HILBERT PROPERTIES UNDER BASE CHANGE IN SMALL EXTENSIONS

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ABSTRACT. We study the preservation of the Hilbert property and of the weak Hilbert property under base change in field extensions. In particular we show that these properties are preserved if the extension is finitely generated or Galois with finitely generated Galois group, and we also obtain some negative results.

1. INTRODUCTION

The Hilbert property HP of Colliot-Thélène–Sansuc and Serre [5, 23] and the weak Hilbert property WHP introduced later by Corvaja–Zannier [7] are derived from Hilbert's irreducibility theorem and have been studied intensively in recent years, see for example the list of references in the introduction of [4].

It is known that if a variety X over a field K of characteristic zero has HP respectively WHP, then so does the base change X_L, for any finite field extension L/K. Building on [6], the work [4] proves that in the case of $X = A$ an abelian variety over a number field, the same holds true (for WHP) also in very specific infinite Galois extensions L/K.

The aim of this note is to provide a few more such base change results, which are less arithmetic in nature but more general:

Theorem 1.1 (cf. Cor. 3.3). *Let K be a field of characteristic zero, X a normal K-variety, and L/K a finitely generated field extension. If X has HP (respectively WHP), then so does X_L.*

As an application we then give a new proof of the fact that the class of varieties with HP is closed under products (Corollary 3.5).

Our second base change result is for field extensions L/K that are *small* in the concrete sense that L/K is algebraic and has only finitely many subextensions of degree n, for every $n \in \mathbb{N}$.

Theorem 1.2 (cf. Cor. 4.3). *Let K be a field of characteristic zero, X a normal K-variety and L/K a small extension. If X has HP (respectively WHP), then so does X_L.*

In fact most of our results are stated for thin and strongly thin sets rather than HP and WHP, adding flexibility. As opposed to many results in the literature we also try to avoid assuming our varieties are proper and smooth.

In the last section we discuss further questions regarding base change of HP and WHP and obtain a few negative results. In particular, HP and WHP do not go *down* finite extensions (in a strong sense, see Proposition 5.6), and neither "not HP" nor "not WHP" is preserved in unions of chains, even if the extension is small (Proposition 5.8).

2. FINITE EXTENSIONS

We begin by fixing notation and definitions for this note. Let K always be a field of characteristic zero[1]. A *K-variety* is a separated integral scheme of finite type over K. The degree $\deg(f)$ of a finite morphism $f \colon X \to Y$ of K-varieties is the degree of the associated function field extension. A *cover* is a finite surjective morphism of normal K-varieties. For a normal K-variety X, a subset T of $X(K)$ is *thin* (resp. *strongly thin*) in X if $T \subseteq C(K) \cup \bigcup_{i=1}^r f_i(Z_i(K))$ for some proper closed subscheme C of X and a finite collection $(f_i \colon Z_i \to X)_{i=1,\dots,r}$ of covers with $\deg(f_i) > 1$

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[1]Some of our results for HP carry over to positive characteristic, but for WHP so far not even a convincing definition in positive characteristic has emerged.

(resp. with f_i ramified[2]) for each i. See [4, §2] for basics on covers. A normal[3] K-variety X has the Hilbert property HP (resp. the weak Hilbert property WHP) if $X(K)$ is not thin (resp. not strongly thin) in X. We can and will always assume that the Z_i are geometrically irreducible, since otherwise $Z_i(K) = \emptyset$. Also note that every normal K-variety with WHP is automatically geometrically irreducible.

The following lemma, which like many of our results in this work treats thin and strongly thin sets simultaneously, should be considered known to experts, and we include the proof only for completeness: For thin sets, this is precisely [23, Prop. 3.2.1], and for strongly thin sets, [6, Prop. 3.15] gives a proof for the case X smooth proper and $T = X(L)$, which we follow closely.

Lemma 2.1. *Let X be a normal K-variety, let L/K be a finite extension and let $T \subseteq X(L)$. If T is thin (resp. strongly thin) in X_L, then $T \cap X(K)$ is thin (resp. strongly thin) in X.*

Proof. It suffices to consider the cases $T = C(L)$ and $T = f(Z(L))$ for C a closed proper subscheme of X_L and $f \colon Z \to X_L$ a cover with $\deg(f) > 1$ (resp. a ramified cover) and Z geometrically irreducible. As the projection $p \colon X_L \to X$ is closed, $C' = p(C)$ is a proper closed subset of X, and so $C(L) \cap X(K) \subseteq C'(K)$ is strongly thin in X. For the case $T = f(Z(L))$, we may assume that X is affine. Then the Weil restriction $\mathrm{Res}_{L/K}(f) \colon \mathrm{Res}_{L/K}(Z) \to \mathrm{Res}_{L/K}(X_L)$ exists and is a cover of K-varieties (see [22, 4.10.2] for all properties except for integrality, which is spelled out in [8, Lemma 1.10], and normality, which can be deduced from [24, Thm. 1.3.1]). With $\Delta \colon X \to \mathrm{Res}_{L/K}(X_L)$ the diagonal morphism (cf. [22, 4.2.5]), the cartesian square

$$\begin{array}{ccc} \mathrm{Res}_{L/K}(Z) & \longleftarrow & V \\ {\scriptstyle \mathrm{Res}_{L/K}(f)} \downarrow & & \downarrow {\scriptstyle g} \\ \mathrm{Res}_{L/K}(X_L) & \xleftarrow{\ \Delta\ } & X \end{array}$$

gives rise to the following commutative diagram in the category of sets:

$$\begin{array}{ccccc} Z(L) & \xleftarrow{\ \cong\ } & \mathrm{Res}_{L/K}(Z)(K) & \longleftarrow & V(K) \\ {\scriptstyle f} \downarrow & & {\scriptstyle \mathrm{Res}_{L/K}(f)} \downarrow & & \downarrow {\scriptstyle g} \\ X(L) & \xleftarrow{\ \cong\ } & \mathrm{Res}_{L/K}(X_L)(K) & \longleftarrow & X(K) \end{array}$$

The right hand square is cartesian and the composition of the lower horizontal maps is just the inclusion, hence $f(Z(L)) \cap X(K) \subseteq g(V(K))$, and so it suffices to prove that $g(V(K))$ is thin (resp. strongly thin) in X, i.e. that $g(W(K))$ is thin (resp. strongly thin) in X for every irreducible component W of V. As g is finite and therefore closed, we can assume without loss of generality that $g|_W \colon W \to X$ is surjective. The normalization $\pi \colon \overline{W} \to W$ is an isomorphism above a nonempty open subset W' of W. Then $h := g \circ \pi \colon \overline{W} \to X$ is a cover and $B := g(W \setminus W')$ is a proper closed subset of X with $g(W(K)) \subseteq h(\overline{W}(K)) \cup B(K)$. Thus it is enough to show that $h(\overline{W}(K))$ is thin (resp. strongly thin) in X. For this we have to check that $\deg(h) > 1$ (resp. h is ramified). Again we can assume that the K-variety $\overline{W}$ is geometrically irreducible. The following diagram commutes, where p' and p'' denote the projections:

$$\begin{array}{ccccc} Z & \xleftarrow{\ p'\ } & \mathrm{Res}_{L/K}(Z)_L & \xleftarrow{\ \delta\ } & \overline{W}_L \\ {\scriptstyle f} \downarrow & & {\scriptstyle \mathrm{Res}_{L/K}(f)} \downarrow & & \downarrow {\scriptstyle h_L} \\ X_L & \xleftarrow{\ p''\ } & \mathrm{Res}_{L/K}(X_L)_L & \xleftarrow{\ \Delta_L\ } & X_L \end{array}$$

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[2] in the sense that there exists $y \in Z_i$ at which f_i is not unramified, equivalently, is not étale, cf. [4, Lemma 2.4(a)]

[3] It makes sense to define HP also for non-normal varieties, but in order for us to treat HP and WHP simultaneously we restrict to normal ones.

As $p'' \circ \Delta_L = \mathrm{id}_{X_L}$ (see again [22, 4.2.5]), the map $\alpha := p' \circ \delta \colon \overline{W}_L \to Z$ is a cover with $h_L = f \circ \alpha$. This implies $\deg(h) = \deg(h_L) \geq \deg(f) > 1$. If f is ramified, then so is h_L (cf. [4, Lemma 2.4(d)]) and thus also h. $\qquad\square$

3. Finitely generated extensions

In this section we discuss HP and WHP – and more precisely thin and strongly thin sets – under base change in finitely generated extensions.

Lemma 3.1. *Let S be a K-variety with $S(K)$ dense in S, and write $L = K(S)$. Let X be a normal K-variety, and let $T \subseteq X(L)$. If T is thin (resp. strongly thin) in X_L, then $T \cap X(K)$ is thin (resp. strongly thin) in X.*

Proof. It suffices to consider the case $T = C(L) \cup f(Z(L))$ with C a closed proper subscheme of X_L and $f \colon Z \to X_L$ a cover with $\deg(f) > 1$ (resp. a ramified cover) and Z geometrically irreducible. There is a nonempty open subscheme U of $X_L \setminus C$ such that $V := f^{-1}(U) \to U$ is étale.

After replacing S by a suitable nonempty open subscheme we can assume that S is normal, and that Z and U extend to S-schemes $\mathscr{Z}$ and $\mathscr{U}$ of finite type (cf. [15, IV.8.8.2]) in such a way that the morphisms $\mathscr{Z} \to S$ and $\mathscr{U} \to S$ are flat, separated and surjective (cf. [15, IV.8.9.4, IV.8.10.5]). Moreover, as Z and U are geometrically irreducible normal varieties, we can assume that all fibers of $\mathscr{Z} \to S$ and of $\mathscr{U} \to S$ are geometrically irreducible normal varieties (cf. [15, IV.9.9.5, IV.9.7.7], [13, III.9.1.13]).

After again replacing S by one of its nonempty open subschemes and replacing the rest accordingly, $f \colon Z \to X_L$ extends to a finite surjective S-morphism $F \colon \mathscr{Z} \to X_S$, and the inclusion $U \to X_L$ extends to an open S-immersion $\mathscr{U} \to X_S$ (cf. [15, IV.8.8.2, IV.8.10.5]). Then V is the generic fiber of $\mathscr{V} := F^{-1}(\mathscr{U})$. For every $s \in S$ we have the following diagrams with cartesian squares.

$$
\begin{array}{ccccc}
Z & \xrightarrow{\ f\ } & X_L & \longrightarrow & \mathrm{Spec}(L) \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{Z} & \xrightarrow{\ F\ } & X_S & \longrightarrow & S \\
\uparrow & & \uparrow & & \uparrow \\
\mathscr{Z}_s & \xrightarrow{F_s} & X_{K(s)} & \longrightarrow & \mathrm{Spec}(K(s))
\end{array}
\qquad
\begin{array}{ccccc}
V & \xrightarrow{f|_V} & U & \longrightarrow & \mathrm{Spec}(L) \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{V} & \xrightarrow{F|_{\mathscr{V}}} & \mathscr{U} & \longrightarrow & S \\
\uparrow & & \uparrow & & \uparrow \\
\mathscr{V}_s & \xrightarrow{F_s|_{\mathscr{V}_s}} & \mathscr{U}_s & \longrightarrow & \mathrm{Spec}(K(s))
\end{array}
$$

Since $f|_V$ is étale, after shrinking S once more we can assume that for every $s \in S$ the morphism $F_s|_{\mathscr{V}_s} \colon \mathscr{V}_s \to \mathscr{U}_s$ is étale and $\deg(F_s|_{\mathscr{V}_s}) = \deg(f)$ (cf. [16, IV.17.7.11]). As $\mathscr{Z}$ is flat over S, it follows that the morphism $F|_{\mathscr{V}} \colon \mathscr{V} \to \mathscr{U}$ is étale (cf. [16, IV.17.8.2]). Finally, if f is ramified, we can assume that F_s is ramified for every $s \in S$ [16, IV.17.7.11].

As $S(K)$ is dense in S there exists $s \in S(K)$, and

$$
T_s := (X(K) \setminus \mathscr{U}_s(K)) \cup F_s(\mathscr{Z}_s(K))
$$

is thin (resp. strongly thin) in X, so it suffices to prove that $T \cap X(K) \subseteq T_s$. Let $x \in T \cap X(K)$. If $x \notin \mathscr{U}_s(K)$ we are done, so assume $x \in \mathscr{U}_s(K)$. We prove that $x \in F_s(\mathscr{Z}_s(K))$. Let

$$
\hat{x} \colon S \to X_S = X \times_K S, \ t \mapsto (x, t)
$$

be the constant S-morphism corresponding to x. The set $\hat{x}^{-1}(\mathscr{U})$ is open in S and contains s. We replace S by $\hat{x}^{-1}(\mathscr{U})$ and can thus view $\hat{x}$ as a section $\hat{x} \colon S \to \mathscr{U}$ of $\mathscr{U} \to S$. Consider the cartesian square

$$
\begin{array}{ccc}
F^{-1}(\hat{x}) & \longrightarrow & S \\
\downarrow & & \downarrow{\scriptstyle \hat{x}} \\
\mathscr{V} & \xrightarrow{F|_{\mathscr{V}}} & \mathscr{U}
\end{array}
$$

and note that $F^{-1}(\hat{x})$ is a finite étale S-scheme whose generic fiber is $f^{-1}(x)$ and whose fiber over s is $F_s^{-1}(x)$. In particular, every connected component of $F^{-1}(\hat{x})$ is an étale cover of S. So since $f^{-1}(x) \to \mathrm{Spec}(L)$ has a section by the assumption $x \in T$, also $F^{-1}(\hat{x}) \to S$ has a section, and this implies that $F_s^{-1}(x) \to \mathrm{Spec}(K)$ has a section. It follows that $x \in F_s(\mathscr{Z}_s(K))$ as desired. $\qquad\square$

Proposition 3.2. *Let X be a normal K-variety, L/K a finitely generated field extension and $T \subseteq X(L)$. If T is thin (resp. strongly thin) in X_L, then $T \cap X(K)$ is thin (resp. strongly thin) in X.*

Proof. Fix an intermediate field $K \subseteq F \subseteq L$ such that F/K is purely transcendental, i.e. $F = K(\mathbb{A}^n)$ for some n, and L/F is algebraic, hence finite. The claim then follows from Lemma 2.1 applied to L/F and Lemma 3.1 applied to $S = \mathbb{A}^n$. $\qquad\square$

Corollary 3.3. *Let X be a normal K-variety and L/K a finitely generated field extension. If X has HP (resp. WHP), then X_L has HP (resp. WHP).*

In the special case $X = \mathbb{A}^n$, Proposition 3.2 for *thin* sets can be deduced from result in the literature on so-called Hilbert sets, see [10, Remark 11.8(b)].

In [23, problem on p. 20-21], Serre had asked whether the product of two varieties with HP has again HP. This was positively answered in [3] by proving and applying a fibration theorem for HP, which was improved by Luger [20] to a fibration theorem for thin sets. We now give another proof of this product theorem by first proving a variant of the fibration theorem for HP using the generic fiber and then combining this with Corollary 3.3.

Proposition 3.4. *Let $p: \mathscr{X} \to S$ be a dominant morphism of normal K-varieties. Assume that S has HP and that the generic fiber X of p is normal and has HP. Then $\mathscr{X}$ has HP.*

Proof. Let $\mathscr{U}$ be a nonempty open subset of $\mathscr{X}$, and let $(F_i \colon \mathscr{Z}_i \to \mathscr{X})_{i=1,\ldots,r}$ be a family of covers with $\deg(F_i) > 1$ and $\mathscr{Z}_i$ geometrically irreducible for all i. We have to show that $\mathscr{U}(K) \setminus \bigcup_{i=1}^r F_i(\mathscr{Z}_i(K))$ is nonempty. After replacing $\mathscr{U}$ by a smaller nonempty open set we can assume that $F_i^{-1}(\mathscr{U}) \to \mathscr{U}$ is étale for all i. Thus, replacing $\mathscr{X}$ by $\mathscr{U}$, we can assume the F_i étale right from the outset and have to prove that $\mathscr{X}(K) \setminus \bigcup_{i=1}^r F_i(\mathscr{Z}_i(K))$ is nonempty. For each i, the generic fiber $f_i: Z_i \to X$ of F_i is an étale cover of X with $\deg(f_i) = \deg(F_i) > 1$. As X has HP, there exists a point

$$x \in X(L) \setminus \bigcup_{i=1}^r f_i(Z_i(L)),$$

where $L = K(S)$. After replacing S by one of its nonempty open subsets (and replacing the rest accordingly) x extends to a section $\sigma: S \to \mathscr{X}$ of p. Form the cartesian square

$$
\begin{array}{ccc}
\mathscr{Z}_{i,\sigma} & \xrightarrow{\;F_{i,\sigma}\;} & S \\
\downarrow & & \downarrow{\scriptstyle \sigma} \\
\mathscr{Z}_i & \xrightarrow{\;F_i\;} & \mathscr{X}
\end{array}
$$

and note that the generic fiber of the finite étale morphism $F_{i,\sigma}$ is $f_i^{-1}(x)$ and that $f_i^{-1}(x) \to \mathrm{Spec}(L)$ does not admit a section by our choice of x. Thus $F_{i,\sigma}$ does not admit a section. It follows that for every i the connected components of $\mathscr{Z}_{i,\sigma}$ are all of degree > 1 over S. As S has HP, there exists $s \in S(K) \setminus \bigcup_{i=1}^r F_{i,\sigma}(\mathscr{Z}_{i,\sigma}(K))$. Thus $\sigma(s) \in \mathscr{X}(K) \setminus \bigcup_{i=1}^r F_i(\mathscr{Z}_i(K))$, as desired. $\qquad\square$

Corollary 3.5 ([3, Cor. 3.4]). *Let X and Y be normal K-varieties. If both X and Y have HP, so does $X \times_K Y$.*

Proof. Since X has HP, by Corollary 3.3 also the generic fiber $X_{K(Y)}$ of the projection $X \times_K Y \to Y$ has HP. Therefore $X \times_K Y$ has HP by Proposition 3.4. $\qquad\square$

By [6, Thm. 1.9], Corollary 3.5 holds for WHP (and X, Y smooth proper K-varieties) instead of HP when K is finitely generated, but the case of general K is open. Such a product theorem for WHP would follow from Corollary 3.3 if one could prove Proposition 3.4 for WHP instead of

HP. Even the following weaker version seems open (but see [18, Theorem 1.3] for another form of fibration theorem mixing HP and WHP):

Question 3.6. In Proposition 3.4, if we assume that X has only WHP instead of HP, does it still follow that $\mathscr{X}$ has WHP?

4. SMALL EXTENSIONS

In this section we discuss HP and WHP – and again more precisely thin and strongly thin sets – in small extensions as defined in the introduction. A *Galois* extension is small if and only if its Galois group is *small* in the sense that it has only finitely many open subgroups of index n for every $n \in \mathbb{N}$. For example, every (topologically) finitely generated profinite group is small. See [11, Chapter 16.10] for a discussion of small profinite groups and their relation to finitely generated profinite groups. An arithmetically important example of a small Galois extension is the maximal Galois extension of a number field unramified outside a given finite set of primes, see [11, Example 16.10.9]. For a combinatorial sufficient condition for a Galois extension to have small Galois group see [1, Proposition 2.5]. Small *absolute* Galois groups play an important role in many model theoretic results, see [9].

Lemma 4.1. *Let X be a normal K-variety and $f\colon Z \to X$ a cover of degree n. Let L/K be an algebraic extension and let L_0 be the compositum of all subextensions of L/K of degree at most n. Then $f(Z(L)) \cap X(K) = f(Z(L_0)) \cap X(K)$.*

Proof. Since X is normal, the finite surjective morphism f is universally open [15, 14.4.4]. As $\operatorname{char}(K) = 0$, [15, 15.5.2] therefore gives for $x \in X(K)$ that

$$\sum_{z \in f^{-1}(x)} [K(z) : K] \leq n,$$

in particular $f^{-1}(x)(L) = f^{-1}(x)(L_0)$. Thus $x \in f(Z(L))$ if and only if $x \in f(Z(L_0))$. $\square$

Proposition 4.2. *Let X be a normal K-variety, L/K a small extension and $T \subseteq X(L)$. If T is thin (resp. strongly thin) in X_L, then $T \cap X(K)$ is thin (resp. strongly thin) in X.*

Proof. It suffices to consider the case $T = C(L) \cup f(Z(L))$ for a proper closed subset C of X_L and $f\colon Z \to X_L$ a cover of degree $n > 1$ (resp. a ramified cover). Since L/K is algebraic, there exists a finite extension K'/K in L, a proper closed subset C' of $X_{K'}$ and a cover $f'\colon Z' \to X_{K'}$ of degree n (resp. a ramified cover $f'\colon Z' \to X_{K'}$) such that $C'_L = C$, $Z'_L = Z$ and $f'_L = f$. We will now show that

$$T' := T \cap X(K') = C'(K') \cup (f'(Z'(L)) \cap X(K'))$$

is thin (resp. strongly thin) in $X_{K'}$, which by Lemma 2.1 will prove that $T \cap X(K) = T' \cap X(K)$ is thin (resp. strongly thin) in X. Clearly $C'(K')$ is strongly thin in $X_{K'}$, and Lemma 4.1 applied to f' gives that

$$f'(Z'(L)) \cap X(K') = f'(Z'(L_0)) \cap X(K'),$$

where L_0 is the compositum of all subextensions of L/K' of degree at most n. As Z'_{L_0} is irreducible (because even Z'_L is irreducible), and in particular f'_{L_0} is a cover of degree n (resp. a ramified cover, cf. [14, IV.2.7.1]), we have that $f'(Z'(L_0))$ is thin (resp. strongly thin) in X_{L_0}. As the extension L_0/K' is finite because L/K is small, this implies that $f'(Z'(L_0)) \cap X(K')$ is thin (resp. strongly thin) in $X_{K'}$ by Lemma 2.1. Hence T' is thin (resp. strongly thin) in $X_{K'}$. $\square$

Corollary 4.3. *Let X be a normal K-variety and L/K a small extension. If X has HP (resp. WHP), then X_L has HP (resp. WHP).*

The case $X = \mathbb{P}^1$ and L/K Galois of this corollary appears already as [11, Prop. 16.11.1]. The proof, however, uses irreducible specializations and therefore does not carry over from HP to WHP, which is why the introduction of [4] stated (in the special case of Galois extensions with finitely generated Galois group) that at that point we did not know how to prove it for WHP.

Example 4.4. For example, if L denotes the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, and E is an elliptic curve over $\mathbb{Q}$ of positive rank, then E has WHP (as follows from Falting's theorem, see [7]), and thus by Corollary 4.3 so has E_L. By [19] (see also [21, Example 1.2]), $E(L)$ is finitely generated. This is in stark contrast with the results from [4], where, for abelian extensions $M/\mathbb{Q}$, in order to prove that E_M has WHP we needed to assume that $E(M)$ has infinite rank. In particular, the results of [4] do not give that E_L has WHP.

5. Counterexamples

It is obvious that neither HP nor WHP is preserved in unions of chains of fields. In this final section we discuss two other potential preservation theorems for these two properties and their negation. Both constructions will make use of the following result for fields K that are Hilbertian (i.e. $\mathbb{P}^1_K$ has HP) and pseudo-algebraically closed (PAC, see e.g. [11, Chapter 11]):

Proposition 5.1. *Let K be a Hilbertian PAC field. Then every normal K-variety X has HP.*

Proof. By a theorem of Fried–Völklein [12], the absolute Galois group of a Hilbertian PAC field is what is called ω-free [17, Theorem 5.10.3], and [11, Prop. 27.3.4] gives that over an ω-free PAC field every variety has HP. $\qquad\square$

Also Proposition 5.5 below, which is a strengthening of [23, p. 20 Exercise 1], will be applied to both our examples:

Lemma 5.2. *Let $g\colon X \to Y$ be a smooth surjective morphism of normal K-varieties, and let $T \subseteq Y(K)$ be strongly thin in Y. If*

(a) the generic fiber of g is geometrically irreducible, <u>or</u>
(b) g is proper,

then $\bar{T} := \{x \in X(K) : g(x) \in T\}$ is strongly thin in X.

Proof. If $T = C(K)$ for some proper closed subset C of Y, then $\bar{C} := g^{-1}(C)$ is a proper closed subset of X containing $\bar{T}$, thus $\bar{T}$ is strongly thin. It hence suffices to consider the case where $T = f(Z(K))$ for a ramified cover $f\colon Z \to Y$ with Z geometrically irreducible. In the cartesian square

$$
\begin{array}{ccc}
Z \times_Y X & \xrightarrow{\;g'\;} & Z \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{\;g\;} & Y
\end{array}
$$

the fiber product $Z \times_Y X$ is normal because g' is smooth and Z is normal. As $\bar{T}$ is contained in $f'((Z \times_Y X)(K))$, it suffices to prove that the latter set is strongly thin in X in both cases (a) and (b).

First assume that the generic fiber $g^{-1}(\eta)$ of g is geometrically irreducible. Then if ζ denotes the generic point of Z, $(g')^{-1}(\zeta) = g^{-1}(\eta) \times_{K(\eta)} K(\zeta)$ is irreducible. Thus as $(g')^{-1}(\zeta) \to Z \times_Y X$ is dominant, because it is the pullback of the dominant map $\mathrm{Spec}(K(\zeta)) \to Z$ by the surjective morphism g', we get that $Z \times_Y X$ is irreducible and thus f' is a cover. Since f is ramified and g is surjective, it follows that f' is ramified [16, 17.7.3(i)]. Thus $f'((Z \times_Y X)(K))$ is strongly thin in X.

Now assume instead that g is proper. Let $Z_1, \dots, Z_r$ be the irreducible components of $Z \times_Y X$, and denote by $f'_j\colon Z_j \to X$ (resp. $g'_j\colon Z_j \to Z$) the restrictions of f' (resp. g') to Z_j. Note that $f \circ g'_j = g \circ f'_j$ for every j. Each Z_j is open in $Z \times_Y X$ because $Z \times_Y X$ is normal. We have to show that $\bar{T}_j := f'_j(Z_j(K))$ is strongly thin in X for every j. As f'_j is finite, hence closed, we can assume that f'_j is surjective, and therefore a cover. It is enough to show that f'_j is ramified. Since g' is smooth and proper, hence open and closed, and Z is connected, $g'_j\colon Z_j \to Z$ is surjective and smooth. In particular g'_j is flat (cf. [16, 17.5.1]). As f is not smooth it follows that $f \circ g'_j$ is not smooth (cf. [16, 17.7.7]). Hence $g \circ f'_j$ is not smooth and f'_j cannot be étale, as desired. $\qquad\square$

Remark 5.3. In the special case where X and Y are smooth and proper over K, and $T = Y(K)$, the previous lemma is implied by [6, Thm. 3.7]; we need it in the non-proper case however. By the proof of [5, Prop. 7.13], Lemma 5.2(a) holds for "thin" instead of "strongly thin". Lemma 5.2(b) though does not hold for "thin" instead of "strongly thin": The set $T = \mathbb{Q}^{\times 2}$ is thin in $X := \mathbb{G}_{m,\mathbb{Q}}$, the morphism $g \colon X \to X, x \mapsto x^2$ is smooth, proper and surjective, but $\{x \in X(\mathbb{Q}) : g(x) \in T\} = \mathbb{Q}^{\times}$ is not thin in X.

Lemma 5.4. *Let $B, C \subseteq \mathbb{P}^1(\overline{K})$ be finite sets. There exists $\tau \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{Q}})$ with $\tau(B) \cap C = \emptyset$.*

Proof. Without loss of generality, $B, C \subseteq \mathbb{A}^1(\overline{K})$. Choose $a \in \mathbb{Q} \setminus (B - C)$ and let τ be the translation by a. $\qquad\square$

Proposition 5.5. *If some normal K-variety X has WHP, then K is Hilbertian.*

Proof. Let $(f_j \colon Z_j \to \mathbb{P}^1_K)_{j=1,\dots,r}$ be a family of covers with $\deg(f_j) > 1$ and Z_j geometrically irreducible for every j. Let $T = \bigcup_{j=1}^r f_j(Z_j(K))$, and let U be a nonempty open subset of $\mathbb{P}^1_K$. We have to show that $U(K) \setminus T$ is nonempty. By the Riemann–Hurwitz formula, f_j is ramified for every j. In particular, T is strongly thin in $\mathbb{P}^1_K$.

As $\mathrm{char}(K) = 0$, there exists a nonempty open subset X' of X and an étale cover $p \colon X' \to V$ where V is some nonempty open subscheme of $\mathbb{A}^n_K$, $n = \dim(X)$. Let $q \colon V \to \mathbb{A}^1_K \subseteq \mathbb{P}^1_K$ be the projection onto the first coordinate followed by the inclusion. Then $(q \circ p)(X')$ is open in $\mathbb{P}^1$. By Lemma 5.4 there exists an automorphism τ of $\mathbb{P}^1_K$ such that $\tau(q(p(X')))$ contains the finitely many branch points of each f_j. Let $g = \tau \circ q \circ p$. Note that $\tau \circ q$ is smooth and has geometrically connected fibers, and recall that p is an étale cover, in particular smooth and proper. By Lemma 5.2 we conclude that $\bar{T} := \{x \in X'(K) : g(x) \in T\}$ is strongly thin in X'. It follows that $\bar{T}$ is strongly thin in X. Since X has WHP, $g^{-1}(U)(K) \setminus \bar{T}$ contains a point x_0, and $g(x_0) \in U(K) \setminus T$ as desired. $\qquad\square$

Our first example shows that HP and WHP do not go *down* finite extensions. More precisely, "not HP" and "not WHP" are not preserved under base change in finite extensions L/K, in a strong sense:

Proposition 5.6. *Let $K = \mathbb{Q}^{\mathrm{tr}}$ be the maximal totally real Galois extension of $\mathbb{Q}$ and let $L = \mathbb{Q}^{\mathrm{tr}}(i)$. Then every normal L-variety X has HP, but no normal K-variety has WHP.*

Proof. The first claim follows from Proposition 5.1 since L is Hilbertian by a theorem of Weissauer, and PAC by results of Moret-Bailly and Green–Pop–Roquette, see [17, Example 5.10.7]. The second claim follows from Proposition 5.5 since K is not Hilbertian, see e.g. [2, Thm. 1]. $\qquad\square$

Our second example is motivated by attempts to prove a general product theorem for WHP (see discussion before Question 3.6) at least over small extensions L of $\mathbb{Q}$: If X, Y are L-varieties with WHP, one could try to descend X and Y to varieties with WHP over a number field $K \subseteq L$, apply the product theorem of [6] there, and then use Corollary 4.3 for the small extension L/K to conclude that $X \times_L Y$ has WHP. The following Proposition 5.8 shows that this strategy fails.

Lemma 5.7. *Let K be a countable Hilbertian field. There exists a PAC field that is a small extension of K.*

Proof. Let $(C_i)_{i \in \mathbb{N}}$ be an enumeration of the geometrically integral K-curves. For each i we apply the stabilizing basis theorem [11, Theorem 18.9.3] to obtain an integer n_i and a non-constant rational map $f_i \colon C_i \to \mathbb{P}^1$ of degree n_i with the following properties:

(1) $5 \leq n_1 < n_2 < n_3 < \dots$, and
(2) The monodromy group of f_i (that is, the Galois group of the Galois closure of $K(C_i)/K(\mathbb{P}^1)$ viewed as a permutation group acting on the generic fiber) and the geometric monodromy group of f_i (i.e., the monodromy group of $(f_i)_{\overline{K}} \colon (C_i)_{\overline{K}} \to \mathbb{P}^1_{\overline{K}}$) are both S_{n_i}, for each i.

As in the proof of [11, Lemma 16.2.6], since K is Hilbertian, we may inductively construct $a_i \in \mathbb{P}^1(K)$ such that, for each i,

(3) The fiber $f_i^{-1}(a_i)$ of f_i above a_i is integral. In particular, $f_i^{-1}(a_i) = \mathrm{Spec}(E_i)$ for some field extension E_i/K of degree n_i. (In the notation of [11, Lemma 16.2.6], $E_i = K(b_i)$.)

(4) If L_i denotes the Galois closure of E_i/K, we have $\mathrm{Gal}(L_i/K) = S_{n_i}$ and L_i is linearly disjoint over K from the compositum $L_1 \cdots L_{i-1}$.

Let $E = \prod_{i=1}^{\infty} E_i$ and $L = \prod_{i=1}^{\infty} L_i$ be the field composita. By construction, $C_i(E) \neq \emptyset$ for every i, hence E is PAC [11, Theorem 11.2.3]. Also, by construction, L/K is Galois, $\mathrm{Gal}(L/K) = \Gamma := \prod_{i=1}^{\infty} S_{n_i}$, and $\mathrm{Gal}(L/E) = \Gamma_1 := \prod_{i=1}^{\infty} S_{n-1}$, where $S_{n-1} \leq S_n$ is the stabilizer of a point. The profinite group $\Delta := \prod_{i=1}^{\infty} A_{n_i} \leq \Gamma$ is finitely generated, hence small, i.e. the number $N_r(\Delta)$ of open subgroups of Δ of index r is finite, for every $r \in \mathbb{N}$. Indeed, each A_{n_i} is generated by two elements, say $A_{n_i} = \langle \sigma_i, \tau_i \rangle$, so the finitely generated subgroup $\langle (\sigma_i)_{i \in \mathbb{N}}, (\tau_i)_{i \in \mathbb{N}} \rangle \leq \Delta$ projects onto each A_{n_i}, therefore by the Jordan–Hölder theorem also onto $\prod_{i=1}^{m} A_{n_i}$ for every m (as the A_{n_i} are distinct simple groups), and thus it is dense in Δ. On the other hand, since $n_i \geq 5 \geq 3$, $S_{n_i-1} \cdot A_{n_i} = S_{n_i}$ (where we denote $H_1 \cdot H_2 = \{h_1 h_2 : h_1 \in H_1, \ h_2 \in H_2\}$). So $\Gamma_1 \cdot \Delta = \Gamma$.

Let $\mathcal{G}$ be the set of open subgroups of Γ containing Γ_1. For $G \in \mathcal{G}$, the inclusion $\Delta \to \Gamma$ induces a bijection $\Delta/(G \cap \Delta) \to \Gamma/G$, hence $(\Gamma : G) = (\Delta : G \cap \Delta)$. In particular, the map $\mathcal{G} \ni G \mapsto G \cap \Delta$ is injective, which implies that there are at most $N_r(\Delta)$ many open subgroups of Γ of index r containing Γ_1. So by the Galois correspondence, there at most $N_r(\Delta)$ many subextensions of E/K of degree r, for every r, hence E/K is small.   □

Proposition 5.8. *There exists a small extension $L/\mathbb{Q}$ and a smooth proper $\mathbb{Q}$-variety X such that X_L has HP but X_K does not have WHP for any number field K.*

Proof. Let X be any smooth proper $\mathbb{Q}$-curve of genus at least 2, e.g. the Fermat curve $x^4 + y^4 = z^4$. By Lemma 5.7 there exists a small extension L of $\mathbb{Q}$ which is PAC. As $L/\mathbb{Q}$ is small and $\mathbb{Q}$ is Hilbertian, so is L (cf. Proposition 4.3). So by Proposition 5.1, X_L has HP. By Falting's theorem, $X(K)$ is not dense in X, so X_K does not have WHP, for any number field K.   □

In particular, "not HP" and "not WHP" do not go up in chains of number fields even when the union of the chain is a small extension of $\mathbb{Q}$. We remark that the extension $L/\mathbb{Q}$ constructed here has Galois closure which is not small over $\mathbb{Q}$.

Question 5.9. Does there exist a small *Galois* extension $L/\mathbb{Q}$ with L PAC?

Acknowledgements

References

[1] L. Bary-Soroker and A. Fehm, *Random Galois extensions of Hilbertian fields*, J. Théorie Nombres Bordeaux 25 (2013), 31–42.

[2] L. Bary-Soroker and A. Fehm, *On fields of totally S-adic numbers. With an appendix by Florian Pop*, In: "Valuation Theory in Interaction", EMS Series of Congress Reports, A. Campillo, F.-V. Kuhlmann, B. Teissier (eds.), 2014.

[3] L. Bary-Soroker, A. Fehm and S. Petersen, *On varieties of Hilbert type*, Ann. Inst. Fourier, Grenoble 64(5) (2014), 1893–1901.

[4] L. Bary-Soroker, A. Fehm and S. Petersen, *Ramified covers of abelian varieties over torsion fields*, J. reine angew. Math. 805 (2023), 185–211.

[5] J.-L. Colliot-Thélène and J.-J. Sansuc, *Principal homogeneous spaces under flasque tori: applications*, J. Algebra 106 (1987), 148–205.

[6] P. Corvaja, J. L. Demeio, A. Javanpeykar, D. Lombardo and U. Zannier, *On the distribution of rational points on ramified covers of abelian varieties*, Compositio Math. 158(11) (2022), 2109–2155.

[7] P. Corvaja and U. Zannier, *On the Hilbert property and the fundamental group of algebraic varieties*, Math. Z. 286 (2017), 579–602.

[8] C. Diem, A study on theoretical and practical aspects of Weil-restrictions of varieties, PhD thesis, Frankfurt, 2001.

[9] A. Fehm and F. Jahnke, *Fields with almost small absolute Galois group*, Israel J. Math. 214 (2016), 193–207.

[10] M. D. Fried and M. Jarden, "Field Arithmetic", First Edition, Springer, 1986.

[11] M. D. Fried and M. Jarden, "Field Arithmetic", Third Edition, Springer, 2008.

[12] M. D. Fried and H. Völklein, *The Embedding Problem Over A Hilbertian PAC-Field*, Ann. of Math. 135(3) (1992), 469–481.

[13] A. Grothendieck, *Éléments de géométrie algébrique (rédigé avec la cooperation de Jean Dieudonné): III. Étude cohomologique des faisceaux cohérents, Premiére partie*, Publ. Math. IHES 11 (1961), 5–167.

[14] A. Grothendieck, *Éléments de géométrie algébrique (rédigé avec la cooperation de Jean Dieudonné): IV. Étude locale des schémas et des morphismes des schémas, Seconde partie*, Publ. Math. IHES 24 (1965), 5–231.

[15] A. Grothendieck, *Éléments de géométrie algébrique (rédigé avec la cooperation de Jean Dieudonné): IV. Étude locale des schémas et des morphismes des schémas, Troisième partie*, Publ. Math. IHES 28 (1966), 5–255.

[16] A. Grothendieck, *Éléments de géométrie algébrique (rédigé avec la cooperation de Jean Dieudonné): IV. Étude locale des schémas et des morphismes des schémas, Quatrième partie*, Publ. Math. IHES 32 (1967), 5–361.

[17] M. Jarden, "Algebraic patching", Springer, 2011.

[18] A. Javanpeykar, *Hilbert irreducibility for varieties with a nef tangent bundle*, J. Ramanujan Math. Soc. 39(1) (2024), 33–38.

[19] K. Kato, *p-adic Hodge theory and values of zeta functions of modular forms*, Astérisque 295 (2004), 117–290.

[20] C. Luger, *The Hilbert property for arithmetic schemes*, Acta Arithmetica 206 (2022), 115–126.

[21] B. Mazur and K. Rubin, *Studying the growth of Mordell-Weil*, Doc. Math., Extra volume: Kato's Fiftieth Birthday (2003), 585–608.

[22] C. Scheiderer, "Real and Étale Cohomology", Springer, 1994.

[23] J.-P. Serre, "Topics in Galois theory", Second edition, Springer, 2008.

[24] A. Weil, "Adeles and Algebraic Groups", Birkhäuser, 1982.

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