CLUSTERING PHENOMENA IN LOW DIMENSIONS FOR A BOUNDARY YAMABE PROBLEM

SERGIO CRUZ-BLÁZQUEZ, ANGELA PISTOIA, AND GIUSI VAIRA

Abstract. The main scope of this paper is to obtain non-compactness results for a linearly perturbed version of the classical problem of prescribing the scalar and boundary mean curvatures of a compact Riemannian manifold via conformal changes of the metric. One of the principal difficulties for the study of the geometric problem is the loss of compactness due to bubbling of solutions. However, in dimension three, under additional assumptions on the geometry of the manifold and the prescribed curvatures, it has been possible to recover compactness by showing that blow-up points are isolated and simple. In this work we prove that this property becomes false under arbitrarily small linear perturbations of the boundary equation as soon as the restriction on the dimension is lifted. More precisely, on a generic manifold of positive conformal class and dimension between four and seven, we construct a solution for the case of negative prescribed scalar curvature that exhibits a clustering blow-up point at the boundary which is non-umbilic and a local minimizer of the squared norm of the trace-free second fundamental form.

1. Introduction

Given a compact Riemannian manifold (M, g) of dimension $n \geq 3$ with boundary ∂M, a widely studied geometric problem is the following one: *given two smooth functions K and H find a metric conformal to g whose scalar curvature is K and boundary mean curvature is H.*
As it is well known, the geometric problem can be rephrased into the following one: *given two smooth functions K and H find a positive solution to the PDE*

$$\begin{cases} -\dfrac{4(n-1)}{n-2}\Delta_g u + \mathcal{S}_g u = K u^{\frac{n+2}{n-2}} & \text{in } M \\ \dfrac{2}{n-2}\dfrac{\partial u}{\partial \nu} + h_g u = H u^{\frac{n}{n-2}} & \text{on } \partial M. \end{cases} \tag{1.1}$$

Here $\Delta_g := \mathrm{div}_g(\nabla)$ is the Laplace-Beltrami operator, $\mathcal{S}_g$ is the scalar curvature and h_g the boundary mean curvature associated to the metric g and ν is the outward unit normal vector to ∂M. The metric $\tilde{g} = u^{\frac{4}{n-2}} g$ is conformal to g and its scalar and boundary mean curvatures are nothing but K and H, respectively.

The study began with the work of Cherrier [13] who gave a first criterion for the existence and regularity of solution of (1.1). Successively, Escobar in a series of papers [21, 23, 22] found a solution to (1.1) when either $K = 0$ (i.e. scalar flat metric) and H constant or $H = 0$ (i.e. minimal boundary) and K is constant. The proof strongly relies on the dimensions of the manifold, on the properties of the boundary (e.g. being or not umbilic) and on vanishing properties of the Weyl tensor (e.g. being identically zero or not on the boundary or on the whole manifold). Important contributions in this framework are due to the work of Marques in [34, 33], Almaraz [3], Brendle & Chen [9] and Mayer & Ndiaye [35]. The case when $K > 0$ and H is an arbitrary constant, has been successfully treated by Han & Li in [28, 27] and Chen, Ruan & Sun [12].

Date: August 23, 2024.
2010 *Mathematics Subject Classification.* 35B44, 58J32.
Key words and phrases. Prescribed curvature problem, conformal metric, clustering blow-up point.

There are a few results concerning the general case in which K and H are functions (not necessarily constants) and all of them have been obtained for special manifolds (e.g. typically the unit ball or the half sphere). In particular, we refer to the works of Ben Ayed, El Mehdi & Ould Ahmedou [7, 8] and Li [29] when $H = 0$ and Abdelhedi, Chtioui & Ould Ahmedou [1], Chang, Xu & Yang [10], Djadli, Malchiodi & Ould Ahmedou [19] and Xu & Zhang [44] when $K = 0$. The case when both K and H do not vanish, has been studied by Ambrosetti, Li & Malchiodi [5] in a perturbative setting on the $n-$dimensional unit ball and by Djadli, Malchiodi & Ould Ahmedou [18] on the three-dimensional half sphere. Finally, we quote the result of Chen, Ho & Sun [11] where they found a solution to (1.1) when H and K are negative functions provide the manifold has a boundary of negative Yamabe invariant.

Recently, Cruz-Blázquez, Malchiodi and Ruiz [15] considered a manifold whose scalar curvature $\mathcal{S}_g \leq 0$ and the case K negative and H of arbitrary sign. They introduce the *scaling invariant* quantity

$$\mathfrak{D}_n(p) = \sqrt{n(n-1)}\frac{H(p)}{\sqrt{|K(p)|}},\ p \in \partial M$$

and established the existence of a solution to (1.1) whenever $\mathfrak{D}_n < 1$ along the whole boundary. On the other hand, if $\mathfrak{D}_n > 1$ at some boundary points they got a solution only in a three dimensional manifold, for a generic choice of K and H. Let us describe more carefully their result. First of all, via the conformal change of metric due to Escobar [23], one can assume that the mean curvature $h_g = 0$ and $\mathcal{S}_g$ has constant sign (this will be also assumed understood in the rest of our paper), so problem (1.1) reads as

$$\begin{cases} -\dfrac{4(n-1)}{n-2}\Delta_g u + \mathcal{S}_g u = K u^{\frac{n+2}{n-2}} & \text{in } M \\ \dfrac{2}{n-2}\dfrac{\partial u}{\partial \nu} = H u^{\frac{n}{n-2}} & \text{on } \partial M. \end{cases} \tag{1.2}$$

Problem (1.2) is variational in nature, i.e. the solutions of (1.2) are critical points of the energy functional defined on $H^1(M)$

$$J(u) = \frac{2(n-1)}{n-2}\int_M |\nabla_g u|^2 + \frac{1}{2}\int_M \mathcal{S}_g u^2 - \frac{1}{2^*}\int_M K(u^+)^{2^*} - (n-2)\int_{\partial M} H(u^+)^{2^\sharp}$$

where $2^* = \frac{2n}{n-2}$ and $2^\sharp = \frac{2(n-1)}{n-2}$ are the critical Sobolev exponent for M and the critical trace embedding exponent for ∂M, respectively. In [15] the authors show that if $\mathcal{S}_g \leq 0$ and $\mathfrak{D}_n < 1$ along the whole boundary, the functional becomes coercive and they found a global minimizer. On the other hand, if there exists $p \in \partial M$ such that $\mathfrak{D}_n(p) > 1$, they construct a sequence of functions u_i such that the energy $J(u_i) \to -\infty$ and the minimum point does not exist anymore. However, on a $3-$dimensional manifold they recover the existence of a positive solution by using a mountain pass type argument. Their proof relies on a careful blow-up analysis: first they show that the blow-up phenomena occurs at boundary points p with $\mathfrak{D}_n(p) \geq 1$, with different behaviors depending on whether $\mathfrak{D}_n(p) = 1$ or $\mathfrak{D}_n(p) > 1$. To deal with the loss of compactness at points with $\mathfrak{D}_n(p) > 1$, where *bubbling* of solutions occurs, it is shown that in dimension three all the blow-up points are isolated and simple (the classification of blow-up points is given in Remark 1.1 below). The same strategy was also used in [18] for the case $K > 0$ in the three dimensional half sphere. As a consequence, the number of blow-up points is finite and the blow-up is excluded via integral estimates that hold true when $\mathcal{S}_g \leq 0$. In that regard, $n = 3$ is the maximal dimension for which one can prove that the blow-up points with $\mathfrak{D}_n > 1$ are isolated and simple for generic choices of K and H. In the closed case such a property is assured up to dimension four (see [30]) but, as observed in [18], the presence of the boundary produces a stronger interaction of the *bubbling* solutions with the function K.

Remark 1.1. Following standard terminology, it is useful to review the classical classes of blow-ups. We say that $p_0 \in M$ is a *blow-up point* for a sequence of solutions u_i if there exists a sequence p_i in M such that $p_i \to p_0$ and $u_i(p_i) \to +\infty$. Blow-up points $p \in M$ can be classified according to the definitions introduced by Schoen in [43] (see also [18, Definitions 4.3, 4.4, 4.5]). $p_0 \in M$ is said to be an *isolated blow-up point* for u_i if there exists a sequence p_i of local maxima of u_i with $p_i \to p_0$, $u_i(p_i) \to +\infty$ and such that there exist $c > 0$ and $R > 0$ in such a way that

$$0 < u_i(x) \le \frac{c}{d_g(x,p_i)^{\frac{n-2}{2}}} \text{ if } x \in B(p_i, R).$$

Moreover, $p_0 \in M$ is said to be an *isolated and simple blow-up point* for u_i if it is an isolated blow-up point and the radial average

$$\hat{u}_i(r) := r^{\frac{n-2}{2}} \frac{1}{|\partial B(p_i,r)|_g} \int\limits_{B(p_i,r)} u_i d\sigma_g$$

has a exactly one critical point in $(0, R)$.

Motivated by the previous observations, in the present paper we choose to perturb linearly the mean curvature boundary term, i.e. to study the problem

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta_g u + \mathcal{S}_g u = K u^{\frac{n+2}{n-2}} & \text{in } M, \\ \frac{2}{n-2}\frac{\partial u}{\partial \nu} + \varepsilon u = H u^{\frac{n}{n-2}} & \text{on } \partial M, \end{cases} \tag{1.3}$$

where ε is a small and positive parameter, and to address the following question:

(Q) If $n \ge 4$, does (1.3) admit solutions with a non-isolated (i.e. *clustering*) blow-up point?

Our main result provides a partial positive answer.

Theorem 1.2. *Assume $4 \le n \le 7$. Let π be the second fundamental form of ∂M. Assume*

(i) $\mathcal{S}_g > 0$,
(ii) *$H > 0$ and $K < 0$ are constant functions such that $\mathfrak{D}_n > 1$,*
(iii) *There exists $p \in \partial M$ which is non-umbilic (i.e. $\pi(p) \neq 0$) and a non-degenerate minimum point of $\|\pi(\cdot)\|^2$.*

Then for any $k \in \mathbb{N}$, there exist $p^j_\varepsilon \in \partial M$ for $j = 1, \dots, k$ and $\varepsilon_k > 0$ such that for all $\varepsilon \in (0, \varepsilon_k)$ the problem (1.3) has a solution u_ε with k positive peaks at p^j_ε and $p^j_\varepsilon \to p$ as $\varepsilon \to 0$, i.e., p is a clustering blow-up point.

Remark 1.3. We recall that a point $p \in \partial M$ is non-umbilic if the trace-free part of the second fundamental form of ∂M does not vanish at p.. Since $h_g = 0$, the tensor $T_{ij} = h_{ij} - h_g g_{ij}$ reduces to the second fundamental form π whose components are h_{ij} and so p is non-umbilic if $\|\pi(p)\| > 0$. Recently, Cruz-Blázquez and Pistoia in [16] proved that the non-degeneracy assumption (iii) is satisfied for generic Riemannian metrics with minimal boundary, which can also be taken within the conformal class of the original metric on M.

The main ingredients of our construction are the so-called *bubbles*, i.e. the solutions of the problem

$$\begin{cases} -c_n \Delta u = K u^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n_+ \\ \frac{2}{n-2}\frac{\partial u}{\partial \nu} = H u^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+ \end{cases} \tag{1.4}$$

where $c_n := \frac{4(n-1)}{n-2}$ and $\mathfrak{D}_n := \sqrt{n(n-1)}\frac{H}{\sqrt{|K|}} > 1$ (ν is the exterior normal vector to $\partial\mathbb{R}^n_+$). Solutions to (1.4) are completely classified in [14] (see also [31]) . These are given by

$$U_{\delta,y}(x) := \frac{1}{\delta^{\frac{n-2}{2}}} U\left(\frac{x-y}{\delta}\right), \; U(x) := \frac{\alpha_n}{|K|^{\frac{n-2}{4}}} \frac{1}{(|\tilde{x}|^2 + (x_n + \mathfrak{D}_n)^2 - 1)^{\frac{n-2}{2}}} \tag{1.5}$$

where $\alpha_n := (4n(n-1))^{\frac{n-2}{4}}$, $x = (\tilde{x}, x_n)$, $y = (\tilde{y}, 0)$ and $\delta > 0$. The solutions we are looking for are the sum of k positive bubbles which concentrate at the same boundary point p with the same speeds, i.e. in local coordinates (see (3.1) and (4.4)) around p

$$u_\varepsilon(x) \sim \sum_{j=1}^{k} \frac{1}{\delta_j^{\frac{n-2}{2}}} U\left(\frac{x-\eta_j}{\delta_j}\right)$$

where the all concentration parameters δ_j have the same speed with respect to ε and all the concentration points η_j collapse to 0 as $\varepsilon \to 0$ (see (4.5), (4.6) and (4.7)).

Unfortunately this first approximation is not as good as one can expect. We need to refine it adding some extra terms which solve the linear problem (3.2). To find these extra terms, it is crucial the study of the linear theory developed in Section 2. The novelty is Theorem 2.1 which states the non-degeneracy of the bubble (1.5), i.e. all the solution of the linearized problem

$$\begin{cases} -c_n \Delta v - \dfrac{n+2}{n-2} K U^{\frac{4}{n-2}} v = 0 \text{ in } \mathbb{R}^n_+, \\ \dfrac{2}{n-2}\dfrac{\partial v}{\partial \nu} - \dfrac{n}{n-2} H U^{\frac{2}{n-2}} v = 0 \text{ on } \partial\mathbb{R}^n_+, \end{cases} \tag{1.6}$$

are a linear combination of the functions

$$\mathfrak{z}_i(x) := \frac{\partial U}{\partial x_i}(x), \ i = 1, \ldots, n-1, \quad \text{and } \mathfrak{z}_n(x) := \left(\frac{2-n}{2} U(x) - \nabla U(x) \cdot (x + \mathfrak{D}_n \mathfrak{e}_n) + \mathfrak{D}_n \frac{\partial U}{\partial x_n}\right).$$

The proof relies on some new ideas which allow a comparison among solutions to the linear problem (1.6) and the eigenfunctions of the Neumann problem on the ball equipped with the hyperbolic metric (see Lemma 2.3). It is worthwhile to point out that Han & Lin in [27] and Almaraz in [2] related similar linear problems in the case $K \geq 0$ with some eigenvalue problems on spherical caps with standard metric.

Once the refinement of the ansatz is made, we argue using a Ljapunov-Schmidt procedure. As it is usual, the last step consists in finding a critical point of the so-called *reduced energy* and to achieve this goal it is necessary to know the energy of each bubble together with its correction. The contribution of the correction to the energy is relevant and, to capture it, it is necessary to know the exact expression of the correction itself. This part is new and requires a lot of work. This is done in Section 3. Finally, we can write the main terms of the reduced energy which come from the contribution of each peak η_j, the interaction between different peaks η_j and η_i and the linear perturbation $\epsilon-$term. For example, in dimension $n \geq 5$ (up to some constants) it looks like

$$\sum_{j=1}^{k} \left[\delta_j^2 \left(\|\pi(p)\|^2 + \mathfrak{Q}(p)(\eta_i, \eta_j) \right) + \sum_{i \neq j} \frac{(\delta_i \delta_j)^{\frac{n-2}{2}}}{|\eta_i - \eta_j|^{n-2}} - \epsilon \delta_j \right] + h.o.t. \tag{1.7}$$

here $\mathfrak{Q}(p)$ is the quadratic form associated with the second derivative of $\|\pi(\cdot)\|^2$ at the point p which is supposed to be positively definite (remind that p is a minimum point of π). Now, if we choose

$$\delta_j \sim \epsilon \text{ and } |\eta_j| \sim \eta \text{ with } \epsilon^2 \eta^2 \sim \frac{\epsilon^{n-2}}{\eta^{n-2}}$$

we can minimize the leading term in (1.7) as soon as the term *"h.o.t."* is really a higher order term and this is true only in low dimensions $4 \leq n \leq 7$. We believe that this is not merely a technical issue. It would be extremely interesting to understand if in higher dimensions the clustering phenomena appears if the blow-up point is *umbilic*, i.e. $\pi(p) = 0$. It is clear that in this case building the clustering configuration is even more difficult than in the *non-umbilic* case, because the ansatz must be refined at an higher order.

Remark 1.4. Even if our result holds true in low dimensions we decide to write all the steps of the Ljapunov-Schmidt procedure in any dimensions because it would be useful in studying some related problems. In particular, our argument allows to prove that if $n \geq 4$ the problem (1.3) has always a solution with one blow-up boundary point p which is non-umbilic and minimizes $\|\pi(\cdot)\|$. In fact, if

$k = 1$ the expansion of the reduced energy in (1.7) holds true in any dimensions. We also refer to the recent paper [17], where Cruz-Blázquez and Vaira construct a single blowing-up solution for a suitable choice of non-constants K and H. The existence of solutions with a single blow-up point was studied by Ghimenti, Micheletti & Pistoia in [25, 26, 24] when $K = 0$ and $H = 1$ in presence of a linear non-autonomus perturbation.

Remark 1.5. We remark that very recently Ben Ayed & Ould Ahmedou [6] found solutions with *clustering* blow-up points on half spheres of dimension greater than five for a subcritical approximation of the geometric problem (1.1), with a nonconstant function $K > 0$ and $H = 0$. As far as we know, our result is a pioneering work in the construction of solutions with *clustering* blow-up points for the problem (1.3) with K and H not identically zero. In particular, it is the first time that this argument is carried out with $K < 0$ and $H > 0$, which has been proved to be especially challenging due to the existing competition between the critical terms of the energy functional.

Remark 1.6. Finally, we point out that the *clustering* phenomena for Yamabe-type equations have been largely studied in the literature, although most of the results available concern the problem on closed compact manifolds. Consider for instance the linear perturbation of the classical Yamabe equation,

$$-\Delta_g u + \mathcal{S}_g u + \varepsilon u = u^{\frac{n+2}{n-2}} \text{ in } M. \tag{1.8}$$

It is known that in 3−dimensional manifolds all the solutions to (1.8) have isolated and simple blow-up points (see Li and Zhu [32]). However, this property is lost in higher dimensions.
If $n \geq 7$, Pistoia & Vaira [37] build a solution to (1.8) with a *clustering* (i.e. non-isolated) blow-up point at a non-degenerate and non-vanishing minimum point of the Weyl's tensor. In any dimensions $n \geq 4$ the clustering phenomena appears if the linear perturbation term ϵu is replaced with a function h_ε converging to a suitable function h_0 as showed by Druet & Hebey [20] and Robert & Vétois [40] if $n \geq 6$ and by Thizy & Vétois [42] if $n = 4, 5$.
The existence of solutions to (1.8) with a *towering* (i.e. isolated but non-simple) blow-up point has been proved in dimensions $n \geq 7$, by Morabito, Pistoia & Vaira [36] on symmetric nonlocally conformally flat manifolds and by Premoselli [38] in the locally flat case.
In the spirit of [20, 40] it would be interesting to replace the linear perturbation term in (1.3) with some functions h_ε in order to build a solution with a clustering blow-up point in any dimensions $n \geq 4$. Moreover, inspired by the above results we strongly believe that it would be possible to build solutions to problem (1.3) with a *towering* blow-up point in any dimensions $n \geq 4$. This will be the topic (at least in a symmetric setting) of a forthcoming paper.

The paper is organized as follows. In Section 2 we study the linear problem (1.6). In Section 3 we find out the correction term. In Section 4 we sketch the main steps of the proof, which relies on standard arguments typical of the Ljapunov-Schmidt procedure. However, since it involves a lot of new delicate and quite technical estimates, in order to streamline the reading of the work, we have decided to postpone them in the appendices.

In what follows we agree that $f \lesssim g$ means $|f| \leq c|g|$ for some positive constant c which is independent on f and g and $f \sim g$ means $f = g(1 + o(1))$.

Acknowledgements. The first author whishes to acknowledge financial support from FEDER-MINECO Grant PID2021-122122NB-I00 and by J. Andalucia (FQM-116). The second author acknowledges support of INDAM-GNAMPA project "Problemi di doppia curvatura su varietà a bordo e legami con le EDP di tipo ellittico" and of the project "Pattern formation in nonlinear phenomena" funded by the MUR Progetti di Ricerca di Rilevante Interesse Nazionale (PRIN) Bando 2022 grant 20227HX33Z. The first and the third authors have been partially supported by INdAM – GNAMPA Project "Fenomeni di blow-up per equazioni non lineari", E55F22000270001S and by PRIN 2017JP-CAPN.

2. The key linear problem

First of all, it is necessary to study the set of the solutions for the linearized problem:

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta v + \frac{n+2}{n-2}|K|\,U^{\frac{4}{n-2}}v = 0 & \text{in } \mathbb{R}^n_+, \\ \frac{2}{n-2}\frac{\partial v}{\partial \nu} - \frac{n}{n-2}HU^{\frac{2}{n-2}}v = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{2.1}$$

where $\nu = -e_n$ is the exterior normal vector to $\partial\mathbb{R}^n_+$ and

$$U(x) = U_{1,x_0(1)}(\tilde{x}, x_n) = \frac{\alpha_n}{|K|^{\frac{n-2}{4}}}\frac{1}{(|\tilde{x}|^2 + (x_n + \mathfrak{D}_n)^2 - 1)^{\frac{n-2}{2}}} \tag{2.2}$$

where $\tilde{x} = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $x_n \geq 0$, stands for the simplest solution to the boundary Yamabe problem defined in (1.4) when $\mathfrak{D}_n(p) > 1$.

Theorem 2.1. *Let $v \in H^1(\mathbb{R}^+_n)$ be a solution of (2.1). Then v is a linear combination of the functions*

$$\mathfrak{z}_i(x) := \frac{\partial U}{\partial x_i}(x) = \frac{\alpha_n}{|K|^{\frac{n-2}{4}}}\frac{(2-n)x_i}{(|\tilde{x}|^2 + (x_n + \mathfrak{D}_n)^2 - 1)^{\frac{n}{2}}}, \quad i = 1, \dots, n-1 \tag{2.3}$$

and

$$\begin{aligned} \mathfrak{z}_n(x) &:= \left(\frac{2-n}{2}U(x) - \nabla U(x)\cdot(x + \mathfrak{D}_n\mathfrak{e}_n) + \mathfrak{D}_n\frac{\partial U}{\partial x_n}\right) \\ &= \frac{\alpha_n}{|K|^{\frac{n-2}{4}}}\frac{n-2}{2}\frac{|x|^2 + 1 - \mathfrak{D}_n^2}{(|\tilde{x}|^2 + (x_n + \mathfrak{D}_n)^2 - 1)^{\frac{n}{2}}} \end{aligned} \tag{2.4}$$

The proof of Theorem 2.1 will require some preliminary results. In particular it is useful to recall the properties of the conformal Laplacian and boundary operator. For a given metric g, they are defined as

$$L_g v = -\frac{4(n-1)}{n-2}\Delta_g v + S_g v \;\text{ and, }\; B_g v = \frac{2}{n-2}\frac{\partial v}{\partial \nu} + h_g v,$$

being S_g and h_g the scalar and boundary mean curvatures. If we choose a conformal metric of the form $\rho^{\frac{4}{n-2}}g$, then L_g and B_g are *conformally invariant* in the following sense:

$$\begin{aligned} L_g v &= \rho^{\frac{n+2}{n-2}}L_{\rho^{\frac{4}{n-2}}g}(\rho^{-1}v) \quad \text{and} \\ B_g v &= \rho^{\frac{n}{n-2}}B_{\rho^{\frac{4}{n-2}}g}(\rho^{-1}v). \end{aligned} \tag{2.5}$$

Lemma 2.2. *For every $i = 0, 1, \dots,$ let us consider the following boundary eigenvalue problem:*

$$\begin{cases} \gamma_i'' + (n-1)\coth t\,\gamma_i' - \left(\frac{i(i+n-2)}{\sinh^2 t} + n\right)\gamma_i = 0, \; \text{for } 0 < t < T, \\ \gamma_i'(T) - \mu\gamma_i(T) = 0, \end{cases} \tag{2.6}$$

with $\mu \in \mathbb{R}$. Then the following hold true:

- (i) *If $i = 0$, the only bounded solutions are of the form $\gamma_0(t) = c_1\cosh t$ for $c_1 \in \mathbb{R}$, and satisfy (2.6) with $\mu = \mu_0 := \tanh T$.*
- (ii) *If $i = 1$, the only bounded solutions can be written in the form $\gamma_1(t) = c_2\sinh t$ with $c_2 \in \mathbb{R}$, and solve (2.6) with $\mu = \mu_1 := (\tanh T)^{-1}$.*
- (iii) *If $i \geq 2$ and $\mu \leq \mu_1$, (2.6) does not admit bounded solutions.*

Proof. The proofs for (i) and (ii) use the exact same argument, so for the sake of brevity we will only show the proof for (ii)

Firstly, observe that $\sinh t$ solves the first equation of (2.6) with $i = 1$, and it is positive and bounded in $[0, T]$. Therefore, by linear ODE theory, we can write any solution to the equation in the form $\gamma_1(t) = c(t)\sinh t$ for some function $c(t)$. Straightforward computations show that $c(t)$ must solve the following relation:

$$c''(t)\sinh t + (2\cosh t + (n-1)\coth t \sinh t)\, c'(t) = 0. \tag{2.7}$$

If $c(t)$ is non-constant, (2.7) can be integrated and its solutions can be calculated explicitly. For t small enough, they present the asymptotic behavior

$$c(t) = c_3 \left(\frac{1}{t} - (n-1)\ln t + O(t) \right), \quad \text{with } c_3 \neq 0.$$

Thus, $c(t)$ must be constant. The second part of (ii) can be proved by direct computation.

Finally, let us prove (iii). We will consider the unique solution to (2.6) with $\gamma_i(T) = 1$, so we study the following situation:

$$\begin{cases} \gamma_i'' + (n-1)\coth t \gamma_i' - \left(\frac{i(i+n-2)}{\sinh^2 t} + n\right)\gamma_i = 0, \text{ for } 0 < t < T, \\ \gamma_i'(T) = \mu, \\ \gamma_i(T) = 1. \end{cases}$$

Let $i \geq 2$ and define $u_i = \gamma_i - \gamma_1$, with γ_1 denoting the unique solution to (2.6) with $i = 1$, $\mu = \mu_1$ and $\gamma_1(T) = 1$. Then u_i satisfies

$$\begin{cases} u_i'' + (n-1)\coth t u_i' - \left(\frac{i(i+n-2)}{\sinh^2 t} + n\right) u_i = \frac{(i-1)(i+n-1)}{\sinh^2 t}\gamma_1, \\ u_i'(T) = \mu - \mu_1, \\ u_i(T) = 0. \end{cases} \tag{2.8}$$

Firstly, we will show that $u_i'(T) \geq 0$, proving that $\mu \geq \mu_1$. Assume by contradiction that $u_i'(T) < 0$. Then since $u(T) = 0$, there exists a small interval (t_0, T) where $u(t) > 0$. By the first equation of (2.8), since $i \geq 2$,

$$u_i''(t) + (n-1)\frac{\cosh t}{\sinh t} u_i'(t) \geq 0, \quad \text{for } t_0 < t < T. \tag{2.9}$$

Inequality (2.9) can be written in the more convenient way

$$\left((\sinh t)^{n-1} u_i'\right)' \geq 0, \quad \text{for } t_0 < t < T.$$

Consequently, $(\sinh T)^{n-1} u_i'(T) \geq (\sinh t_0)^{n-1} u_i'(t_0)$. In view of this, if $t_0 = 0$, then $u_i'(T) \geq 0$, a contradiction. However, if $t_0 > 0$, then $u_i(t_0) = u_i(T) = 0$ so there exists $t_1 \in (t_0, T)$ with $u_i'(t_1) = 0$, again a contradiction.

To see that the inequality is strict we only need to show that there are no solutions for $i \geq 2$ and $\mu = \mu_1$. Let us define the sequence of linear operators

$$A_i(\phi)(t) = -\phi''(t) - (n-1)\coth t\, \phi'(t) + \left(\frac{i(i+n-2)}{\sinh^2 t} \right) \phi(t),$$

subject to the boundary conditions $\phi(T) = 1$ and $\phi'(T) = \mu_1$. (i) implies that A_0 admits no solution, while (ii) gives us a positive function ϕ_1 satisfying $A_1(\phi_1) = 0$. Therefore, A_1 is a non-negative operator. Now, notice that the following relation holds:

$$A_i = A_1 + (i-1)(i+n-1).$$

Consequently, A_i is a positive operator if $i \geq 2$ and $A_i(\phi) = 0$ only admits the trivial solution. □

Lemma 2.3. *Let $n \geq 3$. Denote by B_R the ball of radius $0 < R < 1$ centered at the origin of $\mathbb{R}^n$, equipped with the hyperbolic metric*

$$g_{\mathbb{H}} = \frac{4\,|dx|^2}{\left(1 - |x|^2\right)^2}.$$

The first eigenvalue of the Neumann boundary problem

$$\begin{cases} \Delta_{\mathbb{H}}\phi - n\phi = 0 & \text{in } B_R, \\ \frac{\partial\phi}{\partial\nu} = \mu\phi & \text{on } \partial B_R. \end{cases} \tag{2.10}$$

is $\mu_0 = \frac{2R}{1+R^2}$, *with corresponding eigenfunction given by* $\phi_0(x) = \frac{1+|x|^2}{1-|x|^2}$. *The second eigenvalue is* $\mu_1 = \frac{1+R^2}{2R}$ *and the corresponding eigenspace is* $n-$*dimensional and generated by the family of eigenfunctions*

$$\left\{\phi_1^i(x) = \frac{|x|\, x_i}{1-|x|^2} : i = 1, \ldots, n\right\}.$$

Proof. Let $d_{\mathbb{H}}$ denote the geodesic distance from the origin, given by $d_{\mathbb{H}}(x) = \ln \frac{1+|x|}{1-|x|}$, and let (t, θ) be the geodesic polar coordinates of a point in $B_R \backslash \{0\}$, where $0 < t < T = \ln \frac{1+R}{1-R}$ and $\theta \in \mathbb{S}^{n-1}$. In these coordinates, the hyperbolic metric takes the form

$$g_{\mathbb{H}} = dt^2 + \sinh^2 t g_{\mathbb{S}^{n-1}},$$

where $g_{\mathbb{S}^{n-1}}$ is the standard metric on $\mathbb{S}^{n-1}$, and (2.10) is equivalent to the following problem:

$$\begin{cases} \frac{\partial^2\phi}{\partial t^2} + (n-1)\coth t \frac{\partial\phi}{\partial t} + \frac{\Delta_{\mathbb{S}^{n-1}}\phi}{\sinh^2 t} - n\phi = 0 & \text{in } B_T, \\ \frac{\partial\phi}{\partial t} = \mu\phi & \text{on } \partial B_T. \end{cases} \tag{2.11}$$

See [39] for more details. Using the fact that spherical harmonics generate $L^2(\mathbb{S}^{n-1})$, we write $\phi(t, \theta) = \sum_i \gamma_i(t)\xi_i(\theta)$, with ξ_i satisfying the equation

$$-\Delta_{\mathbb{S}^{n-1}}\xi_i = i(i+n-2)\xi_i, \quad i = 0, 1, \ldots$$

Therefore, separating variables, we can rewrite (2.11) in the following form:

$$\begin{cases} \sum_i \left(\gamma_i'' + (n-1)\coth t \gamma_i' - \left(\frac{i(i+n-2)}{\sinh^2 t} + n\right)\gamma_i\right)\xi_i = 0, \\ \sum_i \left(\gamma_i'(T) - \mu\gamma_i(T)\right)\xi_i = 0. \end{cases}$$

Since the functions ξ_i are orthogonal, the consequence is that each γ_i is a solution of (2.6). By Lemma 2.2, if $\mu = \mu_0 = \tanh T = \frac{2R}{1+R^2}$, there exists a solution for (2.6) associated to $i = 0$, and consequently a solution for (2.11):

$$\phi_0(t, \theta) = \cosh t.$$

ϕ_0 is non-negative in $[0, T]$, so μ_0 must be the first eigenvalue of (2.10). Again by Lemma 2.2, for $\mu = \mu_1 = (\tanh T)^{-1} = \frac{1+R^2}{2R}$ there exists a solution for (2.6) associated to $i = 1$, which produces the family of solutions for (2.11):

$$\left\{\phi_1^i(t, \theta) = \xi_i(\theta)\sinh t : i = 1, \ldots, n\right\}.$$

The same result guarantees that any other solution of (2.6) must have $\mu > \mu_1$, finishing the proof. □

Finally, we are in position to prove Theorem 2.1.

Proof of Theorem 2.1. This proof follows the ideas of [2, Lemma 2.2], with the fundamental difference that our problem is equivalent to one on a geodesic ball in the Hyperbolic space and not in the Euclidean sphere.

Let us denote $g_\star = |K|\, U^{\frac{4}{n-2}} g_0$. The scalar and boundary mean curvatures of $\mathbb{R}^n_+$ with respect to $g_\star$ are given by (1.1):

$$S_\star = -1, \quad h_\star = \frac{\mathfrak{D}_n(p)}{\sqrt{n(n-1)}}.$$

By means of (2.5), it is possible to rewrite (2.1) as follows:

$$\begin{cases} \Delta_\star \bar{v} - \frac{1}{n-1}\bar{v} = 0 & \text{in } \mathbb{R}^n_+, \\ \frac{\partial \bar{v}}{\partial \nu_\star} - \frac{\mathfrak{D}_n(p)}{\sqrt{n(n-1)}}\bar{v} = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

with $\bar{v} = |K|^{-\frac{n-2}{4}} U^{-1} v$. The differential operators are explicit and their expressions are given by:

$$\Delta_\star \bar{v} = \frac{\left(1 - |\tilde{x}|^2 - (x_n + \mathfrak{D}_n(p))^2\right)^2}{4n(n-1)}\Delta \bar{v} + \frac{n-2}{2n(n-1)}\left(1 - |\tilde{x}|^2 - (x_n + \mathfrak{D}_n(p))^2\right)\nabla \bar{v} \cdot (x + \mathfrak{D}_n(p)e_n)\,, \tag{2.12}$$

$$\frac{\partial \bar{v}}{\partial \nu_\star} = \frac{1 - |\tilde{x}|^2 - (x_n + \mathfrak{D}_n(p))^2}{2\sqrt{n(n-1)}}\frac{\partial \bar{v}}{\partial \eta}. \tag{2.13}$$

Now let us denote by Φ the map given by

$$\Phi = \mathcal{K}^{-1} \circ \tau_{\mathfrak{D}_n(p)} : \mathbb{R}^n_+ \to B_1(0) \subset \mathbb{R}^n, \tag{2.14}$$

where $\tau_{\mathfrak{D}_n(p)}$ is the translation $x \to x + \mathfrak{D}_n(p)\mathfrak{e}_n$ and $\mathcal{K}$ is the *Cayley transform*, which maps conformally the ball of radius 1 centered at the origin of $\mathbb{R}^n$ to the half-space $\mathbb{R}^n_+$. It can be proved that, up to composing with a certain isometry of $\mathbb{H}^n$, $\mathrm{Im}(\Phi) = B_R(0)$ with $R = \mathfrak{D}_n(p) - \sqrt{\mathfrak{D}_n(p)^2 - 1}$. Moreover, Φ is a conformal map and satisfies

$$\Phi^* g_{\mathbb{H}} = \frac{|K|}{n(n-1)} U^{\frac{4}{n-2}} g_0, \quad \text{where } \; g_{\mathbb{H}} = \frac{4\,|dx|^2}{(1-|x|^2)^2} \; \text{ on } \; B_R. \tag{2.15}$$

Multiplying (2.12) by $n(n-1)$ and (2.13) by $\sqrt{n(n-1)}$ and applying (2.15), one can see that $\hat{v} = (\bar{U}^{-1}v) \circ \Phi^{-1}$ is in $H^1(B_R)$ (see [26, Lemma 6]) and satisfies the following problem:

$$\begin{cases} \Delta_{\mathbb{H}} \hat{v} - n\hat{v} = 0 & \text{in } B_R, \\ \frac{\partial \hat{v}}{\partial \nu_{\mathbb{H}}} = \mathfrak{D}_n(p)\hat{v} & \text{on } \partial B_R, \end{cases}$$

being

$$\Delta_{\mathbb{H}} \hat{v} = \frac{\left(1 - |x|^2\right)^2}{4}\Delta \hat{v} + \frac{n-2}{2}\nabla \hat{v} \cdot x, \quad \text{and}$$
$$\frac{\partial \hat{v}}{\partial \nu_{\mathbb{H}}} = \frac{1 - |x|^2}{2}\frac{\partial \hat{v}}{\partial \eta}$$

the Laplace-Beltrami operator and normal derivative on B_R considered with respect to the hyperbolic metric $g_{\mathbb{H}}$. Theorem 2.1 follows from Lemma 2.3, taking into account that $\mathfrak{D}_n(p) = \frac{1+R^2}{2R}$ and

$$\hat{\mathfrak{z}}_i = c_i \phi_1^i \quad \text{for every } \; i = 1, \ldots, n. \tag{2.16}$$

□

3. The building block

Let $p \in \partial M$. The main ingredient to cook up our solutions are the bubbles defined in (1.5) together with the correction found out in Proposition 3.1, i.e. the *building block* of the solutions we are looking for is

$$\mathcal{W}_p(\xi) := \chi\left(\left(\psi_p^\partial\right)^{-1}(\xi)\right)\left[\frac{1}{\delta^{\frac{n-2}{2}}}U\left(\frac{\left(\psi_p^\partial\right)^{-1}(\xi)}{\delta}\right) + \frac{1}{\delta^{\frac{n-4}{2}}}V_p\left(\frac{\left(\psi_p^\partial\right)^{-1}(\xi)}{\delta}\right)\right] \tag{3.1}$$

where $\psi_p^\partial : \mathbb{R}^n_+ \to M$ are the Fermi coordinates in a neighborhood of p and χ is a radial cut-off function, with support in a ball of radius R. Here U is the bubble defined in (2.2) and V_p solves (3.2).

3.1. **The correction of the bubble.** Let us introduce the correction term as the function $V_p : \mathbb{R}^n_+ \to \mathbb{R}$ which is defined below.

Proposition 3.1. *Let U be as in (2.2) and set*

$$\mathtt{E}_p(x) = \sum_{i,j=1}^{n-1} \frac{8(n-1)}{n-2} h^{ij}(p) \frac{\partial^2 U(x)}{\partial x_i \partial x_j} x_n, \ x \in \mathbb{R}^n_+$$

where $h^{ij}(p)$ are the coefficients of the second fundamental form of M at the point $p \in \partial M$. Then the problem

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta V + \frac{n+2}{n-2}|K| U^{\frac{4}{n-2}} V = \mathtt{E}_p & \text{in } \mathbb{R}^n_+, \\ \frac{2}{n-2}\frac{\partial V}{\partial \nu} - \frac{n}{n-2} H U^{\frac{2}{n-2}} V = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{3.2}$$

admits a solution V_p satisfying the following properties:

(i) $\int_{\mathbb{R}^n_+} V_p(x)\mathfrak{z}_i(x)dx = 0$ *for any $i = 1, \dots, n$ (see (2.3) and (2.4))*

(ii) $|\nabla^\alpha V_p|(x) \lesssim \frac{1}{(1+|x|)^{n-3+\alpha}}$ *for any $x \in \mathbb{R}^n_+$ and $\alpha = 0, 1, 2$*

(iii)
$$|K| \int_{\mathbb{R}^n_+} U^{\frac{n+2}{n-2}} V_p dx = (n-1)H \int_{\partial\mathbb{R}^n_+} U^{\frac{n}{n-2}} V_p \, d\tilde{x}.$$

(iv) *if $n \geq 5$*
$$\int_{\mathbb{R}^n_+} \left(-\frac{4(n-1)}{n-2}\Delta V_p + \frac{n+2}{n-2}|K| U^{\frac{4}{n-2}} V_p \right) V_p \geq 0,$$

(v) *the map $p \mapsto V_p$ is $C^2(\partial M)$.*

Proof. First, we will introduce some notation to reduce ourselves to the study of a problem similar to (2.1). Let $\bar{U} = |K|^{\frac{n-2}{4}} U$, then we can rewrite (3.2) as:

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta v + \frac{n+2}{n-2}\bar{U}^{\frac{4}{n-2}} v = f & \text{in } \mathbb{R}^n_+, \\ \frac{2}{n-2}\frac{\partial v}{\partial \nu} - \frac{n}{n-2}\frac{\mathfrak{D}_n(p)}{\sqrt{n(n-1)}}\bar{U}^{\frac{2}{n-2}} v = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

Let Φ be as in (2.14). We set

$$\hat{f}(\Phi^{-1}(x)) = \frac{n(n-2)}{4} f(x) \bar{U}(x)^{-\frac{n+2}{n-2}}$$

Arguing as in the proof of Theorem 2.1, we see that it is enough to consider the following problem for $\hat{v} = (\bar{U}^{-1}v) \circ \Phi^{-1}$:

$$\begin{cases} \Delta_{\mathbb{H}}\hat{v} - n\hat{v} = \hat{f} & \text{in } B_R, \\ \frac{\partial \hat{v}}{\partial \nu_{\mathbb{H}}} = \mathfrak{D}_n(p)\hat{v} & \text{on } \partial B_R, \end{cases} \tag{3.3}$$

By the area formula and (2.16):

$$\begin{aligned} \int_{B_r} \phi_1^k(z)\hat{f}(z)d\mu_{g_{\mathbb{H}}} &= c_n \int_{\mathbb{R}^n_+} \phi_1^k(\Phi^{-1}(x)) h^{ij}(p) \frac{\partial^2 U(x)}{\partial x_i \partial x_j} x_n U^{-\frac{n+2}{n-2}} \left|\mathrm{Jac}\,\Phi^{-1}\right| dx \\ &= c_n \int_{\mathbb{R}^n_+} \mathfrak{z}_k(x) h^{ij}(p) \frac{\partial^2 U(x)}{\partial x_i \partial x_j} x_n dx \\ &= c_n \sum_{\substack{i,j=1 \\ i\neq j}}^{n-1} \int_0^{+\infty} \int_{\mathbb{R}^{n-1}} \frac{x_i x_j p_k(\tilde{x}, x_n)}{\left(|x|^2 - 1\right)^{n+1}} d\tilde{x} dx_n, \end{aligned}$$

being p_k a polynomial in x with $\deg p_k = 1$ if $k = 1, \ldots, n-1$, and $\deg p_n = 2$. To get the last identity we have also used definitions (2.2), (2.3), (2.4) and the condition $\sum_i h^{ii}(p) = 0$. Now, if we take polar coordinates in $\mathbb{R}^{n-1}$ and use the fact that

$$\int_{\mathbb{S}_r^{n-2}} p^\gamma = \frac{r^2}{\gamma(\gamma+n-3)} \int_{\mathbb{S}_r^{n-2}} \Delta p^\gamma$$

for every homogeneous polynomial p^γ of degree γ, we can check that

$$\int_{B_r} \phi_1^k(z)\hat{f}(z)d\mu_{g_{\mathbb{H}}} = 0 \text{ for all } k = 1, \ldots, n.$$

By elliptic linear theory, there exists a solution $\hat{v}$ to (3.3) which is orthogonal to $\{\phi_1^k\}_{k=1}^n$. Consequently, $v = \bar{U}(\hat{v}\circ\Phi)$ is a solution of (3.2) orthogonal to $\{\mathfrak{z}_k\}_{k=1}^n$.

Given $z \in B_R$, let G_{z_0} denote the Green's function solving the problem

$$\begin{cases} \Delta_{\mathbb{H}} G_z - nG_z = \delta_z - \sum_{k=1}^n \frac{\phi_1^k(z)\phi_1^k}{\|\phi_1^k\|_{L^2}} & \text{in } B_R, \\ \frac{\partial G_z}{\partial \nu_{\mathbb{H}}} - \mathfrak{D}_n(p) G_z = 0 & \text{on } \partial B_R. \end{cases}$$

Then, by Green's representation formula

$$\begin{aligned} \psi(z) = \sum_{k=1}^n \int_{B_R} \frac{\phi_1^k(z)\phi_1^k(w)}{\|\phi_1^k\|_{L^2}} \psi(w) d\mu_{g_{\mathbb{H}}}(w) - \int_{B_R} G_z(w)\Delta_{\mathbb{H}}\psi(w) d\mu_{g_{\mathbb{H}}}(w) \\ - \int_{\partial B_R} G_z(w)\left(\frac{\partial}{\partial\nu_{\mathbb{H}}} - \mathfrak{D}_n(p)\right)\psi(w) d\mu_{g_{\mathbb{H}}}(w) \end{aligned} \tag{3.4}$$

Choosing $\psi = \hat{v}$ in (3.4),

$$\hat{v}(z) = -\int_{B_R} G_z(w)\hat{f}(w) d\mu_{g_{\mathbb{H}}}(w),$$

then

$$|\hat{v}(z)| \le c_n \left|h^{ij}(p)\right| \int_{B_R} |w-z|^{2-n} |w + \mathfrak{D}_n(p)e_n|^{-3} d\mu_{g_{\mathbb{H}}}(w).$$

By [4, Proposition 4.12] with $\alpha = 2$ and $\alpha = n-3$,

$$\hat{v}(z) \le c_n \left|h^{ij}(p)\right| |z + \mathfrak{D}_n(p)e_n|^{-1}.$$

Hence, $u = \bar{U}(\hat{u}\circ\Phi)$ satisfies the estimate (ii). To prove (iii), integrate by parts (3.3) to obtain:

$$n\int_{B_R} \hat{v} d\mu_{g_{\mathbb{H}}} - \int_{B_R} \hat{f} d\mu_{g_{\mathbb{H}}} = \mathfrak{D}_n(p) \int_{\partial B_R} \hat{v} ds_{g_{\mathbb{H}}}. \tag{3.5}$$

By (2.15),

$$d\mu_{g_{\mathbb{H}}} = (n(n-1))^{-\frac{n}{2}} |K|^{\frac{n}{2}} U^{\frac{2n}{n-2}} |dx|^2, \tag{3.6}$$

$$ds_{g_{\mathbb{H}}} = (n(n-1))^{-\frac{n-1}{2}} |K|^{\frac{n-1}{2}} U^{\frac{2(n-1)}{n-2}} |d\tilde{x}|^2. \tag{3.7}$$

Therefore, by the area formula:

$$\int_{B_R} \hat{f} d\mu_{g_{\mathbb{H}}} = c_n \int_{\mathbb{R}_+^n} h^{ij}(p) \frac{\partial^2 U(x)}{\partial x_i \partial x_j} x_n U(x) dx = 0. \tag{3.8}$$

Combining (3.5) and (3.8) with the relations (3.6) and (3.7), we get the desired equality.

Finally, integrating by parts we obtain

$$-\int_{B_R} (\Delta_{\mathbb{H}}\hat{v})\hat{v} d\mu_{g_{\mathbb{H}}} = \int_{B_R} |\nabla_{\mathbb{H}}\hat{v}|^2 d\mu_{g_{\mathbb{H}}} - \mathfrak{D}_n(p)\int_{\partial B_R} \hat{v}^2 ds_{g_{\mathbb{H}}}. \tag{3.9}$$

By Lemma 2.3, we know that

$$\inf\left\{\frac{\int_{B_R}\left(|\nabla_{\mathbb{H}}\psi|^2+n\psi^2\right)d\mu_{g_{\mathbb{H}}}}{\int_{\partial B_R}\psi^2 ds_{g_{\mathbb{H}}}}:\int_{\partial B_R}\psi\phi_0 ds_{g_{\mathbb{H}}}=0\right\}=\mathfrak{D}_n(p).$$

If we showed that $\hat{v}$ is orthogonal to ϕ_0 in $L^2(\partial B_R)$, we would get:

$$\int_{B_R}|\nabla_{\mathbb{H}}\hat{v}|^2 d\mu_{g_{\mathbb{H}}}+n\int_{B_R}\hat{v}^2 d\mu_{g_{\mathbb{H}}}\geq\mathfrak{D}_n(p)\int_{\partial B_R}\hat{v}^2 ds_{g_{\mathbb{H}}}. \tag{3.10}$$

Then, combining (3.9) and (3.10), we obtain

$$-\int_{B_R}(\Delta_{\mathbb{H}}\hat{v})\hat{v}d\mu_{g_{\mathbb{H}}}+n\int_{B_R}\hat{v}^2 d\mu_{g_{\mathbb{H}}}\geq 0. \tag{3.11}$$

By the properties of the conformal Laplacian, we know that

$$L_{\mathbb{H}}\psi=\frac{-4(n-1)}{n-2}\Delta_{\mathbb{H}}\psi-n(n-1)\psi=n(n-1)L_\star\phi,$$

with $\psi\circ\Phi^{-1}=\phi$. Thus, multiplying (3.11) by $\frac{4(n-1)}{n-2}$, we obtain

$$\begin{aligned}0&\leq n(n-1)\int_{\mathbb{R}^n_+}L_\star\left(\bar{U}^{-1}v\right)\bar{U}^{-1}v\,d\mu_\star+n(n-1)\left(1+\frac{4}{n-2}\right)\int_{\mathbb{R}^n_+}\bar{U}^{\frac{4}{n-2}}v^2dx\\&=n(n-1)\left(\frac{4(n-1)}{n-2}\int_{\mathbb{R}^n_+}(\Delta v)v\,dx+\frac{n+2}{n-2}\int_{\mathbb{R}^n_+}|K|\,U^{\frac{4}{n-2}}v^2\,dx\right).\end{aligned}$$

We conclude the proof by showing that $\int_{\partial B_R}\hat{v}\phi_0 ds_{g_{\mathbb{H}}}=0$. We will use the fact that ϕ_0 solves (2.10) for $\mu=\mathfrak{D}_n(p)^{-1}$ and that $\hat{v}$ is a solution of (3.3). Integrating by parts:

$$\begin{aligned}0=\int_{B_R}\hat{f}\phi_0 d\mu_{g_{\mathbb{H}}}&=\int_{B_R}\left(\phi_0\Delta_{\mathbb{H}}\hat{v}-\hat{v}\Delta_{\mathbb{H}}\phi_0\right)d\mu_{g_{\mathbb{H}}}\\&=\int_{\partial B_R}\left(\frac{\partial\hat{v}}{\partial\nu_{\mathbb{H}}}\phi_0-\frac{\partial\phi_0}{\partial\nu_{\mathbb{H}}}\hat{v}\right)ds_{g_{\mathbb{H}}}=\left(\mathfrak{D}_n(p)-\frac{1}{\mathfrak{D}_n(p)}\right)\int_{\partial B_R}\hat{v}\phi_0 ds_{g_{\mathbb{H}}},\end{aligned}$$

where the first identity can be proved using the same argument as in (i).
For the proof of (v) we can reason as in Proposition 7 of [26]. □

We end this section by giving a more careful description of the function V_p. In particular, we need to detect the leading part of V_p and since its decay changes as $n=4$ or $n\geq 5$ we have to distinguish the two cases.

Case $n=4$**.** We decompose V_p into three parts: the main part $\bar{w}_p$ is almost a rational function, the second part ζ_p is a harmonic function with prescribed boundary condition and the third one ψ_p is an higher order term. More precisely, let

$$V_p=\bar{w}_p+\zeta_p+\psi_p \tag{3.12}$$

where $\bar{w}_p$, ζ_p and ψ_p solve respectively the following problems

$$-6\Delta\bar{w}_p=\mathtt{E}_p(x),\qquad\text{in }\ \mathbb{R}^4_+ \tag{3.13}$$

$$\begin{cases}-6\Delta\zeta_p=0 & \text{in }\mathbb{R}^4_+\\ \dfrac{\partial\zeta_p}{\partial\nu}=2HU\zeta_p+\left(2HU\bar{w}_p-\dfrac{\partial\bar{w}_p}{\partial\nu}\right), & \text{on }\ \partial\mathbb{R}^4_+\end{cases} \tag{3.14}$$

and

$$\begin{cases}-6\Delta\psi_p+3|K|U^2\psi_p=-3|K|U^2(\bar{w}_p+\zeta_p) & \text{in }\mathbb{R}^4_+\\ \dfrac{\partial\psi_p}{\partial\nu}=2HU\psi_p, & \text{on }\ \partial\mathbb{R}^4_+\end{cases} \tag{3.15}$$

The following holds:

Lemma 3.2. *Set*

$$\bar{w}_p^0(x) := \sum_{\substack{i,j=1\\ i\neq j}}^{3} M_{ij}(p)\frac{x_4 x_i x_j}{\left(|\tilde{x}|^2+(x_4+\mathfrak{D}_4)^2-1\right)^2},\ \textit{with}\ M_{ij}(p)=\frac{2h^{ij}(p)\alpha_4}{|K|^{\frac12}}.$$

Then

$$\bar{w}_p(x)-\bar{w}_p^0(x)=\mathcal{O}\left(\frac{1}{1+|x|^2}\right)\ \textit{and}\ |\nabla\bar{w}_p(x)-\nabla\bar{w}_p^0(x)|=\mathcal{O}\left(\frac{1}{1+|x|^3}\right), \tag{3.16}$$

$$|\zeta_p|\lesssim\frac{1}{1+|x|}\quad \textit{and}\quad |\nabla\zeta_p|\lesssim\frac{1}{1+|x|^2} \tag{3.17}$$

$$|\psi_p|\lesssim\frac{1}{1+|x|^3}\quad \textit{and}\quad |\nabla\psi_p|\lesssim\frac{1}{1+|x|^4}. \tag{3.18}$$

Proof. First we observe that the estimates (3.17) and (3.18) follows by using the same arguments of Proposition 3.1 applied to problems (3.14) and (3.15).
Now it remains to show (3.16).
We remark that we can write

$$\bar{w}_p=2h^{ij}(p)\partial^2_{ij}z_p,\qquad i,j=1,\dots,3\ \ i\neq j$$

where z_p solves the problem

$$-\Delta z_p=U(x)x_4=\frac{\alpha_4}{|K|^{\frac12}}\frac{x_4}{|\tilde{x}|^2+(x_4+\mathfrak{D}_4)^2-1}\qquad \text{in}\ \ \mathbb{R}^4_+. \tag{3.19}$$

The aim is then to understand the main term of the solution z_p of (3.19).
It holds that

$$\frac{x_4+\mathfrak{D}_4}{|\tilde{x}|^2+(x_4+\mathfrak{D}_4)^2-1}=\frac12\ln(|\tilde{x}|^2+(x_4+\mathfrak{D}_4)^2-1).$$

Thus if we take Φ_0 a solution of

$$-\Delta\Phi_0=\ln\left(|\tilde{x}|^2+(x_4+\mathfrak{D}_4)^2-1\right),\quad \text{in}\ \ \mathbb{R}^4_+ \tag{3.20}$$

and Φ_1 a solution of

$$-\Delta\Phi_1=\frac{\mathfrak{D}_4}{|\tilde{x}|^2+(x_4+\mathfrak{D}_4)^2-1}\quad \text{in}\ \ \mathbb{R}^4_+ \tag{3.21}$$

then,

$$z_p=\frac{\alpha_4}{|K|^{\frac12}}\left(\frac12\frac{\partial\Phi_0}{\partial x_4}-\Phi_1\right)$$

solves (3.19).
The advantage of (3.20) and (3.21) over (3.19) is that, under a change of variables

$$-\Delta[\Phi_0(\tilde{x},x_4-\mathfrak{D}_4)]=\ln(|\tilde{x}|^2+x_4^2-1)=\ln(|x|^2-1)\quad \text{in}\ \mathbb{R}^4_+$$

and

$$-\Delta[\Phi_1(\tilde{x},x_4-\mathfrak{D}_4)]=\frac{\mathfrak{D}_4}{|\tilde{x}|^2+x_4^2-1}=\frac{\mathfrak{D}_4}{|x|^2-1}\quad \text{in}\ \mathbb{R}^4_+.$$

If we assume that $\Phi_0(\tilde{x},x_4-\mathfrak{D}_4)$ and $\Phi_1(\tilde{x},x_4-\mathfrak{D}_4)$ are radially symmetric, i.e. $\tilde{\Phi}_0(|x|)=\Phi_0(\tilde{x},x_4-\mathfrak{D}_4)$ and $\tilde{\Phi}_1(|x|)=\Phi_1(\tilde{x},x_4-\mathfrak{D}_4)$ then it is reduced to solve the equations

$$-\tilde{\Phi}_0''-\frac{N-1}{r}\tilde{\Phi}_0'=\ln(r^2-1)\quad \text{in}\ (1,+\infty)$$

and

$$-\tilde{\Phi}_1''-\frac{N-1}{r}\tilde{\Phi}_1'=\frac{\mathfrak{D}_4}{r^2-1}\quad \text{in}\ (1,+\infty).$$

The general solutions are expressed as

$$\tilde{\Phi}_0(r) = \frac{c_1 + 3r^4 - 2(r^2-1)^2\ln(r^2-1)}{16r^2},$$
$$\tilde{\Phi}_1(r) = \frac{c_2}{r^2} + \frac{\mathfrak{D}_4\ln(r^2-1)}{4r^2} - \frac{\mathfrak{D}_4\ln(r^2-1)}{4},$$

with $c_1, c_2 \in \mathbb{R}$. Using the symmetries of the coefficients h^{ij} (with the aid of computer assisted proof), we get

$$\bar{w}_p(x) = \frac{2x_4 \sum_{\substack{i,j=1\\ i<j}}^{3} M_{ij}x_ix_j}{\Big(|\tilde{x}|^2 + (x_4+\mathfrak{D}_4)^2 - 1\Big)^2} + \mathcal{O}\left(\frac{1}{(1+|x|)^2}\right) \text{ in a } C^1\text{−sense.}$$

That concludes the proof.

□

Case $n \geq 5$**.** We can decompose $V_p = w_p + \psi_p$ where w_p solves

$$-c_n\Delta w_p + c_n\frac{n(n+2)}{(|\tilde{x}|^2 + (x_n+\mathfrak{D}_n)^2-1)^2}w_p = \mathtt{E}_p(x) \quad \text{in } \mathbb{R}^n_+, \tag{3.22}$$

and ψ_p solves

$$\begin{cases} -c_n\Delta\psi_p + c_n\dfrac{n(n+2)}{(|\tilde{x}|^2 + (x_n+\mathfrak{D}_n)^2-1)^2}\psi_p = 0, & \text{in } \mathbb{R}^n_+ \\ \dfrac{\partial\psi_p}{\partial\nu} = \dfrac{n\mathfrak{D}_n}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)}\psi_p + \left(\dfrac{n\mathfrak{D}_n}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)}w_p - \dfrac{\partial w_p}{\partial\nu}\right) & \text{on } \partial\mathbb{R}^n_+ \end{cases} \tag{3.23}$$

We claim that

$$w_p(x) = \frac{\beta_n}{4n}\frac{\sum_{\substack{i,j=1\\ j\neq i}}^{n-1} h^{ij}(p)x_ix_j(x_n-\mathfrak{D}_n)}{(|\tilde{x}|^2 + (x_n+\mathfrak{D}_n)^2-1)^{\frac{n}{2}}}. \tag{3.24}$$

Indeed, we look for a solution of (3.22) of the form

$$w_p(x) = \frac{q(x)}{(|\tilde{x}|^2 + (x_n+\mathfrak{D}_n)^2-1)^{\frac{n}{2}}},$$

with $q(x)$ a polynomial function. Straightforward computations show that $q(x)$ has to verify the equation

$$\mathcal{L}(q(x)) = \beta_n x_n \sum_{\substack{i,j=1\\ i\neq j}}^{n-1} h^{ij}(p)x_ix_j,$$

being

$$\mathcal{L}(q) = -(|\tilde{x}|^2 + (x_n+\mathfrak{D}_n)^2 - 1)\Delta q + 2n\nabla q\cdot(x+\mathfrak{D}_n\mathfrak{e}_n) - 2nq$$

with $\beta_n := \frac{2n(n-2)\alpha_n}{|K|^{\frac{n-2}{4}}}$.
Observe that it is possible to write

$$q(x) = \beta_n\sum_{\substack{i,j=1\\ i\neq j}}^{n-1} h^{ij}(p)q^{ij}(x),$$

where every q^{ij} is a polynomial solving $\mathcal{L}(q^{ij}) = x_ix_jx_n$. We note that $\mathcal{L}(x_ix_jx_n) = 4nx_ix_jx_n + 2nx_ix_j\mathfrak{D}_n$ and $\mathcal{L}(x_ix_j) = 2nx_ix_j$, so

$$\mathcal{L}\left(\frac{1}{4n}x_ix_j(x_n-\mathfrak{D}_n)\right) = x_ix_jx_n.$$

Therefore, (3.24) follows.

3.2. **The energy of the building block.** Let us define the energy $J_\varepsilon : H^1(M) \to \mathbb{R}$

$$J_\varepsilon(u) := \int_M \left(\frac{c_n}{2} |\nabla_g u|^2 + \frac{1}{2}\mathcal{S}_g u^2 - K \mathfrak{G}(u) \right) d\nu_g - c_n \frac{n-2}{2} \int_{\partial M} H \mathfrak{F}(u)\, d\sigma_g \\ + (n-1)\varepsilon \int_{\partial M} u^2 \, d\sigma_g \tag{3.25}$$

where

$$\mathfrak{G}(s) := \int_0^s \mathfrak{g}(t)\, dt, \ \mathfrak{g}(t) := (t^+)^{\frac{n+2}{n-2}} \quad \text{and} \quad \mathfrak{F}(s) := \int_0^s \mathfrak{f}(t)\, dt, \ \mathfrak{f}(t) := (t^+)^{\frac{n}{n-2}}.$$

It is useful to introduce the integral quantities whose properties are listed in Appendix A:

$$I_m^\alpha := \int_0^{+\infty} \frac{\rho^\alpha}{(1+\rho^2)^m}\, d\rho \tag{3.26}$$

and if $p \in \partial M$

$$\varphi_m(p) := \int_{\mathfrak{D}_n(p)}^{+\infty} \frac{1}{(t^2-1)^m}\, dt \quad \text{and} \quad \hat{\varphi}_m(p) := \int_{\mathfrak{D}_n(p)}^{+\infty} \frac{(t-\mathfrak{D}_n(p))^2}{(t^2-1)^m}\, dt. \tag{3.27}$$

We will assume that H and K are constant functions. We remark that $\mathfrak{D}_n$, φ_m and $\hat{\varphi}_m$ are also constant functions, so we will omit the dependence on p.

In the following result we compute the energy of the *building block* (3.1) (the proof is quite technical and is postponed in Appendix B).

Proposition 3.3. *It holds true that*

$$J_\epsilon(\mathcal{W}_p) = \mathfrak{E} - \zeta_n(\delta)\Big[\mathfrak{b}_n \|\pi(p)\|^2 + o_n'(1)\Big] + \varepsilon\delta\Big[\mathfrak{c}_n + o_n''(1)\Big]$$

where (see (3.26) *and* (3.27)*)*

$$\mathfrak{E} := \frac{\mathfrak{a}_n}{|K|^{\frac{n-2}{2}}} \left[-(n-1)\varphi_{\frac{n+1}{2}} + \frac{\mathfrak{D}_n}{(\mathfrak{D}_n^2 - 1)^{\frac{n-1}{2}}} \right], \ \mathfrak{a}_n := \alpha_n^{2^\sharp} \omega_{n-1} I_{n-1}^n \frac{n-3}{(n-1)\sqrt{n(n-1)}}, \tag{3.28}$$

moreover (see Proposition B.2 for the definition of $\mathfrak{f}_n$*)*

$$\mathfrak{b}_n := \frac{1}{2}\mathfrak{f}_n + \frac{n-2}{n-1}\alpha_n^2 \omega_{n-1} I_{n-1}^n \frac{1}{|K|^{\frac{n-2}{2}}} \left(4(n-3)\hat{\varphi}_{\frac{n-1}{2}} + \varphi_{\frac{n-3}{2}} \right), \ n \geq 5 \tag{3.29}$$

$$\mathfrak{b}_4 := \frac{192\pi^2}{|K|} + \frac{\alpha_4^2 \omega_3 I_3^4}{|K|} \tag{3.30}$$

and

$$\mathfrak{c}_n := 2(n-2)\omega_{n-1}\alpha_n^2 \frac{1}{|K|^{\frac{n-2}{2}} (\mathfrak{D}_n^2 - 1)^{\frac{n-3}{2}}} I_{n-1}^n. \tag{3.31}$$

Moreover

$$\zeta_4(\delta) := \delta^2 |\ln \delta| \quad \text{and} \quad \zeta_n(\delta) := \delta^2 \text{ if } n \geq 5.$$

and

$$o_n'(1) = \begin{cases} \mathcal{O}(\delta) \text{ if } n \geq 6, \\ \mathcal{O}(\delta |\ln \delta|) \text{ if } n = 5, \\ \mathcal{O}(|\ln \delta|^{-1}) \text{ if } n = 4 \end{cases} \quad \text{and } o_n''(1) = \begin{cases} \mathcal{O}(\delta) \text{ if } n \geq 5, \\ \mathcal{O}(\delta |\ln \delta|) \text{ if } n = 4. \end{cases} \tag{3.32}$$

4. Proof of Theorem 1.2

4.1. **Preliminaries.** Since (M, g) belongs to the positive Escobar class (i.e. the quadratic part of the Euler functional associated to the problem is positive definite), we can provide the Sobolev space $H^1(M)$ with the scalar product

$$\langle u, v\rangle := \int_M (c_n \nabla_g u \nabla_g v + \mathcal{S}_g uv)\, d\nu_g$$

where $d\nu_g$ is the volume element of the manifold. We let $\|\cdot\|$ be the norm induced by $\langle\cdot,\cdot\rangle$.
Moreover, for any $u \in L^q(M)$ (or $u \in L^q(\partial M)$) we denote the L^q- norm of u by $\|u\|_{L^q(M)} := (\int_M |u|^q\, d\nu_g)^{\frac{1}{q}}$ (respectively $\|u\|_{L^q(\partial M)} := (\int_{\partial M} |u|^q\, d\sigma_g)^{\frac{1}{q}}$ where $d\sigma_g$ is the volume element of ∂M.)
We have the well-known embedding continuous maps

$$\mathfrak{i}_{\partial M} : H^1(M) \to L^t(\partial M) \qquad \mathfrak{i}_M : H^1(M) \to L^{\frac{2n}{n-2}}(M)$$
$$\mathfrak{i}^*_{\partial M} : L^{t'}(\partial M) \to H^1(M) \qquad \mathfrak{i}^*_M : L^{\frac{2n}{n+2}}(M) \to H^1(M)$$

for $1 \le t \le \frac{2(n-1)}{n-2}$.
Now given $\mathfrak{f} \in L^{\frac{2(n-1)}{n}}(\partial M)$ the function $w_1 = \mathfrak{i}^*_{\partial M}(\mathfrak{f})$ in $H^1(M)$ is the unique solution of the equation

$$\begin{cases} -c_n\Delta_g w_1 + \mathcal{S}_g w_1 = 0 & \text{in } M\\ \dfrac{\partial w_1}{\partial \nu} = \mathfrak{f} & \text{on } \partial M. \end{cases} \tag{4.1}$$

Moreover, if we let $\mathfrak{g} \in L^{\frac{2n}{n+2}}(M)$, the function $w_2 = \mathfrak{i}^*_M(\mathfrak{g})$ is the unique solution of the equation

$$\begin{cases} -c_n\Delta_g w_2 + \mathcal{S}_g w_2 = \mathfrak{g} & \text{in } M\\ \dfrac{\partial w_2}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{4.2}$$

By continuity of $\mathfrak{i}_M, \mathfrak{i}_{\partial M}$ we get

$$\|\mathfrak{i}^*_{\partial M}(\mathfrak{f})\| \le C_1\|\mathfrak{f}\|_{L^{\frac{2(n-1)}{n}}(\partial M)} \qquad \|\mathfrak{i}^*_M(\mathfrak{g})\| \le C_2\|\mathfrak{g}\|_{L^{\frac{2n}{n+2}}(M)}$$

for some $C_1 > 0$ and independent of $\mathfrak{f}$ and some $C_2 > 0$ and independent of $\mathfrak{g}$.
Then we rewrite the problem (1.3) as

$$u = \mathfrak{i}^*_M(K\mathfrak{g}(u)) + \mathfrak{i}^*_{\partial M}\left(\frac{n-2}{2}(H\mathfrak{f}(u) - \varepsilon u)\right), \quad \text{with } \mathfrak{g}(u) := (u^+)^{\frac{n+2}{n-2}} \text{ and } \mathfrak{f}(u) = (u^+)^{\frac{n}{n-2}}$$

4.2. **The ansatz.** Having in mind Proposition 3.3, we fix a non-umbilic and non-degenerate minimum point $p \in \partial M$ of the function $\|\pi(\cdot)\|^2$ with and we choose

$$d_0 := \frac{\mathfrak{c}_n}{2\mathfrak{b}_n\|\pi(p)\|^2} \tag{4.3}$$

where $\mathfrak{b}_n$ and $\mathfrak{c}_n$ are positive constants defined in (3.29), (3.30) and (3.31). For any integer $k \ge 1$, we look for solutions of (1.3) of the form

$$u_\varepsilon(\xi) := \underbrace{\sum_{j=1}^k \mathcal{W}_j(\xi)}_{:=\mathcal{W}(\xi)} + \Phi_\varepsilon(\xi) \quad \xi \in M \tag{4.4}$$

where

$$\mathcal{W}_j(\xi) = \chi\left(\left(\psi^\partial_p\right)^{-1}(\xi)\right) W_j(\xi)$$

and

$$W_j(\xi) := \frac{1}{\delta_j^{\frac{n-2}{2}}} U\left(\frac{\left(\psi_p^\partial\right)^{-1}(\xi) - \eta(\varepsilon)\tau_j}{\delta_j}\right) + \delta_j \frac{1}{\delta_j^{\frac{n-2}{2}}} V_p\left(\frac{\left(\psi_p^\partial\right)^{-1}(\xi) - \eta(\varepsilon)\tau_j}{\delta_j}\right).$$

Here χ is a radial cut-off function with support in a ball of radius R, the bubble U is defined in (2.2) and V_p solves (3.2). Moreover,

$$\tau_j \in \mathcal{C} := \left\{(\tau_1, \dots, \tau_k) \in \mathbb{R}^{(n-1)k} \ : \ \tau_i \neq \tau_j \text{ if } i \neq j\right\} \tag{4.5}$$

and given d_0 as in (4.3) the concentration parameters δ_j and the rate of the concentration points $\eta(\varepsilon)$ are choosen as follows:

$$\delta_j := \varepsilon\left(d_0 + \eta(\varepsilon) d_j\right), \ d_j \in [0, +\infty) \text{ and } \eta(\varepsilon) := \varepsilon^\alpha \text{ with } \alpha := \frac{n-4}{n} \quad \text{if } n \geq 5 \tag{4.6}$$

or

$$\delta_j := \rho(\varepsilon)\left(d_0 + \eta(\varepsilon) d_j\right) \ d_j \in [0, +\infty) \text{ and } \eta(\varepsilon) := \frac{1}{|\ln \rho(\varepsilon)|^{\frac{1}{4}}} \quad \text{if } n = 4 \tag{4.7}$$

where ρ is the inverse function of $\ell : (0, e^{-\frac{1}{2}}) \to \left(0, \frac{e^{-1}}{2}\right)$ defined by $\ell(s) = -s \ln s$. We remark that $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$.
Finally, the remainder term $\Phi_\varepsilon(\xi)$ belongs to $\mathcal{K}^\perp$ defined as follows.
Let us define for i$= 1, \dots, n$, and $j = 1, \dots, k$

$$\mathcal{Z}_{j,i}(\xi) = \chi\left(\left(\psi_p^\partial\right)^{-1}(\xi)\right) Z_{j,i}(\xi), \text{ with } Z_{j,i}(\xi) := \frac{1}{\delta_j^{\frac{n-2}{2}}} \mathfrak{z}_i\left(\frac{\left(\psi_p^\partial\right)^{-1}(\xi) - \eta(\varepsilon)\tau_j}{\delta_j}\right)$$

where $\mathfrak{z}_i$ are given in (2.3)and (2.4).
We decompose $H^1(M)$ in the direct sum of the following two subspaces

$$\mathcal{K} = \text{span}\left\{\mathcal{Z}_{j,i} \ : \ i = 1, \dots, n, \ j = 1, \dots, k\right\}$$

and

$$\mathcal{K}^\perp := \left\{\psi \in H^1(M) \, : \, \langle \psi, \mathcal{Z}_{j,i}\rangle = 0, \quad i = 1, \dots, n, \ j = 1, \dots, k\right\}.$$

4.3. **The reduction process.** We define the projections

$$\Pi : H^1(M) \to \mathcal{K} \qquad \Pi^\perp : H^1(M) \to \mathcal{K}^\perp.$$

Therefore solving (4.1) is equivalent to solve the couple of equations

$$\Pi^\perp\left\{u_\varepsilon - \mathfrak{i}_M^*(K\mathfrak{g}(u_\varepsilon)) - \mathfrak{i}_{\partial M}^*\left(\frac{n-2}{2}\left(H\mathfrak{f}(u_\varepsilon) - \varepsilon u_\varepsilon\right)\right)\right\} = 0 \tag{4.8}$$

$$\Pi\left\{u_\varepsilon - \mathfrak{i}_M^*(K\mathfrak{g}(u_\varepsilon)) - \mathfrak{i}_{\partial M}^*\left(\frac{n-2}{2}\left(H\mathfrak{f}(u_\varepsilon) - \varepsilon u_\varepsilon\right)\right)\right\} = 0 \tag{4.9}$$

where u_ε is defined in (4.4).

4.4. **Solving the equation (4.8): the remainder term.** We shall find the remainder term $\Phi_\varepsilon \in \mathcal{K}^\perp$ in (4.4). Let us rewrite the equation (4.8) as

$$\mathcal{E} + \mathcal{L}(\Phi_\varepsilon) + \mathcal{N}(\Phi_\varepsilon) = 0 \tag{4.10}$$

where the error term $\mathcal{E}$ is

$$\mathcal{E} := \Pi^\perp\left\{\mathcal{W} - \mathfrak{i}_M^*\left(K\mathfrak{g}\left(\mathcal{W}\right)\right) - \mathfrak{i}_{\partial M}^*\left(\frac{n-2}{2}\left(H\mathfrak{f}\left(\mathcal{W}\right) - \varepsilon\mathcal{W}\right)\right)\right\},$$

the linear operator $\mathcal{L}$ is

$$\mathcal{L}(\Phi_\varepsilon) := \Pi^\perp\left\{\Phi_\varepsilon - \mathfrak{i}_M^*\left(K\mathfrak{g}'\left(\mathcal{W}\right)\Phi_\varepsilon\right) - \mathfrak{i}_{\partial M}^*\left(\frac{n-2}{2}\left(H\mathfrak{f}'\left(\mathcal{W}\right)\Phi_\varepsilon - \varepsilon\Phi_\varepsilon\right)\right)\right\}$$

and the quadratic term $\mathcal{N}(\Phi_\varepsilon)$ is

$$\mathcal{N}(\Phi_\varepsilon) := \Pi^\perp\Big\{ -\mathfrak{i}^*_M\left[K\left(\mathfrak{g}\left(\mathcal{W}+\Phi_\varepsilon\right)-\mathfrak{g}\left(\mathcal{W}\right)-\mathfrak{g}'\left(\mathcal{W}\right)\Phi_\varepsilon\right)\right] \\ -\mathfrak{i}^*_{\partial M}\left[\frac{n-2}{2}H\left(\mathfrak{f}\left(\mathcal{W}+\Phi_\varepsilon\right)-\mathfrak{f}\left(\mathcal{W}\right)-\mathfrak{f}'\left(\mathcal{W}\right)\Phi_\varepsilon\right)\right]\Big\}.$$

The following result holds true.

Proposition 4.1. *For any compact subset $\mathcal{A}\subset(0,+\infty)^k\times\mathcal{C}$ (see (4.5)) there exists $\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$ and for any $(d_1,\dots,d_k,\tau_1,\dots,\tau_k)\in\mathcal{A}$ there exists a unique function $\Phi_\varepsilon\in\mathcal{K}^\perp$ which solves equation (4.8). Moreover, the map $(d_1,\dots,d_k,\tau_1,\dots,\tau_k)\mapsto\Phi_\varepsilon(d_1,\dots,d_k,\tau_1,\dots,\tau_k)$ is of class C^1 and*

$$\|\Phi_\varepsilon\|\lesssim\begin{cases}\varepsilon^2 & \text{if } n\ge 7\\ \varepsilon^2|\ln\varepsilon|^{\frac{2}{3}} & \text{if } n=6\\ \varepsilon^{\frac{3}{2}} & \text{if } n=5\\ \rho(\varepsilon) & \text{if } n=4.\end{cases}$$

We omit the proof because it is standard and relies on the following two key results. First, we estimate the size of the error term $\mathcal{E}$. The proof is postponed in Appendix C.

Lemma 4.2. *Let $n\ge 4$. For any compact subset $\mathcal{A}\subset(0,+\infty)^k\times\mathcal{C}$ there exists $\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$ and for any $(d_1,\dots,d_k,\tau_1,\dots,\tau_k)\in\mathcal{A}$ it holds*

$$\|\mathcal{E}\|\lesssim\begin{cases}\varepsilon^2 & \text{if } n\ge 7\\ \varepsilon^2|\ln\varepsilon|^{\frac{2}{3}} & \text{if } n=6\\ \varepsilon^{\frac{3}{2}} & \text{if } n=5\\ \rho(\varepsilon) & \text{if } n=4.\end{cases}$$

Next, we study the invertibility of the linear operator $\mathcal{L}$. The proof relies on Theorem 2.1 and can be carried out as in [41].

Lemma 4.3. *Let $n\ge 4$. For any compact subset $\mathcal{A}\subset(0,+\infty)^k\times\mathcal{C}$ there exist a positive constant $C>0$ and $\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$ and any $(d_1,\dots,d_k,\tau_1,\dots,\tau_k)\in\mathcal{A}$ it holds*

$$\|\mathcal{L}(\phi)\|\ge C\|\phi\|.$$

4.5. **Solving equation (4.9): the reduced problem.** We know that solutions to problem (1.3) are critical points of the energy functional J_ε defined in (3.25). Let us introduce the so-called *reduced energy*

$$\mathfrak{J}_\varepsilon(d_1,\dots,d_k,\tau_1,\dots,\tau_k) := J_\varepsilon(\mathcal{W}+\Phi_\varepsilon) \tag{4.11}$$

where the remainder term Φ_ε has been found in Proposition 4.1.
We shall prove that a critical point of the reduced energy provides a solution to our problem.

Proposition 4.4. *Assume $4\le n\le 7$. It holds true that*

1. *If $(d_1,\dots,d_k,\tau_1,\dots,\tau_k)\in[0,+\infty)^k\times(\mathbb{R}^n)^k$ is a critical point of the reduced energy (4.11), then $\mathcal{W}+\Phi_\varepsilon$ is a critical point of J_ε and so it solves (1.3).*
2. *The following expansion holds true*

$$\mathfrak{J}_\varepsilon(d_1,\dots,d_k,\tau_1,\dots,\tau_k) = k\mathfrak{E}+k\theta_n(\varepsilon)\left(\mathfrak{c}_n d_0-\mathfrak{b}_n\|\pi(p)\|^2d_0^2\right) \\ +\Theta_n(\varepsilon)\underbrace{\left(-\mathfrak{b}_n\sum_{i=1}^k\mathfrak{Q}(p)(\tau_i,\tau_i)-\mathfrak{b}_n\|\pi(p)\|^2\sum_{i=1}^k d_i^2-\frac{\mathfrak{d}_n}{|K|^{\frac{n-2}{2}}}\sum_{i<j}\frac{d_0^{n-2}}{|\tau_i-\tau_j|^{n-2}}\right)}_{=:\mathfrak{F}_n(d_1,\dots,d_k,\tau_1,\dots,\tau_k)} \\ +o\left(\Theta_n(\varepsilon)\right),$$

C^0- uniformly with respect to $(d_1,\dots,d_k,\tau_1,\dots,\tau_k)$ in a compact set of $(0,+\infty)^k\times\mathcal{C}$. Here $\mathfrak{Q}(p)$ is the quadratic form associated with the second derivative of $p\to\|\pi(p)\|^2$ (being zero the first derivative), $\mathfrak{E}$, $\mathfrak{c}_n$ are constants defined in (3.28) and (3.31), respectively, and

$$\mathfrak{d}_n := \alpha_n^{2^\star}\omega_{n-1}I^{n-2}_{\frac{n+2}{2}}.$$

Moreover

$$\theta_4(\varepsilon)=\Theta_4(\varepsilon):=\rho^2(\varepsilon)|\ln\rho(\varepsilon)| \text{ if } n=4 \quad\text{and}\quad \theta_n(\varepsilon):=\varepsilon^2,\ \Theta_n(\varepsilon):=\varepsilon^{\frac{4(n-2)}{n}} \text{ if } n=5,6,7.$$

As we claimed above, we shall postpone in Appendix D the proof because even if it relies on standard arguments, it requires a lot of new elaborated and technical estimates.

4.6. **Proof of Theorem 1.2: completed.** The claim immediately follows by Proposition 4.4 taking into account that the function $\mathfrak{F}_n(d_1,\dots,d_k,\tau_1,\dots,\tau_k)$ has a maximum point which is stable under C^0- perturbations.

Appendix A. Auxiliary results

We have (see [2] Lemmas 9.4 and 9.5) the following results:

$$I_m^\alpha := \int_0^{+\infty}\frac{\rho^\alpha}{(1+\rho^2)^m}\,d\rho = \frac{2m}{\alpha+1}I^{\alpha+2}_{m+1},\quad \text{for } \alpha+1<2m$$

$$I_m^\alpha = \frac{2m}{2m-\alpha-1}I^\alpha_{m+1},\quad \text{for } \alpha+1<2m+2$$

$$I_m^\alpha = \frac{2m-\alpha-3}{\alpha+1}I^{\alpha+2}_{m},\quad \text{for } \alpha+3<2m.$$

In particular, if $n\ge 4$

$$I_n^n = I_n^{n-2} = \frac{n-3}{2(n-1)}I^n_{n-1},\quad I^{n-2}_{n-1}=\frac{n-3}{n-1}I^n_{n-1},\quad I^{n-2}_{n-2}=\frac{2(n-2)}{n-1}I^n_{n-1}. \tag{A.1}$$

We also set

$$\varphi_m := \int_{\mathfrak{D}_n}^{+\infty}\frac{1}{(t^2-1)^m}\,dt \quad\text{and}\quad \hat\varphi_m := \int_{\mathfrak{D}_n}^{+\infty}\frac{(t-\mathfrak{D}_n)^2}{(t^2-1)^m}\,dt.$$

By straightforward computations we deduce the following results.

Lemma A.1. *It holds true that*

$$\varphi_{\frac{n+1}{2}} = \frac{1}{n-1}\frac{\mathfrak{D}_n}{(\mathfrak{D}_n^2-1)^{\frac{n-1}{2}}} - \frac{n-2}{n-1}\varphi_{\frac{n-1}{2}},$$

$$\varphi_{\frac{n-1}{2}} = \frac{1}{n-3}\frac{\mathfrak{D}_n}{(\mathfrak{D}_n^2-1)^{\frac{n-3}{2}}} - \frac{n-4}{n-3}\varphi_{\frac{n-3}{2}}$$

and

$$\hat\varphi_m := \varphi_{m-1} + (\mathfrak{D}_n^2+1)\varphi_m - \frac{1}{m-1}\frac{\mathfrak{D}_n}{(\mathfrak{D}_n^2-1)^{m-1}}.$$

Moreover

$$\int_{\mathbb{R}^n_+}\frac{|\tilde x|^\alpha}{(|\tilde x|^2+(x_n+\mathfrak{D}_n)^2-1)^m}\,dx = \omega_{n-1}I_m^{n-2+\alpha}\varphi_{\frac{2m-n-\alpha+1}{2}},\qquad \textit{for } n+\alpha<2m \tag{A.2}$$

$$\int_{\mathbb{R}^{n-1}}\frac{|\tilde x|^\alpha}{(|\tilde x|^2+\mathfrak{D}_n^2-1)^m}\,dx = \omega_{n-1}(\mathfrak{D}_n^2-1)^{\frac{n+\alpha-1-2m}{2}}I_m^{n-2+\alpha},\qquad \textit{for } n-1+\alpha<2m \tag{A.3}$$

$$\int_{\mathbb{R}^n_+}\frac{x_n^2|\tilde x|^\alpha}{(|\tilde x|^2+(x_n+\mathfrak{D}_n)^2-1)^m}\,dx = \omega_{n-1}I_m^{n-2+\alpha}\hat\varphi_{\frac{2m-n-\alpha+1}{2}},\qquad \textit{for } n+2+\alpha<2m \tag{A.4}$$

We remind the expansion of the metric given in [21].

Lemma A.2. *Let (M,g) be a compact Riemannian manifold with boundary. If $x = (\tilde{x}, x_n) = (x_1 \ldots, x_n)$ are the Fermi coordinates centered at a point $p \in \partial M$, then the following expansion holds:*

- $\sqrt{|g(x)|} = 1 - \frac{1}{2}\left(\|\pi(p)\|^2 + Ric_\nu(p)\right) x_n^2 - \frac{1}{6}\bar{R}_{ij}(p)x_i x_j + O(|x|^3)$,
- $g^{ij}(x) = \delta_{ij} + 2h^{ij}(p)x_n + \frac{1}{3}\bar{R}_{ikjl}(p)x_k x_l + 2\frac{\partial h^{ij}}{\partial x_k}(p)x_k x_n + \left(R_{injn}(p) + 3h_{ik}(p)h_{kj}(p)\right){x_n}^2 + O(|x|^3)$
- $g^{an}(x) = \delta_{an}$
- $\Gamma^k_{ij}(x) = \mathcal{O}(|x|)$

where $\pi(p)$ is the second fundamental form at p, $h^{ij}(p)$ are its coefficients, $\bar{R}_{ikjl}(p)$ and $R_{abcd}(p)$ are the curvature tensor of the boundary ∂M and M, respectively, $\bar{R}_{ij}(p) = \bar{R}_{ikjk}(p)$ are the coefficients of the Ricci tensor, and $Ric_\nu(p) = R_{inin}(p) = R_{nn}(p)$. Here the indices $i,j,k = 1,\ldots,n-1$ and $a,b = 1,\ldots,n$.

Finally, we recall the estimate:

$$||s+t|^q - s^q| \lesssim \begin{cases} \min\{s^{q-1}|t|, |t|^q\} \text{ if } 0 < q \le 1 \\ s^{q-1}|t| + |t|^q \text{ if } q > 1 \end{cases} \quad \text{for any } s > 0 \text{ and } t \in \mathbb{R}. \tag{A.5}$$

Appendix B. Proof of Proposition 3.3

First we need two technical propositions in which we compute the contribution of correction term V_p to the energy.

Proposition B.1. *Let $n = 4$ and $\bar{w}_p$ the solution of (3.13) (the first term of the expansion of V_p in (3.12)). Then*

$$\int_{B^+_{\frac{R}{\delta}}} |\nabla \bar{w}_p|^2 = \frac{64\pi^2}{|K|}\|\pi(p)\|^2|\ln\delta| + \mathcal{O}(1). \tag{B.1}$$

Proof. We first reduce the integral into (B.1) to one in a simpler domain. Let Q_r^+ denote the upper half of the ball of radius r in the $\|\cdot\|_\infty$ of $\mathbb{R}^4$, that is,

$$Q_r^+ = \left\{x \in \mathbb{R}^4 : x_4 \ge 0 \text{ and } -r \le x_i \le r,\ i = 1,2,3.\right\}$$

and let $A^+(\frac{R}{4\delta}, \frac{R}{\delta})$ the upper half of the annulus with radii $r_1 = \frac{R}{4\delta}$ and $r_2 = \frac{R}{\delta}$.
Then, we can write

$$B^+_{\frac{R}{\delta}} = Q^+_{\frac{R}{2\delta}} \sqcup \Omega_\delta,$$

with $\Omega_\delta := B^+_{\frac{R}{\delta}} \setminus Q^+_{\frac{R}{2\delta}}$. Notice that Ω_δ satisfies $\Omega_\delta \subset A^+(\frac{R}{4\delta}, \frac{R}{\delta})$. Then, by using also Lemma 3.2, we get

$$\int_{B^+_{\frac{R}{\delta}}} |\nabla \bar{w}_p|^2 = \int_{Q^+_{\frac{R}{2\delta}}} |\nabla \bar{w}_p|^2 + \int_{\Omega_\delta} |\nabla \bar{w}_p|^2 = \int_{Q^+_{\frac{R}{2\delta}}} \left|\nabla \bar{w}^0_p\right|^2 + \int_{\Omega_\delta} |\nabla \bar{w}_p|^2 + \mathcal{O}(1),$$

and

$$\int_{\Omega_\delta} |\nabla \bar{w}_p|^2 \le \int_{A^+(\frac{R}{4\delta}, \frac{R}{\delta})} |\nabla \bar{w}_p|^2 \le C\int_{\frac{R}{4\delta}}^{\frac{R}{\delta}} (1+r)^{-4} r^3 dr = \mathcal{O}(1).$$

Then

$$\int_{B^+_{\frac{R}{\delta}}} |\nabla \bar{w}_p|^2 = \int_{Q^+_{\frac{R}{2\delta}}} \left|\nabla \bar{w}^0_p\right|^2 + \mathcal{O}(1).$$

The latter integral can be calculated explicitly with the help of mathematical software. Firstly, we compute

$$\int_0^{\frac{R}{2\delta}} \int_{\frac{-R}{2\delta}}^{\frac{R}{2\delta}} \int_{\frac{-R}{2\delta}}^{\frac{R}{2\delta}} \int_{\frac{-R}{2\delta}}^{\frac{R}{2\delta}} \left|\frac{\partial \bar{w}^0_p}{\partial x_i}\right|^2 dx_1 dx_2 dx_3 dx_4 = \frac{\pi^2}{30}\left(\sum_{\substack{k>j\\k\neq i}}^{3} \sum_{\substack{j=1\\j\neq i}}^{3} \left(3M_{ij}^2 + M_{jk}^2\right)\right)|\ln\delta| + \mathcal{O}(1),$$

for $i = 1,2,3$. Similarly,

$$\int_0^{\frac{R}{2\delta}} \int_{\frac{-R}{2\delta}}^{\frac{R}{2\delta}} \int_{\frac{-R}{2\delta}}^{\frac{R}{2\delta}} \int_{\frac{-R}{2\delta}}^{\frac{R}{2\delta}} \left| \frac{\partial \bar{w}_p^0}{\partial x_4} \right|^2 dx_1 dx_2 dx_3 dx_4 = \frac{\pi^2}{30} \left(\sum_{\substack{i,j=1 \\ i<j}}^{3} 3M_{ij}^2 \right) |\ln \delta| + \mathcal{O}(1).$$

Hence, by the definition of $M_{ij}(p) = \frac{8\sqrt{3}}{|K|^{\frac{1}{2}}} h^{ij}(p)$:

$$\int_{Q^+_{\frac{R}{2\delta}}} \left| \nabla \bar{w}_p^0 \right|^2 = \frac{64\pi^2}{|K|} \left\| \pi(p) \right\|^2 |\ln \delta| + \mathcal{O}(1),$$

being $\|\pi(p)\|^2 = h_{12}^2(p) + h_{13}^2(p) + h_{23}^2(p)$. □

Proposition B.2. *Let $n \geq 5$. Let V_p a solution of (3.2), then there exists a non-negative constant $\mathfrak{f}_n$ depending only on n and $\mathfrak{D}_n$ such that*

$$\int_{\mathbb{R}^n_+} \left(-c_n \Delta V_p + \frac{n+2}{n-2} |K| U^{\frac{4}{n-2}} V_p \right) V_p = \mathfrak{f}_n \left\| \pi(p) \right\|^2.$$

Proof. Let decompose $V_p = w_p + \psi_p$ where w_p solves (3.22) and ψ_p solves (3.23).
For sake of convenience, let us define

$$b(\tilde{x}, 0) = \frac{n \mathfrak{D}_n}{(|\tilde{x}|^2 + \mathfrak{D}_n^2 - 1)} w_p(\tilde{x}, 0) + \left. \frac{\partial w_p}{\partial x_n} \right|_{x_n = 0}.$$

Since $V_p = w_p + \psi_p$ (w_p and ψ_p are defined in (3.22) and in (3.23) respectively) then

$$\int_{\mathbb{R}^n_+} \left(-\Delta V_p + \frac{n(n+2)}{(|\tilde{x}|^2 + (x_n + \mathfrak{D}_n)^2 - 1)^2} V_p \right) V_p \, dx = \beta_n h^{ij}(p) \int_{\mathbb{R}^n_+} \frac{x_i x_j x_n}{(|\tilde{x}|^2 + (x_n + \mathfrak{D}_n)^2 - 1)^{\frac{n+2}{2}}} V_p \, dx$$
$$= \underbrace{\beta_n h^{ij}(p) \int_{\mathbb{R}^n_+} \frac{x_i x_j x_n}{(|\tilde{x}|^2 + (x_n + \mathfrak{D}_n)^2 - 1)^{\frac{n+2}{2}}} w_p \, dx}_{I_{w_p}} + \underbrace{\beta_n h^{ij}(p) \int_{\mathbb{R}^n_+} \frac{x_i x_j x_n}{(|\tilde{x}|^2 + (x_n + \mathfrak{D}_n)^2 - 1)^{\frac{n+2}{2}}} \psi_p \, dx}_{I_{\psi_p}}$$

Let us evaluate separately I_{w_p} and I_{ψ_p}.

$$\begin{aligned} I_{w_p} =& \beta_n h^{ij}(p) \int_{\mathbb{R}^n_+} \frac{x_i x_j x_n}{(|\tilde{x}|^2 + (x_n + \mathfrak{D}_n)^2 - 1)^{\frac{n+2}{2}}} w_p \, dx \\ =& \frac{\beta_n^2}{4n} h^{ij}(p) h^{k\ell}(p) \underbrace{\int_{\mathbb{R}^n_+} \frac{x_i x_j x_k x_\ell x_n (x_n - \mathfrak{D}_n)}{(|\tilde{x}|^2 + (x_n + \mathfrak{D}_n)^2 - 1)^{n+1}} dx}_{I_{ijk\ell}} \end{aligned} \tag{B.2}$$

Notice that by symmetry reasons and the fact that $h^{ii}(p) = 0$ for every $i = 1, \ldots, n-1$, we can write

$$\frac{\beta_n^2}{4n} \sum_{\substack{i,j=1 \\ i \neq j}}^{n-1} \sum_{\substack{k,\ell=1 \\ k \neq l}}^{n-1} h^{ij}(p) h^{k\ell}(p) I_{ijk\ell} = \frac{\beta_n^2}{n} \sum_{\substack{i,j=1 \\ i<j}}^{n-1} \sum_{\substack{k,\ell=1 \\ k<l}}^{n-1} h^{ij}(p) h^{k\ell}(p) I_{ijk\ell}. \tag{B.3}$$

In view of (B.3), if $(i,j) \neq (k,\ell)$, there exists an index, let say i, such that $i \notin \{j,k,\ell\}$. In that case, it is easy to see that

$$h^{ij}(p) h^{k\ell}(p) \int_{\mathbb{R}^n_+} \frac{x_i x_j x_k x_\ell x_n (x_n - \mathfrak{D}_n)}{(|\tilde{x}|^2 + (x_n + \mathfrak{D}_n)^2 - 1)^{n+1}} dx$$
$$= h^{ij}(p) h^{k\ell}(p) \int_{\mathbb{R}^{n-1}_+} \int_{-\infty}^{+\infty} \frac{\partial}{\partial x_i} \left(\frac{-1}{2n} \frac{x_j x_k x_l x_n (x_n - \mathfrak{D}_n)}{(|\tilde{x}|^2 + (x_n + \mathfrak{D}_n)^2 - 1)^{n+1}} \right) dx_i \prod_{\substack{\alpha=1 \\ \alpha \neq i}}^{n} dx_\alpha = 0.$$

Consequently,

$$h^{ij}(p)h^{k\ell}(p)I_{ijk\ell}=\sum_{i,j=1}^{n-1}h^{ij}(p)^2\int_{\mathbb{R}^n_+}\frac{x_i^2x_j^2x_n(x_n-\mathfrak{D}_n)}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^{n+1}}dx.$$

For every $i\neq j$, using polar coordinates in $\mathbb{R}^{n-3}$, we can see that

$$\int_{\mathbb{R}^n_+}\frac{x_i^2x_j^2x_n(x_n-\mathfrak{D}_n)}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^{n+1}}dx=\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}x_i^2x_j^2x_n(x_n-\mathfrak{D}_n)$$

$$\times\int_0^{+\infty}\frac{\omega_{n-4}r^{n-4}dr}{(r^2+{x_i}^2+{x_j}^2+(x_n+\mathfrak{D}_n)^2-1)^{n+1}}dx_idx_jdx_n$$

$$=\frac{\omega_{n-4}\Gamma\left(\frac{n-3}{2}\right)\Gamma\left(\frac{n+5}{2}\right)}{2n!}\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{{x_i}^2{x_j}^2x_n(x_n-\mathfrak{D}_n)}{(x_i^2+x_j^2+(x_n+\mathfrak{D}_n)^2-1)^{\frac{n+5}{2}}}dx_idx_jdx_n$$

$$=\frac{\omega_{n-4}\pi\Gamma\left(\frac{n-3}{2}\right)\Gamma\left(\frac{n+5}{2}\right)}{n!(n-1)(n+1)(n+3)}\underbrace{\int_0^{+\infty}\frac{x_n(x_n-\mathfrak{D}_n)}{((x_n+\mathfrak{D}_n)^2-1)^{\frac{n-1}{2}}}}_{\mathfrak{f}_1(n,\mathfrak{D}_n)}.$$

Substituting in (B.2):

$$I_{w_p}=\frac{\omega_{n-4}\pi{\beta_n}^2\Gamma\left(\frac{n-3}{2}\right)\Gamma\left(\frac{n+5}{2}\right)}{4n(n-1)(n+3)(n+1)!}\mathfrak{f}_1(n,\mathfrak{D}_n)\left\|\pi(p)\right\|^2.$$

Let us now study the term I_{ψ_p}. Multiplying (3.2) by ψ_p and integrating by parts we obtain:

$$\beta_nh^{ij}(p)\int_{\mathbb{R}^n_+}\frac{x_ix_jx_n}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^{\frac{n+2}{2}}}\psi_p\,dx$$

$$=\int_{\mathbb{R}^n_+}\left(\nabla v_p\nabla\psi_p+\frac{n(n+2)}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^2}v_p\psi_p\right)dx-\int_{\partial\mathbb{R}^n_+}\frac{\partial v_p}{\partial\eta}\psi_p\,d\tilde{x}$$

$$=\int_{\mathbb{R}^n_+}\left(\nabla w_p\nabla\psi_p+\frac{n(n+2)}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^2}w_p\psi_p\right)dx$$

$$+\int_{\mathbb{R}^n_+}\left(|\nabla\psi_p|^2+\frac{n(n+2)}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^2}\psi_p^2\right)dx-\int_{\partial\mathbb{R}^n_+}\frac{n\mathfrak{D}_n}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)}w_p\psi_p\,d\tilde{x}$$

$$-\int_{\partial\mathbb{R}^n_+}\frac{n\mathfrak{D}_n}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)}\psi_p^2\,d\tilde{x}$$

$$=\int_{\partial\mathbb{R}^n_+}\frac{\partial\psi_p}{\partial\eta}w_p-\int_{\partial\mathbb{R}^n_+}\frac{n\mathfrak{D}_n}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)}w_p\psi_p\,d\tilde{x}$$

$$+\int_{\mathbb{R}^n_+}\left(|\nabla\psi_p|^2+\frac{n(n+2)}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^2}\psi_p^2\right)dx-\int_{\partial\mathbb{R}^n_+}\frac{n\mathfrak{D}_n}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)}\psi_p^2\,d\tilde{x}$$

$$=\int_{\partial\mathbb{R}^n_+}\frac{n\mathfrak{D}_n}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)}w_p\psi_p\,d\tilde{x}+\int_{\partial\mathbb{R}^n_+}b(\tilde{x},0)w_p(\tilde{x},0)\,d\tilde{x}-\int_{\partial\mathbb{R}^n_+}\frac{n\mathfrak{D}_n}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)}w_p\psi_p\,d\tilde{x}$$

$$+\int_{\mathbb{R}^n_+}\left(|\nabla\psi_p|^2+\frac{n(n+2)}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^2}\psi_p^2\right)dx-\int_{\partial\mathbb{R}^n_+}\frac{n\mathfrak{D}_n}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)}\psi_p^2\,d\tilde{x}$$

We will study separately the terms with and without ψ_p. By (3.24)

$$b(\tilde{x},0)=\frac{\beta_n}{4n}\frac{\sum_{\substack{i,j=1\\i\neq j}}^{n-1}h^{ij}(p)x_ix_j}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)^{\frac{n}{2}}}$$

and arguing as before:

$$\int_{\partial\mathbb{R}^n_+} b(\tilde{x},0)w_p(\tilde{x},0)\,d\tilde{x} = -\frac{\mathfrak{D}_n\beta_n^2}{16n^2}\sum_{i,j=1}^{n-1} h^{ij}(p)^2\int_{\mathbb{R}^{n-1}}\frac{{x_i}^2{x_j}^2}{(|\tilde{x}|^2+{\mathfrak{D}_n}^2-1)^n}d\tilde{x}.$$

Now, for $i\neq j$ fixed,

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}}\frac{x_i^2x_j^2}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)^n}d\tilde{x}\\
&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}x_i^2x_j^2\int_0^{+\infty}\frac{\omega_{n-4}r^{n-4}dr}{(r^2+x_i^2+x_j^2+\mathfrak{D}_n^2-1)}dx_idx_j\\
&=\frac{\omega_{n-4}\Gamma\left(\frac{n-3}{2}\right)\Gamma\left(\frac{n+3}{2}\right)}{2(n-1)!}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{{x_i}^2{x_j}^2dx_idx_j}{(x_i^2+x_j^2+{\mathfrak{D}_n}^2-1)^{\frac{n+3}{2}}}\\
&=\frac{\omega_{n-4}\pi\Gamma\left(\frac{n-3}{2}\right)\Gamma\left(\frac{n+3}{2}\right)(\mathfrak{D}_n^2-1)^{\frac{3-n}{2}}}{(n-1)!(n-3)(n-1)(n+1)}.
\end{aligned}$$

Finally,

$$\int_{\partial\mathbb{R}^n_+} b(\tilde{x},0)w_p(\tilde{x},0)\,d\tilde{x} = -\frac{\mathfrak{D}_n{\beta_n}^2\omega_{n-4}\pi\Gamma\left(\frac{n-3}{2}\right)\Gamma\left(\frac{n+3}{2}\right)(\mathfrak{D}_n^2-1)^{\frac{3-n}{2}}}{16n(n-1)(n-3)(n+1)!}\left\|\pi(p)\right\|^2.$$

Finally we address the terms with ψ_p.
Since ψ_p solves (3.23) we can write

$$\psi_p=\frac{\beta_n}{4n}\sum_{\substack{i,j=1\\ i\neq j}}^{n-1}h^{ij}(p)\psi_{ij}$$

where ψ_{ij} solves

$$\begin{cases}
-\Delta\psi_{ij}+\dfrac{n(n+2)}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^2}\psi_{ij}=0, & \text{in } \mathbb{R}^n_+\\
\dfrac{\partial\psi_{ij}}{\partial\eta}-\dfrac{n\mathfrak{D}_n}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)}\psi_{ij}=\dfrac{x_ix_j}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)^{\frac{n}{2}}} & \text{on } \partial\mathbb{R}^n_+
\end{cases}$$

It is not difficult to check that ψ_{ij} is odd in x_i and x_j and even in all the other variables x_ℓ, $\ell=1,\dots,n-1$, and so

$$\int\limits_{\mathbb{R}^{n-1}}\frac{x_ix_j}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)^{\frac{n}{2}}}\psi_{\ell\kappa}(\tilde{x},0)d\tilde{x}=0 \text{ if } (i,j)\neq(\ell,\kappa). \tag{B.4}$$

Moreover it holds that $\psi_{ij}=\psi_{ji}$ and $\psi_{ij}=\psi_{12}(\sigma_{ij}x)$, where σ_{ij} permutes the x_i and x_j variables, i.e.

$$\sigma_{ij}(x_1,\dots,x_i,\dots,x_j,\dots,x_n)=(x_1,\dots,x_j,\dots,x_i,\dots,x_n).$$

Multiplying by ψ_p in (3.23), integrating by parts and using (B.4) we immediately see that

$$\begin{aligned}
&\int_{\mathbb{R}^n_+}|\nabla\psi_p|^2+\int_{\mathbb{R}^n_+}\frac{n(n+2)}{(|\tilde x|^2+(x_n+\mathfrak{D}_n)^2-1)^2}\psi_p^2-\int_{\partial\mathbb{R}^n_+}\frac{n\mathfrak{D}_n}{|\tilde x|^2+(x_n+\mathfrak{D}_n)^2-1}\psi_p^2\\
&=\frac{\beta_n}{4n}\int_{\partial\mathbb{R}^n_+}\frac{h^{ij}(p)x_ix_j}{(|\tilde x|^2+(x_n+\mathfrak{D}_n)^2-1)^{\frac n2}}\psi_p\\
&=\frac{\beta_n}{4n}\sum_{\substack{i,j=1\\ i\neq j}}^{n-1}h^{ij}(p)\sum_{\substack{\ell,\kappa=1\\ \ell\neq\kappa}}^{n-1}h^{\ell\kappa}(p)\int\limits_{\mathbb{R}^{n-1}}\frac{x_ix_j}{(|\tilde x|^2+\mathfrak{D}_n^2-1)^{\frac n2}}\psi_{\ell\kappa}(\tilde x,0)d\tilde x\\
&=\sum_{\substack{i,j=1\\ i\neq j}}^{n-1}h^{ij}(p)^2\int\limits_{\mathbb{R}^{n-1}}\frac{x_ix_j}{(|\tilde x|^2+\mathfrak{D}_n^2-1)^{\frac n2}}\psi_{ij}(\tilde x,0)d\tilde x\\
&=\sum_{\substack{i,j=1\\ i\neq j}}^{n-1}h^{ij}(p)^2\underbrace{\left(\int\limits_{\mathbb{R}^{n-1}}\frac{x_1x_2}{(|\tilde x|^2+\mathfrak{D}_n^2-1)^{\frac n2}}\psi_{12}(\tilde x,0)d\tilde x\right)}_{\mathfrak{t}_2(n,\mathfrak{D}_n)}\\
&=\mathfrak{f}_2(n,\mathfrak{D}_n)\|\pi(p)\|^2,
\end{aligned}$$

where $\mathfrak{f}_2$ only depends on n and $\mathfrak{D}_n$ because the functions ψ_{ij} do not depend on the point p.

Collecting all the previous estimates, we get a constant $\mathfrak{f}_n$, which only depends on n and $\mathfrak{D}_n$, such that

$$\int_{\mathbb{R}^n_+}\left(-c_n\Delta V_p+\frac{n+2}{n-2}|K|U^{\frac{4}{n-2}}V_p\right)V_p=\mathfrak{f}_n\|\pi(p)\|^2.$$

Notice that $\mathfrak{f}_n$ needs to be non-negative because V_p satisfies Proposition 3.1-(iv). □

Proof of Proposition 3.3. We write $\mathcal{W}_p=\chi\left(\left(\psi_p^\partial\right)^{-1}(\xi)\right)W$ with

$$W(\xi):=\underbrace{\frac{1}{\delta^{\frac{n-2}{2}}}U\left(\frac{\left(\psi_p^\partial\right)^{-1}(\xi)}{\delta}\right)}_{:=U}+\delta\underbrace{\frac{1}{\delta^{\frac{n-2}{2}}}V_p\left(\frac{\left(\psi_p^\partial\right)^{-1}(\xi)}{\delta}\right)}_{:=V}.$$

and also $\mathcal{W}_p=\mathcal{W}=\mathcal{U}+\delta\mathcal{V}$ with

$$\mathcal{U}(\xi)=\chi\left(\left(\psi_p^\partial\right)^{-1}(\xi)\right)U(\xi)\text{ and }\mathcal{V}(\xi)=\chi\left(\left(\psi_p^\partial\right)^{-1}(\xi)\right)V(\xi).$$

We have

$$\begin{aligned}
J_\varepsilon(\mathcal{W})=&\underbrace{\frac{c_n}{2}\int_M|\nabla_g(\mathcal{U}+\delta\mathcal{V})|^2}_{I_1}+\underbrace{\frac12\int_M\mathcal{S}_g(\mathcal{U}+\delta\mathcal{V})^2}_{I_2}+\underbrace{(n-1)\varepsilon\int_{\partial M}(\mathcal{U}+\delta\mathcal{V})^2}_{I_3}\\
&\underbrace{-(n-2)\int_{\partial M}H\left[\left((\mathcal{U}+\delta\mathcal{V})^+\right)^{\frac{2(n-1)}{n-2}}-\mathcal{U}^{\frac{2(n-1)}{n-2}}\right]}_{I_4}\underbrace{-(n-2)\int_{\partial M}H\mathcal{U}^{\frac{2(n-1)}{n-2}}}_{I_5}\\
&\underbrace{-\frac{n-2}{2n}\int_MK\left[\left((\mathcal{U}+\delta\mathcal{V})^+\right)^{\frac{2n}{n-2}}-\mathcal{U}^{\frac{2n}{n-2}}\right]}_{I_6}\underbrace{-\frac{n-2}{2n}\int_MK\mathcal{U}^{\frac{2n}{n-2}}}_{I_7}
\end{aligned}$$

Estimate of I_2 By (A.2) (with $\alpha=0$ and $m=n-2$) and (A.1), if $n\ge 5$

$$\begin{aligned}
I_2 &:= \frac{1}{2}\delta^2\int_{\mathbb{R}^n_+}\mathcal{S}_g(\delta x)\left(U(x)\chi(\delta x)+\delta V_p(x)\chi(\delta x)\right)^2|g(\delta x)|^{\frac12}\,dx\\
&=\frac12\delta^2\mathcal{S}_g(p)\int_{\mathbb{R}^n_+}U^2(x)\,dx+\begin{cases}\mathcal{O}(\delta^3) & \text{if } n\ge 6\\ \mathcal{O}(\delta^3|\ln\delta|) & \text{if } n=5\end{cases}\\
&=\frac12\delta^2\frac{\mathcal{S}_g(p)}{|K|^{\frac{n-2}{2}}}\int_{\mathbb{R}^n_+}\frac{\alpha_n^2}{(|\tilde x|^2+(x_n+\mathfrak{D}_n)^2-1)^{n-2}}\,dx+\begin{cases}\mathcal{O}(\delta^3) & \text{if } n\ge 6\\ \mathcal{O}(\delta^3|\ln\delta|) & \text{if } n=5\end{cases}\\
&=\delta^2\frac12\alpha_n^2\omega_{n-1}\frac{2(n-2)}{n-1}I^n_{n-1}\frac{\mathcal{S}_g(p)}{|K|^{\frac{n-2}{2}}}\varphi_{\frac{n-3}{2}}+\begin{cases}\mathcal{O}(\delta^3) & \text{if } n\ge 6\\ \mathcal{O}(\delta^3|\ln\delta|) & \text{if } n=5\end{cases}
\end{aligned}$$

and if $n=4$

$$\begin{aligned}
I_2 &= \frac{\alpha_4^2}{2}\delta^2\frac{\mathcal{S}_g(p)}{|K|}\int_{B^+_{\frac{R}{\delta}}}\frac{1}{(|\tilde x|^2+(x_4+\mathfrak{D}_4(p))^2-1)^2}\,dx+\mathcal{O}(\delta^2)\\
&=\frac{\alpha_4^2\omega_3}{2}\delta^2\frac{\mathcal{S}_g(p)}{|K|}I_2^2\int_{\sqrt{\mathfrak{D}_4^2(p)-1}}^{\sqrt{\left(\frac{R}{\delta}+\mathfrak{D}_4(p)\right)^2-1}}\frac{1}{\sqrt{\lambda^2+1}}\,d\lambda+\mathcal{O}(\delta^2)\\
&=-\frac{2\alpha_4^2\omega_3}{3}\frac{\mathcal{S}_g(p)}{|K|}I_3^4\delta^2\ln\delta+\mathcal{O}(\delta^2)
\end{aligned}$$

Estimate of I_3 By (A.3) (with $\alpha=0$ and $m=n-2$) and (A.1), if $n\ge 4$ that

$$\begin{aligned}
I_3 &:= (n-1)\varepsilon\delta\int_{\mathbb{R}^{n-1}}\left(U(\tilde x,0)\chi(\delta\tilde x,0)+\delta V_p(\tilde x,0)\chi(\delta\tilde x,0)\right)^2|g(\delta\tilde x,0)|^{\frac12}\,d\tilde x\\
&=(n-1)\varepsilon\delta\int_{\mathbb{R}^{n-1}}U^2(\tilde x,0)\,d\tilde x+\begin{cases}\mathcal{O}(\varepsilon\delta^2) & \text{if } n\ge 5\\ \mathcal{O}(\varepsilon\delta^2|\ln\delta|) & \text{if } n=4.\end{cases}\\
&=(n-1)\varepsilon\delta\frac{1}{|K|^{\frac{n-2}{2}}}\alpha_n^2\int_{\mathbb{R}^{n-1}}\frac{1}{(|\tilde x|^2+\mathfrak{D}_n^2-1)^{n-2}}\,d\tilde x+\begin{cases}\mathcal{O}(\varepsilon\delta^2) & \text{if } n\ge 5\\ \mathcal{O}(\varepsilon\delta^2|\ln\delta|) & \text{if } n=4.\end{cases}\\
&=\varepsilon\delta\underbrace{2(n-2)\omega_{n-1}\alpha_n^2\frac{1}{|K|^{\frac{n-2}{2}}(\mathfrak{D}_n^2-1)^{\frac{n-3}{2}}}I^n_{n-1}}_{:=\mathfrak{c}_n}+\begin{cases}\mathcal{O}(\varepsilon\delta^2) & \text{if } n\ge 5\\ \mathcal{O}(\varepsilon\delta^2|\ln\delta|) & \text{if } n=4.\end{cases}
\end{aligned}$$

Estimate of I_5 By (A.3) (with $\alpha=2$ and $m=n-1$) and Lemma A.2, if $n\ge 4$

$$\begin{aligned}
I_5 &:= -(n-2)\int_{\mathbb{R}^{n-1}}H\left(U(\tilde x,0)\chi(\delta\tilde x,0)\right)^{2^\sharp}|g(\delta\tilde x,0)|^{\frac12}\,d\tilde x\\
&=-(n-2)H\int_{\partial\mathbb{R}^n_+}U^{2^\sharp}(\tilde x,0)\,d\tilde x+\frac{n-2}{6}\delta^2\overline{R}_{ij}(p)H\int_{\mathbb{R}^{n-1}}U^{2^\sharp}(\tilde x,0)\tilde x_i\tilde x_j\,d\tilde x+\mathcal{O}(\delta^3)\\
&=-(n-2)H\int_{\partial\mathbb{R}^n_+}U^{2^\sharp}(\tilde x,0)\,d\tilde x\\
&\quad+\delta^2\frac{\alpha_n^{2^\sharp}(n-2)}{6(n-1)}\frac{H\overline{R}_{ii}(p)}{|K|^{\frac{n-1}{2}}}\int_{\mathbb{R}^{n-1}}\frac{|\tilde x|^2}{(|\tilde x|^2+\mathfrak{D}_n^2-1)^{n-1}}\,d\tilde x+\mathcal{O}(\delta^3)\\
&=-(n-2)H\int_{\partial\mathbb{R}^n_+}U^{2^\sharp}(\tilde x,0)\,d\tilde x\\
&\quad+\delta^2\frac{\alpha_n^{2^\sharp}(n-2)}{6(n-1)}\omega_{n-1}\frac{H\overline{R}_{ii}(p)}{|K|^{\frac{n-1}{2}}(\mathfrak{D}_n^2-1)^{\frac{n-3}{2}}}I^n_{n-1}+\mathcal{O}(\delta^3).
\end{aligned}$$

Estimate of I_7 By (A.2) (with $\alpha = 2$ and $m = n$), (A.4) (with $\alpha = 0$ and $m = n$), Lemma A.2 and (A.1), if $n \geq 4$

$$
\begin{aligned}
I_7 &:= -\frac{n-2}{2n}\int_{\mathbb{R}^n_+} K\left(U(x)\chi(\delta x)\right)^{2^*}|g(\delta x)|^{\frac{1}{2}}\,dx \\
&= \frac{n-2}{2n}|K|\int_{\mathbb{R}^n_+} U^{2^*}(x)\,dx - \frac{n-2}{4n}\left(\|\pi(p)\|^2 + \mathrm{Ric}_\nu(p)\right)\delta^2|K|\int_{\mathbb{R}^n_+} U^{2^*}x_n^2\,dx \\
&\quad - \frac{n-2}{12n}\overline{R}_{ij}(p)\delta^2|K|\int_{\mathbb{R}^n_+} U^{2^*}(x)\tilde{x}_i\tilde{x}_j\,dx + \mathcal{O}(\delta^3) \\
&= \frac{n-2}{2n}|K|\int_{\mathbb{R}^n_+} U^{2^*}(x)\,dx \\
&\quad - \delta^2\frac{n-2}{4n}\alpha_n^{2^*}\frac{\left(\|\pi(p)\|^2 + \mathrm{Ric}_\nu(p)\right)|K|}{|K|^{\frac{n}{2}}}\int_{\mathbb{R}^n_+}\frac{x_n^2}{(|\tilde{x}|^2 + (x_n + \mathfrak{D}_n)^2 - 1)^n}\,dx \\
&\quad - \delta^2\alpha_n^{2^*}\frac{n-2}{12n(n-1)}\frac{\overline{R}_{ii}(p)|K|}{|K|^{\frac{n}{2}}}\int_{\mathbb{R}^n_+}\frac{|\tilde{x}|^2}{(|\tilde{x}|^2 + (x_n + \mathfrak{D}_n)^2 - 1)^n}\,dx + \mathcal{O}(\delta^3) \\
&= \frac{n-2}{2n}|K|\int_{\mathbb{R}^n_+} U^{2^*}(x)\,dx \\
&\quad - \delta^2\frac{n-2}{4n}\frac{n-3}{2(n-1)}I^n_{n-1}\omega_{n-1}\alpha_n^{2^*}\frac{\left(\|\pi(p)\|^2 + \mathrm{Ric}_\nu(p)\right)}{|K|^{\frac{n-2}{2}}}\hat{\varphi}_{\frac{n+1}{2}} \\
&\quad - \delta^2\alpha_n^{2^*}\frac{n-3}{2(n-1)}I^n_{n-1}\omega_{n-1}\frac{n-2}{12n(n-1)}\frac{\overline{R}_{ii}(p)}{|K|^{\frac{n-2}{2}}}\varphi_{\frac{n-1}{2}} + \mathcal{O}(\delta^3)
\end{aligned}
$$

Estimate of I_4 *and* I_6 By Lemma A.2-(i), if $n \geq 4$

$$
\begin{aligned}
I_4 &= -(n-2)\int_{\partial\mathbb{R}^n_+} H\left[\left((U + \delta V_p)^+\right)^{\frac{2(n-1)}{n-2}} - U^{\frac{2(n-1)}{n-2}}\right](\tilde{x},0)|g(\delta\tilde{x},0)|^{\frac{1}{2}}\,d\tilde{x} \\
&= -2(n-1)\delta H\int_{\partial\mathbb{R}^n_+} U^{\frac{n}{n-2}}V_p\,d\tilde{x} - \frac{n(n-1)}{n-2}\delta^2 H\int_{\partial\mathbb{R}^n_+} U^{\frac{2}{n-2}}V_p^2\,d\tilde{x} + \mathcal{O}(\delta^3)
\end{aligned}
$$

and similarly

$$
I_6 = |K|\delta\int_{\mathbb{R}^n_+} U^{\frac{n+2}{n-2}}V_p + \frac{n+2}{2(n-2)}\delta^2|K|\int_{\mathbb{R}^n_+} U^{\frac{4}{n-2}}V_p^2 + \mathcal{O}(\delta^3).
$$

Estimate of I_1 First we have

$$
I_1 := \underbrace{\frac{c_n}{2}\int_M |\nabla_g \mathcal{U}_{\delta,p}|^2\,d\nu_g}_{:=I_1^1} + \underbrace{c_n\delta\int_M \nabla_g\mathcal{U}_{\delta,p}\nabla_g\mathcal{V}_{\delta,p}\,d\nu_g}_{:=I_1^2} + \underbrace{\frac{c_n}{2}\delta^2\int_M |\nabla_g\mathcal{V}_{\delta,p}|^2\,d\nu_g}_{:=I_1^3}
$$

and we separately estimate the terms I_1^i with $i = 1, 2, 3$. Set $B_\delta := \{x \in \mathbb{R}^n_+ \ : \ |\delta x| \leq R\}$.

Estimate of I_1^1 By Lemma A.2-(ii)-(iii) we get

$$
\begin{aligned}
I_1^1 &= \frac{c_n}{2}\int_{\mathbb{R}^n_+} g^{ab}(\delta x)\frac{\partial}{\partial x_a}(U(x)\chi(\delta x))\frac{\partial}{\partial x_b}(U(x)\chi(\delta x))|g(\delta x)|^{\frac12}\,dx\\
&= c_n\int_{B_\delta}\left[\frac{|\nabla U|^2}{2}+\left(\delta h_{ij}x_n+\frac{\delta^2}{6}\overline{R}_{ikj\ell}x_kx_\ell+\delta^2\frac{\partial h_{ij}}{\partial x_k}x_kx_n+\frac{\delta^2}{2}\left(R_{injn}+3h_{ik}h_{kj}\right)x_n^2\right)\frac{\partial U}{\partial x_i}\frac{\partial U}{\partial x_j}\right]\\
&\quad\times\left(1-\frac{\delta^2}{2}\left(\|\pi\|^2+\mathrm{Ric}_\nu\right)x_n^2-\frac{\delta^2}{6}\overline{R}_{\ell m}x_\ell x_m\right)d\tilde{x}\,dx_n+\mathcal{O}(\delta^3)\\
&= \int_{B_\delta}\left(\frac{c_n}{2}|\nabla U|^2+\delta^2\frac{c_n}{2}\left(R_{injn}+3h_{ik}h_{kj}\right)x_n^2\frac{\partial U}{\partial x_i}\frac{\partial U}{\partial x_j}\right)\\
&\quad\times\left(1-\frac{\delta^2}{2}\left(\|\pi\|^2+\mathrm{Ric}_\nu\right)x_n^2-\frac{\delta^2}{6}\overline{R}_{\ell m}x_\ell x_m\right)d\tilde{x}\,dx_n+\mathcal{O}(\delta^3).
\end{aligned}
$$

Moreover, by (A.4) (with $\alpha=0$ or $\alpha=2$ and $m=n-1$ and with $\alpha=0$ and $m=n$ secondly), (A.2) (with $\alpha=2$ and $m=n-1$ first and with $\alpha=2$ and $m=n$ secondly) and (A.1), if $n\geq 5$

$$
\begin{aligned}
I_1^1 &= \frac{c_n}{2}\int_{\mathbb{R}^n_+}|\nabla U|^2-\frac{c_n}{4}\delta^2\left(\|\pi(p)\|^2+\mathrm{Ric}_\nu(p)\right)\int_{\mathbb{R}^n_+}x_n^2|\nabla U|^2-\frac{c_n}{12}\delta^2\frac{\overline{R}_{\ell\ell}(p)}{n-1}\int_{\mathbb{R}^n_+}|\tilde{x}|^2|\nabla U|^2\\
&\quad+\frac{c_n\alpha_n^2}{2}\frac{(n-2)^2}{n-1}\delta^2\frac{\left(3\|\pi(p)\|^2+\mathrm{Ric}_\nu(p)\right)}{|K|^{\frac{n-2}{2}}}\int_{\mathbb{R}^n_+}\frac{x_n^2|\tilde{x}|^2}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^n}\,dx+\mathcal{O}(\delta^3)\\
&= \frac{c_n}{2}\int_{\mathbb{R}^n_+}|\nabla U|^2-\frac{c_n\alpha_n^2(n-2)^2}{4}\delta^2\frac{(\|\pi(p)\|^2+\mathrm{Ric}_\nu(p))}{|K|^{\frac{n-2}{2}}}\int_{\mathbb{R}^n_+}\frac{x_n^2}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^{n-1}}\,dx\\
&\quad-\frac{c_n\alpha_n^2(n-2)^2}{4}\delta^2\frac{(\|\pi(p)\|^2+\mathrm{Ric}_\nu(p))}{|K|^{\frac{n-2}{2}}}\int_{\mathbb{R}^n_+}\frac{x_n^2}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^{n}}\\
&\quad-\frac{c_n\alpha_n^2(n-2)^2}{12(n-1)}\delta^2\frac{\overline{R}_{\ell\ell}(p)}{|K|^{\frac{n-2}{2}}}\int_{\mathbb{R}^n_+}\frac{|\tilde{x}|^2}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^{n-1}}\\
&\quad-\frac{c_n\alpha_n^2(n-2)^2}{12(n-1)}\delta^2\frac{\overline{R}_{\ell\ell}(p)}{|K|^{\frac{n-2}{2}}}\int_{\mathbb{R}^n_+}\frac{|\tilde{x}|^2}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^{n}}\\
&\quad+\frac{c_n\alpha_n^2}{2}\frac{(n-2)^2}{n-1}\delta^2\frac{\left(3\|\pi(p)\|^2+\mathrm{Ric}_\nu(p)\right)}{|K|^{\frac{n-2}{2}}}\int_{\mathbb{R}^n_+}\frac{x_n^2|\tilde{x}|^2}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^n}\,dx+\mathcal{O}(\delta^3)\\
&= \frac{c_n}{2}\int_{\mathbb{R}^n_+}|\nabla U|^2-\frac{c_n\alpha_n^2(n-2)^2(n-3)}{4(n-1)}\omega_{n-1}I_{n-1}^n\delta^2\frac{(\|\pi(p)\|^2+\mathrm{Ric}_\nu(p))}{|K|^{\frac{n-2}{2}}}\hat{\varphi}_{\frac{n-1}{2}}\\
&\quad-\frac{c_n\alpha_n^2(n-2)^2(n-3)}{8(n-1)}\omega_{n-1}I_{n-1}^n\delta^2\frac{(\|\pi(p)\|^2+\mathrm{Ric}_\nu(p))}{|K|^{\frac{n-2}{2}}}\hat{\varphi}_{\frac{n+1}{2}}\\
&\quad-\frac{c_n\alpha_n^2(n-2)^2}{12(n-1)}\omega_{n-1}I_{n-1}^n\delta^2\frac{\overline{R}_{\ell\ell}(p)}{|K|^{\frac{n-2}{2}}}\varphi_{\frac{n-3}{2}}-\frac{c_n\alpha_n^2(n-2)^2(n-3)}{24(n-1)^2}\omega_{n-1}I_{n-1}^n\delta^2\frac{\overline{R}_{\ell\ell}(p)}{|K|^{\frac{n-2}{2}}}\varphi_{\frac{n-1}{2}}\\
&\quad+\frac{c_n\alpha_n^2}{4}\frac{(n-2)^2(n-3)}{(n-1)^2}\omega_{n-1}I_{n-1}^n\delta^2\frac{\left(3\|\pi(p)\|^2+\mathrm{Ric}_\nu(p)\right)}{|K|^{\frac{n-2}{2}}}\hat{\varphi}_{\frac{n-1}{2}}+\mathcal{O}(\delta^3)
\end{aligned}
$$

and if $n=4$

$$
\begin{aligned}
I_1^1 &= \frac{c_4}{2}\int_{\mathbb{R}^4_+}|\nabla U|^2 - c_4\alpha_4^2\frac{\left(\|\pi(p)\|^2+\mathrm{Ric}_\nu(p)\right)}{|K|}\delta^2\int_{B_\delta}\frac{x_4^2}{(|\tilde{x}|^2+(x_4+\mathfrak{D}_4)^2-1)^3}\,dx\\
&- \frac{1}{9}c_4\alpha_4^2\frac{\overline{R}_{\ell\ell}(p)}{|K|}\delta^2\int_{B_\delta}\frac{|\tilde{x}|^2}{(|\tilde{x}|^2+(x_4+\mathfrak{D}_4)^2-1)^3}\,dx\\
&+ \frac{2}{3}c_4\alpha_4^2\frac{\left(3\|\pi(p)\|^2+\mathrm{Ric}_\nu(p)\right)}{|K|}\delta^2\int_{B_\delta}\frac{x_4^2|\tilde{x}|^2}{(|\tilde{x}|^2+(x_4+\mathfrak{D}_4)^2-1)^4}\,dx+\mathcal{O}(\delta^2)\\
&= \frac{c_4}{2}\int_{\mathbb{R}^4_+}|\nabla U|^2+\frac{1}{3}c_4\omega_3\alpha_4^2 I_3^4\frac{\left(\|\pi(p)\|^2+\mathrm{Ric}_\nu(p)\right)}{|K|}\delta^2\ln\delta+\frac{1}{9}c_4\omega_3\alpha_4^2 I_3^4\frac{\overline{R}_{\ell\ell}(p)}{|K|}\delta^2\ln\delta\\
&- \frac{1}{54}c_4\omega_3\alpha_4^2\frac{\left(3\|\pi(p)\|^2+\mathrm{Ric}_\nu(p)\right)}{|K|}I_3^4\delta^2\ln\delta+\mathcal{O}(\delta^2).
\end{aligned}
$$

Estimate of I_1^2 and I_1^3 if $n\geq 5$

We have

$$
\begin{aligned}
I_1^2 &= \delta c_n\int_{\mathbb{R}^n_+} g^{ab}(\delta x)\frac{\partial}{\partial x_a}(U(x)\chi(\delta x))\frac{\partial}{\partial x_b}(V_p(x)\chi(\delta x))|g(\delta x)|^{\frac{1}{2}}\,dx\\
&= \delta c_n\int_{\mathbb{R}^n_+}\nabla U\nabla V_p\,dx+\delta^2 2c_n h^{ij}(p)\int_{\mathbb{R}^n_+}x_n\frac{\partial U}{\partial x_i}\frac{\partial V_p}{\partial x_j}\,dx+\mathcal{O}(\delta^3)\\
&= -\delta\frac{n+2}{n-2}|K|\int_{\mathbb{R}^n_+}U^{\frac{n+2}{n-2}}V_p\,dx+c_n\frac{n}{2}\delta H\int_{\partial\mathbb{R}^n_+}U^{\frac{n}{n-2}}V_p\\
&- c_n\delta^2\underbrace{\int_{\mathbb{R}^n_+}|\nabla V_p|^2}_{<\infty\text{ for }n\geq 5}+c_n\frac{n}{2}\delta^2 H\int_{\partial\mathbb{R}^n_+}U^{\frac{2}{n-2}}V_p^2-\frac{n+2}{n-2}|K|\delta^2\int_{\mathbb{R}^n_+}U^{\frac{4}{n-2}}V_p^2+\mathcal{O}(\delta^3)
\end{aligned}
$$

since

$$
\begin{aligned}
c_n\int_{\mathbb{R}^n_+}\nabla U\nabla V_p &= -c_n\int_{\mathbb{R}^n_+}U\Delta V_p+c_n\int_{\partial\mathbb{R}^n_+}U(\tilde{x},0)\frac{\partial V_p}{\partial\nu}\\
&= -\frac{n+2}{n-2}|K|\int_{\mathbb{R}^n_+}U^{\frac{n+2}{n-2}}V_p\,dx+c_n\frac{n}{2}H\int_{\partial\mathbb{R}^n_+}U^{\frac{n}{n-2}}(\tilde{x},0)V_p\,d\tilde{x}\\
&+\underbrace{8\frac{n-1}{n-2}\int_{\mathbb{R}^n_+}h^{ij}(p)\frac{\partial^2 U}{\partial x_i\partial x_j}x_n U}_{=0}
\end{aligned}
$$

and

$$
\begin{aligned}
2c_n h^{ij}(p)\int_{\mathbb{R}^n_+}x_n\frac{\partial U}{\partial x_i}\frac{\partial V_p}{\partial x_j}\,dx &= -2c_n h^{ij}(p)\int_{\mathbb{R}^n_+}x_n\frac{\partial^2 U}{\partial x_i\partial x_j}V_p\,dx\\
&= -\int_{\mathbb{R}^n_+}\left(-c_n\Delta V_p+\frac{n+2}{n-2}|K|U^{\frac{4}{n-2}}V_p\right)V_p\,dx\\
&= -c_n\int_{\mathbb{R}^n_+}|\nabla V_p|^2+c_n\int_{\partial\mathbb{R}^n_+}V_p\frac{\partial V_p}{\partial\nu}-\frac{n+2}{n-2}|K|\int_{\mathbb{R}^n_+}U^{\frac{4}{n-2}}V_p^2\\
&= -c_n\int_{\mathbb{R}^n_+}|\nabla V_p|^2+c_n\frac{n}{2}H\int_{\partial\mathbb{R}^n_+}U^{\frac{2}{n-2}}V_p^2-\frac{n+2}{n-2}|K|\int_{\mathbb{R}^n_+}U^{\frac{4}{n-2}}V_p^2
\end{aligned}
$$

and also

$$
I_1^3 := \frac{c_n}{2}\delta^2\int_{\mathbb{R}^n_+}|\nabla V_p|^2+\mathcal{O}(\delta^3).
$$

Estimate of I_1^2 and I_1^3 if $n=4$
Let $V_p=\bar w_p+\zeta_p+\psi_p$ as in (3.12) and set $w_p=\bar w_p+\zeta_p$ so that w_p solves the problem

$$\begin{cases} -6\Delta w_p=\mathtt{E}_p(x) & \text{in } \mathbb{R}^4_+\\ \dfrac{\partial w_p}{\partial\nu}=2HUw_p, & \text{on } \partial\mathbb{R}^4_+\end{cases}\tag{B.5}$$

Then

$$\begin{aligned} I_1^2&=\delta c_4\int_{\mathbb{R}^4_+}g^{ab}(\delta x)\frac{\partial}{\partial x_a}(U(x)\chi(\delta x))\frac{\partial}{\partial x_b}(V_p(x)\chi(\delta x))|g(\delta x)|^{\frac12}\,dx\\ &=-\delta\frac{n+2}{n-2}|K|\int_{\mathbb{R}^4_+}U^3V_p\,dx+2c_4\delta H\int_{\partial\mathbb{R}^4_+}U^2V_p\\ &+\delta^2 12h^{ij}(p)\int_{B^+_{\frac{R}{\delta}}}x_4\frac{\partial U}{\partial x_i}\frac{\partial V_p}{\partial x_j}\,dx+\mathcal{O}(\delta^2)\\ &=-\delta\frac{n+2}{n-2}|K|\int_{\mathbb{R}^4_+}U^3V_p\,dx+2c_4\delta H\int_{\partial\mathbb{R}^4_+}U^2V_p\\ &+\delta^2 12h^{ij}(p)\int_{B^+_{\frac{R}{\delta}}}x_4\frac{\partial U}{\partial x_i}\frac{\partial w_p}{\partial x_j}\,dx+\mathcal{O}(\delta^2).\end{aligned}\tag{B.6}$$

Here we have used the fact that

$$\int_{B^+_{\frac{R}{\delta}}}x_4\frac{\partial U}{\partial x_i}\frac{\partial \psi_p}{\partial x_j}\,dx=\mathcal{O}(1).$$

Let us study the last integral term of (B.6). Integrating by parts in x_j and using the equation (B.5):

$$\begin{aligned} 12h^{ij}(p)\int_{B^+_{\frac{R}{\delta}}}x_4\frac{\partial U}{\partial x_i}\frac{\partial w_p}{\partial x_j}dx&=12h^{ij}(p)\int_{\partial^+B^+_{\frac{R}{\delta}}}x_4\frac{\partial U}{\partial x_i}\frac{x_j}{|x|}w_p-\int_{B^+_{\frac{R}{\delta}}}\mathtt{E}_pw_p\\ &=12h^{ij}(p)\int_{\partial^+B^+_{\frac{R}{\delta}}}x_4\frac{\partial U}{\partial x_i}\frac{x_j}{|x|}w_p-6\int_{B^+_{\frac{R}{\delta}}}|\nabla w_p|^2\\ &+12\int_{\partial' B^+_{\frac{R}{\delta}}}HUw_p^2+6\int_{\partial^+B^+_{\frac{R}{\delta}}}\nabla w_p\cdot\frac{x}{|x|}w_p\\ &=-6\int_{B^+_{\frac{R}{\delta}}}|\nabla w_p|^2+\mathcal{O}(1),\end{aligned}\tag{B.7}$$

since

$$\int_{\partial' B^+_{\frac{R}{\delta}}}HUw_p^2=\mathcal{O}(1),$$

$$\int_{\partial^+B^+_{\frac{R}{\delta}}}x_4\frac{\partial U}{\partial x_i}\frac{x_j}{|x|}w_p\lesssim\frac{R}{\delta}\left(1+\frac{1}{\delta}\right)^{-3}\left(1+\frac{1}{\delta}\right)^{-1}\frac{\omega_{n-1}}{2}\frac{R^3}{\delta^3}=\mathcal{O}(1),$$

$$\int_{\partial^+B^+_{\frac{R}{\delta}}}\nabla w_p\cdot\frac{x}{|x|}w_p\lesssim\left(1+\frac{R}{\delta}\right)^{-2}\left(1+\frac{R}{\delta}\right)^{-1}\frac{\omega_{n-1}}{2}\frac{R^3}{\delta^3}=\mathcal{O}(1).$$

By (B.6) and (B.7),

$$I_1^2=-6\delta^2\int_{B^+_{\frac{R}{\delta}}}|\nabla w_p|^2+\mathcal{O}(\delta^2).\tag{B.8}$$

Analogously,

$$I_1^3 := \frac{c_4}{2}\delta^2 \int_{B^+_{\frac{R}{\delta}}} |\nabla V_p|^2 + \mathcal{O}(\delta^2) = 3\delta^2 \int_{B^+_{\frac{R}{\delta}}} |\nabla w_p|^2 + \mathcal{O}(\delta^2). \tag{B.9}$$

Finally, combining (B.8) and (B.9), we obtain

$$I_1^2 + I_1^3 = -3\delta^2 \int_{B^+_{\frac{R}{\delta}}} |\nabla w_p|^2 + \mathcal{O}\left(\delta^2\right). \tag{B.10}$$

Now, using the fact that $w_p = \bar{w}_p + \zeta_p$ and the decay estimate (3.17) we get

$$\begin{aligned}\int_{B^+_{\frac{R}{\delta}}} |\nabla w_p|^2 &= \int_{B^+_{\frac{R}{\delta}}} |\nabla \bar{w}_p|^2 + \int_{B^+_{\frac{R}{\delta}}} |\nabla \zeta_p|^2 + \int_{B^+_{\frac{R}{\delta}}} \nabla \bar{w}_p \nabla \zeta_p \\ &= \int_{B^+_{\frac{R}{\delta}}} |\nabla \bar{w}_p|^2 + \int_{B^+_{\frac{R}{\delta}}} |\nabla \zeta_p|^2 + \int_{\partial^+ B^+_{\frac{R}{\delta}}} \nabla \zeta_p \cdot \frac{x}{|x|} \bar{w}_p \\ &+ \int_{\partial' B^+_{\frac{R}{\delta}}} \left(2HU\zeta_p + \left(2HU\bar{w}_p - \frac{\partial \bar{w}_p}{\partial \nu}\right)\right) \bar{w}_p \\ &= \int_{B^+_{\frac{R}{\delta}}} |\nabla \bar{w}_p|^2 + \mathcal{O}(1)\end{aligned}$$

since again, by using the decay properties, we get

$$\int_{\partial' B^+_{\frac{R}{\delta}}} \left(2HU\zeta_p + \left(2HU\bar{w}_p - \frac{\partial \bar{w}_p}{\partial \nu}\right)\right) \bar{w}_p = \mathcal{O}(1)$$

and

$$\int_{\partial^+ B^+_{\frac{R}{\delta}}} \nabla \zeta_p \cdot \frac{x}{|x|} \bar{w}_p = \mathcal{O}(1)$$

and by using the problem solved by ζ_p, i.e. (3.14), we get

$$\begin{aligned}0 &= \int_{B^+_{\frac{R}{\delta}}} |\nabla \zeta_p|^2 - \int_{\partial^+ B^+_{\frac{R}{\delta}}} \nabla \zeta_p \cdot \frac{x}{|x|} \zeta_p + \int_{\partial' B^+_{\frac{R}{\delta}}} \left(2HU\zeta_p + \left(2HU\bar{w}_p - \frac{\partial \bar{w}_p}{\partial \nu}\right)\right) \zeta_p \\ &= \int_{B^+_{\frac{R}{\delta}}} |\nabla \zeta_p|^2 + \mathcal{O}(1)\end{aligned}$$

from which it follows that

$$\int_{B^+_{\frac{R}{\delta}}} |\nabla \zeta_p|^2 = \mathcal{O}(1).$$

By Proposition B.1 we get

$$\int_{B^+_{\frac{R}{\delta}}} |\nabla \bar{w}_p|^2 = \frac{64\pi^2}{|K|} \|\pi(p)\|^2 |\ln \delta| + \mathcal{O}(1)$$

Then (B.10) reduces to

$$I_1^2 + I_1^3 = -3\delta^2 \int_{Q^+_{\frac{R}{2\delta}}} |\nabla \bar{w}_p|^2 + \mathcal{O}(\delta^2) = -3\delta^2 \int_{Q^+_{\frac{R}{2\delta}}} \left|\nabla \bar{w}_p^0\right|^2 + \mathcal{O}(\delta^2) = -3\frac{64\pi^2}{|K|} \|\pi(p)\|^2 \delta^2 |\ln \delta| + \mathcal{O}(\delta^2).$$

Conclusion.

We collect all the previous estimates and we take into account that

- the terms of order δ cancel because of Proposition 3.1-(iii)

- the higher order terms which contain $\mathrm{Ric}_\nu(p)$ and $\overline{R}_{\ell\ell}(p)$ (di order δ^2 if $n \geq 5$ and $\delta^2|\ln\delta|$ if $n=4$) cancel, because by Lemma A.1 and the fact that $\mathcal{S}_g(p) = 2\mathrm{Ric}_\nu(p) + \overline{R}_{\ell\ell}(p) + \|\pi(p)\|^2$

$$\delta^2\alpha_n^2\omega_{n-1}I_{n-1}^n\frac{n-2}{n-1}\frac{\mathrm{Ric}_\nu(\mathrm{p})}{|K|^{\frac{n-2}{2}}}\left(2\varphi_{\frac{n-3}{2}} - (n-3)(n-1)\hat{\varphi}_{\frac{n+1}{2}}\right) = 0$$

and

$$\delta^2\alpha_n^2\frac{n-2}{3(n-1)}\omega_{n-1}I_{n-1}^n\frac{\overline{R}_{\ell\ell}(p)}{|K|^{\frac{n-2}{2}}}\left(-(n-4)\varphi_{\frac{n-3}{2}} - (n-3)\varphi_{\frac{n-1}{2}} + \frac{\mathfrak{D}_n}{(\mathfrak{D}_n-1)^{\frac{n-3}{2}}}\right) = 0.$$

Finally, we have if $n=4$

$$J_\varepsilon(\mathcal{W}) = \mathfrak{E} - \delta^2|\ln\delta|\underbrace{\left(\frac{192\pi^2}{|K|} + \alpha_4^2\omega_3 I_3^4\frac{1}{|K|}\right)}_{:=\mathfrak{b}_4}\|\pi(p)\|^2 + \varepsilon\delta\mathfrak{c}_4 + \mathcal{O}(\delta^2).$$

and if $n \geq 5$

$$J_\varepsilon(\mathcal{W}) = \mathfrak{E} - \delta^2(\underbrace{\frac{1}{2}\mathfrak{f}_n + \mathfrak{f}_n^1}_{:=\mathfrak{b}_n})\|\pi(p)\|^2 + \varepsilon\delta\mathfrak{c}_n + \begin{cases}\mathcal{O}(\delta^3) & \text{if } n \geq 6\\ \mathcal{O}(\delta^3|\ln\delta|) & \text{if } n=5\end{cases},$$

because the higher order terms which contain $\|\pi(p)\|^2$ reduces to

$$\delta^2\alpha_n^2\frac{n-2}{n-1}\omega_{n-1}I_{n-1}^n\frac{\|\pi(p)\|^2}{|K|^{\frac{n-2}{2}}}\left(\varphi_{\frac{n-3}{2}} - (n-1)(n-3)\hat{\varphi}_{\frac{n+1}{2}} - (n-1)(n-3)\hat{\varphi}_{\frac{n-1}{2}} + 3(n-3)\hat{\varphi}_{\frac{n-1}{2}}\right)$$
$$= -\delta^2\underbrace{\alpha_n^2\frac{n-2}{n-1}\omega_{n-1}I_{n-1}^n\frac{1}{|K|^{\frac{n-2}{2}}}\left(4(n-3)\hat{\varphi}_{\frac{n-1}{2}} + \varphi_{\frac{n-3}{2}}\right)}_{:=\mathfrak{f}_n^1}\|\pi(p)\|^2$$

and, by Proposition B.2,

$$\frac{1}{2}\int\limits_{\mathbb{R}^n_+}\left(-c_n\Delta V_p + \frac{n+2}{n-2}|K|U^{\frac{4}{n-2}}V_p\right)V_p = \frac{1}{2}\mathfrak{f}_n\|\pi(p)\|^2$$

Here the energy of the bubble $\mathfrak{E}$ is constant and is computed in the remark below. □

Remark B.3. The energy of the bubble is given by

$$\mathfrak{E} = \frac{c_n}{2}\int_{\mathbb{R}^+_n}|\nabla U|^2 - \frac{n-2}{2n}K\int_{\mathbb{R}^n_+}U^{2^*} - (n-2)H\int_{\partial\mathbb{R}^n_+}U^{2^\sharp}$$

where $c_n := \frac{4(n-1)}{n-2}$ and

$$U(\tilde{x}, x_n) := \frac{\alpha_n}{|K|^{\frac{n-2}{4}}}\frac{1}{(|\tilde{x}|^2 + (x_n + \mathfrak{D}_n)^2 - 1)^{\frac{n-2}{2}}}$$

and $\alpha_n := (4n(n-1))^{\frac{n-2}{4}}$ and $\mathfrak{D}_n := \sqrt{n(n-1)}\frac{H}{\sqrt{|K|}}$.
We recall that U satisfies (1.4). Hence

$$\frac{c_n}{2}\int_{\mathbb{R}^n_+}|\nabla U|^2 = \frac{c_n(n-2)}{4}H\int_{\partial\mathbb{R}^n_+}U^{2^\sharp} - \frac{1}{2}|K|\int_{\mathbb{R}^n_+}U^{2^*}$$

Then

$$\begin{aligned}\mathfrak{E} &= -\frac{1}{n}|K|\int_{\mathbb{R}^n_+} U^{2^*} + H\int_{\partial\mathbb{R}^n_+} U^{2^\sharp}\\ &= -\frac{1}{n}|K|\frac{\alpha_n^{2^*}}{|K|^{\frac{n}{2}}}\int_{\mathbb{R}^n_+}\frac{1}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^n}\,d\tilde{x}\,dx_n\\ &+ H\frac{\alpha_n^{2^\sharp}}{|K|^{\frac{n-1}{2}}}\int_{\partial\mathbb{R}^n_+}\frac{1}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)^{n-1}}\,d\tilde{x}\end{aligned}$$

Now, by using (A.3) with $\alpha=0$ and $m=n-1$ we get for $n\geq 4$

$$\int_{\partial\mathbb{R}^n_+}\frac{1}{(|\tilde{x}|^2+\mathfrak{D}_n^2-1)^{n-1}}\,d\tilde{x} = \omega_{n-1}\frac{n-3}{n-1}\frac{I^n_{n-1}}{(\mathfrak{D}_n^2-1)^{\frac{n-1}{2}}}.$$

Instead, by using (A.2) with $\alpha=0$ and $m=n$ we get

$$\int_{\mathbb{R}^n_+}\frac{1}{(|\tilde{x}|^2+(x_n+\mathfrak{D}_n)^2-1)^n}\,d\tilde{x}\,dx_n = \omega_{n-1}\frac{n-3}{2(n-1)}I^n_{n-1}\varphi_{\frac{n+1}{2}}.$$

Collecting all the previous terms, by Lemma A.1

$$\begin{aligned}\mathfrak{E} &= -\frac{\alpha_n^{2^*}(n-3)}{2n(n-1)}\omega_{n-1}I^n_{n-1}\frac{\varphi_{\frac{n+1}{2}}}{|K|^{\frac{n-2}{2}}} + \frac{\alpha_n^{2^\sharp}(n-3)}{n-1}\omega_{n-1}I^n_{n-1}\frac{H}{|K|^{\frac{n-1}{2}}(\mathfrak{D}_n^2-1)^{\frac{n-1}{2}}}\\ &= \alpha_n^{2^\sharp}\omega_{n-1}I^n_{n-1}\frac{n-3}{n-1}\frac{1}{|K|^{\frac{n-2}{2}}}\left[-\frac{\alpha_n^{2^*-2^\sharp}}{2n}\varphi_{\frac{n+1}{2}} + \frac{H}{|K|^{\frac{1}{2}}(\mathfrak{D}_n^2-1)^{\frac{n-1}{2}}}\right]\\ &= \underbrace{\alpha_n^{2^\sharp}\omega_{n-1}I^n_{n-1}\frac{n-3}{(n-1)\sqrt{n(n-1)}}}_{:=\mathfrak{a}_n}\frac{1}{|K|^{\frac{n-2}{2}}}\left[-(n-1)\varphi_{\frac{n-1}{2}} + \frac{\mathfrak{D}_n}{(\mathfrak{D}_n^2-1)^{\frac{n-1}{2}}}\right].\end{aligned}$$

Appendix C. Proof of Lemma 4.2

In the following we use the following notation

$$W_j(\xi) := \underbrace{\frac{1}{\delta_j^{\frac{n-2}{2}}}U\left(\frac{\left(\psi_p^\partial\right)^{-1}(\xi)-\eta(\varepsilon)\tau_j}{\delta_j}\right)}_{:=U_j} + \delta_j\underbrace{\frac{1}{\delta_j^{\frac{n-2}{2}}}V_p\left(\frac{\left(\psi_p^\partial\right)^{-1}(\xi)-\eta(\varepsilon)\tau_j}{\delta_j}\right)}_{:=V_j}.$$

and

$$\mathcal{U}_j(\xi) = \chi\left(\left(\psi_p^\partial\right)^{-1}(\xi)\right)U_j(\xi), \qquad \mathcal{V}_j(\xi) = \chi\left(\left(\psi_p^\partial\right)^{-1}(\xi)\right)V_j(\xi).$$

Proof. Let

$$\gamma_M := \mathfrak{i}_M^*\left(K\mathfrak{g}\left(\mathcal{W}\right)\right) \quad \text{and } \gamma_{\partial M} := \mathfrak{i}_{\partial M}^*\left(\frac{n-2}{2}\left(H\mathfrak{f}\left(\mathcal{W}\right)-\varepsilon\mathcal{W}\right)\right).$$

By using the equations that γ_M and $\gamma_{\partial M}$ satisify (see (4.1), (4.2)) we get

$$
\begin{aligned}
\|\mathcal{E}\|^2 &= c_n \int_M \left|\nabla_g \left(\mathcal{W}-\gamma_M-\gamma_{\partial M}\right)\right|^2 d\nu_g + \int_M \mathcal{S}_g \left(\mathcal{W}-\gamma_M-\gamma_{\partial M}\right)^2 d\nu_g \\
&= -c_n \int_M \left[\Delta_g \left(\mathcal{W}-\gamma_M-\gamma_{\partial M}\right)\left(\mathcal{W}-\gamma_M-\gamma_{\partial M}\right)\right] d\nu_g \\
&+ \int_M \mathcal{S}_g \left(\mathcal{W}-\gamma_M-\gamma_{\partial M}\right)^2 d\nu_g \\
&+ c_n \int_{\partial M} \frac{\partial}{\partial \nu}\left(\mathcal{W}-\gamma_M-\gamma_{\partial M}\right)\left(\mathcal{W}-\gamma_M-\gamma_{\partial M}\right) d\sigma_g \\
&= \underbrace{\sum_{j=1}^k \left[\int_M \left(-c_n\Delta_g\mathcal{W}_j+\mathcal{S}_g\mathcal{W}_j-K\mathfrak{g}(\mathcal{W}_j)\right)\mathcal{E}\, d\nu_g\right.}_{} \\
&\qquad \underbrace{\left. +c_n \int_{\partial M}\left(\frac{\partial}{\partial\nu}\mathcal{W}_j-\frac{n-2}{2}Hf(\mathcal{W}_j)+\frac{n-2}{2}\varepsilon\mathcal{W}_j\right)\mathcal{E}\, d\nu_g\right]}_{=:(I)} \\
&+ \underbrace{\int_M K\left(\sum_{j=1}^k \mathfrak{g}(\mathcal{W}_j)-\mathfrak{g}\left(\sum_{j=1}^k\mathcal{W}_j\right)\right)\mathcal{E}\, d\nu_g}_{=:(II)} \\
&+ \underbrace{\frac{n-2}{2}c_n\int_{\partial M} H\left(\sum_{j=1}^k \mathfrak{f}(\mathcal{W}_j)-\mathfrak{f}\left(\sum_{j=1}^k\mathcal{W}_j\right)\right)\mathcal{E}\, d\nu_g}_{=:(III)}
\end{aligned}
$$

- Let us estimate (I), which is the sum of the contribution of each peak. We estimate each term in the sum and for the sake of simplicity, we replace $\mathcal{W}_j$ by $\mathcal{W}$. Each term looks like

$$
\begin{aligned}
&\underbrace{\int_M \left(-c_n\Delta_g\mathcal{U}-K\mathcal{U}^{\frac{n+2}{n-2}}+K\left(\mathfrak{g}(\mathcal{U})-\mathfrak{g}(\mathcal{U}+\delta\mathcal{V})\right)-c_n\delta\Delta_g\mathcal{V}\right)\mathcal{E}}_{(I_1)} \\
&+ \underbrace{\int_{\partial M}\left(c_n\frac{\partial\mathcal{U}}{\partial\nu}-2(n-1)H\mathcal{U}^{\frac{n}{n-2}}\right)\mathcal{E}}_{(I_2)} \\
&+ \underbrace{2(n-1)\int_{\partial M}\left(H\left(\mathfrak{f}(\mathcal{U})-\mathfrak{f}(\mathcal{U}+\delta\mathcal{V})\right)+\frac{2\delta}{n-2}\frac{\partial\mathcal{V}}{\partial\nu}\right)\mathcal{E}}_{(I_3)} \\
&+ \underbrace{\int_M \mathcal{S}_g\mathcal{W}\mathcal{E}}_{(I_4)} \\
&+ \underbrace{2(n-1)\varepsilon\int_{\partial M}\mathcal{W}\mathcal{E}}_{(I_5)}.
\end{aligned}
$$

Estimate of (I_1). We have

$$
|(I_1)| \lesssim \|\mathcal{E}\|_{H^1(M)}\|A_\delta\|_{L^{\frac{2n}{n+2}}(M)}
$$

with

$$A_\delta = -c_n\Delta_g\left(\mathcal{U}+\delta\mathcal{V}\right) - K\,\mathcal{U}^{\frac{n+2}{n-2}} + K\left(\mathfrak{g}(\mathcal{U}) - \mathfrak{g}(\mathcal{U}+\delta\mathcal{V})\right).$$

Now, in local coordinates, the Laplace-Beltrami operator reads as:

$$\Delta_g\,\phi = \Delta\phi + \left(g^{ij}-\delta^{ij}\right)\partial^2_{ij}\phi - g^{ij}\Gamma^k_{ij}\partial_k\phi. \tag{C.1}$$

and so by the decay of U and V_p (see Proposition 3.1) and by Lemma A.2 in variables $x=\delta y$ with $|\delta y|\le R$

$$\begin{aligned}
A_\delta(y) &= -\delta^{-\frac{n+2}{2}}c_n\Delta U(y)\chi(\delta y) - \frac{8(n-1)}{n-2}\delta^{-\frac{n}{2}}h^{ij}(p)\partial^2_{ij}U(y)\chi(\delta y)y_n - \delta^{-\frac{n+2}{2}}K\chi^{\frac{n+2}{n-2}}(\delta y)U^{\frac{n+2}{n-2}}(y)\\
&+\delta^{-\frac{n+2}{2}}K\chi^{\frac{n+2}{n-2}}(\delta y)\left(\mathfrak{g}(U)-\mathfrak{g}(U+\delta V_p)\right) - \delta^{-\frac{n}{2}}c_n\Delta V_p + \delta^{-\frac{n-2}{2}}\Lambda(y)\\
&= \delta^{-\frac{n+2}{2}}K\left(\chi(\delta y)-\chi^{\frac{n+2}{n-2}}(\delta y)\right)U^{\frac{n+2}{n-2}}(y)\\
&+\delta^{-\frac{n+2}{2}}K\chi^{\frac{n+2}{n-2}}(\delta y)\left(\mathfrak{g}(U)-\mathfrak{g}(U+\delta V_p)\right) + \delta^{-\frac{n}{2}}K\chi(\delta y)\mathfrak{g}'(U)V_p(y) + \delta^{-\frac{n-2}{2}}\Lambda(y)
\end{aligned}$$

where

$$|\Lambda(y)| \lesssim \frac{1}{1+|y|^{n-2}} \text{ if } |\delta y|\le R.$$

Finally,

$$\begin{aligned}
\|A_\delta\|_{L^{\frac{2n}{n+2}}(M)} &\lesssim \left\|K\left(\chi(\delta y)-\chi^{\frac{n+2}{n-2}}(\delta y)\right)U^{\frac{n+2}{n-2}}(y)\right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)}\\
&+\delta\left\|K\left(\chi(\delta y)-\chi^{\frac{n+2}{n-2}}(\delta y)\right)U^{\frac{4}{n-2}}(y)V_p(y)\right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)}\\
&+\left\|K\chi^{\frac{n+2}{n-2}}(\delta y)\left(\mathfrak{g}(U+\delta V_p)-\mathfrak{g}(U)-\delta\mathfrak{g}'(U)V_p(y)\right)\right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)}\\
&+\delta^2\|\Lambda\|_{L^{\frac{2n}{n+2}}(B(0,R/\delta))}\\
&\lesssim \begin{cases}\delta^2 \text{ if } n\ge 7\\ \delta^2|\ln\delta|^{\frac{2}{3}} \text{ if } n=6\\ \delta^{\frac{n-2}{2}} \text{ if } n=4,5,\end{cases}
\end{aligned}$$

because by by (A.5)

$$\left|\left(\mathfrak{g}(U+\delta V_p)-\mathfrak{g}(U)-\delta\mathfrak{g}'(U)V_p(y)\right)\right| \lesssim \begin{cases}U^{\frac{6-n}{n-2}}\left(\delta V_p\right)^2 \text{ if } n\ge 6\\ U^{\frac{6-n}{n-2}}\left(\delta V_p\right)^2 + \left(\delta V_p\right)^{\frac{n+2}{n-2}} \text{ if } n=4,5\end{cases}$$

which implies

$$\left\|\left(\mathfrak{g}(U+\delta V_p)-\mathfrak{g}(U)-\delta\mathfrak{g}'(U)V_p(y)\right)\right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} \lesssim \begin{cases}\delta^{\frac{n+2}{n-2}} \text{ if } n\ge 6\\ \delta^2 \text{ if } n=4,5.\end{cases}$$

and also

$$\delta^2\|\Lambda\|_{L^{\frac{2n}{n+2}}(B(0,R/\delta))} \lesssim \begin{cases}\delta^2 \text{ if } n\ge 7\\ \delta^2|\ln\delta|^{\frac{2}{3}} \text{ if } n=6\\ \delta^{\frac{n-2}{2}} \text{ if } n=4,5.\end{cases}$$

Estimate of (I_2). We have

$$(I_2) = 2(n-1)\int_{\partial M}\left(\frac{2}{n-2}\frac{\partial \mathcal{U}}{\partial \nu} - H\mathcal{U}^{\frac{n}{n-2}}\right)\mathcal{E}$$
$$\lesssim \|\mathcal{E}\|_{H^1(M)}\left\|\frac{2}{n-2}\frac{\partial \mathcal{U}}{\partial \nu} - H\mathcal{U}^{\frac{n}{n-2}}\right\|_{L^{\frac{2(n-1)}{n}}(\partial M)}.$$

and

$$\left\|\frac{2}{n-2}\frac{\partial \mathcal{U}}{\partial \nu} - H\mathcal{U}^{\frac{n}{n-2}}\right\|_{L^{\frac{2(n-1)}{n}}(\partial M)}$$
$$= \left(\int_{\partial M}\left(\frac{2}{n-2}\frac{\partial \mathcal{U}}{\partial \nu} - H\mathcal{U}^{\frac{n}{n-2}}\right)^{\frac{2(n-1)}{n}}\right)^{\frac{n}{2(n-1)}}$$
$$\lesssim \left(\int_{\partial \mathbb{R}^n_+}\left(\frac{2}{n-2}\frac{\partial U(\tilde y)}{\partial \nu}\chi(\delta\tilde y) - HU^{\frac{n}{n-2}}(\tilde y)\chi^{\frac{n}{n-2}}(\delta\tilde y)\right)\sqrt{|g(\delta\tilde y)|}d\tilde y\right)^{\frac{n}{2(n-1)}}$$
$$\lesssim \left(\int_{\partial \mathbb{R}^n_+} HU^{\frac{2(n-1)}{n-2}}(\tilde y)\left(\chi(\delta\tilde y) - \chi^{\frac{n}{n-2}}(\delta\tilde y)\right)^{\frac{2(n-1)}{n}} d\tilde y\right)^{\frac{n}{2(n-1)}}$$
$$\lesssim \delta^2$$

Estimate of (I_3). We have

$$|(I_3)| \lesssim \|\mathcal{E}\|_{H^1(M)}\left\|H\left(\mathfrak{f}(\mathcal{U}) - \mathfrak{f}(\mathcal{U}+\delta\mathcal{V})\right) + \frac{2\delta}{n-2}\frac{\partial \mathcal{V}}{\partial \nu}\right\|_{L^{\frac{2(n-1)}{n}}(\partial M)}$$

and by the decay of V_p and (A.5)

$$\left\|H\left(\mathfrak{f}(\mathcal{U}) - \mathfrak{f}(\mathcal{U}+\delta\mathcal{V})\right) + \frac{2\delta}{n-2}\frac{\partial \mathcal{V}_{\delta,p}}{\partial \nu}\right\|_{L^{\frac{2(n-1)}{n}}(\partial M)}$$
$$\lesssim \left\|H\left(\mathfrak{f}(U) - \mathfrak{f}(U+\delta V_p)\right)\chi^{\frac{n}{n-2}}(\delta\tilde y) + \frac{2\delta}{n-2}\frac{\partial V_p}{\partial \nu}\chi(\delta\tilde y)\right\|_{L^{\frac{2(n-1)}{n}}(\partial \mathbb{R}^n_+)}$$
$$\lesssim \delta\left\|H\mathfrak{f}'(U+\delta\theta V_p)V_p\chi^{\frac{n}{n-2}}(\delta\tilde y) - \frac{n}{n-2}HU^{\frac{2}{n-2}}V_p\chi(\delta\tilde y)\right\|_{L^{\frac{2(n-1)}{n}}(\partial \mathbb{R}^n_+)}$$
$$\lesssim \delta\left\|H\left(\chi^{\frac{n}{n-2}}(\delta\tilde y) - \chi(\delta\tilde y)\right)\mathfrak{f}'(U+\delta\theta V_p)V_p\right\|_{L^{\frac{2(n-1)}{n}}(\partial \mathbb{R}^n_+)}$$
$$+ \delta\left\|H\chi(\delta\tilde y)\left(\mathfrak{f}'(U+\delta\theta V_p)) - \mathfrak{f}'(U)\right)V_p\right\|_{L^{\frac{2(n-1)}{n}}(\partial \mathbb{R}^n_+)}$$
$$\lesssim \begin{cases}\delta^2 \text{ if } n\geq 5\\ \delta^2|\ln\delta|^{\frac{2}{3}} \text{ if } n=4.\end{cases}$$

because by (A.5)

$$\left|\left(\mathfrak{f}'(U+\delta\theta V_p)) - \mathfrak{f}'(U)\right)V_p\right| \lesssim \delta U^{\frac{4-n}{n-2}}V_p^2.$$

Estimate of (I_4). By Hölder's inequality

$$|(I_4)| \lesssim \int_M |\mathcal{U}||\mathcal{E}|\ d\nu_g + \delta\int_M |\mathcal{V}||\mathcal{E}|\,d\nu_g \lesssim \|\mathcal{E}\|\left(\|\mathcal{U}\|_{L^{\frac{2n}{n+2}}(M)} + \delta\,\|\mathcal{V}\|_{L^2(M)}\right),$$

with

$$\|\mathcal{U}\|_{L^{\frac{2n}{n+2}}(M)} \lesssim \begin{cases} \delta^2 & \text{if } n \geq 7 \\ \delta^2 |\ln \delta|^{\frac{2}{3}} & \text{if } n = 6 \\ \delta^{\frac{n-2}{2}} & \text{if } n = 4, 5 \end{cases}$$

and

$$\|\mathcal{V}\|_{L^2(M)} \lesssim \begin{cases} \delta^2 & \text{if } n \geq 7 \\ \delta^2 |\ln \delta|^{\frac{1}{2}} & \text{if } n = 6 \\ \delta^{\frac{n-2}{2}} & \text{if } n = 4, 5 \end{cases}$$

Estimate of (I_5). By Hölder's inequality

$$|(I_5)| \lesssim \varepsilon \left(\|\mathcal{U}\|_{L^{\frac{2(n-1)}{n}}(\partial M)} + \delta \|\mathcal{V}\|_{L^2(\partial M)} \right) \|\mathcal{E}\|,$$

with

$$\|\mathcal{U}\|_{L^{\frac{2(n-1)}{n}}(\partial M)} \lesssim \begin{cases} \delta & \text{if } n \geq 5 \\ \delta |\ln \delta|^{\frac{2}{3}} & \text{if } n = 4 \end{cases}$$

and

$$\|\mathcal{V}\|_{L^2(\partial M)} \lesssim \begin{cases} \delta^{\frac{3}{2}} & \text{if } n \geq 6 \\ \delta^{\frac{3}{2}} |\ln \delta|^{\frac{1}{2}} & \text{if } n = 5 \\ \delta & \text{if } n = 4. \end{cases}$$

Finally, collecting all the previous estimates, by the choice of δ_j in (4.6) and (4.7)

$$|(I)| \lesssim \begin{cases} \varepsilon^2 & \text{if } n \geq 7 \\ \varepsilon^2 |\ln \varepsilon|^{\frac{2}{3}} & \text{if } n \geq 6 \\ \varepsilon^{\frac{3}{2}} & \text{if } n = 5 \\ \rho(\varepsilon) & \text{if } n = 4. \end{cases}$$

- Let us estimate the interaction terms (II) and (III).
Set for any $h = 1, \dots, k$ $B_h^+ = B^+(\eta(\varepsilon)\tau_h, \eta(\varepsilon)\sigma/2)$ where $\sigma > 0$ is small enough and $\partial' B_h^+ = B_h^+ \cap \partial \mathbb{R}_+^n$. Since σ is small then $B_h^+ \subset B_R^+$ and $\partial' B_h^+ \subset \partial' B_R^+$ and they are disjoints.
We remark that in $B_{2R}^+(0)$ we get

$$W_i(x) \lesssim \frac{\delta_i^{\frac{n-2}{2}}}{|x - \eta(\varepsilon)\tau_i|^{n-2}}.$$

Then

$$\left| \int_M K \left(\sum_j \mathfrak{g}(\mathcal{W}_j) - \mathfrak{g}\left(\sum_j \mathcal{W}_j \right) \right) \mathcal{E} \right| \lesssim \left\| \mathfrak{g}\left(\sum_j \mathcal{W}_j \right) - \sum_j \mathfrak{g}(\mathcal{W}_j) \right\|_{L^{\frac{2n}{n+2}}(M)} \|\mathcal{E}\|$$

Hence

$$
\begin{aligned}
\|\ldots\ldots\|_{L^{\frac{2n}{n+2}}(M)} &\lesssim \left[\int_{B^+_{2R}\setminus B^+_R} |\ldots\ldots|^{\frac{2n}{n+2}} |g(x)|^{\frac12}\,dx\right]^{\frac{n+2}{2n}}\\
&+ \left[\int_{B^+_R\setminus \bigcup_h B^+_h} |\ldots\ldots|^{\frac{2n}{n+2}} |g(x)|^{\frac12}\,dx\right]^{\frac{n+2}{2n}} + \sum_{h=1}^k \left[\int_{B^+_h} |\ldots\ldots|^{\frac{2n}{n+2}} |g(x)|^{\frac12}\,dx\right]^{\frac{n+2}{2n}}\\
&\le \sum_{i=1}^k \left[\int_{B^+_R} (1-\chi^{2^*}(|x|))|W_i|^{2^*} |g(x)|^{\frac12}\,dx\right]^{\frac{n+2}{2n}} + \left[\int_{B^+_R\setminus \bigcup_h B^+_h} |W_i|^{2^*} |g(x)|^{\frac12}\,dx\right]^{\frac{n+2}{2n}}\\
&+ \sum_{h=1}^k \left[\int_{B^+_h} \left||W_h|^{2^*-2}\sum_{i\neq h} W_i\right|^{\frac{2n}{n+2}} |g(x)|^{\frac12}\,dx\right]^{\frac{n+2}{2n}} + \sum_{h=1}^k \left[\int_{B^+_h} \left|\sum_{i\neq h} W_i\right|^{2^*} |g(x)|^{\frac12}\,dx\right]^{\frac{n+2}{2n}}
\end{aligned}
$$

Let us estimate each term.

$$
\begin{aligned}
&\sum_{i=1}^k \left[\int_{B^+_{2R}\setminus B^+_R} (1-\chi^{2^*}(|x|))|W_i|^{2^*} |g(x)|^{\frac12}\,dx\right]^{\frac{n+2}{2n}} \lesssim \sum_{i=1}^k \left[\int_{\mathbb{R}^n_+\setminus B^+_R} \frac{\delta_i^n}{|x-\eta(\varepsilon)\tau_i|^{2n}}\,dx\right]^{\frac{n+2}{2n}}\\
&\lesssim \sum_{i=1}^k \frac{\delta_i^{\frac{n+2}{2}}}{\eta(\varepsilon)^{\frac{n+2}{2}}} \left[\int_{\mathbb{R}^n_+\setminus B^+_{\frac{R}{\eta(\varepsilon)}}} \frac{1}{|y-\tau_i|^{2n}}\,dy\right]^{\frac{n+2}{2n}} \lesssim \begin{cases} \varepsilon^{(1-\alpha)\frac{n+2}{2}} & \text{if } n\ge 5\\ (\rho(\varepsilon))^3|\ln\rho(\varepsilon)|^{\frac34} & \text{if } n=4.\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\left[\int_{B^+_R\setminus \bigcup_h B^+_h} |W_i|^{2^*} |g(x)|^{\frac12}\,dx\right]^{\frac{n+2}{2n}} \lesssim \left[\int_{B^+_R\setminus \bigcup_h B^+_h} \frac{\delta_i^n}{|x-\eta(\varepsilon)\tau_i|^{2n}}\,dx\right]^{\frac{n+2}{2n}}\\
&\lesssim \sum_{i=1}^k \frac{\delta_i^{\frac{n+2}{2}}}{\eta(\varepsilon)^{\frac{n+2}{2}}} \left[\int_{\mathbb{R}^n_+\setminus B^+_{\frac{R}{\eta(\varepsilon)}}} \frac{1}{|y-\tau_i|^{2n}}\,dy\right]^{\frac{n+2}{2n}} \lesssim \begin{cases} \varepsilon^{(1-\alpha)\frac{n+2}{2}} & \text{if } n\ge 5\\ (\rho(\varepsilon))^3|\ln\rho(\varepsilon)|^{\frac34} & \text{if } n=4.\end{cases}
\end{aligned}
$$

Now for $n\ge 7$

$$
\begin{aligned}
&\sum_{h=1}^k \left[\int_{B^+_h} \left||W_h|^{2^*-2}\sum_{i\neq h} W_i\right|^{\frac{2n}{n+2}} |g(x)|^{\frac12}\,dx\right]^{\frac{n+2}{2n}}\\
&\lesssim \sum_{h=1}^k\sum_{i\neq h} \left[\int_{B^+_h} \frac{\delta_h^{\frac{4n}{n+2}}}{|x-\varepsilon^\alpha\tau_h|^{\frac{8n}{n+2}}} \frac{\delta_i^{\frac{n(n-2)}{n+2}}}{|x-\varepsilon^\alpha\tau_i|^{\frac{2n(n-2)}{n+2}}}\,dx\right]^{\frac{n+2}{2n}}\\
&\lesssim \sum_{h=1}^k\sum_{i\neq h} \delta_h^2\delta_i^{\frac{n-2}{2}} \left(\int_{B^+_h} \frac{1}{|x-\varepsilon^\alpha\tau_h|^{\frac{8n}{n+2}}}\,dx\right)^{\frac{n+2}{2n}}\\
&\lesssim \sum_{h=1}^k\sum_{i\neq h} \delta_h^2\delta_i^{\frac{n-2}{2}} \left(\int_{B^+_{\varepsilon^\alpha\sigma/2}} \frac{1}{|y|^{\frac{8n}{n+2}}}\,dy\right)^{\frac{n+2}{2n}} \lesssim \varepsilon^{\frac{n+2}{2}+\alpha\frac{n-6}{2}}
\end{aligned}
$$

while for $n = 4, 5, 6$

$$
\begin{aligned}
&\sum_{h=1}^{k} \left[\int_{B_h^+} \left| |W_h|^{2^*-2} \sum_{i \neq h} W_i \right|^{\frac{2n}{n+2}} |g(x)|^{\frac{1}{2}}\, dx \right]^{\frac{n+2}{2n}} \\
&\lesssim \sum_{h=1}^{k} \sum_{i \neq h} \frac{\delta_h^2 \delta_i^{\frac{n-2}{2}}}{\eta(\varepsilon)^{\frac{6-n}{2}}} \left(\int_{B^+_{\frac{\sigma}{2}}} \frac{1}{(|\tilde{y}|^2 + (y_n + \mathfrak{D}_n \delta_h \eta(\varepsilon)^{-1})^2 - \delta_h^2 \eta(\varepsilon)^{-2})^{\frac{4n}{n+2}}}\, dx \right)^{\frac{n+2}{2n}} \\
&\lesssim \sum_{h=1}^{k} \sum_{i \neq h} \frac{\delta_h^2 \delta_i^{\frac{n-2}{2}}}{\eta(\varepsilon)^{\frac{6-n}{2}}} \left(\int_{B^+_{\frac{\sigma}{2}}} \frac{1}{(|y|^2 + (\delta_h \eta(\varepsilon)^{-1})^2 (\mathfrak{D}_n^2(p) - 1))^{\frac{4n}{n+2}}}\, dy \right)^{\frac{n+2}{2n}} \\
&\lesssim \begin{cases} \varepsilon^4 |\ln \varepsilon|^{\frac{2}{3}} & \text{if} \quad n = 6 \\ \varepsilon^3 & \text{if} \quad n = 5 \\ (\rho(\varepsilon))^3 |\ln \rho(\varepsilon)| & \text{if} \quad n = 4. \end{cases}
\end{aligned}
$$

At the end

$$
\begin{aligned}
\sum_{h=1}^{k} \left[\int_{B_h^+} \left| \sum_{i \neq h} W_i \right|^{2^*} |g(x)|^{\frac{1}{2}}\, dx \right]^{\frac{n+2}{2n}} &\lesssim \sum_{h=1}^{k} \sum_{i \neq h} \left[\int_{B_h^+} \frac{\delta_i^n}{|x - \eta(\varepsilon)\tau_i|^{2n}}\, dx \right]^{\frac{n+2}{2n}} \\
&\lesssim \begin{cases} \varepsilon^{(1-\alpha)\frac{n+2}{2}} & \text{if } n \geq 5 \\ (\rho(\varepsilon))^3 |\ln \rho(\varepsilon)|^{\frac{3}{4}} & \text{if } n = 4. \end{cases}
\end{aligned}
$$

For the term (III) we get

$$
|(III)| \lesssim \left\| \sum_j \mathfrak{f}(\mathcal{W}_j) - \mathfrak{f}\left(\sum_j \mathcal{W}_j \right) \right\|_{L^{\frac{2(n-1)}{n}}(\partial M)} \|\mathcal{E}\|
$$

Hence

$$
\begin{aligned}
\|\ldots\ldots\|_{L^{\frac{2(n-1)}{n}}(\partial M)} &\lesssim \left[\int_{\partial' B^+_{2R} \setminus \partial' B^+_R} |\ldots\ldots|^{\frac{2(n-1)}{n}} |g(\tilde{x}, 0)|^{\frac{1}{2}}\, dx \right]^{\frac{n}{2(n-1)}} \\
&+ \left[\int_{\partial' B^+_R \setminus \bigcup_h \partial' B^+_h} |\ldots\ldots|^{\frac{2(n-1)}{n}} |g(\tilde{x}, 0)|^{\frac{1}{2}}\, dx \right]^{\frac{n}{2(n-1)}} + \sum_{h=1}^{k} \left[\int_{\partial' B^+_h} |\ldots\ldots|^{\frac{2(n-1)}{n}} |g(\tilde{x}, 0)|^{\frac{1}{2}}\, dx \right]^{\frac{n}{2(n-1)}} \\
&\leq \sum_{i=1}^{k} \left[\int_{\partial' B^+_R} (1 - \chi^{2^\sharp}(\tilde{x}, 0)) |W_i|^{2^\sharp} |g(\tilde{x}, 0)|^{\frac{1}{2}}\, dx \right]^{\frac{n}{2(n-1)}} + \left[\int_{\partial' B^+_R \setminus \bigcup_h \partial' B^+_h} |W_i|^{2^\sharp} |g(\tilde{x}, 0)|^{\frac{1}{2}}\, dx \right]^{\frac{n}{2(n-1)}} \\
&+ \sum_{h=1}^{k} \left[\int_{\partial' B^+_h} \left| |W_h|^{2^\sharp - 2} \sum_{i \neq h} W_i \right|^{\frac{2(n-1)}{n}} |g(\tilde{x}, 0)|^{\frac{1}{2}}\, dx \right]^{\frac{n}{2(n-1)}} + \sum_{h=1}^{k} \left[\int_{\partial' B^+_h} \left| \sum_{i \neq h} W_i \right|^{2^\sharp} |g(\tilde{x}, 0)|^{\frac{1}{2}}\, dx \right]^{\frac{n}{2(n-1)}}
\end{aligned}
$$

Now

$$\sum_{i=1}^{k}\left[\int_{\partial' B^+_{2R}\setminus\partial' B^+_R}|W_i|^{2^\sharp}|g(\tilde x,0)|^{\frac12}\,dx\right]^{\frac{n}{2(n-1)}}\lesssim\sum_{i=1}^{k}\left[\int_{\mathbb{R}^{n-1}\setminus\partial' B^+_R}\frac{\delta_i^{n-1}}{|\tilde x-\eta(\varepsilon)\tilde\tau_i|^{\frac{2(n-1)(n-3)}{n-2}}}\,d\tilde x\right]^{\frac{n}{2(n-1)}}$$

$$\lesssim\sum_{i=1}^{k}\frac{\delta_i^{\frac n2}}{\eta(\varepsilon)^{\frac n2}}\left[\int_{\mathbb{R}^{n-1}\setminus\partial' B^+_{\frac{R}{\eta(\varepsilon)}}}\frac{1}{|\tilde y-\tilde\tau_i|^{2(n-1)}}\,d\tilde x\right]^{\frac{n}{2(n-1)}}$$

$$\lesssim\begin{cases}\varepsilon^{(1-\alpha)\frac n2} & \text{if } n\ge5\\ (\rho(\varepsilon))^2|\ln\rho(\varepsilon)|^{\frac23} & \text{if } n=4\end{cases}$$

Similarly

$$\left[\int_{\partial' B^+_R\setminus\bigcup_h\partial' B^+_h}|W_i|^{2^\sharp}|g(\tilde x,0)|^{\frac12}\,dx\right]^{\frac{n}{2(n-1)}}\lesssim\begin{cases}\varepsilon^{(1-\alpha)\frac n2} & \text{if } n\ge5\\ (\rho(\varepsilon))^2|\ln\rho(\varepsilon)|^{\frac23} & \text{if } n=4\end{cases}$$

Now for $n\ge5$

$$\sum_{h=1}^{k}\left[\int_{\partial' B^+_h}\left||W_h|^{2^\sharp-2}\sum_{i\ne h}W_i\right|^{\frac{2(n-1)}{n}}|g(\tilde x,0)|^{\frac12}\,dx\right]^{\frac{n}{2(n-1)}}$$

$$\lesssim\sum_{h=1}^{k}\sum_{i\ne h}\delta_h\delta_i^{\frac{n-2}{2}}\left[\int_{\partial' B^+_h}\frac{1}{|\tilde x-\eta(\varepsilon)\tilde\tau_h|^{\frac{4(n-1)}{n}}}\,d\tilde x\right]^{\frac{n}{2(n-1)}}$$

$$\lesssim\sum_{h=1}^{k}\sum_{i\ne h}\delta_h\delta_i^{\frac{n-2}{2}}\left[\int_{\partial' B^+_{\eta(\varepsilon)\sigma/2}}\frac{1}{|\tilde y|^{\frac{4(n-1)}{n}}}\,d\tilde x\right]^{\frac{n}{2(n-1)}}\lesssim\varepsilon^{\frac n2+\alpha\frac{n-4}{2}}$$

and for $n=4$

$$\sum_{h=1}^{k}\left[\int_{\partial' B^+_h}\left||W_h|\sum_{i\ne h}W_i\right|^{\frac32}|g(\tilde x,0)|^{\frac12}\,dx\right]^{\frac23}\lesssim\rho(\varepsilon).$$

Finally

$$\sum_{h=1}^{k}\left[\int_{\partial' B^+_h}\left|\sum_{i\ne h}W_i\right|^{2^\sharp}|g(\tilde x,0)|^{\frac12}\,dx\right]^{\frac{n}{2(n-1)}}\lesssim\begin{cases}\varepsilon^{(1-\alpha)\frac n2} & \text{if } n\ge5\\ (\rho(\varepsilon))^2|\ln\rho(\varepsilon)|^{\frac23} & \text{if } n=4\end{cases}.$$

We collect all the above estimates and the claim follows. □

Appendix D. Proof of Proposition 4.4

The proof of (1) is standard. It is also standard to prove that

$$J_\varepsilon(\mathcal{W}+\Phi_\varepsilon)=J_\varepsilon(\mathcal{W})+\mathcal{O}(\|\Phi_\varepsilon\|^2)$$

C^0- uniformly with respect to $(d_1,\dots,d_k,\tau_1,\dots,\tau_k)$ in a compact subsets of $[0,+\infty)^k\times\mathcal{C}$. We only need to estimate the leading term $J_\varepsilon(\mathcal{W})$.

We claim that

$$
\begin{aligned}
J_\varepsilon(\mathcal{W}) &:= \underbrace{\sum_{i=1}^{k} J_\varepsilon(\mathcal{W}_i)}_{(I)} - \underbrace{\sum_{j<i}\int_M K\mathfrak{g}(\mathcal{W}_i)\mathcal{W}_j\, d\nu_g}_{(II)} - \underbrace{\sum_{j<i} c_n\frac{n-2}{2}\int_{\partial M} H\mathfrak{f}(\mathcal{W}_i)\mathcal{W}_j\, d\sigma_g}_{(III)} \\
&+ \sum_{i<j}\int_M \left(c_n\nabla_g\mathcal{W}_i\nabla_g\mathcal{W}_j + \mathcal{S}_g\mathcal{W}_i\mathcal{W}_j - K\mathfrak{g}(\mathcal{W}_i)\mathcal{W}_j\right)\, d\nu_g \\
&- c_n\frac{n-2}{2}\int_{\partial M} H\left(\mathfrak{F}\left(\sum_{i=1}^{k}\mathcal{W}_i\right) - \sum_{i=1}^{k}\mathfrak{F}(\mathcal{W}_i) - \mathfrak{f}(\mathcal{W}_i)\mathcal{W}_j\right) d\sigma_g \\
&- \int_M K\left(\mathfrak{G}\left(\sum_{i=1}^{k}\mathcal{W}_i\right) - \sum_{i=1}^{k}\mathfrak{G}(\mathcal{W}_i) - \mathfrak{g}(\mathcal{W}_i)\mathcal{W}_j\right) d\nu_g + (n-1)\varepsilon\sum_{i\neq j}\int_{\partial M}\mathcal{W}_i\mathcal{W}_j\, d\sigma_g \\
&= k\mathfrak{E} - \sum_{i=1}^{k}\zeta_n(\delta_i)\left[\mathfrak{b}_n\|\pi(p_i)\|^2 + o_n'(1)\right] - \varepsilon\sum_{i=1}^{k}\delta_i(\mathfrak{c}_n + o_n''(1)) \\
&- \sum_{j<i}\mathfrak{d}_n\frac{1}{|K|^{\frac{n-2}{2}}}\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\frac{1}{|\tau_i-\tau_j|^{n-2}} + o\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\right).
\end{aligned}
\tag{D.1}
$$

The contribution of each single bubble is encoded in the first term (I) whose expansion is given in Proposition 3.3. All the other terms come from the interaction among different bubble. First we estimate the leading term $(II)+(III)$.
For any $h=1,\dots,k$ let $B_h^+ := B^+\left(\eta(\varepsilon)\tau_h, \eta(\varepsilon)\frac{\sigma}{2}\right)\subset B_R^+$ provide σ is small enough and moreover B_h^+ are disjoint each other and $\partial' B_h^+ = B_h^+\cap\partial\mathbb{R}^n_+$.

$$
\begin{aligned}
(II) &= \int_{B_i^+} K\mathfrak{g}\left(\mathcal{W}_i(x)\right)\mathcal{W}_j(x)|g(x)|^{\frac12}\,dx + \int_{B_R^+\setminus B_i^+} K\mathfrak{g}\left(\mathcal{W}_i(x)\right)\mathcal{W}_j(x)|g(x)|^{\frac12}\,dx \\
&+ \int_{B_R^+}(1-\chi^{2^*}(|x|))K\mathfrak{g}\left(\mathcal{W}_i(x)\right)\mathcal{W}_j(x)|g(x)|^{\frac12}\,dx
\end{aligned}
$$

Now, the main term in (II) is given by

$$\int_{B_i^+} K\mathfrak{g}\left(\mathcal{W}_i(x)\right)\mathcal{W}_j(x)|g(x)|^{\frac12}\,dx =$$

$$\int_{B_i^+} K\mathfrak{g}\left(\delta_i^{-\frac{n-2}{2}}U\left(\frac{x-\eta(\varepsilon)\tau_i}{\delta_i}\right)+\delta_i\delta_i^{-\frac{n-2}{2}}V_p\left(\frac{x-\eta(\varepsilon)\tau_i}{\delta_i}\right)\right)\times$$

$$\times\left(\delta_j^{-\frac{n-2}{2}}U\left(\frac{x-\eta(\varepsilon)\tau_j}{\delta_j}\right)+\delta_j\delta_j^{-\frac{n-2}{2}}V_p\left(\frac{x-\eta(\varepsilon)\tau_j}{\delta_j}\right)\right)|g(x)|^{\frac12}\,dx$$

$$=\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}\frac{\alpha_n}{|K|^{\frac{n-2}{4}}}\int_{B^+_{\frac{\eta(\varepsilon)\sigma}{2\delta_i}}} K\mathfrak{g}(U+\delta_iV_p)\times$$

$$\times\left(\frac{1}{\left(|\delta_i\tilde y+\eta(\varepsilon)(\tilde\tau_i-\tilde\tau_j)|^2+(\delta_iy_n+\delta_j\mathfrak{D}_n)^2-\delta_j^2\right)^{\frac{n-2}{2}}}\right)|g(\delta_iy+\eta(\varepsilon)\tau_i)|^{\frac12}\,dy$$

$$+\mathcal{O}\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-3}}\int_{B^+_{\frac{\eta(\varepsilon)\sigma}{2\delta_i}}}\mathfrak{g}(U+\delta_iV_p)\,dy\right)$$

$$=\frac{\alpha_n}{|K|^{\frac{n-2}{4}}}\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}|\tau_i-\tau_j|^{n-2}}K\int_{\mathbb{R}^n_+}U^{2^*-1}\,dy+o\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\right)$$

$$=-\frac{\alpha_n^{2^*}}{|K|^{\frac{n-2}{2}}}\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}|\tau_i-\tau_j|^{n-2}}\int_{\mathbb{R}^n_+}\frac{1}{(|\tilde y|^2+(y_n+\mathfrak{D}_n)^2-1)^{\frac{n+2}{2}}}\,dy+o\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\right)$$

$$=\underbrace{-\alpha_n^{2^*}\omega_{n-1}\int_0^{+\infty}\frac{r^{n-2}}{(1+r^2)^{\frac{n+2}{2}}}\,dr}_{:=\mathfrak{d}_n}\,\varphi_{\frac32}\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}|\tau_i-\tau_j|^{n-2}}\frac{1}{|K|^{\frac{n-2}{2}}}+o\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\right)$$

$$=-\mathfrak{d}_n\left(\frac{\mathfrak{D}_n}{(\mathfrak{D}_n^2-1)^{\frac12}}-1\right)\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}|\tau_i-\tau_j|^{n-2}}\frac{1}{|K|^{\frac{n-2}{2}}}+o\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\right).$$

Now

$$\left|\int_{B_R^+\setminus B_i^+}K\mathfrak{g}\left(\mathcal{W}_i(x)\right)\mathcal{W}_j(x)|g(x)|^{\frac12}\,dx\right|$$

$$\lesssim\int_{\mathbb{R}^n_+\setminus B^+(\eta(\varepsilon)\tau_i,\eta(\varepsilon)\frac{\sigma}{2})}\frac{\delta_i^{\frac{n+2}{2}}}{|x-\eta(\varepsilon)\tau_i|^{n+2}}\frac{\delta_j^{\frac{n-2}{2}}}{|x-\eta(\varepsilon)\tau_j|^{n-2}}\,dx=(\text{setting }x=\eta(\varepsilon)y)$$

$$=c\frac{\delta_i^{\frac{n+2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^n}\int_{\mathbb{R}^n_+\setminus B^+(\tau_i,\frac{\sigma}{2})}\frac{1}{|y-\tau_i|^{n+2}}\frac{1}{|y-\tau_j|^{n-2}}\,dx$$

$$=o\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\right)$$

and similarly

$$\left|\int_{B_R^+}(1-\chi^{2^*}(|x|))K\mathfrak{g}\left(\mathcal{W}_i(x)\right)\mathcal{W}_j(x)|g(x)|^{\frac12}\,dx\right|=o\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\right).$$

For the interaction terms (III), we argue as before, obtaining that

$$\begin{aligned}
&2(n-1)\int_{\partial M} H\mathfrak{f}(\mathcal{W}_i)\mathcal{W}_j \\
&= 2(n-1)\alpha_n \frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}|\tau_i-\tau_j|^{n-2}} \frac{H}{|K|^{\frac{n-2}{4}}}\int_{\mathbb{R}^{n-1}} U^{2^\sharp-1}\,d\tilde{x}(1+o(1)) \\
&= \underbrace{\frac{2(n-1)\alpha_n^{2^\sharp}\omega_{n-1}}{\sqrt{n(n-1)}}\int_0^{+\infty}\frac{r^{n-2}}{(1+r^2)^{\frac{n}{2}}}\,dr}_{:=\mathfrak{h}_n} \frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}|\tau_i-\tau_j|^{n-2}} \frac{\mathfrak{D}_n}{|K|^{\frac{n-2}{2}}(\mathfrak{D}_n^2-1)^{\frac{1}{2}}}(1+o(1)).
\end{aligned}$$

Then,

$$(II)+(III)=\sum_{j<i}\mathfrak{d}_n\frac{1}{|K|^{\frac{n-2}{2}}}\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\frac{1}{|\tau_i-\tau_j|^{n-2}}+o\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\right)$$

since a simple computation shows that $\mathfrak{d}_n-\mathfrak{h}_n=0$.
Now we evaluate the remaining terms.
For $i\neq j$

$$\left|\varepsilon\int_{\partial M}\mathcal{W}_i\mathcal{W}_j\,d\nu_g\right|\lesssim\varepsilon\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-3}}\int_{\mathbb{R}^{n-1}}\frac{1}{|\tilde{y}-\tilde{\tau}_i|^{n-2}}\frac{1}{|\tilde{y}-\tilde{\tau}_j|^{n-2}}\,d\tilde{y}=o\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\right)$$

Now

$$\begin{aligned}
&\sum_{i<j}\left[\int_M\nabla_g\mathcal{W}_i\nabla_g\mathcal{W}_j+\mathcal{S}_g\mathcal{W}_i\mathcal{W}_j-K\mathfrak{g}(\mathcal{W}_i)\mathcal{W}_j\right]d\nu_g-2(n-1)\sum_{i<j}\int_{\partial M}H\mathfrak{f}(\mathcal{W}_i)\mathcal{W}_j\,d\sigma_g \\
&=\sum_{i<j}\left[\int_M\left(-c_n\Delta_g\mathcal{W}_i+\mathcal{S}_g\mathcal{W}_i-K\mathfrak{g}(\mathcal{W}_i)\right)\mathcal{W}_j\,d\nu_g\right]+\sum_{i<j}c_n\int_{\partial M}\left(\frac{\partial\mathcal{W}_i}{\partial\nu}-\frac{n-2}{2}H\mathfrak{f}(\mathcal{W}_i)\right)\mathcal{W}_j\,d\sigma_g
\end{aligned}$$

and by using (C.1) we get that

$$|-c_n\Delta_g\mathcal{W}_i+\mathcal{S}_g\mathcal{W}_i-Kg(\mathcal{W}_i)|\lesssim\frac{\delta_i^{\frac{n-2}{2}}}{(|\tilde{x}-\eta(\varepsilon)\tilde{\tau}_i|^2+(x_n-\eta(\varepsilon)\tau_{i,n}+\mathfrak{D}_n\delta_i)^2-\delta_i^2)^{\frac{n-2}{2}}}$$

Hence

$$\begin{aligned}
&\sum_{i<j}\left[\int_M\left(-c_n\Delta_g\mathcal{W}_i+\mathcal{S}_g\mathcal{W}_i-Kg(\mathcal{W}_i)\right)\mathcal{W}_j\,d\nu_g\right]\lesssim\sum_{i<j}\int_{B_R^+}\frac{\delta_i^{\frac{n-2}{2}}}{|x-\eta(\varepsilon)\tau_i|^{n-2}}\frac{\delta_j^{\frac{n-2}{2}}}{|x-\eta(\varepsilon)\tau_j|^{n-2}} \\
&\lesssim\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{2(n-2)}}\eta(\varepsilon)^n\int_{\mathbb{R}^n_+}\frac{1}{|y-\tau_i|^{n-2}}\frac{1}{|y-\tau_j|^{n-2}}=o\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\right)
\end{aligned}$$

and similary

$$\sum_{i<j}c_n\int_{\partial M}\left(\frac{\partial\mathcal{W}_i}{\partial\nu}-\frac{n-2}{2}Hf(\mathcal{W}_i)\right)\mathcal{W}_j\,d\sigma_g=o\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\right)$$

Now

$$\begin{aligned}&\int_M K\left[\mathfrak{G}\left(\sum_{j=1}^k \mathcal{W}_j\right)-\sum_{j=1}^k \mathfrak{G}(\mathcal{W}_j)-\sum_{i\neq j}\mathfrak{g}(\mathcal{W}_i)\mathcal{W}_j\right]\\
&=\sum_{h=1}^k\int_{B_h^+} K\left[\mathfrak{G}\left(\sum_{j=1}^k W_j\right)-\sum_{j=1}^k \mathfrak{G}(W_j)-\sum_{i\neq j}\mathfrak{g}(W_i)W_j\right]|g(x)|^{\frac12}\,dx\\
&+\int_{B_R^+\setminus\bigcup_h B_h^+} K\left[\mathfrak{G}\left(\sum_{j=1}^k W_j\right)-\sum_{j=1}^k \mathfrak{G}(W_j)-\sum_{i\neq j}\mathfrak{g}(W_i)W_j\right]|g(x)|^{\frac12}\,dx\\
&+\int_{B_R^+}(1-\chi^{2^*}(|x|))K\left[\mathfrak{G}\left(\sum_{j=1}^k W_j\right)-\sum_{j=1}^k \mathfrak{G}(W_j)-\sum_{i\neq j}\mathfrak{g}(W_i)W_j\right]|g(x)|^{\frac12}\,dx\end{aligned}$$

It is immediate that

$$\begin{aligned}&\left|\int_{B_R^+}(1-\chi^{2^*}(|x|))K\left[\mathfrak{G}\left(\sum_{j=1}^k W_j\right)-\sum_{j=1}^k \mathfrak{G}(W_j)-\frac{2n}{n-2}\sum_{i\neq j}\mathfrak{g}(W_i)W_j\right]|g(x)|^{\frac12}\,dx\right|\\
&=\mathcal{O}\left(\delta_j^n+\delta_i^2\frac{\delta_j^{\frac{n-2}{2}}\delta_i^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\right).\end{aligned}$$

Now, outside the $k-$ balls

$$\begin{aligned}&\left|\int_{B_R^+\setminus\bigcup_h B_h^+} K\left[\mathfrak{G}\left(\sum_{j=1}^k W_j\right)-\sum_{j=1}^k \mathfrak{G}(W_j)-\sum_{i\neq j}\mathfrak{g}(W_i)W_j\right]|g(x)|^{\frac12}\,dx\right|\\
&\lesssim\sum_{i\neq j}\int_{B_R^+\setminus\bigcup_h B_h^+}\left(|W_i|^{2^*-2}W_j^2+|W_j|^{2^*-2}W_i^2\right)dx=o\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\right)\end{aligned}$$

because if $i\neq j$

$$\begin{aligned}\int_{B_R^+\setminus\bigcup_h B_h^+}|W_i|^{2^*-2}W_j^2&\lesssim\int_{B_R^+\setminus\bigcup_h B_h^+}\frac{\delta_i^2}{|x-\eta(\varepsilon)\tau_i|^4}\frac{\delta_j^{n-2}}{|x-\eta(\varepsilon)\tau_j|^{2(n-2)}}\,dx\\
&\lesssim\frac{\delta_i^2\delta_j^{n-2}}{\eta(\varepsilon)^n}=o\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}}\right).\end{aligned}$$

On each ball B_h^+ we also have

$$\begin{aligned}&\int_{B_h^+}\left|K\left[\mathfrak{G}\left(\sum_{j=1}^k W_j\right)-\sum_{j=1}^k \mathfrak{G}(W_j)-\sum_{i\neq j}\mathfrak{g}(W_i)W_j\right]|g(x)|^{\frac12}\right|dx\\
&\le\int_{B_h^+}\left|\mathfrak{G}\left(W_h+\sum_{i\neq h}W_i\right)-\mathfrak{G}(W_h)-\sum_{j\neq h}\mathfrak{g}(W_h)W_j\right|dx+\sum_{i\neq h}\int_{B_h^+}|\mathfrak{G}(W_i)|\,dx\\
&+\sum_{\substack{i\neq h\\ j\neq i}}\int_{B_h^+}|\mathfrak{g}(W_h)W_j|\,dx\lesssim\sum_{i\neq h}\int_{B_h^+}W_h^{2^*-2}W_i^2+c\sum_{i\neq h}\int_{B_h^+}W_i^{2^*}\,dx\\
&+c\sum_{\substack{i\neq h\\ j\neq i}}\int_{B_h^+}W_i^{2^*-1}W_j\,dx\end{aligned}$$

Now if $i \neq h$ and $n \geq 5$ then

$$\int_{B_h^+} W_h^{2^*-2} W_i^2 \lesssim \int_{B_h^+} \frac{\delta_h^2}{|x-\eta(\varepsilon)\tau_h|^4} \frac{\delta_i^{n-2}}{|x-\eta(\varepsilon)\tau_i|^{2(n-2)}}\, dx$$
$$\lesssim \frac{\delta_h^2 \delta_i^{n-2}}{\eta(\varepsilon)^n} \int_{B^+(\tau_h,\sigma/2)} \frac{1}{|y-\tau_h|^4} \frac{1}{|y-\tau_i|^{2(n-2)}} = o\left(\frac{\delta_i^{\frac{n-2}{2}} \delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}} \right).$$

If, instead, $n = 4$ then

$$\int_{B_h^+} W_h^{2^*-2} W_i^2 \lesssim \frac{\delta_h^{n-2} \delta_i^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}} \int_{B^+_{\frac{\eta(\varepsilon)}{\delta_h}}} \frac{1}{(|x|^2+1)^2}\, dx$$
$$\lesssim \frac{\delta_h^{n-2} \delta_i^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}} \left| \ln \frac{\eta(\varepsilon)}{\delta_h} \right| = o\left(\frac{\delta_i^{\frac{n-2}{2}} \delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}} \right).$$

If $i, j \neq h$ then

$$\int_{B_h^+} W_i^{2^*-1} W_j \leq \int_{B_h^+} \frac{\delta_i^{\frac{n+2}{2}}}{|x-\eta(\varepsilon)\tau_i|^{n+2}} \frac{\delta_j^{\frac{n-2}{2}}}{|x-\eta(\varepsilon)\tau_j|^{n-2}}$$
$$\leq \frac{\delta_i^{\frac{n+2}{2}} \delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^n} = o\left(\frac{\delta_i^{\frac{n-2}{2}} \delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}} \right).$$

If $i \neq h$ then

$$\int_{B_h^+} W_i^{2^*-1} W_h \leq \int_{B_h^+} \frac{\delta_i^{\frac{n+2}{2}}}{|x-\eta(\varepsilon)\tau_i|^{n+2}} \frac{\delta_h^{\frac{n-2}{2}}}{|x-\eta(\varepsilon)\tau_h|^{n-2}}$$
$$\leq \frac{\delta_i^{\frac{n+2}{2}} \delta_h^{\frac{n-2}{2}}}{\eta(\varepsilon)^n} = o\left(\frac{\delta_i^{\frac{n-2}{2}} \delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}} \right)$$

Finally, if $i \neq h$

$$\int_{B_h^+} W_i^{2^*} \leq \int_{B_h^+} \frac{\delta_i^n}{|x-\eta(\varepsilon)\tau_i|^n} \leq \frac{\delta_i^n}{\eta(\varepsilon)^n} = o\left(\frac{\delta_i^{\frac{n-2}{2}} \delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}} \right)$$

In a similar way

$$\int_{\partial M} \left[\mathfrak{F}\left(\sum_i \mathcal{W}_i \right) - \sum_i \mathfrak{F}(\mathcal{W}_i) - \sum_{i \neq j} \mathfrak{f}(\mathcal{W}_i)\mathcal{W}_j \right] d\nu_g$$
$$= \sum_{h=1}^k \int_{\partial' B_h^+} [\ldots\ldots] \, |g(x)|^{\frac{1}{2}}\, dx + \int_{\partial' B_R^+ \setminus \bigcup_h \partial' B_h^+} [\ldots\ldots] \, |g(x)|^{\frac{1}{2}}\, dx$$
$$+ \int_{\partial' B_R^+} (1-\chi^{2^\sharp}) [\ldots\ldots] \, |g(x)|^{\frac{1}{2}}$$
$$= o\left(\frac{\delta_i^{\frac{n-2}{2}} \delta_j^{\frac{n-2}{2}}}{\eta(\varepsilon)^{n-2}} \right).$$

Let us look at the main term of (D.1). Let $\mathfrak{Q}(p)$ be the quadratic form associated with the second derivative of $p \to \|\pi(p)\|^2$ (being zero the first derivative) If $n=4$ by (4.7)

$$\begin{aligned}
&\sum_{i=1}^{k}(\delta_i^2|\ln\delta_i|)\left[\mathfrak{b}_4\|\pi(p_i)\|^2+o_n'(1)\right]-\varepsilon\sum_{i=1}^{k}\delta_i(\mathfrak{c}_n+o_n''(1))\\
&-\sum_{j<i}\mathfrak{d}_n\frac{1}{|K|^{\frac{n-2}{2}}}\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta^{n-2}}\frac{1}{|\tau_i-\tau_j|^{n-2}}+o\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta^{n-2}}\right)\\
&=\sum_{i=1}^{k}\rho^2\left(d_0+\eta d_j\right)^2\left(|\ln\rho|+\mathcal{O}(1)\right)\mathfrak{b}_4\Big[\|\pi(p)\|^2+\frac{1}{2}\eta^2\mathfrak{Q}(\tau_i,\tau_i)+\mathcal{O}(\eta^3)+o_n'(1)\Big]\\
&-\varepsilon\sum_{i=1}^{k}\delta_i(\mathfrak{c}_n+o_n''(1))\\
&-\frac{\rho^2}{\eta^2}\sum_{j<i}\mathfrak{d}_n\frac{1}{|K|}\frac{d_0^2}{|\tau_i-\tau_j|^2}+o\left(\frac{\rho^2}{\eta^2}\right)
\end{aligned}$$

and the claim follows because of the choice of d_0 in (4.3) and the fact that since $\eta=|\ln\rho|^{-\frac{1}{4}}$ (see (3.32))

$$o_n'(1)=\mathcal{O}(|\ln\rho|)=o(\eta^2).$$

If $5\le n\le 7$ by (4.6)

$$\begin{aligned}
&\sum_{i=1}^{k}\delta_i^2\left[\mathfrak{b}_n\|\pi(p_i)\|^2+o_n'(1)\right]-\varepsilon\sum_{i=1}^{k}\delta_i(\mathfrak{c}_n+o_n''(1))\\
&-\sum_{j<i}\mathfrak{d}_n\frac{1}{|K|^{\frac{n-2}{2}}}\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta^{n-2}}\frac{1}{|\tau_i-\tau_j|^{n-2}}+o\left(\frac{\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}}{\eta^{n-2}}\right)\\
&=\sum_{i=1}^{k}\varepsilon^2(d_0+\eta d_j)^2\mathfrak{b}_n\Big[\|\pi(p)\|^2+\frac{1}{2}\eta^2\mathfrak{Q}(p)(\tau_i,\tau_i)+\mathcal{O}(\eta^3)+o_n'(1)\Big]\\
&-\varepsilon\sum_{i=1}^{k}\delta_i(\mathfrak{c}_n+o_n''(1))\\
&-\frac{\varepsilon^{n-2}}{\eta^{n-2}}\sum_{j<i}\mathfrak{d}_n\frac{1}{|K|^{\frac{n-2}{2}}}\frac{d_0^{n-2}}{|\tau_i-\tau_j|^{n-2}}o\left(\frac{\varepsilon^{n-2}}{\eta^{n-2}}\right)
\end{aligned}$$

and the claim follows because of the choice of d_0 in (4.3) and the fact that since $\eta=\varepsilon^{\frac{n-4}{n}}$ (see (3.32))

$$o_n'(1)=\mathcal{O}(\varepsilon)=o(\eta^2)\text{ if }n=6,7\text{ and }o_n'(1)=\mathcal{O}(\varepsilon|\ln\varepsilon|)=o(\eta^2)\text{ if }n=5.$$

We point out that in higher dimensions $n\ge 8$ this is not true anymore.

References

[1] W. Abdelhedi, H. Chtioui and M.O. Ahmedou, *A Morse theoretical approach for the boundary mean curvature problem on B^4*, J. Funct. Anal. **254** (2008), no. 5, 1307–1341.

[2] S.M. Almaraz, *A compactness theorem for scalar-flat metrics on manifolds with boundary.* Calc. Var. Part. Differ. Equ. **41** (2011), no. 3–4, 341–386.

[3] S.M. Almaraz, *An existence theorem of conformal scalar flat metrics on manifolds with boundary*, Pacific J. Math. **248** (2010), no. 1, 1–22.

[4] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer Monographs in Mathematics. Springer, Berlin (1998).

[5] A. Ambrosetti, Y.Y. Li and A. Malchiodi, *On the Yamabe problem and the scalar curvature problems under boundary conditions*, Math. Ann. **322** (2002), no. 4, 667–699.

[6] M. Ben Ayed and M.O. Ahmedou, *Non simple blow ups for the Nirenberg problem on half spheres*, https://arxiv.org/abs/2108.08608

[7] M. Ben Ayed, K. El Mehdi and M.O. Ahmedou, *Prescribing the scalar curvature under minimal boundary conditions on the half sphere*, Adv. Nonlinear Stud. **2** (2002), no. 2, 93–116.

[8] M. Ben Ayed, K. El Mehdi and M.O. Ahmedou, *The scalar curvature problem on the four dimensional half sphere*, Calc. Var. Part. Differ. Equ. **22** (2005), no. 4, 465–482.

[9] S. Brendle and S.Y.S. Chen, *An existence theorem for the Yamabe problem on manifolds with boundary*, J. Eur. Math. Soc. **16** (2014), no. 5, 991–1016.

[10] A. Chang, X. Xu and P. Yang, *A perturbation result for prescribing mean curvature*, Math. Ann. **310** (1998), no. 3, 473–496.

[11] X. Chen, P.T. Ho and L. Sun, *Liming Prescribed scalar curvature plus mean curvature flows in compact manifolds with boundary of negative conformal invariant*, Ann. Global Anal. Geom. **53** (2018), no. 1, 121–150.

[12] X. Chen, Y. Ruan and L. Sun, *The Han-Li conjecture in constant scalar curvature and constant boundary mean curvature problem on compact manifolds*, Adv. Math. **358** (2019), 56 pp.

[13] P. Cherrier, *Problemes de Neumann non lineaires sur les varietes riemanniennes*, J. of Funct. Anal. **57** (1984), 154–206.

[14] M. Chipot, M. Fila and I. Shafrir, *On the Solutions to some Elliptic Equations with Nonlinear Neumann Boundary Conditions.* Adv. in Diff. Eqs, 1, **1** (1996), 91–110.

[15] S. Cruz-Blázquez, A. Malchiodi and D. Ruiz, *Conformal metrics with prescribed scalar and mean curvature*, to appear on J. für die Reine und Ang. Math.

[16] S. Cruz-Blázquez and A. Pistoia, *Non-degeneracy of critical points of the squared norm of the second fundamental form on manifolds with minimal boundary.* J. Geom. Anal. 33 (2023), no. 10, Paper No. 332, 16 pp.

[17] S. Cruz-Blázquez and G. Vaira *Positive Blow-up Solutions for a Linearly Perturbed Boundary Yamabe Problem* arXiv:2301.07396 (to appear on Adv. Diff. Eq.)

[18] Z. Djadli, A. Malchiodi and M.O. Ahmedou, *Prescribing Scalar and Boundary Mean Curvature on the Three Dimensional Half Sphere*, The Journal of Geom. Anal. **13** (2003), no. 2. 255–289.

[19] Z. Djadli, A. Malchiodi and M.O. Ahmedou, *The prescribed boundary mean curvature problem on B^4*, J. Diff. Eqs **206** (2004), no. 2, 373–398.

[20] O. Druet and E. Hebey, *Blow-up examples for second order elliptic PDEs of critical Sobolev growth*, Trans. Amer. Math. Soc. **357** (5) (2005) 1915–1929.

[21] J. Escobar, *Conformal deformation of a Riemannian metric to a scalar at metric with constant mean curvature on the boundary*, Ann. Math. **136** (1992), 1–50.

[22] J. Escobar, *Conformal metrics with prescribed mean curvature on the boundary*, Calc. Var. **4** (1996), 559–592.

[23] J. Escobar, *The Yamabe problem on manifolds with boundary*, J. Differ. Geom., **35** (1992), 21–84.

[24] M. Ghimenti, A.M. Micheletti and A. Pistoia, *Blow-up phenomena for linearly perturbed Yamabe problem on manifolds with umbilic boundary*, J. Differential Equations **267** (2019), no. 1, 587–618.

[25] M. Ghimenti, A.M. Micheletti and A. Pistoia, *Blow-up solutions concentrated along minimal submanifolds for some supercritical elliptic problems on Riemannian manifolds*, J. Fixed Point Theory Appl. 14 (2013), no. 2, 503–525.

[26] M. Ghimenti, A.M. Micheletti and A. Pistoia, *Linear Perturbation of the Yamabe Problem on Manifolds with Boundary*, J. Geom. Anal. **28** (2018) 1315–1340.

[27] Z.C. Han and Y.Y. Li, *The existence of conformal metrics with constant scalar curvature and constant boundary mean curvature*, Comm. Anal. Geom. **8** (2000), 809–869.

[28] Z.C. Han and Y.Y. Li, *The Yamabe problem on manifolds with boundaries: existence and compactness results*, Duke Math. J. **99** (1999) 489–542.

[29] Y.Y. Li, *The Nirenberg problem in a domain with boundary*, Topol. Meth. Nonlinear Anal. **6** (1995), no. 2, 309–329.

[30] Y.Y. Li, *Prescribing scalar curvature on S^n and related problems. II. Existence and compactness,* Comm. Pure Appl. Math. **49** (1996), no. 6, 541–597.

[31] Y.Y. Li and M. Zhu, *Uniqueness theorems through the method of moving spheres*, Duke Math. J. **80** (1995), no. 2, 383–417.

[32] Y.Y. Li and M. Zhu, *Yamabe type equations on three-dimensional Riemannian manifolds*, Commun. Contemp. Math. **1** (1) (1999) 1–50.

[33] F.C. Marques, *Conformal deformations to scalar-flat metrics with constant mean curvature on the boundary*, Comm. Anal. Geom. **15** (2007), no. 2, 381–405.

[34] F.C. Marques, *Existence results for the Yamabe problem on manifolds with boundary*, Indiana Univ. Math. J. (2005), 1599–1620.

[35] M. Mayer and C.B. Ndiaye, *Barycenter technique and the Riemann mapping problem of Cherrier-Escobar*, J. Differential Geom. **107** (2017), no. 3, 519–560.

[36] F. Morabito, A. Pistoia and G. Vaira, *Towering phenomena for the Yamabe equation on symmetric manifolds*, Potential Anal. **47** (1) (2017), 53–102.

[37] A. Pistoia and G. Vaira, *Clustering phenomena for linear perturbation of the Yamabe equation*, Partial Differential Equations Arising from Physics and Geometry, London Mathematical Society Lecture Note Series **450** (2019), 311–331. Cambridge University Press, Cambridge.

[38] B. Premoselli, *Towers of bubbles for Yamabe-type equations and for the Brézis-Nirenberg problem in dimensions* $n \geq 7$, J. Geom. Anal. **32** (2022), no. 3, paper no. 73, 65 pp.

[39] F. Punzo, *On well-posedness of semilinear parabolic and elliptic problems in the hyperbolic space*, J. Differential Equations **251** (2011), no. 7, 1972-1989.

[40] F. Robert and J. Vétois, *Examples of non-isolated blow-up for perturbations of the scalar curvature equation on non-locally conformally flat manifolds*, J. Differential Geom. 98 (2) (2014) 349–356.

[41] F. Robert and J. Vétois, *A general theorem for the construction of blowing-up solutions to some elliptic nonlinear equations via Lyapunov-Schmidt's finite-dimensional reduction.* Concentration analysis and applications to PDE, 85–116, Trends Math., Birkhäuser/Springer, Basel, 2013.

[42] P.D. Thizy and J. Vétois,*Positive clusters for smooth perturbations of a critical elliptic equation in dimensions four and five*, J. Funct. Anal. **275** (2018), no. 1, 170–195.

[43] R.M. Schoen, Notes from graduates lecture in Stanford University (1988). http://www.math.washington.edu/pollack/research/Schoen-1988-notes.html.

[44] X. Xu and H. Zhang, *Conformal metrics on the unit ball with prescribed mean curvature*, Math. Ann. **365** (2016), no. 1–2, 497–557.

Sergio Cruz-Blázquez (Corresponding author), Dipartimento di Matematica, Università degli Studi di Bari, Via E. Orabona, 4, 70125 Bari (Italy)

E-mail address: `sergio.cruz@uniba.it`

Present address: Universidad de Granada, Departamento de Análisis Matemático, Campus Fuentenueva s/n, 18071 Granada (Spain)

E-mail address: sergiocruz@ugr.es

Angela Pistoia, Dipartimento SBAI, Sapienza Università di Roma, Building RM 002, Via Antonio Scarpa 16 00161 Roma (Italy)

E-mail address: `angela.pistoia@uniroma1.it`

Giusi Vaira, Dipartimento di Matematica, Università degli Studi di Bari, Via E. Orabona, 4, 70125 Bari (Italy)

E-mail address: `giusi.vaira@uniba.it`