BLOWING-UP HERMITIAN YANG–MILLS CONNECTIONS

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Abstract. We investigate hermitian Yang–Mills connections for pullback vector bundles on blow-ups of Kähler manifolds along submanifolds. Under some technical asumptions on the graded object of a simple and semi-stable vector bundle, we provide a necessary and sufficent numerical criterion for the pullback bundle to admit a sequence of hermitian Yang–Mills connections for polarisations that make the exceptional divisor sufficiently small, and show that those connections converge to the pulled back hermitian Yang-Mills connection of the graded object.

1. Introduction

A cornerstone in gauge theory is the Hitchin–Kobayashi correspondence ([10, 13, 21, 8]). This celebrated generalisation of the Narasimhan and Seshadri theorem asserts that a holomorphic vector bundle over a Kähler manifold carries an Hermite–Einstein metric if and only if it is polystable in the sense of Mumford and Takemoto ([15, 20]). The interplay between the differential geometric side, in the form of the hermitian Yang–Mills connections (HYM for short) that originated from physics, and the algebro-geometric side, that of stability notions coming from moduli constructions, has had many applications and become a very fertile source of inspiration. Given that HYM connections are canonically attached to polystable vector bundles, it is natural to investigate their relations to natural maps between vector bundles, such as pullbacks. In this paper, we address the problem of pulling back HYM connections along blow-ups. While the similar problem for extremal Kähler metrics has seen many developments in the past ten years [1, 2, 3, 19, 18, 6], relatively little seems to be known about the behaviour of HYM connections under blow-ups [5, 7]. In this paper, under some mild asumptions, we solve the problem for pullback of *semi-stable* vector bundles on blow-ups along smooth centers.

Let $\pi : X' \to X$ be the blow-up of a polarised Kähler manifold $(X, [\omega])$ along a submanifold $Z \subset X$, and $E' = \pi^*E$ the pullback of a holomorphic vector bundle $E \to X$. For $0 < \varepsilon \ll 1$, $L_\varepsilon := \pi^*[\omega] - \varepsilon c_1(Z')$ defines a polarisation on X', where we denote by $Z' = \pi^{-1}(Z)$ the exceptional divisor. There are obstructions for E' to admit irreducible HYM connections with respect to $\omega_\varepsilon \in c_1(L_\varepsilon)$, with $0 < \varepsilon \ll 1$. In particular, E should be *simple* and *semi-stable* with respect to $[\omega]$. In the latter case, E admits a Jordan–Holder filtration by semi-stable sheaves with polystable graded object $\mathrm{Gr}(E)$ (see Section 2.2 for definitions). A further obstruction comes then from subsheaves of E arising from $\mathrm{Gr}(E)$. While those sheaves have the same slope as E, their pullbacks to X' could destabilise E'. Our main result asserts that, under some technical asumptions on $\mathrm{Gr}(E)$ and on the dimension of Z, these are actually the only obstructions for E' to carry a HYM connection. More precisely,

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from now on, and *until the end of the paper, we will assume that $Z \subset X$ satisfies*

$$\mathrm{codim}(Z) \geq 3.$$

(this hypothesis is used in the proofs of Lemma 5.5 and Proposition 5.1).

Recall that a semi-stable holomorphic vector bundle $E \to (X, [\omega])$ is said to be *sufficiently smooth* if its graded object $\mathrm{Gr}(E)$ is locally free. We will assume further that the Jordan–Hölder filtration with locally-free subsheaves

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_\ell = E$$

is unique. For $1 \leq i \leq \ell$, denote by

$$\mu_{L_\varepsilon}(\mathcal{F}_i) = \frac{c_1(\pi^*\mathcal{F}_i) \cdot L_\varepsilon^{n-1}}{\mathrm{rank}(\mathcal{F}_i)}$$

the slope of $\pi^*\mathcal{F}_i$ on (X', L_ε). In the following statements, when referring to a HYM connection A on E (resp. on $\mathrm{Gr}(E)$, or π^*E), we will implicitly assume that $A^{0,1}$ is gauge equivalent to the holomorphic connection, or Dolbeault operator, of E (resp. of $\mathrm{Gr}(E)$, or π^*E).

Theorem 1.1. *Let $E \to X$ be a sufficiently smooth semi-stable holomorphic vector bundle on $(X, [\omega])$. Assume that the stable components of $\mathrm{Gr}(E)$ are pairwise non-isomorphic and that E admits a unique Jordan–Holder filtration by locally free subsheaves. Then, there exists $\varepsilon_0 > 0$ and a sequence of HYM connections $(A_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)}$ on π^*E with respect to $(\omega_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)}$ if and only if*[1]

$$\forall\, i \in [\![1, \ell-1]\!],\ \mu_{L_\varepsilon}(\mathcal{F}_i) \underset{\varepsilon\to 0}{<} \mu_{L_\varepsilon}(E). \tag{1.1}$$

In that case, if A denotes a HYM connection on $\mathrm{Gr}(E)$ with respect to ω, then $(A_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)}$ can be chosen so that $A_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \pi^*A$ in any Sobolev norm.

The expression $\mu_{L_\varepsilon}(\mathcal{F}) \underset{\varepsilon\to 0}{<} \mu_{L_\varepsilon}(E)$ means that the first non-zero term in the ε-expansion for $\mu_{L_\varepsilon}(E) - \mu_{L_\varepsilon}(\mathcal{F})$ is strictly positive. The conclusion of Theorem 1.1 can be rephrased as follows. If $\overline{\partial}_{\mathrm{Gr}(E)}$ (resp. $\overline{\partial}_E$) stands for the Dolbeault operator of $\mathrm{Gr}(E)$ (resp. of E) and if h is a Hermite–Einstein metric on $\mathrm{Gr}(E)$ with respect to ω, then there exist gauge transformations $(f_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ on X' such that the Chern connections of $(f_\varepsilon^*\pi^*\overline{\partial}_E, \pi^*h)$ are HYM with respect to $(\omega_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ and converge to the Chern connection of $(\pi^*\overline{\partial}_{\mathrm{Gr}(E)}, \pi^*h)$ in any Sobolev norm.

Remark 1.2. The hypothesis $\mathrm{codim}(Z) \geq 3$ is technical and used in the proofs of Lemma 5.5 and Proposition 5.1. It seems purely technical, and the case of blowing-up points on surfaces suggests that it should not be necessary (cf. [5, 7]).

Remark 1.3. The simplicity of E is implied by the hypotheses made on $\mathrm{Gr}(E)$ (see [17, Lemma 36]). Semi-stability and condition (1.1) are also necessary to produce the connections (A_ε) from Theorem 1.1. The other three asumptions on $\mathrm{Gr}(E)$ are technical. Assuming $\mathrm{Gr}(E)$ to be locally free enables one to regard E as a smooth complex deformation of $\mathrm{Gr}(E)$ and to work with the various connections on the same underlying complex vector bundle. We should warn the reader though that if one drops this asumption, Condition (1.1) may not be enough to ensure semi-stability of π^*E on (X', L_ε) (see the extra conditions in [16, Theorem 1.10]). On the other hand, the asumption on $\mathrm{Gr}(E)$ having no pairwise isomorphic components is purely

[1] For $m \in \mathbb{N}^*$, we denote by $[\![1, m]\!]$ the set of integers $\{1, \ldots, m\}$.

technical, and ensures that its automorphism group, which will provide obstructions in the perturbative theory, is abelian. Finally, the fact that the Jordan–Holder filtration is unique provides a special shape in the extension matrix from $\mathrm{Gr}(E)$ to E, with non-zero terms above the diagonal. This is used to produce approximate solutions by induction on the number of stable components of $\mathrm{Gr}(E)$.

We now list some corollaries of Theorem 1.1. First, the stable case :

Corollary 1.4. *Let $E \to X$ be a stable holomorphic vector bundle on $(X, [\omega])$ and let A be a HYM connection on E with respect to ω. Then, there exists $\varepsilon_0 > 0$ and a sequence of HYM connections $(A_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)}$ on π^*E with respect to $(\omega_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)}$ such that $A_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \pi^*A$ in any Sobolev norm.*

For the semi-stable case, Condition (1.1) reduces to a finite number of intersection product computations. One interesting feature comes from the second term in the expansion of $\mu_{L_\varepsilon}(E)$. It is the opposite of the slope of the restriction of E to Z. The following formula is proved in [16, Section 4.1], where $m = \dim(Z)$:

$$\mu_{L_\varepsilon}(E) = \mu_L(E) - \binom{n-1}{m-1} \mu_{L_{|Z}}(E_{|Z}) \varepsilon^{n-m} + O(\varepsilon^{n-m+1}). \tag{1.2}$$

We then have :

Corollary 1.5. *Let $E \to X$ be a sufficiently smooth semi-stable holomorphic vector bundle on $(X, [\omega])$. Assume that the stable components of $\mathrm{Gr}(E)$ are pairwise non-isomorphic and that the Jordan–Holder filtration with locally free subsheaves is unique. Denote by A an HYM connection on $\mathrm{Gr}(E)$ with respect to ω. If*

$$\forall\, i \in [\![1, \ell - 1]\!],\ \mu_{L_{|Z}}(E_{|Z}) < \mu_{L_{|Z}}((\mathcal{F}_i)_{|Z}), \tag{1.3}$$

then, there exists $\varepsilon_0 > 0$ and a sequence of HYM connections $(A_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)}$ on π^*E with respect to $(\omega_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)}$ converging to π^*A in any Sobolev norm.

Condition (1.3) was checked on explicit examples in [16, Section 4.5] to produce stable perturbations of tangent sheaves by blow-ups, and our result provides information on the associated connections and their asymptotic behaviour. Note that by the Mehta–Ramanathan theorem [14], if $[\omega] = c_1(L)$ is integral, and if Z is a generic intersection of divisors in linear systems $|L^k|$, then $E_{|Z}$ is semi-stable as soon as E is. In that case, Condition (1.3) cannot be satisfied, and it seems unlikely that Condition (1.1) will hold true. Hence, blowing-up such subvarieties tend to destabilise a semi-stable bundle.

In general, we expect that it should not be too hard to obtain stability of sufficiently smooth pulled back bundles under condition (1.1) with purely algebraic methods. However, we emphasize that the Hitchin–Kobayashi correspondence doesn't provide any information on the asymptotic behaviour of the associated HYM connections, which is then the main content of Theorem 1.1. Nevertheless, we state the following corollary, that extends [16, Theorem 1.10] to a non-equivariant situation:

Corollary 1.6. *Let $E \to X$ be a sufficiently smooth semi-stable holomorphic vector bundle on $(X, [\omega])$. Assume that the stable components of $\mathrm{Gr}(E)$ are pairwise non-isomorphic and that the Jordan–Holder filtration with locally free subsheaves is unique. Then, there exists $\varepsilon_0 > 0$ such that $\pi^*E \to (X', L_\varepsilon)$ is*

(i) stable if and only if for all $i \in [\![1, \ell - 1]\!]$, $\mu_{L_\varepsilon}(\mathcal{F}_i) \underset{\varepsilon \to 0}{<} \mu_{L_\varepsilon}(E)$,

(ii) semi-stable if and only if for all $i \in [\![1, \ell - 1]\!]$, $\mu_{L_\varepsilon}(\mathcal{F}_i) \underset{\varepsilon \to 0}{\leq} \mu_{L_\varepsilon}(E)$,

(iii) unstable otherwise.

Finally, we comment on previous related works. Theorem 1.1 extends results from [5, 7] where blow-ups of HYM connections along points are considered. In the present paper, we consider blow-ups along any smooth subvariety, and also cover the semi-stable situation, which is technically more involved due to the presence of automorphisms of the graded object that obstruct the linear theory. While [7] is a gluing construction of a similar vein to the various solutions to the analogous problem of producing extremal Kähler metrics on blow-ups [2, 3, 19, 18, 6], one of the key feature in our approach is to directly apply the quantitative implicit function theorem, closely following the method developed in [17]. The main new technical input is in Section 3, where we perform a precise study of the linear operators involved in relation to the geometry of the blow-up. The rest of the argument, in Sections 4 and 5, is more standard, and follows the work of Sektnan and the second author [17].

Outline: In Section 2, we recall basic material about HYM connections and stability. We then perform in Section 3 the analysis of the linear theory on the blow-up. Relying on this, in Section 4 we explain how to produce approximate solutions to the HYM equations. This section, and the one that follows, rely on the treatment in [17]. Then, we perturb those approximate solutions in Section 5 to actual solutions, which concludes the proof of Theorem 1.1. The corollaries are addressed in Section 5.3.

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2. Preliminaries

In Sections 2.1 and 2.2 we introduce the notions of HYM connections and slope stability, together with some general results, and refer the reader to [11] and [9]. From Section 3 we start to specialise the discussion to blow-ups. In particular, in Section 3.2, we provide various asymptotic expressions for the linearisation of the HYM equation on the blow-up. Those results will be used in Section 5.

2.1. **The hermitian Yang–Mills equation.** Let $E \to X$ be a holomorphic vector bundle over a compact Kähler manifold X. A hermitian metric on E is *Hermite–Einstein* with respect to a Kähler metric with Kähler form ω if the curvature $F_h \in \Omega^2(X, \operatorname{End} E)$ of the corresponding Chern connection satisfies

$$\Lambda_\omega (iF_h) = c \operatorname{Id}_E \tag{2.1}$$

for some real constant c. Equivalently, if h is some hermitian metric on the smooth complex vector bundle underlying E, a hermitian connection A on (E, h) is said to be *hermitian Yang–Mills* if it satisfies

$$\begin{cases} F_A^{0,2} &= 0, \\ \Lambda_\omega (iF_A) &= c\,\mathrm{Id}_E . \end{cases}$$

The first equation of this system implies that the $(0,1)$-part of A determines a holomorphic structure on E, while the second that h is Hermite–Einstein for this holomorphic structure. We will try to find hermitian Yang–Mills connections within the complex gauge group orbit, which we now define. The *complex gauge group* is

$$\mathscr{G}^{\mathbb{C}}(E) = \Gamma(\mathrm{GL}(E, \mathbb{C})) .$$

We note that the set of unitary gauge transformations preserves the metric h so, taking account of the fibre-wise polar decomposition of an element of $\mathscr{G}^{\mathbb{C}}(E)$, we consider the bundle $\mathrm{End}_H(E,h)$ of Hermitian endomorphisms of E and the set

$$\mathscr{G}^{\mathbb{C}}(E,h) := \mathscr{G}^{\mathbb{C}}(E) \cap \Gamma(\mathrm{End}_H(E,h)) = \{e^s \ : \ s \in \Gamma(\mathrm{End}_H(E,h))\}.$$

If $\bar\partial$ is the Dolbeault operator defining the holomorphic structure on E, then for any $f \in \mathscr{G}^{\mathbb{C}}(E)$, $f \circ \bar\partial \circ f^{-1}$ defines an equivalent holomorphic structure on E. Let $d_A = \partial_A + \bar\partial_A$ be the Chern connection of (E,h) with respect to the original complex structure (that is $\bar\partial_A = \bar\partial$). Then the Chern connection A^f of h with respect to $f \circ \bar\partial \circ f^{-1}$ is

$$d_{A^f} = (f^*)^{-1} \circ \partial_A \circ (f^*) + f \circ \bar\partial \circ f^{-1}.$$

Solving the hermitian Yang–Mills equation is equivalent to solving

$$\Psi(s) = c\,\mathrm{Id}_E$$

where

$$\begin{array}{rccc} \Psi : & \Gamma(\mathrm{End}_H(E,h)) & \longrightarrow & \Gamma(\mathrm{End}_H(E,h)), \\ & s & \longmapsto & i\Lambda_\omega(F_{A^{\exp(s)}}), \end{array}$$

and where $\Gamma(\mathrm{End}_H(E,h))$ is the tangent space to $\mathscr{G}^{\mathbb{C}}(E,h)$ at the identity. For a connection A on E, the Laplace operator Δ_A is

$$\Delta_A = i\Lambda_\omega \left(\bar\partial_A \partial_A - \partial_A \bar\partial_A\right) . \tag{2.2}$$

If $A_{\mathrm{End}\,E}$ denote the connection induced by A on $\mathrm{End}\,E$, then :

Lemma 2.1. *If A is the Chern connection of $(E, \overline{\partial}, h)$, the differential of Ψ at identity is*

$$d\Psi_{\mathrm{Id}_E} = \Delta_{A_{\mathrm{End}\,E}} .$$

If moreover A is assumed to be hermitian Yang–Mills, then the kernel of $\Delta_{A_{\mathrm{End}\,E}}$ acting on $\Gamma(\mathrm{End}(E))$ is given by the Lie algebra $\mathfrak{aut}(E)$ of the space of automorphisms $\mathrm{Aut}(E)$ of $(E, \overline{\partial})$.

The last statement about the kernel follows from the Kähler identities and the Akizuki-Nakano identity that imply $\Delta_{A_{\mathrm{End}\,E}} = \partial_A^*\partial_A + \bar\partial_A^*\bar\partial_A$, the two terms of which are equal if A is Hermitian Yang-Mills. The operator $\Delta_{A_{\mathrm{End}\,E}}$ being elliptic and self-adjoint, $\mathfrak{aut}(E)$ will then appear as a cokernel in the linear theory for perturbations of hermitian Yang–Mills connections.

2.2. **Slope stability.** We recall some basic facts about slope stability, as introduced by [15, 20], and refer the interested reader to [9] for a detailed treatment. We denote here $L := [\omega]$ the polarisation of the n-dimensional Kähler manifold X.

Definition 2.2. For $\mathcal{E}$ a torsion-free coherent sheaf on X, the slope $\mu_L(\mathcal{E}) \in \mathbb{Q}$ (with respect to L) is given by the intersection formula

$$\mu_L(\mathcal{E}) = \frac{\deg_L(\mathcal{E})}{\operatorname{rank}(\mathcal{E})}, \tag{2.3}$$

where $\operatorname{rank}(\mathcal{E})$ denotes the rank of $\mathcal{E}$ while $\deg_L(\mathcal{E}) = c_1(\mathcal{E}) \cdot L^{n-1}$ stands for its degree. Then, $\mathcal{E}$ is said to be *slope semi-stable* (resp. *slope stable*) with respect to L if for any coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$, one has

$$\mu_L(\mathcal{F}) \leq \mu_L(\mathcal{E}) \text{ (resp. } \mu_L(\mathcal{F}) < \mu_L(\mathcal{E})).$$

A direct sum of slope stable sheaves of the same slope is said to be *slope polystable*.

In this paper, we will often omit "slope" and simply refer to stability of a sheaf, the polarisation being implicit. We will make the standard identification of a holomorphic vector bundle E with its sheaf of sections, and thus talk about slope stability notions for vector bundles as well. In that case slope stability relates nicely to differential geometry via the Hitchin–Kobayashi correspondence :

Theorem 2.3 ([10, 13, 21, 8])**.** *There exists a Hermite–Einstein metric on E with respect to ω if and only if E is polystable with respect to L*

We will be mostly interested in semi-stable vector bundles. A *Jordan–Hölder filtration* for a torsion-free sheaf $\mathcal{E}$ is a filtration by coherent saturated subsheaves:

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_\ell = \mathcal{E}, \tag{2.4}$$

such that the corresponding quotients,

$$\mathcal{G}_i = \frac{\mathcal{F}_i}{\mathcal{F}_{i-1}}, \tag{2.5}$$

for $i = 1, \ldots, \ell$, are stable with slope $\mu_L(\mathcal{G}_i) = \mu_L(\mathcal{E})$. In particular, the graded object of this filtration

$$\operatorname{Gr}(\mathcal{E}) := \bigoplus_{i=1}^{l} \mathcal{G}_i \tag{2.6}$$

is polystable. From [9, Section 1], we have the standard existence and uniqueness result:

Proposition 2.4. *Any semi-stable coherent torsion-free sheaf $\mathcal{E}$ on (X, L) admits a Jordan–Hölder filtration, and the double dual of the graded object $\operatorname{Gr}(\mathcal{E})^{**}$ of such filtrations is unique up to isomorphism.*

When E is locally-free and semi-stable, we say that it is *sufficiently smooth* if $\operatorname{Gr}(E)$ is locally-free. In that case, we denote $\mathfrak{E}_{[\omega]}$ the set of holomorphic subbundles of E built out of successive extensions of some of the stable components of $\operatorname{Gr}(E)$. Equivalently, $\mathfrak{E}_{[\omega]}$ is the set of holomorphic subbundles of E arising in a Jordan-Holder filtration for E. Finally, we recall that a necessary condition for E to be stable is simplicity, that is $\operatorname{Aut}(E) = \mathbb{C}^* \cdot \operatorname{Id}_E$.

3. Geometry of the blow-up

We consider now $Z \subset X$ an m-dimensional complex submanifold of codimension $r = n - m \geq 2$ and the blow-up map

$$\pi : \mathrm{Bl}_Z(X) \to X.$$

We will denote by $X' = \mathrm{Bl}_Z(X)$ the blown-up manifold and by $Z' = \pi^{-1}(Z) \subseteq X'$ the exceptional divisor. We denote by

$$L_\varepsilon := \pi^* L - \varepsilon [Z']$$

a polarisation on X', for $0 < \varepsilon \ll 1$. Let $E \to X$ be a holomorphic vector bundle, and denote by $E' = \pi^* E$ the pulled back bundle. For any holomorphic subbundle $F \subset E$, the intersection numbers

$$\mu_{L_\varepsilon}(\pi^* E) - \mu_{L_\varepsilon}(\pi^* F) = \frac{c_1(\pi^* E) \cdot L_\varepsilon^{n-1}}{\mathrm{rank}(E)} - \frac{c_1(\pi^* F) \cdot L_\varepsilon^{n-1}}{\mathrm{rank}(F)}$$

admit expansions in ε, with first term given by $\mu_L(E) - \mu_L(F)$ (this can be seen using the polynomial expression in ε for $(\pi^* L - \varepsilon [Z'])^{n-1}$). For that reason, given the Hitchin–Kobayashi correspondence in Theorem 2.3, semi-stability of E on (X, L) is a necessary condition for its pullback E' to admit an HYM connection with respect to a Kähler metric in L_ε, for all $0 < \varepsilon \ll 1$. Restricting to irreducible connections, another necessary condition is simplicity of E', which, by Hartogs' theorem, is equivalent to simplicity of E. Then, natural candidates to test for stability of E' are given by the pullbacks of elements in $\mathfrak{E}_{[\omega]}$, and Condition (1.1) clearly is necessary for E' to be stable in the polarisations we consider, and thus to admit an irreducible HYM connection. Hence, we will assume E to be simple, semi-stable, and to satisfy (1.1).

We now turn back to the differential geometry of the blow-up. Our main goal in this section is to obtain a decomposition for the space of sections of $\pi^* E$ as a direct sum of pulled-back sections and a natural complementary space. Then, denoting by Δ_ϵ the Laplace operator of a pulled-back connection on $\pi^* E$ with respect to Kähler representatives ω_ε of L_ϵ, as described below, we will accordingly obtain a decomposition of Δ_ϵ, acting on decomposed sections, of the form

$$\begin{pmatrix} \Delta_X & 0 \\ \mathcal{L} & \varepsilon^{-1} \Delta_{\mathcal{V}} \end{pmatrix} \tag{3.1}$$

plus higher order terms in ϵ, where Δ_X will act as the Laplace operator on X for pulled-back terms and $\Delta_{\mathcal{V}}$ will stand for some invertible linear operator on the supplementary space of sections, and for some second order operator $\mathcal{L}$ (see Equation 3.5 in Proposition 3.8 below).

This expansion will then be used in Section 4 to inductively produce approximate solutions to the HYM equation on X' by using the invertibility of the first order term in the ϵ-expansion of Δ_ϵ.

3.1. Decomposition of spaces of sections.

We have a commutative diagram:

$$\begin{array}{ccc} Z' & \xrightarrow{\iota} & X' \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\iota_0} & X \end{array}$$

where ι_0 and ι denote the inclusions, while the vertical arrows are given by the projection map π. We then have a pullback map on sections

$$\pi^* : \Gamma(X, \mathrm{End}(E)) \longrightarrow \Gamma(X', \mathrm{End}(\pi^* E))$$

as well as a restriction map :

$$\iota^* : \Gamma(X', \mathrm{End}(\pi^* E)) \longrightarrow \Gamma(Z', \mathrm{End}(\iota^* \pi^* E)).$$

Our goal now is to fit these maps in a short exact sequence, that will in the end split the space $\Gamma(X', \mathrm{End}(\pi^* E))$. If $N_Z = TX_{|Z}/TZ$ denotes the normal bundle of Z in X, then $Z' \simeq \mathbb{P}(N_Z)$, and we can fix a $(1,1)$-form $\lambda \in c_1(\mathcal{O}_{\mathbb{P}(N_Z)}(1))$ that restricts to Kähler metrics on the fibers of $\mathbb{P}(N_Z) \to Z$. We also fix a Kähler form $\omega \in c_1(L)$ on X, and consider its restriction to Z. We then have a Kähler $\mathbb{CP}^{r-1}$-fibration :

$$\pi : (Z', \lambda) \longrightarrow (Z, \omega).$$

By averaging along fibers as described in [17, Section 2.3], we obtain a splitting

$$\Gamma(Z', \mathrm{End}(\iota^* \pi^* E)) = \pi^*(\Gamma(Z, \mathrm{End}(\iota_0^* E))) \oplus \Gamma_0(Z', \mathrm{End}(\iota^* \pi^* E)), \tag{3.2}$$

where the space $\Gamma_0(Z', \mathrm{End}(\iota^* \pi^* E))$ is obtained as follows. First, the space of smooth functions can be decomposed as

$$\mathcal{C}^\infty(Z') = \pi^* \mathcal{C}^\infty(Z) \oplus \mathcal{C}_0^\infty(Z')$$

by setting for any $f \in \mathcal{C}^\infty(Z')$:

$$\forall z \in Z,\ f_Z(z) := \frac{\int_{\pi^{-1}(z)} f_{|\pi^{-1}(z)}\, \lambda^{r-1}}{\int_{\pi^{-1}(z)} \lambda^{r-1}} \tag{3.3}$$

and

$$f_0 := f - \pi^* f_Z \in \mathcal{C}_0^\infty(Z'),$$

where $\mathcal{C}_0^\infty(Z')$ is the space of fibrewise average 0 functions. In the same way, we decompose sections $s \in \Gamma(Z', \mathrm{End}(\iota^* \pi^* E)) \simeq \Gamma(Z', \pi^*(\mathrm{End}(\iota_0^* E)))$ by using local trivialisations for $\mathrm{End}(\iota_0^* E))$ on Z. For such a local trivialisation on some open set $U \subset Z$, s is represented by a smooth function $s : \pi^{-1}(U) \to \mathbb{C}^{\mathrm{rank}(\mathrm{End}(E))}$, and we can define a section s_Z of $\mathrm{End}(\iota_0^* E))$ over U by averaging as in (3.3) over the fibers of π each coordinate function for s. It is a direct computation to see that those local sections patch together and define a global one on Z (see [17, Section 2.3, first page]), denoted s_Z, and we then set $s_0 = s - \pi^* s_Z$. The space $\Gamma_0(Z', \mathrm{End}(\iota^* \pi^* E))$ is then the space of sections $s \in \mathrm{End}(\iota^* \pi^* E)$ over Z' such that $s_Z = 0$.

From now on, we will sometimes omit the ι^* and π^* to simplify notation. Using the projection on the second factor

$$p_0 : \Gamma(Z', \mathrm{End}(E)) \to \Gamma_0(Z', \mathrm{End}(E))$$

in (3.2), we deduce a short exact sequence :

$$0 \longrightarrow \Gamma(X, \mathrm{End}(E)) \xrightarrow{\pi^*} \Gamma(X', \mathrm{End}(E)) \xrightarrow{p_0 \circ \iota^*} \Gamma_0(Z', \mathrm{End}(E)) \longrightarrow 0.$$

We can actually split this sequence by mean of a linear extension operator

$$\iota_* : \Gamma_0(Z', \mathrm{End}(E)) \longrightarrow \Gamma(X', \mathrm{End}(E))$$

such that

$$p_0 \circ \iota^* \circ \iota_* = \mathrm{Id}.$$

This can be done using bump functions and a standard partition of unity argument. The end result is the following proposition.

Proposition 3.1. *Let $Z \subseteq X$ be a closed codimension $r \geq 2$ complex submanifold of the compact Kähler manifold X, and let $X' = \mathrm{Bl}_Z(X)$ the blow-up of X along Z. Let E be a holomorphic vector bundle on X and $E' = \pi^* E$. Then we have the isomorphism*

$$\begin{array}{ccc} \Gamma(X', \mathrm{End}(E)) & \longrightarrow & \Gamma(X, \mathrm{End}(E)) \oplus \Gamma_0(Z', \mathrm{End}(E)) \\ s & \longmapsto & (s - \iota_* \circ p_0 \circ \iota^* s \ , \ p_0 \circ \iota^* s), \end{array} \tag{3.4}$$

with inverse map $(s_X, s_Z) \mapsto (\pi^* s_X + \iota_* s_Z)$.

Moreover, this splits the Lie algebra of gauge transformations, and will be used to identify contributions coming from X and from Z' in the ε-expansion of the linearisation, which we describe in the next section. From now on, by abuse of notations, we will consider the spaces $\Gamma(X, \mathrm{End}(E))$ and $\Gamma_0(Z', \mathrm{End}(E))$ as subspaces of $\Gamma(X', \mathrm{End}(\pi^* E))$, and denote $s = s_X + s_Z$ the decomposition of an element $s \in \Gamma(X', \mathrm{End}(E))$.

3.2. **Decomposition of the Laplace operator.** To begin, we note that the decomposition arising from the isomorphism (3.4) is independent of any metric or connection defined on E and E'. We extend λ to a closed $(1,1)$-form over X' as in [22, Section 3.3] and consider the family of Kähler metrics on X':

$$\omega_\varepsilon = \pi^* \omega + \varepsilon \lambda \in c_1(L_\varepsilon), \ 0 < \varepsilon \ll 1.$$

In this section we study how the Laplace operator of a pulled-back connection behaves, with respect to the decomposition of (3.4) and with respect to the parameter $\varepsilon > 0$.

Let A be a Hermitian connection on E, which we pull back to X' and extend to the bundle $\mathrm{End}(\pi^* E)$. We will now study the Laplace operator

$$\Delta_\varepsilon s = i \Lambda_\varepsilon (\bar\partial_A \partial_A - \partial_A \bar\partial_A) s$$

acting on the various components of $s = s_X + s_Z \in \Gamma(X', \mathrm{End}(E))$, where Λ_ε is the Lefschetz operator for the metric ω_ε. For this, we need to introduce an operator on Z' (recall that $r = \mathrm{codim}(Z)$).

Definition 3.2. The *vertical Laplace operator*, denoted

$$\Delta_{\mathcal{V}} : \Gamma_0 \left(Z', \mathrm{End}(E) \right) \to \Gamma_0 \left(Z', \mathrm{End}(E) \right),$$

is the operator

$$\Delta_{\mathcal{V}} = i \Lambda_{\mathcal{V}} \big(\partial_F \bar\partial_F - \bar\partial_F \partial_F \big),$$

where

$$\Lambda_{\mathcal{V}} : \Omega^{1,1} \left(Z', \mathrm{End}(E) \right) \to \Gamma_0 \left(Z', \mathrm{End}(E) \right)$$

is the *vertical contraction operator* defined on each fiber $F_z = \pi^{-1}(z)$ of $\pi : Z' \to Z$ by

$$(\Lambda_{\mathcal{V}} \alpha)_{|F_z} \, \lambda_{|F_z}^{r-1} = (r-1) \alpha_{|F_z} \wedge \lambda_{|F_z}^{r-2}$$

and where $d_F = \partial_F + \bar\partial_F$ is the flat connection along the fibres of $\pi : Z' \to Z$.

The following lemma relies on [12] and was already observed in [17, Lemma 20].

Lemma 3.3. *The vertical Laplacian*

$$\Delta_{\mathcal{V}} : \Gamma_0 \left(Z', \mathrm{End}(E) \right) \to \Gamma_0 \left(Z', \mathrm{End}(E) \right)$$

is an invertible linear map.

In the following statements, an expression of the form $O(\varepsilon^j)$ is to be understood as holding pointwise. Convergence considerations of those expressions with respect to various Sobolev space norms will be addressed in Section 5.

Lemma 3.4. *If $s_Z = \iota_*\sigma_Z$ for $\sigma_Z \in \Gamma(Z', \mathrm{End}(E))$, then*

$$(p_0 \circ \iota^*)\Delta_\varepsilon(\iota_*\sigma_Z) = \varepsilon^{-1}\Delta_{\mathcal{V}}\sigma_Z + \mathcal{O}(1).$$

Proof. We introduce the operator D given by

$$Ds_Z = i(\bar\partial_A\partial_A - \partial_A\bar\partial_A)s_Z.$$

The Laplacian Δ_ε satisfies on X' :

$$\Delta_\varepsilon s_Z\, \omega_\varepsilon^n = nDs_Z \wedge \omega_\varepsilon^{n-1},$$

or equivalently

$$\Delta_\varepsilon s_Z = \frac{n\ Ds_Z \wedge (\omega + \varepsilon\lambda)^{n-1}}{(\omega + \varepsilon\lambda)^n}.$$

We note that ω is a Kähler form on X, but on X' is degenerate along the fibre directions of the submanifold Z'. Then $(i^*\omega)^{m+1} = 0 \in \Omega^{2(m+1)}(Z')$, and at $x \in Z' \subseteq X'$, $\omega^{m+2} = 0$. Then, expanding $(\omega + \varepsilon\lambda)^{n-1}$ and $(\omega + \varepsilon\lambda)^n$ gives

$$\iota^*\Delta_\varepsilon s_Z = (n - m - 1)\varepsilon^{-1}\frac{Ds_Z \wedge \omega^{m+1} \wedge \lambda^{n-m-2}}{\omega^{m+1} \wedge \lambda^{n-m-1}} + \mathcal{O}(1).$$

Restricting to Z', the connection 1-forms of A vanish, so $\iota^*Ds_Z = i\partial\bar\partial\sigma_Z$, acting on the coefficient functions of σ_Z. On the other hand, by considering a convenient orthonormal frame at $x \in Z'$, we see that $\iota^*\Delta_\varepsilon\iota_*\sigma_Z = \varepsilon^{-1}\Delta_{\mathcal{V}}\sigma_Z + \mathcal{O}(1)$. □

In the next lemma, we denote $\Delta_\varepsilon s_Z = (\Delta_\varepsilon s_Z)_X + (\Delta_\varepsilon s_Z)_Z$ the decomposition according to (3.4).

Lemma 3.5. *For $s_Z = \iota_*\sigma_Z$ with $\sigma_Z \in \Gamma(Z', \mathrm{End}(E))$, we have*

$$(\Delta_\varepsilon s_Z)_X = \mathcal{O}(1).$$

Proof. By definition, $(\Delta_\varepsilon s_Z)_X = \pi^*\phi$ for some $\phi \in \Gamma(X, \mathrm{End}(E))$. As we also have

$$(\Delta_\varepsilon s_Z)_X \quad = \quad (\mathrm{Id} - \iota_*(p_0 \circ \iota^*))\Lambda_\varepsilon Ds_Z,$$

we deduce that the section ϕ is the continuous extension of $\pi_*(\mathrm{Id} - \iota_*(p_0 \circ \iota^*))\Lambda_\varepsilon Ds_Z$ across $Z \subseteq X$. On $X' \setminus Z'$ we have

$$\Lambda_\varepsilon Ds_Z \quad = \quad n\frac{Ds_Z \wedge (\omega^{n-1} + \mathcal{O}(\varepsilon))}{\omega^n + \mathcal{O}(\varepsilon)} = \mathcal{O}(1).$$

As $\pi_*(\mathrm{Id} - \iota_*(p_0 \circ \iota^*))$ is $\mathcal{O}(1)$, the result follows. □

From the previous two lemmas, in the decomposition

$$s = s_X + s_Z,$$

$\Delta_\varepsilon s_Z$ also lies in the subspace $\Gamma_0(Z', \mathrm{End}(E)) \subseteq \Gamma(X', \mathrm{End}(E))$ to higher order in ε. For $s_X \in \Gamma(X, \mathrm{End}(E))$,

$$\Delta_\varepsilon s_X = (\Delta_\varepsilon s_X)_X + (\Delta_\varepsilon s_X)_Z$$

where $(\Delta_\varepsilon s_X)_Z = \iota_*(p_0 \circ \iota^*)\Delta_\varepsilon s_X$. We first consider $\iota^*\Delta_\varepsilon s_X$.

Lemma 3.6. *For $s_X = \pi^*\sigma_X \in \Gamma(X, \mathrm{End}(E)) \subseteq \Gamma(X', \mathrm{End}(E))$,*

$$\iota^*\Delta_\varepsilon s_X = (m+1)\frac{Ds_X \wedge \omega^m \wedge \lambda^{n-m-1}}{\omega^{m+1}\wedge\lambda^{n-m-1}} + \mathcal{O}(\varepsilon).$$

Proof. Firstly, $s_X = \pi^*\sigma_X$, and the connection A is pulled back from X, so Ds_X is basic for the projection to X and $Ds \wedge \omega^{m+1} = 0$ at points in Z'. Secondly, we note that $\omega^{m+1}\wedge\lambda^{n-m-1}$ is a volume form on X', in a neighbourhood of Z'. Then, the result follows similarly to the previous lemma. □

For the final term $(\Delta_\varepsilon s_X)_X$, we introduce Δ_X the Laplace operator of A on $\mathrm{End}(E) \to (X, \omega)$:

$$\begin{array}{cccc} \Delta_X: & \Gamma(X, \mathrm{End}(E)) & \to & \Gamma(X, \mathrm{End}(E)) \\ & \sigma & \mapsto & i\Lambda_\omega(\bar\partial_A\partial_A - \partial_A\bar\partial_A)\sigma. \end{array}$$

Lemma 3.7. *For $s_X = \pi^*\sigma_X \in \Gamma(X, \mathrm{End}(E)) \subseteq \Gamma(X', \mathrm{End}(E))$,*

$$(\Delta_\varepsilon s_X)_X = \pi^*(\Delta_X\sigma_X) + \mathcal{O}(\varepsilon).$$

Proof. There is $\phi \in \Gamma(X, \mathrm{End}(E))$ such that $(\Delta_\varepsilon s_X)_X = \pi^*\phi$. The element ϕ can be identified as the lowest order term in the asymptotic expansion in ε of $(\Delta_\varepsilon\pi^*\sigma_X)_X$. However, we have at $x \in X' \setminus Z'$:

$$\Delta_\varepsilon\pi^*\sigma_X = n\frac{D\pi^*\sigma_X \wedge (\omega+\varepsilon\lambda)^{n-1}}{(\omega+\varepsilon\lambda)^n} = n\pi^*\frac{D\sigma_X\wedge\omega^{n-1}}{\omega^n} + \mathcal{O}(\varepsilon)$$

so we see that the lowest order term in the expansion of $(\Delta_\varepsilon\pi^*\sigma_X)_X$ is $\Delta_X\sigma_X$. □

Summarizing the above calculations, we obtained :

Proposition 3.8. *With respect to the decomposition $s = s_X + s_Z$ produced by (3.4), the operator Δ_ε takes the form*

$$\begin{pmatrix} \Delta_X & 0 \\ \mathcal{L} & \varepsilon^{-1}\Delta_{\mathcal{V}} \end{pmatrix} \tag{3.5}$$

plus higher order terms, for some second order operator $\mathcal{L}$.

4. The approximate solutions

The aim of this section is to produce approximate solutions to the HYM equations, with a precise understanding of the failure to satisfy the equations in terms of the parameter ε. As in the previous section, expressions of the form $\mathcal{O}(\varepsilon^j)$ are to be understood as holding pointwise until Section 5.

In this section, the methods and proofs follow [17] very closely, so at certain points we will only sketch some of the steps, providing full details only when major differences with that article occur. We note that in our work, the parameter ε plays the role of k^{-1} in [17], and in the decomposition $\Gamma(X', \mathrm{End}\,E) = \Gamma(X, \mathrm{End}\,E) \oplus \Gamma_0(Z', \mathrm{End}\,E)$, the space $\Gamma(X, \mathrm{End}\,E)$ (resp. $\Gamma_0(Z', \mathrm{End}\,E)$) plays the role of $\Gamma(B, \mathrm{End}(E))$ (resp. of $\Gamma_0(X, \mathrm{End}\,E)$) in [17].

4.1. **Fixing the complex deformation parameters.** The aim of this section is Proposition 4.2, where we obtain a good control on the error terms of our contracted curvature operator that live in the kernel of its linearisation. This will be achieved by using deformation theory. We refer to [11, Section 7.2] and [4] for the deformation theory techniques that will be used. We start from a semi-stable and sufficiently smooth holomorphic vector bundle E on (X, L), with $L = [\omega]$. We assume that the Jordan–Holder filtration with locally-free quotients is unique and denote it by :

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_\ell = E.$$

We introduce the graded object

$$\mathrm{Gr}(E) = \bigoplus_{i=1}^{\ell} \mathcal{G}_i$$

which is locally free, and the $\mathcal{G}_i$'s which are the successive stable quotients in the Jordan–Holder filtration for E. In particular, the $\mathcal{G}_i$'s are stable holomorphic vector bundles on (X, L). The bundle E is then obtained by a sequence of extensions of vector bundles:

$$0 \to \mathcal{F}_{i-1} \to \mathcal{F}_i \to \mathcal{G}_i \to 0,$$

$i = 1, \ldots, \ell$, with $\mathcal{F}_0 = 0$ and $\mathcal{F}_\ell = E$, and where $\mathrm{Gr}(E)$ is the same underlying smooth vector bundle as E, but with a different holomorphic structure. Thus the Dolbeault operator $\overline{\partial}_E$ on E is of the form

$$\overline{\partial}_E = \overline{\partial}_0 + \gamma$$

where $\overline{\partial}_0$ is the Dolbeault operator on $\mathrm{Gr}(E)$ and $\gamma \in \Omega^{0,1}(B, \mathrm{Gr}(E)^* \otimes \mathrm{Gr}(E))$ can be written

$$\gamma = \sum_{i<j} \gamma_{ij}$$

with (possibly vanishing) $\gamma_{ij} \in \Omega^{0,1}(B, \mathcal{G}_j^* \otimes \mathcal{G}_i)$. The integrability condition $\overline{\partial}_E^2 = 0$ imposes the Maurer-Cartan equation

$$\overline{\partial}_0 \gamma + \gamma \wedge \gamma = 0$$

where we will use the notation $\overline{\partial}_0$ to denote the induced operator $\overline{\partial}_{0,\mathrm{End}(E)}$ when no confusion should arise. Note that in the matrix block decomposition induced by the splitting $\mathrm{Gr}(E) = \mathcal{G}_1 \oplus \ldots \oplus \mathcal{G}_\ell$, the representation of γ is upper-diagonal:

$$\gamma = \begin{bmatrix} 0 & \gamma_{1,2} & \cdots & \gamma_{1,\ell} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \gamma_{l-1,\ell} \\ 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

Assume now that $\mathrm{Gr}(E)$ has non-isomorphic stable quotients. We now explain the technical implications of the hypotheses made on E and its graded object. First, by [17, Lemma 36], E is simple. The automorphism group $G := \mathrm{Aut}(\mathrm{Gr}(E))$ is a reductive Lie group with Lie algebra $\mathfrak{g} := \mathfrak{aut}(\mathrm{Gr}(E))$ and compact form $K \subset G$, with $\mathfrak{k} := \mathrm{Lie}(K)$. We have

$$\mathfrak{g} = \bigoplus_{i,j} H^0(X, \mathrm{Hom}(\mathcal{G}_i, \mathcal{G}_j)).$$

As the $\mathcal{G}_i$'s are all stable, and non isomorphic, we deduce that

$$\mathfrak{g} = \bigoplus_{i=1}^{\ell} \mathbb{C} \cdot \mathrm{Id}_{\mathcal{G}_i},$$

and in particular $\mathfrak{g}$ is abelian. The upshot for us is that the elements in $\mathfrak{k}$ that obstruct the linear theory all live on the diagonal in the matrix block decomposition induced by the decomposition $\mathrm{Gr}(E) = \oplus_{i=1}^{\ell} \mathcal{G}_i$. From the uniqueness hypothesis on the Jordan–Hölder filtration, we deduce that the terms $\gamma_{i,i+1}$, for $1 \leq i \leq \ell - 1$, are all non-zero, see [17, Lemma 33]. This will enable us to use induction on ℓ, the number of stable components, in the construction of the approximate solutions.

Our starting point to produce HYM connections on $\pi^* E \to (X', \omega_\varepsilon)$ will be a product Hermite–Einstein metric $h = h_1 \oplus \ldots \oplus h_\ell$ on $\mathrm{Gr}(E)$ (we can fix one by polystability), and the associated HYM connection A_0 on $(\mathrm{Gr}(E), \overline{\partial}_0)$ with contracted curvature form

$$\Lambda_\omega i F_{A_0} = c_0 \cdot \mathrm{Id}\,.$$

We then introduce the L^2 projection

(4.1)
$$\begin{array}{cccc} \Pi_{i\mathfrak{k}}: & \Gamma(X, \mathrm{End}_H(E,h)) & \to & i\mathfrak{k} \\ & s & \mapsto & \displaystyle\frac{1}{\mathrm{Vol}(\omega)} \sum_{i=1}^{\ell} \frac{1}{\mathrm{rank}(\mathcal{G}_i)} \Big(\int_X \mathrm{trace}_{\mathcal{G}_i}(s)\, \omega^n\Big)\ \mathrm{Id}_{\mathcal{G}_i} \end{array}$$

and the induced orthogonal decomposition:

$$\Gamma(X, \mathrm{End}_H(E,h)) = i\mathfrak{k} \oplus \Gamma_{i\mathfrak{k}^\perp}(X, \mathrm{End}_H(E,h))$$

with respect to the pairing

$$(s_1, s_2) \mapsto \int_X \mathrm{trace}(s_1 \cdot s_2)\, \omega^n.$$

Denoting by $\Delta_{\mathrm{Gr}(E),0}$ the Laplace operator of the Chern connection of $(\overline{\partial}_0, h)$ with respect to ω, from Lemma 2.1 and self-adjointness, we have

Lemma 4.1. *The following operator is invertible :*

$$\Delta_{\mathrm{Gr}(E),0} : \Gamma_{i\mathfrak{k}^\perp}(X, \mathrm{End}_H(E,h)) \to \Gamma_{i\mathfrak{k}^\perp}(X, \mathrm{End}_H(E,h)).$$

As stated earlier, the hypothesis of $\mathrm{Gr}(E)$ being locally free implies that E can be considered a complex deformation of the holomorphic vector bundle $\mathrm{Gr}(E)$. This can be seen through the gauge action of G, as we now explain. First gauge fix γ by imposing

$$\overline{\partial}_0^* \gamma = 0, \tag{4.2}$$

where the adjoint is with respect to the Kähler structure ω. Then, there is a natural action of G on elements in

$$\mathfrak{Def}(\mathrm{Gr}(E)) := \{\gamma' \in \Omega^{0,1}(Z, \mathrm{Gr}(E)^* \otimes \mathrm{Gr}(E)),\ \overline{\partial}_0 \gamma' + \gamma' \wedge \gamma' = 0,\ \overline{\partial}_0^* \gamma' = 0\}$$

parametrising small complex deformations of $\mathrm{Gr}(E)$. On γ, the action of an element $\mathbf{g} \in G \simeq (\mathbb{C}^*)^\ell$ of the form

$$\mathbf{g} = g_1\, \mathrm{Id}_{\mathcal{G}_1} + \ldots + g_\ell\, \mathrm{Id}_{\mathcal{G}_\ell}$$

is given by

$$\mathbf{g}^*\gamma = \mathbf{g}\cdot\gamma\cdot\mathbf{g}^{-1} = \sum_{i<j} g_i g_j^{-1}\gamma_{ij}. \tag{4.3}$$

Then, γ is gauge-conjugated to all elements of this form. To parameterise a family of Dolbeault operators from $\overline{\partial}_0$ to $\overline{\partial}_E$, we consider for any $\underline{\beta} = (\beta_1,\ldots,\beta_{\ell-1}) \in (\mathbb{C}^*)^{\ell-1}$,

$$\mathbf{g}_{\underline{\beta}} := \Pi_{i=1}^{\ell-1}\beta_i \in G$$

such that for all i,

$$\beta_i = (\mathbf{g}_{\underline{\beta}})_i^{-1}(\mathbf{g}_{\underline{\beta}})_{i+1}.$$

Setting

$$\gamma_{\underline{\beta}} := \mathbf{g}_{\underline{\beta}}\cdot\gamma\cdot\mathbf{g}_{\underline{\beta}}^{-1},$$

the family of Dolbeault operators

$$\overline{\partial}_{\underline{\beta}} := \overline{\partial}_0 + \gamma_{\underline{\beta}}$$

can be extended across $\underline{\beta} = 0$ and gives a complex family of holomorphic vector bundles $E_{\underline{\beta}}$ with $E_{\underline{\beta}}$ isomorphic to E for $\underline{\beta} \neq 0$ and isomorphic to $\mathrm{Gr}(E)$ for $\underline{\beta} = 0$. Thus we see that E can be obtained as a complex deformation of $\mathrm{Gr}(E)$. We will soon use this freedom given by the G-action to calibrate the complex deformation parameter γ with the metric variation parameter ε.

While $\mathrm{Gr}(E)$ is polystable, the *asymptotic slopes* $\mu_{L_\varepsilon}(\pi^*\mathcal{G}_i)$ may have different expansions in ε (although they all share the same leading order term $\mu_L(E)$), preventing $\pi^*\mathrm{Gr}(E)$ from being polystable, and the existence of an HYM connection on $\pi^*\mathrm{Gr}(E) \to (X', L_\varepsilon)$. We then assume from now on that for $1 \leq i \leq \ell - 1$,

$$\mu_{L_\varepsilon}(\mathcal{F}_i) \underset{\varepsilon\to 0}{<} \mu_{L_\varepsilon}(E).$$

This provides a *signed* numerical obstruction for $\pi^*\mathrm{Gr}(E)$ to be polystable. This will relate to the contracted curvature as follows. Let's introduce q_i to be the *discrepancy order* of $\mathcal{F}_i$ for each $i \in [\![1, \ell-1]\!]$, that is the order $q_i \in \mathbb{N}$ such that

$$\mu_{L_\varepsilon}(E) - \mu_{L_\varepsilon}(\mathcal{F}_i) = \nu_i\varepsilon^{q_i} + \mathcal{O}(\varepsilon^{q_i+1})$$

for a *positive* constant $\nu_i > 0$. The *discrepancy order* q of E will be

$$q = \max(q_i)_{1\leq i\leq \ell-1}.$$

Define now $\underline{m} = (m_i)_{1\leq i\leq\ell-1}$ where for all i, m_i satisfies[2]

$$2m_i = q_i.$$

We then define $\mathbf{g}_{\underline{\beta},\underline{m}} \in G$ by[3]

$$\mathbf{g}_{\underline{\beta},\underline{m}} = \mathrm{Id}_{\mathcal{G}_1} + \beta_1\varepsilon^{-m_1}\,\mathrm{Id}_{\mathcal{G}_2} + \beta_1\beta_2\varepsilon^{-m_1-m_2}\,\mathrm{Id}_{\mathcal{G}_3} + \ldots + (\Pi_{i=1}^{\ell-1}\beta_i)\varepsilon^{-m_1-\ldots-m_{\ell-1}}\,\mathrm{Id}_{\mathcal{G}_\ell},$$

where the constants $\underline{\beta} = (\beta_i)_{1\leq i\leq\ell-1} \in (\mathbb{R}^*)^{\ell-1}$ will be determined soon. Denote by

$$\gamma_\varepsilon := \mathbf{g}_{\underline{\beta},\underline{m}}\cdot\gamma\cdot\mathbf{g}_{\underline{\beta},\underline{m}}^{-1}. \tag{4.4}$$

[2]The formula differs from [17], where it was $2m_i + 1 = q_i$, as here, when contracting upon Λ_ε, we don't obtain an extra ε-term (as was the case when contracting basic terms in [17])

[3]Note that our convention for the gauge action is opposite to that in [17], which accounts for the negative powers in ε here.

Then, the operators $\overline{\partial}_E$ and $\overline{\partial}_\varepsilon = \overline{\partial}_0 + \gamma_\varepsilon$ are gauge equivalent for any $\underline{\beta}$. We denote by A_ε the Chern connection on $(\pi^*E, \pi^*(\overline{\partial}_\varepsilon), \pi^*h)$ (and we will drop from now the π^* to ease notations). We will consider projections onto various components of $\mathfrak{g}$. For $\psi \in \mathfrak{g}$ and $s \in \Gamma(X', \mathrm{End}(E))$, with

$$s = s_X + s_Z \in \Gamma(X', \mathrm{End}(E)) = \Gamma(X, \mathrm{End}(E)) \oplus \Gamma_0(Z', \mathrm{End}(E)),$$

we will denote by $\Pi_{\langle\psi\rangle}(s)$ the orthogonal projection (as in Equation (4.1)) of s_X onto the subspace spanned by ψ. By Chern-Weil theory, the constants that one obtains by expanding $\Pi_{\langle \mathrm{Id}_{\mathcal{G}_i}\rangle}\Lambda_{\omega_\varepsilon}(iF_{A_0})$ are (up to multiplicative constants) the constant that appear in the ε expansion of $\mu_{L_\varepsilon}(\pi^*\mathcal{G}_i)$. Thus, these terms give the topological defect of $(\pi^*\mathrm{Gr}(E), \pi^*h)$ from being Hermite–Einstein. Considering instead the projections of the contracted curvature onto the $\mathrm{Id}_{\mathcal{F}_i}$'s, we see that those topological deffects appear in the ε expansion precisely at the discrepancy order q_i. On the other hand, using the extra contribution provided by the complex deformation parameters, we can get rid of those obstructions for A_ε, up to the discrepancy order on each $\mathcal{F}_i$. Exactly as in [17, Proposition 49], using Chern-Weil theory together with the facts that the constants ν_i are all strictly positive and the $\gamma_{i,i+1}$ all non zero, we obtain :

Proposition 4.2. *There exist $\underline{\beta} = (\beta_1, \ldots, \beta_{\ell-1}) \in (\mathbb{R}^*)^{\ell-1}$ and a positive constant $\mathcal{C}$ such that for all $j \in [\![1, \ell-1]\!]$,*

$$\Pi_{\langle \mathrm{Id}_{\mathcal{F}_j}\rangle}\Lambda_{\omega_\varepsilon}(iF_{A_\varepsilon}) = \mathcal{C}\,\mu_{L_\varepsilon}(E)\,\mathrm{Id}_{\mathcal{F}_j} + O(\varepsilon^{q_j+\frac{1}{2}}).$$

In what follows, we assume that the constants $\underline{\beta}$ are fixed to satisfy the conclusion in Proposition 4.2. The next step is to produce, for any order $p \geq q$, gauge perturbations $f_\varepsilon \cdot A_\varepsilon$ of A_ε satisfying

$$\Lambda_{\omega_\varepsilon}(iF_{f_\varepsilon\cdot A_\varepsilon}) = \mathcal{C}\,\mu_{L_\varepsilon}(E)\,\mathrm{Id}_E + O(\varepsilon^p).$$

4.2. **Inductive process and approximate solutions.** To construct the approximate solutions to any desired order, we will perturb iteratively the connections A_ε by gauge transformations of the form $\exp(\varepsilon^i s_i)$ and use the properties of the associated Laplace operators. We denote by Δ_ε (resp. $\Delta_{\mathrm{Gr}(E),\varepsilon}$) the associated Laplacian of A_ε (resp. of the Chern connection of $(\pi^*\overline{\partial}_0, \pi^*h)$) with respect to ω_ε. We also set

$$\Delta_X := \Delta_{\mathrm{Gr}(E),0}$$

the Laplacian of $(\overline{\partial}_0, h)$ on X and $\mathcal{L}$ the second order operator associated to $(\overline{\partial}_0, h)$ as in the matrix Expression (3.5).

Proposition 4.3. *Under the above hypotheses, there are expansions*

$$\Delta_\varepsilon = \Delta_{\mathrm{Gr}(E),\varepsilon} + O(\varepsilon), \tag{4.5}$$

and for $s_X + s_Z \in \Gamma(X, \mathrm{End}\,E) \oplus \Gamma_0(Z', \mathrm{End}\,E)$,

$$\Delta_{\mathrm{Gr}(E),\varepsilon}(s_X + \varepsilon s_Z) = \Delta_X(s_X) + \mathcal{L}(s_X) + \Delta_{\mathcal{V}}(s_Z) + \mathcal{O}(\varepsilon). \tag{4.6}$$

The same expansions also hold at a Chern connection on $\pi^*E \to X'$ coming from a complex structure $f_\varepsilon \cdot \pi^*(\overline{\partial}_0 + \gamma_\varepsilon)$ provided $f_\varepsilon = \mathrm{Id}_E + s_\varepsilon$ for some $s_\varepsilon \in \Gamma(X', \mathrm{End}\,\pi^*E)$ whose X-component $s_{X,\varepsilon}$ satisfies $s_{X,\varepsilon} = O(\varepsilon)$ and whose Z'-component $s_{Z,\varepsilon}$ satisfies $s_{Z,\varepsilon} = O(\varepsilon^2)$.

Proof. For the proof of (4.5), we use the formula for the change of curvature induced by the change in complex structure $\overline{\partial}_0 \mapsto \overline{\partial}_0 + \gamma_\varepsilon$:

$$F_{A_\varepsilon} = F_A + d_{A_0}(\gamma_\varepsilon - \gamma_\varepsilon^*) + (\gamma_\varepsilon - \gamma_\varepsilon^*) \wedge (\gamma_\varepsilon - \gamma_\varepsilon^*). \tag{4.7}$$

Upon contraction with ω_ε, we obtain

$$\Lambda_\varepsilon(iF_{A_\varepsilon}) = \Lambda_\varepsilon(iF_{A_0}) + \Lambda_\varepsilon\left(\partial_0\gamma_\varepsilon - \overline{\partial}_0\gamma_\varepsilon^*\right) - 2\Lambda_\varepsilon\left(\gamma_\varepsilon \wedge \gamma_\varepsilon^*\right).$$

Now, by choice of $g_{\underline{\beta},\underline{m}}$ and definition of γ_ε in Equation (4.4), we see that

$$\gamma_\varepsilon = \mathcal{O}(\varepsilon^{\min(m_i)}).$$

By definition, $m_i = \frac{q_i}{2}$. By Formula (1.2), for all $i \in [\![1, \ell - 1]\!]$, we have $q_i \geq n - m$, and as we are blowing-up a submanifold, we must have $m \leq n - 2$. Then, $q_i \geq 2$, and thus $m_i \geq 1$ for all i. Then,

$$\gamma_\varepsilon = \mathcal{O}(\varepsilon),$$

which gives

$$\Lambda_\varepsilon(iF_{A_\varepsilon}) = \Lambda_\varepsilon(iF_{A_0}) + \mathcal{O}(\varepsilon).$$

Note that all the terms in $\left(\partial_0\gamma_\varepsilon - \overline{\partial}_0\gamma_\varepsilon^*\right) - 2\left(\gamma_\varepsilon \wedge \gamma_\varepsilon^*\right)$ are pulled back terms, and arguing as in Section 3.2, we obtain that this estimate is preserved for first order variations, which gives

$$\Delta_\varepsilon = \Delta_{\mathrm{Gr}(E),\varepsilon} + O(\varepsilon).$$

The proof of (4.6) is a direct consequence of the various lemmas in Section 3.2. Finally, we consider the perturbed connections under gauge transformations f_ε as described in the statement of the lemma. The expansion (4.5) is trivially preserved for such connections, as $f_\varepsilon = \mathrm{Id}_E + \mathcal{O}(\varepsilon)$. Then, the X-part of $\partial_{A_\varepsilon}\bar{\partial}_{A_\varepsilon} - \bar{\partial}_{A_\varepsilon}\partial_{A_\varepsilon}$ changes at order ε while its Z'-part changes at order ε^2, which is enough to compensate for the ε^{-1}-contribution that comes upon restriction to Z' and contraction, as in the proof of Lemma 3.4. This concludes the proof. □

When producing the approximate HYM connections, we will use the mapping properties of Δ_X and $\Delta_{\mathcal{V}}$. This will allow us to remove all error terms modulo elements in the cokernel of $\Delta_X = \Delta_{\mathrm{Gr}(E),0}$, which is $\mathfrak{g}$. By Chern-Weil theory, those remaining terms will be controlled by the expansions of $\mu_{L_\varepsilon}(\mathcal{F}_i)$, and it is important in the argument that upon removing the other errors, they remain fixed. This is the content of the following lemma.

Lemma 4.4. *Let $s \in \Gamma(\mathrm{End}(\mathcal{G}_i))$ and $j \in [\![1, \ell]\!]$. Then*

$$\Pi_{\langle \mathrm{Id}_{\mathcal{F}_j}\rangle}\Delta_\varepsilon(s) = \mathcal{O}(\varepsilon^{q_j})$$

and

$$\Pi_{\langle \mathrm{Id}_{\mathcal{F}_j}\rangle}\Lambda_\varepsilon\left((\partial_\varepsilon s - \bar{\partial}_\varepsilon s) \wedge (\partial_\varepsilon s - \bar{\partial}_\varepsilon s)\right) = \mathcal{O}(\varepsilon^{q_j}).$$

Let $\sigma \in \Gamma(\mathrm{Hom}(\mathcal{G}_p, \mathcal{G}_i))$, with $i < p$, and $j \in [\![1, \ell]\!]$. Then

$$\Pi_{\langle \mathrm{Id}_{\mathcal{F}_j}\rangle}\Delta_\varepsilon(\varepsilon^{m_i + \ldots + m_{p-1} + 1}\sigma) = \mathcal{O}(\varepsilon^{q_j + 1}).$$

and

$$\Pi_{\langle \mathrm{Id}_{\mathcal{F}_j}\rangle}\Lambda_\varepsilon\left(\varepsilon^{2(m_i + \ldots + m_{p-1} + 1)}\left((\partial_\varepsilon\sigma - \bar{\partial}_\varepsilon\sigma) \wedge (\partial_\varepsilon\sigma - \bar{\partial}_\varepsilon\sigma)\right)\right) = \mathcal{O}(\varepsilon^{q_j + 1}),$$

and similarily for $p < i$.

Proof. Note that we only need to consider the $\Gamma(X, \operatorname{End} E)$ components of the sections in the splitting (3.4), as we are only interested in the projection onto $\operatorname{Id}_{\mathcal{F}_j}$. Then, the proof follows exactly as in [17, Lemma 46 and Lemma 47]. □

The only errors that will then remain to remove are those in $\mathfrak{g}$. This is where we will use the following lemma, whose proof is a direct adaptation of [17, proof of Lemma 48].

Lemma 4.5. *For all $j \in [\![1, \ell-1]\!]$, there is a negative constant $a_{j,j+1}$ such that*

$$\Pi_{\mathfrak{g}} \Delta_\varepsilon(\operatorname{Id}_{\mathcal{F}_j}) = a_{j,j+1} \varepsilon^{q_j} \operatorname{Id}_{j,j+1} + \mathcal{O}(\varepsilon^{q_j+1}), \tag{4.8}$$

where for $(p, l) \in [\![1, \ell]\!]^2$:

$$\operatorname{Id}_{pl} = \frac{1}{\operatorname{rank} \mathcal{G}_p} \operatorname{Id}_{\mathcal{G}_p} - \frac{1}{\operatorname{rank} \mathcal{G}_l} \operatorname{Id}_{\mathcal{G}_l}.$$

Note that as in Proposition 4.3, the conclusions of Lemma 4.4 and 4.5 also hold for perturbed connections of the form $f_\varepsilon \cdot A_\varepsilon$ for $f_\varepsilon = \operatorname{Id}_E + s_{\varepsilon,X} + s_{\varepsilon,Z}$ with $s_{\varepsilon,X} = \mathcal{O}(\varepsilon)$ and $s_{\varepsilon,Z} = \mathcal{O}(\varepsilon^2)$. We are now ready to prove the main proposition of this section.

Proposition 4.6. *Let $p \geq q$. Then, there exists gauge transformations $f_\varepsilon^p = \operatorname{Id}_E + s_{\varepsilon,X}^p + s_{\varepsilon,Z}^p$ with $s_{\varepsilon,X}^p = \mathcal{O}(\varepsilon)$ and $s_{\varepsilon,Z}^p = \mathcal{O}(\varepsilon^2)$, and constants c_ε^p, such that for all $0 < \varepsilon \ll 1$, we have*

$$\Lambda_\varepsilon \left(F_{f_\varepsilon^p \cdot A_\varepsilon} \right) = c_\varepsilon^p \operatorname{Id}_E + \mathcal{O}(\varepsilon^p).$$

Proof. The proof follows the strategy developped in [17, Section 5.2 and Section 5.3.2], so we refer to those relevant sections and will only sketch the argument. We can expand in ε each term on the right hand side of the following identity :

$$\Lambda_\varepsilon(iF_{A_\varepsilon}) = \Lambda_\varepsilon(iF_{A_0}) + \Lambda_\varepsilon \left(\partial_0 \gamma_\varepsilon - \overline{\partial}_0 \gamma_\varepsilon^* \right) - 2\Lambda_\varepsilon \left(\gamma_\varepsilon \wedge \gamma_\varepsilon^* \right).$$

As A_0 is HYM, we have

$$\Lambda_\varepsilon(iF_{A_0}) = c \operatorname{Id}_E + \mathcal{O}(\varepsilon).$$

By the gauge fixing condition $\overline{\partial}_0^* \gamma = \Lambda_0 \partial_0 \gamma = 0$, the following off-diagonal terms have entries

$$(\Lambda_\varepsilon \left(\partial_0 \gamma_\varepsilon - \overline{\partial}_0 \gamma_\varepsilon^* \right))_{ip} = \mathcal{O}(\varepsilon^{m_i + \ldots + m_{p-1} + 1}),$$

while the remaining terms will contribute to higher orders. Following [17, Proposition 23 and Section 5.2], we start by perturbing inductively A_ε to remove all errors that live in $\Gamma_{i\mathfrak{k}^\perp}(X, \operatorname{End}_H(E, h)) \oplus \Gamma_0(Z', \operatorname{End}_H(E, h))$. This can be achieved by considering perturbations $\exp(\varepsilon^i (s_X + \varepsilon s_Z) \cdot A_\varepsilon)$ thanks to Proposition 4.3, together with Lemmas 3.3 and 4.1. What remains are errors in $i\mathfrak{k}$. By Lemma 4.4, and from the formula

$$\Lambda F_{f \cdot A} = \Lambda F_A + \Delta_A(f) + \Lambda \left((\partial_A f - \overline{\partial}_A f) \wedge (\partial_A f - \overline{\partial}_A f) \right),$$

the previous perturbations won't affect any of the $\operatorname{Id}_{\mathcal{F}_j}$ projections of the contracted curvatures. By proposition 4.2, we then obtain a connection whose contracted curvature satisfies, for all $j \in [\![1, \ell-1]\!]$,

$$\Lambda_{\omega_\varepsilon} \left(iF_{A_\varepsilon} \right) = \mathcal{C} \, \mu_{L_\varepsilon}(E) \operatorname{Id}_{\mathcal{F}_j} + O(\varepsilon^{q_j + \frac{1}{2}}).$$

The end of the proof is then done by induction on ℓ the number of stable components of $\operatorname{Gr}(E)$. As discussed in [17, Section 5.3.2], using Lemma 4.5, we can remove errors in $\mathfrak{g}$ beyond the orders q_j, by induction on ℓ, and eventually obtain the result. □

5. The perturbation argument

We keep notations from the last section. We will now prove Theorem 1.1, and its corollaries. Again, we follow closely [17]. The first step is to obtain an upper bound for the operator norm of the inverse of Δ_ε. This is where the argument from [17] has to be adapted, given the different geometric settup that we address. Then, based on this bound and on the construction of approximate solutions, a quantitative version of the implicit function theorem is used in Section 5.2.

For a Riemannian metric g on X', let $L^2_d(g)$ denote the Sobolev space $W^{d,2}(X',g)$ of order 2 and d derivatives, with respect to the metric g. If g is Kähler with Kähler form η, we may write $L^2_d(\eta)$ instead of $L^2_d(g)$. When $d=0$, we omit the subscript. For Sobolev spaces associated to π^*E or $\operatorname{End}\pi^*E$, we use the metric h (and the metric it induces on $\operatorname{End}\pi^*E$) on the bundle.

5.1. Estimating the linearisation.

Our main goal in this section is to prove:

Proposition 5.1. *There exists $C>0$ such that for all $s\in\Gamma(X',\operatorname{End}E)$ whose trace is average zero with respect to ω_ε,*

$$\|\Delta_\varepsilon(s)\|_{L^2_d(\omega_\varepsilon)} \geq C\varepsilon^q\|s\|_{L^2_{d+2}(\omega_\varepsilon)}. \tag{5.1}$$

The same estimate also holds for the Laplacian of the perturbed connections built in Proposition 4.6.

This will be done by steps. We first work out the case of a single stable component $\mathcal{G}:=\mathcal{G}_i\subset\operatorname{Gr}(E)$.

Lemma 5.2. *There exists $C>0$ such that for all $s\in\Gamma(X',\operatorname{End}\mathcal{G})$ that is $L^2(\omega_\varepsilon)$-orthogonal to $\operatorname{Id}_{\mathcal{G}}$,*

$$\|\Delta_\varepsilon(s)\|_{L^2_d(\omega_\varepsilon)} \geq C\varepsilon\|s\|_{L^2_{d+2}(\omega_\varepsilon)}. \tag{5.2}$$

The proof of Lemma 5.2 will require to compare various L^2-norms on X', X and Z'. We gather those comparisons in the following lemmas.

Lemma 5.3. *There are positive constants C and C' independent on ε such that for any $\sigma\in\Gamma(X,\operatorname{End}\mathcal{G})$ (or $\sigma\in\Omega^1(X,\operatorname{End}\mathcal{G})$), we have*

$$C\|\pi^*\sigma\|_{L^2(X',\omega_\varepsilon)} \leq \|\sigma\|_{L^2(X,\omega)} \leq C'\|\pi^*\sigma\|_{L^2(X',\omega_\varepsilon)}. \tag{5.3}$$

Proof. The result follows from the facts that the 2-forms ω_ε vary in a bounded family ($\varepsilon\in[0,\varepsilon_0]$), and that a pulled back section is constant on the fibers of $\pi: Z'\to Z$. □

Lemma 5.4. *There are positive constants C and C' independent on ε such that for any $\sigma\in\Gamma_0(Z',\operatorname{End}\mathcal{G})$ (or $\sigma\in\Omega^1(Z',\operatorname{End}\mathcal{G})$), we have*

$$C\|\iota_*\sigma\|^2_{L^2(X',\omega_\varepsilon)} \leq \|\sigma\|^2_{L^2(Z',\omega_\varepsilon)} \leq C'\|\iota_*\sigma\|^2_{L^2(X',\omega_\varepsilon)}. \tag{5.4}$$

Proof. This can be done by a local and then patching argument. We may fix $W\subset X'$ an open set with $Z'\subset W$, and such that $\iota_*\sigma$ vanishes away from W. Then,

$$\|\iota_*\sigma\|^2_{L^2(X',\omega_\varepsilon)} = \int_W \|\iota_*\sigma\|^2_\varepsilon\,\omega_\varepsilon^n.$$

We may also consider an open finite covering of W by sets U of the form $U\simeq B(0,1)\times V$ where the sets $V\subset Z'$ cover Z' and such that $\iota_*\sigma$ is given locally by

$\rho\cdot\sigma$ for a bump function ρ on $B(0,1)$. Then, from the facts that ε varies in a compact family, and that for any $v\in V$, the metrics $(\omega_\varepsilon(b,v))_{b\in B(0,1)}$ on $B(0,1)\times\{v\}$ are mutually bounded, by applying Fubini's theorem, we obtain bounds :

$$C\int_U \|\iota_*\sigma\|^2_\varepsilon\,\omega_\varepsilon^n \leq \int_V \|\sigma\|^2_\varepsilon\,\omega_\varepsilon^{n-1}\leq C'\int_U \|\iota_*\sigma\|^2_\varepsilon\,\omega_\varepsilon^n.$$

A patching argument then provides the result. □

Lemma 5.5. *There exists a positive constant C independent on ε such that for any $s_X\in\Gamma(X,\operatorname{End}\mathcal{G})$ and $s_Z\in\Gamma_0(Z',\operatorname{End}\mathcal{G})$, we have*

$$|\langle s_X,s_Z\rangle_{L^2(\omega_\varepsilon)}|\leq C\varepsilon^3||s_X||_{L^2(\omega_\varepsilon)}\,||s_Z||_{L^2(\omega_\varepsilon)}.$$

Proof. We keep the notations from the proof of Lemma 5.4. Note that we can restrict to the open set W as s_Z vanishes away from W. Then, locally, we estimate

$$\int_U \operatorname{trace}(s_X\cdot s_Z)\,\omega_\varepsilon^n.$$

Take coordinates $(b,f,v)\in B(0,1)\times F\times V_Z$ so that $U\simeq B(0,1)\times F\times V_Z$ where this time F denote fibers of $\pi: Z'\to Z$, $V_Z\subset Z$ is an open set in Z and $B(0,1)$ stands for normal coordinates. We may assume that s_X is independent on $f\in F$:

$$s_X=s_X(b,v)$$

and s_Z is of the form

$$s_Z=\rho(b)s_Z(f,w)$$

for ρ some cut off function used to produce the extension operator. Then, we compute with Fubini's :

$$\begin{aligned}\int_U \operatorname{trace}(s_X\cdot s_Z)\,\omega_\varepsilon^n &= \int_{B\times F\times V_Z}\rho(b)\operatorname{trace}(s_X(b,w)\cdot s_Z(f,w))\,\omega_\varepsilon^n\\ &= \int_{B\times V_Z}\rho(b)\int_F \operatorname{trace}(s_X(b,w)\cdot s_Z(f,w))\,\omega_\varepsilon^n.\end{aligned}$$

But to higher order in ε,

$$\int_F \operatorname{trace}(s_X(b,w)\cdot s_Z(f,w))\,\omega_\varepsilon^n =$$

$$\varepsilon^{n-m-1}\omega^{m+1}\int_F \operatorname{trace}(s_X(b,w)\cdot s_Z(f,w))\,\lambda^{n-m-1}+\mathcal{O}(\varepsilon^{n-m})$$

and as s_Z is of average zero w.r.t. λ^{n-m-1}, we get

$$\int_F \operatorname{trace}(s_X(b,w)\cdot s_Z(f,w))\omega_\varepsilon^n=\mathcal{O}(\varepsilon^{n-m}),$$

which is $\mathcal{O}(\varepsilon^3)$ as $\operatorname{codim}(Z)=n-m\geq 3$. The result then follows from Cauchy-Schwarz inequality, for a constant C that depends on ω,λ and ε_0. □

Proof of Lemma 5.2. The positive constants C,C',C_i used in the proof might vary at several stages. We follow the strategy from [17, Section 4.2]. The key to obtain this estimate is the analogue of [17, Lemma 26], as the rest of the argument follows as in [17, Section 4.2]. So our goal is to obtain the following Poincaré type inequality: there exists $C>0$ such that for any $\varepsilon\in[0,\varepsilon_0]$, and all $s\in\Gamma(X',\operatorname{End}\mathcal{G})$ that is $L^2(\omega_\varepsilon)$-orthogonal to $\operatorname{Id}_{\mathcal{G}}$, we have :

$$\|d_{A_\varepsilon}(s)\|^2_{L^2(\omega_\varepsilon)}\geq C\varepsilon\|s\|^2_{L^2(\omega_\varepsilon)}.$$

As $A_\varepsilon = A_0 + \mathcal{O}(\varepsilon)$, it is actually enough to obtain this Poincaré inequality for A_0:

$$\|d_{A_0}(s)\|^2_{L^2(\omega_\varepsilon)} \geq C\varepsilon \|s\|^2_{L^2(\omega_\varepsilon)}.$$

We will then prove this in three steps. First, for sections $s \in \Gamma(X, \operatorname{End}\mathcal{G})$, then for sections $s \in \Gamma_0(Z', \operatorname{End}\mathcal{G})$ and last for sums of such sections.

Step 1 : For $\sigma \in \Gamma(X, \operatorname{End}\mathcal{G})$ and $s = \pi^*\sigma \in \Gamma(X', \operatorname{End}\mathcal{G})$, let

$$\begin{aligned} \alpha &= \frac{1}{\operatorname{rank}\mathcal{G}}\,\frac{1}{\operatorname{vol}(X,\omega)}\int_X \operatorname{trace}(\sigma)\,\omega^n, \\ \alpha' &= \frac{1}{\operatorname{rank}\mathcal{G}}\,\frac{1}{\operatorname{vol}(X',\omega_\varepsilon)}\int_{X'} \operatorname{trace}(s)\,\omega_\varepsilon^n. \end{aligned}$$

Then, s is orthogonal to Id on X' if and only if $\alpha' = 0$. Moreover,

$$\alpha' - \alpha = \mathcal{O}(\varepsilon)\int_X \operatorname{trace}(\sigma)\,\omega^n.$$

Then, noting that $d_{A_0}\sigma = d_{A_0}(\sigma - \alpha\,\mathrm{Id})$ as Id is parallel, we have

$$\begin{aligned} \|d_{A_0}(\pi^*\sigma)\|^2_{L^2(X',\omega_\varepsilon)} &\geq C_1\|\pi^* d_{A_0}\sigma\|^2_{L^2(X',\omega_\varepsilon)}, \\ &\geq C_2\|d_{A_0}\sigma\|^2_{L^2(X,\omega)}, \\ &\geq C_3\|\sigma - \alpha\,\mathrm{Id}\,\|^2_{L^2(X,\omega)}, \\ &\geq C_4\|\sigma\|^2_{L^2(X,\omega)}, \\ &\geq C_5\|s\|^2_{L^2(X',\omega_\varepsilon)}, \end{aligned}$$

where the second and fifth inequalities follow from Lemma 5.3, the third one from Poincaré inequality for A_0 on (X, ω), and the fourth inequality follows since α is $\mathcal{O}(\varepsilon\|\sigma\|_{L^2(\omega)})$.

Step 2 : We next consider $s = \iota_*\sigma \in \Gamma_0(Z', \operatorname{End}\mathcal{G})$. We set this time α'' to be the constant given by the $L^2(Z', \omega_\varepsilon)$-orthogonal projection of σ onto Id_E :

$$\alpha'' = \frac{1}{\operatorname{rank}\mathcal{G}\operatorname{vol}(Z',\omega_\varepsilon)}\int_{Z'} \operatorname{trace}(\sigma)\,\omega_\varepsilon^{n-1}.$$

In general, for a given $\sigma \in \Gamma(Z', \operatorname{End}\mathcal{G})$, $\alpha'' = \mathcal{O}(1)$ with respect to ε. However, by definition of the subspace $\Gamma_0(Z', \operatorname{End}\mathcal{G}) \subset \Gamma(Z', \operatorname{End}\mathcal{G})$, since $\sigma \in \Gamma_0(Z', \operatorname{End}\mathcal{G})$ is orthogonal to the identity on all fibres of $\pi : Z' \to Z$, an argument similar to that of Lemma 5.5, using the same system of coordinates $(b, f, v) \in B(0,1) \times F \times V_Z$, shows that $\alpha'' = \mathcal{O}(\varepsilon\|\sigma\|_{L^2(Z',\omega_\varepsilon)})$. Then we find positive constants C_i such that

$$\begin{aligned} \|d_{A_0}(\iota_*\sigma)\|^2_{L^2(\omega_\varepsilon)} &\geq C_1\|\iota_*(d_{A_0}\sigma)\|^2_{L^2(X',\omega_\varepsilon)} \\ &\geq C_2\|d_{A_0}\sigma\|^2_{L^2(Z',\omega_\varepsilon)} \\ &\geq \varepsilon\, C_3\|\sigma - \alpha''\mathrm{Id}\|^2_{L^2(Z',\omega_\varepsilon)} \\ &\geq \varepsilon\, C_4\|\sigma\|^2_{L^2(Z',\omega_\varepsilon)}, \\ &\geq \varepsilon\, C_5\|\iota_*\sigma\|^2_{L^2(X',\omega_\varepsilon)}, \end{aligned}$$

where this time the first inequality follows from construction of the operator ι_*, the second and last inequalities come from Lemma 5.4, the third one follows from the corresponding [17, Lemma 26], and the fourth one from the fact that $\alpha'' = \mathcal{O}(\varepsilon\|\sigma\|_{L^2(Z',\omega_\varepsilon)})$.

Step 3 : We need now to obtain similar estimates for sections $s = s_X + s_Z$. Note however that the splitting of (3.4) is *not* orthogonal with respect to the $L^2(\omega_\varepsilon)$ inner product. What remains is to estimate

$$\langle d_{A_0} s_X, d_{A_0} s_Z\rangle_{L^2(\omega_\varepsilon)} = \langle \Delta_{A_0,\varepsilon}(s_X), s_Z\rangle_{L^2(\omega_\varepsilon)} = \langle s_X, \Delta_{A_0,\varepsilon}(s_Z)\rangle_{L^2(\omega_\varepsilon)},$$

for $\Delta_{A_0,\varepsilon}$ the Laplacian of A_0 with respect to ω_ε. We use the expansion of Δ_ε from Section 3.2. From Lemma 3.6, Lemma 3.7 and formula (3.5) :

$$\langle \Delta_{A_0,\varepsilon}(s_X), s_Z\rangle_{L^2(\omega_\varepsilon)} = \langle \Delta_X s_X, s_Z\rangle_{L^2(\omega_\varepsilon)} + \langle \mathcal{L}(s_X), s_Z\rangle_{L^2(\omega_\varepsilon)} + \mathcal{O}(\varepsilon)$$

Then, from Lemma 5.5,

$$|\langle \Delta_X s_X, s_Z\rangle_{L^2(\omega_\varepsilon)}| \leq C_1\,\varepsilon^3 ||\Delta_X s_X||_{L^2(\omega_\varepsilon)}\,||s_Z||_{L^2(\omega_\varepsilon)}.$$

Setting $s_X = \pi^*\sigma$, and using Lemma 5.3, together with the continuity of the Green operator of Δ_X, we obtain

$$\begin{aligned} |\langle \Delta_X s_X, s_Z\rangle_{L^2(\omega_\varepsilon)}| &\leq C_2\,\varepsilon^3 ||\Delta_X \sigma||_{L^2(\omega)}\,||s_Z||_{L^2(\omega_\varepsilon)},\\ &\leq C_3\,\varepsilon^3 ||\sigma||_{L^2(\omega)}\,||s_Z||_{L^2(\omega_\varepsilon)},\\ &\leq C_4\,\varepsilon^3 ||s_X||_{L^2(\omega_\varepsilon)}\,||s_Z||_{L^2(\omega_\varepsilon)}, \end{aligned}$$

where again we dealt with the $L^2(X,\omega)$-projection of σ onto $\mathrm{Id}_{\mathcal{G}}$ as in Step 1. For the term $\langle \mathcal{L}(s_X), s_Z\rangle_{L^2(\omega_\varepsilon)}$, we use local coordinates $(b, f, v) \in B \times F \times V_Z$ as in the proof of Lemma 5.5. Then, the expression of $\mathcal{L}$ can be locally written

$$\begin{aligned} \mathcal{L}(s_X) &= \iota_* \circ p_0 \circ \iota^* \left((m+1) \frac{D s_X \wedge \omega^m \wedge \lambda^{n-m-1}}{\omega^{m+1} \wedge \lambda^{n-m-1}} \right)\\ &= (m+1) \frac{D_b s_X \wedge \omega^m \wedge \lambda^{n-m-1}}{\omega^{m+1} \wedge \lambda^{n-m-1}} \end{aligned}$$

where D_b stands for derivatives in the $B(0,1)$-direction. We then compute the local contribution on U of the term $\langle \mathcal{L}(s_X), s_Z\rangle_{L^2(\omega_\varepsilon)}$:

$$\int_U \mathrm{trace}(\mathcal{L}(s_X)\cdot s_Z)\,\omega_\varepsilon^n = \int_{B\times F\times V_Z} \rho(b)\,\mathrm{trace}(\mathcal{L}(s_X(b,w))\cdot s_Z(f,w))\,\omega_\varepsilon^n,$$

and by Fubini's, Stokes theorem, and integration by parts, the relevant higher order term in ε is

$$\int_U D_b(\rho(b))\,\mathrm{trace}(s_X \cdot s_Z)\omega_\varepsilon^n,$$

which is $\mathcal{O}(\varepsilon^3\,||s_X||_{L^2(\omega_\varepsilon)}\,||s_Z||_{L^2(\omega_\varepsilon)})$ by a similar argument as in Lemma 5.5. Gathering those estimates, we see that the mixed terms satisfy

$$\langle d_{A_0} s_X, d_{A_0} s_Z\rangle_{L^2(\omega_\varepsilon)} = \langle \Delta_{A_0,\varepsilon}(s_X), s_Z\rangle_{L^2(\omega_\varepsilon)} = \mathcal{O}(\varepsilon^3\,||s_X||_{L^2(\omega_\varepsilon)}\,||s_Z||_{L^2(\omega_\varepsilon)}),$$

and together with steps 1 and 2, using Lemma 5.5 again, we obtain

$$\|d_{A_0}(s_X + s_Z)\|^2_{L^2(\omega_\varepsilon)} \geq C\varepsilon \|s_X + s_Z\|^2_{L^2(\omega_\varepsilon)}.$$

Then, from this Poincaré inéquality, the result follows as in [17, Section 4.2], using an analogous uniform Schauder estimate as in [17, Proposition 27]. The latter estimate can be obtained by patching local Schauder estimates (as in [17, Lemma 28]) that are easily derived away from the exceptional divisor, and can be obtained around a point in the exceptional divisor by adapting [17, Proof of Lemma 28] using local coordinates as in Lemma 5.5. □

Once this bound settled, we can deal with the general case.

Proof of Proposition 5.1. The proof for the estimate (5.1) can be established exactly as in [17, Proposition 50], following [17, Lemma 51, Lemma 52]. First, to obtain the estimate for elements in $\mathfrak{g}^\perp$, one uses the full expansion in the linearisation :

$$\begin{array}{rcl}\Delta_\varepsilon & = & \Delta_{\mathrm{Gr}(E),\varepsilon} \\ & + & i\Lambda_\varepsilon\left(\overline{\partial}_0([\gamma_\varepsilon^*,\cdot]) - [\gamma_\varepsilon^*,\overline{\partial}_0\cdot]\right) \\ & + & i\Lambda_\varepsilon\left(\partial_0([\gamma_\varepsilon,\cdot]) - [\gamma_\varepsilon,\partial_0\cdot]\right) \\ & + & i\Lambda_\varepsilon([\gamma_\varepsilon,[\gamma_\varepsilon^*,\cdot]] - [\gamma_\varepsilon^*,[\gamma_\varepsilon,\cdot]]).\end{array} \tag{5.5}$$

As γ_ε is a pulled back term, arguing as in Section 3.2, we see that the X and Z' contributions of the operator $\Lambda_\varepsilon\left(\partial_0([\gamma_\varepsilon,\cdot])\right)$ in the splitting (3.4) will be of the same order as γ_ε. As argued before, from the choice of $g_{\underline{\beta},\underline{m}}$ and definition of γ_ε in Equation (4.4), we see that

$$(\gamma_\varepsilon)_{ip} = \mathcal{O}(\varepsilon^{m_i}).$$

Now, $m_i = \frac{q_i}{2} \geq \frac{\mathrm{codim}(Z)}{2}$ by Formula (1.2). Arguing similarly for the other terms in (5.5), we see that the contributions from $\Delta_\varepsilon - \Delta_{\mathrm{Gr}(E),\varepsilon}$ will all come at order at least $\varepsilon^{\frac{\mathrm{codim}(Z)}{2}}$, and will be absorbed by the estimate in Lemma 5.2, because we assumed $\mathrm{codim}(Z) \geq 3$. Arguing as in [17, Lemma 51], we obtain the bound :

$$\|\Delta_\varepsilon(s)\|_{L^2_d(\omega_\varepsilon)} \geq C\varepsilon\|s\|_{L^2_{d+2}(\omega_\varepsilon)},$$

for $s \in \mathfrak{g}^\perp$. Then, from here, the proof for the estimate for sections in $\mathfrak{g}$, or for sums of sections, follows as in [17, proof of Lemma 52 and proof of Proposition 50]. Finally, the result for perturbed connections follows as in [17, Proposition 53]. □

5.2. **Perturbing to solutions.** To conclude the proof of Theorem 1.1, we refer to [17, Section 4.2 and Section 5.4.2]. It relies on a quantitative version of the implicit function theorem, applied to the operator

$$\begin{array}{rccc}\Psi_{p,\varepsilon}: & L^2_{d+2}(X',\omega_\varepsilon)\times\mathbb{R} & \to & L^2_d(X',\omega_\varepsilon) \\ & (s,\alpha) & \mapsto & i\Lambda_{\omega_\varepsilon}(F_{\exp(s)\cdot f^p_\varepsilon\cdot A_\varepsilon}) - \alpha\,\mathrm{Id}_E,\end{array}$$

where $f^p_\varepsilon \cdot A_\varepsilon$ are the connections built in Proposition 4.6. Proposition 4.6 is the analogue of [17, Proposition 23 and Section 5.3.2] where approximate solutions to $\Psi_{p,\varepsilon} = 0$ are constructed, while Proposition 5.1 plays the role of [17, Propositions 50 and 53] and provides the required estimate on the linearisation of $\Psi_{p,\varepsilon}$, for p large enough, at the approximate solutions. Then, the implicit function theorem, as stated in [17, Theorem 24], enables to conclude the existence of zeros for $\Psi_{p,\varepsilon}$, for p large and ε small, which ends the proof of Theorem 1.1.

5.3. **Proof of the corollaries.** We comment now on the various corollaries stated in the introduction. First, Corollary 1.4 is a direct application of Theorem 1.1, where $E = \mathrm{Gr}(E)$ as a single stable component. Corollary 1.5 also follows directly, using Formula (1.2). What remains is to show Corollary 1.6. The only remaing case to study is when for all $i \in [\![1,\ell-1]\!]$, $\mu_{L_\varepsilon}(\mathcal{F}_i) \underset{\varepsilon\to 0}{\leq} \mu_{L_\varepsilon}(E)$, with at least one equality. This case can be dealt with exactly as for [17, Corollary 32], proved in [17, Section 5.5].

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