LOCAL BEHAVIOR FOR SOLUTIONS TO ANISOTROPICALLY WEIGHTED QUASILINEAR DEGENERATE PARABOLIC EQUATIONS

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ABSTRACT. This paper aims to study the local behavior of solutions to quasilinear degenerate parabolic equations with a class of anisotropic weights, which comprise two power-type weights of different dimensions. We first capture the asymptotic behavior of the solution near the singular or degenerate point of the weights. In particular, we find an explicit upper bound on the decay rate exponent determined by the structures of the equations and weights. This exponent can be achieved under certain condition and reflects the damage effect of the weights on the regularity of solution. Furthermore, we prove the local Hölder regularity of solutions to non-homogeneous parabolic p-Laplace equations with single power-type weights.

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1. Introduction and main results

Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 2$. For $T > 0$, denote $\Omega_T := \Omega \times (-T, 0]$. In this paper, we focus on a class of anisotropic weighted quasilinear parabolic equations as follows:

$$\begin{cases} w_1 \partial_t u - \operatorname{div}(w_2 \mathbf{a}(x,t,u,\nabla u)) = w_2 b(x,t,u,\nabla u), & \text{in } \Omega_T, \\ \|u\|_{L^\infty(\Omega_T)} \leq \mathcal{M} < \infty, \end{cases} \tag{1.1}$$

where $w_1 = |x'|^{\theta_1}|x|^{\theta_2}$, $w_2 = |x'|^{\theta_3}|x|^{\theta_4}$, $x' = (x_1, ..., x_{n-1})$, the ranges of θ_i, $i = 1, 2, 3, 4$ are prescribed in the following theorems, the functions $\mathbf{a} : \Omega_T \times \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $b : \Omega_T \times \mathbb{R}^{n+1} \to \mathbb{R}$ are only measurable and subject to the following structure conditions: for $p > 1$ and $(x.t) \in \Omega_T$,

(H1) $\mathbf{a}(x,t,u,\nabla u) \cdot \nabla u \geq \lambda_1 |\nabla u|^p - \phi_1(x,t)$,
(H2) $|\mathbf{a}(x,t,u,\nabla u)| \leq \lambda_2 |\nabla u|^{p-1} + \phi_2(x,t)$,
(H3) $|b(x,t,u,\nabla u)| \leq \lambda_3 |\nabla u|^{p-1} + \phi_3(x,t)$.

Here λ_i, $i = 1, 2, 3$ are given positive constants and ϕ_i, $i = 1, 2, 3$ are nonnegative functions such that

$$\|\phi\|_{L^{l_0}(\Omega_T, w_2)} := \left(\int_{\Omega_T} |\phi|^{l_0} w_2 \, dx dt \right)^{\frac{1}{l_0}} < \infty, \quad \phi := \phi_1 + \phi_2^{\frac{p}{p-1}} + \phi_3^{\frac{p}{p-1}}, \tag{1.2}$$

where l_0 satisfies

$$l_0 > \frac{n + p + \theta_1 + \theta_2}{p}. \tag{1.3}$$

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Remark that the anisotropy of the weight $|x'|^{\theta_1}|x|^{\theta_2}$ comes from $|x'|^{\theta_1}$. From the geometric point of view, $|x'|^{\theta_1}$ represents a degenerate or singular line, while $|x|^{\theta_2}$ only exhibits singularity or degeneracy at the origin. In fluid mechanics, these weights have been utilized to describe the singularities of (-1)-homogeneous axisymmetric no-swirl solutions to the three-dimensional stationary Navier-Stokes equations. Starting from Landau solution [35] with single point singularity of order $O(|x|^{-1})$, Li and Yan [40] recently classified those other homogeneous solutions found in [38, 39] into two types of singular solutions, which exhibit the singularities of orders $O(|x|^{-1}\ln|x'|^{-1}|x|)$ and $O(|x'|^{-1})$, respectively. With regard to their properties and more applications in weighted Sobolev spaces and relevant PDEs, see e.g. [6, 41, 43, 44].

The structure conditions in (**H1**)–(**H3**) were first proposed and studied in the well-known work [19] completed by DiBenedetto, where the method of intrinsic scaling was developed to establish the Hölder estimates of solutions to degenerate parabolic equations. We here would like to point out that the assumed conditions in (1.2)–(1.3) imposed on ϕ_i, $i = 1, 2, 3$ are different from that in [19] and have obvious advantages in simplifying the computations and presenting the following proofs in a more concise manner. Given these structure conditions, the degeneracy and singularity of (1.1) are of the same nature to the prototype as follows:

$$w_1\partial_t u - \mathrm{div}(w_2|\nabla u|^{p-2}\nabla u) = 0.$$

When $w_1 = w_2 = 1$, it is the classical parabolic p-Laplace equation, which is frequently used to describe a large variety of diffusion phenomena occurring in natural sciences and engineering applications such as nonlinear porous medium flows and chemical concentration. For more relevant physical models and explanations, see [3, 9, 49] and the references therein. On one hand, according to the diffused feature of the flows, the equation can be divided into three types as follows. If $1 < p < 2$, it is termed fast diffusion equation and its solution will undergo extinction in finite time. The equation in the case of $p > 2$, by contrast, is called slow diffusion equation and its solution always decays in the form of power-function to the stable state. The borderline case of $p = 2$ corresponds to heat equation and its solution decays exponentially in the time variable. On the other hand, from the view of analysis, if $p > 2$, the equation is said to be degenerate since its modulus of ellipticity $|\nabla u|^{p-2}$ degenerates to be zero at points where $|\nabla u| = 0$, while if $1 < p < 2$, the equation is singular because $|\nabla u|^{p-2}$ blows up at points where $|\nabla u| = 0$. In contrast to the case of $p = 2$, these singular or degenerate nature in the case of $p \neq 2$ weaken the smoothness of the solution to be of $C^{1,\alpha}$ for some $0 < \alpha < 1$. With regard to the regularity for quasilinear elliptic and parabolic equations without weights, the literature is very wide, see e.g. [1,3,4,7,8,15–25,36,37,48] and the references therein.

For the weighted case, the situation becomes more complex. From the perspective of physical phenomena, the weights play a role in enhancing or reducing the diffusion rate of the flows. This fact can be observed by using the standard separation of variables method to obtain exact solution for heat equation with monomial weight $|x|^{\theta_2}$. The problem of studying their enhancement and weakening effects on the diffusion may possess a potential application in manufacturing porous medium materials with special permeation rates and dominating diffusion processes according to the requirements of the industry. From the angle of the structure of the above weighted equations, the modulus of ellipticity consisting of $|\nabla u|^{p-2}$ and the weight

w_2 exhibits more complicated singular and degenerate behavior. This leads to that the solution will become worse and its regularity may further fall to be only of C^α. In fact, the damage effect induced by the weights is stronger than that of $|\nabla u|^{p-2}$. Even when $p=2$, the smoothness of the solution has always been reduced to be of C^α due to their damage effect, except for some solutions of special structures such as even solutions found in [45]. The weakening effect of the weight on the regularity has been clearly presented in the recent work [10, 11], where Dong, Li and Yang utilized spherical harmonic expansion to solve the sharp decay rate exponents of solutions near the origin for the following weighted elliptic equations

$$\begin{cases}\operatorname{div}(\kappa(x)|x|^2\nabla u)=0, & \text{in } B_R,\\ 0\leq u\leq M<\infty, & \text{in } B_R,\\ \int_{\mathbb{S}^{n-1}}\kappa x_i=0,\ i=1,2,...,n,\ \Lambda^{-1}\leq\kappa(x)\leq\Lambda, & \text{in } B_R,\end{cases}$$

where $R,\Lambda>0$ and $\mathbb{S}^{n-1}$ denotes the $(n-1)$-dimensional sphere. Specifically, they obtained

$$u(x)=u(0)+O(1)|x|^\alpha,\quad \alpha=\frac{-n+\sqrt{n^2+4\lambda_1}}{2},\quad \text{in } B_{R/2},$$

where $|O(1)|\leq C=C(n,\Lambda,M)$, $\lambda_1\leq n-1$ is the first nonzero eigenvalue of the eigenvalue problem as follows:

$$-\operatorname{div}_{\mathbb{S}^{n-1}}(\kappa(\xi)\nabla_{\mathbb{S}^{n-1}}u(\xi))=\lambda\kappa(\xi)u(\xi),\quad \xi\in\mathbb{S}^{n-1}.$$

Especially when κ is a constant, $\lambda_1=n-1$. See Lemma 2.2 in [10] and Lemmas 2.2 and 5.1 in [11] for more details. More importantly, Dong, Li and Yang [10, 11] used these exponents to find the optimal gradient blow-up rates for the insulated conductivity problem arising from composite materials, which has been previously regarded as a challenging problem.

The study on the weighted equations can date back to the famous work [26], where Fabes, Kenig and Serapioni [26] proved the local Hölder continuity of solutions to second-order elliptic equations of divergence form with the weight $|x|^{\theta_2}$ for any $\theta_2>-n$. Recently, Miao and Zhao [44] studied a class of anisotropic Muckenhoupt weights having more general form of $|x'|^{\theta_1}|x|^{\theta_2}|x_n|^{\theta_3}$ and raised a couple of intriguing problems involving anisotropic weighted interpolation and Poincaré inequalities and the classification for their enhancement and weakening effects on the diffusion. One of the main motivations in [44] originates from the weight $|x_n|^{\theta_3}$, since this weight plays a significant role in the establishment of the global regularity for fast diffusion equations in [31, 32] completed by Jin and Xiong. Their results especially answered an open question raised by Berryman and Holland [2]. Subsequently, Jin, Ros-Oton and Xiong [33] further extended the results to porous medium equations. For more investigations related to weighted elliptic and parabolic equations, we refer to [5, 12–14, 27–29, 34, 43, 45–47] and the references therein.

Before giving the definition of weak solution to problem (1.1), we first list the required weighted spaces. Given a weight w, we use $L^p(\Omega,w)$, $L^p(\Omega_T,w)$ and $W^{1,p}(\Omega,w)$ to denote the weighted L^p spaces and weighted Sobolev spaces with their norms, respectively, given by

$$\begin{cases}\|u\|_{L^p(\Omega,w)}=\left(\int_\Omega|u|^pwdx\right)^{\frac{1}{p}},\quad \|u\|_{L^p(\Omega_T,w)}=\left(\int_{\Omega_T}|u|^pwdxdt\right)^{\frac{1}{p}},\\ \|u\|_{W^{1,p}(\Omega,w)}=\left(\int_\Omega|u|^pwdx\right)^{\frac{1}{p}}+\left(\int_\Omega|\nabla u|^pwdx\right)^{\frac{1}{p}}.\end{cases}$$

A function $u \in C((-T,0];L^2(\Omega,w_1)) \cap L^p((-T,0);W^{1,p}(\Omega,w_2))$ is called a weak solution of (1.1), provided that for any $-T < t_1 < t_2 \leq 0$,

$$\int_\Omega u\varphi w_1 dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega (-u\partial_t\varphi w_1 + w_2\mathbf{a}(x,t,u,\nabla u)\cdot\nabla\varphi)dxdt$$
$$= \int_{t_1}^{t_2}\int_\Omega b(x,t,u,\nabla u)\varphi w_2 dxdt,$$

for all $\varphi \in W^{1,2}((0,T);L^2(\Omega,w_1)) \cap L^p((0,T);W_0^{1,p}(\Omega,w_2))$.

Define

$$\begin{cases}\mathcal{A} = \{(\theta_1,\theta_2) : \theta_1 > -(n-1),\, \theta_2 \geq 0\},\\ \mathcal{B} = \{(\theta_1,\theta_2) : \theta_1 > -(n-1),\, \theta_2 < 0,\, \theta_1+\theta_2 > -n\},\\ \mathcal{C}_p = \{(\theta_1,\theta_2) : \theta_1 < (n-1)(p-1),\, \theta_2 \leq 0\},\\ \mathcal{D}_p = \{(\theta_1,\theta_2) : \theta_1 < (n-1)(p-1),\, \theta_2 > 0,\, \theta_1+\theta_2 < n(p-1)\}.\end{cases}$$

Introduce the following exponent conditions:

(**K1**) $(\theta_1,\theta_2) \in [(\mathcal{A}\cup\mathcal{B})\cap(C_p\cup\mathcal{D}_p)] \cup \{\theta_1 = 0,\, \theta_2 \geq n(p-1)\}$, $(\theta_3,\theta_4) \in \mathcal{A}\cup\mathcal{B}$,

(**K2**) $\theta_1+\theta_2 > p-n,\ \theta_1 \geq \theta_3,\ \theta_1+\theta_2 \geq \theta_3+\theta_4,\ \theta_3 + \min\{0,\theta_4\} > 1-n$.

We here give some explanations for the implications of (**K1**)–(**K2**). First, $\mathcal{A}\cup\mathcal{B}$ is called the measure condition, which makes $w_i dx$, $i=1,2$ become two Radon measures. Second, $(\mathcal{A}\cup\mathcal{B})\cap(C_p\cup\mathcal{D}_p)$ is introduced to guarantee that the considered weight w_1 belongs to the Muckenhoupt class A_p (see [42]) and we then obtain anisotropic weighted Poincaré inequality, which is critical to the establishment of the isoperimetric inequality of De Giorgi type. By contrast, the range of $\{\theta_1 = 0,\, \theta_2 \geq n(p-1)\}$ is added by using the theories of quasiconformal mappings for the same purpose. See Section 2 in [43] and Corollary 15.35 in [30] for these statements. As for (**K2**), the range of $\theta_1+\theta_2 > p-n$ is required to establish anisotropic weighted parabolic Sobolev inequality in Proposition 2.4 below, while other ranges are used to ensure the validity of the switch from the measures w_1dxdt to w_2dxdt, see the proofs in Section 4.

Throughout this paper, let C be a universal constant depending only on the data including $n,p,l_0,\mathcal{M},\|\phi\|_{L^{l_0}(\Omega_T,w_2)}$, λ_i, $i=1,2,3$ and θ_i, $i=1,2,3,4$, whose value may change from line to line.

For later use, define two constants as follows:

$$\varepsilon_0 := \frac{p(l_0-1)-n-\theta_1-\theta_2}{p(l_0-1)+2}, \quad \vartheta := \theta_1+\theta_2-\theta_3-\theta_4. \tag{1.4}$$

The first main result is concerned with the asymptotic behavior of solution to problem (1.1) near the singular or degenerate point of the weights.

Theorem 1.1. *Assume that $p>2$, $n\geq 2$, (**H1**)–(**H3**), (1.2)–(1.3) and (**K1**)–(**K2**) hold. Let u be a weak solution of problem (1.1) with $\Omega\times(-T,0] = B_1\times(-1,0]$. Then there exists a small constant $0<\alpha\leq\varepsilon_0$ depending only on the above data, such that for any fixed $t_0 \in (-1/2,0]$ and all $(x,t)\in B_{1/2}\times(-1/2,t_0]$,*

$$u(x,t) = u(0,t_0) + O(1)\big(|x| + |t-t_0|^{\frac{1}{p+\vartheta}}\big)^\alpha, \tag{1.5}$$

where ε_0 and ϑ are defined by (1.4), $|O(1)| \leq C$ for some positive constant C depending only on the data.

Remark 1.2. Fix the values of n, p, l_0. The upper bound exponent ε_0 tends to zero, as $\theta_1 + \theta_2 \to p(l_0 - 1) - n$. Moreover, we see from Remark 4.12 below that when ε_0 is a sufficiently small positive constant, the value of α in (1.5) can attain the upper bound ε_0. In this case, if we let the value of $\theta_1 + \theta_2$ increase towards to $p(l_0 - 1) - n$ and meanwhile decrease the value of $\theta_3 + \theta_4$, then the regularity exponents ε_0 and $\frac{\varepsilon_0}{p+\vartheta}$ corresponding to the space and time variables all decrease, which shows the weakening effect of the weights on the regularity of u and their reduction effect on the diffusion rate of the flows.

When the weights w_i, $i = 1, 2$ are chosen to be the same type of single power-type weights and the considered equation becomes non-homogeneous weighted parabolic p-Laplace equation as follows:

$$\begin{cases} w_1\partial_t u - \mathrm{div}(w_2|\nabla u|^{p-2}\nabla u) = w_2(\lambda_3|\nabla u|^{p-1} + \phi_3(x,t)), & \text{in } \Omega_T, \\ \|u\|_{L^\infty(\Omega_T)} \leq \mathcal{M} < \infty, \end{cases} \tag{1.6}$$

we further establish the Hölder estimates for the solution as follows.

Theorem 1.3. *Set $p > 2$, $n \geq 2$. Let* (1.2)–(1.3) *and* (**K1**)–(**K2**) *hold. Suppose that u is a weak solution of problem* (1.6) *with $\Omega \times (-T, 0] = B_1 \times (-1, 0]$. Then there exists a small constant $0 < \tilde{\alpha} < \frac{\alpha}{1+\alpha}$ with α determined by Theorem 1.1, such that if $(w_1, w_2) = (|x'|^{\theta_1}, |x'|^{\theta_3})$ or $(w_1, w_2) = (|x|^{\theta_2}, |x|^{\theta_4})$,*

$$|u(x,t) - u(y,s)| \leq C\big(|x - y| + |t - s|^{\frac{1}{p+\vartheta}}\big)^{\tilde{\alpha}}, \tag{1.7}$$

for any $(x,t), (y,s) \in B_{1/4} \times (-1/2, 0]$, where ϑ is given by (1.4).

Remark 1.4. According to the classical regularity theory for quasilinear parabolic equations, we know that the regularity of solution should be of $C^{1,\alpha}$ at the points away from the degenerate or singular points of the weights. This means that the regularity of solution to the considered weighted equation is actually determined by the behavior of the solution near these singular or degenerate points. Therefore, although we have provided a clear control relationship between the decay rate exponent α and the Hölder regularity exponent $\tilde{\alpha}$ in Theorem 1.3, it is worth expecting that the value of $\tilde{\alpha}$ in (1.7) can be further improved to be sufficiently close to the value of α in (1.5). The reason why we don't achieve this improvement is purely technical, and the problem on how to achieve the improvement remains open.

Remark 1.5. For any given $R_0 > 0$, when we replace $B_1 \times (-1, 0]$ with $B_{R_0} \times (-R_0^{p+\vartheta}, 0]$ in Theorems 1.1 and 1.3, we deduce from their proofs with minor modification that the results in (1.5) and (1.7) hold with $(-R_0^{p+\vartheta}/2, 0]$, $B_{R_0/2} \times (-R_0^{p+\vartheta}/2, t_0]$ and $B_{R_0/4} \times (-R_0^{p+\vartheta}/2, 0]$ substituting for $(-1/2, 0]$, $B_{1/2} \times (-1/2, t_0]$ and $B_{1/4} \times (-1/2, 0]$, respectively. A difference is that the constant C will depend upon R_0, but the exponents α and $\tilde{\alpha}$ are independent of R_0.

In order to complete the proofs of Theorems 1.1 and 1.3, the key lies in making clear the behavior of the solution near the singular or degenerate point of the weights, which will be achieved by combining intrinsic scaling technique developed in [19] and exponential variable substitution introduced in [22]. The first step is to establish local energy estimates in Section 3, which are the building blocks of the method of intrinsic scaling. Before that, we do some preliminary work in Section

2. Subsequently, Theorems 1.1 and 1.3 are proved in Section 4. The proofs rely on the establishments of three main ingredients required in the parabolic version of De Giorgi truncation method (see e.g. [32, 43]), which consist of the expansion of the distribution function of solution in time, the decay estimates and the oscillation improvement of the solution, see Lemmas 4.5, 4.7 and 4.8 below for finer details. Remark that the idea of intrinsic scaling is to choose suitably rescaled cylinders whose dimensions accommodate the singularity and degeneracy of the equations and weights. By working with these cylinders, we recover the homogeneity between the space and time variables such that anisotropic weighted parabolic Sobolev inequality in Proposition 2.4 can be applied to the improvement on oscillation of the solution in Lemma 4.8. By comparison, the purpose of exponential variable substitution is to produce sufficiently large time interval for the transformed solution, which makes intrinsic scaling technique be successfully used to establish the desired decay estimates and improve the oscillation of the solution in a large cylinder. Finally, we give an alternative proof for the expansion in time by making use of the logarithmic estimates in the Appendix.

It is worth emphasizing that the anisotropic weights considered here will greatly increase the difficulties of analyses and computations due to their sophisticated forms. Especially it leads to distinct differences in the process of the technical implementation compared to the unweighted case in [19, 22]. In this paper, we optimize the proof procedures as much as possible by picking the concise conditions in (1.2)–(1.3) and presenting the proofs in the style resembling the classical De Giorgi truncation method of parabolic version. These improvements contribute to deepening the readers' understanding on intrinsic scaling technique and exponential variable substitution. More importantly, we capture an explicit upper bound ε_0 on the decay rate exponent α in Theorem 1.1, which can be attained under certain condition and meanwhile reveals the damage effect of the weights on the regularity of the solution and their weakening effect on the diffusion of the flows. Our results are new and allow for further generalizations to the doubly non-linear parabolic equations with anisotropic weights in future work.

2. Preliminaries

We start by stating the following inequality.

Lemma 2.1. *For any $a, b \geq 0$ and $p > 1$, we obtain that for any $\varepsilon > 0$,*

(i) if $p > 1$ is an integer,

$$(a+b)^p \leq (1+\varepsilon)a^p + \frac{C(p)}{\varepsilon^{p-1}} b^p;$$

(ii) if $p > 1$ is not an integer,

$$(a+b)^p \leq (1+\varepsilon)a^p + \frac{C(p)}{\varepsilon^{\frac{[p]}{p-[p]}}} b^p,$$

where $[p]$ denotes the integer part of p.

For readers' convenience, the proof of Lemma 2.1 is provided in the Appendix.

For $r,p,q>1$, let $f\in C((-1,0];L^p(B_1,w_1))\cap L^r((-1,0);L^q(B_1,w_2))$. Introduce the Steklov average as follows: for $0<h\ll 1$,

$$f_h(x,t)=\begin{cases}\frac{1}{h}\int_{t-h}^{t}f(x,s)ds, & t\in(-1+h,0),\\ 0, & t\leq -1+h,\end{cases}$$

and

$$f_{\bar{h}}(x,t)=\begin{cases}\frac{1}{h}\int_{t}^{t+h}f(x,s)ds, & t\in(-1,-h),\\ 0, & t\geq -h.\end{cases}$$

Lemma 2.2. *Assume that* $(\theta_1,\theta_2),(\theta_3,\theta_4)\in\mathcal{A}\cup\mathcal{B}$*. For* $r,p,q>1$*, if*

$$f\in C((-1,0];L^p(B_1,w_1))\cap L^r((-1,0);L^q(B_1,w_2)),$$

then for any $\delta\in(0,1)$*,*

$$\sup_{t\in[-1+\delta,0]}\|(f_h-f)(\cdot,t)\|_{L^p(B_1,w_1)}\to 0,\quad \text{as } h\to 0,$$

and

$$\left(\int_{-1+\delta}^{0}\|(f_h-f)(\cdot,t)\|^r_{L^q(B_1,w_2)}dt\right)^{\frac{1}{r}}\to 0,\quad \text{as } h\to 0.$$

Proof. From Minkowski's inequality, we obtain that as $h\to 0$,

$$\begin{aligned}\|(f_h-f)(\cdot,t)\|_{L^p(B_1,w_1)}\leq&\frac{1}{h}\int_{t-h}^{t}\|f(\cdot,s)-f(\cdot,t)\|_{L^p(B_1,w_1)}ds\\ \leq&\sup_{t-h\leq s\leq t}\|f(\cdot,s)-f(\cdot,t)\|_{L^p(B_1,w_1)}\to 0.\end{aligned}$$

Similarly, using Minkowski's inequality twice, we have

$$\begin{aligned}&\left(\int_{-1+\delta}^{0}\|(f_h-f)(\cdot,t)\|^r_{L^q(B_1,w_2)}dt\right)^{\frac{1}{r}}\\ &\leq\frac{1}{h}\int_{-h}^{0}\left(\int_{-1+\delta}^{0}\|f(\cdot,t+s)-f(\cdot,t)\|^r_{L^q(B_1,w_2)}dt\right)^{\frac{1}{r}}ds\\ &\leq\sup_{-h\leq s\leq 0}\left(\int_{-1+\delta}^{0}\|f(\cdot,t+s)-f(\cdot,t)\|^r_{L^q(B_1,w_2)}dt\right)^{\frac{1}{r}}\to 0,\quad \text{as } h\to 0,\end{aligned}$$

where in the last inequality we utilized the continuity of Lebesgue integrals with respect to translations.

□

Remark that Lemmas 2.1 and 2.2 will be used in the process of establishing local energy estimates in Section 3. We next state anisotropic weighted isoperimetric inequality and parabolic Sobolev embedding theorem.

Lemma 2.3. *For* $n\geq 2$ *and* $1<p<\infty$*, let* $(\theta_1,\theta_2)\in[(\mathcal{A}\cup\mathcal{B})\cap(C_p\cup\mathcal{D}_p)]\cup\{\theta_1=0,\ \theta_2\geq n(p-1)\}$*. Then there exists some constant* $1<\tilde{p}=\tilde{p}(n,p,\theta_1,\theta_2)<p$ *such*

that for any $R>0$, $l>k$ and $u\in W^{1,\tilde{p}}(B_R,w_1)$,

$$(l-k)^{\tilde{p}}\left(\int_{\{u\geq l\}\cap B_R}w_1dx\right)^{\tilde{p}}\int_{\{u\leq k\}\cap B_R}w_1dx$$
$$\leq C(n,p,\theta_1,\theta_2)R^{\tilde{p}(n+\theta_1+\theta_2+1)}\int_{\{k<u<l\}\cap B_R}|\nabla u|^{\tilde{p}}w_1dx, \tag{2.1}$$

and

$$(l-k)^{\tilde{p}}\left(\int_{\{u\leq k\}\cap B_R}w_1dx\right)^{\tilde{p}}\int_{\{u\geq l\}\cap B_R}w_1dx$$
$$\leq C(n,p,\theta_1,\theta_2)R^{\tilde{p}(n+\theta_1+\theta_2+1)}\int_{\{k<u<l\}\cap B_R}|\nabla u|^{\tilde{p}}w_1dx,$$

where $w_1=|x'|^{\theta_1}|x|^{\theta_2}dx$.

Proof. To begin with, making use of Theorem 2.6 in [43] and the open-end property in Theorem 15.13 of [30], we obtain that if $(\theta_1,\theta_2)\in(\mathcal{A}\cup\mathcal{B})\cap(C_p\cup\mathcal{D}_p)$, there exists some constant $1<\tilde{p}=\tilde{p}(n,p,\theta_1,\theta_2)<p$ such that the weight w_1 belongs to the Muckenhoupt class $A_{\tilde{p}}$. This, together with Theorem 15.21 and Corollary 15.35 in [30], gives the following weighted Poincaré inequality: if $(\theta_1,\theta_2)\in[(\mathcal{A}\cup\mathcal{B})\cap(C_p\cup\mathcal{D}_p)]\cup\{\theta_1=0,\ \theta_2\geq n(p-1)\}$, for any $R>0$, $u\in W^{1,\tilde{p}}(B_R,w_1)$,

$$\int_{B_R}|u-\bar{u}_{B_R}|^{\tilde{p}}w_1dx\leq C(n,\tilde{p},\theta_1,\theta_2)R^{\tilde{p}}\int_{B_R}|\nabla u|^{\tilde{p}}w_1dx, \tag{2.2}$$

where $\bar{u}_{B_R}:=\frac{\int_{B_R}uw_1dx}{\int_{B_R}w_1dx}$. Applying (2.2) to the proof of Proposition 2.10 in [43], we derive that Lemma 2.3 holds.

□

Let $n\geq 2$, $p>1$, $R>0$, and $-1\leq t_1<t_2\leq 0$. For $u\in C((t_1,t_2);L^p(B_R,w_1))\cap L^p((t_1,t_2);W_0^{1,p}(B_R,w_2))$, write

$$\|u\|_{V_0^p(B_R\times(t_1,t_2),w_1,w_2)}=\left(\sup_{t\in(t_1,t_2)}\int_{B_R}|u|^pw_1dx+\int_{t_1}^{t_2}\int_{B_R}|\nabla u|^pw_2dxdt\right)^{\frac{1}{p}},$$

where w_1 and w_2 are given in (1.1). We now state anisotropic weighted parabolic Sobolev inequality as follows.

Proposition 2.4. *For $n\geq 2$, $p>1$, $R>0$, $\theta_1+\theta_2>p-n$, and $-1\leq t_1<t_2\leq 0$, let $u\in C((t_1,t_2);L^p(B_R,w_1))\cap L^p((t_1,t_2);W_0^{1,p}(B_R,w_2))$. Then*

$$\|u\|_{L^{p\chi}(B_R\times(t_1,t_2),w_2)}\leq C(n,p,\theta_1,\theta_2)\|u\|_{V_0^p(B_R\times(t_1,t_2),w_1,w_2)},$$

where $\chi=\frac{n+p+\theta_1+\theta_2}{n+\theta_1+\theta_2}$.

Proof. Observe from the anisotropic Caffarelli-Kohn-Nirenberg inequality in [41] that for any $u\in W_0^{1,p}(B_R,w_2)$,

$$\left(\int_{B_R}|u|^{\frac{p(n+\theta_1+\theta_2)}{n+\theta_1+\theta_2-p}}|x'|^{\frac{\theta_3(n+\theta_1+\theta_2)-p\theta_1}{n+\theta_1+\theta_2-p}}|x|^{\frac{\theta_4(n+\theta_1+\theta_2)-p\theta_2}{n+\theta_1+\theta_2-p}}dx\right)^{\frac{n+\theta_1+\theta_2-p}{n+\theta_1+\theta_2}}$$
$$\leq C\int_{B_R}|\nabla u|^p|x'|^{\theta_3}|x|^{\theta_4}dx,$$

which, together with Hölder's inequality, reads that

$$\begin{aligned}
&\int_{B_R} |u|^{p\chi}|x'|^{\theta_3}|x|^{\theta_4}dx\\
&=\int_{B_R} |u|^p|x'|^{\theta_3-\theta_1(\chi-1)}|x|^{\theta_4-\theta_2(\chi-1)}|u|^{p(\chi-1)}|x'|^{\theta_1(\chi-1)}|x|^{\theta_2(\chi-1)}dx\\
&\le\left(\int_{B_R} |u|^{\frac{p}{2-\chi}}|x'|^{\frac{\theta_3-\theta_1(\chi-1)}{2-\chi}}|x|^{\frac{\theta_4-\theta_2(\chi-1)}{2-\chi}}dx\right)^{2-\chi}\left(\int_{B_R}|u|^p|x'|^{\theta_1}|x|^{\theta_2}dx\right)^{\chi-1}\\
&\le C\int_{B_R}|\nabla u|^p|x'|^{\theta_3}|x|^{\theta_4}dx\left(\int_{B_R}|u|^p|x'|^{\theta_1}|x|^{\theta_2}dx\right)^{\chi-1}. \qquad (2.3)
\end{aligned}$$

Integrating (2.3) from t_1 to t_2 and using Young's inequality, we obtain

$$\begin{aligned}
\left(\int_{t_1}^{t_2}\int_{B_R}|u|^{p\chi}w_2\right)^{\frac{1}{\chi}} &\le C\left(\sup_{t\in(t_1,t_2)}\int_{B_R}|u|^pw_1dx\right)^{\frac{\chi-1}{\chi}}\left(\int_{B_R}|\nabla u|^pw_2dx\right)^{\frac{1}{\chi}}\\
&\le C\left(\sup_{t\in(t_1,t_2)}\int_{B_R}|u|^pw_1dx+\int_{t_1}^{t_2}\int_{B_R}|\nabla u|^pw_2dxdt\right).
\end{aligned}$$

The proof is finished.

□

3. Local energy estimates

For later use, we first fix some notations. For $x_0\in\mathbb{R}^n$, $t_0\in\mathbb{R}$ and $\rho,\tau>0$, let $B_\rho(x_0)$ be the ball of centre x_0 and radius ρ. We use $[(x_0,t_0)+Q(\rho,\tau)]:=B_\rho(x_0)\times(t_0-\tau,t_0)$ to denote the backward cylinder of radius ρ and height τ with vertex at (x_0,t_0). For brevity, write $B_\rho:=B_\rho(0)$ and $Q(\rho,\tau):=[(0,0)+Q(\rho,\tau)]$. For $k\in\mathbb{R}$ and $u\in C((-1,0];L^2(B_1,w_1))\cap L^p((-1,0);W^{1,p}(B_1,w_2))$, define

$$(u-k)_+=\max\{u-k,0\},\quad (u-k)_-=\max\{k-u,0\}.$$

For $E\subset B_1$ and $\widetilde{E}\subset B_1\times(-1,0]$, let

$$|E|_{\mu_{w_i}}=\int_E w_idx,\quad |\widetilde{E}|_{\nu_{w_i}}=\int_{\widetilde{E}}w_idxdt,\quad i=1,2. \qquad (3.1)$$

Local energy estimates for the truncated solution are now listed as follows.

Lemma 3.1. *Suppose that u is the solution to problem* (1.1) *with $\Omega_T=B_1\times(-1,0]$. Let $v_\pm:=(u-k)_\pm$ with $k\in\mathbb{R}$. For any $[(x_0,t_0)+Q(\rho,\tau)]\subset B_1\times(-1,0]$ and $\xi\in C^\infty([(x_0,t_0)+Q(\rho,\tau)])$ which vanishes on $\partial B_\rho(x_0)\times(t_0-\tau,t_0)$ and satisfies that $0\le\xi\le1$, we derive*

$$\begin{aligned}
&\sup_{s\in(t_0-\tau,t_0)}\left\{\int_{B_\rho(x_0)}v_\pm^2\xi^p(x,s)w_1dx+\frac{\lambda_1}{3}\int_{B_\rho(x_0)\times(t_0-\tau,s)}|\nabla(v_\pm\xi)|^pw_2dxdt\right\}\\
&\le\int_{B_\rho(x_0)}v_\pm^2\xi^p(x,t_0-\tau)w_1dx\\
&\quad+C\int_{[(x_0,t_0)+Q(\rho,\tau)]}(v_\pm^2\xi^{p-1}|\partial_t\xi|w_1+v_\pm^p(|\nabla\xi|^p+|\xi|^p)w_2)dxdt\\
&\quad+C\|\phi\|_{L^{l_0}(B_1\times(-1,0),w_2)}|[(x_0,t_0)+Q(\rho,\tau)]\cap\{v_\pm>0\}|_{\nu_{w_2}}^{1-\frac{1}{l_0}},
\end{aligned}$$

where $\phi=\phi_1+\phi_2^{\frac{p}{p-1}}+\phi_3^{\frac{p}{p-1}}$.

Proof. Without loss of generality, let $(x_0,t_0)=(0,0)$. Set $\varphi=\pm(u_h-k)_\pm\xi^p$. Then picking the test function $\varphi_{\bar{h}}$, we obtain that for any $-\tau\leq s\leq 0$,

$$\int_{B_\rho}u\varphi_{\bar{h}}w_1dx\Big|_{-\tau}^{s}+\int_{-\tau}^{s}\int_{B_\rho}(-u\partial_t\varphi_{\bar{h}}w_1+w_2\mathbf{a}(x,t,u,\nabla u)\cdot\nabla\varphi_{\bar{h}})dxdt$$
$$=\int_{-\tau}^{s}\int_{B_\rho}b(x,t,u,\nabla u)\varphi_{\bar{h}}w_2dxdt. \tag{3.2}$$

By a direct calculation, we deduce

$$\int_{-\tau}^{s}\int_{B_\rho}(\partial_tu_h\varphi w_1+w_2[\mathbf{a}(x,t,u,\nabla u)]_h\cdot\nabla\varphi)dxdt$$
$$=\int_{-\tau}^{s}\int_{B_\rho}[b(x,t,u,\nabla u)]_h\varphi w_2dxdt.$$

First, we have from integration by parts and Lemma 2.2 that

$$\int_{-\tau}^{s}\int_{B_\rho}\partial_tu_h\varphi w_1dxdt=\frac{1}{2}\int_{-\tau}^{s}\int_{B_\rho}\partial_t[(u_h-k)_\pm]^2\xi^pw_1dxdt$$
$$=\frac{1}{2}\int_{B_\rho}((u_h-k)_\pm^2\xi^p(x,s)-(u_h-k)_\pm^2\xi^p(x,-\tau))w_1dx$$
$$-\frac{p}{2}\int_{B_\rho\times(-\tau,s)}(u_h-k)_\pm^2\xi^{p-1}\partial_t\xi w_1dxdt$$
$$\to\frac{1}{2}\int_{B_\rho}((u-k)_\pm^2\xi^p(x,s)-(u-k)_\pm^2\xi^p(x,-\tau))w_1dx$$
$$-\frac{p}{2}\int_{B_\rho\times(-\tau,s)}(u-k)_\pm^2\xi^{p-1}\partial_t\xi w_1dxdt,\quad\text{as } h\to 0.$$

With regard to the remaining parts in (3.2), by first sending $h\to 0$ and then making use of the structure conditions in (**H1**)–(**H3**), we obtain from Young's inequality and Lemma 2.2 that

$$\int_{-\tau}^{s}\int_{B_\rho}w_2[\mathbf{a}(x,t,u,\nabla u)]_h\cdot\nabla\varphi dxdt$$
$$\to\int_{-\tau}^{s}\int_{B_\rho}\mathbf{a}(x,t,u,\nabla u)\cdot\big[\pm\nabla(u-k)_\pm\xi^p\pm p(u-k)_\pm\xi^{p-1}\nabla\xi\big]w_2dxdt$$
$$\geq\lambda_1\int_{-\tau}^{s}\int_{B_\rho}|\nabla(u-k)_\pm|^p\xi^pw_2dxdt-\int_{(B_\rho\times(-\tau,s))\cap\{(u-k)_\pm>0\}}\phi_1\xi^pw_2dxdt$$
$$-\lambda_2p\int_{B_\rho\times(-\tau,s)}|\nabla(u-k)_\pm|^{p-1}(u-k)_\pm\xi^{p-1}|\nabla\xi|w_2dxdt$$
$$-p\int_{B_\rho\times(-\tau,s)}(u-k)_\pm\phi_2\xi^{p-1}|\nabla\xi|w_2dxdt$$
$$\geq\frac{5\lambda_1}{6}\int_{-\tau}^{s}\int_{B_\rho}|\nabla(u-k)_\pm|^p\xi^pw_2dxdt-C\int_{B_\rho\times(-\tau,s)}(u-k)_\pm^p|\nabla\xi|^pw_2dxdt$$
$$-C\int_{(B_\rho\times(-\tau,s))\cap\{(u-k)_\pm>0\}}\big(\phi_1+\phi_2^{\frac{p}{p-1}}\big)w_2dxdt, \tag{3.3}$$

and

$$\begin{aligned}
&\int_{-\tau}^{s}\int_{B_\rho}[b(x,t,u,\nabla u)]_h\varphi w_2 \to \int_{-\tau}^{s}\int_{B_\rho}\pm b(x,t,u,\nabla u)(u-k)_\pm\xi^p w_2\\
&\leq \lambda_3\int_{B_\rho\times(-\tau,s)}|\nabla(u-k)_\pm|^{p-1}(u-k)_\pm\xi^p w_2+\int_{B_\rho\times(-\tau,s)}(u-k)_\pm\phi_3\xi^p w_2\\
&\leq \frac{\lambda_1}{6}\int_{B_\rho\times(-\tau,s)}|\nabla(u-k)_\pm|^p\xi^p w_2+C\int_{B_\rho\times(-\tau,s)}(u-k)_\pm^p\xi^p w_2\\
&\quad+\int_{(B_\rho\times(-\tau,s))\cap\{(u-k)_\pm>0\}}\phi_3^{\frac{p}{p-1}}w_2. \qquad (3.4)
\end{aligned}$$

Combining these above facts, we deduce from Hölder's inequality that for any $-\tau\leq s\leq 0$,

$$\begin{aligned}
&\int_{B_\rho}v_\pm^2\xi^p(x,s)w_1dx+\frac{2\lambda_1}{3}\int_{B_\rho\times(-\tau,s)}|\xi\nabla v_\pm|^p w_2dxdt\\
&\leq\int_{B_\rho}v_\pm^2\xi^p(x,-\tau)w_1dx+C\int_{Q(\rho,\tau)}(v_\pm^2\xi^{p-1}|\partial_t\xi|w_1+v_\pm^p(|\nabla\xi|^p+|\xi|^p)w_2)dxdt\\
&\quad+C\int_{Q(\rho,\tau)\cap\{(u-k)_\pm>0\}}\big(\phi_1+\phi_2^{\frac{p}{p-1}}+\phi_3^{\frac{p}{p-1}}\big)w_2dxdt\\
&\leq\int_{B_\rho}v_\pm^2\xi^p(x,-\tau)w_1dx+C\int_{Q(\rho,\tau)}(v_\pm^2\xi^{p-1}|\partial_t\xi|w_1+v_\pm^p(|\nabla\xi|^p+|\xi|^p)w_2)dxdt\\
&\quad+C\|\phi\|_{L^{l_0}(B_1\times(-1,0),w_2)}|Q(\rho,\tau)\cap\{v_\pm>0\}|_{\nu_{w_2}}^{1-\frac{1}{l_0}}, \qquad (3.5)
\end{aligned}$$

where $v_\pm=(u-k)_\pm$ and $\phi=\phi_1+\phi_2^{\frac{p}{p-1}}+\phi_3^{\frac{p}{p-1}}$. Then applying Lemma 2.1 with $\varepsilon=1$ to (3.5), we complete the proof of Lemma 3.1.

□

4. Local regularity for weak solutions

Denote

$$\bar{\varepsilon}_0:=(p-2)\varepsilon_0,$$

where ε_0 is given by (1.4). In light of $p>2$ and using the exponent conditions in (1.3) and (**K1**)–(**K2**), we obtain that for any $R\in(0,\frac{1}{2}]$ and $t_0\in[-\frac{1}{2},0]$, there holds

$$[(0,t_0)+Q(2R,R^{p+\vartheta-\bar{\varepsilon}_0})]\subset B_1\times(-1,0].$$

By a translation, in the following we assume without loss of generality that $(0,t_0)=(0,0)$. Let

$$\mu^+=\sup_{Q(2R,R^{p+\vartheta-\bar{\varepsilon}_0})}u,\quad \mu^-=\inf_{Q(2R,R^{p+\vartheta-\bar{\varepsilon}_0})}u,$$

and

$$\omega=\underset{Q(2R,R^{p+\vartheta-\bar{\varepsilon}_0})}{osc}u=\mu^+-\mu^-.$$

Construct the cylinder

$$Q(R,a_0R^{p+\vartheta}),\quad a_0=\left(\frac{\omega}{A}\right)^{2-p},$$

where the constant A will be determined later, which depends only upon the data. Observe that one of the following two relations must hold: either $\omega \le AR^{\varepsilon_0}$, or $\omega > AR^{\varepsilon_0}$. In the case when $\omega > AR^{\varepsilon_0}$, we have

$$Q(R, a_0 R^{p+\vartheta}) \subset Q(2R, R^{p+\vartheta-\bar{\varepsilon}_0}),$$

and thus,

$$\underset{Q(R,a_0R^{p+\vartheta})}{osc} u \le \omega. \tag{4.1}$$

Remark that this fact is generally not verifiable for such a given box since its dimensions would have to be intrinsically matched with regard to the essential oscillation of the solution inside it. So we introduce the cylinder $Q(R, R^{p+\vartheta-\bar{\varepsilon}_0})$ to guarantee that (4.1) holds under the condition of $\omega > AR^{\varepsilon_0}$. We further construct subcylinders of smaller size inside $Q(R, a_0R^{p+\vartheta})$ as follows:

$$Q(R, mR^{p+\vartheta}), \quad m = \left(\frac{\omega}{M}\right)^{2-p},$$

where $0 < M \le A$. This implies that $Q(R, mR^{p+\vartheta}) \subset Q(R, a_0R^{p+\vartheta})$.

The key to the proofs of Theorems 1.1 and 1.3 lies in achieving the following desired oscillation improvement of the solution u.

Proposition 4.1. *The constants A, M (and then a_0, m) can be determined and there exists two constants $\kappa_* > 1$ and $1 < c_* \le A^{p-2}$, both depending only on the data such that we have either $\omega \le AR^{\varepsilon_0}$, or*

(i) if

$$|B_R \cap \{u(\cdot, -c_*\omega^{2-p}R^{p+\vartheta}) > \mu^+ - 2^{-1}\omega\}|_{\mu_{w_1}} \le 2^{-1}|B_R|_{\mu_{w_1}}, \tag{4.2}$$

then

$$u(x,t) \le \mu^+ - 2^{-\kappa_*}\omega, \quad \text{for } (x,t) \in Q(R/2, m(R/2)^{p+\vartheta}); \tag{4.3}$$

(ii) if

$$|B_R \cap \{u(\cdot, -c_*\omega^{2-p}R^{p+\vartheta}) < \mu^- + 2^{-1}\omega\}|_{\mu_{w_1}} \le 2^{-1}|B_R|_{\mu_{w_1}}, \tag{4.4}$$

then

$$u(x,t) \ge \mu^- + 2^{-\kappa_*}\omega, \quad \text{for } (x,t) \in Q(R/2, m(R/2)^{p+\vartheta}). \tag{4.5}$$

Remark 4.2. First, we would like to point out that the values of the constants A, M are given by (4.20)–(4.21) below. Second, since $\frac{p+\vartheta}{p-2} > \varepsilon_0$ and $A \ge c_*^{\frac{1}{p-2}}$, then we have

$$c_*\omega^{2-p}R^{p+\vartheta} < 1, \quad \text{if } \omega > AR^{\varepsilon_0} > c_*^{\frac{1}{p-2}} R^{\frac{p+\vartheta}{p-2}},$$

which indicates that the starting time $-c_*\omega^{2-p}R^{p+\vartheta}$ lies in the interval $(-1, 0]$ and then the assumed conditions in (4.2) and (4.4) are valid.

4.1. **The proof of Proposition 4.1.** In the following we only give the proof of (4.3). The proof of (4.5) is similar and thus omitted. We now begin with recalling a measure lemma as follows.

Lemma 4.3 (see Lemma 2.1 in [43])**.** *$d\mu := |x'|^{\theta_1}|x|^{\theta_2}dx$ is a Radon measure if $(\theta_1, \theta_2) \in \mathcal{A} \cup \mathcal{B}$. Moreover, $C^{-1}R^{n+\theta_1+\theta_2} \le \mu(B_R) \le CR^{n+\theta_1+\theta_2}$ for any $R > 0$ and some constant $C = C(n, \theta_1, \theta_2) > 0$.*

Making use of Lemma 4.3, we prove the following switch lemma from the measures w_1dxdt to w_2dxdt.

Lemma 4.4. *Assume that* $(\theta_1,\theta_2),(\theta_3,\theta_4)\in\mathcal{A}\cup\mathcal{B}$ *and* $\theta_3+\min\{0,\theta_4\}>1-n$. *There exists a positive constant* $C_0=C_0(n,p,\theta_1,\theta_2,\theta_3,\theta_4)$ *such that for any* $\varepsilon,\rho\in(0,1/2]$ *and* $\widetilde{E}\subset Q(\rho,\rho^{p+\vartheta})$, *if*

$$\frac{|\widetilde{E}|_{\nu_{w_1}}}{|Q(\rho,\rho^{p+\vartheta})|_{\nu_{w_1}}}\leq\varepsilon^{\beta},\tag{4.6}$$

then

$$\frac{|\widetilde{E}|_{\nu_{w_2}}}{|Q(\rho,\rho^{p+\vartheta})|_{\nu_{w_2}}}\leq C_0\varepsilon^{n-1+\theta_3+\min\{0,\theta_4\}},$$

where $\beta=n-1+\theta_3+\min\{0,\theta_4\}+\max\{0,\theta_1-\theta_3\}+\max\{0,\theta_2-\theta_4\}$, *the measures* ν_{w_i}, $i=1,2$ *are defined by* (3.1).

Proof. Observe from Lemma 4.3 that

$$\frac{|\widetilde{E}|_{\nu_{w_2}}}{|Q(\rho,\rho^{p+\vartheta})|_{\nu_{w_2}}}\leq\frac{C}{\rho^{n+p+\theta_1+\theta_2}}\bigg(\int_{\widetilde{E}\cap\{|x'|<\varepsilon\rho\}}w_2dxdt+\int_{\widetilde{E}\cap\{|x'|\geq\varepsilon\rho\}}w_2dxdt\bigg).$$

For brevity, write

$$I_1:=\int_{\widetilde{E}\cap\{|x'|<\varepsilon\rho\}}w_2dxdt,\quad I_2:=\int_{\widetilde{E}\cap\{|x'|\geq\varepsilon\rho\}}w_2dxdt.$$

For the first term I_1, we discuss as follows:

(i) if $\theta_4\geq0$, then

$$I_1\leq C\rho^{\theta_4}\int_{\widetilde{E}\cap\{|x'|<\varepsilon\rho\}}|x'|^{\theta_3}dxdt\leq C\varepsilon^{n-1+\theta_3}\rho^{n+p+\theta_1+\theta_2};$$

(ii) if $\theta_4<0$, then

$$I_1\leq\int_{\widetilde{E}\cap\{|x'|<\varepsilon\rho\}}|x'|^{\theta_3+\theta_4}dxdt\leq C\varepsilon^{n-1+\theta_3+\theta_4}\rho^{n+p+\theta_1+\theta_2}.$$

As for the second term I_2, we deduce from (4.6) that

(i) if $\theta_1\geq\theta_3$, then

$$I_2\leq(\varepsilon\rho)^{\theta_3-\theta_1}\int_{\widetilde{E}\cap\{|x'|\geq\varepsilon\rho\}}|x'|^{\theta_1}|x|^{\theta_4}dxdt,$$

which implies that for $\theta_2\geq\theta_4$,

$$I_2\leq(\varepsilon\rho)^{-\vartheta}|\widetilde{E}|_{\nu_{w_1}}\leq C\varepsilon^{\beta-\vartheta}\rho^{n+p+\theta_1+\theta_2},$$

and, for $\theta_2<\theta_4$,

$$I_2\leq\varepsilon^{\theta_3-\theta_1}\rho^{-\vartheta}|\widetilde{E}|_{\nu_{w_1}}\leq C\varepsilon^{\beta-\theta_1+\theta_3}\rho^{n+p+\theta_1+\theta_2};$$

(ii) if $\theta_1<\theta_3$, then

$$I_2\leq\rho^{\theta_3-\theta_1}\int_{\widetilde{E}\cap\{|x'|\geq\varepsilon\rho\}}|x'|^{\theta_1}|x|^{\theta_4}dxdt,$$

which leads to that for $\theta_2\geq\theta_4$,

$$I_2\leq\varepsilon^{\theta_4-\theta_2}\rho^{-\vartheta}|\widetilde{E}|_{\nu_{w_1}}\leq C\varepsilon^{\beta-\theta_2+\theta_4}\rho^{n+p+\theta_1+\theta_2},$$

and, for $\theta_2 < \theta_4$,

$$I_2 \leq \rho^{-\vartheta} |\widetilde{E}|_{\nu_{w_1}} \leq C\varepsilon^{\beta} \rho^{n+p+\theta_1+\theta_2}.$$

Therefore, combining these above facts, we have

$$\frac{|\widetilde{E}|_{\nu_{w_2}}}{|Q(\rho, \rho^{p+\vartheta})|_{\nu_{w_2}}} \leq C\varepsilon^{n-1+\theta_3+\min\{0,\theta_4\}}.$$

□

In order to complete the proof of Proposition 4.1, we need to establish the following three main lemmas required in the De Giorgi truncation method of parabolic version. To begin with, we make use of the energy estimates to prove that the distribution function of u expands in the time variable, which is also the first step in the technical implementation for the expansion of positivity developed in [22], see Chapter 4 in [24] for more detailed explanations.

Lemma 4.5. *Then there exists a constant $\delta_0 \in (0, 1)$ depending only on the data and independent of ω such that we obtain either $\omega \leq R^{\varepsilon_0}$, or if for some $\bar{t} \in [-1, -\delta_0 \omega^{2-p} R^{p+\vartheta}]$,*

$$|B_R \cap \{u(\cdot, \bar{t}) > \mu^+ - 2^{-1}\omega\}|_{\mu_{w_1}} \leq 2^{-1} |B_R|_{\mu_{w_1}}, \tag{4.7}$$

then

$$|B_R \cap \{u(\cdot, t) > \mu^+ - 2^{-c_0}\omega\}|_{\mu_{w_1}} \leq \frac{3}{4} |B_R|_{\mu_{w_1}}, \quad \forall\, t \in [\bar{t}, \bar{t} + \delta_0 \omega^{2-p} R^{p+\vartheta}],$$

where c_0 is given by

$$c_0 := 1 - \frac{\ln \frac{\sqrt{5}-2}{2\sqrt{5}}}{\ln 2}. \tag{4.8}$$

Remark 4.6. An alternative proof for the similar expansion in time, which involves application of the logarithmic estimates, is left in the Appendix.

Proof. For $\delta_0 \in (0, 1]$ and $k \in [\mu^-, \mu^+]$, write

$$A^{\delta_0}(k, R) = (B_R \times [\bar{t}, \bar{t} + \delta_0 \omega^{2-p} R^{p+\vartheta}]) \cap \{u > k\}.$$

Choose a smooth cut-off function $\zeta \in C_0^\infty(B_R)$ satisfying that $\zeta = 1$ in $B_{(1-\varrho)R}$, and $0 \leq \zeta \leq 1$, $|\nabla \zeta| \leq \frac{2}{\varrho R}$ in B_R, where $\varrho \in (0, 1)$ is to be determined later. For brevity, denote $v = (u - (\mu^+ - 2^{-1}\omega))_+$. In light of (4.7) and $\delta_0 \in (0, 1]$, it follows from Lemmas 3.1 and 4.3 that if $\omega > R^{\varepsilon_0}$,

$$\begin{aligned}
&\sup_{t\in(\bar t,\bar t+\delta_0 R^{p+\vartheta})}\int_{B_{(1-\varrho)R}} v^2 w_1 dx\\
&\leq \int_{B_R} v^2(x,\bar t)\zeta^p(x) w_1 dx + C\int_{B_R\times[\bar t,\bar t+\delta_0\omega^{2-p}R^{p+\vartheta}]} v^p(|\nabla\zeta|^p+|\zeta|^p)w_2 dxdt\\
&\quad + C\big|A^{\delta_0}(\mu^+-2^{-1}\omega,R)\big|_{\nu_{w_2}}^{1-\frac{1}{l_0}}\\
&\leq \frac{\omega^2}{8}|B_R|_{\mu_{w_1}} + \frac{C\omega^p}{(\varrho R)^p}\big|A^{\delta_0}(\mu^+-2^{-1}\omega,R)\big|_{\nu_{w_2}} + C\big|A^{\delta_0}(\mu^+-2^{-1}\omega,R)\big|_{\nu_{w_2}}^{1-\frac{1}{l_0}}\\
&\leq \frac{\omega^2}{8}|B_R|_{\mu_{w_1}} + C\left(\frac{\omega^{\frac{p(l_0-1)+2}{l_0}}}{\varrho^p R^{\frac{p(l_0-1)-n-\theta_1-\theta_2}{l_0}}}+1\right)\big|A^{\delta_0}(\mu^+-2^{-1}\omega,R)\big|_{\nu_{w_2}}^{1-\frac{1}{l_0}}\\
&\leq \frac{\omega^2}{4}|B_R|_{\mu_{w_1}}\left(\frac12+\frac{C}{\varrho^p}\left(\frac{|A^{\delta_0}(\mu^+-2^{-1}\omega,R)|_{\nu_{w_2}}}{|B_R\times[\bar t,\bar t+\omega^{2-p}R^{p+\vartheta}]|_{\nu_{w_2}}}\right)^{1-\frac{1}{l_0}}\right). \qquad (4.9)
\end{aligned}$$

For every $t\in[\bar t,\bar t+\delta_0\omega^{2-p}R^{p+\vartheta}]$, we have

$$\begin{aligned}
&\int_{B_{(1-\varrho)R}} v^2(x,t)w_1 dx\\
&\geq \frac{\omega^2}{4}\big(1-2^{-(c_0-1)}\big)^2|B_{(1-\varrho)R}\cap\{u(\cdot,t)>\mu^+-2^{-c_0}\omega\}|_{\mu_{w_1}},
\end{aligned}$$

where c_0 is defined by (4.8). Substituting this into (4.9), we obtain

$$\begin{aligned}
&|B_{(1-\varrho)R}\cap\{u(\cdot,t)>\mu^+-2^{-c_0}\omega\}|_{\mu_{w_1}}\\
&\leq \frac{|B_R|_{\mu_{w_1}}}{(1-2^{-(c_0-1)})^2}\left(\frac12+\frac{C}{\varrho^p}\left(\frac{|A^{\delta_0}(\mu^+-2^{-1}\omega,R)|_{\nu_{w_2}}}{|B_R\times[\bar t,\bar t+\omega^{2-p}R^{p+\vartheta}]|_{\nu_{w_2}}}\right)^{1-\frac{1}{l_0}}\right).
\end{aligned}$$

Observe that $|B_R\setminus B_{(1-\varrho)R}|_{\mu_{w_1}}\leq C\varrho|B_R|_{\mu_{w_1}}$. Take

$$\varrho=\left(\frac{|A^{\delta_0}(\mu^+-2^{-1}\omega,R)|_{\nu_{w_2}}}{|B_R\times[\bar t,\bar t+\omega^{2-p}R^{p+\vartheta}]|_{\nu_{w_2}}}\right)^{\frac{l_0-1}{l_0(p+1)}}.$$

In view of the value of c_0 in (4.8), we have

$$\begin{aligned}
&\frac{|B_R\cap\{u(\cdot,t)>\mu^+-2^{-c_0}\omega\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}}\\
&\leq \frac{1}{2(1-2^{-(c_0-1)})^2}+\frac{C}{\varrho^p}\left[\varrho^{p+1}+\left(\frac{|A^{\delta_0}(\mu^+-2^{-1}\omega,R)|_{\nu_{w_2}}}{|B_R\times[\bar t,\bar t+\omega^{2-p}R^{p+\vartheta}]|_{\nu_{w_2}}}\right)^{1-\frac{1}{l_0}}\right]\\
&\leq \frac58+C_*\left(\frac{|A^{\delta_0}(\mu^+-2^{-1}\omega,R)|_{\nu_{w_2}}}{|B_R\times[\bar t,\bar t+\omega^{2-p}R^{p+\vartheta}]|_{\nu_{w_2}}}\right)^{\frac{l_0-1}{l_0(p+1)}}\leq\frac58+C_*\delta_0^{\frac{l_0-1}{l_0(p+1)}}.
\end{aligned}$$

By picking $\delta_0=(8C_*)^{-\frac{l_0-1}{l_0(p+1)}}$, we obtain

$$\frac{|B_R\cap\{u(\cdot,t)>\mu^+-2^{-c_0}\omega\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}}\leq\frac34.$$

The proof is complete.

□

From the assumed condition in (4.7), we obtain that for any $\sigma \in (0,1]$,

$$|B_R \cap \{u(\cdot,\bar{t}) > \mu^+ - 2^{-1}\sigma\omega\}|_{\mu_{w_1}} \leq 2^{-1}|B_R|_{\mu_{w_1}}. \tag{4.10}$$

Since the constant δ_0 captured in Lemma 4.5 is independent of ω, then by using (4.10) and replacing ω with $\sigma\omega$ in the above proof, we have either $\omega \leq \sigma^{-1}R^{\varepsilon_0}$, or

$$|B_R \cap \{u(\cdot,\bar{t}+\delta_0(\sigma\omega)^{2-p}R^{p+\vartheta}) > \mu^+ - 2^{-c_0}\sigma\omega\}|_{\mu_{w_1}} \leq \frac{3}{4}|B_R|_{\mu_{w_1}}, \tag{4.11}$$

where c_0 is defined by (4.8) and $\bar{t} \in [-1, -\delta_0(\sigma\omega)^{2-p}R^{p+\vartheta}]$. Introduce the following exponential variable substitution: for $\tau \in [0,\infty)$,

$$\sigma = e^{-\frac{\tau}{p-2}}, \quad \tilde{u}(x,\tau) = \frac{e^{\frac{\tau}{p-2}}}{\omega}(\delta_0 R^{p+\vartheta})^{\frac{1}{p-2}} u(x,\bar{t}+e^{\tau}\omega^{2-p}\delta_0 R^{p+\vartheta}). \tag{4.12}$$

Define $\tilde{\mu}^{\pm} := \frac{e^{\frac{\tau}{p-2}}}{\omega}(\delta_0 R^{p+\vartheta})^{\frac{1}{p-2}}\mu^{\pm}$. Then (4.11) implies that for any $\tau \geq 0$,

$$|B_R \cap \{\tilde{u}(\cdot,\tau) > \tilde{\mu}^+ - \kappa_0\}|_{\mu_{w_1}} \leq \frac{3}{4}|B_R|_{\mu_{w_1}}, \quad \kappa_0 := \frac{(\delta_0 R^{p+\vartheta})^{\frac{1}{p-2}}}{2^{c_0}}. \tag{4.13}$$

For simplicity, denote

$$\mathcal{G} := \mathcal{G}(\tau) = \frac{e^{\frac{\tau}{p-2}}}{\omega}(\delta_0 R^{p+\vartheta})^{\frac{1}{p-2}}.$$

A direct calculation shows that

$$\begin{aligned} w_1\partial_\tau\tilde{u} =& \mathcal{G}^{p-1}w_1\partial_t u + \frac{\mathcal{G}w_1}{p-2}u \\ =& \mathcal{G}^{p-1}[\operatorname{div}(w_2\mathbf{a}(x,t,u,\nabla u)) + w_2\tilde{b}(x,t,u,\nabla u)] + \frac{\tilde{u}w_1}{p-2} \\ =&: \operatorname{div}(w_2\tilde{\mathbf{a}}(x,\tau,\tilde{u},\nabla\tilde{u})) + w_2\tilde{b}(x,\tau,\tilde{u},\nabla\tilde{u}) + \frac{\tilde{u}w_1}{p-2}, \end{aligned}$$

where $\tilde{\mathbf{a}} : B_1 \times \mathbb{R}^+ \times \mathbb{R}^- \times \mathbb{R}^n \to \mathbb{R}^n$ and $\tilde{b} : B_1 \times \mathbb{R}^+ \times \mathbb{R}^- \times \mathbb{R}^n \to \mathbb{R}$ are subject to the following structure conditions:

$$\begin{cases} \tilde{\mathbf{a}}(x,\tau,\tilde{u},\nabla\tilde{u}) \cdot \nabla\tilde{u} \geq \lambda_1|\nabla\tilde{u}|^p - \tilde{\phi}_1(x,\tau), \\ |\tilde{\mathbf{a}}(x,\tau,\tilde{u},\nabla\tilde{u})| \leq \lambda_2|\nabla\tilde{u}|^{p-1} + \tilde{\phi}_2(x,\tau), \\ |\tilde{b}(x,\tau,\tilde{u},\nabla\tilde{u})| \leq \lambda_3|\nabla\tilde{u}|^{p-1} + \tilde{\phi}_3(x,\tau). \end{cases}$$

Here $\tilde{\phi}_i$, $i = 1,2,3$ are given by

$$\tilde{\phi}_1(x,\tau) = \mathcal{G}^p\bar{\phi}_1(x,\tau), \quad \tilde{\phi}_i(x,\tau) = \mathcal{G}^{p-1}\bar{\phi}_i(x,\tau), \quad i = 2,3,$$

where

$$\bar{\phi}_i(x,\tau) = \phi_i(x,\bar{t}+e^{\tau}\omega^{2-p}\delta_0 R^{p+\vartheta}), \quad i = 1,2,3.$$

Similarly as before, write $\tilde{\phi} := \tilde{\phi}_1 + \tilde{\phi}_2^{\frac{p}{p-1}} + \tilde{\phi}_3^{\frac{p}{p-1}}$. Remark that the admissible time interval corresponding to the transformed solution $\tilde{u}$ becomes the infinite interval $[0,\infty)$, which allows us to establish the decay estimates and achieve the pointwise oscillation improvement of the solution over a large cylinder in the following.

Define $\widetilde{m} := (\frac{\kappa_0}{2^{j_*}})^{2-p}$, where $j_* \geq 1$ will be chosen later. For $R \in (0,\frac{1}{2}]$, we introduce the forward cylinders as follows:

$$Q^+(2R,\widetilde{m}(3R)^{p+\vartheta}) := B_{2R} \times (0,\widetilde{m}(3R)^{p+\vartheta}],$$

and

$$\mathcal{Q}_R^+(\widetilde{m}) := B_R \times (\widetilde{m}R^{p+\vartheta}, \widetilde{m}(3R)^{p+\vartheta}], \quad \text{for } 0 < R \leq \frac{1}{2}.$$

We now establish the decaying estimates for the distribution function of $\tilde{u}$ as follows.

Lemma 4.7. *There exists a constant $\overline{C}_0 > 1$ depending only on the data and independent of j_*, ω such that we have either $\omega \leq \mathcal{N}_* R^{\varepsilon_0}$ for some constant $\mathcal{N}_* := \mathcal{N}_*(j_*, \mathrm{data}) > 1$, or*

$$\frac{|\mathcal{Q}_R^+(\widetilde{m}) \cap \{\tilde{u} > \tilde{\mu}^+ - \frac{\kappa_0}{2^{j_*}}\}|_{\nu_{w_1}}}{|\mathcal{Q}_R^+(\widetilde{m})|_{\nu_{w_1}}} \leq \frac{\overline{C}_0}{j_*^{\frac{p-\tilde{p}}{p\tilde{p}}}},$$

where $\tilde{p} \in (1, p)$ is given by Lemma 2.3, κ_0 is defined by (4.13).

Proof. For $i \geq 0$, write $k_i = \tilde{\mu}^+ - \frac{\kappa_0}{2^i}$ and

$$A_i(\tau) = B_R \cap \{\tilde{u}(\cdot, \tau) > k_i\}, \quad A_i = \mathcal{Q}_R^+(\widetilde{m}) \cap \{\tilde{u} > k_i\}.$$

Making use of (4.13), we have from Lemma 4.3 that

$$|B_R \setminus A_i(\tau)|_{\mu_{w_1}} \geq \frac{1}{4}|B_R|_{\mu_{w_1}} \geq C(n, \theta_1, \theta_2) R^{n+\theta_1+\theta_2}. \tag{4.14}$$

Using (2.1), we have

$$\begin{aligned}
&(k_{i+1} - k_i)^{\tilde{p}} |A_{i+1}(\tau)|_{\mu_{w_1}}^{\tilde{p}} |B_R \setminus A_i(\tau)|_{\mu_{w_1}} \\
&\leq C(n, p, \theta_1, \theta_2) R^{\tilde{p}(n+\theta_1+\theta_2+1)} \int_{A_i(\tau)\setminus A_{i+1}(\tau)} |\nabla \tilde{u}|^{\tilde{p}} w_1 dx,
\end{aligned}$$

which, together with (4.14), reads that

$$|A_{i+1}(\tau)|_{\mu_{w_1}} \leq \frac{C2^i}{\kappa_0} R^{\frac{(n+\theta_1+\theta_2)(\tilde{p}-1)}{\tilde{p}}+1} \left(\int_{A_i(\tau)\setminus A_{i+1}(\tau)} |\nabla \tilde{u}|^{\tilde{p}} w_1 dx \right)^{\frac{1}{\tilde{p}}}.$$

Integrating this from $\widetilde{m}R^{p+\vartheta}$ to $\widetilde{m}(3R)^{p+\vartheta}$ and utilizing Hölder's inequality, we deduce

$$|A_{i+1}|_{\nu_{w_1}} \leq \frac{C2^i \widetilde{m}^{\frac{\tilde{p}-1}{\tilde{p}}}}{\kappa_0} R^{\frac{(n+p+\vartheta+\theta_1+\theta_2)(\tilde{p}-1)}{\tilde{p}}+1} \left(\int_{A_i\setminus A_{i+1}} |\nabla \tilde{u}|^{\tilde{p}} w_1 dx d\tau \right)^{\frac{1}{\tilde{p}}}.$$

In view of $1 < \tilde{p} < p$ and using Hölder's inequality again, we derive

$$\begin{aligned}
&\left(\int_{A_i\setminus A_{i+1}} |\nabla \tilde{u}|^{\tilde{p}} w_1 dx d\tau \right)^{\frac{1}{\tilde{p}}} \\
&\leq \left(\int_{A_i\setminus A_{i+1}} |\nabla \tilde{u}|^p w_2 dx d\tau \right)^{\frac{1}{p}} \left(\int_{A_i\setminus A_{i+1}} |x'|^{\frac{p\theta_1-\tilde{p}\theta_3}{p-\tilde{p}}} |x|^{\frac{p\theta_2-\tilde{p}\theta_4}{p-\tilde{p}}} dx d\tau \right)^{\frac{p-\tilde{p}}{p\tilde{p}}} \\
&\leq R^{\frac{\vartheta}{p}} |A_i \setminus A_{i+1}|_{\nu_{w_1}}^{\frac{p-\tilde{p}}{p\tilde{p}}} \left(\int_{\mathcal{Q}_R^+(\widetilde{m})} |\nabla(\tilde{u} - k_i)_+|^p w_2 dx d\tau \right)^{\frac{1}{p}}.
\end{aligned}$$

For simplicity, denote

$$\rho_0 = 2R, \quad \varsigma_0 = \widetilde{m}(3R)^{p+\vartheta}, \quad \text{and then } Q^+(\rho_0, \varsigma_0) = Q^+(2R, \widetilde{m}(3R)^{p+\vartheta}).$$

Take a smooth cutoff function $\xi \in C^\infty(Q^+(\rho_0,\varsigma_0))$ such that

$$\begin{cases} 0 \le \xi \le 1, & \text{in } Q^+(\rho_0,\varsigma_0), \\ \xi = 1, & \text{in } \mathcal{Q}_R^+(\widetilde{m}), \\ \xi = 0, & \text{on } \partial_{pa} Q^+(\rho_0,\varsigma_0), \\ |\nabla\xi| \le \frac{2}{R}, \quad |\partial_\tau \xi| \le \frac{2}{\widetilde{m}R^{p+\vartheta}}, \end{cases}$$

where $\partial_{pa}Q^+(\rho_0,\varsigma_0)$ represents the parabolic boundary of $Q^+(\rho_0,\varsigma_0)$. Applying the proof of Lemma 3.1 to $\tilde{u}$, we deduce from (**K2**) and Lemma 4.3 that if $\omega > \mathcal{N}_*(j_*,\mathrm{data})R^{\varepsilon_0}$,

$$\begin{aligned} &\int_{\mathcal{Q}_R^+(\widetilde{m})} |\nabla(\tilde{u}-k_i)_+|^p w_2 dx d\tau \le \int_{Q^+(\rho_0,\varsigma_0)} |\nabla((\tilde{u}-k_i)_+\xi)|^p w_2 dx d\tau \\ &\le C\int_{Q^+(\rho_0,\varsigma_0)} \big((\tilde{u}-k_i)_+^2 |\partial_\tau\xi||x'|^{\theta_1-\theta_3}|x|^{\theta_2-\theta_4} + (\tilde{u}-k_i)_+^p(|\nabla\xi|^p+|\xi|^p)\big) w_2 dx d\tau \\ &\quad + C\|\tilde{\phi}\|_{L^{l_0}(Q^+(\rho_0,\varsigma_0),w_2)} |Q^+(\rho_0,\varsigma_0)\cap\{\tilde{u}>k_i\}|_{\nu_{w_2}}^{1-\frac{1}{l_0}} \\ &\le \frac{C\kappa_0^p}{2^{pi}R^p}|Q^+(\rho_0,\varsigma_0)\cap\{\tilde{u}>k_i\}|_{\nu_{w_2}} \\ &\quad + C\mathcal{G}(\widetilde{m}(3R)^{p+\vartheta})\|\bar{\phi}\|_{L^{l_0}(Q^+(\rho_0,\varsigma_0),w_2)} |Q^+(\rho_0,\varsigma_0)\cap\{\tilde{u}>k_i\}|_{\nu_{w_2}}^{1-\frac{1}{l_0}} \\ &\le \left(\frac{C2^{\frac{j_*(p-2)}{l_0}}}{2^{pi}R^{\frac{p(l_0-1)-n-\theta_1-\theta_2}{l_0}}} + \left(\frac{\mathcal{N}_*}{\omega}\right)^{\frac{p(l_0-1)+2}{l_0}}\right) \frac{|Q^+(\rho_0,\varsigma_0)\cap\{\tilde{u}>k_i\}|_{\nu_{w_2}}^{1-\frac{1}{l_0}}}{R^{-\frac{(p+\vartheta)(p(l_0-1)+2)}{l_0(p-2)}}} \\ &\le \frac{C\widetilde{m}\kappa_0^p}{2^{pi}}R^{n+\theta_1+\theta_2}, \end{aligned}$$

where $C := C(\mathrm{data})$, $\mathcal{N}_* := \mathcal{N}_*(j_*,\mathrm{data})$, and we also used the fact that

$$\mathcal{G}(\widetilde{m}(3R)^{p+\vartheta}) \le \frac{\mathcal{C}_* R^{\frac{p(p+\vartheta)}{p-2}}}{\omega^p}, \quad \mathcal{C}_* := \mathcal{C}_*(j_*,\mathrm{data}) > 1, \tag{4.15}$$

and

$$\|\bar{\phi}\|_{L^{l_0}(Q^+(\rho_0,\varsigma_0),w_2)} \le \left(\frac{\omega^{p-2}}{\delta_0 R^{p+\vartheta}}\right)^{\frac{1}{l_0}} \|\phi\|_{L^{l_0}(Q(1,1),w_2)}. \tag{4.16}$$

A combination of these above facts shows that

$$|A_{i+1}|_{\nu_{w_1}} \le C|A_i\setminus A_{i+1}|_{\nu_{w_1}}^{\frac{p-\tilde{p}}{p\tilde{p}}} |\mathcal{Q}_R^+(\widetilde{m})|_{\nu_{w_1}}^{\frac{p\tilde{p}+\tilde{p}-p}{p\tilde{p}}}.$$

Hence we obtain that for $j \ge 1$,

$$j|A_j|_{\nu_{w_1}}^{\frac{p\tilde{p}}{p-\tilde{p}}} \le \sum_{i=0}^{j-1} |A_{i+1}|_{\nu_{w_1}}^{\frac{p\tilde{p}}{p-\tilde{p}}} \le C|\mathcal{Q}_R^+(\widetilde{m})|_{\nu_{w_1}}^{\frac{p\tilde{p}}{p-\tilde{p}}}.$$

The proof is complete.

□

Utilizing Lemma 4.7, we obtain the following pointwise oscillation improvement for the solution $\tilde{u}$.

Lemma 4.8. *The constant j_* can be chosen depending only on the data and independent of ω such that we derive either $\omega \leq A_0 R^{\varepsilon_0}$ for some large constant $A_0 := A_0(\text{data}) > 1$, or*

$$\tilde{u}(x,\tau) \leq \tilde{\mu}^+ - \frac{\kappa_0}{2^{j_*+1}}, \quad \text{for } (x,\tau) \in B_{R/2} \times (\widetilde{m}(2R)^{p+\vartheta}, \widetilde{m}(3R)^{p+\vartheta}],$$

where κ_0 is defined by (4.13).

Proof. For $i = 0, 1, 2, ...,$ write

$$r_i = \frac{R}{2} + \frac{R}{2^{i+1}}, \quad \tilde{r}_i = \big(2^{-i} + 2^{p+\vartheta}(1 - 2^{-i})\big)R^{p+\vartheta},$$

and

$$k_i = \tilde{\mu}^+ - \frac{\kappa_0}{2^{j_*+1}} - \frac{\kappa_0}{2^{j_*+1+i}}.$$

Denote $\widetilde{\mathcal{Q}}_i^+(\widetilde{m}) := B_{r_i} \times (\widetilde{m}\tilde{r}_i, \widetilde{m}(3R)^{p+\vartheta}]$. Choose a cutoff function $\xi_i \in C^\infty(\widetilde{\mathcal{Q}}_i^+(m))$ such that

$$\begin{cases} 0 \leq \xi_i \leq 1, & \text{in } \widetilde{\mathcal{Q}}_i^+(\widetilde{m}),\\ \xi_i = 1, & \text{in } \widetilde{\mathcal{Q}}_{i+1}^+(\widetilde{m}),\\ \xi_i = 0, & \text{on } \partial_{pa}\widetilde{\mathcal{Q}}_i^+(\widetilde{m}),\\ |\nabla \xi_i| \leq \frac{2^{i+3}}{R}, \quad |\partial_\tau \xi_i| \leq \frac{2^{i+2}}{(2^{p+\vartheta}-1)\widetilde{m}R^{p+\vartheta}}, \end{cases}$$

where $\partial_{pa}\widetilde{\mathcal{Q}}_i^+(\widetilde{m})$ denotes the parabolic boundary of $\widetilde{\mathcal{Q}}_i^+(\widetilde{m})$. In light of (**K2**), Lemma 4.3 and (4.15)–(4.16), it follows from the proof of Lemma 3.1 with a slight modification that

$$\begin{aligned}
&\sup_{\tau \in (\widetilde{m}r_i^{p+\vartheta}, \widetilde{m}(3R)^{p+\vartheta})} m\int_{B_{r_i}} (\tilde{u} - k_i)_+^p \xi_i^p w_1 dx + \frac{\lambda_1}{3}\int_{\widetilde{\mathcal{Q}}_i^+(\widetilde{m})} |\nabla((\tilde{u} - k_i)_+ \xi_i)|^p w_2 dx d\tau\\
&\leq \sup_{\tau \in (\widetilde{m}r_i^{p+\vartheta}, \widetilde{m}(3R)^{p+\vartheta})} \int_{B_{r_i}} (\tilde{u} - k_i)_+^2 \xi_i^p w_1 dx + \frac{\lambda_1}{3}\int_{\widetilde{\mathcal{Q}}_i^+(\widetilde{m})} |\nabla((\tilde{u} - k_i)_+ \xi_i)|^p w_2 dx d\tau\\
&\leq C\int_{\widetilde{\mathcal{Q}}_i^+(\widetilde{m})} \big((\tilde{u} - k_i)_+^2 |\partial_\tau \xi_i||x'|^{\theta_1-\theta_3}|x|^{\theta_2-\theta_4} + (\tilde{u} - k_i)_+^p |\nabla \xi_i|^p\big) w_2 dx d\tau\\
&\quad + C\|\tilde{\phi}\|_{L^{l_0}(\widetilde{\mathcal{Q}}_i^+(\widetilde{m}), w_2)} |\widetilde{\mathcal{Q}}_i^+(\widetilde{m}) \cap \{\tilde{u} > k_i\}|_{\nu_{w_2}}^{1-\frac{1}{l_0}}\\
&\leq \frac{C2^{pi}}{R^p}\left(\frac{\kappa_0}{2^{j_*}}\right)^p |\widetilde{\mathcal{Q}}_i^+(\widetilde{m}) \cap \{\tilde{u} > k_i\}|_{\nu_{w_2}}\\
&\quad + C\mathcal{G}(\widetilde{m}(3R)^{p+\vartheta})\|\bar{\phi}\|_{L^{l_0}(\widetilde{\mathcal{Q}}_i^+(\widetilde{m}), w_2)} |\widetilde{\mathcal{Q}}_i^+(\widetilde{m}) \cap \{\tilde{u} > k_i\}|_{\nu_{w_2}}^{1-\frac{1}{l_0}}\\
&\leq \frac{C2^{pi}}{R^p}\left(\frac{\kappa_0}{2^{j_*}}\right)^p |\widetilde{\mathcal{Q}}_i^+(\widetilde{m}) \cap \{\tilde{u} > k_i\}|_{\nu_{w_2}}\\
&\quad + \frac{\overline{\mathcal{C}}_* R^{\frac{(p+\vartheta)(p(l_0-1)+2)}{l_0(p-2)}}}{\omega^{\frac{p(l_0-1)+2}{l_0}}} |\widetilde{\mathcal{Q}}_i^+(\widetilde{m}) \cap \{\tilde{u} > k_i\}|_{\nu_{w_2}}^{1-\frac{1}{l_0}},
\end{aligned} \tag{4.17}$$

where $C = C(\text{data})$ and $\overline{\mathcal{C}}_* = \overline{\mathcal{C}}_*(j_*, \text{data})$. Define $\hat{u}(x,\hat{\tau}) = \tilde{u}(x, \widetilde{m}\hat{\tau})$, $\hat{\xi}_i(x,\hat{\tau}) = \xi_i(x, \widetilde{m}\hat{\tau})$, and

$$\hat{A}_i(\hat{\tau}) = B_{r_i} \cap \{\hat{u}(\cdot,\hat{\tau}) > k_i\}, \quad \hat{A}_i = \widetilde{\mathcal{Q}}_i^+(1) \cap \{\hat{u} > k_i\}.$$

Combining Proposition 2.4 and (4.17), we obtain

$$\begin{aligned}
&2^{-p(i+2)}\left(\frac{\kappa_0}{2^{j_*}}\right)^p|\hat{A}_{i+1}|_{\nu_{w_2}}^{\frac{1}{\chi}}=(k_{i+1}-k_i)^p|\hat{A}_{i+1}|_{\nu_{w_2}}^{\frac{1}{\chi}}\\
&\le\|(\hat{u}_i-k_i)_+\hat{\xi}_i\|^p_{L^{p\chi}(\widetilde{\mathcal{Q}}_i^+(1),w_2)}\le C\|(\hat{u}_i-k_i)_+\hat{\xi}_i\|^p_{V_0^p(\widetilde{\mathcal{Q}}_i^+(1),w_1,w_2)}\\
&\le\frac{C2^{pi}}{R^p}\left(\frac{\kappa_0}{2^{j_*}}\right)^p|\hat{A}_i|_{\nu_{w_2}}+\frac{\widetilde{\mathcal{C}}_*R^{\frac{p(p+\vartheta)}{p-2}}}{\omega^{\frac{p(l_0-1)+2}{l_0}}}|\hat{A}_i|_{\nu_{w_2}}^{1-\frac{1}{l_0}},
\end{aligned}$$

where $\chi=\frac{n+p+\theta_1+\theta_2}{n+\theta_1+\theta_2}$ and $\widetilde{\mathcal{C}}_*=\widetilde{\mathcal{C}}_*(j_*,\text{data})$. Consequently, if $\omega>\overline{\mathcal{N}}_*(j_*,\text{data})R^{\varepsilon_0}$, we have

$$\begin{aligned}
|\hat{A}_{i+1}|_{\nu_{w_2}}\le&\left(\frac{C4^{pi}}{R^p}|\hat{A}_i|_{\nu_{w_2}}+\Big(\frac{\overline{\mathcal{N}}(j_*,\text{data})}{\omega}\Big)^{\frac{p(l_0-1)+2}{l_0}}|\hat{A}_i|_{\nu_{w_2}}^{1-\frac{1}{l_0}}\right)^\chi\\
\le&\left[\left(\frac{C4^{pi}}{R^{\frac{p(l_0-1)-n-\theta_1-\theta_2}{l_0}}}+\Big(\frac{\overline{\mathcal{N}}(j_*,\text{data})}{\omega}\Big)^{\frac{p(l_0-1)+2}{l_0}}\right)|\hat{A}_i|_{\nu_{w_2}}^{1-\frac{1}{l_0}}\right]^\chi\\
\le&\left(\frac{C4^{pi}}{R^{\frac{p(l_0-1)-n-\theta_1-\theta_2}{l_0}}}|\hat{A}_i|_{\nu_{w_2}}^{1-\frac{1}{l_0}}\right)^\chi.
\end{aligned}$$

Denote

$$F_i:=\frac{|\hat{A}_i|_{\nu_{w_2}}}{|B_R\times(R^{p+\vartheta},(3R)^{p+\vartheta}]|_{\nu_{w_2}}}.$$

Therefore, we obtain

$$\begin{aligned}
F_{i+1}\le&(C4^{pi})^\chi F_i^{\frac{\chi(l_0-1)}{l_0}}\le\prod_{s=0}^{i}\left[(C4^{p(i-s)})^\chi\right]^{\left(\frac{\chi(l_0-1)}{l_0}\right)^s}F_0^{\left(\frac{\chi(l_0-1)}{l_0}\right)^{i+1}}\\
\le&(\widetilde{C}_0F_0)^{\left(\frac{\chi(l_0-1)}{l_0}\right)^{i+1}},
\end{aligned}$$

where $\widetilde{C}_0=\widetilde{C}_0(\text{data})$. Fix the value of j_* such that

$$\frac{\overline{C}_0}{j_*^{\frac{p-\tilde{p}}{p\tilde{p}}}}\le(2C_0\widetilde{C}_0)^{-\frac{\beta}{n-1+\theta_3+\min\{0,\theta_4\}}},\tag{4.18}$$

where β and C_0 are determined by applying Lemma 4.4 to the domain $B_R\times(R^{p+\vartheta},(3R)^{p+\vartheta}]$, $\overline{C}_0$ is given by Lemma 4.7. Hence, it follows that

$$F_{i+1}\le2^{-\left(\frac{\chi(l_0-1)}{l_0}\right)^{i+1}}\to0,\quad\text{as } i\to\infty.$$

The proof is then finished by letting $A_0:=A_0(\text{data})=\overline{\mathcal{N}}_*(j_*,\text{data})$.

□

Based on Lemma 4.8, we are now ready to complete the proof of Proposition 4.1 by rescaling back to u.

Proof of Proposition 4.1. Let

$$t=\bar{t}+e^\tau\omega^{2-p}\delta_0R^{p+\vartheta}.$$

When $\tau\in(\widetilde{m}(2R)^{p+\vartheta},\widetilde{m}(3R)^{p+\vartheta}]$, we have

$$\bar{t}+\delta_0e^{\delta_0^{-1}2^{p+\vartheta+(p-2)(j_*+c_0)}}\omega^{2-p}R^{p+\vartheta}<t\le\bar{t}+\delta_0e^{\delta_0^{-1}3^{p+\vartheta}2^{(p-2)(j_*+c_0)}}\omega^{2-p}R^{p+\vartheta},$$

where c_0, δ_0, j_* are, respectively, given by Lemma 4.5 and (4.18). Pick

$$\bar{t} = -c_*\omega^{2-p}R^{p+\vartheta}, \quad c_* = \delta_0 e^{\delta_0^{-1}3^{p+\vartheta}2^{(p-2)(j_*+c_0)}}. \tag{4.19}$$

Remark that the value of $\bar{t}$ chosen in (4.19) satisfies the requirement in (4.11), that is,

$$\bar{t} \in [-1, -\delta_0(\sigma\omega)^{2-p}R^{p+\vartheta}], \quad \text{as } \tau \in (\widetilde{m}(2R)^{p+\vartheta}, \widetilde{m}(3R)^{p+\vartheta}] \text{ and } \omega > \sigma^{-1}R^{\varepsilon_0},$$

where $\sigma = e^{-\frac{\tau}{p-2}}$. Choose

$$M = (b_* 2^{p+\vartheta})^{\frac{1}{p-2}}, \quad \text{and thus } m = \left(\frac{\omega}{M}\right)^{2-p} = 2^{p+\vartheta}b_*\omega^{2-p}, \tag{4.20}$$

where

$$b_* = \delta_0 e^{\delta_0^{-1}3^{p+\vartheta}2^{(p-2)(j_*+c_0)}} - \delta_0 e^{\delta_0^{-1}2^{p+\vartheta+(p-2)(j_*+c_0)}}.$$

Then we have $t \in (-m(R/2)^{p+\vartheta}, 0]$, as $\tau \in (\widetilde{m}(2R)^{p+\vartheta}, \widetilde{m}(3R)^{p+\vartheta}]$. From (4.12) and Lemma 4.8, we have either

$$\omega \leq AR^{\varepsilon_0} := \max\{A_0, c_*^{\frac{1}{p-2}}\}R^{\varepsilon_0}, \tag{4.21}$$

or

$$u(x,t) \leq \mu^+ - \frac{\omega}{2^{c_0+j_*+1}e^{\frac{\widetilde{m}(3R)^{p+\vartheta}}{p-2}}} =: \mu^+ - \frac{\omega}{2^{\kappa_*}},$$

for any $(x,t) \in Q(R/2, m(R/2)^{p+\vartheta})$, where

$$\kappa_* = c_0 + j_* + 1 + \frac{3^{p+\vartheta}2^{(c_0+j_*)(p-2)}}{\delta_0 \ln 2}.$$

Therefore, (4.3) is proved. By the same argument, we obtain that (4.5) also holds. □

4.2. **The proofs of Theorems 1.1 and 1.3.** First, a direct application of Proposition 4.1 gives the following result.

Lemma 4.9. *Assume as in Theorems 1.1 and 1.3. Then we have either* $\omega \leq AR^{\varepsilon_0}$*, or*

$$\operatorname*{osc}_{Q(R/2, m(R/2)^{p+\vartheta})} u \leq \eta^*\omega, \quad \eta^* = 1 - 2^{-\kappa_*},$$

where A, m, κ_* *are determined by Proposition 4.1.*

Proof. Note that one of the following two inequalities must hold:

$$|B_R \cap \{u(\cdot, -c_*\omega^{2-p}R^{p+\vartheta}) > \mu^+ - 2^{-1}\omega\}|_{\mu_{w_1}} \leq 2^{-1}|B_R|_{\mu_{w_1}}, \tag{4.22}$$

and

$$|B_R \cap \{u(\cdot, -c_*\omega^{2-p}R^{p+\vartheta}) < \mu^- + 2^{-1}\omega\}|_{\mu_{w_1}} \leq 2^{-1}|B_R|_{\mu_{w_1}}. \tag{4.23}$$

From Proposition 4.1, it follows that if $\omega > AR^{\varepsilon_0}$,

$$\sup_{Q(R/2, m(R/2)^{p+\vartheta})} u \leq \mu^+ - 2^{-\kappa_*}\omega, \quad \text{if (4.22) holds},$$

and

$$\inf_{Q(R/2, m(R/2)^{p+\vartheta})} u \geq \mu^- + 2^{-\kappa_*}\omega, \quad \text{if (4.23) holds}.$$

In either case, we all obtain

$$\underset{Q(R/2,m(R/2)^{p+\vartheta})}{osc} u \le (1-2^{-k_*})\omega.$$

The proof is finished.

□

We proceed to use Lemma 4.9 to construct a series of nested and shrinking cylinders with the same vertex such that the essential oscillation of u in these cylinders goes to zero as the radius of the cylinder tends to zero. Denote

$$\omega_0 := \max\{\omega, AR^{\varepsilon_0}\}, \tag{4.24}$$

and, for $k \ge 0$,

$$R_k := A^{-k}R, \quad \omega_{k+1} := \max\{\eta^*\omega_k, AR_k^{\varepsilon_0}\}, \quad \tilde{a}_k := \left(\frac{\omega_k}{A}\right)^{2-p}. \tag{4.25}$$

Since A is a large constant, we have

$$\tilde{a}_{k+1}R_{k+1}^{p+\vartheta} = \left(\frac{\omega_{k+1}}{A}\right)^{2-p}\frac{R_k^{p+\vartheta}}{A^{p+\vartheta}} \le \frac{\tilde{a}_k R_k^{p+\vartheta}}{(\eta^*)^{p-2}A^{p+\vartheta}} < \tilde{a}_k R_k^{p+\vartheta},$$

which implies that $Q(R_{k+1}, \tilde{a}_{k+1}R_{k+1}^{p+\vartheta}) \subset Q(R_k, \tilde{a}_k R_k^{p+\vartheta})$. Remark that $\tilde{a}_0 \le a_0$.

Lemma 4.10. *Assume as in Theorems 1.1 and 1.3. Then for any $k = 0, 1, 2, ...,$*

$$\underset{Q(R_k,\tilde{a}_k R_k^{p+\vartheta})}{osc} u \le \omega_k. \tag{4.26}$$

Proof. Observe first that (4.26) holds obviously for $k = 0$ in virtue of $\tilde{a}_0 \le a_0$. We now suppose that (4.26) holds in the case of $k = i$ for any given $i \ge 1$. Then we prove that it also holds for $k = i+1$. Since $\underset{Q(R_i,\tilde{a}_i R_i^{p+\vartheta})}{osc} u \le \omega_i$, it then follows from the proof of Lemma 4.9 with minor modification that

$$\underset{Q(R_i/2,m_i(R_i/2)^{p+\vartheta})}{osc} u \le \max\{\eta^*\omega_i, AR_i^{\varepsilon_0}\} = \omega_{i+1}, \quad m_i := \left(\frac{M}{\omega_i}\right)^{p-2}, \tag{4.27}$$

where M is given by (4.20). Due to the fact that $\omega_{i+1} \ge \eta^*\omega_i$ and A is a large constant, we obtain

$$\begin{aligned} m_i\left(\frac{R_i}{2}\right)^{p+\vartheta} &= \left(\frac{M}{\omega_i}\right)^{p-2}\left(\frac{R_i}{2}\right)^{p+\vartheta} \ge \left(\frac{A}{\omega_{i+1}}\right)^{p-2}\left(\frac{\eta^* M}{A}\right)^{p-2}\left(\frac{R_i}{2}\right)^{p+\vartheta} \\ &= \tilde{a}_{i+1}R_{i+1}^{p+\vartheta}\frac{(\eta^*M)^{p-2}A^{2+\vartheta}}{2^{p+\vartheta}} \ge \tilde{a}_{i+1}R_{i+1}^{p+\vartheta}, \end{aligned}$$

which, in combination with (4.27), shows that

$$\underset{Q(R_{i+1},\tilde{a}_{i+1}R_{i+1}^{p+\vartheta})}{osc} u \le \omega_{i+1}.$$

The proof is finished.

□

Based on the result in Lemma 4.10, we now give a more precise characterization for the oscillation decay property of the solution u.

Proposition 4.11. *Assume as in Theorems 1.1 and 1.3. Then for any* $0 < \rho \leq R \leq \frac{1}{2}$,

$$\underset{Q(\rho.\tilde{a}_0\rho^{p+\vartheta})}{osc} u \leq \max\left\{(\eta^*)^{-1}\omega_0, A^{1+2\varepsilon_*}R^{\varepsilon_*}\right\}\left(\frac{\rho}{R}\right)^{\varepsilon_*},$$

where η^* *is given in Lemma 4.9,* ω_0 *and* $\tilde{a}_0$ *are, respectively, defined by* (4.24)–(4.25), *and*

$$\varepsilon_* := \min\left\{\varepsilon_0, -\frac{\ln\eta^*}{\ln A}\right\}. \tag{4.28}$$

Remark 4.12. Observe that the value of ε_0 tends to zero, as $\theta_1+\theta_2 \nearrow p(l_0-1)-n$ or $l_0 \searrow \frac{n+p+\theta_1+\theta_2}{p}$. Then if $\varepsilon_0 \leq -\frac{\ln\eta^*}{\ln A}$, ε_* becomes the explicit exponent ε_0, whose value is clearly determined by the structure of the considered equation and the weights. By contrast, if $\varepsilon_0 > -\frac{\ln\eta^*}{\ln A}$, the effects of the weights on the regularity of the solution will be concealed beneath the inexplicit constant $-\frac{\ln\eta^*}{\ln A}$.

Remark 4.13. Recall that for any $R \in (0, \frac{1}{2}]$ and $t_0 \in [-\frac{1}{2}, 0]$, if $\omega > AR^{\varepsilon_0}$, we have

$$[(0,t_0)+Q(R,a_0R^{p+\vartheta})] \subset [(0,t_0)+Q(2R,R^{p+\vartheta-\bar{\varepsilon}_0})] \subset B_1\times(-1,0].$$

Therefore, by a translation, it follows from the proof of Proposition 4.11 with a slight modification that for any $0<\rho\leq R\leq\frac{1}{2}$, there holds either $\omega\leq AR^{\varepsilon_0}$, or

$$\underset{[(0,t_0)+Q(\rho,\tilde{a}_0\rho^{p+\vartheta})]}{osc} u \leq \max\left\{(\eta^*)^{-1}\omega_0, A^{1+2\varepsilon_*}R^{\varepsilon_*}\right\}\left(\frac{\rho}{R}\right)^{\varepsilon_*}.$$

Proof of Proposition 4.11. For any $0<\rho\leq R\leq\frac{1}{2}$, there exists an integer $k\geq 0$ such that $R_{k+1} = A^{-(k+1)}R \leq \rho \leq A^{-k}R = R_k$. In light of (4.28), we have $A^{\varepsilon_*}\eta^* \leq 1$. This, together with (4.26), shows that the conclusion obviously holds for $k=0$. So in the following it suffices to consider the case when $k\geq 1$. Utilizing the fact of $A^{\varepsilon_*}\eta^*\leq 1$ again, we deduce

$$\begin{aligned}\omega_k \leq&\max\{\eta^*\omega_{k-1}, AR_{k-1}^{\varepsilon_*}\} \leq \max\left\{(\eta^*)^k\omega_0, \max_{0\leq i\leq k-1} A(\eta^*)^i R_{k-1-i}^{\varepsilon_*}\right\}\\ \leq&\max\left\{(\eta^*)^k\omega_0, \max_{0\leq i\leq k-1} R^{\varepsilon_*}(\eta^*)^i A^{1-(k-1-i)\varepsilon_*}\right\}\\ \leq&\max\left\{(\eta^*)^k\omega_0, A\left(\frac{R}{A^{k-1}}\right)^{\varepsilon_*}\right\}.\end{aligned} \tag{4.29}$$

Since $\varepsilon_*\leq-\frac{\ln\eta^*}{\ln A}$ and $R_{k+1}\leq\rho$, then

$$(\eta^*)^k \leq (\eta^*)^{-1}A^{-(k+1)\varepsilon_*} \leq (\eta^*)^{-1}\left(\frac{\rho}{R}\right)^{\varepsilon_*}, \quad A\left(\frac{R}{A^{k-1}}\right)^{\varepsilon_*} \leq A^{1+2\varepsilon_*}\rho^{\varepsilon_*}.$$

Inserting this into (4.29), we have

$$\omega_k \leq \max\left\{(\eta^*)^{-1}\omega_0, A^{1+2\varepsilon_*}R^{\varepsilon_*}\right\}\left(\frac{\rho}{R}\right)^{\varepsilon_*}.$$

Since

$$\begin{aligned}\omega_k =&\max\{\eta^*\omega_{k-1}, AR_{k-1}^{\varepsilon_0}\} = \max\left\{(\eta^*)^k\omega_0, \max_{0\leq i\leq k-1} A(\eta^*)^i R_{k-1-i}^{\varepsilon_0}\right\}\\ =&\max\left\{(\eta^*)^k\omega_0, \max_{0\leq i\leq k-1} R^{\varepsilon_0}(\eta^*)^i A^{1-(k-1-i)\varepsilon_0}\right\} \leq \omega_0,\end{aligned}$$

then we have $\tilde{a}_0 \le \tilde{a}_k$. This implies that

$$\underset{Q(\rho.\tilde{a}_0\rho^{p+\vartheta})}{osc} u \le \underset{Q(R_k,\tilde{a}_k R_k^{p+\vartheta})}{osc} u \le \max\left\{(\eta^*)^{-1}\omega_0, A^{1+2\varepsilon_*}R^{\varepsilon_*}\right\}\left(\frac{\rho}{R}\right)^{\varepsilon_*}.$$

The proof is complete.

□

In the following we make use of Proposition 4.11 to complete the proofs of Theorems 1.1 and 1.3, respectively.

Proof of Theorem 1.1. Denote $\hat{a}_0 := 2^{\varepsilon_0(p-2)}$. Then we obtain that for any $0 < R \le \frac{1}{2}$,

$$\hat{a}_0 = 2^{\varepsilon_0(p-2)} = \left(\frac{A}{\max\{2\mathcal{M}, A2^{-\varepsilon_0}\}}\right)^{p-2} \le \left(\frac{A}{\max\{\omega, AR^{\varepsilon_0}\}}\right)^{p-2} = \tilde{a}_0.$$

Then using Proposition 4.11 and Remark 4.13, we deduce that there exists a constant $0 < \alpha \le \varepsilon_0$ depending only on the data such that for any $t_0 \in (-1/2, 0]$ and $\rho \in (0, \frac{1}{2}]$,

$$\underset{[(0,t_0)+Q(\rho,\hat{a}_0\rho^{p+\vartheta})]}{osc} u \le C\rho^\alpha,$$

which leads to that for every $(x,t) \in B_{1/2} \times (-1/2, t_0]$,

(*i*) if $|t - t_0| \le \hat{a}_0 2^{-(p+\vartheta)}$,

$$\begin{aligned}|u(x,t) - u(0,t_0)| \le& |u(x,t) - u(x,t_0)| + |u(x,t_0) - u(0,t_0)|\\ \le& C\big((\hat{a}_0^{-1}|t-t_0|)^{\frac{\alpha}{p+\theta}} + |x|^\alpha\big) \le C\big(|x| + |t-t_0|^{\frac{1}{p+\vartheta}}\big)^\alpha;\end{aligned}$$

(*ii*) if $|t - t_0| > \hat{a}_0 2^{-(p+\theta)}$, there exists an increasing set $\{t_i\}_{i=1}^N$, $1 \le N \le \left[\hat{a}_0^{-1}2^{p+\vartheta-1}\right] + 1$ such that $t < t_1 \le \cdots \le t_N < t_0$,

$$\begin{aligned}|u(x,t) - u(0,t_0)| \le& |u(x,t) - u(x,t_1)| + |u(x,t_1) - u(x,t_0)| + |u(x,t_0) - u(0,t_0)|\\ \le& C\big((\hat{a}_0^{-1}|t-t_1|)^{\frac{\alpha}{p+\vartheta}} + (\hat{a}_0^{-1}|t_1-t_0|)^{\frac{\alpha}{p+\vartheta}} + |x|^\alpha\big)\\ \le& C\big(|x| + |t-t_0|^{\frac{1}{p+\vartheta}}\big)^\alpha, \quad \text{if } N = 1,\end{aligned}$$

and

$$\begin{aligned}&|u(x,t) - u(0,t_0)|\\ &\le |u(x,t) - u(x,t_1)| + \sum_{i=1}^{N-1}|u(x,t_i) - u(x,t_{i+1})|\\ &\quad + |u(x,t_N) - u(x,t_0)| + |u(x,t_0) - u(0,t_0)|\\ &\le C\Big((\hat{a}_0^{-1}|t-t_1|)^{\frac{\alpha}{p+\vartheta}} + \sum_{i=1}^{N-1}(\hat{a}_0^{-1}|t_i - t_{i+1}|)^{\frac{\alpha}{p+\vartheta}} + (\hat{a}_0^{-1}|t_N - t_0|)^{\frac{\alpha}{p+\vartheta}} + |x|^\alpha\Big)\\ &\le C\big(|x| + |t-t_0|^{\frac{1}{p+\vartheta}}\big)^\alpha, \quad \text{if } N \ge 2.\end{aligned}$$

The proof is finished.

□

Proof of Theorem 1.3. In the following we take the proof in the case of $(w_1, w_2) = (|x'|^{\theta_1}, |x'|^{\theta_3})$ for example. Another case is the same and thus omitted. In this case, we have $\theta_2 = \theta_4 = 0$ and $\vartheta = \theta_1 - \theta_3$. By a translation, we deduce from the proof of Theorem 1.1 with a slight modification that there exists two constants $0 < \alpha \leq \varepsilon_0$ and $C > 1$, both depending only on the data, such that for any given $\bar{x} \in \{(0', \bar{x}_n) : |\bar{x}_n| \leq 1/2\}$ and $t_0 \in (-1/2, 0]$,

$$|u(x,t) - u(\bar{x}, t_0)| \leq C\big(|x - \bar{x}| + |t - t_0|^{\frac{1}{p+\vartheta}}\big)^{\alpha}, \tag{4.30}$$

for all $(x,t) \in B_{1/4}(\bar{x}) \times (-1/2, t_0]$. For $R \in (0, 1/4)$ and $(y,s) \in Q(1/R, 1/R^{p+\vartheta})$, let $u_R(y,s) = u(Ry, R^{p+\vartheta}s)$ and $\phi_{3,R}(y,s) = \phi_3(Ry, R^{p+\vartheta}s)$. Then u_R solves

$$|y'|^{\theta_1}\partial_s u_R - \mathrm{div}(|y'|^{\theta_3}|\nabla u_R|^{p-2}\nabla u_R) = |y'|^{\theta_3}\left(\lambda_3|\nabla u_R|^p + R^p\phi_{3,R}\right),$$

for $(y,s) \in Q(1/R, 1/R^{p+\vartheta})$.

For any given $(x,t), (\tilde{x}, \tilde{t}) \in B_{1/4} \times (-1/2, 0]$, suppose without loss of generality that $|\tilde{x}'| \leq |x'|$. Let $R = |x'|$. Then applying the proof of Theorem 1.1 with minor modification, we obtain that there exist two constants $0 < \gamma < 1$ and $C > 1$, both depending only upon the data, such that for any fixed $\bar{y} \in B_{1/(2R)} \cap \{|y'| = 1\}$ and $\bar{s} \in (-2^{-1}R^{-(p+\vartheta)}, 0]$,

$$|u_R(y,s) - u_R(\bar{y}, \bar{s})| \leq C\Big(|y - \bar{y}| + \big(\hat{a}_0^{-1}|s - \bar{s}|\big)^{1/p}\Big)^{\gamma}, \tag{4.31}$$

for any (y,s) satisfying that $|y - \bar{y}| + (\hat{a}_0^{-1}|s - \bar{s}|)^{1/p} \leq 1/4$. For later use, we limit the range of γ to be in $(0, \alpha]$. Otherwise, if $\gamma > \alpha$, then (4.31) also holds for any $\gamma \in (0, \alpha]$.

Note that

$$\begin{aligned}|u(x,t) - u(\tilde{x}, \tilde{t})| \leq& |u(x,t) - u(x, \tilde{t})| + |u(x, \tilde{t}) - u(x', \tilde{x}_n, \tilde{t})| \\ &+ |u(x', \tilde{x}_n, \tilde{t}) - u(\tilde{x}, \tilde{t})|.\end{aligned} \tag{4.32}$$

Set $c_1 \geq 1 + \frac{p}{p+\vartheta}$. If $|t - \tilde{t}| \leq \hat{a}_0 R^{c_1(p+\vartheta)}$, it follows from (4.31) that

$$\begin{aligned}&|u(x,t) - u(x, \tilde{t})| = \big|u_R(x/R, t/R^{p+\vartheta}) - u_R(x/R, \tilde{t}/R^{p+\vartheta})\big| \\ &\leq C|(t - \tilde{t})/(\hat{a}_0 R^{p+\vartheta})|^{\frac{\gamma}{p}} \leq C|t - \tilde{t}|^{\frac{(c_1-1)\gamma}{c_1 p}}.\end{aligned}$$

By contrast, if $|t - \tilde{t}| > \hat{a}_0 R^{c_1(p+\vartheta)}$, using (4.30), we obtain

$$\begin{aligned}&|u(x,t) - u(x, \tilde{t})| \\ &\leq |u(x,t) - u(0', x_n.t)| + |u(0', x_n, t) - u(0', x_n, \tilde{t})| + |u(0', x_n, \tilde{t}) - u(x, \tilde{t})| \\ &\leq C\big(R^{\alpha} + (\hat{a}_0^{-1}|t - \tilde{t}|)^{\frac{\alpha}{p+\vartheta}}\big) \leq C|t - \tilde{t}|^{\frac{\alpha}{c_1(p+\vartheta)}}.\end{aligned}$$

We now proceed to deal with the remaining two terms in the right hand side of (4.32). Choose $c_2 \geq 2$. If $|x_n - \tilde{x}_n| \leq R^{c_2}$, we have from (4.31) that

$$\begin{aligned}&|u(x, \tilde{t}) - u(x', \tilde{x}_n, \tilde{t})| = \big|u_R(x/R, \tilde{t}/R^{p+\vartheta}) - u_R(x'/R, \tilde{x}_n/R, \tilde{t}/R^{p+\vartheta})\big| \\ &\leq C|(x_n - \tilde{x}_n)/R|^{\gamma} \leq C|x_n - \tilde{x}_n|^{\frac{(c_2-1)\gamma}{c_2}},\end{aligned}$$

while, if $|x_n - \tilde{x}_n| > R^{c_2}$, it follows from (4.30) that

$$\begin{aligned}&|u(x, \tilde{t}) - u(x', \tilde{x}_n, \tilde{t})| \\ &\leq |u(x, \tilde{t}) - u(0', x_n, \tilde{t})| + |u(0', x_n, \tilde{t}) - u(0', \tilde{x}_n, \tilde{t})| + |u(0', \tilde{x}_n, \tilde{t}) - u(x', \tilde{x}_n, \tilde{t})| \\ &\leq C(|x'|^{\alpha} + |x_n - \tilde{x}_n|^{\alpha}) \leq C|x_n - \tilde{x}_n|^{\frac{\alpha}{c_2}}.\end{aligned}$$

Similarly, if $|x' - \tilde{x}'| \le R^{c_2}$, we obtain from (4.31) that

$$\begin{aligned}|u(x', \tilde{x}_n, \tilde{t}) - u(\tilde{x}, \tilde{t})| &= \big|u_R(x'/R, \tilde{x}_n/R, \tilde{t}/R^{p+\vartheta}) - u_R(\tilde{x}/R, \tilde{t}/R^{p+\vartheta})\big| \\ &\le C|(x' - \tilde{x}')/R|^{\gamma} \le C|x' - \tilde{x}'|^{\frac{(c_2-1)\gamma}{c_2}},\end{aligned}$$

while, if $|x' - \tilde{x}'| > R^{c_2}$, utilizing (4.30), it follows that

$$\begin{aligned}|u(x', \tilde{x}_n, \tilde{t}) - u(\tilde{x}, \tilde{t})| &\le |u(x', \tilde{x}_n, \tilde{t}) - u(0', \tilde{x}_n, \tilde{t})| + |u(0', \tilde{x}_n, \tilde{t}) - u(\tilde{x}, \tilde{t})| \\ &\le C\big(R^{\alpha} + |\tilde{x}'|^{\alpha}\big) \le C|x' - \tilde{x}'|^{\frac{\alpha}{c_2}}.\end{aligned}$$

Based on these above facts, we pick $c_1 = 1 + \frac{p\alpha}{\gamma(p+\vartheta)}$ and $c_2 = 1 + \frac{\alpha}{\gamma}$ such that

$$\frac{(c_1-1)\gamma}{c_1 p} = \frac{\alpha}{c_1(p+\vartheta)}, \quad \frac{(c_2-1)\gamma}{c_2} = \frac{\alpha}{c_2}. \tag{4.33}$$

Remark that since $\frac{c_i-1}{c_i}$ increases in c_i and c_i^{-1} decreases in c_i for $i = 1, 2$, then the values of c_1 and c_2 taken in (4.33) maximize the Hölder regularity exponent and thus the choice is best. Therefore, we obtain that for any $(x, t), (\tilde{x}, \tilde{t}) \in B_{1/4} \times (-1/2, 0)$,

$$|u(x, t) - u(\tilde{x}, \tilde{t})| \le C\big(|x - \tilde{x}| + |t - \tilde{t}|^{\frac{1}{p+\vartheta}}\big)^{\frac{\alpha\gamma}{\alpha+\gamma}}.$$

This leads to that Theorem 1.3 holds.

□

5. Appendix

Proof of Lemma 2.1. On one hand, if $p > 1$ is an integer, we deduce from the binomial theorem and Young's inequality that

$$\begin{aligned}(a+b)^p =& a^p + \sum_{j=1}^{p} C_p^j a^{p-j} b^j \le (1+\varepsilon)a^p + C(p)b^p \sum_{j=1}^{p} \varepsilon^{-\frac{p-j}{j}} \\ \le& (1+\varepsilon)a^p + \frac{C(p)}{\varepsilon^{p-1}} b^p, \quad C_p^j = \frac{p!}{j!(p-j)!},\end{aligned} \tag{5.1}$$

where $j!$ represents the factorial of j.

On the other hand, if $p > 1$ is not an integer, using (5.1), we obtain

$$\begin{aligned}(a+b)^p =& (a+b)^{[p]+(p-[p])} \le \left((1+\varepsilon)a^{[p]} + \frac{C(p)}{\varepsilon^{[p-1]}} b^{[p]}\right)\left(a^{p-[p]} + b^{p-[p]}\right) \\ =& (1+\varepsilon)a^p + (1+\varepsilon)a^{[p]} b^{p-[p]} + \frac{C(p)}{\varepsilon^{[p-1]}} a^{p-[p]} b^{[p]} + \frac{C(p)}{\varepsilon^{[p-1]}} b^{p-2}.\end{aligned} \tag{5.2}$$

A consequence of Young's inequality gives that

$$(1+\varepsilon)a^{[p]} b^{p-[p]} \le \varepsilon a^p + \frac{C(p)}{\varepsilon^{\frac{[p]}{p-[p]}}} b^p, \quad \frac{C(p)}{\varepsilon^{[p-1]}} a^{p-[p]} b^{[p]} \le \varepsilon a^p + \frac{C(p)}{\varepsilon^{p-1}} b^p.$$

Substituting the above inequalities into (5.2), we have

$$(a+b)^p \le (1+3\varepsilon)a^p + \frac{C(p)}{\varepsilon^{\frac{[p]}{p-[p]}}} b^p.$$

The proof is complete.

□

In the following, we provide an alternative proof in terms of the expansion of the distribution function of u in time.

Proposition 5.1. *Assume as in Lemma 4.5. Then there exists two constants* $A_1, j_0 > 2$ *depending only on the data such that we have either* $\omega \leq A_1 R^{\varepsilon_0}$, *or if for some* $\bar{t} \in [-1, -\omega^{2-p}R^{p+\vartheta}]$,

$$|B_R \cap \{u(\cdot,\bar{t}) > \mu^+ - 2^{-1}\omega\}|_{\mu_{w_1}} \leq 2^{-1}|B_R|_{\mu_{w_1}}, \tag{5.3}$$

then

$$|B_{R/2} \cap \{u(\cdot,t) > \mu^+ - 2^{-(j_0+1)}\omega\}|_{\mu_{w_1}} \leq \frac{3}{4}|B_{R/2}|_{\mu_{w_1}}, \ \forall\, t \in [\bar{t}, \bar{t} + \omega^{2-p}R^{p+\vartheta}].$$

We prepare to use the following logarithmic estimates to prove Proposition 5.1. For that purpose, we first introduce the logarithmic function as follows:

$$\Psi^{\pm}_{k,\delta}(u) := \Psi(H^{\pm}_k, (u-k)_{\pm}, \delta) = \ln^+ \frac{H^{\pm}_k}{H^{\pm}_k - (u-k)_{\pm} + \delta}, \quad 0 < \delta < H^{\pm}_k, \tag{5.4}$$

where $H^{\pm}_k = \sup\limits_{[(x_0,t_0)+Q(\rho,\tau)]} (u-k)_{\pm}$, $\ln^+$ means that $\ln^+ s = \max\{\ln s, 0\}$ for $s > 0$. For any fixed $B_\rho(x_0) \subset B_1$, let $\zeta \in C^\infty(B_1)$ be a smooth cutoff function satisfying that

$$0 \leq \zeta \leq 1, \ |\nabla\zeta| < \infty \text{ in } B_1, \quad \text{and } \zeta = 0 \text{ in } B_1 \setminus B_\rho(x_0). \tag{5.5}$$

The required logarithmic inequalities are now stated as follows.

Lemma 5.2. *Let* u *be the solution to problem* (1.1) *with* $\Omega \times (-T, 0] = B_1 \times (-1, 0]$. *Then for any cylinder* $[(x_0,t_0) + Q(\rho,\tau)] \subset B_1 \times (-1, 0]$, *we obtain*

$$\begin{aligned}
&\sup_{t_0-\tau<t<t_0} \int_{B_\rho(x_0)} [\Psi^{\pm}_{k,\delta}(u)]^2(x,t)\zeta^p(x)w_1 dx \\
&\leq \int_{B_\rho(x_0)} [\Psi^{\pm}_{k,\delta}(u)]^2(x,t_0-\tau)\zeta^p(x)w_1 dx \\
&\quad + C\int_{[(x_0,t_0)+Q(\rho,\tau)]} \Psi^{\pm}_{k,\delta}(u)|\partial_u \Psi^{\pm}_{k,\delta}(u)|^{2-p}(|\nabla\zeta|^p + |\zeta|^p)w_2 dxdt \\
&\quad + \frac{C}{\delta^2}\left(1 + \ln\frac{H^{\pm}_k}{\delta}\right)\|\phi\|_{L^{l_0}(B_1\times(-1,0),w_2)}|[(x_0,t_0)+Q(\rho,\tau)] \cap \{v_{\pm} > 0\}|^{1-\frac{1}{l_0}}_{\nu_{w_2}},
\end{aligned}$$

where ζ *is defined by* (5.5) *and* $\phi = \phi_1 + \phi_2^{\frac{p}{p-1}} + \phi_3^{\frac{p}{p-1}}$.

Proof. Without loss of generality, assume that $(x_0,t_0) = (0,0)$. For simplicity, denote

$$\psi(f) := \Psi^{\pm}_{k,\delta}(f), \quad \psi' := \partial_f \psi, \quad f = u \text{ or } u_h.$$

Set $\varphi = [\psi^2(u_h)]'\zeta^p$. Let χ_Σ represent the characteristic function of the set Σ. It follows from a straightforward computation that

$$\psi'(u_h) = \frac{\pm\chi_{\{(u_h-k)_\pm>0\}}}{H^{\pm}_k - (u_h-k)_{\pm} + \delta}, \quad \psi''(u_h) = [\psi'(u_h)]^2,$$

and

$$[\psi^2(u_h)]'' = 2(1+\psi(u_h))\psi'^2(u_h) \in L^\infty_{loc}(B_1 \times (-1,0)),$$

which implies that φ is an admissible test function. Similarly as before, we obtain that for any $-\tau \le s \le 0$,

$$\begin{aligned}&\int_{-\tau}^{s}\int_{B_\rho}(\partial_t u_h\varphi w_1 + w_2[\mathbf{a}(x,t,u,\nabla u)]_h\cdot\nabla\varphi)dxdt\\ &=\int_{-\tau}^{s}\int_{B_\rho}[b(x,t,u,\nabla u)]_h\varphi w_2 dxdt. \end{aligned}\tag{5.6}$$

To begin with, integrating by parts and using Lemma 2.2, we have

$$\begin{aligned}&\int_{-\tau}^{s}\int_{B_\rho}\partial_t u_h\varphi w_1 = \int_{B_\rho\times(-\tau,s)}\partial_t\psi^2(u_h)\zeta^p w_1\\ &=\int_{B_\rho}\psi^2(u_h)(x,s)\zeta^p(x)w_1dx - \int_{B_\rho}\psi^2(u_h)(x,-\tau)\zeta^p(x)w_1dx\\ &\to\int_{B_\rho}\psi^2(u)(x,s)\zeta^p(x)w_1dx - \int_{B_\rho}\psi^2(u)(x,-\tau)\zeta^p(x)w_1dx,\quad \text{as } h\to 0.\end{aligned}$$

For simplicity, denote

$$Q_{s,k} := (B_\rho\times(-\tau,s))\cap\{(u-k)_\pm > 0\}.$$

As for the remaining two terms in (5.6), by first letting $h\to 0$, it then follows from Lemma 2.2, (**H1**)–(**H3**) and Young's inequality that

$$\begin{aligned}&\int_{-\tau}^{s}\int_{B_\rho}[\mathbf{a}(x,t,u,\nabla u)]_h\cdot\nabla\varphi w_2dxdt\\ &\to\int_{B_\rho\times(-\tau,s)}\mathbf{a}(x,t,u,\nabla u)\cdot\big([\psi^2(u)]''\nabla u\zeta^p + p[\psi^2(u)]'\zeta^{p-1}\nabla\zeta\big)w_2dxdt\\ &\ge 2\lambda_1\int_{Q_{s,k}}(1+\psi)\psi'^2|\nabla u|^p\zeta^p w_2dxdt - 2\int_{Q_{s,k}}(1+\psi)\psi'^2\phi_1\zeta^p w_2dxdt\\ &\quad - 2p\lambda_2\int_{Q_{s,k}}\psi|\psi'|\zeta^{p-1}|\nabla\zeta||\nabla u|^{p-1}w_2dxdt\\ &\quad - 2p\int_{Q_{s,k}}\psi|\psi'|\zeta^{p-1}|\nabla\zeta|\phi_2w_2dxdt\\ &\ge\lambda_1\int_{Q_{s,k}}(1+\psi)\psi'^2|\nabla u|^p\zeta^p w_2dxdt - 2\int_{Q_{s,k}}(1+\psi)\psi'^2\phi_1\zeta^p w_2dxdt\\ &\quad - C\int_{Q_{s,k}}\psi|\psi'|^{2-p}|\nabla\zeta|^p w_2dxdt - C\int_{Q_{s,k}}\psi\psi'^2\phi_2^{\frac{p}{p-1}}\zeta^p w_2dxdt,\end{aligned}$$

and

$$\begin{aligned}&\int_{-\tau}^{s}\int_{B_\rho}[b(x,t,u,\nabla u)]_h\varphi w_2dxdt \to \int_{B_\rho\times(-\tau,s)}b(x,t,u,\nabla u)[\psi^2(u)]'\zeta^p w_2dxdt\\ &\le 2\lambda_3\int_{Q_{s,k}}\psi|\psi'||\nabla u|^{p-1}\zeta^p w_2dxdt + 2\int_{Q_{s,k}}\psi|\psi'|\zeta^p\phi_3w_2dxdt\\ &\le\frac{2\lambda_1}{3}\int_{Q_{s,k}}(1+\psi)\psi'^2\zeta^p|\nabla u|^p w_2dxdt + C\int_{Q_{s,k}}\psi|\psi'|^{2-p}|\zeta|^p w_2dxdt\\ &\quad+\frac{2}{\delta}\ln\left(\frac{H_k^\pm}{\delta}\right)\int_{Q_{s,k}}\phi_3\zeta^p w_2dxdt,\end{aligned}$$

where we also used the facts of $\psi \leq \ln \frac{H_k^{\pm}}{\delta}$ and $|\psi'| \leq \frac{1}{\delta}$. Combining these above facts, we have from Hölder's inequality that

$$\begin{aligned}
&\sup_{-\tau<t<0} \int_{B_\rho} [\Psi_{k,\delta}^{\pm}(u)]^2(x,t)\zeta^p(x)w_1 dx \\
&\leq \int_{B_\rho} [\Psi_{k,\delta}^{\pm}(u)]^2(x,-\tau)\zeta^p(x)w_1 dx + C\int_{Q(\rho,\tau)} \Psi_{k,\delta}^{\pm}(u)|\partial_u \Psi_{k,\delta}^{\pm}(u)|^{2-p}(|\nabla\zeta|^p + |\zeta|^p)w_2 \\
&\quad + \frac{C}{\delta^2}\left(1+\ln\frac{H_k^{\pm}}{\delta}\right)\int_{Q(\rho,\tau)\cap\{(u-k)_\pm>0\}} \phi w_2 dxdt \\
&\leq \int_{B_\rho} \Psi_{k,\delta}^{\pm}(u)(x,-\tau)\zeta^p(x)w_1 dx + C\int_{Q(\rho,\tau)} \Psi_{k,\delta}^{\pm}(u)|\partial_u \Psi_{k,\delta}^{\pm}(u)|^{2-p}(|\nabla\zeta|^p + |\zeta|^p)w_2 \\
&\quad + \frac{C}{\delta^2}\left(1+\ln\frac{H_k^{\pm}}{\delta}\right)\|\phi\|_{L^{l_0}(B_1\times(-1,0),w_2)}|Q(\rho,\tau)\cap\{(u-k)_\pm>0\}|_{\nu_{w_2}}^{1-\frac{1}{l_0}}.
\end{aligned}$$

The proof is complete. □

We are now ready to use the logarithmic estimates in Lemma 5.2 to complete the proof of Proposition 5.1.

Proof of Proposition 5.1. Let $k = \mu^+ - \frac{\omega}{2}$ and take $\delta = \frac{\omega}{2^{j_0+1}}$ in (5.4) for some positive constant $j_0 > 2$ to be determined later. For brevity, write

$$\Psi := \ln^+ \frac{H_k^+}{H_k^+ - (u-(\mu^+ - \frac{\omega}{2}))_+ + \frac{\omega}{2^{j_0+1}}},$$

where

$$H_k^+ = \sup_{Q_R^+(\omega)}\left(u - \left(\mu^+ - \frac{\omega}{2}\right)\right)_+ \leq \frac{\omega}{2}, \quad Q_R^+(\omega) := B_R \times [\bar{t}, \bar{t} + \omega^{2-p}R^{p+\vartheta}].$$

Let $\zeta \in C^\infty(B_R)$ be a smooth cutoff function satisfying that $\zeta = 1$ in $B_{(1-\varrho)R}$, $0 \leq \zeta \leq 1$ and $|\nabla\zeta| \leq \frac{4}{\varrho R}$ in B_R, where $\varrho \in (0,1)$ is to be chosen later. Observe that

$$\Psi \leq \ln\left(\frac{\frac{\omega}{2}}{\frac{\omega}{2^{j_0+1}}}\right) = j_0 \ln 2,$$

and

$$|\partial_u \Psi|^{2-p} = \left|H_k^+ - (u-k)_+ + \frac{\omega}{2^{j_0+1}}\right|^{p-2} \leq \omega^{p-2}.$$

Based on these above facts, it follows from (5.3), Lemmas 4.3 and 5.2 that for any $t \in [\bar{t}, \bar{t} + \omega^{2-p} R^{p+\vartheta}]$, if $\omega > A_1 R^{\varepsilon_0}$ with $A_1 := 4^{\frac{j_0 l_0}{p(l_0-1)+2}}$,

$$\begin{aligned}
\int_{B_{(1-\varrho)R}} \Psi^2(x,t) w_1 dx &\le \int_{B_R} \Psi^2(x,\bar{t})\zeta(x) w_1 dx + \frac{C}{R^p}\int_{Q_R^+(\omega)} \Psi |\partial_u \Psi|^{2-p} w_2 dx dt \\
&\quad + C\Big(\frac{2^{j_0}}{\omega}\Big)^2 \left(1 + \ln \frac{H_k^+ 2^{j_0}}{\omega}\right) \Big| Q_R^+(\omega) \cap \Big\{u > \mu^+ - \frac{\omega}{2}\Big\}\Big|_{\nu_{w_2}}^{1-\frac{1}{l_0}} \\
&\le \frac{(j_0 \ln 2)^2}{2} |B_R|_{\mu_{w_1}} + C j_0 \Big(\varrho^{-p} + 4^{j_0} R^{\frac{p(l_0-1)-n-\theta_1-\theta_2}{l_0}} \omega^{-\frac{p(l_0-1)+2}{l_0}}\Big) |B_R|_{\mu_{w_1}} \\
&\le (j_0 \ln 2)^2 |B_R|_{\mu_{w_1}} \left(\frac{1}{2} + \frac{C}{j_0 \varrho^p}\right). \qquad (5.7)
\end{aligned}$$

Note that for $(x,t) \in \{x \in B_{(1-\varrho)R} : u > \mu^+ - 2^{-(j_0+1)}\omega\} \times [\bar{t}, \bar{t} + \omega^{2-p} R^{p+\vartheta}]$,

$$\Psi^2(x,t) \ge \ln^2 \frac{H_k^+}{H_k^+ - \frac{\omega}{2} + \frac{\omega}{2^{j_0}}} \ge \ln^2 \left(\frac{\frac{\omega}{2}}{\frac{\omega}{2^{j_0}}}\right) = (j_0 - 1)^2 (\ln 2)^2,$$

where we also utilized the fact that $\ln \frac{H_k^+}{H_k^+ - \frac{\omega}{2} + \frac{\omega}{2^{j_0}}}$ is decreasing in H_k^+. Then we deduce that for any $t \in [\bar{t}, \bar{t} + \omega^{2-p} R^{p+\vartheta}]$,

$$\int_{B_{(1-\varrho)R}} \Psi^2(x,t) w_1 dx \ge (j_0 - 1)^2 (\ln 2)^2 \Big| B_{(1-\varrho)R} \cap \Big\{u(\cdot,t) > \mu^+ - \frac{\omega}{2^{j_0+1}}\Big\}\Big|_{\mu_{w_1}}.$$

Substituting this into (5.7), we obtain that for any $t \in [\bar{t}, \bar{t} + \omega^{2-p} R^{p+\vartheta}]$,

$$\Big| B_{(1-\varrho)R} \cap \Big\{u(\cdot,t) > \mu^+ - \frac{\omega}{2^{j_0+1}}\Big\}\Big|_{\mu_{w_1}} \le \left(\frac{j_0}{j_0 - 1}\right)^2 |B_R|_{\mu_{w_1}} \left(\frac{1}{2} + \frac{C}{j_0 \varrho^p}\right).$$

Observe that $|B_R \setminus B_{(1-\varrho)R}|_{\mu_{w_1}} \le C\varrho |B_R|_{\mu_{w_1}}$. Then by choosing $\varrho = j_0^{-\frac{1}{p+1}}$ and $j_0 = (24\overline{C}_*)^{p+1}$, we deduce that for any $t \in [\bar{t}, \bar{t} + \omega^{2-p} R^{p+\vartheta}]$,

$$\begin{aligned}
&\frac{\big| B_R \cap \big\{u(\cdot,t) > \mu^+ - \frac{\omega}{2^{j_0+1}}\big\}\big|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}} \\
&\le \frac{j_0^2}{2(j_0-1)^2} + \frac{\overline{C}_*}{\varrho^p}\left(\varrho^{p+1} + \frac{1}{j_0}\right) \le \frac{j_0^2}{2(j_0-1)^2} + \frac{2\overline{C}_*}{\sqrt[p+1]{j_0}} \le \frac{3}{4}.
\end{aligned}$$

□

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