

L^p Hardy inequality on $C^{1,\gamma}$ domains

PIER DOMENICO LAMBERTI AND YEHUDA PINCHOVER

Abstract. We consider the L^p Hardy inequality involving the distance to the boundary of a domain in the n -dimensional Euclidean space with nonempty compact boundary. We extend the validity of known existence and non-existence results, as well as the appropriate tight decay estimates for the corresponding minimizers, from the case of domains of class C^2 to the case of domains of class $C^{1,\gamma}$ with $\gamma \in (0, 1]$. We consider both bounded and exterior domains. The upper and lower estimates for the minimizers in the case of exterior domains and the corresponding related non-existence result seem to be new even for C^2 -domains.

Mathematics Subject Classification (2010): 49R05 (primary); 35B09, 35J92 (secondary).

1. Introduction

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$ with nonempty boundary, and let $\delta(x) = d(x, \partial\Omega)$ denote the distance of a point $x \in \mathbb{R}^n$ to the boundary of Ω . Fix $p \in]1, \infty[$. We say that the L^p Hardy inequality is satisfied in Ω if there exists $c > 0$ such that

$$\int_{\Omega} |\nabla u|^p \, dx \geq c \int_{\Omega} \frac{|u|^p}{\delta^p} \, dx \quad \text{for all } u \in C_c^\infty(\Omega). \quad (1.1)$$

The L^p Hardy constant of Ω is the best constant for inequality (1.1) which is denoted here by $H_p(\Omega)$. It is a classical result that goes back to Hardy himself (see for example [3, 22]) that if $n = 1$ and Ω is a bounded or unbounded interval, then the L^p Hardy inequality holds and $H_p(\Omega)$ coincides with the widely known constant

$$c_p = \left(\frac{p-1}{p} \right)^p.$$

Both authors acknowledge the financial support received by each other's institution in the frame of the *Visiting Scientist Program* of the University of Padova. This research was also supported by the INDAM - GNAMPA project 2017. The second author acknowledges the support of the Israel Science Foundation (grant No. 970/15).

Received November 2, 2016; accepted in revised form December 5, 2017.

Published online September 2019.

It is also well-known that if Ω is bounded and has a sufficiently regular boundary in $\mathbb{R}^n$, then the L^p Hardy inequality holds and $H_p(\Omega) \leq c_p$ [1, 24]. Moreover, if Ω is convex, and more generally if it is weakly mean convex, *i.e.*, if $\Delta d \leq 0$ in the distributional sense in Ω , then $H_p(\Omega) = c_p$ [6, 11, 24]. On the other hand, it is also well-known (see for example [3, 22]) that if $\Omega = \mathbb{R}^n \setminus \{0\}$ and $p \neq n$, then the L^p -Hardy inequality holds and $H_p(\Omega)$ coincides with the other widely known constant

$$c_{p,n}^* = \left| \frac{p-n}{p} \right|^p,$$

which indicates that the L^p Hardy inequality does not hold for $\mathbb{R}^n \setminus \{0\}$ if $p = n$ (for a short proof of this inequality see [14, 15]). It also follows (see [10, 24]) that if Ω is an exterior domain (*i.e.*, an unbounded domain with nonempty compact boundary) with sufficiently regular boundary and $p \neq n$, then the L^p Hardy inequality holds with $H_p(\Omega) \leq c_{p,n}$, where

$$c_{p,n} = \min\{c_p, c_{p,n}^*\}.$$

The L^p Hardy constant can be seen as the infimum of a Rayleigh quotient, namely

$$H_p(\Omega) = \inf_{\substack{u \in \tilde{W}^{1,p}(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} |\nabla u|^p \, dx}{\int_{\Omega} \frac{|u|^p}{\delta^p} \, dx}, \quad (1.2)$$

where

$$\tilde{W}^{1,p}(\Omega) := \left\{ u \in W_{\text{loc}}^{1,p}(\Omega) \mid \|u\|_{L^p(\Omega; \delta^{-p})} + \|\nabla u\|_{L^p(\Omega)} < \infty \right\},$$

and $\|u\|_{L^p(\Omega; \delta^{-p})} := (\int_{\Omega} |u|^p \delta^{-p} \, dx)^{1/p}$ is the natural weighted L^p norm associated with this problem. Note that if Ω is a bounded domain with regular boundary, say of class C^1 , then $\tilde{W}^{1,p}(\Omega) = W_0^{1,p}(\Omega)$ (one can use the same argument as in [24, Appendix B]), while the two spaces do not coincide if, for example, Ω is an exterior domain and $p > n$, in which case the first space contains functions that are constant or even unbounded at infinity.

It is important to note that if the infimum for (1.2) is achieved at a function u , then u satisfies the corresponding Euler-Lagrange equation

$$-\Delta_p u - \frac{H_p(\Omega)}{\delta^p} \mathcal{I}_p u = 0, \quad (1.3)$$

in Ω , where $-\Delta_p v := -\operatorname{div}(|\nabla v|^{p-2} \nabla v)$ is the celebrated p -Laplace operator, and the operator $\mathcal{I}_p$ is defined by $\mathcal{I}_p v := |v|^{p-2} v$. In this case, $H_p(\Omega)$ can be considered as the principal weighted eigenvalue of the p -Laplacian with respect to the Hardy weight, and u is a corresponding principal eigenfunction. In particular, it turns out that if the minimizer u exists, then it is unique up to scalar multiples, and u does not change its sign in Ω .

We refer to [24] for an introduction to this topic and to [1,3,6,9–13,22,23,25] and references therein for more information. We refer also to [2,4,5,7,8,14,16] for recent developments in this subject.

The focus of the present paper is on the problem of the existence of minimizers for (1.2). In the case of bounded domains of class C^2 this problem was solved in [24,25] where, among other results, it was proved that a minimizer exists for (1.2) if and only if $H_p(\Omega) < c_p$. In the case of exterior C^2 -domains, it was proved in [10] that if $H_p(\Omega) < c_{p,n}$, then a minimizer exists for (1.2). Importantly, in [10,24,25] the assumption that Ω is of class C^2 is used in a substantial way, and weakening this assumption seems highly nontrivial. Indeed, many arguments used in such papers are based on the well-known *tubular neighbourhood theorem* which allows to use tubular coordinates near the boundary of a domain of class C^2 . Moreover, in [24,25] the assumption that Ω is of class C^2 is used also to guarantee that the distance function δ is of class C^2 in a neighbourhood of the boundary, which in turn allows to use δ for the construction of suitable positive subsolutions and supersolutions of equation (1.3). However, the tubular neighbourhood theorem does not hold if Ω is of class $C^{1,\gamma}$ with $0 < \gamma < 1$ and the distance function δ is not guaranteed to be differentiable near the boundary (the classical example is given by the parabolic open set $\Omega = \{(x, y) \in \mathbb{R}^2 : y > |x|^{1+\gamma}\}$, in which case δ is not differentiable at all points $(0, y)$ of Ω close to $(0, 0)$).

In the present paper, we prove that the existence and non-existence results in [10,24,25] hold under the assumption that Ω is of class $C^{1,\gamma}$ with $\gamma \in (0, 1]$, and we prove that the decay estimates for the minimizers in [24,25] still hold. Moreover, we provide decay and growth estimates for the minimizers also in the case of exterior domains near the boundary and infinity. Our approach develops some ideas used in [24] for the case $p = 2$. In particular, we use the notion of *spectral gap* and *Agmon ground state*, and elaborate the constructions of appropriate subsolutions and supersolutions which replace those considered in [24,25]. To do so, we first compute the so-called *Hardy constant at infinity*, i.e., the constant

$$\lambda_{p,\infty}(\Omega) = \sup \left\{ \lambda \in \mathbb{R} \mid \exists K \Subset \Omega \text{ and } u \in W_{\text{loc}}^{1,p}(\Omega \setminus \bar{K}) \text{ such that} \right. \\ \left. u > 0 \text{ and } -\Delta_p u - \frac{\lambda}{\delta^p} \mathcal{I}_p u \geq 0 \text{ in } \Omega \setminus \bar{K} \right\}, \quad (1.4)$$

where we write $A \Subset O$ if O is open, $\bar{A}$ is compact and $\bar{A} \subset O$.

We prove that if Ω is a $C^{1,\gamma}$ -domain with compact boundary, then $\lambda_{p,\infty}(\Omega) = c_p$ if Ω is bounded, and $\lambda_{p,\infty}(\Omega) = c_{p,n}$ if Ω is unbounded.

By a criterion in [28, Lemma 4.6] (proved there only for the linear case), it follows that if $H_p(\Omega) < \lambda_{p,\infty}(\Omega)$, then the operator $-\Delta_p - \frac{H_p(\Omega)}{\delta^p} \mathcal{I}_p$ is critical in Ω , which means that it admits an Agmon ground state, i.e., a positive solution of equation (1.3) in Ω which has minimal growth near $\partial\Omega$ and infinity. We note that the quantity $\lambda_{p,\infty}(\Omega) - H_p(\Omega)$ is also referred to as the spectral gap, since in the linear case ($p = 2$), $\lambda_{2,\infty}(\Omega)$ is the bottom of the essential spectrum of the operator

$-\delta^2\Delta$, see [24, Section 3]. Thus, the condition $\lambda_{2,\infty}(\Omega) - H_2(\Omega) > 0$ implies that $H_2(\Omega)$ belongs to the discrete spectrum, and hence, it is an eigenvalue whose eigenfunction is the required minimizer.

The last step in the proof of the existence result, consists in proving that in the case of a spectral gap, the above mentioned Agmon ground state u belongs to the space $\tilde{W}^{1,p}(\Omega)$ and this is done by constructing a supersolution v to equation (1.3) which belongs to $L^p(\Omega; \delta^{-p})$: indeed, u being of minimal growth, it follows that $0 < u \leq Cv$ near the boundary and infinity, for some constant $C > 0$, hence $u \in L^p(\Omega; \delta^{-p})$.

In a similar way, the non-existence of minimizers follows by a comparison principle proved in [24,25] combined with the construction of a suitable subsolution which does not belong to $L^p(\Omega; \delta^{-p})$.

It is clear that one of the major ingredients of our arguments is the construction of subsolutions and supersolutions with the appropriate growth and this is used not only to provide the required estimates for the minimizers, but also for computing $\lambda_{p,\infty}(\Omega)$. In [24,25] the construction of subsolutions and supersolutions was done by using the so-called *Agmon trick*, namely, the subsolutions and supersolutions were given by functions of the type $\delta^\alpha + \delta^\beta$ and $\delta^\alpha - \delta^\beta$, respectively, for suitable constants $\alpha, \beta > 0$. As we have mentioned above, if Ω is of class $C^{1,\gamma}$ with $0 < \gamma < 1$ such functions cannot be used. In this paper, we replace them by functions of the form $G^\alpha + G^\beta$ and $G^\alpha - G^\beta$, where G is a p -harmonic function defined in a neighbourhood of the boundary and infinity. At infinity, the function G is simply given by $G(x) = |x|^\beta$ for an appropriate β . Near the boundary of Ω , the function G can be any positive p -harmonic function vanishing at $\partial\Omega$ (for example, one may consider the positive minimal Green function of the p -Laplacian, see Section 2 for details). It is exactly at this point that the regularity of Ω plays a crucial role. First, the assumption $\partial\Omega \in C^{1,\gamma}$ guarantees that the p -harmonic function G is of class $C^{1,\tilde{\gamma}}$ up to $\partial\Omega$ for some $\tilde{\gamma} \in (0, \gamma)$. Second, the same assumption allows to use the Hopf lemma and to conclude that $\nabla G(x) \neq 0$ for all $x \in \partial\Omega$. The condition $\nabla G(x) \neq 0$ is of fundamental importance for our analysis, since it allows to control the asymptotic behaviour of $\nabla G(x)/G(x)$ as $x \rightarrow \partial\Omega$ in a precise way as it is explained in Lemma 3.2. We believe that Lemma 3.2 is of independent interest since it is proved without the use of the tubular neighbourhood theorem and actually allows to bypass it. We note that the Hopf lemma holds also if $\partial\Omega$ is of class $C^{1,\text{Dini}}$ (see, [26]) and this allows to gain some generality as it is explained in Remark 4.2.

Finally, we point out that our results could be of help in relaxing the boundary regularity assumptions of those statements in [4,5] the proofs of which require the existence of a minimizer for the variational problem (1.2).

The paper is organized as follows. In Section 2 we recall a number of notions concerning critical and subcritical operators and in particular, we reformulate and generalize the criterion [28, Lemma 4.6] in Lemma 2.3. Section 3 is devoted to the construction of subsolutions and supersolutions, and in particular it contains the technical Lemma 3.2 which is applied to prove the key Lemmas 3.4 and 3.5. In Section 4, we prove the existence of minimizers and the corresponding decay and

growth estimates, see Theorems 4.1 and 4.4, and also Theorem 4.3 for a further relaxation of the boundary conditions. We conclude the paper in Section 5, where we prove the lower estimates and the corresponding non-existence results, namely Theorems 5.1 and 5.4.

ACKNOWLEDGEMENTS. The authors are grateful to the anonymous referee for reading the paper very carefully and pointing out one blunder of technical type which was possible to fix.

2. Preliminaries

In this section we recall the notions of *positive minimal Green function*, *Agmon ground state*, and *subcritical and critical operators*. Moreover, we discuss a criterion for ensuring the existence of Agmon ground states for equation (1.3). We refer to [15, 16, 28–30] and references therein for details and proofs, and for extensive discussions on this subject.

Fix $p \in]1, \infty[$, and let Ω be a domain (*i.e.*, an open connected set) in $\mathbb{R}^n$, where $n \geq 2$. Let $V \in L_{\text{loc}}^\infty(\Omega)$ (in fact, this assumption is not optimal, and we may assume that V belongs to an appropriate local Morrey space, see [29]). Consider the operator

$$Q_V(u) := -\Delta_p u + V|u|^{p-2}u,$$

and the corresponding form $\mathcal{Q}$ defined by

$$\mathcal{Q}_V(u, \varphi) := \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx + \int_{\Omega} V|u|^{p-2}u\varphi \, dx,$$

for all $u \in W_{\text{loc}}^{1,p}(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$. As customary in the theory of quasilinear equations, we say that u is a (*weak*) *solution* for the equation $Q_V(v) = 0$ in Ω (or simply that $Q_V(u) = 0$ in Ω) if $u \in W_{\text{loc}}^{1,p}(\Omega)$ and $\mathcal{Q}_V(u, \varphi) = 0$ for all $\varphi \in C_c^\infty(\Omega)$. We also say that u is a *subsolution* (respectively *supersolution*) for the equation $Q_V(v) = 0$ in Ω if $\mathcal{Q}_V(u, \varphi) \leq 0$ (respectively $\mathcal{Q}_V(u, \varphi) \geq 0$) for all $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$; in these cases, we also simply write $Q_V(u) \leq 0$ (respectively $Q_V(u) \geq 0$) in Ω . Recall that by Allegretto-Piepenbrink theory [29, 30], there exists a positive solution (or equivalently, a positive supersolution) for the equation $Q_V(u) = 0$ in Ω if and only if $\mathcal{Q}_V(\varphi, \varphi) \geq 0$ for all $\varphi \in C_c^\infty(\Omega)$, in which case the operator Q_V is called non-negative in Ω (and we write $Q_V \geq 0$ in Ω). Obviously, the operator Q_V is non-negative if $V \geq 0$.

We recall briefly some basic regularity results concerning solutions of the quasilinear equations appearing in the present paper. It is well known that if $V \in L_{\text{loc}}^\infty(\Omega)$, then solutions of the equation

$$(-\Delta_p + V\mathcal{I}_p)u = 0$$

in Ω are in $C^{1,\alpha}(\Omega)$, and positive solutions satisfies the local Harnack inequality (see for example [20,21,27]). Moreover, if the potential V is bounded up to a $C^{1,\gamma}$ portion of $\partial\Omega$, then solutions of the above equation are $C^{1,\alpha}$ up to this portion. Furthermore, the Hopf lemma holds for p -harmonic functions in $C^{1,\gamma}$ domains and even under weaker assumptions (see Remark 4.2).

Definition 2.1. Given a compact set K contained in Ω , we say that a positive solution u to the equation $Q_V(u) = 0$ in $\Omega \setminus K$ is a *positive solution of minimal growth in a neighbourhood of infinity in Ω* (briefly $u \in \mathcal{M}_{\Omega,K}^{Q_V}$) if for any bounded smooth open set $\mathcal{K}$ with $K \subset \mathcal{K} \Subset \Omega$ and any positive supersolution v to the equation $Q_V(u) = 0$ in $\Omega \setminus \mathcal{K}$ we have that the condition $u \leq v$ on $\partial\mathcal{K}$ implies that $u \leq v$ on $\Omega \setminus \mathcal{K}$.

We have the following theorem which includes the definitions of the notions mentioned above.

Theorem 2.2 ([19,29,30]). Assume that the operator Q_V is non-negative in Ω and fix $x_0 \in \Omega$. Then there exists a solution $u \in \mathcal{M}_{\Omega,\{x_0\}}^{Q_V}$ of $Q_V(u) = 0$, and it is unique up to a multiplicative constant. Moreover, the following alternative holds:

(A) either u has a singularity at the point x_0 with the following asymptotic behaviour

$$u(x) \sim \begin{cases} |x - x_0|^{\frac{p-n}{p-1}} & \text{if } 1 < p < n \\ -\log |x - x_0| & \text{if } p = n \\ 1 & \text{if } p > n, \end{cases} \quad (2.1)$$

as $x \rightarrow x_0$, in which case u is called a positive minimal Green function with pole at x_0 for Q_V in Ω , and the operator Q_V is called subcritical in Ω ;

(B) or u is a global positive solution of the equation $Q_V(u) = 0$ in Ω , in which case u is called Agmon ground state for Q_V and the operator Q_V is called critical in Ω .

Let Ω' be a subdomain of a domain Ω such that $\overline{\Omega'} \subset \Omega$. If Q_V is non-negative in Ω , then Q_V is subcritical in Ω' [30]. Therefore, if Ω is a domain with nonempty compact boundary and $\partial\Omega$ is sufficiently regular, then the p -Laplacian (that is, Q_V with $V = 0$) is subcritical in Ω , and hence, the corresponding function $u \in \mathcal{M}_{\Omega,\{x_0\}}^{Q_V}$ is a positive minimal Green function. Such a minimal Green function G provides us with a positive p -harmonic function defined in a relative neighbourhood of $\partial\Omega$ which will be used in the sequel. Importantly, if $\partial\Omega$ is of class $C^{1,\gamma}$ with $0 < \gamma < 1$, then G is $C^{1,\alpha}$ up to the boundary, $G(x) = 0$ and $\nabla G(x) \neq 0$ for all $x \in \partial\Omega$ since the Hopf lemma holds, see [26].

In the case of linear elliptic equations, it was stated and proved in [28, Lemma 4.6] that the existence of a spectral gap implies the existence of an Agmon ground state. The statement and the proof in [28] can be adapted to our case and for the convenience of the reader we indicate here how to do it.

Lemma 2.3. *Let Ω be a domain in $\mathbb{R}^n$ such that the L^p Hardy inequality holds, and let $V := -H_p(\Omega)/\delta^p$. If the operator Q_V has a spectral gap, i.e., $H_p(\Omega) < \lambda_{p,\infty}(\Omega)$, then Q_V is critical.*

Proof. Following [28, Lemma 4.6], we set

$$S := \{t \in \mathbb{R} \mid Q_{-t\delta^{-p}} \geq 0 \text{ in } \Omega\}, \quad S_\infty := \{t \in \mathbb{R} \mid Q_{-t\delta^{-p}} \geq 0 \text{ in } \Omega \setminus \bar{K} \text{ for some } K \Subset \Omega\}.$$

Clearly, S and S_∞ are intervals, and since Q_V has a spectral gap, it follows that

$$S =]-\infty, H_p(\Omega)] \subsetneq S_\infty \subseteq]-\infty, \lambda_{\infty,p}(\Omega)].$$

For simplicity, we set $\lambda_0 = H_p(\Omega)$. Let $\lambda_1 \in S_\infty \setminus S$.

Claim. There exists a nonzero non-negative potential $\mathcal{V} \in L^\infty(\Omega)$ with compact support in Ω such that $Q_{-\lambda_1\delta^{-p}+\mathcal{V}} \geq 0$ in Ω .

Since $\lambda_0 < \lambda_1 < \lambda_{\infty,p}$, there exists a smooth open set $K_0 \Subset \Omega$ such that the equation $Q_{-\lambda_1\delta^{-p}}(u) = 0$ in $\Omega \setminus \bar{K}_0$ admits a positive solution.

Fix a smooth open set K satisfying $K_0 \Subset K \Subset \Omega$. We first show that there exists a positive solution v of the equation $Q_{-\lambda_1\delta^{-p}}(u) = 0$ in $\Omega \setminus \bar{K}$ satisfying $v = 0$ on ∂K .

To this end, consider a smooth exhaustion $\{\Omega_i\}_{i \in \mathbb{N}}$ of Ω by smooth relatively compact subdomains such that $x_0 \in \Omega_1 \setminus \bar{K}$ and such that $\bar{K} \subset \Omega_{i-1} \Subset \Omega_i$ for all $i > 1$. Let v_i be the unique positive solution of the Dirichlet problem

$$\begin{cases} Q_{-\lambda_1\delta^{-p}}(u) = f_i & \text{in } \Omega_i \setminus \bar{K} \\ u = 0 & \text{on } \partial(\Omega_i \setminus \bar{K}), \end{cases}$$

where f_i is a nonzero nonnegative function in $C_c^\infty(\Omega_i \setminus \overline{\Omega_{i-1}})$ normalized in such a way that $v_i(x_0) = 1$. The existence and uniqueness of such a solution is guaranteed by [29, Theorem 3.10] combined with the fact that $Q_{-\lambda_1\delta^{-p}}(u) = 0$ admits a positive solution in $\Omega \setminus \bar{K}_0$.

By the Harnack principle and elliptic regularity (see for example, [29]) the sequence $\{v_i\}_{i \in \mathbb{N}}$ admits a subsequence converging locally uniformly to a positive solution v of the equation $Q_{-\lambda_1\delta^{-p}}(u) = 0$ in $\Omega \setminus \bar{K}$ satisfying $v = 0$ on ∂K . Note that by classical regularity theory we have that v is of class $C^{1,\alpha}$ up to ∂K .

Let K_1 be an open set such that $K \Subset K_1 \Subset \Omega$ and let $\min_{x \in \partial K_1} v(x) = m > 0$. Let $\varepsilon > 0$ be fixed in such a way that $8\varepsilon < m$. Let F be a C^2 function from $[0, +\infty[$ to $[0, +\infty[$ such that $F(t) = \varepsilon$ for all $0 \leq t \leq 2\varepsilon$ and $F(t) = t$ for all $t \geq 4\varepsilon$ and such that $F'(t) \neq 0$ for all $t > 2\varepsilon$. Assume also that $|F'(t)|^{p-2}F''(t) \rightarrow 0$ as $t \rightarrow 2\varepsilon$, hence the function $t \rightarrow |F'(t)|^{p-2}F''(t)$ (defined identically equal to zero on $[0, 2\varepsilon]$) is continuous on $[0, +\infty[$ (for this purpose, it is enough for example that F is chosen to be of the type $\varepsilon + (t - 2\varepsilon)^\beta$ for all $t > 2\varepsilon$ sufficiently close to 2ε and $\beta > \max\{p/(p-1), 2\}$). We set $\bar{v}(x) = F(v(x))$ for all $x \in K_1 \cap (\Omega \setminus \bar{K})$. By definition, it follows that there exists an open neighborhood U of ∂K such that

$\bar{v}(x) = \varepsilon$ for all $x \in U \cap (\Omega \setminus \bar{K})$, and there exists an open neighborhood U_1 of ∂K_1 such that $\bar{v}(x) = v(x)$ for all $x \in U_1 \cap K_1$. Thus $\bar{v}(x)$ can be extended continuously into the whole of Ω by setting $\bar{v}(x) = \varepsilon$ for all $x \in \bar{K}$ and $\bar{v}(x) = v(x)$ for all $x \in \Omega \setminus K_1$. By [16, Lemma 2.10] we have that

$$-\Delta_p \bar{v}(x) = -|F'(v(x))|^{p-2}[(p-1)F''(v(x))|\nabla v(x)|^p + F'(v(x))\Delta_p v(x)] \quad (2.2)$$

for all $x \in K_1 \setminus \bar{K}$. By our assumptions on F and v , it follows that $-\Delta_p \bar{v}(x)$ is a continuous function which vanishes on $U \cap (\Omega \setminus \bar{K})$ and equals $\lambda_1 \delta^{-p} v$ in $U_1 \cap K_1$. In particular, it makes sense to compute $\Delta_p \bar{v}$ in Ω , and it turns out that $\Delta_p \bar{v} = 0$ in K and $-\Delta_p \bar{v} = \lambda_1 \delta^{-p} v$ in $\Omega \setminus \bar{K}_1$. We can now define the potential $\mathcal{V}$ by setting

$$\mathcal{V} = \frac{|Q_{-\lambda_1 \delta^{-p}}(\bar{v})|}{\bar{v}^{p-1}}.$$

By construction, $\mathcal{V}$ is a bounded function with compact support in Ω and $\bar{v}$ is a positive supersolution of the equation $Q_{-\lambda_1 \delta^{-p} + \mathcal{V}}(u) = 0$ in Ω . Hence, $Q_{-\lambda_1 \delta^{-p} + \mathcal{V}} \geq 0$ in Ω , and the Claim is proved.

We set $\lambda_t = t\lambda_1 + (1-t)\lambda_0$. By using [16, Lemma 4.3] (see also [30, Proposition 4.3]), it follows that the set

$$\{(t, s) \in [0, 1] \times \mathbb{R} : Q_{-\lambda_t \delta^{-p} + s\mathcal{V}} \geq 0 \text{ in } \Omega\}$$

is a convex set. Hence, the function $\nu : [0, 1] \rightarrow \mathbb{R}$ defined by

$$\nu(t) := \min\{s \in \mathbb{R} : Q_{-\lambda_t \delta^{-p} + s\mathcal{V}} \geq 0 \text{ in } \Omega\}$$

is convex. Since $\mathcal{V}$ has compact support it follows by [29, Proposition 4.19] that $Q_{-\lambda_t \delta^{-p} + \nu(t)\mathcal{V}}$ is critical for all $t \in [0, 1]$. We note that by definition $\nu(t) > 0$ for all $t \in]0, 1]$, while $\nu(0) \leq 0$. Since ν is convex, we must have $\nu(0) = 0$, and hence $Q_{-\lambda_0 \delta^{-p}}$ is critical. $\square$

3. Construction of subsolutions and supersolutions

The proofs of our main theorems are based on the construction of suitable subsolutions and supersolutions to equations of the form $-\Delta_p u - \lambda \delta^{-p} |u|^{p-2} u = 0$, which is carried out in this section. To do so, we need a number of preliminary results.

Recall our notation $\mathcal{I}_p u = |u|^{p-2} u$. By $\mathbb{R}^n \cup \{\infty\}$ we denote the standard one-point compactification of $\mathbb{R}^n$ (note that in this paper the symbol ∞ will not be used with reference to the one point compactification of a bounded domain Ω , as often is done in the related literature). Finally, for $\alpha \in [0, 1]$ we set

$$\lambda_\alpha := (p-1)\alpha^{p-1}(1-\alpha). \quad (3.1)$$

Observe that $\lambda_\alpha = c_p$ if $\alpha = (p-1)/p$, the function λ_α is increasing with respect to $\alpha \in [0, (p-1)/p]$ and decreasing for $\alpha \in [(p-1)/p, 1]$.

The first part of the following lemma is taken from [16, Proposition 4.5].

Lemma 3.1. *Let U be an open set in $\mathbb{R}^n$. Let G be a positive function defined on U such that $-\Delta_p G = 0$ in U . Let $W := |\nabla G/G|^p$. Then for every $\alpha \in (0, 1)$ we have*

$$(-\Delta_p - \lambda_\alpha W \mathcal{I}_p) G^\alpha = 0, \quad \text{in } U. \quad (3.2)$$

Moreover, if $x_0 \in \overline{U}$, where the closure $\overline{U}$ of U is taken in $\mathbb{R}^n \cup \{\infty\}$, and

$$\lim_{x \rightarrow x_0} \frac{|\nabla G(x)|}{G(x)} d(x) = c \quad (3.3)$$

for some $c > 0$, where d is a positive function defined in a relative punctured neighborhood of x_0 . Then for every $\varepsilon > 0$ there exists an open neighbourhood U_ε of x_0 such that

$$\left(-\Delta_p - \frac{c^p \lambda_\alpha - \varepsilon}{d^p} \mathcal{I}_p\right) G^\alpha \geq 0 \quad \text{in } (U_\varepsilon \cap U) \setminus \{x_0\}. \quad (3.4)$$

Proof. For the proof of (3.2) we refer to [16, Proposition 4.5]. In order to prove (3.4) we note that

$$\begin{aligned} \left(-\Delta_p - \frac{c^p \lambda_\alpha - \varepsilon}{d^p} \mathcal{I}_p\right) G^\alpha &= \left(-\Delta_p - \lambda_\alpha W \mathcal{I}_p\right) G^\alpha \\ &\quad + c^p \lambda_\alpha \left(\frac{W}{c^p} - \frac{1}{d^p}\right) \mathcal{I}_p G^\alpha + \varepsilon \frac{\mathcal{I}_p G^\alpha}{d^p} \\ &= (\lambda_\alpha (W d^p - c^p) + \varepsilon) \frac{\mathcal{I}_p G^\alpha}{d^p}. \end{aligned} \quad (3.5)$$

By (3.3), it follows that there exists an open neighbourhood U_ε of x_0 such that $\lambda_\alpha (W(x) d(x)^p - c^p) \geq -\varepsilon$ for all $x \in (U_\varepsilon \cap U) \setminus \{x_0\}$ which combined with (3.5) yields (3.4). $\square$

The proof of the following lemma would be straightforward for open sets Ω of class C^2 , in which case the tubular neighbourhood theorem holds and no boundary point can be approached by points from the cut locus of Ω . However, assuming that Ω is of class $C^{1,\gamma}$ with $0 < \gamma < 1$, or even just of class C^1 as we do here, requires a more detailed analysis.

As usual, by modulus of continuity of a real or vector-valued function f defined on a subset A of $\mathbb{R}^n$ we mean an increasing function $\omega : [0, \infty[\rightarrow [0, \infty[$ such that $w(t) \rightarrow 0$ as $t \rightarrow 0$ and such that $|f(x) - f(y)| \leq \omega(|x - y|)$ for all $x, y \in A$.

Lemma 3.2. *Let Ω be an open set in $\mathbb{R}^n$ of class C^1 , $x_0 \in \partial\Omega$ and U be an open neighbourhood of x_0 . Let $G \in C^1(\overline{\Omega \cap U})$ be a non-negative function such that $G(x) = 0$, $\nabla G(x) \neq 0$ for all $x \in U \cap \partial\Omega$. Then*

$$\lim_{x \rightarrow x_0} \frac{|\nabla G(x)|}{G(x)} \delta(x) = 1. \quad (3.6)$$

Moreover, if ω is a modulus of continuity of ∇G in a neighbourhood of x_0 , then

$$\left| \frac{\nabla G(x)}{G(x)} \right| = \frac{1}{\delta(x)} + \frac{O(\omega(\delta(x)))}{\delta(x)} \quad \text{as } x \rightarrow x_0. \quad (3.7)$$

Proof. Since Ω is of class C^1 , it can be represented locally around x_0 as the sub-graph of a C^1 function. This means that there exists an open neighbourhood B of x_0 and an isometry R such that $R(B) = \Pi_{i=1}^n]a_i, b_i[$ for $a_i, b_i \in \mathbb{R}$ and $R(\Omega \cap B) = \{x \in \Pi_{i=1}^n]a_i, b_i[: x_n < \varphi(x_1, \dots, x_{n-1})\}$ where φ is a suitable C^1 function from $\Pi_{i=1}^{n-1}]a_i, b_i[$ to $]a_n, b_n[$. To shorten our notation, in the sequel we write $\bar{x}$ for $(x_1, \dots, x_{n-1})$. Moreover, we may assume directly that the isometry R is the identity and that $B \subseteq U$. We now proceed dividing the proof in three steps.

Step 1. We prove that there exists an open neighbourhood $\tilde{B} \subset B$ of x_0 and $c > 0$ such that

$$c\delta(x) \leq G(x) \leq c^{-1}\delta(x) \quad (3.8)$$

for all $x \in \tilde{B} \cap \Omega$. Since ∇G is continuous up to $\partial\Omega$, G vanishes on $\partial\Omega$ and ∇G does not vanish at any point of $\partial\Omega$, it follows that if $x \in \Omega \cap B$ is sufficiently close to $\partial\Omega$, then $\frac{\partial G(x)}{\partial x_n} \neq 0$, hence there exists $c_1 > 0$ such that

$$c_1 \leq \left| \frac{\partial G(x)}{\partial x_n} \right| \leq c_1^{-1} \quad (3.9)$$

for all $x \in \tilde{B} \cap \Omega$, where $\tilde{B}$ is an open neighbourhood of x_0 with $\tilde{B} \subset B$. Now, by the Lagrange's mean value theorem, we have $G(\bar{x}, x_n) = \frac{\partial G(\bar{x}, \xi_x)}{\partial x_n}(x_n - \varphi(\bar{x}))$ where $\xi_x \in]x_n, \varphi(\bar{x})[$, hence

$$c_1(\varphi(\bar{x}) - x_n) \leq G(x) \leq c_1^{-1}(\varphi(\bar{x}) - x_n) \quad (3.10)$$

for all $x \in \tilde{B} \cap \Omega$. By standard arguments and by possibly shrinking $\tilde{B}$, we have that there exists $c_2 > 0$ such that

$$\delta(x) \leq \varphi(\bar{x}) - x_n \leq c_2\delta(x) \quad (3.11)$$

for all $x \in \tilde{B} \cap \Omega$, which combined with (3.10) yields (3.8).

Step 2. Let ω be a modulus of continuity of ∇G on $\overline{\Omega \cap B}$ as in the statement. For every $x \in \Omega$ we denote by $P(x)$ a point in $\partial\Omega$ of minimal distance of x from $\partial\Omega$, which means that $\delta(x) = |x - P(x)|$. We prove that

$$G(x) = \nabla G(x) \cdot (x - P(x)) + O(\omega(\delta(x)))\delta(x) \quad \text{as } x \rightarrow x_0. \quad (3.12)$$

By the Lagrange's mean value theorem applied to the function

$$t \mapsto G(P(x) + t(x - P(x))), \quad \text{where } t \in [0, 1],$$

and x is fixed in $\Omega \cap B$, we obtain

$$\begin{aligned} G(x) &= G(P(x)) + \nabla G(P(x) + \eta_x(x - P(x))) \cdot (x - P(x)) \\ &= \nabla G(x) \cdot (x - P(x)) \\ &\quad + (\nabla G(P(x) + \eta_x(x - P(x))) - \nabla G(x)) \cdot (x - P(x)), \end{aligned} \quad (3.13)$$

for some $\eta_x \in]0, 1[$. Then we have

$$\begin{aligned} &|(\nabla G(P(x) + \eta_x(x - P(x))) - \nabla G(x)) \cdot (x - P(x))| \\ &\leq \omega(|(\eta_x - 1)(x - P(x))|)|x - P(x)| \leq \omega(\delta(x))\delta(x), \end{aligned} \quad (3.14)$$

for all $x \in \Omega \cap B$. By combining (3.13) and (3.14) we obtain (3.12).

Step 3. We note that

$$\lim_{x \rightarrow x_0} P(x) = x_0 \quad \text{and} \quad \frac{x - P(x)}{|x - P(x)|} = \nu(P(x)), \quad (3.15)$$

where $\nu(P(x))$ is the unit inner normal to $\partial\Omega$ at the point $P(x)$. By (3.12) and the second equality in (3.15) we have

$$\frac{\nabla G(x)}{G(x)} \cdot \nu(P(x)) = \frac{1}{\delta(x)} + \frac{O(\omega(\delta(x)))}{G(x)}. \quad (3.16)$$

Consequently, by (3.8) and using the fact that $\omega(\delta(x)) \rightarrow 0$ as $x \rightarrow x_0$, we deduce that

$$\lim_{x \rightarrow x_0} \frac{\nabla G(x)}{G(x)} \cdot \nu(P(x))\delta(x) = 1. \quad (3.17)$$

Thus, by (3.17)

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{|\nabla G(x)|}{G(x)}\delta(x) &= \lim_{x \rightarrow x_0} \frac{|\nabla G(x)|\nabla G(x) \cdot \nu(P(x))}{G(x)\nabla G(x) \cdot \nu(P(x))}\delta(x) \\ &= \frac{|\nabla G(x_0)|}{\nabla G(x_0) \cdot \nu(x_0)} = 1, \end{aligned} \quad (3.18)$$

where in the last equality we have used the fact that $\nabla G(x_0) = \nabla G(x_0) \cdot \nu(x_0)\nu(x_0)$ and $\nabla G(x_0) \cdot \nu(x_0) > 0$ since ν points inwards. This completes the proof of (3.6).

Step 4. For $x \in U \cap \Omega$ we consider an orthonormal basis

$$\{V_1(P(x)), \dots, V_{n-1}(P(x))\}$$

of the tangent hyperplane to $\partial\Omega$ at the point $P(x)$. Since G vanishes on $U \cap \partial\Omega$ we have $\nabla G(P(x)) \cdot V_i(P(x)) = 0$ for all $i = 1, \dots, n-1$ hence

$$\begin{aligned} \nabla G(x) &= \sum_{i=1}^{n-1} \nabla G(x) \cdot V_i(P(x))V_i(P(x)) + \nabla G(x) \cdot \nu(P(x))\nu(P(x)) \\ &= \sum_{i=1}^{n-1} (\nabla G(x) - \nabla G(P(x))) \cdot V_i(P(x))V_i(P(x)) \\ &\quad + \nabla G(x) \cdot \nu(P(x))\nu(P(x)) \\ &= O(\omega(\delta(x))) + \nabla G(x) \cdot \nu(P(x))\nu(P(x)), \end{aligned} \quad (3.19)$$

which combined with (3.8) and (3.16) yields (3.7). $\square$

We also need the following lemma which represents a special case of a general statement proved in [16, Lemma 2.10]. Formula (3.20) has to be understood in the distributional sense.

Lemma 3.3. *Let U be an open set in $\mathbb{R}^n$, and let G be a positive function of class $C^1(U)$. Then for all $\alpha, \beta > 0$ we have*

$$\begin{aligned} \Delta_p(G^\alpha \pm G^\beta) &= \left| \alpha G^{\alpha-1} \pm \beta G^{\beta-1} \right|^{p-2} \left[\left(\alpha G^{\alpha-1} \pm \beta G^{\beta-1} \right) \Delta_p G \right. \\ &\quad \left. + (p-1) |\nabla G|^p \left[(\alpha^2 - \alpha) G^{\alpha-2} \pm (\beta^2 - \beta) G^{\beta-2} \right] \right]. \end{aligned} \quad (3.20)$$

We are now ready to prove the following theorem which guarantees the existence of the above mentioned subsolutions and supersolutions in a neighbourhood of a compact boundary.

Lemma 3.4. *Let Ω be a domain in $\mathbb{R}^n$ with nonempty compact boundary of class C^1 and U be an open neighbourhood of $\partial\Omega$. Let $\gamma \in]0, 1]$ and $G \in C^{1,\gamma}(\overline{\Omega} \cap U)$ be a positive function such that $\Delta_p G = 0$ in $\Omega \cap U$ and $G(x) = 0$ for all $x \in \partial\Omega$. Let $\alpha, \beta \in (0, 1)$ be such that $(p-1)/p \leq \alpha < \beta < \alpha + \gamma$. Then there exists an open neighbourhood $\mathcal{U}$ of $\partial\Omega$, $\mathcal{U} \subset U$, such that the functions $G^\alpha + G^\beta$ and $G^\alpha - G^\beta$ are a subsolution and a supersolution, respectively, for the equation $-\Delta_p v = \lambda_\alpha \mathcal{I}_p v / \delta^p$ in $\Omega \cap \mathcal{U}$, where $\lambda_\alpha = (p-1)\alpha^{p-1}(1-\alpha)$. Moreover, $\mathcal{U}$ can be chosen to be independent of small perturbations of α and β .*

Proof. First we consider the case of the subsolution. By Lemma 3.3 and (3.7), it follows that

$$\begin{aligned} & -\Delta_p(G^\alpha + G^\beta) \\ &= \left(\alpha G^{\alpha-1} + \beta G^{\beta-1} \right)^{p-2} \left(\frac{\lambda_\alpha}{\alpha^{p-2}} G^{\alpha-2} + \frac{\lambda_\beta}{\beta^{p-2}} G^{\beta-2} \right) |\nabla G|^p \\ &= G^{\alpha(p-1)} (\alpha + \beta G^{\beta-\alpha})^{p-2} \left(\frac{\lambda_\alpha}{\alpha^{p-2}} + \frac{\lambda_\beta}{\beta^{p-2}} G^{\beta-\alpha} \right) \left| \frac{\nabla G}{G} \right|^p \\ &\leq G^{\alpha(p-1)} (\alpha + \beta G^{\beta-\alpha})^{p-2} \left(\frac{\lambda_\alpha}{\alpha^{p-2}} + \frac{\lambda_\beta}{\beta^{p-2}} G^{\beta-\alpha} \right) \left(\frac{1}{\delta^p} + O(\delta^{\gamma-p}) \right). \end{aligned} \quad (3.21)$$

By (3.21), in order to guarantee that $G^\alpha + G^\beta$ is a subsolution as required in the statement, it suffices to impose the condition

$$\begin{aligned} & G^{\alpha(p-1)} (\alpha + \beta G^{\beta-\alpha})^{p-2} \left(\frac{\lambda_\alpha}{\alpha^{p-2}} + \frac{\lambda_\beta}{\beta^{p-2}} G^{\beta-\alpha} \right) \left(\frac{1}{\delta^p} + O(\delta^{\gamma-p}) \right) \\ &\leq \frac{\lambda_\alpha}{\delta^p} (G^\alpha + G^\beta)^{p-1} \end{aligned}$$

which can be written in the form

$$(\alpha + \beta G^{\beta-\alpha})^{p-2} \left(\frac{\lambda_\alpha}{\alpha^{p-2}} + \frac{\lambda_\beta}{\beta^{p-2}} G^{\beta-\alpha} \right) (1 + O(\delta^\gamma)) \leq \lambda_\alpha (1 + G^{\beta-\alpha})^{p-1}. \quad (3.22)$$

Since $G^{\beta-\alpha} = 0$ on $\partial\Omega$, by expanding both sides of (3.22) in $G^{\beta-\alpha}$ up to the first order, inequality (3.22) can also be written in the form

$$(\lambda_\alpha + A G^{\beta-\alpha} + o(G^{\beta-\alpha})) (1 + O(\delta^\gamma)) \leq \lambda_\alpha (1 + (p-1)G^{\beta-\alpha} + o(G^{\beta-\alpha})), \quad (3.23)$$

where

$$A := (p-2)\lambda_\alpha\beta/\alpha + \lambda_\beta\alpha^{p-2}/\beta^{p-2}.$$

Note that since $G(x)$ is asymptotic to $\delta(x)$ as $x \rightarrow \partial\Omega$ and $\beta - \alpha < \gamma$, we have that $\delta(x)^\gamma / G(x)^{\beta-\alpha} \rightarrow 0$ as $x \rightarrow \partial\Omega$. Moreover, by a direct computation and by using condition $(p-1)/p \leq \alpha < \beta$, one can easily verify that $A < (p-1)\lambda_\alpha$. Thus, passing to the limit as $x \rightarrow \partial\Omega$ in both sides of (3.23), one can see that condition (3.23) is satisfied in $\Omega \cap \mathcal{U}$, where $\mathcal{U}$ is a suitable neighbourhood of $\partial\Omega$ which can be chosen to be independent of α and β , if α and β are as in the statement and belong to small neighbourhoods of two fixed parameters α_0, β_0 satisfying the conditions $(p-1)/p \leq \alpha_0 < \beta_0$.

We now consider the case of the supersolution. Proceeding as above, we see that in order to guarantee that $G^\alpha - G^\beta$ is a positive supersolution as required in the statement, we clearly may first take a small neighbourhood $\mathcal{U}_1$ of $\partial\Omega$ such that $G^\alpha - G^\beta$ is positive in $\Omega \cap \mathcal{U}_1$. So, it suffices to impose the condition

$$\begin{aligned} & G^{\alpha(p-1)} |\alpha - \beta G^{\beta-\alpha}|^{p-2} \left(\frac{\lambda_\alpha}{\alpha^{p-2}} - \frac{\lambda_\beta}{\beta^{p-2}} G^{\beta-\alpha} \right) \left(\frac{1}{\delta^p} - O(\delta^{\gamma-p}) \right) \\ & \geq \frac{\lambda_\alpha}{\delta^p} (G^\alpha - G^\beta)^{p-1} \end{aligned}$$

in $\Omega \cap \mathcal{U}_2$, where $\mathcal{U}_2$ is a smaller neighbourhood of $\partial\Omega$. The latter inequality can be written in the form

$$(\lambda_\alpha - A G^{\beta-\alpha} + o(G^{\alpha-\beta})) (1 - O(\delta^\gamma)) \geq \lambda_\alpha (1 - (p-1)G^{\beta-\alpha} + o(G^{\alpha-\beta})), \quad (3.24)$$

where A is the same constant defined above. Again, since $A < (p-1)\lambda_\alpha$ we easily deduce as in the case of the subsolution the desired assertion. $\square$

We now construct sub- and super-solutions near ∞ for the operator

$$-\Delta_p - \lambda_\alpha \left| \frac{p-n}{p-1} \right|^p \frac{\mathcal{I}_p}{\delta^p}$$

on an unbounded domain Ω with compact boundary. Recall that if $p = n$, then for such a domain $H_p(\Omega) = 0$. So, for our purpose, we need to consider only the case where $p \neq n$.

Lemma 3.5. *Let Ω be an unbounded domain in $\mathbb{R}^n$ with nonempty compact boundary. Let G be the function defined in $\mathbb{R}^n \setminus \{0\}$ by $G(x) := |x|^{\frac{p-n}{p-1}}$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Then the following statements hold:*

(i) *If $p < n$ and $\alpha, \beta \in (0, 1)$ are such that*

$$\frac{p-1}{p} \leq \alpha < \beta < \alpha + \frac{p-1}{n-p},$$

then there exists $M > 0$ such that the functions $G^\alpha + G^\beta$, $G^\alpha - G^\beta$ are a subsolution and a supersolution, respectively, for the equation $-\Delta_p v = \lambda_\alpha \left| \frac{p-n}{p-1} \right|^p \mathcal{I}_p v / \delta^p$ on $\{x \in \mathbb{R}^n : |x| > M\}$.

(ii) *If $p > n$ and $\alpha, \beta \in (0, 1)$ are such that $\beta < \alpha \leq (p-1)/p$, then there exists $M > 0$ such that the functions $G^\alpha + G^\beta$, $G^\alpha - G^\beta$ are a subsolution and a supersolution respectively, for the equation $-\Delta_p v = \lambda_\alpha \left| \frac{p-n}{p-1} \right|^p \mathcal{I}_p v / \delta^p$ on $\{x \in \mathbb{R}^n : |x| > M\}$.*

Proof. By Lemma 3.3 it follows that

$$\begin{aligned} -\Delta_p(G^\alpha \pm G^\beta) &= \left| \alpha G^{\alpha-1} \pm \beta G^{\beta-1} \right|^{p-2} \left(\frac{\lambda_\alpha}{\alpha^{p-2}} G^{\alpha-2} \pm \frac{\lambda_\beta}{\beta^{p-2}} G^{\beta-2} \right) |\nabla G|^p \\ &= G^{\alpha(p-1)} \left| \alpha \pm \beta G^{\beta-\alpha} \right|^{p-2} \left(\frac{\lambda_\alpha}{\alpha^{p-2}} \pm \frac{\lambda_\beta}{\beta^{p-2}} G^{\beta-\alpha} \right) \left| \frac{\nabla G}{G} \right|^p \\ &= G^{\alpha(p-1)} \left| \alpha \pm \beta G^{\beta-\alpha} \right|^{p-2} \left(\frac{\lambda_\alpha}{\alpha^{p-2}} \pm \frac{\lambda_\beta}{\beta^{p-2}} G^{\beta-\alpha} \right) \left| \frac{p-n}{p-1} \right|^p \frac{1}{|x|^p}. \end{aligned}$$

Since $\partial\Omega$ is compact, it follows that $|\delta^{-p} - |x|^{-p}| \leq O(\delta^{-1})\delta^{-p}$ as $|x| \rightarrow \infty$. Thus, in order to verify that $G^\alpha + G^\beta$ is a subsolution as required in the statement, it suffices to impose the condition

$$\begin{aligned} &(\alpha + \beta G^{\beta-\alpha})^{p-2} \left(\frac{\lambda_\alpha}{\alpha^{p-2}} + \frac{\lambda_\beta}{\beta^{p-2}} G^{\beta-\alpha} \right) (1 + O(\delta^{-1})) \\ &\leq \lambda_\alpha (1 + G^{\beta-\alpha})^{p-1}. \end{aligned} \quad (3.25)$$

Similarly, in order to guarantee that $G^\alpha - G^\beta$ is a supersolution as required in the statement, it suffices to impose the condition

$$\begin{aligned} &|\alpha - \beta G^{\beta-\alpha}|^{p-2} \left(\frac{\lambda_\alpha}{\alpha^{p-2}} - \frac{\lambda_\beta}{\beta^{p-2}} G^{\beta-\alpha} \right) (1 - O(\delta^{-1})) \\ &\geq \lambda_\alpha |1 - G^{\beta-\alpha}|^{p-2} (1 - G^{\beta-\alpha}). \end{aligned} \quad (3.26)$$

By assumptions, in both cases $p < n$ and $n < p$, we have that $G^{\beta-\alpha}(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Thus, condition (3.25) can be written as

$$\begin{aligned} &(\lambda_\alpha + \alpha G^{\beta-\alpha} + o(G^{\beta-\alpha})) (1 + O(\delta^{-1})) \\ &\leq \lambda_\alpha (1 + (p-1)G^{\beta-\alpha} + o(G^{\beta-\alpha})), \end{aligned} \quad (3.27)$$

while condition (3.26) can be written as

$$\begin{aligned} & (\lambda_\alpha - AG^{\beta-\alpha} + o(G^{\beta-\alpha})) (1 - O(\delta^{-1})) \\ & \geq \lambda_\alpha (1 - (p-1)G^{\beta-\alpha} + o(G^{\beta-\alpha})), \end{aligned} \quad (3.28)$$

where in both cases $A = (p-2)\lambda_\alpha\beta/\alpha + \lambda_\beta\alpha^{p-2}/\beta^{p-2}$ is the same constant appearing in the proof of Lemma 3.4. As it was noted in the proof of Lemma 3.4, if $(p-1)/p \leq \alpha < \beta$, then $A < (p-1)\lambda_\alpha$. However, it can be easily seen that $A < (p-1)\lambda_\alpha$ also if $0 < \beta < \alpha \leq (p-1)/p$. It follows that in order to verify the validity of conditions (3.27) and (3.28) for $|x|$ large enough, it suffices to verify that $O(\delta^{-1})G^{\alpha-\beta} = o(1)$ as $|x| \rightarrow \infty$. This condition is satisfied because $G(x)$ is asymptotic to $\delta(x)^{\frac{p-n}{p-1}}$ as $|x| \rightarrow \infty$ and $|\alpha - \beta| < |(p-1)/(p-n)|$. $\square$

4. Upper bounds and existence of minimizers

Using the results of the previous section, we can prove the following existence result for bounded domains. Note that, assuming that Ω is of class $C^{1,\gamma}$ as we do here, would allow to skip a few steps in our proof. However, we prefer to write down more details which explain how our method could be adapted to more general situations as described in Theorem 4.3, see Remark 4.2 below.

Theorem 4.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ of class $C^{1,\gamma}$ with $\gamma \in]0, 1]$. Then $\lambda_{p,\infty}(\Omega) = c_p$. Moreover, if $H_p(\Omega) < c_p$, then there exists a positive minimizer $u \in W_0^{1,p}(\Omega)$ for (1.2). In particular, if $\alpha \in](p-1)/p, 1[$ is such that $\lambda_\alpha = H_p(\Omega)$, then*

$$0 < u(x) \leq C\delta^\alpha(x) \quad \forall x \in \Omega. \quad (4.1)$$

Proof. Let $\tilde{x}_0 \in \Omega$ and G be the positive minimal Green function in Ω of the p -Laplacian with pole at $\tilde{x}_0$. Recall that since Ω is of class $C^{1,\gamma}$, then G is of class $C^{1,\tilde{\gamma}}$ away from $\tilde{x}_0$ and up to $\partial\Omega$, for some $\tilde{\gamma} \in (0, \gamma)$, and $G(x) = 0$, $\nabla G(x) \neq 0$ for all $x \in \partial\Omega$ by the Hopf lemma (see Section 2). Thus, G satisfies equality (3.6) for all $x_0 \in \partial\Omega$. In particular, choosing $\alpha = (p-1)/p$ in Lemma 3.1, we have that G^α satisfies (3.4) with $\lambda_\alpha = c_p$ and $c = 1$.

Since $\partial\Omega$ is compact, it follows that $G^{(p-1)/p}$ is a supersolution to the equation $-\Delta_p v - (c_p - \varepsilon)\mathcal{I}_p v/\delta^p = 0$ in a relative neighbourhood of $\partial\Omega$. By passing to the limit as $\varepsilon \rightarrow 0$ and using definition (1.4), we get that $\lambda_{p,\infty}(\Omega) \geq c_p$.

On the other hand, since Ω is of class C^1 , any point at the boundary has a tangent hyperplane, hence locally around any fixed point at the boundary it is possible to apply the same argument of [24, Theorem 5] and conclude that $\lambda_{p,\infty}(\Omega) \leq c_p$.

More precisely, let $P \in \partial\Omega$ be fixed and Π be the tangent hyperplane at $\partial\Omega$ in P . We claim that condition (2.2) in [24, Theorem 5] is satisfied, that is, for all $x \in \Omega$ in a suitable neighborhood of P we have

$$|d(x, \Pi) - \delta(x)| \leq o(1)d(x, P) \quad (4.2)$$

where $o(1)$ is a quantity which tends to zero as $x \rightarrow P$. To prove (4.2) we argue as follows. Since Ω is of class C^1 , we can assume without loss of generality that $P = 0$ (the origin of the coordinate system), $\Pi = \{x \in \mathbb{R}^n : x_n = 0\}$ and that there exists an open neighborhood U of P such that

$$\Omega \cap U = \left\{ (\bar{x}, x_n) \in \mathbb{R}^n : \bar{x} \in \Pi_{i=1}^{n-1}]a_i, b_i[, a_n < x_n < g(\bar{x}) \right\},$$

for suitable real numbers a_i, b_i , where g is a function of class $C^1(\Pi_{i=1}^{n-1}]a_i, b_i[)$ such that $g(0) = 0$ and $\nabla g(0) = 0$. Given $x = (\bar{x}, x_n) \in \Omega \cap U$, we set $r_x = d(x, P) = |x|$ and $L_x = \sup_{\bar{z} \in B(x, 2r_x) \cap \Omega} |\nabla g(\bar{z})|$. We note that L_x is well-defined for x sufficiently close to P and that $\delta(x) = \inf_{\bar{y} \in B(x, 2r_x) \cap \Omega} d(x, (\bar{y}, g(\bar{y})))$. Thus, for any $y \in B(x, 2r_x) \cap \Omega$ we have

$$\begin{aligned} |x_n - g(\bar{x})| &\leq |x_n - g(\bar{y})| + |g(\bar{y}) - g(\bar{x})| \\ &\leq |x_n - g(\bar{y})| + L_x |\bar{x} - \bar{y}| \\ &\leq (1 + L_x) d(x, (\bar{y}, g(\bar{y}))) \end{aligned}$$

hence

$$|x_n - g(\bar{x})| \leq (1 + L_x) \inf_{\bar{y} \in B(x, 2r_x) \cap \Omega} d(x, (\bar{y}, g(\bar{y}))) = (1 + L_x) \delta(x). \quad (4.3)$$

It follows from (4.3) that

$$|x_n| \leq |x_n - g(\bar{x})| + |g(\bar{x})| \leq (1 + L_x) \delta(x) + L_x |\bar{x}| \leq (1 + L_x) \delta(x) + L_x d(x, P)$$

hence

$$|x_n| - \delta(x) \leq L_x \delta(x) + L_x d(x, P) \leq 2L_x d(x, P). \quad (4.4)$$

On the other hand,

$$\delta(x) \leq |x_n - g(\bar{x})| \leq |x_n| + L_x |\bar{x}| \leq |x_n| + L_x d(x, P). \quad (4.5)$$

In conclusion, combining (4.4) and (4.5) we get

$$||x_n| - \delta(x)| \leq 2L_x d(x, P) \quad (4.6)$$

where $L_x \rightarrow 0$ as $x \rightarrow P$ since ∇g is continuous. Thus condition (4.2) is satisfied.

Now, condition (4.2) (together with the fact that a segment perpendicular to Π is contained in Ω , which is clearly satisfied in our case) is used in [24, Theorem 5] to prove that for any open neighbourhood V of P and any $\varepsilon > 0$ there exists a function $\varphi \in C_c^\infty(V \cap \Omega)$ such that

$$\frac{\int_\Omega |\nabla \varphi|^p dx}{\int_\Omega \frac{|\varphi|^p}{\delta^p} dx} \leq (1 + \varepsilon)(c_p + \varepsilon)$$

which allows us to conclude that the admissible numbers λ in (1.4) satisfy $\lambda \leq (1 + \varepsilon)(c_p + \varepsilon)$ for any $\varepsilon > 0$ (recall the Allegretto-Piepenbrink theory mentioned at the beginning of Section 2). Hence, $\lambda_{p,\infty}(\Omega) \leq c_p$. This proves that $\lambda_{p,\infty}(\Omega) = c_p$.

We assume now that $H_p(\Omega) < c_p$ and prove the existence of a minimizer for (1.2). First of all we note that since $H_p(\Omega) < \lambda_{p,\infty}(\Omega)$, Lemma 2.3 implies that the positive function of minimal growth $u \in \mathcal{M}_{\Omega, \{x_0\}}^{Q-H_p(\Omega)\delta^{-p}}$ is an Agmon ground state.

We now prove that $u \in L^p(\Omega; \delta^{-p})$. Since $\lambda_\alpha = c_p$ if $\alpha = (p-1)/p$ and $H_p(\Omega) < c_p$, we can choose $\tilde{\alpha} > (p-1)/p$ close enough to $(p-1)/p$ so that $\lambda_{\tilde{\alpha}} > H_p(\Omega)$. Note that this choice of $\tilde{\alpha}$ implies that $G^{\tilde{\alpha}} \in L^p(\Omega, \delta^{-p})$. As above, using (3.4) and the compactness of $\partial\Omega$ it follows that the function $G^{\tilde{\alpha}}$ is a supersolution to the equation $-\Delta_p v - (\lambda_{\tilde{\alpha}} - \varepsilon)\mathcal{I}_p v / \delta^p = 0$ in a relative neighbourhood of $\partial\Omega$. Hence, in such a neighbourhood

$$\left(-\Delta_p - H_p(\Omega)\frac{\mathcal{I}_p}{\delta^p}\right)G^{\tilde{\alpha}} \geq \left(-\Delta_p - (\lambda_{\tilde{\alpha}} - \varepsilon)\frac{\mathcal{I}_p}{\delta^p}\right)G^{\tilde{\alpha}} \geq 0, \quad (4.7)$$

provided that $\varepsilon > 0$ is small enough to guarantee that $H_p(\Omega) \leq \lambda_{\tilde{\alpha}} - \varepsilon$. Thus, $G^{\tilde{\alpha}}$ is a positive supersolution to the equation $-\Delta_p v - H_p(\Omega)\frac{\mathcal{I}_p v}{\delta^p} = 0$ in a relative neighbourhood of $\partial\Omega$. Therefore, the ground state u satisfies the condition $0 < u \leq kG^{\tilde{\alpha}}$ in a relative neighbourhood of $\partial\Omega$ for a suitable positive constant k . This implies that $u \in L^p(\Omega, \delta^{-p})$.

We now prove that $\nabla u \in L^p(\Omega)$. Note that since $u \leq kG^{\tilde{\alpha}}$ in a relative neighbourhood of $\partial\Omega$, we have that $u(x) \rightarrow 0$ as $x \rightarrow \partial\Omega$, hence u is continuous up to the boundary of Ω . Then we use a standard truncation argument as follows. For any $\varepsilon > 0$ we consider the real-valued function F_ε defined on $[0, \infty[$ by setting $F_\varepsilon(x) = 0$ if $0 \leq x < \varepsilon/2$, $F_\varepsilon(x) = 2x - \varepsilon$ if $\varepsilon/2 < x < \varepsilon$, $F_\varepsilon(x) = x$ if $x \geq \varepsilon$. Moreover, we set $u_\varepsilon = F_\varepsilon \circ u$. Since u_ε has compact support in Ω , it can be used as a test function in the weak formulation of the problem solved by u , namely

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = H_p(\Omega) \int_{\Omega} \frac{|u|^{p-2} u \varphi}{\delta^p} \, dx, \quad (4.8)$$

where one can see by a standard approximation argument that it is possible to choose not only test functions $\varphi \in C_c^\infty(\Omega)$ but also functions in $W^{1,p}(\Omega)$ with compact support. Plugging u_ε in (4.8) we get

$$\begin{aligned} & \int_{\{x \in \Omega: u(x) \geq \varepsilon\}} |\nabla u|^p \, dx + 2 \int_{\{x \in \Omega: \varepsilon/2 < u(x) < \varepsilon\}} |\nabla u|^p \, dx \\ &= H_p(\Omega) \int_{\Omega} \frac{|u|^{p-2} u u_\varepsilon}{\delta^p} \, dx \end{aligned} \quad (4.9)$$

which in particular yields

$$\int_{\{x \in \Omega: u(x) \geq \varepsilon\}} |\nabla u|^p \, dx \leq H_p(\Omega) \int_{\Omega} \frac{|u|^{p-2} u u_\varepsilon}{\delta^p} \, dx. \quad (4.10)$$

Finally, passing to the limit in (4.10) as $\varepsilon \rightarrow 0$, we get that

$$\int_{\Omega} |\nabla u|^p dx \leq H_p(\Omega) \int_{\Omega} \frac{|u|^p}{\delta^p} dx < \infty \quad (4.11)$$

as required. Thus $u \in W_0^{1,p}(\Omega)$ since the Sobolev norm of u is finite and u vanishes at the boundary of Ω .

In order to prove estimate (4.1) we proceed as follows. Let α be as in the statement and let $\beta \in (0, 1)$ be such that $\alpha < \beta < \alpha + \tilde{\gamma}$. Then we can apply Lemma 3.4 and conclude that in a suitable relative neighbourhood of $\partial\Omega$

$$\left(-\Delta_p - H_p(\Omega) \frac{\mathcal{I}_p}{\delta^p}\right)(G^\alpha - G^\beta) = \left(-\Delta_p - \lambda_\alpha \frac{\mathcal{I}_p}{\delta^p}\right)(G^\alpha - G^\beta) \geq 0. \quad (4.12)$$

Thus $G^\alpha - G^\beta$ is a positive supersolution to the equation $-\Delta_p v - H_p(\Omega) \frac{\mathcal{I}_p v}{\delta^p} = 0$ in a relative neighbourhood of $\partial\Omega$. Since u is a positive solution of minimal growth in a neighbourhood of infinity in Ω , it follows that u satisfies $u \leq C(G^\alpha - G^\beta)$ in a relative neighbourhood of $\partial\Omega$ for a suitable positive constant C . Since $G(x)$ is asymptotic to $\delta(x)$ as $x \rightarrow \partial\Omega$, we deduce the validity of (4.1). $\square$

Remark 4.2. In the proof of Theorem 4.1, the assumption $\Omega \in C^{1,\gamma}$ was used in a substantial way only to prove the validity of (4.1), and to establish the upper bound $\lambda_{p,\infty}(\Omega) \leq c_p$. Note that $\lambda_{p,\infty}(\Omega) \leq c_p$ holds provided there exists one point $z \in \partial\Omega$ which admits a tangent hyperplane in the sense of [24, Theorem 5].

On the other hand, the proof of inequality $\lambda_{p,\infty}(\Omega) \geq c_p$ and the proof of the existence of a minimizer in $W_0^{1,p}(\Omega)$ under the condition $H_p(\Omega) < c_p$, rely only on the assumption that Ω is of class C^1 and on the existence of a p -harmonic function u defined in a relative neighbourhood of $\partial\Omega$ such that $u(x) = 0$ and $\nabla u(x) \neq 0$ for all $x \in \partial\Omega$. Under these weaker assumptions, it was also proved that a slightly weaker estimate holds for the positive minimizer u . Namely, estimate (4.1) holds with the power α replaced by any power $\tilde{\alpha}$ smaller than α . We recall in particular that the condition $\nabla u(x) \neq 0$ for all $x \in \partial\Omega$ is guaranteed by the Hopf lemma which holds under weaker assumptions on $\partial\Omega$, for example under the assumption that Ω is of class $C^{1,\text{Dini}}$, see [26]. Recall also that the Hopf lemma does not hold in general under the sole assumption that Ω is of class C^1 , see e.g., [20, Subsection 3.2].

Following the observations of the previous remark, we can state the following variant of the previous theorem.

Theorem 4.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ of class C^1 such that the Hopf lemma holds for the p -Laplacian (namely, any positive p -harmonic function u defined in a relative neighbourhood of $\partial\Omega$ such that $u = 0$ on $\partial\Omega$ satisfies $\nabla u(x) \neq 0$ for all $x \in \partial\Omega$). Then $\lambda_{p,\infty}(\Omega) \geq c_p$. Moreover, if $H_p(\Omega) < c_p$, then there exists a positive minimizer $u \in W_0^{1,p}(\Omega)$ for (1.2). In particular, if $\alpha \in](p-1)/p, 1[$ is such that $\lambda_\alpha = H_p(\Omega)$, then for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$0 < u(x) \leq C_\varepsilon \delta^{\alpha-\varepsilon}(x) \quad \forall x \in \Omega.$$

We can also consider the case of exterior domains. Recall that $c_{p,n}^* = |\frac{p-n}{p}|^p$ and $c_{p,n} = \min\{c_p, c_{p,n}^*\}$. It is well known that if $p = n$, then $H_p(\Omega) = \lambda_{p,\infty}(\Omega) = 0$ [24]. Therefore, in the following theorem we consider the case $p \neq n$.

Theorem 4.4. *Let Ω be an unbounded domain in $\mathbb{R}^n$ with nonempty compact boundary of class $C^{1,\gamma}$ with $\gamma \in]0, 1]$, and let $p \neq n$. Then $\lambda_{p,\infty}(\Omega) = c_{p,n}^*$.*

Moreover, if $H_p(\Omega) < c_{p,n}$, then there exists a positive minimizer $u \in \tilde{W}^{1,p}(\Omega)$ for (1.2). Finally, let $\alpha, \alpha_1 \in](p-1)/p, 1[$ and $\alpha_2 \in]0, (p-1)/p[$ be such that $\lambda_\alpha = H_p(\Omega)$, $\lambda_{\alpha_1} = \lambda_{\alpha_2} = |(p-1)/(p-n)|^p H_p(\Omega)$. Then there exists $C > 0$, an open neighbourhood $\mathcal{U}$ of $\partial\Omega$, and $M > 0$ such that u satisfies the following estimates:

- (i) $0 < u(x) \leq C\delta^\alpha(x)$ for all $x \in \Omega \cap \mathcal{U}$;
- (ii) If $p < n$, then $0 < u(x) \leq C|x|^{\frac{\alpha_1(p-n)}{p-1}}$ for all $|x| > M$;
- (iii) If $p > n$, then $0 < u(x) \leq C|x|^{\frac{\alpha_2(p-n)}{p-1}}$ for all $|x| > M$.

Proof. Let $\tilde{x}_0 \in \Omega$ and let G be a positive function defined on Ω which coincides in a relative neighbourhood of $\partial\Omega$ with the positive minimal Green function in Ω of the p -Laplacian with pole at $\tilde{x}_0$, and such that $G(x) = |x|^{\frac{p-n}{p-1}}$ for all x in a neighbourhood of ∞ (note that the specific definition of G outside such neighbourhoods is irrelevant here).

Since G satisfies (3.6) for all $x_0 \in \partial\Omega$, we can apply the same argument as in the proof of Theorem 4.1 to conclude that for any $\alpha \in (0, 1)$ and $\varepsilon > 0$ sufficiently small the function G^α is a positive supersolution to the equation

$$-\Delta_p v - (\lambda_\alpha - \varepsilon)\mathcal{I}_p v / \delta^p = 0 \quad (4.13)$$

in a neighbourhood of $\partial\Omega$.

We note now that $\Delta_p G = 0$ also in a neighbourhood of ∞ and that G satisfies (3.3) with $x_0 = \infty$ and $c = |p-n|/(p-1)$. Thus, by (3.4) it follows that for any $\alpha \in (0, 1)$ and $\varepsilon > 0$ sufficiently small the function G^α is a supersolution to the equation

$$-\Delta_p v - \left(\left| \frac{p-n}{p-1} \right|^p \lambda_\alpha - \varepsilon \right) \frac{\mathcal{I}_p v}{\delta^p} = 0 \quad (4.14)$$

in a neighbourhood of ∞ .

Recall that for $\alpha = (p-1)/p$ we have $\lambda_\alpha = c_p$, hence $\left| \frac{p-n}{p-1} \right|^p \lambda_\alpha = c_{p,n}^*$. Thus, choosing $\alpha = (p-1)/p$ and looking at the equations (4.13) and (4.14) we immediately see that for any $\varepsilon > 0$ sufficiently small the function $G^{(p-1)/p}$ is a supersolution of equation $-\Delta_p v - (c_{p,n} - \varepsilon)\mathcal{I}_p v / \delta^p = 0$ in a relative neighbourhood of $\partial\Omega \cup \{\infty\}$. Thus, passing to the limit as $\varepsilon \rightarrow 0$ we conclude that $\lambda_{p,\infty}(\Omega) \geq c_{p,n}$.

As in the proof of Theorem 4.1, we can use the argument of [24, Theorem 5] in a relative neighbourhood of any point of $\partial\Omega$ to prove that $\lambda_{p,\infty} \leq c_p$. Moreover, by [24, Example 2] it also follows that $\lambda_{p,\infty}(\Omega) \leq c_{n,p}^*$. So, $\lambda_{p,\infty}(\Omega) \leq c_{p,n}$. Thus, $\lambda_{p,\infty}(\Omega) = c_{p,n}$.

Assume now that $H_p(\Omega) < c_{p,n}$. We need to prove the existence of a minimizer for (1.2). As in the proof of Theorem 4.1, since $H_p(\Omega) < \lambda_{p,\infty}(\Omega)$, by Lemma 2.3 it follows that the positive function of minimal growth $u \in \mathcal{M}_{\Omega, \{\tilde{x}_0\}}^{Q-H_p(\Omega)\delta^{-p}}$ is an Agmon ground state. Arguing as in the proof of Theorem 4.1 we choose $\beta \in (\alpha, 1)$ such that $\beta < \alpha + \tilde{\gamma}$ where $\tilde{\gamma} \in (0, \gamma)$ is such that G is of class $C^{1,\tilde{\gamma}}$ in a relative neighbourhood of $\partial\Omega$. Exactly as in the proof of Theorem 4.1, it turns out that $G^\alpha - G^\beta$ is a positive supersolution to the equation $-\Delta_p v - H_p(\Omega) \frac{\mathcal{I}_p v}{\delta^p} = 0$ in a relative neighbourhood of $\partial\Omega$. Hence, the Agmon ground state u satisfies the condition $u \leq C(G^\alpha - G^\beta)$ in a relative neighbourhood of $\partial\Omega$ which provides the validity of the estimate in statement (i) for the function u .

In order to analyze the behaviour of u at ∞ , we use Lemma 3.5. We consider first the case $p < n$. Let $\beta \in (0, 1)$ be such $\alpha_1 < \beta < \alpha_1 + (p-1)/(n-p)$. Then by Lemma 3.5 we have

$$-\Delta_p(G^{\alpha_1} - G^\beta) \geq \lambda_{\alpha_1} \left| \frac{p-n}{p-1} \right|^p \mathcal{I}_p(G^{\alpha_1} - G^\beta)/\delta^p = H_p(\Omega) \mathcal{I}_p(G^{\alpha_1} - G^\beta)/\delta^p$$

in a neighbourhood of ∞ , which means that $G^{\alpha_1} - G^\beta$ is a supersolution. Thus u satisfies the condition $u(x) \leq C(G^{\alpha_1} - G^\beta)$, which implies that u satisfies the estimate in statement (ii) in a neighbourhood of ∞ (note that for $p < n$, $G(x) \rightarrow 0$ as $|x| \rightarrow \infty$, hence the leading term in $G^{\alpha_1} - G^\beta$ is given by G^{α_1}).

As far as the case $p > n$ we argue in the same way. We consider $\beta \in (0, 1)$ such that $0 < \beta < \alpha_2$ and we get that $G^{\alpha_2} - G^\beta$ is a supersolution in a neighbourhood of ∞ . Thus the Agmon ground state u satisfies the estimate in statement (iii) in a neighbourhood of ∞ (note that for $p > n$, $G(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, hence the leading term in $G^{\alpha_2} - G^\beta$ is given by G^{α_2}).

In conclusion, we have proved that u satisfies the appropriate estimates in statements (i), (ii), (iii). This implies that $u \in L^p(\Omega; \delta^{-p})$.

It remains to prove that $\nabla u \in L^p(\Omega)$. We can apply the same argument used in the proof of Theorem 4.1 to conclude that $\nabla u \in L^p(U)$, where U is a relative neighbourhood of $\partial\Omega$. On the other hand, since the operator $-\Delta_p - \frac{H_p(\Omega)}{\delta^p} \mathcal{I}_p$ has a Fuchsian type singularity at infinity, it follows from [19, Lemma 2.6] that there exists $r_0 > 0$ such that

$$|\nabla u(x)| \leq C \frac{u(x)}{|x|} \quad \text{for all } |x| > r_0. \quad (4.15)$$

Since $u \in L^p(\Omega; \delta^{-p})$, it follows from (4.15) that $\nabla u \in L^p(\Omega)$. $\square$

5. Lower bounds and non-existence of minimizers

In the present section we prove that the existence of a minimizer to the variational problem implies the existence of a spectral gap (equivalently, the absence of a spec-

tral gap implies the non-existence of minimizers). In the case of a bounded domain, the proof is based on a construction of a suitable subsolution for the equation $-\Delta_p v - c_p \delta^{-p} \mathcal{I}_p v = 0$ and a comparison principle proved in [25, Proposition 3.1]. In the case of unbounded domains, the proof is also based on the use of positive solutions of minimal growth at infinity for equations of the type $-\Delta_p v - \lambda |x|^{-p} \mathcal{I}_p v = 0$.

Theorem 5.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ of class $C^{1,\gamma}$ with $\gamma \in]0, 1]$ and fix $0 < \lambda \leq H_p(\Omega)$. Let $\alpha \in [(p-1)/p, 1)$ be such that $\lambda_\alpha = \lambda$, and let $\mathcal{U}$ be an open neighbourhood of $\partial\Omega$. If $u \in C(\overline{\Omega \cap \mathcal{U}} \setminus \partial\Omega)$ is a positive solution of the equation*

$$-\Delta_p v - \frac{\lambda}{\delta^p} \mathcal{I}_p v = 0 \quad (5.1)$$

in $\Omega \cap \mathcal{U}$, then there exists a constant $C > 0$ such that

$$\delta(x)^\alpha \leq Cu(x) \quad \text{in } \Omega \cap \mathcal{U}. \quad (5.2)$$

Hence, if u is a minimizer in (1.2), then $H_p(\Omega) < c_p$.

Proof. Let $x_0 \in \Omega$ be fixed, and let G be the positive minimal Green function in Ω for the p -Laplacian with pole at x_0 , and note that G vanishes at $\partial\Omega$. Recall that since Ω is of class $C^{1,\gamma}$, by standard regularity theory there exists an open neighbourhood U_0 of $\bar{\Omega}$ and $\tilde{\gamma} \in]0, \gamma]$ such that G is of class $C^{1,\tilde{\gamma}}(\bar{\Omega} \cap U_0)$. Moreover, since Ω is of class $C^{1,\gamma}$, the Hopf lemma holds, and hence, $\nabla G(x) \neq 0$ for all $x \in \partial\Omega$. Fix $\beta \in (0, 1)$ such that $\alpha < \beta < (p-1)/p + \tilde{\gamma}$. In light of Lemma 3.4, there exist $\varepsilon > 0$ and an open neighbourhood $U \subset U_0$ of $\partial\Omega$ such that for all $\alpha < \tilde{\alpha} < \alpha + \varepsilon < \beta$, the function $v := G^{\tilde{\alpha}} + G^\beta$ satisfies $-\Delta_p v \leq \frac{\lambda_{\tilde{\alpha}}}{\delta^p} \mathcal{I}_p v$ in $\Omega \cap U$. Hence,

$$-\Delta_p v \leq \frac{\lambda_\alpha}{\delta^p} \mathcal{I}_p v \quad \text{in } \Omega \cap U.$$

Let C be a positive constant such that $v \leq Cu$ on $\Omega \cap \partial U$ for all $\tilde{\alpha} \in (0, 1)$ sufficiently close to α . Then by the comparison principle proved in [25, Proposition 3.1], we can conclude that

$$v \leq Cu, \quad \text{in } \Omega \cap U \quad (5.3)$$

provided

$$\liminf_{r \rightarrow 0} \frac{1}{r} \int_{D_r} v^p \left(\left| \frac{\nabla v}{v} \right|^{p-1} + \left| \frac{\nabla u}{u} \right|^{p-1} \right) dx = 0, \quad (5.4)$$

where $D_r = \{x \in \Omega : r/2 < \delta(x) < r\}$. Since $\tilde{\alpha} > (p-1)/p$, condition (5.4) can be verified exactly as in the proof of [25, Lemma 5.1], where v is replaced by $\delta^{\tilde{\alpha}} + \delta^\beta$: for this purpose, note in particular that $G(x)$ is asymptotic to $\delta(x)$ as $x \rightarrow \partial\Omega$ and that [25, Proposition 2.1 (ii)] holds true also in the case of $C^{1,\gamma}$ domains (and actually also in the case of $C^{0,1}$ domains) as it can be easily verified.

Since the constant C in (5.3) does not depend on $\tilde{\alpha}$ for $\tilde{\alpha}$ close enough to α , it follows that

$$G^\alpha \leq Cu$$

in a relative neighbourhood of $\partial\Omega$, by which we can immediately deduce (5.2) (here one should take C sufficiently large in order to control the function G^α not only in a small relative neighbourhood of $\partial\Omega$ but also in the whole of $\Omega \cap \mathcal{U}$). If $H_p(\Omega) = c_p$, then $\alpha = (p-1)/p$. Therefore, (5.2) clearly implies that $u \notin W_0^{1,p}(\Omega)$, hence u cannot be a minimizer. $\square$

Combining the results of Lemma 3.4 and Theorem 5.1 we obtain the following tight upper and lower bounds for positive solutions of minimal growth near $\partial\Omega$.

Corollary 5.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ of class $C^{1,\gamma}$ with $\gamma \in]0, 1]$ and fix $0 < \lambda \leq c_p$. Let $\alpha \in [(p-1)/p, 1)$ be such that $\lambda_\alpha = \lambda$, and let $\mathcal{U}$ be an open neighbourhood of $\partial\Omega$. Let $u \in C(\overline{\Omega \cap \mathcal{U}} \setminus \partial\Omega)$ be a positive solution of the equation*

$$-\Delta_p v - \frac{\lambda}{\delta^p} \mathcal{I}_p v = 0 \quad \text{in } \Omega \cap \mathcal{U} \quad (5.5)$$

of minimal growth in a neighbourhood of infinity in Ω .

Then there exists a constant $C > 0$ such that

$$C^{-1} \delta(x)^\alpha \leq u(x) \leq C \delta(x)^\alpha \quad \text{in } \Omega \cap \mathcal{U}. \quad (5.6)$$

Remark 5.3. In the limiting case $\lambda = 0$, and under the mild regularity assumptions of Lemma 3.2, one obtains from estimate (3.8) that (5.6) holds with the limiting exponent $\alpha = 1$. This gives a strong indication to our feeling that in general, estimate (5.6) does not hold if the bounded domain Ω is merely in the C^1 class. Indeed, note that (5.6) does not hold in the class of bounded *Lipschitz domains*. Indeed, for $n = 2$, $p = 2$, $\lambda = 0$ one can take the *Lipschitz domain* $\Omega =]0, 1[\times]0, 1[$, and note that the function $u(x, y) = xy$ is a harmonic function of minimal growth near $(0, 0)$ that does not satisfy estimate (5.6) with $\alpha = 1$. For other examples concerning the case of a general conic point, $p = 2$ and $0 < \lambda \leq c_2$, see [17].

Next, we prove that for a $C^{1,\gamma}$ -exterior domain, the existence of a minimizer to the variational problem implies the existence of a spectral gap.

Theorem 5.4. *Assume that Ω is an unbounded domain in $\mathbb{R}^n$ with nonempty compact boundary of class $C^{1,\gamma}$ with $\gamma \in]0, 1]$. Fix $p \neq n$ and $0 < \lambda \leq H_p(\Omega)$. Let $\alpha, \alpha_1 \in [(p-1)/p, 1[$ and $\alpha_2 \in]0, (p-1)/p]$ be such that $\lambda = \lambda_\alpha := (p-1)\alpha^{p-1}(1-\alpha)$, $\lambda_{\alpha_1} := \lambda_{\alpha_2} = |(p-1)/(p-n)|^p \lambda$.*

If u is a positive solution of the equation (5.1), then there exists $C > 0$, an open neighbourhood $\mathcal{U}$ of $\partial\Omega$ and $M > 0$ such that u satisfies the following estimates:

- (i) $u(x) \geq C \delta^\alpha(x)$ for all $x \in \Omega \cap \mathcal{U}$;

- (ii) If $p < n$, then $u(x) \geq C|x|^{\frac{\alpha_1(p-n)}{p-1}}$ for all $|x| > M$;
 (iii) If $p > n$, then $u(x) \geq C|x|^{\frac{\alpha_2(p-n)}{p-1}}$ for all $|x| > M$.

Hence, if u is a minimizer for (1.2), then $H_p(\Omega) < c_{p,n}$.

Proof. Without loss of generality, we may assume that $0 \in \mathbb{R}^n \setminus \bar{\Omega}$, and let $R > 0$ be such that $\mathbb{R}^n \setminus \Omega \subset B(0, R)$. Recall that by Theorem 4.4, $H_p(\Omega) \leq c_{p,n}$.

Let u be any positive solution of the equation (5.1). Since $H_p(\Omega) \leq c_{p,n} \leq c_p$, estimate (i) follows from the proof of Theorem 5.1.

On the other hand, for any $0 < \mu \leq c_{p,n}^*$ consider the equation

$$-\Delta_p v - \frac{\mu}{|x|^p} \mathcal{I}_p v = 0 \quad (5.7)$$

in $\mathbb{R}^n \setminus \{0\}$. Then $v_\mu(x) := |x|^{\beta(\mu)}$ is a positive solution of (5.7) of minimal growth near ∞ , where $\beta(\mu) \leq (p-n)/p$ is the larger (respectively, smaller) root if $p < n$ (respectively, if $p > n$) of the transcendental equation

$$-\beta|\beta|^{p-2}[\beta(p-1) + n - p] = \mu,$$

see, [18, Example 1.1]. Note that $\beta(\mu) = (p-n)/p$ if and only if $\mu = c_{p,n}^*$. Note also that $\beta(\mu) = \alpha(\mu)(p-n)/(p-1)$ where $\alpha(\mu)$ is the larger (respectively, smaller) positive real number such that $\lambda_{\alpha(\mu)} = |(p-1)/(p-n)|^p \mu$ if $p < n$ (respectively, if $p > n$).

Take, $\mu = \lambda$. Then u is a positive supersolution of (5.7) in $\mathbb{R}^n \setminus B(0, R)$ since u is a solution of (5.1) and $\delta(x) \leq |x|$ for all $x \in \mathbb{R}^n \setminus B(0, R)$. Therefore, there exists a positive constant c such that $c|x|^{\beta(\mu)} \leq u(x)$ in $\mathbb{R}^n \setminus B(0, R)$, and we obtain estimate (ii) if $p < n$ and (iii) if $p > n$.

Estimates (i)-(iii) for u clearly imply that if $\lambda = H_p(\Omega) = c_{p,n}$, then any positive solution u satisfies $u \notin W_0^{1,p}(\Omega)$, and therefore, the variational problem does not admit a minimizer. $\square$

Finally, by combining the results of Lemma 3.5, Corollary 5.2, and Theorem 5.4, we obtain for $C^{1,\gamma}$ -exterior domains Ω tight upper and lower bounds for positive solutions of minimal growth in a neighbourhood of infinity in Ω , that is, a neighbourhood of $\partial\Omega \cup \{\infty\}$.

Corollary 5.5. *Let Ω be an unbounded domain in $\mathbb{R}^n$ with nonempty compact boundary of class $C^{1,\gamma}$ with $\gamma \in]0, 1]$, and fix $p \neq n$ and $0 < \lambda \leq c_{p,n}$. Let $\alpha, \alpha_1 \in [(p-1)/p, 1[$ and $\alpha_2 \in]0, (p-1)/p]$ be such that $\lambda = \lambda_\alpha = (p-1)\alpha^{p-1}(1-\alpha)$, $\lambda_{\alpha_1} = \lambda_{\alpha_2} = |(p-1)/(p-n)|^p \lambda$. If u is a positive solution of the equation (5.1) of minimal growth in a neighbourhood of infinity in Ω , then there exists $C > 0$, an open neighbourhood $\mathcal{U}$ of $\partial\Omega$ and $M > 0$ such that u satisfies the following estimates:*

- (i) $C^{-1}\delta^\alpha(x) \leq u(x) \leq C\delta^\alpha(x)$ for all $x \in \Omega \cap \mathcal{U}$;

- (ii) If $p < n$, then $C^{-1}|x|^{\frac{\alpha_1(p-n)}{p-1}} \leq u(x) \leq C|x|^{\frac{\alpha_1(p-n)}{p-1}}$ for all $|x| > M$;
 (iii) If $p > n$, then $C^{-1}|x|^{\frac{\alpha_2(p-n)}{p-1}} \leq u(x) \leq C|x|^{\frac{\alpha_2(p-n)}{p-1}}$ for all $|x| > M$.

References

- [1] A. ANCONA, *On strong barriers and an inequality of Hardy for domains in $\mathbb{R}^n$* , J. Lond. Math. Soc. (2) **34** (1986), 274–290.
- [2] F. G. AVKHADIEV, *Families of domains with best possible Hardy constant*, Russian Math. (Iz. VUZ) **57** (2013), 49–52.
- [3] A. A. BALINSKY, W. D. EVANS and R. T. LEWIS, “The Analysis and Geometry of Hardy’s Inequality”, Universitext, Springer, Cham, 2015.
- [4] G. BARBATIS and P. D. LAMBERTI, *Shape sensitivity analysis of the Hardy constant*, Non-linear Anal. **103** (2014), 98–112.
- [5] G. BARBATIS and P. D. LAMBERTI, *Monotonicity, continuity and differentiability results for the L^p Hardy constant*, Israel J. Math. **215** (2016), 1011–1024.
- [6] G. BARBATIS, S. FILIPPAS and A. TERTIKAS, *A unified approach to improved L^p Hardy inequalities with best constants*, Trans. Amer. Math. Soc. **356** (2004), 2169–2196.
- [7] G. BARBATIS and A. TERTIKAS, *On the Hardy constant of non-convex planar domains: the case of the quadrilateral*, J. Funct. Anal. **266** (2014), 3701–3725.
- [8] G. BARBATIS and A. TERTIKAS, *On the Hardy constant of some non-convex planar domains*, In: “Geometric Methods in PDE’s”, G. Citti, M. Manfredini, D. Morbidelli, S. Polidoro and F. Uguzzoni (eds.), Vol. 13, Springer INdAM Series, 2015, 15–41.
- [9] H. BREZIS and M. MARCUS, *Hardy’s inequalities revisited*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), 217–237.
- [10] J. CHABROWSKI and M. WILLEM, *Hardy’s inequality on exterior domains*, Proc. Amer. Math. Soc. **134** (2006), 1019–1022.
- [11] L. D’AMBROSIO, *Hardy-type inequalities related to degenerate elliptic differential operators*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), 451–486.
- [12] E. B. DAVIES, *A review of Hardy inequalities*, Operator Theory Adv. Appl. **110** (1998), 511–523.
- [13] E. B. DAVIES, *The Hardy constant*, Quart. J. Math. Oxford Ser. (2) **184** (1995), 417–431.
- [14] B. DEVYVER, *A spectral result for Hardy inequalities*, J. Math. Pures Appl. (9) **102** (2014), 813–853.
- [15] B. DEVYVER, M. FRAAS and Y. PINCHOVER, *Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon*, J. Funct. Anal. **266** (2014), 4422–4489.
- [16] B. DEVYVER and Y. PINCHOVER, *Optimal L^p Hardy-type inequalities*, Ann. Inst. H. Poincaré. Anal. Non Linéaire **33** (2016), 93–118.
- [17] B. DEVYVER, Y. PINCHOVER and G. PSARAKIS, *Optimal Hardy inequalities in cones*, Proc. Roy. Soc. Edinburgh Sect. A **147** (2017), 89–124.
- [18] M. FRAAS and Y. PINCHOVER, *Positive Liouville theorems and asymptotic behavior for p -Laplacian type elliptic equations with a Fuchsian potential*, Confluentes Math. **3** (2011), 291–323.
- [19] M. FRAAS and Y. PINCHOVER, *Isolated singularities of positive solutions of p -Laplacian type equations in $\mathbb{R}^d$* , J. Differential Equations **254** (2013), 1097–1119.
- [20] D. GILBARG and N. S. TRUDINGER, “Elliptic Partial Differential Equations of Second Order”, reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
- [21] J. HEINONEN, T. KILPELÄINEN and O. MARTIO, “Nonlinear Potential Theory of Degenerate Elliptic Equations”, Unabridged republication of the 1993 original, Dover, 2006.

- [22] A. KUFNER, L. MALIGRANDA and L.-E. PERSSON, “The Hardy Inequality: About its History and Some Related Results”, Vydavatelský Servis Publishing House, Pilsen, 2007.
- [23] A. LAPTEV and A. SOBOLEV, *Hardy inequalities for simply connected planar domains*, In: “Spectral Theory of Differential Operators”, T. Suslina (ed.), Amer. Math. Soc. Transl. Ser. 2, Vol. 225, Amer. Math. Soc., Providence, RI, 2008, 133–140.
- [24] M. MARCUS, V. J. MIZEL and Y. PINCHOVER, *On the best constant for Hardy’s inequality in $\mathbb{R}^n$* , Trans. Amer. Math. Soc. **350** (1998), 3237–3255.
- [25] M. MARCUS and I. SHAFRIR, *An eigenvalue problem related to Hardy’s L^p inequality*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29** (2000), 581–604.
- [26] H. MIKAYELYAN and H. SHAHGOLIAN, *Hopf’s lemma for a class of singular/degenerate PDE-s*, Ann. Acad. Sci. Fenn. Math. **40** (2015), 475–484.
- [27] J. MALÝ and W. P. ZIEMER, “Fine Regularity of Solutions of Elliptic Partial Differential Equations”. Math. Surveys Monogr., Vol. 51, Amer. Math. Soc., Providence, RI, 1997.
- [28] Y. PINCHOVER, *Large scale properties of multiparameter oscillation problems*, Comm. Partial Differential Equations **15** (1990), 647–673.
- [29] Y. PINCHOVER and G. PSARADAKIS, *On positive solutions of the (p, A) -Laplacian with a potential in Morrey space*, Anal. PDE (2016), 1317–1358.
- [30] Y. PINCHOVER and K. TINTAREV, *Ground state alternative for p -Laplacian with potential term*, Calc. Var. Partial Differential Equations **28** (2007), 179–201.

Dipartimento di Matematica “Tullio Levi-Civita”
 Università degli Studi di Padova
 Via Trieste, 63
 35121 Padova, Italia
 lamberti@math.unipd.it

Department of Mathematics
 Technion - Israel Institute of Technology
 Haifa 32000, Israel
 pincho@technion.ac.il