The angular Laplacian on the lowest dimensional non-symmetric Damek-Ricci space

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Abstract

We compute the angular Laplacian on the 7-dim non-symmetric Damek-Ricci space $S = NA$, where N is the complexified Heisenberg group.

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1 Introduction

Let $S = NA$ be a Damek-Ricci space, i.e., the semidirect product of a (connected and simply connected) nilpotent Lie group N of Heisenberg type [9] and the one-dimensional Lie group $A \cong \mathbb{R}^+$ acting on N by anisotropic dilations. When S is equipped with a suitable left-invariant Riemannian metric γ_S, S becomes a (noncompact, simply connected) homogeneous harmonic Riemannian space [6, 7]. Conversely, every such space is a Damek-Ricci space if we exclude $\mathbb{R}^n$ and the "degenerate" case of real hyperbolic spaces (see [8], Corollary 1.2). We refer to [10] for a nice introduction to the geometry and harmonic analysis on Damek-Ricci spaces.

We can identify S with the unit ball B in $\mathfrak{s}$ via the Cayley transform $C : S \to B$ [5, 10]:

$$S = NA \overset{C}{\cong} B = \{(V, Z, t) \in \mathfrak{s} : \ |V|^2 + |Z|^2 + t^2 < 1\}.$$

Here $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$ and $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ are the Lie algebras of S and N, respectively, where $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $\mathfrak{v}$ its orthogonal complement in $\mathfrak{n}$, and $\mathfrak{a} = \mathrm{Lie}A \simeq \mathbb{R}$ is a 1-dimensional Lie algebra with a scalar product.

In the ball model, endowed with the transported metric $\gamma_B = C^{-1\,*}\gamma_S$, the geodesics through the origin are the diameters, i.e., $C(\mathrm{Exp}_e\, r\,\omega) = \mathrm{th}\frac{r}{2}\,\omega$ for $r \in \mathbb{R}$ and $\omega \in \mathfrak{s}$, $|\omega| = 1$, and the Riemannian sphere $S(r)$ of radius $r > 0$ (centered at the origin) is just the Euclidean sphere $S(R)$ of radius $R = \mathrm{th}\,\frac{r}{2}$ (see, [10], Thm. 10).

For nonsymmetric S the geodesic spheres $S(r)$ are not homogeneous, i.e., there is no subgroup of isometries acting transitively on them. We do have the analogue of the group M of symmetric space theory, namely the group of orthogonal automorphisms of the H-type Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, which however is not transitive on the unit sphere $S^{p+q} = \partial B$ ($p = \dim \mathfrak{v}$, $q = \dim \mathfrak{z}$).

Let $\gamma_{S(r)}$ be the induced metric on $S(r)$, and let $L_{S(r)}$ be the associated induced Laplacian (the "angular" Laplacian). Since $S(r) \simeq S^{p+q}$ is not homogeneous, an explicit formula for $L_{S(r)}$ is still lacking and its spectrum is unknown. Only partial results have been obtained in the biradial and in the $\mathfrak{v}$-radial case [1, 2].

In this paper we compute the angular Laplacian on the lowest dimensional nonsymmetric Damek-Ricci space, namely the 7-dimensional space $S = NA$ where N is the complexified Heisenberg group. Here the induced metric on $S(r) \simeq S^6$ takes a relatively simple form (albeit non-homogeneous), and one can work out some of the general features of $L_{S(r)}$. The approach is completely different from that in [2]. In section 2 we do a calculation in pure Riemannian geometry by working in bispherical coordinates on S^6. We confirm the general result in [2] about the $\mathfrak{v}$-radial part of $L_{S(r)}$, and we give additional details on the non-$\mathfrak{v}$-radial part. This is quite complicated, even in this simple example, and a full spectral analysis of $L_{S(r)}$ seems out of reach at the moment.

In section 3 we present a more intrinsic approach. Instead of coordinates, we use moving frames adapted to the (local) decomposition of the tangent bundle on $S^6 \simeq S(R)$ into horizontal and vertical parts. The horizontal distribution on S^6 is defined as the kernel of the non-round part of the induced metric. It is a smooth distribution of dimension 4 at generic points of S^6. Using the explicit form of the induced metric, one easily proves that the angular Laplacian is the sum of a term proportional to the round Laplacian plus a differential operator L' which is purely vertical. This gives a nice interpretation of the result obtained in section 2 using coordinates, and it is analogous to the symmetric case situation [4]. The computation of L' requires an explicit basis for vertical vector fields. This presents several subtleties, but it is set up and almost completed in subsection 3.3.

2 Calculation in coordinates

2.1 The induced metric in bispherical coordinates on S^6

Let $S = NA$ be the lowest (=7) dimensional non-symmetric Damek-Ricci space. Then $q = 2$ (dimension of the center $\mathfrak{z}$ of $\mathfrak{n}$), $p = 4$ (dimension of the orthogonal complement $\mathfrak{v}$ of $\mathfrak{z}$ in $\mathfrak{n}$), i.e., $\mathfrak{z} = \mathbb{R}^2$ and $\mathfrak{v} = \mathbb{R}^4$, with commutations (cf. [10], p. 67)

$$\begin{aligned}[V, V'] &= [(a, b, c, d), (a', b', c', d')] \\ &= (ab' - ba' + dc' - cd', ac' - ca' + bd' - db').\end{aligned}$$

N is just the 6-dimensional complexified Heisenberg group. We have the endomorphisms J_1, J_2 of $\mathfrak{v}$ defined by $[V, V'] = (\langle J_1V, V'\rangle, \langle J_2V, V'\rangle)$, and given explicitly as 4×4 matrices in the canonical basis of $\mathbb{R}^4$ by:

$$J_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

whence

$$J_1J_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Write the unit sphere in $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a} = \mathbb{R}^6$ as

$$S^6 = \{(V, Z, t) \in \mathbb{R}^4 \oplus \mathbb{R}^2 \oplus \mathbb{R} : \ |V|^2 + |Z|^2 + t^2 = 1\},$$

and let γ_{S^6} be the round metric on S^6. Then the metric γ_S in geodesic polar coordinates $(r, \omega) = (r, (V, Z, t)) \in [0, +\infty) \times S^6$ is given by $\gamma_S = dr^2 + \gamma_{S(r)}$, where the induced metric on the Riemannian sphere $S(r)$ is the following r-dependent metric on S^6 (cf. [1], Theorem 3.3):

$$\begin{aligned}\gamma_{S(r)} = 4\operatorname{sh}^2\tfrac{r}{2}\,\gamma_{S^6} + 4\operatorname{sh}^4\tfrac{r}{2}\,\Bigg\{&\Big|[V, dV] + tdZ - Zdt\Big|^2 + \Big(\langle J_1J_2V, dV\rangle + z_1dz_2 - z_2dz_1\Big)^2 \\ &- \frac{1}{\big((1 - t\operatorname{th}\frac{r}{2})^2 + \operatorname{th}^2\frac{r}{2}\,|Z|^2\big)^2}\Big(\operatorname{th}^2\tfrac{r}{2}\,|V|^2(z_1dz_2 - z_2dz_1) \\ &+ 2\operatorname{th}\tfrac{r}{2}(1 - t\operatorname{th}\tfrac{r}{2})\big(z_1[V, dV]_2 - z_2[V, dV]_1\big) + \big((1 - t\operatorname{th}\tfrac{r}{2})^2 - \operatorname{th}^2\tfrac{r}{2}\,|Z|^2\big)\langle J_1J_2V, dV\rangle\Big)^2\Bigg\}.\end{aligned} \tag{2.1}$$

Here $Z = (z_1, z_2) \in \mathfrak{z}$ and if $V = (v_1, v_2, v_3, v_4) \in \mathfrak{v}$ then

$$\begin{cases} [V, dV]_1 = \langle J_1 V, dV \rangle = v_1 dv_2 - v_2 dv_1 + v_4 dv_3 - v_3 dv_4, \\ [V, dV]_2 = \langle J_2 V, dV \rangle = v_1 dv_3 - v_3 dv_1 + v_2 dv_4 - v_4 dv_2, \\ \langle J_1 J_2 V, dV \rangle = v_4 dv_1 - v_1 dv_4 + v_2 dv_3 - v_3 dv_2. \end{cases}$$

Of course the differentials dV, dZ, dt in (2.1) are not all independent, namely $\langle V, dV \rangle + \langle Z, dZ \rangle + t dt = 0$.

Remark 2.1. We use the following notation from now on. If ω_1, ω_2 are 1-forms, then obviously $\omega_1 \otimes \omega_2$ is different from $\omega_2 \otimes \omega_1$. To simplify our formulas, we shall always omit the tensor product symbol and write $\omega_1 \omega_2$ for the symmetrized product divided by 2, i.e., $\omega_1 \omega_2 = (\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1)/2$. Then, for instance,

$$(\omega_1 + \omega_2)^2 = \omega_1^2 + \omega_2^2 + 2\omega_1 \omega_2.$$

We interpret the induced metric (2.1) as the sum of a constant curvature term plus a "perturbation" term (the curly bracket).

The problem is to write down this metric in a suitable coordinate system on S^6, $g_{ij} = (\gamma_{S(r)})_{ij}$, compute the inverse metric g^{ij}, and then compute the angular Laplacian $L_{S(r)}$ by the usual Riemannian formula

$$L_{S(r)} = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g}\, g^{ij}\, \partial_j \right) = g^{ij} \partial_i \partial_j + \left(\frac{\partial_i \sqrt{g}}{\sqrt{g}} \right) g^{ij} \partial_j + \left(\partial_i g^{ij} \right) \partial_j, \tag{2.2}$$

where $g = \det g_{ij}$ and summation over repeated indices is understood.

We use bispherical coordinates on S^6

$$(\rho, \phi, \alpha, \omega) \in [0, 1] \times [0, \pi] \times [0, 2\pi] \times S^3$$

defined as follows. Let

$$S^1 = \{(0, Z, 0) : \ |Z| = 1\} = S^6 \cap \mathbb{R}^2,$$

$$S^3 = \{(V, 0, 0) : \ |V| = 1\} = S^6 \cap \mathbb{R}^4,$$

be the unit spheres in $\mathfrak{z}$ and $\mathfrak{v}$, respectively, and let

$$S^2 = \{(0, Z, t) : \ |Z|^2 + t^2 = 1\} = S^6 \cap (\mathbb{R}^2 \oplus \mathbb{R})$$

be the unit sphere in $\mathfrak{z} \oplus \mathfrak{a}$. Any $\xi = (V, Z, t) \in S^6$ can be parametrized in the form

$$\begin{cases} V = \sqrt{1 - \rho^2}\, \omega \\ t = \rho \cos \phi \\ Z = \rho \sin \phi\, \tilde{\omega}, \end{cases} \tag{2.3}$$

where $\rho^2 = t^2 + |Z|^2 = 1 - |V|^2$, and

$$0 \le \rho \le 1, \quad 0 \le \phi \le \pi, \quad \omega \in S^3, \quad \tilde{\omega} = (\cos\alpha, \sin\alpha) \in S^1, \quad 0 \le \alpha \le 2\pi. \tag{2.4}$$

The choices of $\rho, \phi, \alpha, \omega$ are unique except when $V = 0$ or $Z = 0$. We do not specify yet any coordinate system on S^3 but write $\omega = (a_1, a_2, a_3, a_4)$ with $|\omega|^2 = a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$ and round metric

$$\gamma_{S^3} = |d\omega|^2 = da_1^2 + da_2^2 + da_3^2 + da_4^2, \ \text{ where } \ a_1 da_1 + a_2 da_2 + a_3 da_3 + a_4 da_4 = 0.$$

We have:

$$\begin{cases} dV = \sqrt{1-\rho^2}\, d\omega - \dfrac{\rho\, d\rho}{\sqrt{1-\rho^2}}\omega, \\ [V, dV] = (1-\rho^2)[\omega, d\omega], \quad dt = d\rho \cos\phi - \rho \sin\phi\, d\phi, \\ dZ = d\rho \sin\phi\, \tilde{\omega} + \rho\cos\phi\, d\phi\, \tilde{\omega} + \rho \sin\phi\, d\tilde{\omega} = (dz_1, dz_2), \\ tdZ - Zdt = \rho^2 d\phi\, \tilde{\omega} + \rho^2 \sin\phi\cos\phi\, d\tilde{\omega}, \\ z_1 dz_2 - z_2 dz_1 = \rho^2 \sin^2\phi\, d\alpha, \quad \langle J_1 J_2 V, dV\rangle = (1-\rho^2)\langle J_1 J_2 \omega, d\omega\rangle, \\ z_1 [V, dV]_2 - z_2 [V, dV]_1 = \rho(1-\rho^2)\sin\phi\left(\cos\alpha [\omega, d\omega]_2 - \sin\alpha [\omega, d\omega]_1\right). \end{cases} \tag{2.5}$$

Using (2.5), the round metric $\gamma_{S^6} = |dV|^2 + |dZ|^2 + dt^2$ is computed to be

$$\gamma_{S^6} = \frac{d\rho^2}{1-\rho^2} + \rho^2\left(d\phi^2 + \sin^2\phi\, d\alpha^2\right) + (1-\rho^2)\,\gamma_{S^3}. \tag{2.6}$$

Using (2.5) and (2.6) in (2.1) gives the induced metric in bispherical coordinates:

$$\begin{aligned} \gamma_{S(r)} &= 4\,\mathrm{sh}^2 \tfrac{r}{2}\left(\tfrac{d\rho^2}{1-\rho^2} + \rho^2\left(d\phi^2 + \sin^2\phi\, d\alpha^2\right) + (1-\rho^2)\,\gamma_{S^3}\right) \\ &+ 4\,\mathrm{sh}^4 \tfrac{r}{2}\Bigg\{\rho^4\Big(d\phi^2 + \sin^2\phi\, d\alpha^2\Big) + (1-\rho^2)^2\Big([\omega, d\omega]_1^2 + [\omega, d\omega]_2^2 + \langle J_1 J_2 \omega, d\omega\rangle^2\Big) \\ &+ 2\rho^2(1-\rho^2)\Bigg(d\phi\Big(\cos\alpha\,[\omega, d\omega]_1 + \sin\alpha\,[\omega, d\omega]_2\Big) \\ &+ \sin\phi\, d\alpha\Big(\sin\phi\,\langle J_1 J_2\omega, d\omega\rangle - \cos\phi\big(\sin\alpha\,[\omega, d\omega]_1 - \cos\alpha\,[\omega, d\omega]_2\big)\Big)\Bigg) \\ &- \frac{(1-\rho^2)^2}{\left(1 + \rho^2 \mathrm{th}^2 \frac{r}{2} - 2\rho\cos\phi\,\mathrm{th}\frac{r}{2}\right)^2}\Bigg[\mathrm{th}^2 \tfrac{r}{2}\,\rho^2 \sin^2\phi\, d\alpha \\ &+ 2\,\mathrm{th}\tfrac{r}{2}\left(1 - \rho\cos\phi\,\mathrm{th}\tfrac{r}{2}\right)\rho\sin\phi\Big(\cos\alpha\,[\omega, d\omega]_2 - \sin\alpha\,[\omega, d\omega]_1\Big) \\ &+ \langle J_1 J_2 \omega, d\omega\rangle\Big(\left(1 - \rho\cos\phi\,\mathrm{th}\tfrac{r}{2}\right)^2 - \mathrm{th}^2\tfrac{r}{2}\,\rho^2\sin^2\phi\Big)\Bigg]^2\Bigg\}. \end{aligned} \tag{2.7}$$

Here

$$\begin{cases} [\omega, d\omega]_1 = a_1 da_2 - a_2 da_1 + a_4 da_3 - a_3 da_4, \\ [\omega, d\omega]_2 = a_1 da_3 - a_3 da_1 + a_2 da_4 - a_4 da_2, \\ \langle J_1 J_2 \omega, d\omega \rangle = a_4 da_1 - a_1 da_4 + a_2 da_3 - a_3 da_2. \end{cases} \tag{2.8}$$

Remarks.

1) The differential $d\rho$ only occurs in the "unperturbed" part of the metric, i.e., in γ_{S^6}, thus the ρ-coordinate decouples from the other coordinates and

$$g^{\rho\rho} = \frac{1}{g_{\rho\rho}} = \frac{1-\rho^2}{4\operatorname{sh}^2 \frac{r}{2}}, \qquad g^{\rho j} = g_{\rho j} = 0 \ \ \forall j \neq \rho.$$

2) The differential $d\phi$ occurs in the curly bracket only in the r-independent part (not in the square bracket).

3) There are no terms in $d\phi\, d\alpha$ so $g_{\phi\alpha} = 0$.

The metric $\gamma_{S(r)}$ is not homogeneous. However it is invariant under the group M of orthogonal automorphisms of S. This group is trivial on $\mathfrak{a}$, and leaves $\mathfrak{z}$ and $\mathfrak{v}$ invariant. Moreover, it is known that M is transitive on the unit spheres in $\mathfrak{z}$ and $\mathfrak{v}$, i.e., on $S^1 \times S^3$. In view of this invariance, we can, as a first step, compute our metric at any convenient point $(\tilde\omega_0, \omega_0) \in S^1 \times S^3$, for example at $\tilde\omega_0 = (1,0)$ (i.e., $\alpha_0 = 0$) and $\omega_0 = (1,0,0,0)$. Let $P_0 = (\rho, \phi, \alpha_0, \omega_0)$. Since

$$a_1 = \sqrt{1 - a_2^2 - a_3^2 - a_4^2}, \qquad da_1 = -\frac{a_2 da_2 + a_3 da_3 + a_4 da_4}{\sqrt{1 - a_2^2 - a_3^2 - a_4^2}},$$

we take (a_2, a_3, a_4) as coordinates around P_0 on S^3. Note that $\langle J_1 \omega_0, \omega \rangle = [\omega_0, \omega]_1 = a_2$, $\langle J_2 \omega_0, \omega \rangle = [\omega_0, \omega]_2 = a_3$, and $\langle J_1 J_2 \omega_0, \omega \rangle = -a_4$. Setting $a_1 = 1$ and $a_2 = a_3 = a_4 = 0$ in (2.8) gives at P_0:

$$[\omega, d\omega] = (da_2, da_3), \qquad \langle J_1 J_2 \omega, d\omega \rangle = -da_4,$$

and of course $da_1 = 0$, $\gamma_{S^3}|_{P_0} = da_2^2 + da_3^2 + da_4^2$. The metric (2.7) at the point P_0 in the coordinate system $(\rho, \phi, \alpha, a_2, a_3, a_4)$ becomes

$$\begin{aligned} \gamma_{S(r)}|_{P_0} &= 4\operatorname{sh}^2 \tfrac{r}{2} \left(\tfrac{d\rho^2}{1-\rho^2} + \rho^2 \left(d\phi^2 + \sin^2\phi\, d\alpha^2 \right) + (1-\rho^2) \left(da_2^2 + da_3^2 + da_4^2 \right) \right) \\ &+ 4\operatorname{sh}^4 \tfrac{r}{2} \Big\{ \rho^4 \Big(d\phi^2 + \sin^2\phi\, d\alpha^2 \Big) + (1-\rho^2)^2 \Big(da_2^2 + da_3^2 + da_4^2 \Big) \\ &+ 2\rho^2(1-\rho^2) \Big(d\phi\, da_2 + \sin\phi\, d\alpha \Big(-\sin\phi\, da_4 + \cos\phi\, da_3 \Big) \Big) \\ &- \frac{(1-\rho^2)^2}{\left(1 + \rho^2 \operatorname{th}^2 \frac{r}{2} - 2\rho\cos\phi \operatorname{th} \frac{r}{2}\right)^2} \Big[\operatorname{th}^2 \tfrac{r}{2}\, \rho^2 \sin^2\phi\, d\alpha \\ &+ 2\operatorname{th} \tfrac{r}{2} \left(1 - \rho\cos\phi \operatorname{th}\tfrac{r}{2}\right) \rho \sin\phi\, da_3 + da_4 \Big(\operatorname{th}^2 \tfrac{r}{2}\, \rho^2 \sin^2\phi - \left(1 - \rho\cos\phi \operatorname{th}\tfrac{r}{2}\right)^2 \Big) \Big]^2 \Big\}. \end{aligned} \tag{2.9}$$

Note that the coordinates ϕ and a_2 only couple with each other. Expanding the square gives terms in $d\alpha^2$, da_3^2, da_4^2, $d\alpha da_3$, $d\alpha da_4$, $da_3 da_4$. Reordering the coordinates as $(\rho, \phi, a_2, \alpha, a_3, a_4)$, we get from (2.9) the metric $g_{ij} = (\gamma_{S(r)})_{ij}$ at P_0 in the following block-diagonal form:

$$g_{ij}|_{P_0} = \left(\begin{array}{c|cc|ccc} g_{\rho\rho} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & g_{\phi\phi} & g_{\phi a_2} & 0 & 0 & 0 \\ 0 & g_{\phi a_2} & g_{a_2 a_2} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & g_{\alpha\alpha} & g_{\alpha a_3} & g_{\alpha a_4} \\ 0 & 0 & 0 & g_{\alpha a_3} & g_{a_3 a_3} & g_{a_3 a_4} \\ 0 & 0 & 0 & g_{\alpha a_4} & g_{a_3 a_4} & g_{a_4 a_4} \end{array}\right) = 4\,\mathrm{sh}^2\frac{r}{2}\,\gamma_{ij} + 4\,\mathrm{sh}^4\frac{r}{2}\,h_{ij}, \qquad (2.10)$$

where γ_{ij} is the round metric at P_0,

$$\gamma_{ij} = (\gamma_{S^6})_{ij}|_{P_0} = \left(\begin{array}{c|cc|ccc} \frac{1}{1-\rho^2} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \rho^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\rho^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \rho^2\sin^2\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\rho^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\rho^2 \end{array}\right),$$

and the "perturbation" term h_{ij} is given by

$$h_{ij} = \left(\begin{array}{c|cc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \rho^4 & \rho^2(1-\rho^2) & 0 & 0 & 0 \\ 0 & \rho^2(1-\rho^2) & (1-\rho^2)^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & h_{\alpha\alpha} & h_{\alpha a_3} & h_{\alpha a_4} \\ 0 & 0 & 0 & h_{\alpha a_3} & h_{a_3 a_3} & h_{a_3 a_4} \\ 0 & 0 & 0 & h_{\alpha a_4} & h_{a_3 a_4} & h_{a_4 a_4} \end{array}\right).$$

The 2×2 block in the plane (ϕ, a_2) is

$$\begin{pmatrix} g_{\phi\phi} & g_{\phi a_2} \\ g_{\phi a_2} & g_{a_2 a_2} \end{pmatrix} = 4\,\mathrm{sh}^2\tfrac{r}{2}\begin{pmatrix} \rho^2 & 0 \\ 0 & 1-\rho^2 \end{pmatrix} + 4\,\mathrm{sh}^4\tfrac{r}{2}\begin{pmatrix} \rho^4 & \rho^2(1-\rho^2) \\ \rho^2(1-\rho^2) & (1-\rho^2)^2 \end{pmatrix}.$$

The inverse of this block is

$$\begin{pmatrix} g^{\phi\phi} & g^{\phi a_2} \\ g^{\phi a_2} & g^{a_2 a_2} \end{pmatrix} = \frac{1}{4\,\mathrm{sh}^2\frac{r}{2}}\begin{pmatrix} \frac{1}{\rho^2} & 0 \\ 0 & \frac{1}{1-\rho^2} \end{pmatrix} - \frac{1}{4\,\mathrm{ch}^2\frac{r}{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The 3×3 lower block in h_{ij} has entries (here $R = \operatorname{th} \frac{r}{2}$)

$$
\begin{cases}
h_{\alpha\alpha} = \rho^4 \sin^2\phi \left(1 - \dfrac{(1-\rho^2)^2 R^4 \sin^2\phi}{\left(1+\rho^2 R^2 - 2R\rho\cos\phi\right)^2}\right), \\
h_{a_3a_3} = (1-\rho^2)^2 \left(1 - \dfrac{4 R^2 \rho^2 \sin^2\phi (1 - R\rho\cos\phi)^2}{\left(1+\rho^2 R^2 - 2R\rho\cos\phi\right)^2}\right), \\
h_{a_4a_4} = (1-\rho^2)^2 \left(1 - \dfrac{\left(R^2 \rho^2 \sin^2\phi - (1-R\rho\cos\phi)^2\right)^2}{\left(1+\rho^2 R^2 - 2R\rho\cos\phi\right)^2}\right), \\
h_{\alpha a_3} = \rho^2 (1-\rho^2) \sin\phi \left(\cos\phi - \dfrac{2(1-\rho^2) R^3 \rho \sin^2\phi (1 - R\rho\cos\phi)}{\left(1+\rho^2 R^2 - 2R\rho\cos\phi\right)^2}\right), \\
h_{\alpha a_4} = -\rho^2 (1-\rho^2) \sin^2\phi \left(1 + \dfrac{(1-\rho^2) R^2 \left(R^2 \rho^2 \sin^2\phi - (1-R\rho\cos\phi)^2\right)}{\left(1+R^2\rho^2 - 2R\rho\cos\phi\right)^2}\right), \\
h_{a_3a_4} = -\dfrac{2(1-\rho^2)^2 R\rho \sin\phi (1-R\rho\cos\phi) \left(R^2\rho^2 \sin^2\phi - (1-R\rho\cos\phi)^2\right)}{\left(1+\rho^2 R^2 - 2R\rho\cos\phi\right)^2}.
\end{cases}
$$

Note the kernel in the denominators

$$
\left(1+\rho^2 R^2 - 2R\rho\cos\phi\right)^2 = \left((1-Rt)^2 + R^2|Z|^2\right)^2.
$$

2.2 The inverse metric and the induced Laplacian

The first important check is to show that the square root of the metric determinant at P_0 is

$$
\sqrt{g}|_{P_0} = 2^6 \operatorname{sh}^6 \tfrac{r}{2} \operatorname{ch}^2 \tfrac{r}{2} \, \rho^2 \, (1-\rho^2) \, \sin\phi, \tag{2.11}
$$

in agreement with the general formula for the Riemannian volume element in geodesic polar coordinates on a Damek-Ricci space. Indeed this volume element reads Vol $= dr\, d\sigma_{S(r)}$, where the induced measure on the Riemannian sphere $S(r)$ is [10]

$$
d\sigma_{S(r)} = 2^{p+q} \left(\operatorname{sh} \tfrac{r}{2}\right)^{p+q} \left(\operatorname{ch} \tfrac{r}{2}\right)^q \, d\omega_{S^{p+q}},
$$

$d\omega_{S^n}$ denoting the volume element of the round metric on S^n. In bispherical coordinates on S^{p+q} we have [1]

$$
d\omega_{S^{p+q}} = \rho^q \, (1-\rho^2)^{\frac{p}{2}-1} \, (\sin\phi)^{q-1} \, d\rho \, d\phi \, d\omega_{S^{p-1}} \, d\omega_{S^{q-1}}.
$$

In our case $q = 2$, $p = 4$, $d\omega_{S^{q-1}} = d\omega_{S^1} = d\alpha$, and $d\omega_{S^{p-1}}|_{P_0} = d\omega_{S^3}|_{P_0} = da_2 da_3 da_4$.

Now by (2.10):

- the determinant of the 1×1 block is

$$
g_{\rho\rho} = 4(1-\rho^2)^{-1} \operatorname{sh}^2 \tfrac{r}{2};
$$

- the determinant of the 2×2 block is

$$2^4\rho^2(1-\rho^2)\mathrm{sh}^4\tfrac{r}{2}\,\mathrm{ch}^2\tfrac{r}{2};$$

- the determinant of the 3×3 block is computed to be

$$2^6\rho^2(1-\rho^2)^2\mathrm{sh}^6\tfrac{r}{2}\,\mathrm{ch}^2\tfrac{r}{2}\sin^2\phi.$$

This establishes (2.11).

The main calculation is now to find the inverse of the 3×3 block in $g_{ij}|_{P_0}$. Since $\gamma_{S(r)}$ is the sum of a constant curvature metric plus a "perturbation" term, and looking at the inverse of the 2×2 block, we expect that the inverse metric $g^{ij}|_{P_0}$ in the coordinate frame $(\rho, \phi, a_2, \alpha, a_3, a_4)$ will take the form

$$g^{ij}|_{P_0} = \frac{1}{4\,\mathrm{sh}^2\,\frac{r}{2}}\,\gamma^{ij} - \frac{1}{4\,\mathrm{ch}^2\,\frac{r}{2}}\,k^{ij}, \tag{2.12}$$

where γ^{ij} is the inverse of γ_{ij},

$$\gamma^{ij} = \left(\begin{array}{c|cc|ccc} 1-\rho^2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \frac{1}{\rho^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-\rho^2} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{\rho^2\sin^2\phi} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{1-\rho^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1-\rho^2} \end{array}\right), \tag{2.13}$$

and k^{ij} is a block-diagonal matrix of the form

$$k^{ij} = \left(\begin{array}{c|cc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & k^{\alpha\alpha} & k^{\alpha a_3} & k^{\alpha a_4} \\ 0 & 0 & 0 & k^{\alpha a_3} & k^{a_3 a_3} & k^{a_3 a_4} \\ 0 & 0 & 0 & k^{\alpha a_4} & k^{a_3 a_4} & k^{a_4 a_4} \end{array}\right), \tag{2.14}$$

with the k^{ij} in the lower block depending on r, in general. In fact, we have the following result.

Proposition 2.2. *The inverse of the* 3×3 *block in* $g_{ij}|_{P_0}$ *has entries:*

$$\begin{cases} g^{\alpha\alpha}|_{P_0} = \dfrac{1-R^2}{4R^2}\dfrac{1}{\rho^2\sin^2\phi} - \dfrac{1-R^2}{4}\dfrac{1}{\sin^2\phi}, \\ g^{\alpha a_3}|_{P_0} = -\dfrac{1-R^2}{4}\cot\phi, \\ g^{\alpha a_4}|_{P_0} = \dfrac{1-R^2}{4}, \\ g^{a_3a_3}|_{P_0} = \dfrac{1-R^2}{4R^2(1-\rho^2)} - \dfrac{1-R^2}{4}k^{a_3a_3}, \\ g^{a_4a_4}|_{P_0} = \dfrac{1-R^2}{4R^2(1-\rho^2)} - \dfrac{1-R^2}{4}k^{a_4a_4}, \\ g^{a_3a_4}|_{P_0} = -\dfrac{1-R^2}{4}k^{a_3a_4}, \end{cases}$$

where $k^{a_3a_3} = 1 - k^{a_4a_4}$*, and*

$$\begin{cases} k^{a_4a_4} = \dfrac{\rho^2R^2\sin^2\phi\big(3+\rho^2R^2-4\rho R\cos\phi\big)}{(1+\rho^2R^2-2\rho R\cos\phi)^2} \\ \qquad = \rho^2R^2\sin^2\phi\left(\dfrac{1}{1+\rho^2R^2-2\rho R\cos\phi} + \dfrac{2(1-\rho R\cos\phi)}{(1+\rho^2R^2-2\rho R\cos\phi)^2}\right), \\ k^{a_3a_4} = \dfrac{\rho R\sin\phi\Big(2+4\rho^2R^2\cos^2\phi - \rho R\cos\phi\big(5+\rho^2R^2\big)\Big)}{(1+\rho^2R^2-2\rho R\cos\phi)^2} \\ \qquad = \rho R\sin\phi\left(1 - \dfrac{\rho^2R^2}{1+\rho^2R^2-2\rho R\cos\phi} + \dfrac{(1-\rho R\cos\phi)(1-\rho^2R^2)}{(1+\rho^2R^2-2\rho R\cos\phi)^2}\right). \end{cases}$$

Proof. This is just brute force calculation for the 3×3 block of $g_{ij}|_{P_0}$ in (2.10). □

Since $\frac{1-R^2}{R^2} = \frac{1}{\mathrm{sh}^2\frac{r}{2}}$ and $1-R^2 = \frac{1}{\mathrm{ch}^2\frac{r}{2}}$, we see that $g^{ij}|_{P_0}$ has indeed the claimed form (2.12)-(2.14), with the 3×3 block in k^{ij} given by

$$k^{ij} = \left(\begin{array}{c|cc} \frac{1}{\sin^2\phi} & \cot\phi & -1 \\ \hline \cot\phi & k^{a_3a_3} & k^{a_3a_4} \\ -1 & k^{a_3a_4} & k^{a_4a_4} \end{array}\right). \tag{2.15}$$

Note that the entries $k^{\alpha\alpha} = \frac{1}{\sin^2\phi}$, $k^{\alpha a_3} = \cot\phi$ and $k^{\alpha a_4} = -1$ are rather simple and r-independent, but the remaining entries $k^{a_3a_3}$, $k^{a_3a_4}$, $k^{a_4a_4}$ are quite complicated and explicitly depend on r.

By M-invariance, we can now claim that the inverse metric g^{ij} at any point $P \in S^6$ will take the form (2.12), namely

$$g^{ij}|_P = \frac{1}{4\,\mathrm{sh}^2\frac{r}{2}}\,\gamma^{ij} - \frac{1}{4\,\mathrm{ch}^2\frac{r}{2}}\,k^{ij}, \tag{2.16}$$

with γ^{ij} the inverse round metric at P (see below), and k^{ij} a suitable matrix with $k^{\rho i} = 0, \forall i$, and the 5×5 block not necessarily in block-diagonal form, in general. The expression of k^{ij} at a generic point $P \in S^6$ will be quite complicated and will not be given here. On the other hand the round metric on S^6 and its inverse at a point $P = (\rho, \phi, \alpha, a_2, a_3, a_4)$ can be obtained using the following formulae in (2.6).

The round metric on S^3 in the coordinates (a_2, a_3, a_4) is easily seen to be:

$$(\gamma_{S^3})_{ij} = \begin{pmatrix} \frac{1-a_3^2-a_4^2}{1-a_2^2-a_3^2-a_4^2} & \frac{a_2a_3}{1-a_2^2-a_3^2-a_4^2} & \frac{a_2a_4}{1-a_2^2-a_3^2-a_4^2} \\ \frac{a_2a_3}{1-a_2^2-a_3^2-a_4^2} & \frac{1-a_2^2-a_4^2}{1-a_2^2-a_3^2-a_4^2} & \frac{a_3a_4}{1-a_2^2-a_3^2-a_4^2} \\ \frac{a_2a_4}{1-a_2^2-a_3^2-a_4^2} & \frac{a_3a_4}{1-a_2^2-a_3^2-a_4^2} & \frac{1-a_2^2-a_3^2}{1-a_2^2-a_3^2-a_4^2} \end{pmatrix}.$$

From this one computes

$$\det \gamma_{S^3} = \tfrac{1}{1-a_2^2-a_3^2-a_4^2},$$

and inverse metric

$$(\gamma_{S^3}^{-1})^{ij} = \begin{pmatrix} 1-a_2^2 & -a_2a_3 & -a_2a_4 \\ -a_2a_3 & 1-a_3^2 & -a_3a_4 \\ -a_2a_4 & -a_3a_4 & 1-a_4^2 \end{pmatrix}.$$

The round Laplacian on S^3 can then be written in the form

$$L_{S^3} = \partial_{a_2}^2 + \partial_{a_3}^2 + \partial_{a_4}^2 - \big(a_2\partial_{a_2} + a_3\partial_{a_3} + a_4\partial_{a_4}\big)^2 - 2\big(a_2\partial_{a_2} + a_3\partial_{a_3} + a_4\partial_{a_4}\big). \quad (2.17)$$

The round metric on S^6, its inverse, and the square root of the metric determinant at $P = (\rho, \phi, \alpha, a_2, a_3, a_4)$ are obtained from (2.6):

$$\gamma_{ij} = (\gamma_{S^6})_{ij}|_P = \left(\begin{array}{c|cc|ccc} \frac{1}{1-\rho^2} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \rho^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho^2\sin^2\phi & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & (1-\rho^2)(\gamma_{S^3})_{ij} & \\ 0 & 0 & 0 & & & \end{array}\right),$$

$$\gamma^{ij} = (\gamma_{S^6}^{-1})^{ij}|_P = \left(\begin{array}{c|cc|ccc} 1-\rho^2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \frac{1}{\rho^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\rho^2\sin^2\phi} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & \frac{1}{1-\rho^2}(\gamma_{S^3}^{-1})^{ij} & \\ 0 & 0 & 0 & & & \end{array}\right),$$

$$\sqrt{\gamma} = \sqrt{\det \gamma_{ij}} = \frac{\rho^2(1-\rho^2)\sin\phi}{\sqrt{1-a_2^2-a_3^2-a_4^2}}. \quad (2.18)$$

From this one computes the round Laplacian on S^6 (see also [1], (4.12)):

$$\begin{aligned} L_{S^6} &= \frac{1}{\sqrt{\gamma}}\partial_i\left(\sqrt{\gamma}\,\gamma^{ij}\,\partial_j\right) = \gamma^{ij}\partial_i\partial_j + \left(\frac{\partial_i\sqrt{\gamma}}{\sqrt{g}}\right)\gamma^{ij}\partial_j + \left(\partial_i\gamma^{ij}\right)\partial_j \\ &= (1-\rho^2)\partial_\rho^2 + \left(\tfrac{2}{\rho} - 6\rho\right)\partial_\rho + \tfrac{1}{\rho^2}L_{S^2} + \tfrac{1}{1-\rho^2}L_{S^3}, \end{aligned} \quad (2.19)$$

where L_{S^3} is the round Laplacian (2.17) on S^3 (unit sphere in $\mathfrak{v}$), and

$$L_{S^2} = \partial_\phi^2 + \cot\phi\, \partial_\phi + \frac{1}{\sin^2\phi}\partial_\alpha^2$$

is the round Laplacian on S^2 (unit sphere in $\mathfrak{z} \oplus \mathfrak{a}$).

Substitution of (2.16) in (2.2) gives the following result.

Theorem 2.3. *The angular Laplacian at any point $P \in S^6$ can be written in the form*

$$L_{S(r)} = \frac{1}{4\,\mathrm{sh}^2\frac{r}{2}}\, L_{S^6} - \frac{1}{4\,\mathrm{ch}^2\frac{r}{2}}\, L', \tag{2.20}$$

where L_{S^6} is the round Laplacian (2.19) on S^6, and L' is the differential operator on S^6

$$L' = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}\, k^{ij}\,\partial_j\right) = k^{ij}\partial_i\partial_j + \left(\frac{\partial_i\sqrt{g}}{\sqrt{g}}\right) k^{ij}\partial_j + \left(\partial_i k^{ij}\right)\partial_j, \tag{2.21}$$

where k^{ij} is the non-round part of the inverse metric in (2.16). Here $\sqrt{g} = \sqrt{\det g_{ij}} = 2^6\,\mathrm{sh}^6\frac{r}{2}\,\mathrm{ch}^2\frac{r}{2}\sqrt{\gamma}$, with $\sqrt{\gamma} = \sqrt{\det\gamma_{ij}}$ given by (2.18).

We observe that formula (2.20) for $L_{S(r)}$ remarkably reminds the symmetric-case formula (cf. [4], (6.16)), but L' is now r-dependent.

Using (2.14)-(2.15), we can write down the symbol $k^{ij}\partial_i\partial_j$ of L' at P_0. Obviously it depends on R in the terms $k^{a_3a_3}\partial_{a_3}^2 + k^{a_4a_4}\partial_{a_4}^2 + 2k^{a_3a_4}\partial_{a_3}\partial_{a_4}$, but its $\mathfrak{v}$-radial part (the one with only the derivatives ∂_ϕ and ∂_α) is R-independent:

$$k^{\phi\phi}\partial_\phi^2 + k^{\alpha\alpha}\partial_\alpha^2 = \partial_\phi^2 + \tfrac{1}{\sin^2\phi}\partial_\alpha^2,$$

as well as the terms

$$k^{a_2a_2}\partial_{a_2}^2 + 2k^{\phi a_2}\partial_\phi\partial_{a_2} + 2k^{\alpha a_3}\partial_\alpha\partial_{a_3} + 2k^{\alpha a_4}\partial_\alpha\partial_{a_4} = \partial_{a_2}^2 + 2\partial_\phi\partial_{a_2} + 2\cot\phi\,\partial_\alpha\partial_{a_3} - 2\partial_\alpha\partial_{a_4}.$$

For the term $\left(\frac{\partial_i\sqrt{g}}{\sqrt{g}}\right)k^{ij}\partial_j$ in (2.21) we have $(\partial_\phi\sqrt{g})/\sqrt{g} = \cot\phi$ and

$$\partial_\alpha\sqrt{g}|_{P_0} = 0 = \partial_{a_2}\sqrt{g}|_{P_0} = \partial_{a_3}\sqrt{g}|_{P_0} = \partial_{a_4}\sqrt{g}|_{P_0},$$

being $\sqrt{g} \propto \frac{1}{\sqrt{1-a_2^2-a_3^2-a_4^2}}$. Thus

$$\left(\frac{\partial_i\sqrt{g}}{\sqrt{g}}\right) k^{ij}\partial_j = \cot\phi\,\partial_\phi + \cot\phi\,\partial_{a_2}.$$

The last term $\left(\partial_i k^{ij}\right)\partial_j$ in (2.21) cannot be computed directly as it requires the derivatives of k^{ij} (or g^{ij}) at P_0, but we don't have the expression of the inverse metric away from P_0. However, we can proceed indirectly as follows. Recall that the first-order terms in (2.2) can also be written in terms of the Levi-Civita connection coefficients ω_{ij}^k. Let ∇ denote the covariant derivative in the full metric, and define $\nabla_{\partial_i}\partial_j = \omega_{ij}^k\partial_k$. Then

$$L_{S(r)} = g^{ij}\nabla_{\partial_i}\nabla_{\partial_j} = g^{ij}\partial_i\partial_j - g^{ij}\omega_{ij}^k\partial_k. \tag{2.22}$$

The coefficients ω_{ij}^k are given in terms of the derivatives of the metric by

$$\omega_{ij}^k = \tfrac{1}{2} g^{km} \left(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}\right).$$

This can now be computed using the general expression (2.7) of the metric. By separating out the part relative to the round Laplacian, and keeping in mind the first-order term $\left(\frac{\partial_i \sqrt{g}}{\sqrt{g}}\right) k^{ij} \partial_j$ already computed, we get by comparison of (2.22) with (2.2) that

$$\left(\partial_\alpha k^{\alpha\phi} + \partial_\phi k^{\phi\phi} + \partial_{a_2} k^{a_2\phi} + \partial_{a_3} k^{a_3\phi} + \partial_{a_4} k^{a_4\phi}\right)\Big|_{P_0} = 0,$$

$$\left(\partial_\alpha k^{\alpha\alpha} + \partial_\phi k^{\phi\alpha} + \partial_{a_2} k^{a_2\alpha} + \partial_{a_3} k^{a_3\alpha} + \partial_{a_4} k^{a_4\alpha}\right)\Big|_{P_0} = 0.$$

Thus there are no additional first-order terms in ∂_ϕ, and no first-order terms in ∂_α in L'. The $\mathfrak{v}$-radial part of L' at P_0 is then

$$\begin{aligned} L_{S^2} &= \partial_\phi^2 + \cot\phi\,\partial_\phi + \tfrac{1}{\sin^2\phi}\partial_\alpha^2 \\ &= \left(t\partial_{z_1} - z_1\partial_t\right)^2 + \left(t\partial_{z_2} - z_2\partial_t\right)^2 + \left(z_1\partial_{z_2} - z_2\partial_{z_1}\right)^2. \end{aligned} \tag{2.23}$$

By M-invariance, this result remains true at any point $P \in S^6$. Indeed, the operator L_{S^2} is clearly invariant under the action of M on $\mathfrak{z} \oplus \mathfrak{a}$ (just rotations in the $\mathfrak{z}$-plane). More specifically, the operators $\partial_\alpha^2 = (z_1\partial_{z_2} - z_2\partial_{z_1})^2$ and $(t\partial_{z_1} - z_1\partial_t)^2 + (t\partial_{z_2} - z_2\partial_t)^2$ are separately M-invariant.

We will not compute the remaining first-order terms in $(\partial_i k^{ij})\,\partial_j$, as the non-$\mathfrak{v}$-radial part of L' (the one with the derivatives with respect to $a_2, a_3.a_4$) is already quite complicated. We have obtained:

Proposition 2.4. *The operator L' at P_0 is given by*

$$\begin{aligned} L' = \partial_\phi^2 &+ \cot\phi\,\partial_\phi + \frac{1}{\sin^2\phi}\partial_\alpha^2 \\ &+ \partial_{a_2}^2 + 2\partial_\phi\partial_{a_2} + \cot\phi\,\partial_{a_2} + 2\cot\phi\,\partial_\alpha\partial_{a_3} - 2\partial_\alpha\partial_{a_4} \\ &+ k^{a_3a_3}\partial_{a_3}^2 + k^{a_4a_4}\partial_{a_4}^2 + 2k^{a_3a_4}\partial_{a_3}\partial_{a_4} \\ &+ \left(\partial_i k^{ia_2}\right)\partial_{a_2} + \left(\partial_i k^{ia_3}\right)\partial_{a_3} + \left(\partial_i k^{ia_4}\right)\partial_{a_4}. \end{aligned}$$

The $\mathfrak{v}$-radial part of L' is given by (2.23) at every $P \in S^6$.

2.3 The $\mathfrak{v}$-radial part of the spectrum

We can now look for the eigenfunctions of $L_{S(r)}$ on S^6 that depend only on the coordinates (ρ, ϕ, α) in the bispherical coordinate chart $(\rho, \phi, \alpha, \omega)$ of S^6, i.e., independent of $\omega = (a_1, a_2, a_3, a_4) \in S^{p-1} = S^3$, or equivalently, the $\mathfrak{v}$-radial eigenfunctions of $L_{S(r)}$, depending only on (t, Z). By Proposition 2.4, the $\mathfrak{v}$-radial part of the operator $L_{S(r)}$ at any point $P \in S^6$ is given by

$$L_{S(r)} = \frac{1}{4\,\mathrm{sh}^2\,\frac{r}{2}}\left\{(1-\rho^2)\partial_\rho^2 + \left(\tfrac{2}{\rho} - 6\rho\right)\partial_\rho + \tfrac{1}{\rho^2} L_{S^2}\right\} - \frac{1}{4\,\mathrm{ch}^2\,\frac{r}{2}} L_{S^2}. \tag{2.24}$$

Recall that the regular eigenfunctions of the operator $L_{S^2} = \partial_\phi^2 + \cot\phi\,\partial_\phi + \frac{1}{\sin^2\phi}\partial_\alpha^2$ are the usual spherical harmonics $Y_n^m(\phi,\alpha)$, namely

$$\Big(\partial_\phi^2 + \cot\phi\,\partial_\phi + \tfrac{1}{\sin^2\phi}\partial_\alpha^2\Big) Y_n^m = -n(n+1)Y_n^m,$$

where (up to normalization)

$$Y_n^m(\phi,\alpha) = P_n^m(\cos\phi)\,e^{im\alpha}, \quad n \in \mathbb{Z}^+, \;\; m = -n,\dots,n,$$

P_n^m being the associated Legendre polynomials. Looking for (complex) solutions to $L_{S(r)}\chi = \lambda\chi$ in the factored form $\chi(\rho,\phi,\alpha) = f(\rho)Y(\phi,\alpha)$, we find they are given (up to normalization) by

$$\chi_{k,l,m}(\rho,\phi,\alpha) = R_l^{(1,k-l+1/2)}(2\rho^2-1)\rho^{k-l}Y_{k-l}^m(\phi,\alpha), \tag{2.25}$$

where $R_l^{(a,b)}(x)$ is a Jacobi polynomial normalized so that $R_l^{(a,b)}(1) = 1$, and

$$k,l,m \in \mathbb{Z}, \;\; k \geq l \geq 0, \quad m = -(k-l),\dots,k-l.$$

The eigenvalues are the same as in the biradial case, namely

$$\lambda_{k,l} = -\frac{(k+l)(k+l+5)}{4\,\mathrm{sh}^2\frac{r}{2}} + \frac{(k-l)(k-l+1)}{4\,\mathrm{ch}^2\frac{r}{2}}.$$

For $m = 0$ we get back the biradial eigenfunctions $\chi_{k,l} = \chi_{k,l,0}$ (those that depend only on (ρ,ϕ), i.e., on $(t,|Z|)$), see [1].

Formula (2.24) has been generalized to any Damek-Ricci space in [2], Theorem 2.1. We can check that the eigenfunctions $\chi_{k,l,m}$ in (2.25) agree with the functions $\chi_{k,l,j,i}$ in formula (2.22) of [2] for $p = 4$ and $q = 2$. To this end, we just need to take $m = \pm j$. Indeed the associated Legendre polynomials $P_{k-l}^{\pm j}(\cos\phi)$ are proportional to $(\sin\phi)^j R_{k-l-j}^{(j,j)}(\cos\phi)$, as easily proved, and of course $S_{j,i}^{(2)}(\omega_2) = S_{j,\pm}^{(2)}(\alpha) = e^{\pm ij\alpha}$ is a basis for spherical harmonics of degree j on $S^{q-1} = S^1$.

Formula (2.20) appears in [2] as formula (2.17) with $L' = L_{S^q} + L_3$, but the operator L_3 is not explicit. In this section we have obtained a more explicit form of the operator L' at a given point $P_0 \in S^6$ in the 7-dimensional example by working in bispherical coordinates. More insight into the geometrical meaning of L' will be given in the next section.

As regards the non-$\mathfrak{v}$-radial part of the spectrum, this remains largely unknown, due to the complicated form of L'. It is possible that the whole spectrum and eigenfunctions of $L_{S(r)}$ can not be computed in closed form. For example, we don't know if the eigenvalues of $L_{S(r)}$ coincide exactly with the "biradial" ones $\lambda_{k,l}$, or if "new" eigenvalues will appear. In this case the associated eigenspaces will carry no biradial or even $\mathfrak{v}$-radial element, a completely new situation compared to the symmetric case. The general eigenfunctions will be suitable linear combinations of spherical harmonics in $L^2(S^6)$. However nothing prevents, in principle, that the coefficients of these linear combinations depend explicitly on r, in contrast with the symmetric case situation. We refer to [4], subsection 8.5, for a more general discussion of this point for non-symmetric Damek-Ricci spaces.

3 The approach by moving frames

In this section we present a different approach to the computation of the angular Laplacian. It is based on a suitable (local) decomposition of the tangent bundle on $S^6 \simeq S(R)$ into horizontal and vertical parts and on the use of moving frames adapted to this decomposition rather than of coordinates.

3.1 The horizontal-vertical decomposition of the tangent bundle

It is convenient to work in the ball model B of NA. The induced metric $\gamma_{S(R)}$ on the Euclidean sphere $S(R)$ of radius $R < 1$ in B, identified with S^6 by $R\omega \to \omega$, is given by (2.1) with $R = \operatorname{th} \frac{r}{2}$, namely it is the following R-dependent metric on S^6:

$$\gamma_{S(R)} = \frac{4R^2}{1-R^2}\gamma_{S^6} + \frac{4R^4}{(1-R^2)^2}h_R, \tag{3.1}$$

where h_R is the smooth rank-2 tensor on S^6 given by

$$\begin{aligned} h_R|_{(V,Z,t)} = {} & \big|[V,dV] + tdZ - Zdt\big|^2 + \big(z_1dz_2 - z_2dz_1 + \langle J_1J_2V, dV\rangle\big)^2 \\ & - \frac{1}{((1-Rt)^2 + R^2|Z|^2)^2}\bigg(R^2|V|^2\,(z_1dz_2 - z_2dz_1) \\ & + 2R(1-Rt)\big(z_1[V,dV]_2 - z_2[V,dV]_1\big) + \big((1-Rt)^2 - R^2|Z|^2\big)\langle J_1J_2V, dV\rangle\bigg)^2. \end{aligned} \tag{3.2}$$

Recall that for $R = 1$ we have $h_1 = \lim_{R\to 1} h_R = \Theta^2 = |\Theta|^2$, where Θ is the $\mathfrak{z}$-valued 1-form on $S^{6\,*} = S^6 \setminus \{(0,0,1)\}$ given by (cf. [3], (3.20))

$$\begin{aligned} \Theta|_{(V,Z,t)} = {} & [V,dV] + tdZ - Zdt + \frac{2(z_2U_1 - z_1U_2)}{(1-t)^2 + |Z|^2} \times \\ & \times \Big\{ z_1dz_2 - z_2dz_1 + z_1[V,dV]_2 - z_2[V,dV]_1 + (1-t)\langle J_1J_2V, dV\rangle \Big\}, \end{aligned} \tag{3.3}$$

where $U_1 = (1,0)$, $U_2 = (0,1)$. Up to a scalar factor, Θ is the pull-back of the canonical 1-form on the group N by the generalized stereographic projection [3]. It can be extended to a $\mathfrak{z}$-valued 1-form on $\overline{B} \setminus \{(0,0,1)\}$. Moreover, we have a decomposition of the tangent bundle on $S^{6\,*}$ into its horizontal part $HT(S^{6\,*})$, defined as the kernel of Θ (or of $h_1 = \Theta^2$), and the vertical part $VT(S^{6\,*})$ (the orthogonal complement of $\ker\Theta$ with respect to the round metric), with $\dim HT(S^{6\,*}) = 4$, $\dim VT(S^{6\,*}) = 2$. It is proved in [3], Theorems 3.4 and 3.7, that the 1-form Θ, as well as its square, do not extend smoothly at the pole $(0,0,1)$, though they remain bounded there, and the same holds for the horizontal distribution $\ker\Theta$.

For $R < 1$ the non-round part (3.2) of the induced metric (3.1) is smooth, and it is related to h_1 by (cf. [3])

$$R^4 h_R = h_1 \circ \delta_R,$$

where δ_R is the dilation $\delta_R(V,Z,t) = (RV, RZ, Rt)$. Note that h_1 (given by (3.2) with $R=1$) is well defined on B, so the composition $h_1 \circ \delta_R$ is well defined for $R<1$. However, we anticipate that the equality $h_1 = \Theta^2$ only holds on $S^{6\,*}$ but not on B (cf. Corollary 3.4). Thus, in general, the kernel of h_R (which is just $(\ker h_1)\circ\delta_R$ by Proposition 3.1) will be different from $\ker(\Theta^2)\circ\delta_R = (\ker\Theta)\circ\delta_R$ for $R<1$.

We define the horizontal distribution $HT(S(R))$, $R<1$, as the kernel of h_R, i.e., as the following R-dependent distribution on S^6:

$$HT(S(R)) = \ker h_R = \{X \in T(S^6) :\ h_R(X,Y) = 0,\ \forall Y \in T(S^6)\}.$$

Since h_R is smooth, $HT(S(R))$ is a smooth distribution of dimension 4 at generic points of S^6, but it will change dimension at some (zero-measure) point set of S^6. Indeed there are no continuous k-dimensional distributions (continuous fields of k-planes) on S^6 for $1 \le k \le 5$ ([11], Theorem 27.18). For $k=1$, this is the well known result that even spheres do not admit continuous nowhere vanishing vector fields, or 1-forms by duality.

The vertical distribution $VT(S(R))$ is defined as the orthogonal complement of $HT(S(R))$ with respect to the round metric γ_{S^6} (or to the full metric $\gamma_{S(R)}$). Again, this is a smooth distribution of dimension 2 at generic points of S^6. For instance at the poles $x_\pm = (0,0,\pm1)$, we have $h_R|_{(0,0,\pm1)} = |dZ|^2$, $\forall R$, so the horizontal subspace is just $HT_{x_\pm}(S(R)) = \mathfrak{v}$ and the vertical one is $VT_{x_\pm}(S(R)) = \mathfrak{z}$, $\forall R$.

Note that S^6 is not a fibre bundle with fibre $S^q = S^2$ (there is no Hopf fibration for S^6). Thus we do not have an interpretation of the horizontal/vertical distributions in terms of base space/fibres of a global fibration, as in the case of symmetric Damek-Ricci spaces [4]. Nevertheless, the local decomposition

$$T(S^6) \cong HT(S(R)) \oplus VT(S(R)) \qquad (R<1)$$

will serve our purposes of describing the structure of the angular Laplacian.

The following result relates the kernels of h_R on S^6 and of h_1 on $S(R)$.

Proposition 3.1. *Let h_1 be considered as a tensor field defined on $\overline{B} \setminus \{(0,0,1)\}$. For any vector fields X, Y on $S(R)$, $R<1$, we have*

$$\big(h_1 \circ \delta_R\big)\big(X\circ\delta_R, Y\circ\delta_R\big) = \big(h_1(X,Y)\big)\circ\delta_R. \tag{3.4}$$

Consequently, X is in the kernel of h_1 on $S(R)$, $R<1$, if and only if $X\circ\delta_R$ is in the kernel of $h_1\circ\delta_R$ on S^6, and

$$HT(S(R)) = \ker h_R = \ker(h_1\circ\delta_R) = \big(\ker h_1\big)\circ\delta_R, \qquad \forall R<1. \tag{3.5}$$

Proof. We will prove (3.4) for $Y = X$. The general proof is similar but there are more terms to keep track of, due to symmetrization in X and Y. Recall that if ω_1, ω_2 are 1-forms, then

$$(\omega_1+\omega_2)^2(X,Y) = \omega_1(X)\omega_1(Y) + \omega_2(X)\omega_2(Y) + \omega_1(X)\omega_2(Y) + \omega_1(Y)\omega_2(X).$$

For $Y = X$ this simplifies to

$$(\omega_1+\omega_2)^2(X,X) = \omega_1(X)^2 + \omega_2(X)^2 + 2\omega_1(X)\omega_2(X) = \big(\omega_1(X)+\omega_2(X)\big)^2. \tag{3.6}$$

Let $X = \langle X_\mathfrak{v}, \partial_V\rangle + \langle X_\mathfrak{z}, \partial_Z\rangle + X_\mathfrak{a}\partial_t$, where the components $X_\mathfrak{v}$, $X_\mathfrak{z} = (X_{\mathfrak{z}1}, X_{\mathfrak{z}2})$ and $X_\mathfrak{a}$ are $\mathfrak{v}$-valued, $\mathfrak{z}$-valued and $\mathbb{R}$-valued functions, respectively. Then

$$X \circ \delta_R = \tfrac{1}{R}\langle X_\mathfrak{v} \circ \delta_R, \partial_V\rangle + \tfrac{1}{R}\langle X_\mathfrak{z} \circ \delta_R, \partial_Z\rangle + \tfrac{1}{R}(X_\mathfrak{a} \circ \delta_R)\partial_t. \tag{3.7}$$

Using (3.2) with $R = 1$ and (3.6), we have for $(V, Z, t) \in B$

$$\begin{aligned} h_1(X,X)|_{(V,Z,t)} &= \Big(\langle J_1V, X_\mathfrak{v}\rangle + tX_{\mathfrak{z}1} - z_1X_\mathfrak{a}\Big)^2 + \Big(\langle J_2V, X_\mathfrak{v}\rangle + tX_{\mathfrak{z}2} - z_2X_\mathfrak{a}\Big)^2 \\ &+ \Big(z_1X_{\mathfrak{z}2} - z_2X_{\mathfrak{z}1} + \langle J_1J_2V, X_\mathfrak{v}\rangle\Big)^2 - \frac{1}{((1-t)^2 + |Z|^2)^2}\Big(|V|^2\big(z_1X_{\mathfrak{z}2} - z_2X_{\mathfrak{z}1}\big) \\ &+ 2(1-t)\big(z_1\langle J_2V, X_\mathfrak{v}\rangle - z_2\langle J_1V, X_\mathfrak{v}\rangle\big) + \big((1-t)^2 - |Z|^2\big)\langle J_1J_2V, X_\mathfrak{v}\rangle\Big)^2, \end{aligned}$$

where we write $X_\mathfrak{v}, X_\mathfrak{z}$ and $X_\mathfrak{a}$ for $X_\mathfrak{v}(V,Z,t), X_\mathfrak{z}(V,Z,t)$ and $X_\mathfrak{a}(V,Z,t)$, for brevity. Here $h_1(X,X)$ is considered as a function on B in order to compose with δ_R in the next step.

Composing the function $h_1(X,X)$ with δ_R and writing $X_\mathfrak{v} \circ \delta_R$ for $X_\mathfrak{v}(RV, RZ, Rt)$, etc, gives on the one hand

$$\begin{aligned} h_1(X,X) \circ \delta_R|_{(V,Z,t)} &= h_1(X,X)|_{(RV,RZ,Rt)} \\ &= R^2\Big(\langle J_1V, X_\mathfrak{v}\circ\delta_R\rangle + tX_{\mathfrak{z}1}\circ\delta_R - z_1X_\mathfrak{a}\circ\delta_R\Big)^2 \\ &\quad + R^2\Big(\langle J_2V, X_\mathfrak{v}\circ\delta_R\rangle + tX_{\mathfrak{z}2}\circ\delta_R - z_2X_\mathfrak{a}\circ\delta_R\Big)^2 \\ &\quad + R^2\Big(z_1X_{\mathfrak{z}2}\circ\delta_R - z_2X_{\mathfrak{z}1}\circ\delta_R + \langle J_1J_2V, X_\mathfrak{v}\circ\delta_R\rangle\Big)^2 \\ &\quad - \frac{1}{((1-Rt)^2 + R^2|Z|^2)^2}\Big(R^3|V|^2\big(z_1X_{\mathfrak{z}2}\circ\delta_R - z_2X_{\mathfrak{z}1}\circ\delta_R\big) \\ &\quad + 2R^2(1-Rt)\big(z_1\langle J_2V, X_\mathfrak{v}\circ\delta_R\rangle - z_2\langle J_1V, X_\mathfrak{v}\circ\delta_R\rangle\big) \\ &\quad + R\big((1-Rt)^2 - R^2|Z|^2\big)\langle J_1J_2V, X_\mathfrak{v}\circ\delta_R\rangle\Big)^2, \end{aligned}$$

for $(V,Z,t) \in S^6$. On the other hand, when we compute $\big(h_1\circ\delta_R\big)\big(X\circ\delta_R, X\circ\delta_R\big)|_{(V,Z,t)}$ and use (3.7), we obtain exactly the same expression, as easily seen. □

We remark that h_1 must be considered as a function on $\overline{B} \setminus \{(0,0,1)\}$ in the above proposition in order to apply (3.5). In particular, since $h_1|_B \neq \Theta^2|_B$ (cf. Corollary 3.4), we cannot claim equality between $\ker h_R$ and $\ker(\Theta^2)\circ\delta_R = (\ker\Theta)\circ\delta_R$ for $R < 1$.

3.2 The induced Laplacian

Now let a superscript *hor*, *ver* denote restriction of a metric or a differential operator to the horizontal, vertical subspaces. Decompose the round metric as

$$\gamma_{S^6} = \gamma_{S^6}^{hor} + \gamma_{S^6}^{ver}.$$

Then the metric in (3.1) decomposes as

$$\gamma_{S(R)} = \gamma^{hor}_{S(R)} + \gamma^{ver}_{S(R)}, \tag{3.8}$$

where

$$\gamma^{hor}_{S(R)} = \frac{4R^2}{1-R^2}\,\gamma^{hor}_{S^6}, \qquad \gamma^{ver}_{S(R)} = \frac{4R^2}{1-R^2}\,\gamma^{ver}_{S^6} + \frac{4R^4}{(1-R^2)^2}\,h_R. \tag{3.9}$$

(The term h_R is of course vertical by the definition of the horizontal subspace as its kernel.) Thus the horizontal part of the induced metric is proportional to the horizontal part of the round metric, and the vertical part is the sum of a term proportional to $\gamma^{ver}_{S^6}$ and a term proportional to h_R. Let

$$L_{S^6} = L^{hor}_{S^6} + L^{ver}_{S^6}$$

be the decomposition of the round Laplacian into its horizontal and vertical parts. Let $L_{S(R)}$ be the Laplacian of the induced metric (3.1). Then (3.8)-(3.9) imply the following decomposition:

$$L_{S(R)} = L^{hor}_{S(R)} + L^{ver}_{S(R)}, \tag{3.10}$$

where the horizontal part is

$$L^{hor}_{S(R)} = \frac{1-R^2}{4R^2}\,L^{hor}_{S^6} = \frac{1}{4\,\mathrm{sh}^2\,\frac{r}{2}}\,L^{hor}_{S^6},$$

and the vertical part $L^{ver}_{S(R)}$ is the differential operator associated with $\gamma^{ver}_{S(R)}$ in (3.9).

Proposition 3.2. *The induced Laplacian $L_{S(R)}$ can be written in the form*

$$\begin{aligned} L_{S(R)} &= \frac{1-R^2}{4R^2}\,L_{S^6} - \frac{1-R^2}{4}L' \\ &= \frac{1}{4\,\mathrm{sh}^2\,\frac{r}{2}}\,L_{S^6} - \frac{1}{4\,\mathrm{ch}^2\,\frac{r}{2}}\,L', \end{aligned}$$

where L' is the vertical differential operator defined by

$$\frac{1-R^2}{4}L' = \frac{1-R^2}{4R^2}L^{ver}_{S^6} - L^{ver}_{S(R)}. \tag{3.11}$$

Proof. Just add and subtract $\frac{1-R^2}{4R^2}L^{ver}_{S^6}$ in (3.10), to get

$$L_{S(R)} = \frac{1-R^2}{4R^2}\,L_{S^6} + \Big(L^{ver}_{S(R)} - \frac{1-R^2}{4R^2}\,L^{ver}_{S^6}\Big).$$ □

This result gives a different perspective on formula (2.20) obtained in section 2 using the coordinate frame approach. Namely, the induced Laplacian is the sum of a term proportional to the round Laplacian plus a vertical term proportional to L'. This is analogous to the symmetric case formula (see [4], Theorem 6.1), except that L' is now r-dependent.

In order to explicitly compute L' we need a basis X_1, X_2 for vertical vector fields, relative to which we can write

$$\frac{1-R^2}{4}L' = \frac{1-R^2}{4R^2}\sum_1^2 \gamma^{ij}\nabla^{S^6}_{X_i}\nabla^{S^6}_{X_j} - \sum_1^2 g^{ij}\nabla_{X_i}\nabla_{X_j},$$

where g^{ij} and γ^{ij} are the inverse vertical metric and inverse vertical round metric in the vertical frame $\{X_1, X_2\}$, and ∇, ∇^{S^6} denote the covariant derivatives in the full metric and in the round metric, respectively. Of course, the symbol of L' will be determined by

$$\frac{1-R^2}{4}L' = \sum_1^2 \left(\frac{1-R^2}{4R^2}\gamma^{ij} - g^{ij}\right) X_i X_j + \text{ first-order terms in } X_1, X_2. \tag{3.12}$$

Once we know X_1, X_2, we can easily compute the inverse metrics g^{ij} and γ^{ij} and the symbol of L' at all points of $S(R) \simeq S^6$. An interesting question is whether or not L' commutes with the round Laplacian or, more generally, with the Euclidean Laplacian on $\mathfrak{s} = \mathbb{R}^7$. Moreover, the commutator $[X_1, X_2]$, which is needed to find the first-order terms, will allow us to answer the following natural question: is the vertical distribution integrable, i.e., is $[X_1, X_2] = aX_1 + bX_2$ for suitable functions a and b?

3.3 Search for a vertical basis

We now observe that the non-round part of the induced metric $h_1 \circ \delta_R$ is different from $\Theta^2 \circ \delta_R = (\Theta \circ \delta_R)^2$ for $R < 1$. [Note that the equality $\Theta^2 \circ \delta_R = (\Theta \circ \delta_R)^2$ always holds in general, as if $x = R(V, Z, t) \in B$ with $R < 1$, then

$$\begin{aligned}\Theta^2|_x &= \Theta^2|_{(RV,RZ,Rt)} = (\Theta^2 \circ \delta_R)|_{(V,Z,t)} = \theta_1^2|_x + \theta_2^2|_x \\ &= (\theta_1|_x)^2 + (\theta_2|_x)^2 = (\theta_1|_{(RV,RZ,Rt)})^2 + (\theta_2|_{(RV,RZ,Rt)})^2 = \big(\Theta \circ \delta_R\big)^2|_{(V,Z,t)},\end{aligned}$$

where $\theta_{1,2}$ are the component 1-forms of Θ in (3.3), i.e., $\Theta = \theta_1 U_1 + \theta_2 U_2$.]

If we had $h_1 \circ \delta_R = \Theta^2 \circ \delta_R$ for $R < 1$ (i.e., $h_1|_B = \Theta^2|_B$), we could use (3.5) to get $\ker(h_1 \circ \delta_R) = (\ker h_1) \circ \delta_R = \ker(\Theta^2) \circ \delta_R = (\ker \Theta) \circ \delta_R$ for $R < 1$. Moreover, we could immediately claim that a vertical basis is given by $X_{1,2} = T_{1,2} \circ \delta_R$, where $T_{1,2}$ are the vector fields that are dual to $\theta_{1,2}$ in the sense that $\theta_{1,2}(X) = \langle T_{1,2}, X\rangle$ for any vector field X, where $\langle , \rangle$ is the round metric. Indeed, this also implies $\theta_{1,2} \circ \delta_R(X) = R^2\langle T_{1,2} \circ \delta_R, X\rangle$, $\forall X$, so if H satisfies $(\theta_{1,2} \circ \delta_R)(H) = 0$, then $\langle T_{1,2} \circ \delta_R, H\rangle = 0$. Note that $\{\theta_1, \theta_2\}$ is not dual to $\{T_1, T_2\}$ in the sense of $\theta_i(T_j) = \delta_{ij}$, in fact we have $\theta_i(T_j) = \langle T_i, T_j\rangle \neq \delta_{ij}$, the set $\{T_1, T_2\}$ being not orthonormal (or even orthogonal), in general. Now T_1 and T_2 can be obtained from θ_1 and θ_2 (cf. (3.3), see also [3], the formula after (3.20)) by turning all differentials into gradient operators, whence

$$\begin{cases} T_1 \circ \delta_R|_{(V,Z,t)} = [V, \partial_V]_1 + t\partial_{z_1} - z_1\partial_t + \dfrac{2Rz_2}{(1-Rt)^2 + R^2|Z|^2} \times \\ \qquad\qquad \times \Big\{ z_1\partial_{z_2} - z_2\partial_{z_1} + R\big(z_1[V,\partial_V]_2 - z_2[V,\partial_V]_1\big) + (1-Rt)\langle J_1J_2V, \partial_V\rangle \Big\}, \\ T_2 \circ \delta_R|_{(V,Z,t)} = [V, \partial_V]_2 + t\partial_{z_2} - z_2\partial_t - \dfrac{2Rz_1}{(1-Rt)^2 + R^2|Z|^2} \times \\ \qquad\qquad \times \Big\{ z_1\partial_{z_2} - z_2\partial_{z_1} + R\big(z_1[V,\partial_V]_2 - z_2[V,\partial_V]_1\big) + (1-Rt)\langle J_1J_2V, \partial_V\rangle \Big\}. \end{cases} \tag{3.13}$$

However, we have the following result.

Proposition 3.3. *For all $(V, Z, t) \in S^6$ and all $R < 1$, we have*

$$\begin{aligned} &\tfrac{1}{R^4}\big(\Theta \circ \delta_R\big)^2\big|_{(V,Z,t)} = \tfrac{1}{R^4}\big(\theta_1 \circ \delta_R\big)^2\big|_{(V,Z,t)} + \tfrac{1}{R^4}\big(\theta_2 \circ \delta_R\big)^2\big|_{(V,Z,t)} \\ &= \big|[V, dV] + tdZ - Zdt\big|^2 + \big(z_1dz_2 - z_2dz_1 + \langle J_1J_2V, dV\rangle\big)^2 \\ &\quad - \frac{1}{((1-Rt)^2 + R^2|Z|^2)^2}\bigg(\Big(1 - R^2(t^2 + |Z|^2)\Big)(z_1dz_2 - z_2dz_1) \\ &\quad + 2R(1-Rt)\big(z_1[V,dV]_2 - z_2[V,dV]_1\big) + \big((1-Rt)^2 - R^2|Z|^2\big)\langle J_1J_2V, dV\rangle\bigg)^2. \end{aligned} \tag{3.14}$$

Proof. This is a long computation using the formulae for $\theta_{i,R} = \frac{1}{R^2}\theta_i \circ \delta_R$. In the course of the calculation, the following identity (which is easy to prove) is used:

$$\big((1-Rt)^2 + R^2|Z|^2\big)^2 + 4Rt\big((1-Rt)^2 + R^2|Z|^2\big) - 4R^2|Z|^2 = \big(1 - R^2(t^2 + |Z|^2)\big)^2. \qquad \square$$

Comparing (3.14) with (3.2), we see that indeed $h_1 \circ \delta_R \neq \big(\theta_1 \circ \delta_R\big)^2 + \big(\theta_2 \circ \delta_R\big)^2$ for $R < 1$, due to the coefficient of $(z_1dz_2 - z_2dz_1)$ in the round bracket of the third term, namely

$$1 - R^2\big(t^2 + |Z|^2\big) = 1 - R^2 + R^2|V|^2 \neq R^2|V|^2, \quad \forall R < 1.$$

Equivalently, (3.14) implies that $\Theta^2|_B$ is given by

$$\begin{aligned} \Theta^2\big|_{(V,Z,t)} &= \big|[V, dV] + tdZ - Zdt\big|^2 + \big(z_1dz_2 - z_2dz_1 + \langle J_1J_2V, dV\rangle\big)^2 \\ &\quad - \frac{1}{((1-t)^2 + |Z|^2)^2}\bigg(\Big(1 - (t^2 + |Z|^2)\Big)(z_1dz_2 - z_2dz_1) \\ &\quad + 2(1-t)\big(z_1[V,dV]_2 - z_2[V,dV]_1\big) + \big((1-t)^2 - |Z|^2\big)\langle J_1J_2V, dV\rangle\bigg)^2, \end{aligned} \tag{3.15}$$

for all $(V, Z, t) \in B$. (Just let $(RV, RZ, Rt) = (V', Z', t') \in B$ in (3.14), multiply by R^4 and drop primes in the end.) Comparing (3.15) and (3.2) (with $R = 1$) we obtain

Corollary 3.4. *On the domain $\overline{B} \setminus \{(0, 0, 1)\}$ we have*

$$h_1 = \Theta^2 + F, \tag{3.16}$$

where F is the smooth rank-2 tensor field given by

$$\begin{aligned}F|_{(V,Z,t)} = &\frac{\alpha(V,Z,t)}{((1-t)^2+|Z|^2)^2}(z_1dz_2 - z_2dz_1)^2 \\ &+ \frac{4(1-t)\beta(V,Z,t)}{((1-t)^2+|Z|^2)^2}(z_1dz_2 - z_2dz_1)\big(z_1[V,dV]_2 - z_2[V,dV]_1\big) \\ &+ \frac{2\big((1-t)^2-|Z|^2\big)\beta(V,Z,t)}{((1-t)^2+|Z|^2)^2}(z_1dz_2 - z_2dz_1)\langle J_1J_2V, dV\rangle,\end{aligned}$$

$\forall (V,Z,t) \in \overline{B} \setminus \{(0,0,1)\}$, *with*

$$\begin{cases}\alpha(V,Z,t) = \left(1-(t^2+|Z|^2)\right)^2 - |V|^4, \\ \beta(V,Z,t) = 1-(t^2+|Z|^2+|V|^2)\,.\end{cases}$$

Now the functions α and β vanish at the boundary $S^{6\,*}$ of $\overline{B}\setminus\{(0,0,1)\}$ (the sphere "at infinity" for $R=1$, where $F=0$ and we regain $h_1 = \Theta^2$), but they are nonzero on the interior B. Thus $h_1|_B \neq \Theta^2|_B$, and the horizontal distribution $HT(S(R)) = (\ker h_1)\circ\delta_R$ will be different from $(\ker\Theta)\circ\delta_R = (\ker\theta_1\cap\ker\theta_2)\circ\delta_R$ for $R<1$ at generic points of S^6. Consequently, the vectors fields $T_{1,2}\circ\delta_R$ in (3.13) are not vertical for $R<1$, in general.

This makes the search for a vertical basis X_1, X_2 more difficult than expected. The problem is to find two 1-forms ω_1,ω_2 on S^6, depending on R, such that $\omega_1^2+\omega_2^2 = h_R$ in (3.2) for all $R<1$. Then $HT(S(R)) = \ker\omega_1\cap\ker\omega_2$, and $X_{1,2}$ will be obtained from $\omega_{1,2}$ by turning differentials into gradients. We have derived a formula for one of the two 1-forms, say ω_1, which is actually R-independent, but got stuck in the computation of ω_2. Let us briefly elaborate on that.

First we need to fix the basis of the cotangent bundle we shall be working with. Note that we can easily exhibit a vector field which is R-independent and horizontal for all $R\leq 1$. Consider the coordinate vector ∂_ρ from bispherical coordinates in section 2 (recall that $\rho^2 = t^2+|Z|^2 = 1-|V|^2$). Using (2.3), we can rewrite ∂_ρ as

$$\begin{aligned}\partial_\rho &= \langle \tfrac{\partial V}{\partial\rho}, \partial_V\rangle + \langle \tfrac{\partial Z}{\partial\rho}, \partial_Z\rangle + \tfrac{\partial t}{\partial\rho}\partial_t \\ &= -\tfrac{\rho}{1-\rho^2}\langle V,\partial_V\rangle + \tfrac{1}{\rho}\big(\langle Z,\partial_Z\rangle + t\partial_t\big).\end{aligned}$$

To avoid singularities, let us multiply this by $\rho(1-\rho^2)$, to get the vector field

$$H_0 = \rho(1-\rho^2)\partial_\rho = -\rho^2\langle V,\partial_V\rangle + (1-\rho^2)\big(\langle Z,\partial_Z\rangle + t\partial_t\big).$$

It is easily checked that H_0 is in $\ker\Theta\cap\ker F\subset\ker h_1$ on $\overline{B}\setminus\{(0,0,1)\}$ (cf. (3.16)), or equivalently, it is in

$$\ker(\Theta\circ\delta_R)\cap\ker(F\circ\delta_R)\subset\ker h_R, \quad \forall R\leq 1,$$

so H_0 is horizontal for all R. This explains why the ρ-coordinate plays no role in the non-round part h_R of the metric (which is vertical). Note that H_0 vanishes for $\rho = 0,1$, i.e., at the submanifolds S^3 (unit sphere in $\mathfrak{v}$) and S^2 (unit sphere in $\mathfrak{z}\oplus\mathfrak{a}$) of S^6.

We can then use as a basis of the cotangent space $T_x(S^6)^*$ at $x = (V, Z, t)$ the following set of 1-forms:

$$\begin{cases} \alpha_0 = -\rho^2\langle V, dV\rangle + (1-\rho^2)\big(\langle Z, dZ\rangle + tdt\big) \\ \alpha_1 = \langle J_1 V, dV\rangle \\ \alpha_2 = \langle J_2 V, dV\rangle \\ \alpha_3 = \langle J_1 J_2 V, dV\rangle \\ \alpha_4 = tdz_1 - z_1 dt \\ \alpha_5 = tdz_2 - z_2 dt. \end{cases}$$

The 1-form α_0 is dual to H_0 and it is orthogonal to α_j $(1 \le j \le 5)$ with respect to the round metric. Note that the 1-form $z_1 dz_2 - z_2 dz_1$ depends linearly on α_4, α_5, being

$$z_1 dz_2 - z_2 dz_1 = \tfrac{z_1}{t}\alpha_5 - \tfrac{z_2}{t}\alpha_4. \tag{3.17}$$

We assume $\rho \neq 0, 1$ (i.e., $(Z, t) \neq (0, 0)$, $V \neq 0$), and $t \neq 0$, so that α_4 and α_5 are linearly independent.

We expand the tensor field h_R in (3.2) in terms of α_j, $1 \le i, j \le 5$, as

$$h_R = \sum h_{ij}\, \alpha_i \otimes \alpha_j.$$

Comparison with (3.2) gives the following values of the coefficients $h_{ij} = h_{ji}$:

$$\begin{cases} h_{11} = 1 - \frac{4R^2(1-Rt)^2 z_2^2}{((1-Rt)^2+R^2|Z|^2)^2}, & h_{22} = 1 - \frac{4R^2(1-Rt)^2 z_1^2}{((1-Rt)^2+R^2|Z|^2)^2}, \\ h_{33} = 1 - \frac{((1-Rt)^2-R^2|Z|^2)^2}{((1-Rt)^2+R^2|Z|^2)^2}, & h_{12} = \frac{4R^2(1-Rt)^2 z_1 z_2}{((1-Rt)^2+R^2|Z|^2)^2}, \\ h_{13} = \frac{2R(1-Rt)((1-Rt)^2-R^2|Z|^2) z_2}{((1-Rt)^2+R^2|Z|^2)^2}, & h_{23} = -\frac{2R(1-Rt)((1-Rt)^2-R^2|Z|^2) z_1}{((1-Rt)^2+R^2|Z|^2)^2}, \\ h_{44} = 1 + \frac{z_2^2}{t^2}\left(1 - \frac{R^4|V|^4}{((1-Rt)^2+R^2|Z|^2)^2}\right), & h_{55} = 1 + \frac{z_1^2}{t^2}\left(1 - \frac{R^4|V|^4}{((1-Rt)^2+R^2|Z|^2)^2}\right), \\ h_{45} = -\frac{z_1 z_2}{t^2}\left(1 - \frac{R^4|V|^4}{((1-Rt)^2+R^2|Z|^2)^2}\right), & h_{14} = 1 - \frac{2R(1-Rt)R^2|V|^2 z_2^2}{t((1-Rt)^2+R^2|Z|^2)^2}, \\ h_{24} = \frac{2R(1-Rt)R^2|V|^2 z_1 z_2}{t((1-Rt)^2+R^2|Z|^2)^2}, & h_{34} = -\frac{z_2}{t}\left(1 - \frac{R^2|V|^2((1-Rt)^2-R^2|Z|^2)}{((1-Rt)^2+R^2|Z|^2)^2}\right), \\ h_{15} = \frac{2R(1-Rt)R^2|V|^2 z_1 z_2}{t((1-Rt)^2+R^2|Z|^2)^2}, & h_{25} = 1 - \frac{2R(1-Rt)R^2|V|^2 z_1^2}{t((1-Rt)^2+R^2|Z|^2)^2}, \\ h_{35} = \frac{z_1}{t}\left(1 - \frac{R^2|V|^2((1-Rt)^2-R^2|Z|^2)}{((1-Rt)^2+R^2|Z|^2)^2}\right). \end{cases} \tag{3.18}$$

The 5×5 real symmetric matrix $\mathcal{H} = (h_{ij})$ can be diagonalized in $\mathbb{R}^5$. The expected eigenvalues should be of the form $\lambda_1, \lambda_2 \neq 0$, corresponding to the 2-dimensional vertical subspace $VT_x(S(R))$, and $\lambda_3 = \lambda_4 = \lambda_5 = 0$, corresponding to a 3-dimensional horizontal subspace $\widetilde{H}$ such that $HT_x(S(R)) = \widetilde{H} \oplus \mathbb{R}H_0$. (Recall that $\ker h_R$ should be 4-dimensional at generic points, and we have already separated out the 1-dim subspace $\mathbb{R}\alpha_0$ in the cotangent space, so $\mathcal{H}$ will have 3-dim kernel.) Let v_1, v_2, v_3, v_4, v_5 be an orthonormal basis of eigenvectors in $\mathbb{R}^5$, $\mathcal{H}v_j = \lambda_j v_j$ $(1 \le j \le 5)$, and let $\mathcal{R}$ be the matrix whose columns are the v_j. Then $\mathcal{H} = \mathcal{R}\mathcal{D}\mathcal{R}^{-1}$, where $\mathcal{D} = \mathrm{diag}(\lambda_1, \lambda_2, 0, 0, 0)$, and

$$\begin{aligned} h_R &= \sum h_{ij}\, \alpha_i \otimes \alpha_j = \sum \mathcal{R}_{ik}\lambda_k \delta_{kl} (\mathcal{R}^{-1})_{lj}\, \alpha_i \otimes \alpha_j \\ &= \sum \lambda_k \big(\mathcal{R}_{ik}\alpha_i\big) \otimes \big(\mathcal{R}_{jk}\alpha_j\big) = \sum \lambda_k\, \widetilde{\omega}_k \otimes \widetilde{\omega}_k = \lambda_1\, \widetilde{\omega}_1^2 + \lambda_2\, \widetilde{\omega}_2^2, \end{aligned}$$

where $\widetilde{\omega}_k = \sum_1^5 \mathcal{R}_{ik}\alpha_i = \sum(\mathcal{R}^t)_{ki}\alpha_i$. Finally, assuming $\lambda_1, \lambda_2 > 0$, define $\omega_i = \sqrt{\lambda_i}\,\widetilde{\omega}_i$ $(i = 1, 2)$, to get $h_R = \omega_1^2 + \omega_2^2$. To implement this program one needs to compute the eigenvalues $\lambda_{1,2}$ and the eigenvectors v_j $(1 \leq j \leq 5)$. Actually, we only need v_1 and v_2 to compute $\omega_{1,2}$, but v_3, v_4, v_5 will give us, by duality, a set of horizontal vectors that span $HT_x(S(R))$ together with H_0.

The characteristic polynomial of $\mathcal{H}$ will have the form $p(\lambda) = \lambda^3(a\lambda^2 + b\lambda + c)$. To determine this, we use for the matrix $\mathcal{H} - \lambda I$ the elementary rule according to which adding to a row (column) a multiple of another row (column) does not change the value of the determinant. After a few steps, we arrive at a matrix with 3 zeros on the first row and the other entries in that row proportional to $\lambda(2 - \lambda)$. Thus the determinant vanishes for $\lambda = 0$ and $\lambda = 2$. But by (3.18) we easily get

$$\operatorname{trace}\mathcal{H} = 4 + \tfrac{|Z|^2}{t^2}\left(1 - \tfrac{R^4|V|^4}{((1-Rt)^2+R^2|Z|^2)^2}\right) = \lambda_1 + \lambda_2.$$

We conclude that the nonzero eigenvalues are

$$\lambda_1 = 2, \quad \lambda_2 = 2 + \tfrac{|Z|^2}{t^2}\left(1 - \tfrac{R^4|V|^4}{((1-Rt)^2+R^2|Z|^2)^2}\right). \tag{3.19}$$

This is consistent with example 1 below (the case $Z = 0$), where one gets $\lambda_1 = \lambda_2 = 2$. In this case there is a single eigenvalue, and the eigenvectors can be taken to be $v_1 = \frac{1}{\sqrt{2}}(1, 0, 0, 1, 0)$ and $v_2 = \frac{1}{\sqrt{2}}(0, 1, 0, 0, 1)$. The result (3.19) also agrees with example 2 (the degenerate case of $V = 0$, treated as a 5×5 problem), where $\lambda_1 = 2$ and $\lambda_2 = 2 + \frac{|Z|^2}{t^2}$.

To complete our program we need to find the (normalized) eigenvectors v_1, v_2. For $Z \neq 0$, a normalized solution to $\mathcal{H}v_1 = 2v_1$ is given by

$$v_1 = \tfrac{1}{\sqrt{2}|Z|}\big(z_1, z_2, 0, z_1, z_2\big),$$

as easily checked. Thus we get the 1-form

$$\omega_1 = \tfrac{1}{|Z|}\big(z_1\alpha_1 + z_2\alpha_2 + z_1\alpha_4 + z_2\alpha_5\big).$$

This can be rewritten as

$$\begin{aligned}
\omega_1 &= \tfrac{1}{|Z|}\big(z_1\langle J_1V, dV\rangle + z_2\langle J_2V, dV\rangle\big) + \tfrac{1}{|Z|}\big(z_1(tdz_1 - z_1dt) + z_2(tdz_2 - z_2dt)\big)\\
&= \tfrac{1}{|Z|}\big(z_1\langle J_1V, dV\rangle + z_2\langle J_2V, dV\rangle\big) + \tfrac{t}{|Z|}(z_1dz_1 + z_2dz_2) - |Z|dt\\
&= \tfrac{1}{|Z|}(z_1\langle J_1V, dV\rangle + z_2\langle J_2V, dV\rangle) + td|Z| - |Z|dt\\
&= (1-\rho^2)\big(\cos\alpha\langle J_1\omega, d\omega\rangle + + \sin\alpha\langle J_2\omega, d\omega\rangle\big) + \rho^2 d\phi,
\end{aligned}$$

where (ϕ, α) are the angles from bispherical coordinates in (2.3), (2.4), $\omega \in S^3$, and we used $\langle Z, dZ\rangle = |Z|d|Z|$, and $td|Z| - |Z|dt = \rho^2 d\phi$.

We remark that ω_1 is R-independent.

The computation of v_2 is not that simple. Let $v_2 = (a_1, a_2, a_3, a_4, a_5)$. As $\langle v_2, v_1\rangle = 0$, we get $(a_1 + a_4)z_1 + (a_2 + a_5)z_2 = 0$. We have not been able to solve the 5 equations in $\mathcal{H}v_2 = \lambda_2 v_2$. For instance, we tried

$$v_2 = (-az_2, az_1, a_3, -bz_2, bz_1) \quad (a, b \in \mathbb{R}),$$

but it did not work. On the other hand, if we write the 1-form ω_2 as

$$\omega_2 = \sqrt{\lambda_2}\Big(a_1\alpha_1 + a_3\alpha_3 + a_3\alpha_3 + a_4\alpha_4 + a_5\alpha_5\Big),$$

and try to match directly $\omega_1^2+\omega_2^2$ to h_R in (3.2) we run into inconsistencies. For example, if we match the diagonal components h_{ii} in (3.18), we get the following set of coefficients:

$$\begin{cases} \sqrt{\lambda_2}a_1 = \pm\frac{z_2((1-Rt)^2-R^2|Z|^2)}{|Z|((1-Rt)^2+R^2|Z|^2)}, \\ \sqrt{\lambda_2}a_2 = \pm\frac{z_1((1-Rt)^2-R^2|Z|^2)}{|Z|((1-Rt)^2+R^2|Z|^2)}, \\ \sqrt{\lambda_2}a_3 = \pm\frac{2R|Z|(1-Rt)}{(1-Rt)^2+R^2|Z|^2}, \\ \sqrt{\lambda_2}a_4 = \pm\frac{z_2\sqrt{(t^2+|Z|^2)((1-Rt)^2+R^2|Z|^2)^2-R^4|V|^4|Z|^2}}{t|Z|((1-Rt)^2+R^2|Z|^2)}, \\ \sqrt{\lambda_2}a_5 = \pm\frac{z_1\sqrt{(t^2+|Z|^2)((1-Rt)^2+R^2|Z|^2)^2-R^4|V|^4|Z|^2}}{t|Z|((1-Rt)^2+R^2|Z|^2)} \end{cases}$$

where the signs can be arranged using $\langle v_2, v_1\rangle = 0$. However, these values are not consistent with all the non-diagonal components. One can check that h_{12}, h_{13}, h_{23} and h_{45} are correct with these values of a_j, but $h_{1,4}, h_{2,4}, h_{3,4}, h_{1,5}, h_{2,5}, h_{3,5}$ have the wrong values.

Since we do not know how to proceed further, we will leave this open.

Remark 3.5. For $R = 1$, the 1-forms θ_1 and θ_2 do not quite fit into our diagonalization program. Indeed, if we expand $\theta_1 = \sum a_i'\alpha_i$ and $\theta_2 = \sum b_j'\alpha_j$, then the vectors $v_1' = (a_i')$ and $v_2' = (b_j')$ in $\mathbb{R}^5$ are not eigenvectors of the matrix $\mathcal{H}|_{R=1}$, in general. In fact, they are not even orthogonal to each other, $\langle v_1', v_2'\rangle \neq 0$. Nevertheless, $\theta_1^2+\theta_2^2 = h_1 = \omega_1^2+\omega_2^2$. There will be a 2×2 rotation, depending on the coordinates (V,Z,t), relating the two sets of 1-forms.

Besides finding explicit formulae for X_1, X_2 and L', one would also like, for $R < 1$, to get a description of $HT_x(S(R))$ and $VT_x(S(R))$ at generic points $x = (V,Z,t) \in S^6$ analogous to [5], Theorem 7.10 for the symmetric case. This will be quite different in the present case, as the horizontal/vertical subspaces will now depend on R, in general.

We conclude with two examples where the R-dependence has no influence.

1) Let $Z = 0$, i.e., consider the points $x = (V,0,t) \in S^6$. Here $F\circ\delta_R|_x = 0$, $\forall R$, so $h_1\circ\delta_R|_x = \Theta^2\circ\delta_R|_x$, and an orthonormal basis of $VT_x(S(R))$ is

$$\begin{cases} X_1 = T_1\circ\delta_R|_{(V,0,t)} = \langle J_1V, \partial_V\rangle + t\partial_{z_1}, \\ X_2 = T_2\circ\delta_R|_{(V,0,t)} = \langle J_2V, \partial_V\rangle + t\partial_{z_2}. \end{cases}$$

Moreover, an orthogonal basis of $HT_x(S(R))$ is given by H_0 and

$$\begin{cases} H_1 = -\rho^2\langle J_1V, \partial_V\rangle + (1-\rho^2)t\partial_{z_1}, \\ H_2 = -\rho^2\langle J_2V, \partial_V\rangle + (1-\rho^2)t\partial_{z_2}, \\ H_3 = \langle J_1J_2V, \partial_V\rangle. \end{cases}$$

Indeed these vectors are easily seen to be in the kernel of

$$h_R|_x = \big|[V, dV] + tdZ\big|^2 = \big(\langle J_1 V, dV\rangle + tdz_1\big)^2 + \big(\langle J_2 V, dV\rangle + tdz_2\big)^2$$

and they are orthogonal to each other. These bases are independent of R since $h_R|_x$ is independent of R. The subspaces $HT_x(S(R))$ and $VT_x(S(R)) \oplus \mathbb{R}x$ can be described exactly as in [5], Theorem 7.10, case (iii). Indeed, one easily checks that the set $\mathfrak{k}(V)$ of elements of $\mathfrak{v}$ that are orthogonal and commute with V is 1-dimensional and given by $\mathbb{R}J_1J_2V$. The diagonalization program is a bit useless in this case, as h_R is already a sum of squares from the outset. In any case, one computes $\lambda_1 = \lambda_2 = 2$, and of course $\omega_1 = \alpha_1 + \alpha_4$, $\omega_2 = \alpha_2 + \alpha_5$.

2) Let $V = 0$, i.e., consider the points $x = (0, Z, t) \in S^6$. Here $F \circ \delta_R|_x \neq 0$ for $R < 1$, but $h_R|_x$ in (3.2) is still R-independent, namely we have

$$h_R|_x = \big(tdz_1 - z_1dt\big)^2 + \big(tdz_2 - z_2dt\big)^2 + \big(z_1dz_2 - z_2dz_1\big)^2. \tag{3.20}$$

It follows that the horizontal subspace is just $HT_x(S(R)) = \mathfrak{v}$, and the vertical one is the orthogonal complement of x in $\mathfrak{z} \oplus \mathfrak{a}$, i.e., $VT_x(S(R)) \oplus \mathbb{R}x = \mathfrak{z} \oplus \mathfrak{a}$. As a basis of VT_x we can take $X_1 = t\partial_{z_1} - z_1\partial_t$ and $X_2 = t\partial_{z_2} - z_2\partial_t$.

We can also look for two 1-forms $\omega_{1,2}$ such that $\omega_1^2 + \omega_2^2 = h_R$. This is easily solved geometrically by observing that the tensor fields in (3.20) is just the round metric on S^2 (unit sphere in $\mathfrak{z} \oplus \mathfrak{a}$) written in Cartesian coordinates (z_1, z_2, t). In order to write it as a sum of squares, we need to go over to spherical polar coordinates, which are precisely the angles (ϕ, α) from bispherical coordinates in (2.3), (2.4). Thus we get $h_R|_x = d\phi^2 + \sin^2\phi\, d\alpha^2 = \omega_1^2 + \omega_2^2$. We can then take as a basis of VT_x the coordinate vector fields $t\partial_{|Z|} - |Z|\partial_t = \partial_\phi$ and $z_1\partial_{z_2} - z_2\partial_{z_1} = \partial_\alpha$, which are vertical in this case.

We can also use (2.7). The term h_R in this formula is just the one multiplying $4\,\mathrm{sh}^4\frac{r}{2}$, so if we set $\rho = 1$ (i.e., $V = 0$) we get $h_R = d\phi^2 + \sin^2\phi\, d\alpha^2$.

Note that the present case does not fit in the general diagonalization program as we assumed $V \neq 0$ there. (For $V = 0$ we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and the basis becomes degenerate.) One way to deal with this case is to consider only the coefficients $h_{44}, h_{55}, h_{45} = h_{54}$, and just diagonalize this 2×2 matrix. We now have $\operatorname{trace}\mathcal{H} = 2 + \frac{|Z|^2}{t^2} = 1 + \frac{1}{t^2}$. A routine calculation gives the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1/t^2$, and the 1-forms $\omega_1 = d\phi$, $\omega_2 = \sin\phi\, d\alpha$, in agreement with the above.

Another way to deal with this case is to ignore the degeneracy of the basis and diagonalize the 5×5 matrix $\mathcal{H}|_{V=0}$. One gets $\lambda_1 = 2$ and $\lambda_2 = 2 + |Z|^2/t^2$, in agreement with the general result.

3.4 The vertical round metric on the sphere "at infinity"

We now present some calculation on the sphere "at infinity" $S^{6\,*}$ $(R = 1)$ that could generalize to the finite case $(R < 1)$. There is no "induced metric" on $S^{6\,*}$ (the metric (3.1) is infinite for $R = 1$). However, we do have the round metric $\gamma_{S^6} = \langle , \rangle$, and we can study its restriction to the horizontal/vertical distributions. As already mentioned, the horizontal distribution $HT(S^{6\,*}) = \ker(\Theta^2) = \ker h_1$ and the vertical one $VT(S^{6\,*}) =$

$HT(S^{6\,*})^\perp$ do not extend smoothly at the pole $(0,0,1)$ [3], but this will not concern us here.

We start with the computation of the round metric components in the vertical basis $\{T_1, T_2\}$ of $VT(S^{6\,*})$ given by (3.13) with $R = 1$.

Proposition 3.6. *Let $\gamma_{ij} = \langle T_i, T_j\rangle = \gamma_{S^6}(T_i, T_j)$ $(1 \le i, j \le 2)$, then*

$$\gamma_{ij} = \begin{pmatrix} 1 - z_2^2 A & z_1 z_2 A \\ z_1 z_2 A & 1 - z_1^2 A \end{pmatrix}, \tag{3.21}$$

where $A = A(V, Z, t)$ is given for $(V, Z, t) \in S^{6\,*}$ by

$$A(V,Z,t) = \frac{\big(1 - (t^2 + |Z|^2)\big)^2}{\big((1-t)^2 + |Z|^2\big)^2} = \frac{|V|^4}{\big((1-t)^2 + |Z|^2\big)^2}. \tag{3.22}$$

Proof. We have

$$\gamma_{11} = \langle T_1, T_1\rangle = \theta_1(T_1).$$

A routine computation gives

$$\theta_1(T_1)|_{(V,Z,t)} = |V|^2 + |Z|^2 + t^2 - z_2^2\, A(V,Z,t) = 1 - z_2^2\, A(V,Z,t).$$

The result for γ_{22} is obtained in a similar way. For $\gamma_{12} = \langle T_1, T_2\rangle = \theta_1(T_2)$ we compute

$$\theta_1(T_2)|_{(V,Z,t)} = z_1 z_2\, A(V,Z,t).$$

□

Remark 3.7. The functions $z_1^2 A, z_2^2 A, z_1 z_2 A$ do not have a limit at the pole $(0,0,1)$ but they remain bounded there. Just set $1 - t = \rho\cos\phi$, $|Z| = \rho\sin\phi$, $z_1 = |Z|\cos\alpha$, $z_2 = |Z|\sin\alpha$, then $|V|^2 = \rho(2\cos\phi - \rho)$, and the result follows as $\rho \to 0$ (see [3], pp. 931 or 943).

Using (3.21) we obtain

$$\det\gamma_{ij} = 1 - |Z|^2 A, \tag{3.23}$$

and the inverse vertical round metric in the frame $\{T_1, T_2\}$:

$$\gamma^{ij} = \begin{pmatrix} \frac{1 - z_1^2 A}{1 - |Z|^2 A} & -\frac{z_1 z_2 A}{1 - |Z|^2 A} \\ -\frac{z_1 z_2 A}{1 - |Z|^2 A} & \frac{1 - z_2^2 A}{1 - |Z|^2 A} \end{pmatrix}. \tag{3.24}$$

We can then write down the symbol of the vertical round Laplacian.

Proposition 3.8. *The vertical round Laplacian $L^{ver}_{S^6} = L_{S^6}|_{VT(S^{6\,*})}$ is given by*

$$L^{ver}_{S^6} = T_1^2 + T_2^2 + \tfrac{A}{1 - |Z|^2 A}\big(z_1 T_2 - z_2 T_1\big)^2 + \textit{first-order terms in } T_1, T_2. \tag{3.25}$$

Proof. By writing $\gamma^{11} = 1 + \frac{z_2^2 A}{1 - |Z|^2 A}$, $\gamma^{22} = 1 + \frac{z_1^2 A}{1 - |Z|^2 A}$, we have

$$\begin{aligned} L^{ver}_{S^6} &= \sum_1^2 \gamma^{ij} T_i T_j + \text{first-order terms} \\ &= T_1^2 + T_2^2 + \tfrac{A}{1 - |Z|^2 A}\big(z_2^2 T_1^2 + z_1^2 T_2^2 - z_1 z_2\big(T_1 T_2 + T_2 T_1\big)\big) + \text{first-order terms.} \end{aligned}$$

Now the round bracket is precisely $(z_2 T_1 - z_1 T_2)^2$, up to first-order terms. □

Formula (3.25) should be compared to the following formula for the vertical part of the round metric.

Proposition 3.9. *The vertical round metric on $S^{6\,*}$ is given by*

$$\gamma^{ver}_{S^6} = \theta_1^2 + \theta_2^2 + \tfrac{A}{1-|Z|^2 A}\big(z_1\theta_2 - z_2\theta_1\big)^2. \tag{3.26}$$

Proof. We use the general formula $\gamma^{ver}_{S^6} = \sum_1^2 \gamma_{ij}\,\tilde\theta_i \otimes \tilde\theta_j$, where $\tilde\theta_i$ is the vertical dual coframe to T_i, i.e., $\tilde\theta_i$ is a linear combination of θ_j and $\tilde\theta_i(T_j) = \delta_{ij}$. By looking for $\tilde\theta_1 = a\theta_1 + b\theta_2$, etc., we find $a = \gamma^{11}$, $b = \gamma^{12}$, etc., i.e., in general, $\tilde\theta_i = \sum \gamma^{ij}\theta_j$. Thus

$$\begin{aligned}
\gamma^{ver}_{S^6} &= \sum \gamma_{ij}\,\tilde\theta_i \otimes \tilde\theta_j = \sum \gamma^{kl}\theta_k \otimes \theta_l \\
&= \gamma^{11}\theta_1^2 + \gamma^{22}\theta_2^2 + \gamma^{12}(\theta_1 \otimes \theta_2 + \theta_2 \otimes \theta_1) \\
&= \theta_1^2 + \theta_2^2 + \tfrac{A}{1-|Z|^2 A}\Big(z_2^2\theta_1^2 + z_1\theta_2^2 - z_1 z_2(\theta_1 \otimes \theta_2 + \theta_2 \otimes \theta_1)\Big) \\
&= \theta_1^2 + \theta_2^2 + \tfrac{A}{1-|Z|^2 A}\big(z_1\theta_2 - z_2\theta_1\big)^2.
\end{aligned}$$

□

Suppose now we wanted to repeat the analogous calculations for the round metric on the vertical distribution $VT(S(R))$, $R < 1$. We do not have a vertical basis X_1, X_2 yet, so we cannot proceed with the computations. However, we expect that formulae (3.21), (3.23), (3.24), (3.25) and (3.26) could remain valid in a suitable basis X_1, X_2, with the kernel A_R in place of A, where

$$A_R(V, Z, t) = \frac{R^4|V|^4}{\big((1-Rt)^2 + R^2|Z|^2\big)^2}, \quad \forall (V, Z, t) \in S^6.$$

This is just $A \circ \delta_R$ if one takes for A the second formula in (3.22).

If one uses the "false" induced metric obtained by replacing h_R in (3.1) with the right hand side of (3.14), say k_R, then one can work with the "vertical" basis $X_{1,2} = T_{1,2} \circ \delta_R$ in (3.13) and prove all the above results with $A_R = A \circ \delta_R$, but A given by the first formula in (3.22). The two A_R do not coincide, of course, as the two formulae in (3.22) define two different functions on $\overline{B} \setminus \{(0,0,1)\}$, that only agree at the boundary $S^{6\,*}$.

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