rank one}\\
&\qquad \text{and } \nabla \text{ is integrable},\\
\mathrm{Rep}^1_{\pi_1(\Sigma)} &:= \text{the category of complex one dimensional representations of } \pi_1(\Sigma).
\end{aligned}$$

Let $\rho \in \mathrm{Obj}(\mathrm{Rep}^1_{\pi_1(\Sigma)})$. Thus, we have $\rho\colon \pi_1(\Sigma) \to GL(1,\mathbb{C}) = \mathbb{C}^*$ where $\mathbb{C}^*$ is the multiplicative subgroup of $\mathbb{C}$. Since $\mathbb{C}^*$ is abelian,

$$\mathrm{Hom}\,(\pi_1(\Sigma),\mathbb{C}^*) = \mathrm{Hom}\,(\pi_1(\Sigma)_{\mathrm{ab}},\mathbb{C}^*) = \mathrm{Hom}\,(H_1(\Sigma,\mathbb{Z}),\mathbb{C}^*). \tag{5}$$

By the Universal Coefficient Theorem we get

$$\mathrm{Hom}\,(H_1(\Sigma,\mathbb{Z}),\mathbb{C}^*) \cong H^1(\Sigma,\mathbb{C}^*). \tag{6}$$

Hence, by (5) and (6) $\rho \in H^1(\Sigma,\mathbb{C}^*)$. Replacing $\mathbb{C}^*$ by $\mathbb{C}/\mathbb{Z}$, the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^* \to 0,$$

induces the following cohomological long exact sequence

$$\cdots \to H^1(\Sigma,\mathbb{C}) \to H^1(\Sigma,\mathbb{C}/\mathbb{Z}) \xrightarrow{\Phi} H^2(\Sigma,\mathbb{Z}) \to H^2(\Sigma,\mathbb{C}) \to \dots \tag{7}$$

By (7), the image of Φ is torsion. Hence, it has a factorization through

$$\Phi\colon H^1(\Sigma,\mathbb{C}/\mathbb{Z}) \to \mathrm{Tors}\,\big(H^2(\Sigma,\mathbb{Z})\big) \cong H, \quad \rho \mapsto \Phi(\rho) \in H.$$

On the other hand, by the equivalence between $\mathrm{Ref}_X^{1,\nabla}$ and $\mathrm{Rep}^1_{\pi_1(\Sigma)}$, let (M_ρ,∇_ρ) be the pair in $\mathrm{Obj}(\mathrm{Ref}_X^{1,\nabla})$ associated to ρ. Hence, in this reformulation, the *forgetful morphism* is

$$\Psi\colon H^1(\Sigma,\mathbb{C}/\mathbb{Z}) \to \mathrm{Cl}(X,x), \quad \rho = (M_\rho,\nabla_\rho) \mapsto \Psi(\rho) = M_\rho.$$

In Appendix we will prove that the map $\bar{c}_1$ from diagram (4) and the maps Ψ and Φ fit together in a commutative diagram:

Lemma 2.9. $\bar{c}_1 \circ \Psi = \Phi$.

2.7. **Full sheaves.** Let (X,x) be the germ of a normal surface singularity and $\pi\colon \tilde{X} \to X$ be a resolution.

Definition 2.10. *Let $\mathcal{F}$ be a sheaf on $\tilde{X}$. We say that $\mathcal{F}$ is* generically generated by global sections *if it is generated by global sections except in a finite set.*

Recall the following definition of full sheaves from [15, Definition 1.1].

Definition 2.11. *An $\mathcal{O}_{\tilde{X}}$-module $\mathcal{M}$ is called* full *if there is a reflexive $\mathcal{O}_X$-module M such that $\mathcal{M} \cong (\pi^*M)^{\vee\vee}$. We call $\mathcal{M}$ the full sheaf associated to M.*

The following characterization of full sheaves will be very important in the following sections.

Proposition 2.12 ([15, Proposition 1.2])**.** *A locally free sheaf $\mathcal{M}$ on $\tilde{X}$ is full if and only if*

(1) the sheaf $\mathcal{M}$ is generically generated by global sections.

(2) The natural map $H^1_E(\tilde{X},\mathcal{M}) \to H^1(\tilde{X},\mathcal{M})$ is injective.

If $\mathcal{M}$ is the full sheaf associated to M, then $\pi_*\mathcal{M} = M$. Hence, there exists a bijective correspondence between the full sheaves on $\tilde{X}$ and reflexive modules on (X,x) via $\mathcal{M} = (\pi^*M)^{\vee\vee}$ and $\pi_*\mathcal{M} = M$.

In the particular case of a rational singularity the characterization of full sheaves is 'easier':

Proposition 2.13 ([7, Lemma 2.2])**.** *Suppose that (X,x) is a rational normal surface singularity. A locally free sheaf $\mathcal{M}$ on $\tilde{X}$ is full if and only if*

(1) the sheaf $\mathcal{M}$ is generated by global sections,
(2) $H^1_E(\tilde{X},\mathcal{M}) = 0$.

Definition 2.14. *Let $\mathcal{F}$ be a sheaf on $\tilde{X}$. The* base points *of $\mathcal{F}$ are the points $p \in \tilde{X}$ such that $s(p) = 0$ for all $s \in H^0(\tilde{X},\mathcal{F})$. A component E_v is called a* fixed component *of $\mathcal{F}$ if any section $s \in H^0(\tilde{X},\mathcal{F})$ vanishes along E_v.*

In the particular case of *rank one* sheaves, in Proposition 2.12 the condition *(1)*, namely that $\mathcal{M}$ is *generically generated by global sections*, can equivalently be replaced by the condition that $\mathcal{M}$ *has no fixed components*. Obviously, if a sheaf is generically generated by global sections then it cannot have a fixed component. However, if the rank is $r \geq 2$ then it can happen that along an exceptional component we have non-vanishing global sections (thus that component will not be a fixed component), but those non-vanishing sections will not generate the r-dimensional fibers. On the other hand, if the rank is one, in the case of line bundles, if a point is not a fixed point then there exists a global section which does not vanish at that point, and it automatically generates the corresponding one-dimensional fiber. Therefore, we have the following statement.

Proposition 2.15. *Let (X,x) be the germ of a normal surface singularity. Let $\pi\colon \tilde{X} \to X$ be any resolution and let $\mathcal{L}$ be a line bundle. Then, the following conditions are equivalent:*

(1) the sheaf $\mathcal{L}$ is generically generated by global sections.
(2) the sheaf $\mathcal{L}$ has no fixed components.

In general, for a normal surface singularity the generation by global sections of a full sheaf depends on the resolution. Nevertheless, if for some resolution a full sheaf is generated by global sections (i.e. it has no base points) then it satisfies nice properties under pull-back and push-forward.

The following proposition will be used later.

Proposition 2.16. *Let (X,x) be the germ of a normal surface singularity. Let $\pi\colon \tilde{X} \to X$ be any resolution. Let $\sigma\colon \tilde{X}_0 \to \tilde{X}$ be the blow-up in some point $p \in E \subset \tilde{X}$. Let $\mathcal{M}$ be a full $\mathcal{O}_{\tilde{X}}$-sheaf. If $\mathcal{M}$ is generated by global sections, then $\sigma^*\mathcal{M}$ is a full $\mathcal{O}_{\tilde{X}_0}$-sheaf generated by global sections. Moreover, we have the isomorphism $\sigma_*\sigma^*\mathcal{M} \cong \mathcal{M}$.*

Proof. The proof is an adaptation of the proof of [8, Proposition 4.7]. □

3. Rank one full sheaves of rational singularities

In this section we classify all the reflexive modules of rank one over any rational singularity.

Recall that if (X,x) is rational then the link Σ is a rational homology sphere, hence $H_1(\Sigma,\mathbb{Z}) = \mathrm{Tors}(H_1(\Sigma,\mathbb{Z})) = L'/L = H$. Furthermore, since $\mathrm{Pic}^0(\tilde{X}) = \mathbb{C}^{p_g} = 0$, we also have the isomorphism $c_1 : \mathrm{Pic}(\tilde{X}) \to L'$, that is, all the line bundles of $\tilde{X}$ are characterized topologically. In this section, for any $\ell' \in L'$ we denote by $\mathcal{O}_{\tilde{X}}(\ell') \in \mathrm{Pic}(\tilde{X})$ the line bundle which satisfies $c_1(\mathcal{O}_{\tilde{X}}(\ell')) = \ell'$. (Later we will define the bundles $\mathcal{O}_{\tilde{X}}(\ell')$ in a more general non-rational context as well.)

Theorem 3.1. *Let (X,x) be a rational normal surface singularity. Let $\pi\colon \tilde{X}\to X$ be any resolution. Then,*

(1) for any $h\in H$ the line bundle $\mathcal{O}_{\tilde{X}}(-s_h)$ is full.

(2) If a line bundle $\mathcal{L}\in \mathrm{Pic}(\tilde{X})$ is full, then $c_1(\mathcal{L})=-s_h$ for some $h\in H$. Hence $\mathcal{L}=\mathcal{O}_{\tilde{X}}(-s_h)$.

Thus, any full sheaf of rank one is of the form $\mathcal{O}_{\tilde{X}}(-s_h)$ for some $h\in H$.

Proof. *(1)* Let $h\in H$. In order to prove that the line bundle $\mathcal{O}_{\tilde{X}}(-s_h)$ is full, by Proposition 2.13 (and Serre duality) we have to verify that

(i) the sheaf $\mathcal{O}_{\tilde{X}}(-s_h)$ is generated by global sections,

(ii) $H^1(\mathcal{O}_{\tilde{X}}(s_h+K_{\tilde{X}}))=0$.

First, we verify (i). Since $c_1(\mathcal{O}_{\tilde{X}}(-s_h))\in -\mathcal{S}'$ and (X,x) is rational, then by [19, Theorem 12.1] the sheaf $\mathcal{O}_{\tilde{X}}(-s_h)$ is generated by global sections.
Next, we prove (ii). Since $\lfloor r_h\rfloor=0$, we get $r_h\in\{r_h\}-\mathcal{S}_{\mathbb{Q}}$. Thus, by Theorem 2.6

$$h^1(\mathcal{O}_{\tilde{X}}(K_{\tilde{X}}+r_h))=0. \tag{8}$$

Set $\Delta:=s_h-r_h$. If $\Delta=0$, then we are done. Suppose that $\Delta>0$. Then consider the following exact sequence

$$0\to\mathcal{O}_{\tilde{X}}(K_{\tilde{X}}+r_h)\to\mathcal{O}_{\tilde{X}}(K_{\tilde{X}}+s_h)\to\mathcal{O}_{\Delta}(K_{\tilde{X}}+s_h)\to 0. \tag{9}$$

By (8) and the induced long cohomological exact sequence we get

$$h^1(\mathcal{O}_{\tilde{X}}(K_{\tilde{X}}+s_h))=h^1(\mathcal{O}_{\Delta}(K_{\tilde{X}}+s_h)).$$

By Serre duality we have

$$h^1(\mathcal{O}_{\Delta}(K_{\tilde{X}}+s_h))=h^0(\mathcal{O}_{\Delta}(-s_h+\Delta))=h^0(\mathcal{O}_{\Delta}(-r_h)).$$

Therefore, we have to prove that $h^0(\mathcal{O}_{\Delta}(-r_h))=0$. Now, the exact sequence

$$0\to\mathcal{O}_{\tilde{X}}(-s_h)\to\mathcal{O}_{\tilde{X}}(-r_h)\to\mathcal{O}_{\Delta}(-r_h)\to 0,$$

induces the cohomological long exact sequence

$$\begin{array}{l} 0\longrightarrow H^0(\mathcal{O}_{\tilde{X}}(-s_h))\longrightarrow H^0(\mathcal{O}_{\tilde{X}}(-r_h))\longrightarrow H^0(\mathcal{O}_{\Delta}(-r_h))\longrightarrow \\ \longrightarrow H^1(\mathcal{O}_{\tilde{X}}(-s_h))\longrightarrow H^1(\mathcal{O}_{\tilde{X}}(-r_h))\longrightarrow H^1(\mathcal{O}_{\Delta}(-r_h))\longrightarrow 0 \end{array} \tag{10}$$

Since $s(r_h)=s_h$, by Lemma 2.8 (taking $\mathcal{L}=\mathcal{O}_{\tilde{X}}(-r_h)$) we get $H^0(\mathcal{O}_{\tilde{X}}(-s_h))\xrightarrow{\cong}H^0(\mathcal{O}_{\tilde{X}}(-r_h))$. By Lipman's vanishing theorem 2.7, we get $h^1(\mathcal{O}_{\tilde{X}}(-s_h))=0$. Therefore, by (10) we get

$$h^0(\mathcal{O}_{\Delta}(-r_h))=0. \tag{11}$$

This proves the $H^1(\mathcal{O}_{\tilde{X}}(s_h+K_{\tilde{X}}))=0$. Therefore, $\mathcal{O}_{\tilde{X}}(-s_h)$ is a full sheaf.

(2) Let $\mathcal{L}\in\mathrm{Pic}(\tilde{X})$ be a full sheaf. Now we prove that its first Chern class $c_1(\mathcal{L})$ is equal to $-s_h$ for some $h\in H$. Since X is a rational singularity, $\mathcal{L}$ is generated by global sections, hence $(c_1(\mathcal{L}),E_v)\geq 0$ for any $v\in V$. Thus, $\ell':=-c_1(\mathcal{L})\in\mathcal{S}'$.

Set $h:=[\ell']\in H=L'/L$. By the properties of the Lipman cone reviewed in the preliminaries, there exists a unique minimal element $s_h\in\mathcal{S}'_h$ such that $[s_h]=h$. Therefore, $s_h\leq\ell'$. Denote by $\delta:=\ell'-s_h$. If $\delta=0$, we are done. Indeed, $s_h=\ell'$ implies that $\mathcal{L}=\mathcal{O}_{\tilde{X}}(-s_h)$ (recall that X is a rational singularity, thus $\mathrm{Pic}^0(\tilde{X})=0$). Suppose that $\delta>0$. Consider the exact sequence

$$0\to\mathcal{O}_{\tilde{X}}(K_{\tilde{X}}+s_h)\to\mathcal{O}_{\tilde{X}}(K_{\tilde{X}}+\ell')\to\mathcal{O}_{\delta}(K_{\tilde{X}}+\ell')\to 0. \tag{12}$$

Since $\mathcal{L}$ is full, $H^1(\mathcal{O}_{\tilde{X}}(K_{\tilde{X}}+\ell'))=0$. Thus, by the induced cohomological long exact sequence

$$H^1(\mathcal{O}_\delta(K_{\tilde{X}}+\ell'))=0.$$

By Serre duality

$$H^1(\mathcal{O}_\delta(K_{\tilde{X}}+\ell'))=H^0(\mathcal{O}_\delta(-\ell'+\delta))=H^0(\mathcal{O}_\delta(-s_h)),$$

hence

$$h^0(\mathcal{O}_\delta(-s_h))=0. \tag{13}$$

On the other hand, from the exact sequence

$$0\to\mathcal{O}_{\tilde{X}}(-\ell')\to\mathcal{O}_{\tilde{X}}(-s_h)\to\mathcal{O}_\delta(-s_h)\to 0,$$

and $h^1(\mathcal{O}_{\tilde{X}}(-s_h))=0$ by Lipman's vanishing theorem, we obtain

$$h^1(\mathcal{O}_\delta(-s_h))=0. \tag{14}$$

In particular, (13) and (14) provide

$$\chi(\mathcal{O}_\delta(-s_h))=0.$$

This by Riemann-Roch theorem reads as $0=\chi(\mathcal{O}_\delta(-s_h))=\chi(\delta)-(\delta,s_h)$. Since (X,x) is rational and $\delta>0$, by Artin's criterion $\chi(\delta)>0$. Furthermore, since $s_h\in\mathcal{S}'$, we also obtain $-(\delta,s_h)\geq 0$. Therefore, $0=\chi(\delta)-(\delta,s_h)>0$. This is a contradiction. Therefore, δ must be equal to zero. □

Remark 3.2. *Above we proved both parts (1) and (2) independently of each others based on cohomological properties of rational singularities. In this way we wished to creat prototypical proofs for both directions, which might be useful in the context of more general singularities.*

However, using specifically the properties of full sheaves, it turns out that parts (1) and (2) are equivalent, once one of them is proved the other one follows 'automatically'.

Indeed, let us verify (2)⇒(1). Consider the sheaf $\mathcal{L}=\mathcal{O}_{\tilde{X}}(-s_h)$. *Set* $M:=\pi_*(\mathcal{L})\in\mathrm{Cl}(X,x)$ *and* $\mathcal{M}:=(\pi^*M)^{\vee\vee}$. *Then* $\mathcal{M}$ *is full. On the other hand,* $\pi_*(\mathcal{L})=\pi_*(\mathcal{M})$, *hence (by diagram (4))* $\mathcal{L}=\mathcal{M}(\ell)$ *for some* $\ell\in L$. *But* $c_1(\mathcal{L})=c_1(\mathcal{M})+\ell$, *and by (2)* $c_1(\mathcal{M})=-s_{h'}$ *for some* $h'\in H$. *Hence,* $-s_h=-s_{h'}+\ell$, *which implies* $h=h'$ *and* $\ell=0$. *Hence* $\mathcal{L}=\mathcal{M}$ *is full.*

For (1)⇒(2), set $\mathcal{L}$ *a full sheaf. If we set* $[c_1(\mathcal{L})]=-h$, *then both* $\mathcal{L}$ *and* $\mathcal{O}_{\tilde{X}}(-s_h)$ *are full and lift the very same reflexive sheaf* $(\bar{c}_1)^{-1}(-h)$. *Hence they coincide.*

In the minimally elliptic case we will prove only the analogue of part (2), and then the classification follows by a similar argument as the proof of (2)⇒(1).

Remark 3.3. *By definition, there is a one to one correspondence between full sheaves over* $\tilde{X}$ *and reflexive modules over* X. *Since the set of reflexive modules over* X *is independent of the resolution, it is natural to ask the independence of the resolution of the classification of Theorem 3.1. Let us verify this fact directly in the next discussion.*

To verify the naturalness of the sheaves of the theorem we need some notation. Let (X,x) be a rational normal surface singularity. Let $\pi\colon\tilde{X}\to X$ be any resolution. Let $\sigma\colon\tilde{X}_0\to\tilde{X}$ the blow-up at some point $p\in E\subset\tilde{X}$. Denote by

$$L(\tilde{X}):=H_2(\tilde{X},\mathbb{Z}),\qquad L'(\tilde{X}):=H_2(\tilde{X},\partial\tilde{X},\mathbb{Z})\quad\text{and}\qquad H(\tilde{X}):=L'(\tilde{X})/L(\tilde{X}),$$
$$L(\tilde{X}_0):=H_2(\tilde{X}_0,\mathbb{Z}),\qquad L'(\tilde{X}_0):=H_2(\tilde{X}_0,\partial\tilde{X}_0,\mathbb{Z})\quad\text{and}\qquad H(\tilde{X}_0):=L'(\tilde{X}_0)/L(\tilde{X}_0).$$

The map σ induces, via the cohomological pullback and the duality $H_2(\tilde{X},\partial\tilde{X},\mathbb{Z})\cong H^2(\tilde{X},\mathbb{Z})$, the following maps $\sigma^*\colon L'(\tilde{X})\to L'(\tilde{X}_0)$ and $\sigma^*\colon L(\tilde{X})\to L(\tilde{X}_0)$. They induce an isomorphism $\sigma^*\colon H(\tilde{X})\to H(\tilde{X}_0)$.

Proposition 3.4. *Consider the cycles $s_{h,\tilde{X}}$ and $s_{h,\tilde{X}_0}$ in the corresponding resolutions. Set*

$$\begin{aligned}\mathrm{Full}^1(\tilde{X}) &:= \left\{\mathcal{O}_{\tilde{X}}(-s_{h,\tilde{X}})\in \mathrm{Pic}(\tilde{X}) \text{ with } h\in H(\tilde{X})\right\},\\ \mathrm{Full}^1(\tilde{X}_0) &:= \left\{\mathcal{O}_{\tilde{X}_0}(-s_{h,\tilde{X}_0})\in \mathrm{Pic}(\tilde{X}_0) \text{ with } h\in H(\tilde{X}_0)\right\}.\end{aligned}$$

Then, the map σ induces the bijection $\sigma^*\colon \mathrm{Full}^1(\tilde{X})\to\mathrm{Full}^1(\tilde{X}_0)$. Furthermore, the induced map by σ satisfies

$$\sigma^*\mathcal{O}_{\tilde{X}}(-s_{h,\tilde{X}})=\mathcal{O}_{\tilde{X}_0}(-s_{\sigma^*(h),\tilde{X}_0}).$$

Proof. The verification of the statements is fairly immediate, however, we indicate the main point to emphasize the differences between the rational and minimally elliptic case (cf. Remark 4.7(c)).

Since X has a *rational singularity*, by Proposition 2.13 *any full sheaf is generated by its global sections*. Therefore by Proposition 2.16 the map σ induces the following bijection

$$\sigma^*\colon \mathrm{Full}^1(\tilde{X})\to\mathrm{Full}^1(\tilde{X}_0) \tag{15}$$

hence one has the commutative diagram

$$\begin{array}{ccc}\mathrm{Full}^1(\tilde{X}_0) & \longrightarrow & H(\tilde{X}_0)\\ \uparrow \sigma^* & \circlearrowleft & \uparrow\sigma^*\\ \mathrm{Full}^1(\tilde{X}) & \longrightarrow & H(\tilde{X})\end{array} \tag{16}$$

□

Remark 3.5. *Let (X,x) be a rational normal surface singularity. By Theorem 3.1 all the rank one full sheaves are classified by the group H. Since $H=H^2(\Sigma,\mathbb{Z})$, where Σ is the link of (X,x), H is a topological invariant. In particular, Theorem 3.1 provides a topological classification. Thus, the set of full sheaves of rank one only depends on the topology of (X,x).*

The following theorem makes Remark 3.5 even more precise.

Theorem 3.6. *Let (X,x) be a rational normal surface singularity. Let $\pi\colon\tilde{X}\to X$ be any resolution. Then, any full $\mathcal{O}_{\tilde{X}}$-sheaf of rank one is flat.*

Proof. By Lemma 2.9, the following diagram commutes

$$\begin{array}{ccc}\mathrm{Cl}(X,x) & \xrightarrow{\bar{c}_1} & H\\ \uparrow\Psi & & \uparrow =\\ H^1(\Sigma,\mathbb{C}/\mathbb{Z}) & \xrightarrow{\Phi} & H^2(\Sigma,\mathbb{Z})\end{array}$$

Since (X,x) is a rational singularity, by (7) and (4) the maps Φ and $\bar{c}_1$ are isomorphisms. Hence, the map Ψ is also an isomorphism. □

Remark 3.7. *(1) The computation of the set $\{s_h\}_{h\in H}$, even for very concrete families of topological types of singularities (families of links) in general is not easy. Recall that there exists an algorithm which provides s_h, cf. 2.5, however a closed expression in general is missing. In special cases the known formulae are surprisingly technical and arithmetical. For the cyclic quotient (string graphs) and star shaped graphs see [23], for surgery 3-manifolds see [24, 27]. (All these are reported in [26] as well.) For general rational singularities a concrete closed formula is not known (by the authors).*

(2) In [30] Wunram determined the first Chern classes of full sheaves associated with the minimal resolution of cyclic quotient singularities, as cycles in L'. These expressions can be identified with

the expressions of $\{s_h\}_h$ given in [23]. We emphasize that in [30] the universal property of this Chern class, as the minimal element of the Lipman cone with fixed $h \in H$, was not recognized. We believe that this new conceptual point of view brings essentially new perspectives in the theory (besides the generalization of full sheaves to the rational and minimally elliptic cases).

(3) By the McKay's correspondence in the ADE case, non-trivial full shaves (of any rank) are in bijection with the irreducible exceptional curves of the minimal resolution: the Chern classes are of type $\{-E^*_v\}_{v\in V}$ (where each $-E^*_v$ can be represented, as a divisor, by a cut of E_v). The rank of $\mathcal{M}$ with $c_1(\mathcal{M}) = -E^*_v$ is the E_v-multiplicity m_v of Z_{min}. Hence, the rank one non-trivial full sheaves correspond to those components E_v with $m_v = 1$. On the other hand, for other, more general singularities, this correspondence between $\{E_v\}_v$ and the full shaves is broken. (This happens in our rational case too: $\{E_v\}_{v\in V}$ versus $\{s_h\}_{h\in H}$.) In order to keep (at least) part of this correspondence $\{E_v\}_{v\in V} \leftrightarrow \mathrm{Ref}_X$, Wunram in [30] introduced the family of special full shaves (they will be discussed below).

Definition 3.8. *Let (X, x) be a rational normal surface singularity and $\tilde{X} \to X$ a resolution. A full sheaf $\mathcal{M} \in \mathrm{Pic}(\tilde{X})$ is called special if $H^1(\tilde{X}, \mathcal{M}^*) = 0$. Then $\pi_*\mathcal{M}$ is called a special reflexive module.*

For rational (X, x) the trivial sheaf $\mathcal{O}_{\tilde{X}}$ is full and special. Wunram in [30, Theorem 1.2(b)] proved the following fact.

Proposition 3.9. *Let $\tilde{X}$ be the minimal resolution of a rational singularity. Then special non-trivial indecomposable reflexive modules (i.e. special full sheaves of $\tilde{X}$) correspond bijectively with irreducible components $\{E_v\}_{v\in V}$. By this correspondence $\mathcal{M}_v \leftrightarrow E_v$, realized by $c_1(\mathcal{M}_v) = -E^*_v \in L'$, one also has $\mathrm{rank}(\mathcal{M}_v) = m_v$ (the E_v-coeffcient of Z_{min}).*

Next, we combine the statements of Theorem 3.1 and Proposition 3.9.

Theorem 3.10. *Let (X, x) be a rational singularity with a fixed resolution $\tilde{X}$. Then, for some s_h with $h \neq 0$, the following facts are equivalent:*

(1) The full sheaf $\mathcal{L} = \mathcal{O}_{\tilde{X}}(-s_h)$ ($h \neq 0$) is special,

(2) $(-s_h, Z_{min}) = 1$,

(3) $s_h = E^*_v$ for some $v \in V$ and $m_v = 1$.

Proof. *(2)* and *(3)* are obviously equivalent. Indeed, write $s_h \neq 0$ as $\sum_v n_v E^*_v$ with certain $n_v \in \mathbb{Z}_{\geq 0}$. Since $Z_{min} = \sum_v m_v E_v$ is supported on E (i.e. $m_v \in \mathbb{Z}_{>0}$ for any v), $(-s_h, Z_{min}) = \sum_v n_v m_v$.

For the equivalence *(1)*⇔*(3)* we provide two proofs. The first one is analytic, and basically it follows from Proposition 3.9, whenever the resolution is minimal. For a non-minimal resolution one can prove the stability of the statements with respect to a blow-up.

However, we present another topological/combinatorial proof as well, based on the combinatorics of rational graphs and (generalized) Laufer sequences (valid in any resolution). It shows some additional topological bridges, and it can serve as a prototype for further generalizations as well.

The combinatorial proof of (1)⇒(2). If $\mathcal{L} = \mathcal{O}_{\tilde{X}}(-s_h)$ is special then $h^1(\mathcal{L}^*) = 0$. Then we apply Lemma 2.8*(b)* applied for $c = -s_h$ and the bundle $\mathcal{L}^*$.

Since $s_h \neq 0$ and $(,)$ is negative definite, there exists some $E_{v(0)}$ such that $(-s_h, E_{v(0)}) > 0$.

Then, consider the computation sequence $\{z_i\}_{i=1}^t$ which connects $E_{v(0)}$ with Z_{min}. Write also $z_0 = 0$. Since (X, x) is rational, by Laufer's rationality criterion, $(z_i, E_{v_i}) = 1$ for all $0 < i < t$.

Next, consider the sequence $x_i := -s_h + z_i$ for all such $i \geq 0$. It connects $-s_h$ with $-s_h + Z_{min}$. The point is that at each intermediate step $(x_i, E_{v_i}) = (-s_h, E_{v_i}) + (z_i, E_{v_i}) \geq 1$ for $i > 0$. Therefore,

$\{x_i\}_{i\geq 0}$ can be considered as the beginning of a computation sequence, which computes $h^1(\mathcal{L}^*)$ in Lemma 2.8*(b)*. But since $h^1(\mathcal{L}^*)=0$, we must have $\sum_i h^1(\mathcal{O}_{\mathbb{P}^1}(-x_i,E_{v_i}))=0$ for this (partial) sequence, hence $(x_i,E_{v_i})=1$ for all $i\geq 0$. For $i=0$ this reads as $(-s_h,E_{v(0)})=1$ and for all other $i>0$ we must have $(-s_h,E_{v_i})=0$. Since $\sum_i E_{v_i}=Z_{min}$, summation over i gives $(-s_h,Z_{min})=1$.

The combinatorial proof (3)⇒(1) Consider again the computation sequence $\{x_i\}_i$ which connects $c=-s_h$ with $s(-s_h)$. Note that $x_0=-s_h$, and (since $s_h=E^*_v$) the next step is necessarily $x_1=-s_h+E_v$, since E_v is the only base element E_u with $(-s_h,E_u)>0$. Using Lemma 2.8*(b)* $h^1(\mathcal{O}_{\tilde{X}}(s_h))=\sum_i((x_i,E_{v_i})-1)$. Hence we have to show that $(x_i,E_{v_i})=1$ for any i.

As an independent construction, let Γ^e_v be the 'extended' graph constructed as follows. Let Γ be the dual resolution graph of the resolution $\tilde{X}\to X$. Then Γ^e_v consists of Γ and a new vertex v^e, which is glued to v (the vertex which appears in *(3)*) by an edge. Let k be the Euler number of v^e. One sees that if $k\ll 0$ then Γ^e_v is negative definite. Furthermore, in such a case $k\ll 0$, the E_{v^e}-multiplicity m^e of E_{v^e} in $Z_{min}(\Gamma^e_v)$ is one (for details see e.g. [10, Th. 4.1.3]). On the other hand, by [10, Th. 4.1.3], Γ^e_v is rational if and only if m_v (the E_v coefficient of $Z_{min}=Z_{min}(\Gamma)$) is one. Since this appears as an assumption in *(3)* we get that Γ^e_v is rational. Let us consider a computation sequence for $Z_{min}(\Gamma^e_v)$. For the first step we choose E_{v^e}. Then we continue by the Laufer algorithm. (Hence the next added term is necessarily E_v.) In this way we obtain a series $\{y_j\}_j$. By the above discussion (since $k\ll 0$ and $m^e=1$), E_{v^e} will be not chosen again.

Let us compare the two computation sequences after we rewrite them as $x_i=-s_h+z_i=-E^*_v+z_i$ and $y_i=E_{v^e}+\bar{z}_i$. By comparing the two algorithms (and the terms) we see that we can make along the algorithms the common choices $z_i=\bar{z}_i$ (and both sequences end simultaneously). Moreover, the terms $(x_i,E_{v_i})=(y_i,E_{v_i})$ are also equal. Now let us repeat what we get. By [10, Th. 4.1.3], $m_v=1$ implies that Γ^e_v is rational. By Laufer criterion of rationality of this graph, $(y_i,E_{v_i})=1$ for every i, hence by the coincidence of the two sequences $(x_i,E_{v_i})=1$ too. Hence $h^1(\mathcal{O}_{\tilde{X}}(s_h))=0$ by Lemma 2.8*(b)*. Hence $\mathcal{O}_{\tilde{X}}(-s_h)$ is special.

[Though in the proof the next additional info is not visible, it might help the reader. Along the steps of the sequence $\{x_i\}_i$ and $\{y_i\}_i$ we accumulate $\sum_i E_{v_i}=s(-s_h)+s_h=Z_{min}(\Gamma^e_v)-E_{v^e}$. In fact, it can be shown that this is a sum of Artin fundamental cycles of different support in Γ. The first one is exactly Z_{min}, this first subsequence was considered in the proof of *(1)⇒(2)*. The tower of fundamental cycles is explicitly described in the proof of [10, Th. 4.1.3]. See also [18].] □

Theorem 3.10 was formulated from the perspective of the Chern class $-s_h$. In the next statement we reformulate it as an equivalence from the perspective of the exceptional divisors of the resolution. (Maybe it is worth to emphasize that for a graph it can happen that $s_{[E^*_v]}=s_{[E^*_u]}=E^*_v$ for some $v\neq u$, $v,u\in V$, see e.g. Example 3.12.) Note that the condition $s_h=E^*_v$ in part *(3)* says that $h=[E^*_v]$ and $s_{[E^*_v]}=E^*_v$. Hence, *(3)* reads as $s_{[E^*_v]}=E^*_v$ and $m_v=1$ (with $h=[E^*_v]$). A non-obvious point is that in the minimal resolution the condition $m_v=1$ by oneself already guarantees $s_{[E^*_v]}=E^*_v$.

Corollary 3.11. *Let (X,x) be a rational singularity and let $\tilde{X}$ be its minimal resolution. For a fixed vertex $v\in V$ the following facts are equivalent:*

(1) Γ^e_v is rational,

(2) $m_v=1$,

(3) $s_{[E^*_v]}=E^*_v$ and $m_v=1$,

(4) the sheaf $\mathcal{L}=\mathcal{O}_{\tilde{X}}(-E^*_v)$ is non-trivial special full.

If any of these conditions hold then $[E^*_v]\neq 0$.

Hence, the rank one non-trivial special full sheaves are classified in the minimal resolution by vertices with $m_v = 1$.

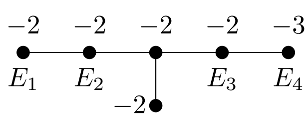
Proof. *(1)⇔(2)* follows from [10, Th. 4.1.3] or [18]. Let us verify that if any of the above conditions hold then $[E_v^*] \neq 0$, hence Theorem 3.10 can be applied. Indeed, assume e.g. that $m_v = 1$. Then by Proposition 3.9 we get that $\mathcal{O}_{\tilde X}(-E_v^*)$ is full. Assume that $[E_v^*] = 0$, that is, $E_v^* \in L_{>0}$. Since $\mathcal{O}_{\tilde X}(-E_v^*)$ is full, $H^1(\tilde X, \mathcal{O}_{\tilde X}(K_{\tilde X} + E_v^*)) = 0$, hence $H^1(E_v^*, \mathcal{O}_{E_v^*}(K_{\tilde X} + E_v^*)) = 0$ too, which by Serre duality implies $H^0(\mathcal{O}_{E_v^*}) = 0$, which cannot hold since $H^0(\mathcal{O}_{E_v^*})$ contains at least the constants.

Hence, for all the cases we can assume that $[E_v^*] \neq 0$. Then *(3)⇔(4)* follows from Theorem 3.10, and *(2)⇒(4)* from Proposition 3.9 and *(3)⇒(2)* is obvious. □

It is interesting to compare (directly) *(1)* and *(4)*. Their equivalence relates a (rational) surgery property of the link with special reflexive modules of the singularity.

In an arbitrary (non-minimal) resolution *(3)⇔(4)* still holds (since $[E_v^*] \neq 0$ and Theorem 3.10 remain valid). However *(2)⇒(3)* and Proposition 3.9 do not necessarily hold, see Example 3.12(e) when we create a redundant vertex with $m_v = 1$. In non-minimal resolutions *(3)* is the right index set for the classification.

Example 3.12. *Consider the singularity E_{12} from the Riemenschneider list [28] with the following resolution graph Γ. By a computation $H = \mathbb{Z}_7$. The universal abelian covering (with Galois group H) is the Poincaré sphere $\Sigma(2,3,5)$ with finite fundamental group, hence Γ is the graph of a quotient singularity.*



We have the following cases regarding the different irreducible exceptional curves.

(a) $s_{[E_1^*]} = E_1^*$ and $m_1 = 1$, hence with Chern class $-E_1^*$ there exists a rank one special full sheaf.

(b) $s_{[E_4^*]} = r_{[E_4^*]} = E_4^*$ and $m_4 = 1$, hence with Chern class $-E_4^*$ there exists a rank one special full sheaf.

(c) $s_{[E_2^*]} = s_{[E_4^*]} = E_4^*$, hence $s_{[E_2^*]} \neq E_2^*$. In fact, E_2^* does not equal with any s_h ($h \in H$). Note also that $m_2 = 2$. Therefore, with Chern class $-E_2^*$ there exists no rank one full sheaf. However, with this Chern class there exists an indecomposable rank 2 special full sheaf.

(d) $s_{[E_3^*]} = E_3^*$ and $m_3 = 2$. Therefore, with Chern class $-E_3^*$ there exists a rank one non-special full sheaf, and an indecomposable rank 2 special full sheaf. (Compare with [30, Example 2] as well.)

(e) Let us blow up $\tilde X$ at a generic point of E_4, and let E_{new} be the new exceptional divisor in $\tilde X_{new}$. Then in the new graph $m_{new} = 1$, however $s_{[E_{new}^*]} = E_4^* \neq E_{new}^*$. That is, $\mathcal{O}_{\tilde X_{new}}(-E_{new}^*)$ is not full.

4. Rank one full sheaves of minimally elliptic singularities

Let (X, x) denote a minimally elliptic singularity. Recall that in any resolution $Z_K \in L$. As usual, C denotes the elliptic cycle.

First, we characterize the full shaves of rank one with non-trivial first Chern class.

Theorem 4.1. *Let (X, x) be a minimally elliptic singularity. Let $\pi\colon \tilde X \to X$ be any resolution such that the support $|C|$ of C is E (this happens e.g. in the minimal resolution, or even in the minimal* *good resolution, cf. [26, p. 300]). If $\mathcal{L}$ is full sheaf of rank one such that $[c_1(\mathcal{L})]$ is non-zero, then $c_1(\mathcal{L}) = -s_h$ for some $h \in H$, $h \neq 0$.*

Proof. The proof follows the same strategy as the proof of Theorem 3.1, however, in this case the package of results that we use from singularity theory should be valid for minimally elliptic germs, and even certain steps should be modified.

Let $\mathcal{L}$ be a full sheaf of rank one such that $[c_1(\mathcal{L})]$ is non-zero, hence $c_1(\mathcal{L})$ is also different from zero. By Proposition 2.12, the sheaf $\mathcal{L}$ is generically generated by global sections. Therefore, the sheaf does not have any fixed component. Thus, $\ell' := -c_1(\mathcal{L}) \in \mathcal{S}'$. Moreover, since $|C| = E$ and ℓ' is a non-trivial element of $\mathcal{S}'$, by [26, Theorem 7.2.31] we get $H^1(\tilde{X}, \mathcal{L}) = 0$. By Proposition 2.12, the natural map from $H^1_E(\tilde{X}, \mathcal{L})$ to $H^1(\tilde{X}, \mathcal{L})$ is injective. But $H^1(\tilde{X}, \mathcal{L}) = 0$, therefore

$$H^1_E(\tilde{X}, \mathcal{L}) = 0. \tag{17}$$

Set $h := [\ell'] \in H = L'/L$. By the special property of the Lipman cone (cf. preliminaries), there exists a unique minimal element $s_h \in \mathcal{S}'$ such that $[s_h] = h$. Therefore, $s_h \leq \ell'$. Denote by $\delta := \ell' - s_h \in L_{\geq 0}$. We have to prove that $\delta = 0$. Suppose that $0 < \delta$. Consider the exact sequence

$$0 \to \mathcal{L}^{\vee}(K_{\tilde{X}}) \otimes \mathcal{O}_{\tilde{X}}(-\delta) \to \mathcal{L}^{\vee}(K_{\tilde{X}}) \to \mathcal{L}^{\vee}(K_{\tilde{X}}) \otimes \mathcal{O}_{\delta} \to 0. \tag{18}$$

By Serre duality and (17)

$$H^1(\tilde{X}, \mathcal{L}^{\vee}(K_{\tilde{X}})) \cong H^1_E(\tilde{X}, \mathcal{L}) = 0. \tag{19}$$

By (19) and considering the long exact sequence of cohomology associated to (18) we obtain

$$H^1(\tilde{X}, \mathcal{L}^{\vee}(K_{\tilde{X}}) \otimes \mathcal{O}_{\delta}) = 0$$

By Serre duality again

$$H^0(\delta, \mathcal{L}(\delta)) = 0. \tag{20}$$

On the other hand, one can consider the following exact sequence as well:

$$0 \to \mathcal{L} \to \mathcal{L}(\delta) \to \mathcal{L}(\delta) \otimes \mathcal{O}_{\delta} \to 0. \tag{21}$$

Since $-c_1(\mathcal{L}(\delta)) = \ell' - \delta = s_h \in \mathcal{S}'$, by [26, Theorem 7.2.31] we get $H^1(\tilde{X}, \mathcal{L}(\delta)) = 0$. Therefore, from the cohomological long exact sequence associated with (21) and the previous vanishing, we get

$$H^1(\delta, \mathcal{L}(\delta)) = 0. \tag{22}$$

In particular, (20) and (22) provide

$$\chi(\mathcal{L}(\delta) \otimes \mathcal{O}_{\delta}) = 0. \tag{23}$$

By Riemann-Roch theorem

$$\chi(\mathcal{L}(\delta) \otimes \mathcal{O}_{\delta}) = \chi(\delta) - (\delta, s_h) = 0. \tag{24}$$

Since (X, x) is elliptic, we have $\min_{Z>0} \chi(Z) = 0$. Therefore, $0 \leq \chi(\delta)$. Since $s_h \in \mathcal{S}'$, then $0 \leq -(\delta, s_h)$. By the previous inequalities and by (24) we get

$$\chi(\delta) = 0 \quad \text{and} \quad (\delta, s_h) = 0. \tag{25}$$

By assumption $\delta \neq 0$. Since $\chi(\delta) = 0$, we get $C \leq \delta$. By hypothesis, the support of C is E. Therefore, the support of δ is E, hence $(\delta, s_h) \neq 0$. This is a contradiction to the second equality of (25). Thus, $\delta = 0$, $-c_1(\mathcal{L}) = \ell' = s_h$, where $h = [-c_1(\mathcal{L})] \in H$. This ends the proof. □

The first part of the main classification result is the following theorem.

Theorem 4.2. *Let (X,x) be a minimally elliptic singularity. Let $\pi\colon \tilde{X} \to X$ be any resolution such that $|C| = E$. Then,*

$$\bigcup_{h\in H\setminus\{0\}} \mathrm{Pic}^{-s_h}(\tilde{X}) = \{\textit{rank one full sheaves with } [c_1(\mathcal{L})] \neq 0\} \subset \mathrm{Full}^1(\tilde{X}).$$

Proof. The proof is identical with the proof of *(2)⇒(1)* in Remark 3.2. □

The missing case is $h = 0$, equivalently $s_h = 0$, cf. Remark 2.1. This case is done in the sequel.

4.1. **The case $h = 0$.** In this section we study the families of full shaves with $[c_1(\mathcal{L})] = 0$. This particular case should be treated via a different strategy. First, we have the following characterization of full sheaves of rank one with trivial first Chern class.

Proposition 4.3. *Let (X,x) be the germ of a normal surface singularity. Let $\pi\colon \tilde{X} \to X$ be any resolution. Let $\mathcal{L}$ be a full sheaf of rank one. Then, the following statements are equivalent:*

(1) the sheaf is trivial, i.e., $\mathcal{L} \cong \mathcal{O}_{\tilde{X}}$,

(2) $c_1(\mathcal{L}) = 0$.

Proof. *(1)⇒(2)* follows trivially. Next we prove *(2)⇒(1)*. Since $\mathcal{L}$ is a full sheaf, it has a non-trivial global section s without fixed components. Since $c_1(\mathcal{L}) = 0$, for any $p \in \tilde{X}$ we get $s(p) \neq 0$. Hence, the section s trivializes the line bundle $\mathcal{L}$. □

Consider the following example:

Example 4.4. *Consider the minimally elliptic singularity*

$$X = \{x^2 + y^3 + z^7 = 0\} \subset \mathbb{C}^3.$$

By [23, p. 7] its link Σ it is an integer homology sphere. Hence, $H = H^2(\Sigma, \mathbb{Z}) = 0$. Denote by $\pi\colon \tilde{X} \to X$ its minimal resolution. Since $H = 0$, for any $\mathcal{L} \in \mathrm{Pic}(\tilde{X})$ we get $[c_1(\mathcal{L})] = 0$. Nevertheless, by [6] there exists non-trivial non-flat full sheaves of rank one.

By Proposition 4.3 we need to study the full sheaves with non-trivial first Chern class but with trivial class in H. By Example 4.4 such sheaves exist.

Let (X,x) be a minimally elliptic singularity and $\pi\colon \tilde{X} \to X$ any resolution such that $|C| = E$. Since Z_{min} is integral, $\chi(Z_{min}) = 0$ and C is the minimally elliptic cycle, we have (see [22, 26])

$$C = Z_K \leq Z_{min}.$$

Theorem 4.5. *(a) Let (X,x) be a minimally elliptic singularity. Let $\pi\colon \tilde{X} \to X$ be any resolution such that $|C| = E$. Let $\mathcal{L}$ be a full sheaf of rank one such that $[c_1(\mathcal{L})] = 0$ and $c_1(\mathcal{L}) \neq 0$. Then $\mathcal{L} \in \mathrm{Pic}^{-Z_{min}}(\tilde{X})$.*

(b) Conversely, any $\mathcal{L} \in \mathrm{Pic}^{-Z_{min}}(\tilde{X}) \setminus \{\mathcal{O}_{\tilde{X}}(-Z_{min})\}$ is full.

Proof. *(a)* Let $\mathcal{L}$ be a full sheaf of rank one such that $[c_1(\mathcal{L})] = 0$ and $-c_1(\mathcal{L}) \neq 0$. Since $\mathcal{L}$ does not have fixed components then $\ell' := -c_1(\mathcal{L}) \in \mathcal{S}'$. Since $[\ell'] = 0$ and $\ell' \neq 0$, then $\ell' \in \mathcal{S}_0 \setminus \{0\}$. Since $Z_{min} = \min\{\mathcal{S}_0 \setminus \{0\}\}$, we get $Z_{min} \leq \ell'$. Now the proof follows exactly as the proof of Theorem 4.1.

(b) Let $\mathcal{L} \in \mathrm{Pic}^{-Z_{min}}(\tilde{X}) \setminus \{\mathcal{O}_{\tilde{X}}(-Z_{min})\}$. Set $M := \pi_*(\mathcal{L})$ and $\mathcal{M} := (\pi^*M)^{\vee\vee}$. Then $\mathcal{M}$ is full. Also, $\mathcal{L} = \mathcal{M}(\ell)$ for some $\ell \in L$. Taking c_1 we get $-Z_{min} = c_1(\mathcal{M}) + \ell$, hence $[c_1(\mathcal{M})] = 0$. Note that $c_1(\mathcal{M})$ cannot be zero (otherwise we would have $\ell = -Z_{min}$, $\mathcal{M} = \mathcal{O}_{\tilde{X}}$ by Proposition 4.3, hence $\mathcal{L} = \mathcal{O}_{\tilde{X}}(-Z_{min})$ which is not the case). Then, by part *(a)*, $c_1(\mathcal{M}) = -Z_{min}$. This shows that $\ell = 0$ and $\mathcal{L} = \mathcal{M}$ is full. □

Thus, the classification statement of rank one full sheaves is the following.

Theorem 4.6. *Let (X,x) be a minimally elliptic singularity. Let $\pi\colon \tilde{X}\to X$ be any resolution such that $|C|=E$. Then*

$$\mathrm{Full}^1(\tilde{X}) = \left(\bigcup_{h\in H\setminus\{0\}} \mathrm{Pic}^{-s_h}(\tilde{X})\right) \cup \left(\mathrm{Pic}^{-Z_{min}}(\tilde{X})\setminus\{\mathcal{O}_{\tilde{X}}(-Z_{min})\}\right)\cup\{\mathcal{O}_{\tilde{X}}\}.$$

Remark 4.7. *(a) In the minimally elliptic case the classification of rank one full sheaves cannot be described only in terms of first Chern classes.*

(b) Consider the minimally elliptic singularity $X=\{x^2+y^3+z^7=0\}\subset\mathbb{C}^3$. In this case, $H=0$, hence by Theorem 4.6

$$\mathrm{Full}^1(\tilde{X}) = \left(\mathrm{Pic}^{-Z_{min}}(\tilde{X})\setminus\{\mathcal{O}_{\tilde{X}}(-Z_{min})\}\right)\cup\{\mathcal{O}_{\tilde{X}}\}.$$

Since Σ is an integer homology 3-sphere, we get $H^1(\Sigma,\mathbb{C}/\mathbb{Z})=0$. Therefore, the only one dimensional representation of $\pi_1(\Sigma)$ is the trivial one. Thus, $\mathcal{O}_{\tilde{X}}$ is the only flat full sheaf. The non-trivial non-flat full sheaves of rank one constructed in [6] belong to $\mathrm{Pic}^{-Z_{min}}(\tilde{X})\setminus\{\mathcal{O}_{\tilde{X}}(-Z_{min})\}$.

(c) For minimally elliptic singularities we do not prove the analogues of Remark 3.3 and Proposition 3.4, which are valid for rational singularities. The point is that in the case of minimally elliptic singularities the fact that a full sheaf is generated by its global sections (i.e. it has no base points) is not guaranteed automatically, hence Proposition 2.16 cannot be immediately applied, and the bijection (15) might be obstructed. Hence the above characterizations are valid (at least at this moment) in resolutions with $|C|=E$.

5. Flat and non-flat families of reflexive modules

In the first part of this section we classify those reflexive modules on a minimally elliptic singularity that admit flat connections. We fix a resolution with $|C|=E$, in which case the theorems of the previous section can be applied, and there is a one-to-one correspondence between reflexive modules and full sheaves. (This can be e.g. the minimal or the minimal good resolution.)

First we study the case when the link is a rational homology sphere. In this case an interesting situation appears. Note that each $\mathrm{Pic}^{\ell'}(\tilde{X})$ is an affine space, a $\mathrm{Pic}^0(\tilde{X})=\mathbb{C}$ torsor. Hence the space of rank one full sheaves consists of one dimensional families. On the other hand, $H_1(\Sigma,\mathbb{Z})$ is finite, hence the set of flat bundles is discrete: for any $h\in H$ we have to choose exactly one concrete bundle from $\mathrm{Pic}^{-s_h}(\tilde{X})$, which is flat. In the $h=0$ case this is easy, it is the trivial bundle $\mathcal{O}_{\tilde{X}}$. For $h\neq 0$ the choice should be done by a precise principle.

In fact, such a choice is already present in the literature, under the name *natural line bundles*, cf. [25,26]. If $\tilde{X}$ is the resolution of a normal surface singularity with rational homology sphere link then the morphism $c_1:\mathrm{Pic}(\tilde{X})\to L'$ has a section (morphism of groups). For any $\ell'\in L'$ there exists a unique line bundle $\mathcal{L}_{\ell'}\in\mathrm{Pic}^{\ell'}(\tilde{X})$ with the following universal property: if $N\ell'$ is an integral cycle for some $N\in\mathbb{Z}_{>0}$ then $(\mathcal{L}_{\ell'})^{\otimes N}=\mathcal{O}_{\tilde{X}}(N\ell')$. This line bundle will be denoted by $\mathcal{O}_{\tilde{X}}(\ell')$.

Theorem 5.1. *Let (X,x) be a minimally elliptic singularity such that its link is a rational homology sphere. Let $\pi\colon\tilde{X}\to X$ be any resolution with $|C|=E$.*

Corresponding to $h=[c_1(\mathcal{L})]=0$, among the rank one full sheaves

$$\left(\mathrm{Pic}^{-Z_{min}}(\tilde{X})\setminus\{\mathcal{O}_{\tilde{X}}(-Z_{min})\}\right)\cup\{\mathcal{O}_{\tilde{X}}\}$$

only the trivial sheaf $\mathcal{O}_{\tilde{X}}$ admits a flat connection.

Corresponding to $h = [c_1(\mathcal{L})] \neq 0$, *among the rank one full sheaves* $\mathrm{Pic}^{-s_h}(\tilde{X})$ *only* $\mathcal{O}_{\tilde{X}}(-s_h)$ *admits a flat connection.*

Proof. The case $h = 0$ is clear. In the sequel assume that $h \neq 0$. By Theorem 4.2 any element of $\mathrm{Pic}^{-s_h}(\tilde{X})$ is a full sheaf.

By Lemma 2.9, the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(\tilde{X},\mathbb{Z}) & \longrightarrow & H^1(\tilde{X},\mathcal{O}_{\tilde{X}}) & \longrightarrow & \mathrm{Cl}(X,x) & \xrightarrow{\bar{c}_1} & H & \longrightarrow & 0 \\ & & \uparrow = & & & & \uparrow \Psi & \circlearrowleft & \uparrow = & & \\ & & H^1(\Sigma,\mathbb{Z}) & \longrightarrow & H^1(\Sigma,\mathbb{C}) & \longrightarrow & H^1(\Sigma,\mathbb{C}^*) & \xrightarrow{\Phi} & H & \longrightarrow & 0 \end{array}$$

Here $H^1(\Sigma,\mathbb{Z}) = 0$ and $H^1(\tilde{X},\mathcal{O}_{\tilde{X}}) \simeq \mathbb{C}$. Furthermore, $H^1(\Sigma,\mathbb{C}) = 0$ too, hence Φ is an isomorphism. Hence the only flat bundle in $(\bar{c}_1)^{-1}(h) \simeq \mathbb{C}$ is $\Psi(\Phi^{-1}(h))$.

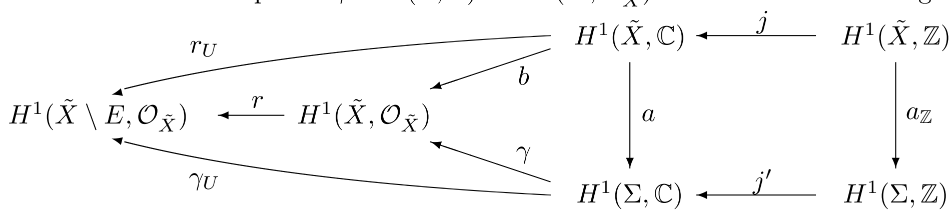
Let us write $\rho_h \in H^1(\Sigma,\mathbb{C}/\mathbb{Z})$ for the representation with $\Phi(\rho_h) = h$. Since H is torsion, there exists $N \in \mathbb{N}$ such that $Nh = 0$. Denote $\rho_h \otimes \rho_h \otimes \cdots \otimes \rho_h$ (the tensor of ρ_h with itself N-times) by $\rho_h^{\otimes N}$. Since $\Phi\left(\rho_h^{\otimes N}\right) = Nh = 0$ and Φ is an isomorphism, we get that $\rho_h^{\otimes N}$ is isomorphic to the trivial representation. Therefore, $\Psi\left(\rho_h^{\otimes N}\right) = \mathcal{O}_X$. Now, set $\mathcal{L}_h := \Psi(\rho_h)$. Therefore, $\mathcal{L}_h^{\otimes N}$ agrees with $\Psi(\rho_h^{\otimes N})$ in $\tilde{X} \setminus E$. Therefore, there exists an integral cycle $\ell \in L$ such that $\mathcal{L}_h^{\otimes N} \cong \mathcal{O}_{\tilde{X}}(\ell)$, i.e. $\mathcal{L}_h$ is a natural line bundle. Thus, in $\mathrm{Pic}^{-s_h}(\tilde{X})$ the flat bundles are the natural line bundles. $\square$

Finally, we treat the cusp singularities. They are defined by the property that their minimal resolution graph is a cyclic (loop) with all $g(E_v) = 0$ [17,22].

Theorem 5.2. *Let* (X,x) *be a cusp singularity. Then, any reflexive module of* (X,x) *of rank one admits a flat connection. In fact,* $\Psi : H^1(\Sigma,\mathbb{C}^*) \to \mathrm{Cl}(X,x)$ *is an isomorphism.*

Proof. The proof uses two facts: the natural map $H^1(\tilde{X},\mathbb{C}) \to H^1(\tilde{X},\mathcal{O}_{\tilde{X}})$ is an isomorphism, and that the restriction map $M \mapsto M|_{X\setminus\{x\}}$ (where $M \in \mathrm{Cl}(X,x)$) is injective (hence several properties can be verified at $U = X \setminus \{x\} = \tilde{X} \setminus E$ level).

First we construct a morphism $\gamma : H^1(\Sigma,\mathbb{C}) \to H^1(\tilde{X},\mathcal{O}_{\tilde{X}})$. Consider the following diagram:



In this diagram all the maps except γ are natural and they satisfy $r_U = r\,b = \gamma_U\,a$ and $a\,j = j'\,a_{\mathbb{Z}}$.

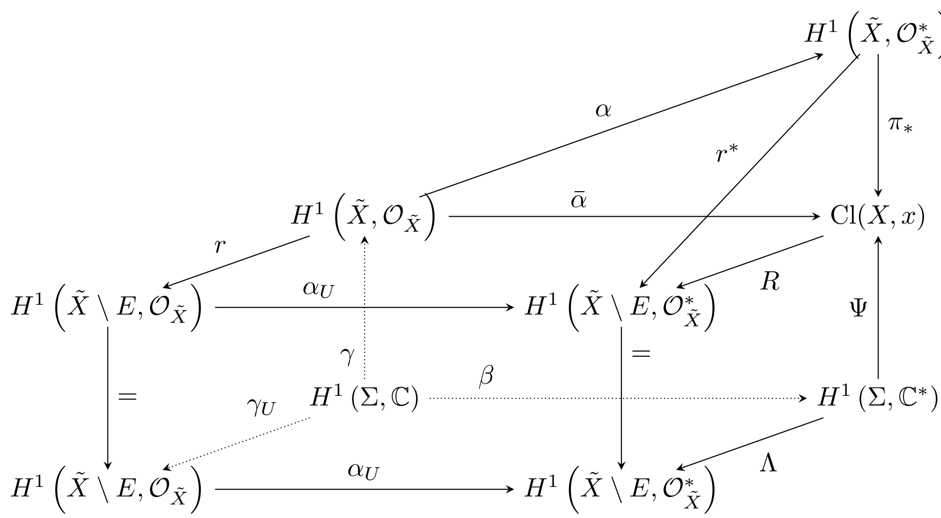
Since (X,x) is a cusp, $H^1(\tilde{X},\mathcal{O}_{\tilde{X}}) \simeq H^1(\tilde{X},\mathbb{C}) \simeq H^1(\Sigma,\mathbb{C}) \simeq \mathbb{C}$, and the morphisms a and b are isomorphisms. We define γ as the composition $b\,a^{-1}$. Clearly, γ is an isomorphism. Moreover, one also has (†) $\gamma_U = r\,\gamma$. Indeed, $r\,\gamma = r\,b\,a^{-1} = r_U\,a^{-1} = \gamma_U$.

Then we consider the following diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^1(\tilde{X},\mathbb{Z}) & \xrightarrow{b\,j} & H^1(\tilde{X},\mathcal{O}_{\tilde{X}}) & \longrightarrow & \mathrm{Cl}(X,x) & \xrightarrow{\bar{c}_1} & H & \longrightarrow & 0 \\ & & \uparrow a_{\mathbb{Z}}^{-1} & & \uparrow \gamma & & \uparrow \Psi & \circlearrowleft & \uparrow = & & \\ 0 & \longrightarrow & H^1(\Sigma,\mathbb{Z}) & \xrightarrow{j'} & H^1(\Sigma,\mathbb{C}) & \longrightarrow & H^1(\Sigma,\mathbb{C}^*) & \xrightarrow{\Phi} & H & \longrightarrow & 0 \end{array} \tag{26}$$

The third 'box' is commutative by Lemma 2.9 (cf. Appendix). The commutativity of the first 'box' follows from the definition of γ and the relation $b\,j = \gamma\,j'\,a_{\mathbb{Z}}$ from the previous diagram.

Next, we wish to verify that the second 'box' commutes. For this, consider the more detailed



Here α, α_U, r and r^* are natural with $r^*\alpha = \alpha_U r$. The morphism π_* is the factorization by L (cf. diagram (4)), and since $r^*(L) = 0$, the morphism R is well–defined with $R\pi_* = r^*$. This R is nothing else then the restriction of a module to $U = \tilde X \setminus E \simeq X\setminus\{x\}$. Hence if we define Λ as $R\Psi$, then this map associates to a representation the corresponding flat bundle on $U = \tilde X\setminus E \simeq X\setminus\{x\}$. Hence one also has $R\overline{\alpha} = \alpha_U r$ and $\Lambda\beta = \alpha_U\gamma_U$. We also set $\overline{\alpha} := \pi_*\alpha$.

Note that R is injective. Indeed, for any module $M \in \mathrm{Cl}(X,x)$ we have $i_*(R(M)) = M$ (where $i \in U \to X$ is the inclusion). This is a key property of any reflexive sheaf, see [13].

Now, we wish to prove the second commutativity in (26), namely $\overline{\alpha}\gamma = \Psi\beta$. Since R is injective, it is enough to verify that $R\overline{\alpha}\gamma = R\Psi\beta$. But, indeed, $R\Psi\beta = \Lambda\beta = \alpha_U\gamma_U \overset{(\dagger)}{=} \alpha_U r\gamma = R\overline{\alpha}\gamma$.

Hence the diagram (26) is commutative with $a_{\mathbb{Z}}$ and γ isomorphisms, hence clearly Ψ is also an isomorphism, and any element of $\mathrm{Cl}(X,x)$ is flat. □

Let us fix now a normal surface singularity and its *minimal good resolution.* We say that (X,x) is (numerically) log-canonical if in this resolution $Z_K \leq E$. Here the fact that we consider minimal *good* resolution is crucial, e.g. the hypersurface singularity $x^2+y^3+z^7 = 0$ (cf. Example 4.4) in the minimal resolution has a single vertex and $Z_K = E$, however it is not log-canonical: in the minimal good resolution $Z_K \leq E$ does not hold.

Corollary 5.3. *Let (X,x) be a log-canonical singularity. Let $\pi\colon \tilde X \to X$ be the minimal resolution. Then, any reflexive module of rank one admits a flat connection.*

Proof. It is known that any log-canonical singularity is either rational, or cusp or simple elliptic (see e.g. [26, Example 6.3.33(d)]). For rational and cusp singularities the statement follows from this note, for simple elliptic singularities by [15]. □

Finally, we note that in the proof of Theorem 5.2 the only fact what we used in the construction of γ is that $a : H^1(\tilde X, \mathbb{C}) \to H^1(\Sigma, \mathbb{C})$ is an isomorphism. This is valid for *any* normal surface singularity. Furthermore, all the commutativity properties also hold. In particular, we obtain the commutative diagram (26) with $a_{\mathbb{Z}}$ isomorphism for any (X,x). From this diagram we read that Ψ is onto if and only if $\gamma = b\,a^{-1}$ is onto. Hence, we obtain the following general theorem.

Theorem 5.4. *Let (X,x) be a normal surface singularity. Then the following facts are equivalent:*

(1) any rank one reflexive sheaf of (X,x) admits a flat connection,

(2) the natural map $b : H^1(\tilde{X}, \mathbb{C}) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ *is onto.*
Moreover, the kernels $\ker[b : H^1(\tilde{X}, \mathbb{C}) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})]$ *and* $\ker[\Psi : H^1(\Sigma, \mathbb{C}^*) \to \mathrm{Cl}(X, x)]$ *are isomorphic for any* (X, x)*.*

6. Appendix

Lemma 6.1. *The following commutative diagram commutes*

$$\begin{array}{ccc} \mathrm{Cl}(X,x) & \xrightarrow{\bar{c}_1} & H \\ \uparrow \Psi & & \uparrow = \\ H^1(\Sigma, \mathbb{C}/\mathbb{Z}) & \xrightarrow{\Phi} & H \end{array}$$

Proof. Let $\rho \in H^1(\Sigma, \mathbb{C}/\mathbb{Z})$. Hence, ρ is a one-dimensional representation of $\pi_1(\Sigma)$. Denote by $L_\rho := \Psi(\rho)$. Let $\pi \colon \tilde{X} \to X$ be any resolution. Denote by $\mathcal{L}_\rho := (\pi^* L_\rho)^{\vee\vee}$. Set

$$h := \Phi(\rho) \quad \text{and} \quad h' := [c_1(\mathcal{L}_\rho)] = \bar{c}_1(\Psi(\rho)).$$

Denote by

$$\begin{aligned} U &:= X_{\mathrm{reg}} = \tilde{X} \setminus E, \\ \mathrm{Pic}^{\mathrm{top}}(\tilde{X}) &:= \text{the topological Picard group of } \tilde{X}, \\ \mathrm{Pic}^{\mathrm{top}}(U) &:= \text{the topological Picard group of } U, \\ \mathrm{Pic}^{\mathrm{top},\nabla}(U) &:= \text{the topological Picard group of } U \text{ of flat shaves.} \end{aligned}$$

Moreover, denote by $\mathrm{top} \colon \mathrm{Pic}(\tilde{X}) \to \mathrm{Pic}^{\mathrm{top}}(\tilde{X})$, $\mathcal{L} \mapsto \mathcal{L}^{\mathrm{top}}$, the natural map. Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}^{\mathrm{top},\nabla}(U) & \longrightarrow & H^1(U, \mathbb{C}/\mathbb{Z}) \\ \downarrow & \circlearrowleft & \downarrow \Phi \\ \mathrm{Pic}^{\mathrm{top}}(U) & \xrightarrow{c_1^{\mathrm{top}}} & \mathrm{Tors}\big(H^2(U, \mathbb{Z})\big) \end{array}$$

where c_1^{top} is the topological Chern class. Recall that the topological first Chern class of a flat vector bundle is always torsion, hence the previous diagram is commutative. Recall that Σ is a deformation retract of U, thus $\mathrm{Tors}\big(H^2(U,\mathbb{Z})\big) \cong \mathrm{Tors}\big(H^2(\Sigma,\mathbb{Z})\big) \cong H$ and $H^1(U, \mathbb{C}/\mathbb{Z}) \cong H^1(\Sigma, \mathbb{C}/\mathbb{Z})$. Hence,

$$c_1^{\mathrm{top}}\left((\mathcal{L}_\rho|_U)^{\mathrm{top}}\right) = \Phi(\rho). \tag{27}$$

Now, by the following commutative diagrams

$$\begin{array}{ccc} \mathrm{Pic}^{\mathrm{top}}(\tilde{X}) & \longrightarrow & \mathrm{Pic}^{\mathrm{top}}(U) \\ \downarrow c_1^{\mathrm{top}} & \circlearrowleft & \downarrow c_1^{\mathrm{top}} \\ H^2(\tilde{X}, \mathbb{Z}) & \longrightarrow & H^2(U, \mathbb{Z}) \end{array} \qquad \begin{array}{ccc} \mathrm{Pic}(\tilde{X}) & \longrightarrow & \mathrm{Pic}^{\mathrm{top}}(\tilde{X}) \\ \downarrow c_1 & \circlearrowleft & \downarrow c_1^{\mathrm{top}} \\ H^2(\tilde{X}, \mathbb{Z}) & \longrightarrow & H^2(\tilde{X}, \mathbb{Z}) \end{array}$$

we have

$$c_1^{\mathrm{top}}\left((\mathcal{L}_\rho|_U)^{\mathrm{top}}\right) = [c_1^{\mathrm{top}}\left(\mathcal{L}_\rho^{\mathrm{top}}\right)] \quad \text{and} \quad c_1^{\mathrm{top}}\left(\mathcal{L}_\rho^{\mathrm{top}}\right) = c_1\left(\mathcal{L}_\rho\right). \tag{28}$$

By the equalities (27) and (28) we get

$$h' = [c_1(\mathcal{L}_\rho)] = \Phi(\rho) = h.$$

This finishes the proof. □

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