ON RIGIDITY PROPERTIES OF TIME-CHANGES OF UNIPOTENT FLOWS

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Abstract. We study time-changes of unipotent flows on finite volume quotients of semisimple linear groups, generalising previous work by Ratner on time-changes of horocycle flows. Any measurable isomorphism between time-changes of unipotent flows gives rise to a non-trivial joining supported on its graph. Under a spectral gap assumption on the groups, we show the following rigidity result: either the only limit point of this graph joining under the action of a one-parameter renormalising subgroup is the trivial joining, or the isomorphism is "affine", namely it is obtained composing an algebraic isomorphism with a (non-constant) translation along the centraliser.

1. Introduction

1.1. **Parabolic and unipotent flows.** Parabolic flows are dynamical systems characterised by a "slow" divergence of nearby points, usually at a polynomial rate. They presents an intermediate chaotic behaviour, in the sense that they have some properties, like mixing, which are typical of highly chaotic systems, but also have zero entropy, a feature of regular (i.e., non chaotic) systems. Fundamental examples of (homogeneous) parabolic flows are unipotent flows on quotients of Lie groups and nilflows on nilmanifolds. For more details and examples of parabolic systems we refer to [13, Chapter 8] and the introduction of [1].

Let G be a connected semisimple Lie group, with Lie algebra $\mathfrak{g}$. We recall that an element $U \in \mathfrak{g} \setminus \{0\}$ is *unipotent* if $\mathrm{ad}(U) = [U, \cdot]$ is a nilpotent linear operator on $\mathfrak{g}$. Given any quotient $M = \Gamma\backslash G$ of G by a discrete subgroup Γ, the unipotent flow $\{\phi_U^t\}_{t\in\mathbb{R}}$ on M generated by U is defined by $\phi_U^t(\Gamma\mathbf{x}) = \Gamma\mathbf{x}\exp(tU)$, with $\mathbf{x} \in G$. One of the prime examples of unipotent flow is the horocycle flow on quotients of $\mathrm{SL}_2(\mathbb{R})$.

Unipotent flows on quotients of Lie groups have been heavily studied by many authors. Their properties are useful in many number theoretic problems, as Margulis' seminal proof of Oppenheim conjecture [22]. Ratner's proof of Raghunathan conjectures [29–31] at the beginning of the 1990s showed that unipotent flows are well-behaved in many ways. In particular, their probability invariant measures are of algebraic nature.

1.2. **Rigidity in unipotent dynamics.** One remarkable feature of unipotent dynamics is the presence of several rigidity phenomena. Roughly speaking, rigidity occurs when a weak form of equivalence implies a stronger one. In the homogeneous setting, it translates to the surprising fact that, under only measure-theoretic assumptions, one can, in some cases, deduce strong, algebraic (in particular, smooth) conclusions. Several results of this type were proved in the 1980s by Marina

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Ratner. In [25], she showed that any mesurable isomorphism ψ between horocycle flows $\{h_i^t\}_{t\in\mathbb{R}}$ on quotients $M_i = \Gamma_i \backslash \mathrm{SL}_2(\mathbb{R})$ for $i = 1, 2$ of $\mathrm{SL}_2(\mathbb{R})$ is in fact algebraic; namely, there exist $\mathbf{x}_0 \in \mathrm{SL}_2(\mathbb{R})$ and $s_0 \in \mathbb{R}$ such that $\mathbf{x}_0\Gamma_1\mathbf{x}_0^{-1} \subset \Gamma_2$ and $\psi(\Gamma_1\mathbf{x}) = h_2^{s_0}(\Gamma_2\mathbf{x}_0\mathbf{x})$. This result was generalised by Witte to unipotent flows in [41,42], and can now be seen as a corollary of the main results in [29].

A strengthening of this isomorphism result for horocycle flows was obtained by providing a complete classification of all possible joinings between them. Let us recall that a *joining* between two probability preserving flows $\phi_i^t\colon (X_i, \mu_i) \to (X_i, \mu_i)$ for $i = 1, 2$ is a probability measure ν on the product $X_1 \times X_2$ which is invariant by the product flow $\phi_1 \times \phi_2$ and projects onto the measures μ_i under the canonical projections onto the factors X_1 and X_2. Clearly, the set of joinings is not empty, since the product measure $\nu = \mu_1 \otimes \mu_2$ is always a joining. In [26], Ratner proved that all joinings of horocycle flows are algebraic. More precisely, using the same notation as above, she showed that, if ν is an ergodic joining of $\{h_1^t\}_{t\in\mathbb{R}}$ and $\{h_2^t\}_{t\in\mathbb{R}}$, then either ν is the product joining or the two lattices are commensurable in the sense that there exist $\mathbf{x}_0 \in \mathrm{SL}_2(\mathbb{R})$ and Γ_0 such that $\Gamma_0 = \Gamma_1 \cap \mathbf{x}_0\Gamma_2\mathbf{x}_0^{-1}$ and the product flow $\{h_1^t \times h_2^t\}_{t\in\mathbb{R}}$ is isomorphic to the horocycle flow on $\Gamma_0 \backslash \mathrm{SL}_2(\mathbb{R})$. This joining result is stronger than the aforementioned one, since any isomorphism between two flows defines a joining supported on the graph, which we call the *graph joining*.

The classification of the ergodic invariant measures for horocycle flows by Dani and Smillie [5] can also be seen as a further rigidity phenomenon. Ratner's proof of Raghunatan conjecture [29] extends this classification to any unipotent action on quotients $\Gamma\backslash G$ of Lie groups G, for which ergodic invariant probability measures are always algebraic; that is, supported on "affine" homogeneous submanifolds (namely, translates of closed subgroups of G intersecting Γ in a lattice).

Remarkably, some of these results extend beyond the homogeneous setting, as we are going to explain.

1.3. **Time-changes of unipotent flows.** Since there is not a precise definition of parabolic flows, it is not clear what properties should be considered typical of the parabolic class, and which should not. In order to gain some insight on this problem, it is natural to look for more examples, especially for non-homogeneous ones. However, it turns out that it is not easy to produce examples of non-homogeneous parabolic flows, since perturbations usually break down the fragile parabolicity and one typically obtains a hyperbolic flow. A simple class of parabolic flows consists of (smooth) *time-changes* of unipotent ones. In this case points move along the same orbits of the original flow, only with a different speed, determined by a smooth function. In particular, time-changes of ergodic flows remain ergodic with respect to an equivalent invariant measure. However, finer dynamical properties can change drastically. For instance, nilflows are never weak-mixing, yet, given any ergodic nilflow, non-trivial time-changes within a class of "trigonometric polynomials" on the nilmanifold are mixing [1,2,7,32]. For Heisenberg nilflows, a stronger dichotomy holds: either a time-change is trivial or is weak-mixing [8]. In the case of the horocycle flow, building on work by Kuschnirenko [19], Marcus proved in [21] that smooth time-changes are mixing of all orders. The rate of mixing was studied by Forni and Ulcigrai in [9]. In the same paper, it was shown that the spectrum is Lebesgue. Independently, at the same time, absolute continuity of the spectrum was proven by Tiedra de Aldecoa in [38].

Much less is known for time-changes of general unipotent flows. Simonelli showed that the spectrum is absolutely continuous for unipotent flows on semisimple Lie groups [35], and the polynomial rate of mixing was studied by the third-named author in [34].

Finally, let us mention that a new kind of parabolic perturbation of unipotent flows on (compact) quotients of $\mathrm{SL}_n(\mathbb{R})$, *not* given by a time-change, was defined and studied by the third-named author in [33].

1.4. **Rigidity of parabolic perturbations of unipotent flows.** A natural question is to ask when a smooth parabolic perturbation of a unipotent flow is measurably isomorphic to the homogeneous unperturbed flow. In the case of time-changes, if the time-change function is measurably cohomologous to a constant, it is easy to see that the two flows are isomorphic and the regularity of the isomorphism is given by the regularity of the transfer function. Ratner proved a rigidity result for time-changes of horocycle flows [27] which, in particular, implies the converse statement. She showed that any measurable isomorphism ψ between two time-changes $\{\widetilde{h}_i^t\}_{t\in\mathbb{R}}$ of horocycle flows on quotients M_i is in fact algebraic in the sense that there exists $\mathbf{x}_0 \in \mathrm{SL}_2(\mathbb{R})$ and a measurable function $\sigma\colon M_2 \to \mathbb{R}$ such that $\mathbf{x}_0\Gamma_1\mathbf{x}_0^{-1} \subset \Gamma_2$ and $\psi(\Gamma_1\mathbf{x}) = h_2^{\sigma(\Gamma_2\mathbf{x}_0\mathbf{x})}(\Gamma_2\mathbf{x}_0\mathbf{x})$. Hence, the problem of establishing the triviality of time-changes is equivalent to solving the cohomological equation for the horocycle flow. By the work of the second author and Forni [6], measurably trivial time-changes are *rare*; namely, they form a closed subspace of countable codimension.

A similar statement holds for time-changes of Heisenberg nilflows. Avila, Forni and Ulcigrai proved that a function is a measurable coboundary if and only if it is a smooth coboundary [2]. Therefore, they were able to provide explicit examples of non-trivial (as a matter of fact, mixing) time-changes.

A weaker form of rigidity than the one discussed above has been showed to hold for the perturbations constructed in [33]. As for time-changes, whenever a certain cocycle associated to the flow is a coboundary, it is easy to see that the perturbation is trivial. The third author showed that assuming that the perturbation is smoothly trivial implies the former statement. Therefore, analogously as for time-changes, the (smooth) triviality of the perturbation is equivalent to a cohomological statement. It would be interesting to extend this result to the measurable setting.

Coming back to time-changes of horocycle flows, their factors and joinings are also algebraic, by a further work of Ratner [28]. Interestingly, non-trivial horocycle time-changes have disjoint rescalings, as shown by Kanigowski, Lemanczyk and Ulcigrai [14], albeit this property is clearly false for the horocycle flow itself.

A partial generalisation of the works of Ratner to the Lorentz group has been recently achieved by Tang [36]. He shows that the existence of a measurable isomorphism between a unipotent flow in $SO(n,1)$ and a time-change of itself generated by a smooth function τ implies that τ and the composition of τ with any element in the centraliser of the unipotent flow are cohomologous. He then deduces a full analougue of Ratner's result under the additional assumption of the transfer function being in L^1. This seems, however, difficult to check in most concrete cases. Under similar assumptions, in [37], he investigates the extension of Ratner's joining result for time-changes of unipotent flows in $SO(n,1)$.

1.5. **Statement of results.** In this paper, we generalise the aforementioned works of Ratner and Tang to general semisimple linear groups satisfying a natural assumption on the spectral gap. Given any two isomorphic time-changes of unipotent flows on any finite-volume quotients, we show that the isomorphism is of a special algebraic form.

In order to state our result precisely, let us introduce some notation. Let G_1 and G_2 be two connected, semisimple, linear groups. Consider two lattices $\Gamma_1 < G_1$ and $\Gamma_2 < G_2$, and consider the quotient manifolds $M_i = \Gamma_i \backslash G_i$ with the probability measure μ_i inherited from the Haar measure on G_i, for $i = 1, 2$. Let $\{\mathbf{u}_i^t\}_{t\in\mathbb{R}}$ be two one-parameter unipotent subgroups of G_i. These define a unipotent flow $\phi_{U_i}^t = \phi_i^t$ on M_i as before. Given two positive measurable functions $\alpha_i \colon M_i \to \mathbb{R}_{>0}$, we consider the time-changes of the unipotent flows ϕ_i^t denoted $\widetilde{\phi}_i^t$, where for simplicity we omit the dependence on α_i.

The Jacobson-Morozov Theorem (see, e.g., [18, Theorem 10.3]), ensure the existence of a ${}_2(\mathbb{R})$-sub-algebra $\mathfrak{s}_i = \langle U_i, A_i, \overline{U_i}\rangle$ inside of $\mathfrak{g}_i$. Exploiting this, we let $\mathbf{a}_i^t = \exp(tA_i)$ be the Cartan one-parameter subgroup which renormalises the unipotent flow defined by $\mathbf{u}_i^t$, meaning that $\mathbf{a}_i^t\mathbf{u}_i^s = \mathbf{u}_i^{se^t}\mathbf{a}_i^t$, for all $s, t \in \mathbb{R}$. Denote by $\phi_{A_1\times A_2}^t \colon M_1 \times M_2 \to M_1 \times M_2$ the diagonal action $\phi_{A_1\times A_2}^t(\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{x}_1\mathbf{a}_1^t, \mathbf{x}_2\mathbf{a}_2^t)$.

Having introduced the necessary notation, we are now ready to state our main result. The precise assumptions about the time-change functions α_i are described in §2, and the definition of *good time-change* is given in Definition 2.1.3. The *strong spectral gap assumption* is also explained in §2. For the definition of graph joining μ_ψ, we refer to §6.

Theorem A. *Let G_i be connected, semisimple, linear groups and Γ_i be irreducible lattices in G_i. Assume that the manifolds $M_i = \Gamma_i\backslash G_i$ satisfy the strong spectral gap condition. Let ϕ_i^t be the unipotent flows on M_i given by $\mathbf{u}_i^t$, $i = 1, 2$. Let $\widetilde{\phi}_i^t$ be the time-changes of the unipotent flows ϕ_i^t obtained by good time-changes α_i. Assume that there is a measurable conjugacy $\psi \colon M_1 \to M_2$ between $\widetilde{\phi}_1^t$ and $\widetilde{\phi}_2^t$, and let μ_ψ be the graph joining defined by ψ.*

Suppose that $(\phi_{A_1\times A_2}^t)_*\mu_\psi$ does not *converge to the trivial joining $\mu_1 \otimes \mu_2$ as $t \to +\infty$. Then, there exist $\overline{\mathbf{g}} \in G_2$, an isomomorphism $\varpi \colon G_1 \to G_2$, such that $\varpi(\mathbf{u}_1) = \mathbf{u}_2$, $\varpi(\Gamma_1) \subset \overline{\mathbf{g}}^{-1}\Gamma_2\overline{\mathbf{g}}$, and, up to passing to a finite quotient, we have that*

$$\psi(\Gamma_1\mathbf{x}) = \Gamma_2\overline{\mathbf{g}}\varpi(\mathbf{x})\mathbf{c}(\Gamma_1\mathbf{x})\mathbf{u}_2^{t(\Gamma_1\mathbf{x})},$$

for μ_1-a.e. $\Gamma_1\mathbf{x} \in M_1$, where $\mathbf{c}(\Gamma_1\mathbf{x})$ commutes with $\mathbf{u}_2$, $t(\Gamma_1\mathbf{x}) \in \mathbb{R}$, and both depend measurably on $\Gamma_1\mathbf{x}$.

If we restrict to a special class of groups and a specific kind of unipotent element, we obtain a rigidity result without any assumption on the graph joining. In particular, this allows us to recover Ratner's original result on time-changes of the horocycle flow [27].

Theorem B. *Let G_i' be connected, semisimple, linear groups with finite centre and without compact factors. Consider the groups $G_i = \mathrm{SL}_2(\mathbb{R}) \times G_i'$, for $i = 1, 2$ and let Γ_i be irreducible lattices in G_i. Let ϕ_i^t be the unipotent flows on M_i given by $\mathbf{u}_i^t = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \times \mathbf{e}_i$, where $\mathbf{e}_i \in G_i'$ is the identity. Let $\widetilde{\phi}_i^t$ be the time-changes of the unipotent flows ϕ_i^t obtained by good time-changes α_i. Assume that there is a measurable conjugacy $\psi \colon M_1 \to M_2$ between $\widetilde{\phi}_1^t$ and $\widetilde{\phi}_2^t$.*

Then, there exist $\overline{\mathbf{g}} \in G_2$*,an isomomorphism* $\varpi\colon G_1 \to G_2$*, such that* $\varpi(\mathbf{u}_1) = \mathbf{u}_2$*,* $\varpi(\Gamma_1) \subset \overline{\mathbf{g}}^{-1}\Gamma_2\overline{\mathbf{g}}$*, and we have that*

$$\psi(\Gamma_1\mathbf{x}) = \Gamma_2\overline{\mathbf{g}}\varpi(\mathbf{x})\mathbf{c}(\Gamma_1\mathbf{x})\mathbf{u}_2^{t(\Gamma_1\mathbf{x})},$$

for μ_1*-a.e.* $\Gamma_1\mathbf{x} \in M_1$*, where* $\mathbf{c}(\Gamma_1\mathbf{x})$ *commutes with* $\mathbf{u}_2$*,* $t(\Gamma_1\mathbf{x}) \in \mathbb{R}$*, and both depend measurably on* $\Gamma_1\mathbf{x}$*.*

Finally, as a consequence of our main technical result, we obtain the following cohomological statement, which generalises [36, Theorem 1.1].

Theorem C. *Let* G_1 *and* G_2 *be two connected, semisimple, linear groups. Let* Γ_i *be irreducible lattices in* G_i *and assume that the manifolds* $M_i = \Gamma_i\backslash G_i$ *satisfy the strong spectral gap assumption, for* $i = 1, 2$*. Let* $\widetilde{\phi}_1^t$ *be the time-change of the unipotent flow obtained from* ϕ_1^t *by a good time-change* α*. Assume that there is a measurable conjugacy* $\psi\colon M_1 \to M_2$ *between* $\widetilde{\phi}_1^t$ *and the unperturbed unipotent flow* ϕ_2^t*. Then* $\alpha(x)$ *and* $\alpha(x\mathbf{c})$ *are* measurably cohomologous *for all* $\mathbf{c}$ *in the centraliser of* $\mathbf{u}_1$*.*

For the precise formulation and the proof of the above result, we refer the reader to Corollary 5.1.2.

Nota Bene. After this article was finished, Lindenstrauss and Wei announced a similar and stronger rigidity result for unipotent flows [20]. Notably, they are able to prove that any measurable isomorphism ψ of time-changes is in fact cohomologous to an algebraic isomorphism (as in our Theorem A, but without the assumption on the graph joining). Furthermore, their result does not require any smoothness assumption on the time-change function.

In both our and Lindenstrauss and Wei's work, the fundamental tool is a version of Ratner's Basic Lemma, which controls the relative distance of the images of points under ψ in terms of the time their orbits stay close. In our proof, we employ geometric arguments which are closer to Ratner's work, whereas Lindenstrauss and Wei' approach is based on the study of Kakutani-Bowen balls under Kakutani equivalence.

The main difference we can see with the arguments outlined in [20] is that they manage to ensure the convergence of the conjugation of ψ by the subgroups generated by A_1 and A_2 to a well-defined measurable map (which will be an isomorphism of the homogeneous flows). This crucial step relies on a $\mathrm{SL}_2(\mathbb{R})$ ergodic theorem [20, §6].

Outline of the paper. The paper is organized as follows.

We begin in §2 by giving all the precise definitions of the objects we deal with. In particular, we explain our assumptions on both the time-changes and the groups we consider.

In §3, using the Lie group structure, we study the geometry of the unipotent flow, defining certain polynomials which measure the divergence of the orbit in the $_2(\mathbb{R})$-sub-algebra given by U_1 and in the remaining part of the Lie algebra. In this section we explain carefully the construction of the blocks, which will be crucial later on. We study the blocks exploiting the polynomial nature of the unipotent flow.

Section 4 is dedicated to the proof of the key technical result we use: Ratner's Basic Lemma (Lemma 4.2.1). This result is applied in §5 to show that the isomorphism maps leaves tangent to the normaliser of U_1 to leaves tangent to the normaliser of U_2. At end of this section, we prove a more general version of Theorem C.

In §6 we define the graph joining and exploit Ratner's classification of joinings of unipotent flows in [29] to prove Theorem A. Finally, Theorem B is proven in §7 by adapting Ratner's original strategy from [27]. The Appendix A, contains, for completeness, a proof of a consequence of Chevalley's Lemma.

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2. Preliminaries

In this section we introduce the class of homogeneous manifolds and times changes of unipotent flow that we shall consider and we highlight their properties. We also introduce two technical conditions to take into account the possibility that the manifold M_2 is not compact.

2.1. **Time-changes and spectral condition.** In general will have groups G_1 and G_2, manifolds M_1 and M_2, flows (ϕ_1^t) and (ϕ_2^t), ... In this section we deal with properties common to these objects. Thus, in order to lighten the notation, we shall drop the indices 1 and 2, so that G, M, (ϕ^t) ... will refer to both groups, manifolds, flows, ...

2.1.1. Recall from the introduction that G is a *connected semisimple linear group* and that the manifold $M = \Gamma\backslash G$ is obtained as a quotient of this group by a lattice Γ. The manifold M is endowed with a probability measure μ locally defined by the Haar measures of G. A point in the group G will be written in boldface characters: $\mathbf{x} \in G$.

We also recall that $\{\mathbf{u}^t\}_{t\in\mathbb{R}}$ is a one-parameter unipotent subgroup of G, generated by an element $U \in \mathfrak{g} = \mathrm{Lie}(G)$. The subgroup $\{\mathbf{u}^t\}_{t\in\mathbb{R}}$ defines a measure-preserving smooth flow on M by letting

$$\phi^t(x) = x\mathbf{u}^t, \qquad (x = \Gamma\mathbf{g} \in M).$$

We identify the vector field on M generating the flow (ϕ^t) with U. In fact, elements of the Lie algebra $\mathfrak{g}$ can be considered as left invariant vector fields on the group G and thus they project to the quotient space M.

We fix, once and for all, a ${}_2(\mathbb{R})$-sub-algebra $\mathfrak{s} = \langle U, A, \bar{U}\rangle$ containing U, that is a triple satisfying the commutation relations

$$[A,U] = U, \quad [A,\bar{U}] = -\bar{U}, \quad [U,\bar{U}] = 2A. \tag{1}$$

The existence of such a sub-algebra $\mathfrak{s}$ is ensured by the Jacobson-Morozov Theorem (see, e.g., [18, Theorem 10.3]). We denote $\mathbf{a}^t = \exp(tA)$ and $\phi_A^t(x) = x\mathbf{a}^t$ the induced flow on M.

Definition 2.1.1. Let $\alpha\colon M \to \mathbb{R}_{>0}$ be a strictly positive measurable function on M. The *time-change of the unipotent flow* (ϕ^t) *with generator* U *determined by* α is the flow $(\widetilde{\phi}^t)$ on M generated by the vector field $\alpha^{-1}U$.

The definition above implies that the time-changed flow $(\widetilde{\phi}^t)$ preserves the measures $\alpha\mu$.

For $x \in M$, let $w(x,t)$ be defined by the equality

$$t = \int_0^{w(x,t)} \alpha(x\mathbf{u}^r)\,\mathrm{d}r. \tag{2}$$

Then

$$\widetilde{\phi}^t(x) = x\mathbf{u}^{w(x,t)}. \tag{3}$$

It is easily verified by its definition that the function w is a *cocycle* for the time-changed flow $(\widetilde{\phi}^t)$, i.e., it satisfies the *cocycle equation*

$$w(x, s+t) = w(x,s) + w\big(\widetilde{\phi}^s(x), t\big). \tag{4}$$

We also define $\xi(x,t)$ to be the inverse function of $w(x,t)$ with respect to the second variable, so that

$$t = w(x, \xi(x,t)), \qquad \forall (x,t) \in M \times \mathbb{R}.$$

The identities (2) and (3) may be rewritten as

$$\xi(x,t) = \int_0^t \alpha(x\mathbf{u}^r)\,\mathrm{d}r \quad \text{and} \quad \widetilde{\phi}^{\xi(x,t)}(x) = x\mathbf{u}^t.$$

By symmetry, the function ξ is a cocycle for the U-flow (ϕ^t).

2.1.2. A measurable cocycle $w(x,t)$ for the time-changed flow $(\widetilde{\phi}^t)$ is a measurable *coboundary* if there exists a function $f\colon M \to \mathbb{R}$, called the *transfer function* such that

$$w(x,t) = f \circ \widetilde{\phi}^t(x) - f(x),$$

for μ-a.e. $x \in M$ and all $t \in \mathbb{R}$. Two cocycles are measurably *cohomologous* if their difference is a measurable coboundary.

Cohomologous cocycles yield isomorphic time-changes, and the isomorphism is exactly flowing along the orbit of the time-changed flow for time $f(x)$. See [15, §9] or [1, §2.1] for more details.

2.1.3. Next we state some of the assumptions on the time-change function α indicating, for justification, their consequences later needed in the proofs.

Assumption 1. We will assume that α is *uniformly bounded*, and we let

$$C_\alpha = \max\{\alpha(x), \alpha^{-1}(x) : x \in M\} > 1.$$

This hypothesis easily implies the following *rough bound* on the cocycle w and its inverse:

$$C_\alpha^{-1} \leq \mathrm{Lip}_t[w(\cdot,t)] \leq C_\alpha, \quad \text{and} \quad C_\alpha^{-1} \leq \mathrm{Lip}_t[\xi(\cdot,t)] \leq C_\alpha, \tag{5}$$

where the symbol Lip_t stands for the Lipschitz constant of a function with respect to the variable denoted by t.

Another immediate consequence of the Assumption 1, is that the measure $\widetilde{\mu} := \alpha\mu$ which is preserved by the time-changed flows $(\widetilde{\phi}^t)$ is equivalent to the measures μ:

$$C_\alpha^{-1}\mu \leq \widetilde{\mu} \leq C_\alpha \mu. \tag{6}$$

Assumption 2. The time-change function α has average 1 with respect to the Haar measure μ.

This assumption is not restrictive, as we can always rescale the function by its average and compose the isomorphism ψ with ϕ_A^t for an appropriate amount.

2.1.4. The rough bound (5) is insufficient for good estimates of the cocycles $w(\cdot, t)$ and $\xi(\cdot, t)$ for large values of t. For precise estimates in this range we need an effective version of the ergodic theorem, which will be derived from a quantitative mixing result. The study of quantitative mixing results is the same of the study of decay of matrix coefficients of the regular representation of G on $L^2(M, \mu)$. This goes back to the work of Harish-Chandra [11, 12] and Warner [39, 40]. For more details we refer the reader to the introduction of [3] and the references therein.

We say that M satisfies the *strong spectral gap assumption* if the restriction of the regular representation of G on $L^2(M, \mu)$ to every noncompact simple factor of G is isolated, in the Fell topology, from the trivial representation.

Assumption 3. The manifolds M_1 and M_2 satisfy the strong spectral gap assumption.

This assumption holds in a number of cases, for example if G has the Kazhdan property (T) [23], or if G is a semisimple group with finite centre and no compact factors and Γ is irreducible [16, p. 285] and [17]. We will use the following quantitative mixing result, stated in [3].

Theorem 2.1.2. *Assume that M satisfies the strong spectral gap assumption. There exist constants $C_M, \widetilde{\eta} > 0$ and a Sobolev norm $\mathcal{S}$ for functions on M such that for all sufficiently smooth $f, g \colon M \to \mathbb{C}$ of average zero and for all $t \geq 1$ we have*

$$|\langle f \circ \phi^t, g \rangle| \leq C_M \mathcal{S}(f) \mathcal{S}(g) t^{-\widetilde{\eta}}.$$

Without loss of generality we may suppose that $\sup_M |f| \leq \mathcal{S}(f)$ for any $f \colon M \to \mathbb{C}$ with $\mathcal{S}(f) < \infty$.

Assumption 4. The time-change function α has finite Sobolev norm $\mathcal{S}(\alpha)$.

Definition 2.1.3 (Good time-change)**.** A time-change function $\alpha \colon M \to \mathbb{R}_{>0}$ is *good* if it satisfies Assumption 1,2, and 4.

Assumption 4 is needed to ensure we can apply Theorem 2.1.2 to α. In the case of $\mathrm{SL}_2(\mathbb{R})$, the polynomial decay in Theorem 2.1.2 holds under weaker assumptions (namely, it is sufficient to require Hölder regularity along the rotation subgroup). In this sense, our assumptions on the time-change function are a natural generalization of Ratner's ones in [27].

Under the assumptions of this quantitative mixing result, we deduce a polynomial estimate on the ergodic averages of (ϕ^t) on a set of large measure.

Lemma 2.1.4. *There exists constants $\eta' \in (0, 1)$ and C'_M, depending only on M, such that the following holds. Let $f \colon M \to \mathbb{C}$ be a function with zero average and finite Sobolev norm $\mathcal{S}(f)$. For all $\omega > 0$ there exists a set $Y = Y(\omega, f) \subset M$ of measure $\mu(Y) \geq 1 - \omega$ and a number $m' = m'(\omega) \geq 1$ such that for all $x \in Y$ and all $t \geq m'$ we have*

$$\left| \int_0^t f(x\mathbf{u}^s) \, \mathrm{d}s \right| \leq C'_M \mathcal{S}(f) t^{1-\eta'}.$$

Proof. Let us denote

$$s_t(x) = \int_0^t f(x\mathbf{u}^s) \, \mathrm{d}s.$$

Using Theorem 2.1.2, we have

$$\begin{aligned}\|s_t\|_2^2 &= \int_M \left(\int_0^t f(x\mathbf{u}^s)\,\mathrm{d}s\right)^2 \mathrm{d}\mu = \int_M \int_0^t \int_0^t f(x\mathbf{u}^r) f(x\mathbf{u}^s)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}\mu \\ &= \int_0^t \int_0^t \langle f(x), f(x\mathbf{u}^{s-r})\rangle\,\mathrm{d}r\,\mathrm{d}s \le \int_0^t \int_{-s}^{t-s} |\langle f(x), f(x\mathbf{u}^r)\rangle|\,\mathrm{d}r\,\mathrm{d}s \\ &\le 2t \int_0^t |\langle f(x), f(x\mathbf{u}^r)\rangle|\,\mathrm{d}r \le \frac{2C_M\,\mathcal{S}(f)^2}{1-\widetilde{\eta}} t^{2-\tilde{\eta}} = C_1\,\mathcal{S}(f)^2\,t^{2-\tilde{\eta}}.\end{aligned}$$

By Chebyshev's Inequality, we have that

$$\mu\{x \in M : |s_t(x)| \ge \mathcal{S}(f)\,t^{1-\frac{\tilde{\eta}}{4}}\} \le C_1\,t^{-\frac{\tilde{\eta}}{2}}.$$

For $t > 0$, we consider the sets $Y_t = \{x \in M : |s_t(x)| < \mathcal{S}(f) t^{1-\tilde{\eta}/4}\}$. Choosing $t_n = n^{4/\tilde{\eta}}$, we have that

$$\mu(Y_{t_n}) \ge 1 - \frac{C_1}{n^2}.$$

Given $\omega > 0$ we define $\bar{k}$ so that $C_1 \sum_{k\ge\bar{k}} \frac{1}{k^2} < \omega$. Hence

$$\mu(Y) > 1-\omega, \qquad \text{where} \qquad Y = \bigcap_{k\ge\bar{k}} Y_{t_k}.$$

Any point $x \in Y$ satisfies $|s_{t_k}(x)| < \mathcal{S}(f) t_k^{1-\tilde{\eta}/4}$ for all $k \ge \bar{k}$. We need to prove a similar estimate for the remaining times $t \ge t_{\bar{k}}$.

Let $k \ge \bar{k}$ such that $t_k < t \le t_{k+1}$. There exists a constant $C_2 = C_2(\tilde{\eta}) > 0$ such that $t_{k+1} - t_k = C_2 t_k^{1-\tilde{\eta}/4}$. Writing $t = t_k + q$, with $0 < q \le C_2 t_k^{1-\tilde{\eta}/4}$, we obtain, for $x \in Y$,

$$|s_t(x)| \le |s_{t_k}(x)| + \left| \int_{t_k}^{t_k+q} f(x\mathbf{u}^s)\,\mathrm{d}s \right| \le C'_M\,\mathcal{S}(f)\,t_k^{1-\frac{\tilde{\eta}}{4}}.$$

with $C'_M = C_1 + C_2$. Setting $m' = t_{\bar{k}}$ and $\eta' = \widetilde{\eta}/4$, we have completed the proof. □

Corollary 2.1.5. *There exists $\eta \in (0, 1/4)$ such that for all $\omega > 0$ there exist a set $Y \subset M$ of measure $\mu(Y) \ge 1-\omega$ and a constant $m \ge 1$ such that for all $x \in Y$ and all $t \ge m$,*

$$|\xi(x,t) - t| \le \frac{1}{C_\alpha^4} t^{1-\eta}, \quad |w(x,t) - t| \le \frac{1}{C_\alpha^4} t^{1-\eta}.$$

Proof. We begin with the estimate for ξ. We have, for all $t \ge m_0$,

$$|\xi(x,t) - t| = \left| \int_0^t (\alpha(x\mathbf{u}^r) - 1)\,\mathrm{d}r \right| \le C'_M \mathcal{S}(\alpha) t^{1-\eta'},$$

where we applied Lemma 2.1.4 to the zero mean function $\alpha - 1$. In order to conclude, it is enough to fix a smaller $\eta < \eta' \le 1/4$ and possibly a larger m_0.

For the second estimate, using the bound on the Lipschitz constants of w seen as functions of time, we have

$$|w(x,t) - t| = |w(x,t) - w(x, \xi(x,t))| \le C_\alpha |\xi(x,t) - t|,$$

and the estimate follows, possibly by again reducing η. □

2.2. **General setting.** We now return to the general setting of Theorem A. Thus G_1 and G_2 are two groups as in §2.1.1 and all objects and constants defined in §2.1.1 have corresponding suffixes. We let $C_\alpha = \max\{C_{\alpha_1}, C_{\alpha_2}\}$, $C_M = \max\{C_{M_1}, C_{M_2}\}$, $\eta = \min\{\eta_1, \eta_2\}$, $m = \max\{m_1, m_2\}$ so that all inequalities stated in §2.1 holds in M_1 and in M_2. Thus we now have

The map $\psi\colon M_1 \to M_2$ is a measurable conjugacy between the time-changes $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$ defined by the good time-change functions α_1 and α_2. We suppose that ψ maps the measure $\alpha_1\mu_1$ to the measure $\alpha_2\mu_2$.

From the inequality (6) we immediately have the following bounds.

Lemma 2.2.1. *For any measurable set $A \subset M_2$ we have*

$$C_\alpha^{-1}\widetilde{\mu}_2(A) \leq \mu_1(\psi^{-1}A) \leq C_\alpha \widetilde{\mu}_2(A). \tag{7}$$

By definition of ψ, for almost all $x \in M_1$,

$$\psi(x\mathbf{u}_1^t) = \psi(\widetilde{\phi}_1^{\xi_1(x,t)}(x)) = \widetilde{\phi}_2^{\xi_1(x,t)}(\psi(x)) = \psi(x)\mathbf{u}_2^{w_2(\psi(x),\xi_1(x,t))}.$$

We define

$$z(x,t) := w_2(\psi(x), \xi_1(x,t)),$$

so that

$$\psi(x\mathbf{u}_1^t) = \psi(x)\mathbf{u}_2^{z(x,t)}, \quad \text{and} \quad C_\alpha^{-2} \leq \mathrm{Lip}_t[z(\cdot,t)] \leq C_\alpha^2.$$

Remark. The identity above exhibits ψ as a measurable isomorphism of the U_1-flow on M_1 and a reparametrization of the U_2-flow on M_2. Observe, however, that we cannot apply directly Corollary 2.1.5 to the function $z(x,t)$ because, a priori, it is not sufficiently smooth.

Lemma 2.2.2. *The map z is a* cocycle over $\{\mathbf{u}_1^t\}_{t\in\mathbb{R}}$, *namely for almost all $x \in M_1$ and all $t, s \in \mathbb{R}$ we have*

$$z(x, t+s) = z(x,s) + z(x\mathbf{u}_1^s, t).$$

Moreover, if for some $x \in M_1$, $m > 0$, and $t' > t \geq 0$ we have

$$z(x,t') - z(x,t) > m,$$

then

$$t' - t > mC_\alpha^{-2}, \quad \textit{and} \quad z(y,t') - z(y,t) > mC_\alpha^{-4} \textit{ for all } y \in M_1.$$

Proof. By definition of z we have

$$\psi(x)\mathbf{u}_2^{z(x,t+s)} = \psi(x\mathbf{u}_1^{t+s}) = \psi(x\mathbf{u}_1^s)\mathbf{u}_2^{z(x\mathbf{u}_1^s,t)} = \psi(x)\mathbf{u}_2^{z(x,s)}\mathbf{u}_2^{z(x\mathbf{u}_1^s,t)},$$

which implies the first claim.

As for the second, by the cocycle relation and the Lipschitz bound on z we get

$$m < z(x,t') - z(x,t) = z(x\mathbf{u}_1^t, t'-t) \leq (t'-t)C_\alpha^2,$$

so that $t' - t > mC_\alpha^{-2}$. Then, for all other $y \in M_1$,

$$z(y,t') - z(y,t) = z(y\mathbf{u}_1^t, t'-t) \geq C_\alpha^{-2}(t'-t) > mC_\alpha^{-4},$$

and the proof is complete. □

2.3. **Two technical conditions.** We now define two conditions which will be relevant later on: the Injectivity Condition (IC) and the Frequently Bounded Radius Condition (FBR).

In general, we show that these conditions hold on a set of arbitrarily large measure.

Definition 2.3.1 (The Injectivity Condition (IC))**.** We say that a point $x \in M_2$ satisfies the *Injectivity Condition* $\mathrm{IC}(\rho, m)$ if, for any lift $\mathbf{x} \in G_2$ of x and any $\mathbf{y} \in B(\mathbf{x}, \rho)$, we have

$$d(\mathbf{x}\mathbf{u}_2^{t_1}, \gamma \mathbf{y}\mathbf{u}_2^{t_2}) > \rho \quad \text{for all} \quad \gamma \in \Gamma_2 \setminus \{\mathbf{e}\} \quad \text{and all} \quad t_1, t_2 \in [-m, m].$$

If the manifold M_2 is compact then, then there exists ρ_0 such that the condition $\mathrm{IC}(\rho, m)$ is satisfied for all $\rho < \rho_0$ and all $m > 0$. Then in the following proposition, we may take $Y_{\mathrm{IC}} = M_2$ and $\zeta = 0$.

Proposition 2.3.2. *Let $m > 1$ and $\zeta > 0$. There exist a compact set $Y_{\mathrm{IC}} = Y_{\mathrm{IC}}(m, \zeta) \subset M_2$, with $\mu_2(Y_{\mathrm{IC}}) > 1 - \zeta$, and $\rho = \rho(m, \zeta) \in (0, 1)$ such that* every $x \in Y_{\mathrm{IC}}$ *satisfies the Injectivity Condition* $\mathrm{IC}(\rho, m)$*.*

Proof. Let $K \subset M_2$ be a compact set of measure $\mu_2(K) \geq 1 - \zeta$. Let $4r$ be the injectivity radius of K. Thus, for any two points $x \in K$ and $y \in M_2$ such that $d(x, y) < r$ and for any lift $\mathbf{x} \in G_2$ of x, there exists a unique lift $\mathbf{y} \in G_2$ of y such that $d(\mathbf{x}\mathbf{u}_2^{t_1}, \mathbf{y}\mathbf{u}_2^{t_2}) < r$ for all t_1, t_2 with $|t_1| < r$ and $|t_2| < r$.

We recall that, if $\mathbf{a}_2^s = \exp(sA_2)$, with $A_2 \in \mathfrak{s}_2$ as in §2.1.1, one has $\mathbf{a}_2^s \mathbf{u}_2^t \mathbf{a}_2^{-s} = \mathbf{u}_2^{e^s t}$. Let s_0 such that $e^{s_0} r > m$ and set $Y_{\mathrm{IC}} = K\mathbf{a}_2^{s_0}$. Clearly $\mu_2(Y_{\mathrm{IC}}) = \mu_2(K) \geq 1 - \zeta$. Let L be the Lipschitz constant of the map $\mathbf{g} \mapsto \mathbf{g}\mathbf{a}_2^{-s_0}$ on G_2. Set $\rho = r/L$.

Arguing by contradiction, let $\mathbf{x} \in \pi_2^{-1}(Y_{\mathrm{IC}})$ with $d(\mathbf{x}, \mathbf{y}) < \rho$ and suppose that $d(\mathbf{x}\mathbf{u}_2^{t_1}, \gamma \mathbf{y}\mathbf{u}_2^{t_2}) \leq \rho$ for some $\gamma \in \Gamma_2 \setminus \{\mathbf{e}\}$ and some $t_1, t_2 \in [-m, m]$.

Then, $d(\mathbf{x}\mathbf{a}_2^{-s_0}, \mathbf{y}\mathbf{a}_2^{-s_0}) \leq L\rho = r$ and $d(\mathbf{x}\mathbf{u}_2^{t_1}\mathbf{a}_2^{-s_0}, \gamma\mathbf{y}\mathbf{u}_2^{t_2}\mathbf{a}_2^{-s_0}) \leq L\rho = r$. Set $\mathbf{x}' := \mathbf{x}\mathbf{a}_2^{-s_0}$ and $\mathbf{y}' := \mathbf{y}\mathbf{a}_2^{-s_0}$. On one hand we have $d(\mathbf{x}', \mathbf{y}') \leq r$ and

$$d(\mathbf{x}'\mathbf{u}_2^{e^{-s_0}t_1}, \gamma\mathbf{y}'\mathbf{u}_2^{e^{-s_0}t_2}) = d(\mathbf{x}\mathbf{u}_2^{t_1}\mathbf{a}_2^{-s_0}, \gamma\mathbf{y}\mathbf{u}_2^{t_2}\mathbf{a}_2^{-s_0}) \leq r.$$

On the other hand, since $e^{-s_0}t_1, e^{-s_0}t_2 \in [-r, r]$, we also have

$$d(\mathbf{x}', \gamma\mathbf{y}') \leq d(\mathbf{x}', \mathbf{x}'\mathbf{u}_2^{e^{-s_0}t_1}) + d(\mathbf{x}'\mathbf{u}_2^{e^{-s_0}t_1}, \gamma\mathbf{y}'\mathbf{u}_2^{e^{-s_0}t_2}) + d(\gamma\mathbf{y}'\mathbf{u}_2^{e^{-s_0}t_2}, \gamma\mathbf{y}') < 3r.$$

This shows that both points $\mathbf{y}'$ and $\gamma\mathbf{y}'$ belong to the ball of radius $3r$ centred in $\mathbf{x}'$. Since $\mathbf{x}' \in \pi_2^{-1}(K)$ and since the radius of injectivity of M_2 at any point in K is strictly greater than $3r$ we conclude that γ is the identity of G_2, a contradiction. □

In the above proof we used the existence of the subgroup $(\mathbf{a}^s)$ to rescale the orbits of the U-flow. In fact the above proposition holds for any volume preserving smooth flow on a finite volume manifold provided that the set of periodic orbit has measure zero.

2.3.1. We now introduce the second condition.

Definition 2.3.3 (The Frequently Bounded Radius Condition (FBR))**.** Given $T_0 \geq 0$, $r_0 > 0$, and $c \in [0, 1]$, we say that a point $x \in M_2$ satisfies the *Frequently Bounded Radius Condition* $\mathrm{FBR}(T_0, c, r_0)$ if, for any lift $\mathbf{x} \in G_2$ of x and any $T \geq T_0$, there exists $t \in [cT, T]$ such that the injectivity radius at $\mathbf{x}\mathbf{a}^t$ is bounded below by r_0.

If the space M_2 is compact, then there is $r_0 > 0$ such that every point $x \in M_2$ satisfies $\mathrm{FBR}(T_0, c, r_0)$ for all $T_0 \geq 0$ and $c \in [0, 1]$. It is also clear that if a point satisfies $\mathrm{FBR}(T_0, c, r_0)$, then it also satisfies $\mathrm{FBR}(T_0', c', r_0')$ for all $T_0' \geq T_0$, $0 \leq c' \leq c$, and $0 < r_0' \leq r_0$.

Lemma 2.3.4. *For every $c \in (0, 1)$ there exists $r_0 > 0$ and $T_0 > 0$ such that almost every point $x \in M_2$ satisfies $\mathrm{FBR}(T_0, c, r_0)$.*

For every $c \in (0, 1)$ and every $\omega > 0$, there exists $r_0 > 0$ and $T_0 \geq 0$ such that the measure of the set of points $x \in M_2$ which satisfy $\mathrm{FBR}(T_0, c, r_0)$ is at least $1 - \omega$.

Proof. It easy to see from the definition that if x satisfies the condition $\mathrm{FBR}(T_0, c, r_0)$, then the point $x\mathbf{a}^{-s}$ satisfies $\mathrm{FBR}(T_0 + s, c, r_0)$, for all $s \geq 0$. Thus, for $c \in (0, 1)$ and $r_0 > 0$ fixed, the set

$$\bigcup_{T'=0}^{\infty} \{x \in M_2 : x \text{ satisfies } \mathrm{FBR}(T', c, r_0)\}$$

is invariant under $\mathbf{a}^t$. By ergodicity, if r_0 is small enough, it has full measure.

Let $A_n = \{x \in M_2 : x \text{ satisfies } \mathrm{FBR}(n, c, r_0)\}$. By the above, the A_n are an increasing sequence of sets whose union has full μ_2-measure. Hence, given ω, there exists an integer T_0 such that

$$\mu_2\left(\bigcup_{n=0}^{T_0} A_n\right) \geq 1 - \omega. \qquad \square$$

3. The geometry of the U-flow

In this section, to ease the notation, we again drop the indices 1 and 2, so that the group G refers to both groups G_1 and G_2.

3.1. Irreducible $_2(\mathbb{R})$-modules and local transverse manifolds.

3.1.1. We recall that we have fixed $_2(\mathbb{R})$-sub-algebras in §2.1.1.

Considering G as the connected component of the identity of the real points of an algebraic linear $\mathbb{R}$-group $\mathbf{G}$ we have that $S = \exp \mathfrak{s}$ is the connected component of the identity of a Zariski closed subgroup $\mathbf{S} < \mathbf{G}$ with Lie algebra $\mathfrak{s}$.

By Chevalley's lemma ([4, 7.9], [43, 3.1.4]) and the simplicity of $\mathbf{S}$ there exists a linear representation ϕ of $\mathbf{G}$ on a finite vector space V and a vector $v_0 \in V$ such that $\mathbf{S} = \mathrm{Stab}_{\mathbf{G}}(v_0)$.

The group S acts on $\mathfrak{g}$ by the adjoint action. By the simplicity of S, the subspace $\mathfrak{s}$ of $\mathfrak{g}$ has an $\mathrm{Ad}(S)$ invariant complementary subspace $\mathfrak{m}$. Thus there exists a neighbourhood $\mathcal{O} \subset G$ of the identity such that any $\mathbf{g} \in \mathcal{O}$ may be written as a product $\mathbf{g} = \mathbf{g}_{\mathfrak{m}}\mathbf{g}_{\mathfrak{s}}$ of elements $\mathbf{g}_{\mathfrak{m}} \in \exp \mathfrak{m}$ and $\mathbf{g}_{\mathfrak{s}} \in \exp \mathfrak{s}$. The following lemma, whose proof is in Appendix A, proves the uniqueness of such a decomposition.

Lemma 3.1.1. *For every neighbourhoods $\mathcal{O}_{\mathfrak{m}}$, $\mathcal{O}_{\mathfrak{s}}$ of the identity, respectively in $\exp \mathfrak{m}$ and in $\exp \mathfrak{s}$, there exists a neighbourhood $\mathcal{O} \subset G$ of the identity such that, if $\mathbf{g} \in \mathcal{O}$ and $\mathbf{g} = \mathbf{g}_{\mathfrak{m}}\mathbf{g}_{\mathfrak{s}}$, with $\mathbf{g}_{\mathfrak{m}} \in \exp \mathfrak{m}$ and $\mathbf{g}_{\mathfrak{s}} \in \exp \mathfrak{s}$, then $\mathbf{g}_{\mathfrak{m}} \in \mathcal{O}_{\mathfrak{m}}$ and $\mathbf{g}_{\mathfrak{s}} \in \mathcal{O}_{\mathfrak{s}}$.*

3.1.2. As a consequence of this lemma, if $\mathbf{g} \in \mathcal{O}$ and

$$\mathbf{g} = \mathbf{g}_{\mathfrak{m}} \mathbf{g}_{\mathfrak{s}}, \quad \text{with } \mathbf{g}_{\mathfrak{m}} \in \exp \mathfrak{m}, \quad \mathbf{g}_{\mathfrak{s}} \in \exp \mathfrak{s}, \tag{8}$$

we may set

$$d(\mathbf{e}, \mathbf{g}) = \max\{d_{\mathfrak{m}}(\mathbf{e}, \mathbf{g}_{\mathfrak{m}}), d_{\mathfrak{s}}(\mathbf{e}, \mathbf{g}_{\mathfrak{s}})\},$$

where $d_{\mathfrak{m}}$, $d_{\mathfrak{s}}$ are distances on $\exp \mathfrak{m}$ and $\exp \mathfrak{s}$ — (see (15) and (17) for their definitions). Thus there exists ε_0 such that $d(\mathbf{e}, \mathbf{g}) < \varepsilon_0$ implies $\mathbf{g} \in \mathcal{O}$ and the decomposition (8) is uniquely determined.

3.1.3. For any $\mathbf{g} \in G$ and any $\varepsilon \in (0, \varepsilon_0)$, let

$$W(\mathbf{g}, \varepsilon) = \Big\{ \mathbf{g}\, \mathbf{g}_{\mathfrak{m}} \exp(aA) \exp(\bar{u}\overline{U}) \mid \mathbf{g}_{\mathfrak{m}} \in \exp(\mathfrak{m}),\ d_{\mathfrak{m}}(\mathbf{e}, \mathbf{g}_{\mathfrak{m}}) \le \varepsilon,\ |a| \le \varepsilon,\ |\bar{u}| \le \varepsilon \Big\}.$$

The set $W(\mathbf{g}, \varepsilon)$ is the *local leaf through* $\mathbf{g}$ transversal to the U-flow. We also define the *(global) leaf through* $\mathbf{g}$ transversal to the U-flow by

$$W(\mathbf{g}) = \{\mathbf{g}\, \mathbf{g}_{\mathfrak{m}} \exp(aA) \exp(\bar{u}\overline{U}) \mid \mathbf{g}_{\mathfrak{m}} \in \exp(\mathfrak{m}),\ a, \bar{u} \in \mathbb{R}\}.$$

Given $\mathbf{x}, \mathbf{y} \in G$ with $d(\mathbf{x}, \mathbf{y}) \le \varepsilon$, we define the *continuous parametrization* $q(s)$ of the U-orbit of $\mathbf{y}$ by the condition $\mathbf{y}\mathbf{u}^{q(s)} \in W(\mathbf{x}\mathbf{u}^s)$. The parametrization $q(s)$ is well-defined at least for all sufficiently small values of $s \ge 0$ (see also Lemma 3.1.2 below).

3.1.4. Let now $\mathbf{x}, \mathbf{y} \in G$, $\varepsilon \in (0, \varepsilon_0)$, and assume that

$$d(\mathbf{x}, \mathbf{y}) \le \varepsilon \quad \text{and} \quad d(\mathbf{x}\mathbf{u}^s, \mathbf{y}\mathbf{u}^t) \le \varepsilon. \tag{9}$$

for some $s > 0$, and $t > 0$. This condition depends only on the relative position of $\mathbf{x}$ and $\mathbf{y}$: if we Setting $\mathbf{g} = \mathbf{x}^{-1}\mathbf{y}$ we may rewrite (9) as

$$d(\mathbf{e}, \mathbf{g}) \le \varepsilon \quad \text{and} \quad d(\mathbf{e}, \mathbf{u}^{-s}\mathbf{g}\mathbf{u}^t) \le \varepsilon. \tag{10}$$

or, with $\mathbf{g} = \mathbf{g}_{\mathfrak{m}}\mathbf{g}_{\mathfrak{s}}$ as in (8),

$$d_{\mathfrak{m}}(\mathbf{e}, \mathbf{u}^{-s}\mathbf{g}_{\mathfrak{m}}\mathbf{u}^s) \le \varepsilon \quad \text{and} \quad d_{\mathfrak{s}}(\mathbf{e}, \mathbf{u}^{-s}\mathbf{g}_{\mathfrak{s}}\mathbf{u}^t) \le \varepsilon. \tag{11}$$

Thus the condition (9) imposes strong restrictions on both $t(s)$ and on the relative position of $\mathbf{x}$ and $\mathbf{y}$. If (9) holds, with an appropriate definition of the distance $d_{\mathfrak{s}}$, there exists $q(s) \in \mathbb{R}$ such that

$$|t - q(s)| \le \varepsilon, \quad \text{and} \quad \mathbf{y}\mathbf{u}^{q(s)} \in W(\mathbf{x}\mathbf{u}^s, \varepsilon) \tag{12}$$

With $\mathbf{x}^{-1}\mathbf{y} = \mathbf{g}_{\mathfrak{m}}\mathbf{g}_{\mathfrak{s}}$ as in (8), this condition may be rewritten as

$$\mathbf{u}^{-s}\mathbf{g}_{\mathfrak{s}}\mathbf{u}^{q(s)} = \exp(a(s)A) \exp(\bar{u}(s)\overline{U}). \tag{13}$$

Thus, the second inequality in formula (11) is equivalent the the requirement that (12) holds true and that $|a(s)| \le \varepsilon/2$ and $|\bar{u}(s)| \le \varepsilon/2$.

It follows that in order to exploit the consequences of the condition (9) we can analyse separately the terms $\mathbf{u}^{-s}\mathbf{g}_{\mathfrak{m}}\mathbf{u}^s$ and $\mathbf{u}^{-s}\mathbf{g}_{\mathfrak{s}}\mathbf{u}^{q(s)}$.

3.1.5. We start by considering the term $\mathbf{u}^{-s}\mathbf{g}_{\mathfrak{m}}\mathbf{u}^{s}$. Since $\mathrm{ad}(U)$ is a nilpotent endomorphism of $\mathfrak{m}$, there exists a basis of the vector space $\mathfrak{m}$ which is a Jordan basis for $\mathrm{ad}(U)$. More precisely, the subspace $\mathfrak{m}$ splits into a direct sum of $\mathrm{Ad}(S)$-invariant subspaces $\mathfrak{m}_\iota$, $(\iota = 1, \dots, I)$, of dimension $d_\iota + 1$, on which $\mathrm{Ad}(S)$ acts irreducibly. By the elementary representation theory of ${}_2(\mathbb{R})$, each subspace $\mathfrak{m}_\iota$ has a basis $(E_{0,\iota}, \dots, E_{d_\iota,\iota})$ such that

$$\mathrm{ad}(U)E_{j,\iota} = E_{j-1,\iota} \quad \text{and} \quad \mathrm{ad}(A)E_{j,\iota} = \left(\frac{d_\iota}{2} - j\right) E_{j,\iota}. \tag{14}$$

We choose $\mathcal{B} = \{E_{j,\iota} : 0 \le j \le d_\iota, 1 \le \iota \le I\}$ as a convenient Jordan basis for $\mathrm{ad}(U)$ and we let, for $\xi = \sum c_{j,\iota}E_{j,\iota}$,

$$\|\xi\| = \max_{\iota,j} |c_{j,\iota}|, \quad \text{and} \quad d_{\mathfrak{m}}(\mathbf{e}, \exp(\xi)) = \|\xi\|. \tag{15}$$

For any $\xi \in \mathfrak{m}$, $\mathrm{Ad}(\mathbf{u}^s)\xi = \exp(s\,\mathrm{ad}(U))\xi$ is a $\mathfrak{m}$-valued polynomial in the variable s. Indeed, the usual computation of the exponential of a Jordan block tells us that if $\xi = \sum_{\iota,j} c_{j,\iota}E_{j,\iota}$ then

$$\exp(s\,\mathrm{ad}(U))\xi = \sum_{\iota,j} c_{j,\iota}(s)E_{j,\iota}, \quad c_{j,\iota}(s) = \sum_{k=0}^{d_\iota - j} c_{k+j,\iota}s^k/k!\,.$$

Since the highest degree polynomials are given by the coefficients of the terms $E_{0,\iota}$, which together with U span the centraliser $\mathfrak{z}(U)$ of U in $\mathfrak{g}$ we are led to define for $\mathbf{g}_{\mathfrak{m}} = \exp\xi$ and $\xi = \sum_{\iota,j} c_{j,\iota}E_{j,\iota}$ the polynomials

$$r_{\mathfrak{m}_\iota}(\mathbf{g}_{\mathfrak{m}}, s) := \sum_{k=0}^{d_\iota} c_{k,\iota}(-1)^k s^k/k!\,,$$

and the vector valued polynomial

$$r_{\mathfrak{m}}(\mathbf{g}_{\mathfrak{m}}, s) := (r_{\mathfrak{m}_1}(\mathbf{g}_{\mathfrak{m}}, s), \dots, r_{\mathfrak{m}_I}(\mathbf{g}_{\mathfrak{m}}, s)) \in \mathbb{R}^I \otimes \mathbb{R}[s]$$

which controls the growth of the term $\mathbf{u}^{-s}\mathbf{g}_{\mathfrak{m}}\mathbf{u}^{s} = \exp \mathrm{Ad}(\mathbf{u}^{-s})\xi$. Clearly

$$|r_{\mathfrak{m}}(\mathbf{g}_{\mathfrak{m}}, s)| := \sup_{1 \le \iota \le I} |r_{\mathfrak{m}_\iota}(\mathbf{g}_{\mathfrak{m}}, s)| \le d_{\mathfrak{m}}(\mathbf{e}, \mathbf{u}^{-s}\mathbf{g}_{\mathfrak{m}}\mathbf{u}^{s}). \tag{16}$$

3.1.6. We now turn to the term $\mathbf{u}^{-s}\mathbf{g}_{\mathfrak{s}}\mathbf{u}^{q(s)} \in \exp\mathfrak{s}$, where we recall that the parametrisation $q(s)$ is defined by the identity (13).

For $\mathbf{x}_{\mathfrak{s}}, \mathbf{y}_{\mathfrak{s}} \in S$ and $\mathbf{g}_{\mathfrak{s}} = \mathbf{x}_{\mathfrak{s}}^{-1}\mathbf{y}_{\mathfrak{s}} = \exp(aA)\exp(\bar{u}\overline{U})\mathbf{u}^u$ it is convenient to define the "distance"

$$d_{\mathfrak{s}}(\mathbf{x}_{\mathfrak{s}}, \mathbf{y}_{\mathfrak{s}}) = d_{\mathfrak{s}}(\mathbf{e}_{\mathfrak{s}}, \mathbf{g}_{\mathfrak{s}}) = |a| + |\bar{u}| + |u|. \tag{17}$$

Writing

$$\mathbf{g}_{\mathfrak{s}} = \exp(aA)\exp(\bar{u}\overline{U})\mathbf{u}^u, \tag{18}$$

the formula (13), after an easy computation, yields

$$e^{a(s)/2} = e^{-a/2}(e^a - \bar{u}s), \qquad \bar{u}(s) = e^{-a}\bar{u}(e^a - \bar{u}s) \tag{19}$$

$$q(s) = s - \frac{\bar{u}s^2 + s(1 - e^a)}{e^a - \bar{u}s} - u. \tag{20}$$

Henceforth $q(s)$, $a(s)$, $\bar{u}(s)$ denote the functions of s defined by the formulas above (depending on the initial parameters a, $\bar{u}$ and u).

The following lemma/definition expresses the condition $\mathbf{y}\mathbf{u}^{q(s)} \in W(\mathbf{x}\mathbf{u}^s)$ in terms of the initial parameters a, $\bar{u}$.

Lemma 3.1.2. *Let $\mathbf{x}, \mathbf{y} \in G$ with $\mathbf{x}^{-1}\mathbf{y} = \mathbf{g}_{\mathfrak{m}}\mathbf{g}_{\mathfrak{s}}$ and $\mathbf{g}_{\mathfrak{s}} = \exp(aA)\exp(\bar{u}\overline{U})\mathbf{u}^u$. Then the condition $\mathbf{y}\mathbf{u}^{q(s)} \in W(\mathbf{x}\mathbf{u}^s)$ is satisfied on the connected interval $[0, s] \cup [s, 0]$ if, and only if, s belongs to the* interval of maximal tracking *defined as*

$$\mathcal{I}(\mathbf{x}, \mathbf{y}) := \mathcal{I}(\mathbf{g}_{\mathfrak{s}}) := \{s \in \mathbb{R} : e^a - \bar{u}s > 0\}.$$

In particular if $0 < \varepsilon < \min\big(\varepsilon_0, 1/(2M)\big)$ and $d(\mathbf{x}, \mathbf{y}) \leq \varepsilon$, the condition $\mathbf{y}\mathbf{u}^{q(s)} \in W(\mathbf{x}\mathbf{u}^s)$ can be fulfilled for all $|s| < M$.

Lemma 3.1.3. *Let $\mathbf{x}, \mathbf{y} \in G$ with $\mathbf{x}^{-1}\mathbf{y} = \mathbf{g}_{\mathfrak{m}}\mathbf{g}_{\mathfrak{s}}$ and $\mathbf{g}_{\mathfrak{s}} = \exp(aA)\exp(\bar{u}\overline{U})\mathbf{u}^u$. Assume $\delta_1, \delta_2 < \min(\varepsilon_0, \ln 2)$. If $d(\mathbf{x}, \mathbf{y}) \leq \delta_1$ and $d(\mathbf{x}\mathbf{u}^{s_0}, \mathbf{y}\mathbf{u}^{t_0}) \leq \delta_2$ for some $s_0 > 0$ and $t_0 \in \mathbb{R}$. Then s_0 belongs to the interval of maximal tracking $\mathcal{I}(\mathbf{x}, \mathbf{y})$ and*

$$|q(s_0) - t_0| \leq \delta_2 \quad \textit{and} \quad |\bar{u}s| \leq 3(\delta_1 + \delta_2), \quad \forall s \in [0, s_0].$$

Furthermore

$$1 - 8\max(\delta_1, \delta_2) \leq e^a - \bar{u}s < 1 + 8\max(\delta_1, \delta_2), \qquad \forall s \in [0, s_0]. \tag{21}$$

Setting $u_0 := q(s_0) - t_0$ we have

$$\mathbf{y}\mathbf{u}^t = (\mathbf{x}\mathbf{u}^{s_0})\exp(a(s_0)A)\exp(\bar{u}(s_0)\overline{U})\mathbf{u}^{u_0}.$$

Furthermore, for all $s \in [0, s_0]$, we have

$$\mathbf{y}\mathbf{u}^{q(s)} = (\mathbf{x}\mathbf{u}^s)\exp(a(s)A)\exp(\bar{u}(s)\overline{U})$$

with

$$|a(s)| \leq 11\delta_1 + 6\delta_2, \qquad |\bar{u}(s)| \leq 13\delta_1.$$

Proof. In the course of the proof we use the estimates $1 - |x|/2 \leq e^x \leq 1 + 2|x|$ for $|x| \leq \log 2 < 1$.

By §3.1.3, the condition $d(\mathbf{x}\mathbf{u}^{s_0}, \mathbf{y}\mathbf{u}^{t_0}) \leq \delta_2$ implies by $d_{\mathfrak{s}}(\mathbf{e}, \mathbf{u}^{-s_0}\mathbf{g}_{\mathfrak{s}}\mathbf{u}^{t_0}) \leq \delta_2$. It follows that the interval of orbit $\{\mathbf{y}\mathbf{u}^{t_0+t'}\}_{|t'| \leq \delta_2}$ crosses the leaf $W(\mathbf{x}\mathbf{u}^{s_0})$ in a point $\mathbf{y}\mathbf{u}^q$ such that $|q - t_0| \leq \delta_2$. Then

$$\mathbf{u}^{-s_0}\mathbf{g}_{\mathfrak{s}}\mathbf{u}^q = \exp(a'A)\exp(\bar{u}'\,\overline{U}),$$

with $|a'| + |\bar{u}'| \leq \delta_2$.

From §3.1.6 the only solution of this identity is given by $q = q(s_0)$, $a' = a(s_0)$, $\bar{u}' = \bar{u}(s_0)$, where $q(s)$, $a(s)$, $\bar{u}(s)$ are the functions defined by the formulas (19)–(20).

Since $|a| + |\bar{u}| + |u| \leq \delta_1$, using the first identity (19) we have

$$e^a - \bar{u}s_0 = e^{(a'+a)/2} \geq 1 - \frac{|a' + a|}{4} > \frac{1}{2}.$$

proving that $s_0 \in \mathcal{I}(\mathbf{x}, \mathbf{y})$. Furtermore, since $e^a \leq 1 + 2|a| \leq 3$ and $e^{(a'-a)/2} - 1 \leq |a' - a|$, we have

$$|\bar{u}s_0| = e^a|1 - e^{(a'-a)/2}| \leq 3(\delta_1 + \delta_2)$$

and consequently $|\bar{u}s| \leq 3(\delta_1 + \delta_2)$ for all $s \in [0, s_0]$. Then, the estimate (21) follows easily.

Finally formula (19) and the inequalities $|\bar{u}s| \leq 3(\delta_1 + \delta_2) \leq 6$, for all $s \in [0, s_0]$, and $e^a - \bar{u}s \leq 1 + 2|a| + |\bar{u}s|$ imply

$$|a(s)| \leq |a| + 2|\log(e^a - \bar{u}s)| \leq 11\delta_1 + 6\delta_2$$

and

$$e^{(a(s)-a)/2} = 1 - e^{-a}\bar{u}s \leq 13.$$

Consequently

$$|\bar{u}(s)| = e^{(a(s)-a)/2}|\bar{u}| \leq 13\delta_1$$

for all $s \in [0, s_0]$. □

3.1.7. In the sequel we shall deal with parametrisations $\tau(s)$ of the orbit of $\mathbf{y}$ satisfying, for sufficiently large s, the inequality

$$|\tau(s) - s| \leq \frac{1}{2}(s^{1-\eta} + \varepsilon),$$

and such that

$$d(\mathbf{x}\mathbf{u}^s, \mathbf{y}\mathbf{u}^{\tau(s)}) \leq \frac{\varepsilon}{2}$$

for some $s > 0$ and $\varepsilon < \min(\varepsilon_0, \log 2)$. By the previous lemma if this second condition occurs we have $s \in \mathcal{I}(\mathbf{x}, \mathbf{y})$ and $|\tau(s) - q(s)| \leq \varepsilon/2$. Then by the first condition $|q(s) - s| \leq 1/2 s^{1-\eta} + \varepsilon$. From formulas (21) and (20), the quantity $\bar{u}s^2 + s(1 - e^a)$ up to a factor close to 1 is the delay $s - q(s)$ of the two orbits. Thus we are led to define

$$r_{\mathfrak{s}}(\mathbf{g}_{\mathfrak{s}}, s) := \bar{u}s^2 + s(1 - e^a), \tag{22}$$

and study the sub-level sets where $|r_{\mathfrak{s}}(\mathbf{g}_{\mathfrak{s}}, s)| \leq s^{1-\eta} + \varepsilon$. When these intervals are not too spaced from each other the following proposition comes to aid.

Proposition 3.1.4. *Let $(V, |\cdot|_V)$ be a finite dimensional normed vector space and $P(s) = c_0 + c_1 s + \cdots + c_k s^k$ be a V-valued polynomial of degree k bounded by $\max\{\varepsilon, Cs^{1-\eta}\}$ on consecutive intervals $J_1 = [s_1 = 0, \bar{s}_1]$, …, $J_h = [s_h, \bar{s}_h]$. (We shall assume $\varepsilon \leq C\bar{s}_1^{1-\eta}$.) Suppose that the intervals $J_1, \ldots, J_h$ are* b-close *in the sense that, for some $b > 0$*

$$d(J_j, J_{j+1}) := s_{j+1} - \bar{s}_j \leq \bar{s}_j^{1+b}, \quad \forall j = 1, \ldots, h-1.$$

Then, for some constants κ_k, κ'_k, depending only on k, we have

$$|c_i|_V \leq \kappa_k(\kappa'_k)^{h-1}\bar{s}_h^{-i+1-\eta+(h-1)kb}.$$

Thus if $(h-1)kb < \eta/2$ we have

$$|c_i|_V \leq \kappa_k(\kappa'_k)^{h-1}\bar{s}_h^{-i+1-\eta/2}.$$

Proof. Since the space of polynomials on $[0, 1]$ of degree lesser or equal than k with values in a finite vector space V has dimension $(k + 1) \times \dim V$, the norms $\|P\|_1 = \sup_{s\in[0,1]}|P(s)|_V$ and $\|P\|_2 = \sup_{0\leq i\leq k}|c_i|_V$ are equivalent.

It follows there exists a constant $\kappa_k > 1$ such that, if a polynomial of degree k such as P is bounded by a constant K on a interval $[0, L]$, then $|c_i|_V \leq \kappa_k K\, L^{-i}$. Thus, with our assumptions, if $h = 1$ we have

$$|c_i|_V \leq \kappa_k C\bar{s}_1^{-i+1-\eta}.$$

This inequality implies that on the interval $[\bar{s}_1, \bar{s}_1 + \bar{s}_1^{1+b}]$, the polynomial $P(s)$ is bounded by $|\sum_{i=0}^k c_i\bar{s}_1^i(1 + \bar{s}_1^b)^i|_V \leq C\sum_{i=0}^k \kappa_k(\bar{s}_1)^{1-\eta}(1 + \bar{s}_1^b)^i \leq C\kappa'_k\bar{s}_1^{1-\eta+kb}$.

In the case $\bar{s}_2^{1-\eta} \leq \kappa'_k\bar{s}_1^{1-\eta+kb}$ we deduce

$$|c_i|_V \leq C\kappa_k\kappa'_k\bar{s}_1^{1-\eta+kb}s_2^{-i} \leq \kappa_k\kappa'_k\bar{s}_2^{-i+1-\eta+kb}. \tag{23}$$

If $\bar{s}_2^{1-\eta} > \kappa'_k \bar{s}_1^{1-\eta+kb}$ then we deduce

$$|c_i|_V \leq C\kappa_k \bar{s}_2^{-i+1-\eta}.$$

Thus the worse case scenario is given by the estimate (23). By induction we obtain that

$$|c_i|_V \leq C\kappa_k (\kappa'_k)^{j-1} \bar{s}_j^{-i+1-\eta+(j-1)kb}. \qquad \square$$

We shall apply this proposition to the $\mathbb{R}^I$-valued polynomial $r_{\mathfrak{m}}(\mathbf{g}_{\mathfrak{m}}, s)$ bounded by ε (in this case in the proposition we choose $C = \varepsilon$ and $\eta = 1$), and to the real valued polynomial $r_{\mathfrak{s}}(\mathbf{g}_{\mathfrak{s}}, s)$ bounded by $\max\{\varepsilon, Cs^{1-\eta}\}$. We remark that the set $\{s \geq 0 : r_{\mathfrak{m}}(\mathbf{g}_{\mathfrak{m}}, s) \leq \varepsilon, r_{\mathfrak{s}}(\mathbf{g}_{\mathfrak{s}}, s) \leq \max\{\varepsilon, Cs^{1-\eta}\}\}$ consists of at most $D = 2\prod_{\iota=1}^{I} d_\iota$ connected components.

Proposition 3.1.5. *There exists a constant $\kappa > 0$ such that the following holds true. Let $\mathbf{x}, \mathbf{y} \in G$ and $\mathbf{x}^{-1}\mathbf{y} = \mathbf{g}_{\mathfrak{m}}\mathbf{g}_{\mathfrak{s}}$ with $\mathbf{g}_{\mathfrak{s}} = \exp(aA)\exp(\bar{u}\overline{U})\mathbf{u}^u$ and $\mathbf{g}_{\mathfrak{m}} = \sum_{j,\iota} c_{j,\iota} E_{j,\iota}$.*

Suppose we have $h \leq D$ intervals $J_1 = [s_1 = 0, \bar{s}_1]$, $\dots$ $J_h = [s_h, \bar{s}_h = L]$ such that

(1) *$r_{\mathfrak{m}}(\mathbf{g}_{\mathfrak{m}}, s) \leq \varepsilon$ and $r_{\mathfrak{s}}(\mathbf{g}_{\mathfrak{s}}, s) \leq \max\{\varepsilon, Cs^{1-\eta}\}$ for all $s \in \bigcup_{j=1}^{h} J_j$.*

(2) *the intervals $J_1, \dots$ J_h are b-close for some $b < \eta/(2D \times \max_\iota d_\iota)$.*

Then,

$$|a| \leq \kappa L^{-\eta/2}, \quad |\bar{u}| \leq \kappa L^{-1-\eta/2},$$

and

$$|c_{j,\iota}| \leq \kappa\varepsilon L^{-j+\eta/2}.$$

The role of the intervals $J_1, \dots, J_h$ will become clear in the next subsection.

3.2. **Blocks.**

3.2.1. We now explain how we are going to apply Proposition 3.1.5. In order to do it, we are going to introduce the notion of (ρ, ε)-*blocks* (or simply *blocks*), which will play a crucial role in the proof of Ratner's Basic Lemma in the next Section.

In this section, for simplicity, we will write $r_{\mathfrak{m}}(\mathbf{g}, s)$ and $r_{\mathfrak{s}}(\mathbf{g}, s)$ instead of $r_{\mathfrak{m}}(\mathbf{g}_{\mathfrak{m}}, s)$ and $r_{\mathfrak{s}}(\mathbf{g}_{\mathfrak{s}}, s)$. We define

$$\begin{aligned} l_\varepsilon(\mathbf{g}_{\mathfrak{m}}) &:= \sup\{s > 0 : |r_{\mathfrak{m}}(\mathbf{g}, t)| \leq \varepsilon \text{ for all } t \in [0, s]\}, \\ l_\varepsilon(\mathbf{g}_{\mathfrak{s}}) &:= \sup\{s > 0 : |r_{\mathfrak{s}}(\mathbf{g}, t)| \leq t^{1-\eta} + \varepsilon \text{ for all } t \in [0, s]\}, \\ l_\varepsilon(\mathbf{g}) &:= \min\{l_\varepsilon(\mathbf{g}_{\mathfrak{m}}), l_\varepsilon(\mathbf{g}_{\mathfrak{s}})\}. \end{aligned}$$

Let $\rho \leq \varepsilon/2$. Let $\mathbf{x}, \mathbf{y} \in G$ with $d(\mathbf{x}, \mathbf{y}) \leq \rho$ be fixed, and let $\tau(s)$ be a parametrization of the U-orbit of $\mathbf{y}$ satisfying $\tau(0) = 0$ and $|\tau(s) - s| \leq \frac{1}{2}(s^{1-\eta} + \varepsilon)$ for all $s \geq l_\varepsilon(\mathbf{g})$. Let now $\mathbf{x}_i = \mathbf{x}\mathbf{u}^{s_i}$ and $\mathbf{y}_i = \mathbf{y}\mathbf{u}^{\tau(s_i)}$ be two points on the U-orbits of $\mathbf{x}$ and $\mathbf{y}$ satisfying $d(\mathbf{x}_i, \mathbf{y}_i) \leq \rho$. A (ρ, ε)-*block* for $\mathbf{x}_i, \mathbf{y}_i$ is an interval

$$J_i = [s_i, \bar{s}_i], \quad \text{with} \quad |J_i| = \bar{s}_i - s_i \leq l_\varepsilon(\mathbf{g}_i),$$

where $\mathbf{g}_i = \mathbf{x}_i^{-1}\mathbf{y}_i$, such that

$$d(\mathbf{x}\mathbf{u}^{\bar{s}_i}, \mathbf{y}\mathbf{u}^{\tau(\bar{s}_i)}) \leq \varepsilon,$$

in other words, if not only the initial points but also the final points are at distance not larger than ρ. By definition, the translated interval $J - s_i = [0, \bar{s}_i - s_i]$ is contained in a sub-level set of the polynomials $r_{\mathfrak{m}}(\mathbf{g}_i, s)$ and $r_{\mathfrak{s}}(\mathbf{g}_i, s)$. Since ρ and ε are fixed, we will simply say *block*.

Let $J_i = [s_i, \bar{s}_i]$ be a block for $\mathbf{x}_i, \mathbf{y}_i$, and assume that there exists $s_j \geq s_i + l_\varepsilon(\mathbf{g}_i) \geq \bar{s}_i$ such that the points $\mathbf{x}_j = \mathbf{x}\mathbf{u}^{s_j}$ and $\mathbf{y}_j = \mathbf{y}\mathbf{u}^{\tau(s_j)}$ are at distance $d(\mathbf{x}_j, \mathbf{y}_j) \leq \varepsilon$. Let J_j be a block for $\mathbf{x}_j, \mathbf{y}_j$, hence we know that $J_j - s_j$ is an interval contained in a sub-level set for $r_{\mathfrak{m}}(\mathbf{g}_j, s)$ and $r_{\mathfrak{s}}(\mathbf{g}_j, s)$ (notice that these polynomials are defined by $\mathbf{g}_j = \mathbf{x}_j^{-1}\mathbf{y}_j$ and not by $\mathbf{g}_i = \mathbf{x}_i^{-1}\mathbf{y}_i$). The following lemma shows that $J_j - s_i$ is contained in a (slightly larger) sub-level set for $r_{\mathfrak{m}}(\mathbf{g}_i, s)$ and $r_{\mathfrak{s}}(\mathbf{g}_i, s)$.

Lemma 3.2.1. *There exists an absolute constant $\tilde{\kappa} > 0$ such that the following holds. Let $J_j = [s_j, \bar{s}_j]$ be a block for $\mathbf{x}_j, \mathbf{y}_j$ as above. Then,*

$$J_j - s_i = [s_j - s_i, \bar{s}_j - s_i] \subset \{|r_{\mathfrak{m}}(\mathbf{g}_i, s)| \leq \tilde{\kappa}\varepsilon\} \cap \{|r_{\mathfrak{s}}(\mathbf{g}_i, s)| \leq \tilde{\kappa}(s^{1-\eta} + \varepsilon)\}.$$

Proof. To simplify the notation, in this proof we will write $\tilde{\mathbf{g}}$ instead of $\mathbf{g}_j$ and $\mathbf{g}$ instead of $\mathbf{g}_i$.

Denote by $|\tilde{J}| = \bar{s}_j - s_j$ the length of the block $J_j = \tilde{J}$. By definition, for all $0 \leq r \leq |\tilde{J}|$, we have that $|r_{\mathfrak{m}}(\tilde{\mathbf{g}}, r)| \leq \varepsilon$. By Proposition 3.1.5 (applied with only one interval), we get that the coefficients $\tilde{c}_{j,\iota}$ in the expression $\tilde{\mathbf{g}}_{\mathfrak{m}} = \sum_{j,\iota} \tilde{c}_{j,\iota} E_{j,\iota}$ satisfy $|\tilde{c}_{j,\iota}| \leq \kappa\varepsilon |\tilde{J}|^{-j+\eta/2}$. Hence, from §3.1.5, it follows that $d_{\mathfrak{m}}(\mathbf{e}, \mathbf{u}^{-r}\tilde{\mathbf{g}}_{\mathfrak{m}}\mathbf{u}^r) \leq \tilde{\kappa}\varepsilon$ for some constant $\tilde{\kappa}$ depending on $\mathfrak{m}$ only. By (16), we deduce that

$$|r_{\mathfrak{m}}(\mathbf{g}, r + s_j - s_i)| \leq d_{\mathfrak{m}}(\mathbf{e}, \mathbf{u}^{-(r+s_j-s_i)}\mathbf{g}_{\mathfrak{m}}\mathbf{u}^{r+s_j-s_i}) = d_{\mathfrak{m}}(\mathbf{e}, \mathbf{u}^{-r}\tilde{\mathbf{g}}_{\mathfrak{m}}\mathbf{u}^r) \leq \tilde{\kappa}\varepsilon,$$

which shows that $\tilde{J} - s_i \subset \{|r_{\mathfrak{m}}(\mathbf{g}, s)| \leq \tilde{\kappa}\varepsilon\}$.

Let us prove the other inclusion. By Lemma 3.1.2, the whole interval $[0, \bar{s}_j]$ is contained in the interval of maximal tracking $\mathcal{I}(\mathbf{x}, \mathbf{y})$. Moreover, by Lemma 3.1.3 and the subsequent discussion, since $d(\mathbf{x}_j, \mathbf{y}_j) \leq \rho \leq \varepsilon/2$, we have that

$$|q(s_j - s_i) - (s_j - s_i)| \leq (s_j - s_i)^{1-\eta} + \varepsilon.$$

On the other hand, by assumption and by (20), we have that

$$|q(r) - r| \leq \frac{|r_{\mathfrak{s}}(\tilde{\mathbf{g}}, r)|}{1 - 10\varepsilon} + \varepsilon \leq 2r^{1-\eta} + 2\varepsilon,$$

for all $0 \leq r \leq |\tilde{J}|$. Therefore, we get

$$\begin{aligned}|q(r + s_j - s_i) - (r + s_j - s_i)| &\leq |q(r) - r| + |q(s_j - s_i) - (s_j - s_i)| \\ &\leq 2r^{1-\eta} + (s_j - s_i)^{1-\eta} + 3\varepsilon \leq 4(r + s_j - s_i)^{1-\eta} + 4\varepsilon.\end{aligned}$$

Finally, again by (20), since

$$\begin{aligned}|q(r + s_j - s_i) - (r + s_j - s_i)| &\geq \frac{|r_{\mathfrak{s}}(\mathbf{g}, r + s_j - s_i)|}{1 + 10\varepsilon} - \varepsilon \\ &\geq \frac{1}{2}|r_{\mathfrak{s}}(\mathbf{g}, r + s_j - s_i)| - \varepsilon,\end{aligned}$$

combining the two inequalities above, we conclude that $\tilde{J} - s_i \subset \{|r_{\mathfrak{s}}(\mathbf{g}, s)| \leq 8(s^{1-\eta} + \varepsilon)\}$, which completes the proof. □

In layman's terms, the previous lemma says that all the blocks that we can find along the future orbits of two fixed points $\mathbf{x}, \mathbf{y}$ must be contained in a certain sub-level set of a polynomial map determined only by the initial points $\mathbf{x}, \mathbf{y}$, whose degree is *bounded independently of the points.*

For any given $\mathbf{x}_i, \mathbf{y}_i$, let $S_1, \dots, S_D$ be the connected components of the sub-level set

$$\mathrm{SubLev}(\mathbf{x}_i, \mathbf{y}_i) := \{|r_{\mathfrak{m}}(\mathbf{g}_i, s)| \leq \tilde{\kappa}\varepsilon\} \cap \{|r_{\mathfrak{s}}(\mathbf{g}_i, s)| \leq \tilde{\kappa}(s^{1-\eta} + \varepsilon)\}.$$

Note again that D is bounded by a constant depending on $\mathfrak{g}$ only. By Lemma 3.2.1, for any block J_j as above, there exists S_{i_j} such that $J_j - s_i \subset S_{i_j}$. We now use Proposition 3.1.5 to deduce the following corollary.

Corollary 3.2.2. *There exists a constant $\kappa > 0$ such that the following holds. Let $\mathbf{x}, \mathbf{y} \in G$ and $\tau(s)$ be as above. Let $J_1 = [s_1 = 0, \bar{s}_1]$ be a (ρ, ε)-block for $\mathbf{x}, \mathbf{y}$. Assume that there exists $s_2 > l_\varepsilon(\mathbf{g}_1)$ such that $\mathbf{x}_2 = \mathbf{x}\mathbf{u}^{s_2}$ and $\mathbf{y}_2 = \mathbf{y}\mathbf{u}^{\tau(s_2)}$ are at distance at most $\rho \leq \varepsilon/2$, and, as before, let $J_2 = [s_2, \bar{s}_2]$ be a (ρ, ε)-block for $\mathbf{x}_2, \mathbf{y}_2$. Assume that this procedure can be repeated $\bar{j}$ times, and let $J_1, \dots, J_{\bar{j}} = [s_{\bar{j}}, \bar{s}_{\bar{j}} = L]$ be the resulting blocks.*

Let $S_1, \dots, S_h$ be the connected components of $\mathrm{SubLev}(\mathbf{x}, \mathbf{y})$ as above, with $h \leq D$ minimal so that $\bigcup_{i=1}^h S_i$ covers $\bigcup_{j=1}^{\bar{j}} J_j$. If the intervals $S_1, \dots, S_h$ are b-close (in the sense of Proposition 3.1.5) for some $b < \eta/(2D \times \max_\iota d_\iota)$, then,

$$|u| \leq \varepsilon, \quad |a| \leq \kappa L^{-\eta/2} \quad |\bar{u}| \leq \kappa L^{-1-\eta/2}, \quad \text{and} \quad |c_{j,\iota}| \leq \kappa\varepsilon L^{-j+\eta/2}.$$

Proof. We apply Proposition 3.1.5 to the intervals $S_1, \dots, S_{h-1}, S_h \cap J_{\bar{j}}$: by assumption they are b-close and by Lemma 3.2.1 the assumption 1 of Proposition 3.1.5 is satisfied as well. □

If the assumptions of Corollary 3.2.2 are satisfied, it is possible to see that the $\mathbf{a}^t$-orbits of $\mathbf{x}$ and $\mathbf{y}$ stay close for all times up to $(1+\eta)\log L$, as the next lemma shows.

Lemma 3.2.3. *Under the assumptions of Corollary 3.2.2, for every $0 < r < 1$ we have*

$$d(\mathbf{x}\mathbf{a}^{(1+r)\log L}, \mathbf{y}\mathbf{a}^{(1+r)\log L}) \leq 3\kappa \left(L^{-\frac{1}{2}+\frac{r+\eta}{2}} + L^{-\frac{\eta}{2}+r} \right).$$

Proof. Call $T = (1+r)\log l$. As in §3.1.4, let us write $\mathbf{x}^{-1}\mathbf{y} = \exp(\xi)\mathbf{g}_{\mathfrak{s}}$, where $\xi \in \mathfrak{m}$ and $\mathbf{g}_{\mathfrak{s}} \in \mathfrak{s}$ satisfy (11). Then,

$$d(\mathbf{x}\mathbf{a}^T, \mathbf{y}\mathbf{a}^T) = d(\mathbf{e}, \mathbf{a}^{-T}\mathbf{x}^{-1}\mathbf{y}\mathbf{a}^T) = \max\{\|\mathbf{a}^{-T}\xi\mathbf{a}^T\|, \ d_{\mathfrak{s}}(\mathbf{e}, \mathbf{a}^{-T}\mathbf{g}_{\mathfrak{s}}\mathbf{a}^T)\}.$$

We now focus on the first term corresponding to $\xi = \sum c_{j,\iota}E_{j,\iota}$. By (14), we have

$$\begin{aligned}\|\mathbf{a}^{-T}\xi\mathbf{a}^T\| &= \max_{j,\iota} |c_{j,\iota}| \cdot \|\mathbf{a}^{-T}E_{j,\iota}\mathbf{a}^T\| = \max_{j,\iota} |c_{j,\iota}| \cdot \|\mathrm{Ad}(\mathbf{a}^{-T})E_{j,\iota}\| \\ &= \max_{j,\iota} |c_{j,\iota}| e^{-(\frac{d_\iota}{2}-j)T}.\end{aligned}$$

By Corollary 3.2.2 and the trivial bounds $j \leq d_\iota$ and $d_\iota \geq 1$, we obtain

$$\|\mathbf{a}^{-T}\xi\mathbf{a}^T\| \leq \kappa\varepsilon L^{-j+\frac{\eta}{2}-(1+r)\left(\frac{d_\iota}{2}-j\right)} \leq \kappa\varepsilon L^{-\frac{1}{2}+\frac{r+\eta}{2}}. \tag{24}$$

We now look at the term corresponding to $\mathbf{g}_{\mathfrak{s}}$. Writing it as in (18), by definition (17), we have

$$d_{\mathfrak{s}}(\mathbf{e}, \mathbf{a}^{-T}\mathbf{g}_{\mathfrak{s}}\mathbf{a}^T) = |e^T\,\bar{u}| + |a| + |e^{-T}\,u|.$$

By Corollary 3.2.2, we conclude

$$d_{\mathfrak{s}}(\mathbf{e}, \mathbf{a}^{-T}\mathbf{g}_{\mathfrak{s}}\mathbf{a}^T) \leq \kappa L^{(1+r)-1-\frac{\eta}{2}} + \kappa L^{-\frac{\eta}{2}} + \varepsilon L^{-(1+r)} \leq 3\kappa L^{-\frac{\eta}{2}+r}.$$

Together with (24), this completes the proof. □

4. Ratner's Basic Lemma

4.1. **Construction of the Good Set $\mathcal{G}$.** Here we construct a set of large measure which satisfies simultaneously the various conditions proved or assumed in the previous section. The first condition obviously guarantees that two orbits, that are geometrically close, are mapped to two orbit segments that are geometrically close. By choosing in the source space initial points sufficiently close, we obtain, both in the source space and the target space, orbit segments of length larger than m_0. Then, the second condition guarantees that the time change on these orbit segments is sub-linear, which yields important consequences for the relative position of the initial points of these orbit segments, thanks to the Lemma 3.2.1. The third condition assures that in the target space the orbit segments with close initial points and close endpoints have continuous lifts to G_2.

These three conditions are sufficient to deal with the conjugacy of the centraliser of $\mathbf{u}_1^t$. The fourth condition is used to control the conjugacy of the normaliser. In that case we need to push orbits along the flow of a normalising element. The fourth condition guarantees that along these pushes we can keep track of the initial positions of orbit segments.

4.1.1. Recall that $\eta \in (0, 1/4)$ is given by Corollary 2.1.5. Let us fix $b > 0$ satisfying

$$b < \frac{\eta}{2D \cdot \max_\iota d_\iota}, \quad \text{and define} \quad c(b) := \frac{1+\eta}{1+\eta-b}. \tag{25}$$

Let $A_2 \in \mathfrak{g}_2$ be the semi-simple element of a $_2(\mathbb{R})$-triple as defined in §2.1.1 for the unipotent element $U_2 \in \mathfrak{g}_2$. As usual, let $\{\mathbf{a}_2^t\}_{t\in\mathbb{R}}$ the associated one-parameter subgroup.

Proposition 4.1.1 (Construction of $\mathcal{G}$). *For every $\omega > 0$, there exist $m_0 > 1$, $\rho \in (0,1)$ and a compact set $\mathcal{G} \subset M_1$ with measure $\mu_1(\mathcal{G}) > 1-\omega$, such that the following properties hold.*

1. Uniform continuity: *the map ψ is uniformly continuous on $\mathcal{G}$.*
2. Sub-polynomial deviations: *for all $x \in \mathcal{G}$ and all $t \geq m_0$ we have*
$$|\xi_1(x,t) - t| \leq C_\alpha^{-4} t^{1-\eta} \quad \text{and} \quad |\xi_2(\psi(x),t) - t| \leq C_\alpha^{-4} t^{1-\eta},$$
where η is given by Corollary 2.1.5.
3. $\mathrm{IC}(\rho, 2C_\alpha^4 m_0)$ on the image $\psi(\mathcal{G})$: *for $\mathbf{x} \in \pi_2^{-1}(\psi(\mathcal{G}))$ and $d(\mathbf{x},\mathbf{y}) < \rho$, then $d(\mathbf{x}\mathbf{u}_2^s, \gamma\mathbf{y}\mathbf{u}_2^t) > \rho$ for all $\gamma \in \Gamma_2 \setminus \{\mathbf{e}\}$ and all $s,t \in [-2C_\alpha^4 m_0, 2C_\alpha^4 m_0]$.*
4. The condition FBR is satisfied on the image $\psi(\mathcal{G})$: *for all $x \in \mathcal{G}$, the point $\psi(x)$ satisfies the condition $\mathrm{FBR}(T_0, c(b), r_0)$ with*
$$T_0 \leq \frac{1}{2}\log\frac{m_0}{C_\alpha}, \quad \text{and} \quad r_0 \geq 20\kappa C_\alpha m_0^{-\frac{\eta-2b}{12}}.$$

Proof. The set $\mathcal{G}$ will be given as the intersection of four sets of measure larger than $1-\omega/4$ on which the conditions 2)–5) hold. Without mentioning it, we use several times the estimate (7).

By Lusin's Theorem we can choose a compact set $K_1 \subset M_1$ of measure larger than $1-\omega/4$ on which ψ is uniformly continuous.

By Lemma 2.3.4, there exist $T_0 \geq 1$ and $r_0 > 0$ such that the set $K_4 \subset M_2$ of points which satisfy the condition $\mathrm{FBR}(T_0, c(b), r_0)$ has measure greater than $1 - \frac{\omega}{4C_\alpha}$. Thus, $\mu_1(\psi^{-1}K_4) > 1-\omega/4$. In order to satisfy the statement (4) of the Proposition, we choose $m_0 > 1$ so large that $T_0 \leq \frac{1}{2}\log\frac{m_0}{C_\alpha}$ and $r_0 \geq 20\kappa C_\alpha m_0^{-\frac{\eta-2b}{12}}$.

By Corollary 2.1.5 (applied twice with ω replaced by $\omega/(4C_\alpha)$), there exist $m' > 1$ and a set $K_2 \subset M_1$ of measure greater than $1-\omega/4$ such that for all $x_1 \in K_2, x_2 \in \psi(K_2)$ and all $t \geq m'$ we have

$$|\xi_1(x_1,t)-t| \leq C_\alpha^{-4}t^{1-\eta} \quad \text{and} \quad |\xi_2(x_2,t)-t| \leq C_\alpha^{-4}t^{1-\eta}.$$

Up to increasing m_0, we can assume $m_0 > m'$.

By Proposition 2.3.2, with $\zeta = \omega/(4C_\alpha^2)$ and $m = 2C^4 m_0$, there exist $\rho > 0$ and a compact set $K_3 \subset M_2$ with $\mu_2(K_3) \geq 1-\omega/(4C_\alpha^2)$ on which the Injectivity Condition $\mathrm{IC}(\rho, 2C^4 m_0)$ holds for any point in K_3.

Then $\mu_1(K_1 \cap K_2 \cap \psi^{-1}K_3 \cap \psi^{-1}K_4) > 1-\omega$. We can choose a compact set

$$\mathcal{G} \subset K_1 \cap K_2 \cap \psi^{-1}K_3 \cap \psi^{-1}K_4$$

of measure $\mu_1(\mathcal{G}) \geq 1-\omega$. The construction is complete, and the properties of $\mathcal{G}$ follow from its definition. □

In the following, we will denote $m = 2C_\alpha^4 m_0 > m_0$.

4.2. **Statement and proof of Ratner's basic lemma.** In this section we will prove the main technical result of this paper, which is an adaptation of Ratner's Basic Lemma in [27] for $G = \mathrm{SL}_2(\mathbb{R})$. A similar result was also proved by Tang in [36] for $G = \mathrm{SO}(n,1)$.

For a given $\omega > 0$ (which will be fixed in the next section), let m_0, ρ, T_0, r_0, and $\mathcal{G}$ be given by Proposition 4.1.1.

4.2.1. *Standing assumptions.* We fix $\varepsilon < 1/10$ (the specific choice of ε plays no role). Up to possibly taking a smaller $\rho \leq \varepsilon/2$, we can assume that if $d(\mathbf{g},\mathbf{e}) \leq \rho$ then $l_\varepsilon(\mathbf{g}) > m$, where, we recall, $l_\varepsilon(\mathbf{g})$ was defined in §3.2. For any such $\mathbf{g}$, we write

$$\mathbf{g} = \exp\Big(\sum_{\iota,j} c_{j,\iota}E_{j,\iota}\Big)\exp(aA)\exp(\bar{u}\bar{U})\,\mathbf{u}^u$$

as in the previous section.

Lemma 4.2.1 (Ratner's Basic Lemma)**.** *There exists $\theta \in (0,1)$ such that the following holds. Let $x, y \in M_2$. Assume that there exist $\lambda > m$, a compact set $A = A(x,y) \subset [0,\lambda]$, and a strictly increasing Lipschitz function $\tau = \tau_{x,y}\colon [0,\lambda] \to \mathbb{R}_+$, with $\tau(0) = 0$, such that*

1. *$0, \lambda_s \in A$;*
2. *If $r \in A$ then the point $x\mathbf{u}_2^r$ satisfies the Injectivity Condition* $\mathrm{IC}(\rho, m)$ *and the Frequently Bounded Radius Condition* $\mathrm{FBR}(T_0, c(b), r_0)$.
3. *If $r \in A$ then*
$$d\left(x\mathbf{u}_2^r, y\mathbf{u}_2^{\tau(r)}\right) < \rho,$$
4. *if $r \in A$ and $r'-r > m$ or $\tau(r')-\tau(r) > m$, then*
$$\left|\big(\tau(r')-\tau(r)\big)-(r'-r)\right| \leq 4(r'-r)^{1-\eta},$$
for some $\eta > 0$.

If the relative density of A in $[0,\lambda]$ is greater than $1-\theta/8$, then there exists $\bar{s} \in A$ such that $x\mathbf{u}_2^{\bar{s}}\mathbf{g} = y\mathbf{u}_2^{\tau(\bar{s})}$, where $\mathbf{g}$ satisfies

$$|u| \leq \varepsilon, \quad |a| \leq \kappa\lambda^{-\eta/2}, \quad |\bar{u}| \leq \kappa\lambda^{-1-\eta/2}, \quad |c_{\iota,j}| \leq \kappa\varepsilon\lambda^{-j+\eta/2}.$$

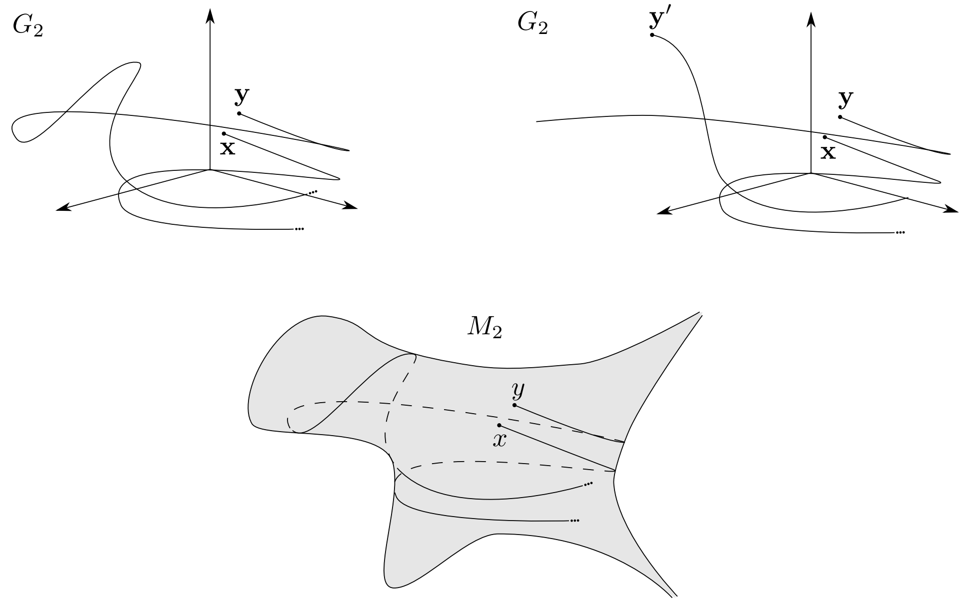

FIGURE 1. A schematic drawing of the relation between blocks. Below are the orbits, under $\mathbf{u}_2^t$, of x and y on M_2, which are close for two blocks of times. The lifts of these orbits to G_2 could either be the orbit of $\mathbf{x}$ and $\mathbf{y}$ as in the top left figure, or be the orbit of $\mathbf{x}$ and several lifts of y, as in the top right figure.

Moreover, if the assumptions above hold for all $\lambda \geq \lambda_0 > m$, then $y = x\mathbf{g}$, where $\mathbf{g}$ commutes *with $\mathbf{u}_2^t$ for all $t \in \mathbb{R}$ and satisfies $d(\mathbf{e}, \mathbf{u}_2^{-\bar{s}}\mathbf{g}\mathbf{u}_2^{\tau(\bar{s})}) \leq \varepsilon$ for some $0 \leq \bar{s} \leq \lambda_0$.*

4.3. **Comments on the proof of the Basic Lemma 4.2.1.** Before diving into the details of the proof of the Basic Lemma 4.2.1, let us sketch the general strategy. Let $x, y \in M_2$ two points at distance lesser than ρ, and fix two lifts $\mathbf{x}, \mathbf{y} \in G_2$ at the same distance. From the assumption that the orbits of x and y (parametrized by r and $\tau = \tau(r)$) stay close in M_2 for a large portion of times, we construct a collection of what we call *blocks*: lifts to G_2 of segments of $\mathbf{u}_2^t$-orbits of x and y which are close together. The lifts are chosen so that the one for x lies on the $\mathbf{u}_2^t$-orbits of $\mathbf{x}$, and the one for y so that the two lifts are close in G_2; these conditions uniquely determine such lifts. Note that two cases are possible: either the lift of the orbit segment of y also lies in the orbit of $\mathbf{y}$, or not, see Figure 1. This type of dichotomy will result in a *relation* between blocks, which we define below.

The goal is to apply Solovay's Lemma 4.3.1 to the collection of time intervals determined by the blocks (namely, the intervals of time for which $\mathbf{x}\mathbf{u}_2^r$ belong to some block in the collection), in order to deduce that the orbits of $\mathbf{x}$ and $\mathbf{y}$ stay close for all future times. The main difficulty is to check the first condition in the statement of Lemma 4.3.1. We will argue as follows: if two blocks *are not in relation*, then this condition is automatically satisfied, see Lemma 4.3.3 below. On the other hand, if two blocks are *related*, this may not necessarily be true. In this case, we "force" the condition by "gluing" together related blocks which are too close to each other. We will then show that the *new* collection of *superblocks* we obtain in this way satisfies the assumptions of Lemma 4.3.1, and hence the proof of the Basic Lemma will follow.

4.3.1. *Solovay's Lemma.* In the proof of Ratner's Basic Lemma 4.2.1, we will use the following result about a collection of intervals on the real line. This result first appeared in [24], the proof being attributed to Solovay.

For an interval $I \subset \mathbb{R}$, we denote its length by $|I|$. Let $\mathcal{J} = \{J_1, \dots, J_n\}$ be a collection of disjoint subintervals of I. The *density* of the collection $\mathcal{J}$ in the interval I is the ratio of the Lebesgue measures of $\bigcup_{i=1}^n J_i$ and I. The *distance* of two intervals I and J is the number $d(I, J) = \inf_{x \in I, y \in J} |x - y|$.

Lemma 4.3.1 (Solovay)**.** *Given a $b > 0$ there exists a real number $\theta = \theta(b) \in (0, 1)$ such that, for any collection $\mathcal{J} = \{J_1, \dots, J_n\}$ of disjoint subintervals of an interval $I \subset \mathbb{R}$, if the following conditions hold true*

1. *$d(J', J'') \geq \min\{|J'|, |J''|\}^{1+b}$, for all distinct $J', J'' \in \mathcal{J}$;*
2. *the density of the collection $\mathcal{J}$ in I is larger than $1 - \theta$;*
3. *$|C| \geq 1$ for each connected component C of $I \setminus \bigcup_{i=1}^n J_i$;*

then there exists an interval $J \in \mathcal{J}$ with $|J| \geq 3/4|I|$.

4.3.2. The proof of Lemma 4.2.1 will be divided in several steps and concluded on page 28. For the rest of the section, again to lighten the notation, we will again drop the indices 2 that refer to objects in the target space.

4.3.3. Fix $\omega > 0$ and let $\theta \in (0, 1)$ be given by Solovay's Lemma (Lemma 4.3.1) for the choice of b in (25). Let now m_0, ρ, T_0, r_0, and $\mathcal{G}$ be given by Proposition 4.1.1, and recall that, if $d(\mathbf{g}, \mathbf{e}) \leq \rho$ then $l_\varepsilon(\mathbf{g}) > m$. Observe also that if $d(\mathbf{x}\mathbf{u}^s, \mathbf{y}\mathbf{u}^{\tau(s)}) \leq \rho$ then there is $t \in [\tau(s) - \rho, \tau(s) + \rho]$ with $d(\mathbf{x}\mathbf{u}^s, \mathbf{y}\mathbf{u}^t) \leq \rho$ and $\mathbf{y}\mathbf{u}^t \in W(\mathbf{x}\mathbf{u}^s, 2\rho) \subset W(\mathbf{x}\mathbf{u}^s, \varepsilon)$.

Let now $x, y \in M$ be given as in the statement of the Basic Lemma. We are going to follow the U-orbits of x and y on M, and construct (ρ, ε)-*blocks* for appropriate *lifts* of x and y to G. Again, since ρ and ε are fixed, here and henceforth we will simply say blocks.

4.3.4. Let x and y be as in the statement of the Basic Lemma. Since $0 \in A_s$, and $\tau(0) = 0$ we have that $d(x, y) < \rho$. Let $\mathbf{x}_1$ and $\mathbf{y}_1$ be lifts of x_1 and y_1 with $d(\mathbf{x}_1, \mathbf{y}_1) \leq \rho$. Since the point x satisfies the Injectivity Condition IC(ρ, m), given $\mathbf{x}_1$, the lift $\mathbf{y}_1$ is unique.

Set $s_1 = 0$. We define the interval $J_1 = [0, \bar{s}_1] = [s_1, \bar{s}_1]$ where

$$\bar{s}_1 = \sup\left\{r \in A_s \cap [0, s] \mid \ r \leq l_\varepsilon(\mathbf{x}_1^{-1}\mathbf{y}_1),\ d(\mathbf{x}_1\, \mathbf{u}^r, \mathbf{y}_1\, \mathbf{u}^{\tau(r)}) \leq \rho\right\}.$$

Notice that J_1 is a (ρ, ε)-*block* for $\mathbf{x}_1$ and $\mathbf{y}_1$, in particular the points $\mathbf{x}_1\mathbf{u}^{\bar{s}_1}$ and $\mathbf{y}_1\mathbf{u}^{\tau(\bar{s}_1)}$ are at distance not larger than ρ. We stress that, however, it is not true that $d(\mathbf{x}_1\, \mathbf{u}^r, \mathbf{y}_1\, \mathbf{u}^{\tau(r)}) \leq \rho$ for all times $r \in J_1$.

Assuming that we have defined, for $j > 1$, $J_1 = [s_1, \bar{s}_1], \dots, J_{j-1} = [s_{j-1}, \bar{s}_{j-1}]$ and $\mathbf{x}_1, \mathbf{y}_1, \dots, \mathbf{x}_{j-1}, \mathbf{y}_{j-1}$, we let

$$s_j = \inf\{r > \bar{s}_{j-1} \mid r \in A_s\}.$$

As $s_j \in A_s$, setting

$$x_j = x_1\mathbf{u}^{s_j}, \quad y_j = y_1\mathbf{u}^{\tau(s_j)},$$

we have $d(x_j, y_j) \le \rho$. Thus if $\mathbf{x}_j := \mathbf{x}_1 \mathbf{u}^{s_j}$ we may define $\mathbf{y}_j$, in a unique manner, asking that

$$\pi_2(\mathbf{y}_j) = y_1 \mathbf{u}^{\tau(s_j)}, \quad d(\mathbf{x}_j, \mathbf{y}_j) \le \rho. \tag{26}$$

Finally, we define

$$\bar{s}_j := \sup \big\{ r \in A_s \cap [s_j, s] \mid r - s_j \le l_\varepsilon(\mathbf{x}_j^{-1}\mathbf{y}_j),\, d(\mathbf{x}_j\, \mathbf{u}^{r-s_j}, \mathbf{y}_j\, \mathbf{u}^{\tau(r)-\tau(s_j)}) \le \rho \big\}. \tag{27}$$

This means that we have lifted the orbit $x_1 \mathbf{u}^r$ to an orbit $\mathbf{x}_1 \mathbf{u}^r$, and we have chosen, at the times s_j, the unique lift $\mathbf{y}_j$ of $y_1 \mathbf{u}^{\tau(s_j)}$ satisfying $d(\mathbf{x}_j, \mathbf{y}_j) \le \rho$. The uniqueness of this lift is justified by the fact that $s_j \in A_s$ and therefore the point $\mathbf{x}_j = \mathbf{x}_1 \mathbf{u}^{s_j}$ satisfies the injectivity condition IC(ρ, m).

With these definitions we have

$$\forall i \le j: \quad \mathbf{x}_j = \mathbf{x}_i \mathbf{u}^{s_j - s_i}, \quad x_j = \pi_2(\mathbf{x}_i). \tag{28}$$

Moreover, setting

$$t_i = \tau(s_i), \quad \bar{t}_i = \tau(\bar{s}_i),$$

we also have

$$x_j = x_i\, \mathbf{u}^{s_j - s_i} \quad y_j = y_i\, \mathbf{u}^{t_j - t_i}. \tag{29}$$

We stress that it *does not* necessarily hold that $\mathbf{y}_j = \mathbf{y}_i\, \mathbf{u}^{t_j - t_i}$. We have therefore a collection $\mathcal{J} = (J_1, \dots, J_j, \dots)$ of closed intervals $J_j = [s_j, \bar{s}_j]$ with disjoint interiors and whose union covers A_s. These intervals J_i are blocks for the lifts $\mathbf{x}_i$, $\mathbf{y}_i$ constructed above of the points $x\mathbf{u}^{s_i}, y\mathbf{u}^{\tau(s_i)} \in M$.

4.3.5. Our previous remark leads to the following definition. Given two blocks J_j and J_j we say that

$$J_j \sim J_k \iff \mathbf{y}_k = \mathbf{y}_j\, \mathbf{u}^{t_k - t_j}. \tag{30}$$

4.3.6. Suppose that for some $j < k$ we have $J_j \sim J_k$. Then

$$s_k - s_j \ge l_\varepsilon(\mathbf{y}_j{}^{-1}\mathbf{x}_j). \tag{31}$$

If not, considering that $d(\mathbf{x}_k, \mathbf{y}_k) \le \rho$ and that $\mathbf{y}_k = \mathbf{y}_j\, \mathbf{u}^{t_k - t_j}$ we would have $\bar{s}_j \ge s_k$, by the definition of $\bar{s}_j$.

The above inequality implies

$$s_k - s_j > m, \tag{32}$$

by the choice of ε.

4.3.7. If $J_j \not\sim J_k$, with $j < k$, we claim that we have

$$\max(s_k - \bar{s}_j, t_k - \bar{t}_j) > m. \tag{33}$$

Assume by contradiction that $\max(s_k - \bar{s}_j, t_k - \bar{t}_j) \le m$, and let $\gamma \in \Gamma_2 \setminus \{\mathbf{e}\}$ such that $\mathbf{y}_k = \gamma\, \mathbf{y}_j\, \mathbf{u}^{t_k - t_j}$. Let us notice that, by definition, $\bar{s}_j - s_j \in A_s$ and thus the point $\mathbf{x}_j \mathbf{u}^{\bar{s}_j - s_j}$ satisfies the injectivity condition IC(ρ, m). Applying this property to this point and noticing that we have $d(\mathbf{x}_j\, \mathbf{u}^{\bar{s}_j - s_j}, \mathbf{y}_j\, \mathbf{u}^{\bar{t}_j - t_j}) \le \rho$ we get

$$\begin{aligned} \rho \ge d(\mathbf{x}_k, \mathbf{y}_k) &= d(\mathbf{x}_j\, \mathbf{u}^{s_k - s_j}, \gamma \mathbf{y}_j \mathbf{u}^{t_k - t_j}) \\ &= d(\mathbf{x}_j\, \mathbf{u}^{\bar{s}_j - s_j}\, \mathbf{u}^{s_k - \bar{s}_j}, \gamma\, \mathbf{y}_j\, \mathbf{u}^{\bar{t}_j - t_j}\, \mathbf{u}^{t_k - \bar{t}_j}) > \rho, \end{aligned}$$

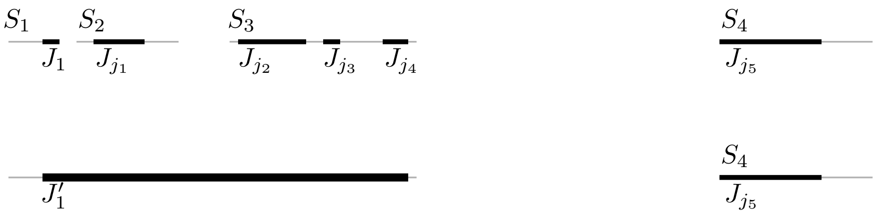

FIGURE 2. The blocks $J_1, J_{j_1}, \dots, J_{j_5}$, in thick black, are equivalent. The levels S_1, S_2, and S_3, in grey, are b-close. Hence $k_{\bar{i}} = 3$, and $j_{\bar{i}} = j_4$. We define the superblock $J'_1 = \mathrm{Conv}(J_1, J_{j_4})$.

which is our desired contradiction. As a consequence of the formula (33) and of the fact $\bar{s}_j \in A_s$ we have

$$|(t_k - \bar{t}_j) - (s_k - \bar{s}_j)| \leq 4(s_k - \bar{s}_j)^{1-\eta} \tag{34}$$

Remark. By the (32) and (33) the number of blocks in the interval $[0, s]$ is finite.

4.3.8. *Construction of the superblocks.* The above constructions yields a collection of disjoint blocks $\mathcal{J} = (J_1, \dots, J_n)$. We can construct a new collection of intervals $\mathcal{J}' = (J'_1, \dots, J'_k)$ that we will call superblocks and later show that this collection satisfies the assumptions of Solovay's Lemma 4.3.1.

We will always assume that a collection of intervals is *ordered* in the obvious way. Moreover, given two intervals J' and J'' we will denote with $\mathrm{Conv}(J', J'')$ their convex hull.

To construct $\mathcal{J}'$ from $\mathcal{J}$ we will do the following, see Figure 2:

- If there is no interval $J_j > J_1$ with $J_1 \sim J_j$ then we set $J'_1 = J_1$ and we next consider J_2.
- If $J_{j_1}, J_{j_2}, \dots$ with $1 < j_1 < j_2 < \dots$ are intervals equivalent to J_1 we define J'_1 as follows:
  - Let $S_1, \dots, S_D$ (ordered in the obvious way) be the connected components of $\mathrm{SubLev}(\mathbf{x}_1, \mathbf{y}_1)$ defined as in §3.2. By Lemma 3.2.1, any J_{j_i} is contained in one of these S_k's.
  - Let $k_{\bar{i}}$ be maximal with the property that $S_{k_{\bar{i}}}$ contains one of the J_{j_i} and the intervals $S_1 = S_{k_1}, \dots, S_{k_{\bar{i}}}$ are b-close. Let $j_{\bar{i}}$ be maximal with the property that $J_{j_{\bar{i}}} \subset S_{k_{\bar{i}}}$ (note that there could be several blocks inside a single interval S_k)
  - We define
  $$J'_1 = \mathrm{Conv}(J_1, J_{j_{\bar{i}}}) = \mathrm{Conv}(J_1, J_2, \dots, J_{j_{\bar{i}}}).$$
  - We restart the procedure from $J_{j_{\bar{i}}+1}$.

Given any ordered list of blocks $\mathcal{J} = (J_1, \dots, J_n)$ and an equivalence relation among blocks the above algorithm produces a ordered list of closed intervals $\mathcal{J}' = (J'_1, \dots, J'_k)$ such that each $J' \in \mathcal{J}'$ is the convex hull of some intervals in $\mathcal{J}$.

We say that *two intervals* J', J'' *are* b*-separated* if $d(J'_j, J'_k) \geq \min\{|J'|, |J''|\}^{1+b}$. Applying the algorithm above to the list of blocks $\mathcal{J} = (J_1, \dots, J_n)$ defined in § 4.3.4, we obtain

Lemma 4.3.2. *The collection of super-blocks* $\mathcal{J}' = (J'_1, \dots, J'_k)$ *satisfies the following:*

(1) *it covers the set* A_s*:* $\bigcup_{J' \in \mathcal{J}'} |J'| \supset \bigcup_{J \in \mathcal{J}} |J| \supset A_s$.
(2) *Any super-block* $J' = \mathrm{Conv}(J_i, J_j)$ *can be written as the convex hull of* $h \leq D$ *intervals contained in* $\mathrm{SubLev}(\mathbf{x}_i, \mathbf{y}_i)$ *which are* b*-close.*
(3) *Any two equivalent super-blocks are* b*-separated.*
(4) $d(J'_j, J'_k) \geq m/5 > 1$ *for any* $J'_j \neq J'_k \in \mathcal{J}'$.

Proof. For the first part, it is enough to notice that every $J' \in \mathcal{J}'$ is defined as a convex hull of elements in $\mathcal{J}$.

For the first superblock, we have

$$J'_1 = \mathrm{Conv}(S_1 \cap J_1, S_2, \dots, S_{k_{\bar{i}}-1}, S_{k_{\bar{i}}} \cap J_{j_{\bar{i}}}),$$

with the convention that the term on the right is simply $\mathrm{Conv}(S_1 \cap J_1, S_1 \cap J_{j_{\bar{i}}})$ if $k_{\bar{i}} = 1$. In any case, we can write J'_1 as a convex hull of at most D intervals contained in $\mathrm{SubLev}(\mathbf{x}_1, \mathbf{y}_1)$, which are b-close by the definition of the algorithm. This proves 2 for the first superblock, the proof for the others is the same.

Part 3 follows from the definition of the algorithm, in particular from the definition of $k_{\bar{i}}$.

For the last statement, part 4, we reason as follows. If $J'_j \sim J'_k$, then the estimate follows from (32) and the proven estimate $d(J'_j, J'_k) \geq |J'_j|^{1+b}$. If $J'_j \not\sim J'_k$, then the estimate follows from (33) and (34).

□

Next we prove that also non-equivalent super-blocks are b-separated.

Lemma 4.3.3. *Let* $J_p \not\sim J_q$*, with* $p < q$ *be two non-equivalent super-blocks. Assume that*

(i) $x_p \, \mathbf{u}^{\bar{s}_p - s_p}$ *satisfies* $\mathrm{IC}(\rho, m)$*,*
(ii) x_p *and* x_q *satisfy* $\mathrm{FBR}(T_0, c(b), r_0)$*.*

Then, J_p *and* J_q *are* b-separated*, namely*

$$d(J_p, J_q) \geq \min\{|J_p|, |J_q|\}^{1+b}.$$

Proof. Let us start by recalling that, by definition,

$$c(b) = \frac{1+\eta}{1+\eta-b}, \quad T_0 \leq \frac{1}{2} \log \frac{m}{C_\alpha}, \quad r_0 > 20\kappa\sqrt{C_\alpha} m^{-\frac{\eta-2b}{12}}$$

Let us call $L = \min\{|J_p|, |J_q|\}$; we will show that if $d(J_p, J_q) \leq L^{1+b}$, then $J_p \sim J_q$. Assume that this is not the case. Then, there exists $\gamma \in \Gamma_2 \setminus \{\mathbf{e}\}$ such that $\mathbf{y}_q = \gamma \, \mathbf{y}_p \, \mathbf{u}^{t_q - t_p}$. Note that by the definition of $\bar{s}_p$ we have $d(\mathbf{x}_p \, \mathbf{u}^{\bar{s}_p - s_p}, \mathbf{y}_p \, \mathbf{u}^{\bar{t}_p - t_p}) \leq \rho$. The fact that the point $x_p \, \mathbf{u}^{\bar{s}_p - s_p}$ satisfies the $\mathrm{IC}(\rho, m)$ injectivity condition implies that either $\bar{s}_p - s_p > m$ or $\bar{t}_p - t_p > m$. In either case, we conclude that $\bar{s}_p - s_p = d(J_p, J_q) > m/C_\alpha$, from which we get $L > (m/C_\alpha)^{1/(1+b)} > \sqrt{m/C_\alpha}$. Let

$$T'_0 = \left(1 + \frac{1}{3}\eta + \frac{1}{3}b\right) \log L > \frac{1}{2} \log \frac{m}{C_\alpha} \geq T_0, \qquad \text{and}$$

$$c' = \frac{1 + \frac{1}{6}\eta + \frac{2}{3}b}{1 + \frac{1}{3}\eta + \frac{1}{3}b} < \frac{1+\eta}{1+\eta-b} = c(b) < 1.$$

Since, by assumption, x_q satisfies the Frequently Bounded Radius Condition $\mathrm{FBR}(T'_0, c', r_0)$, there exists $T \in \left[\left(1 + \frac{1}{6}\eta + \frac{2}{3}b\right) \log L, \left(1 + \frac{1}{3}\eta + \frac{1}{3}b\right) \log L\right]$ such that the injectivity radius at $\mathbf{x}_q \mathbf{a}^T$ is bounded below by r_0.

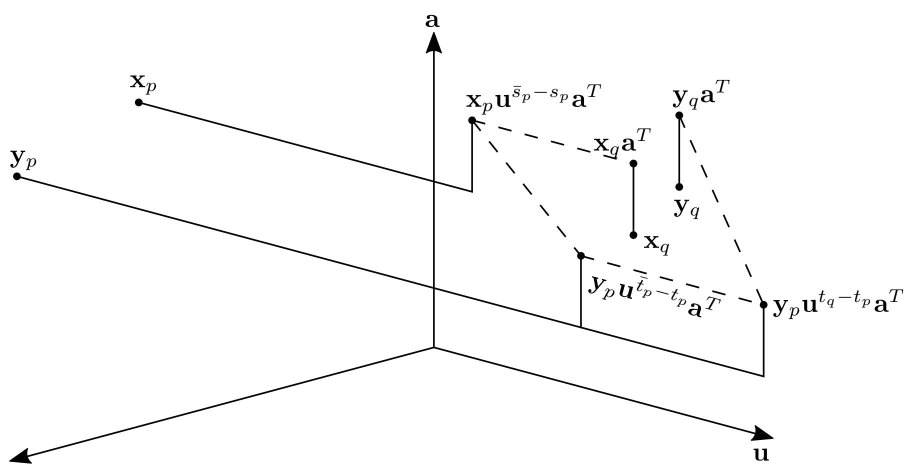

FIGURE 3. The points involved in the proof of equation (35). The dashed lines are the distances we are estimating. Distances have exagerated in order to make the picture clear.

We will show that

$$d(\gamma^{-1}\mathbf{x}_q\mathbf{a}^T, \mathbf{x}_q\mathbf{a}^T) \leq 20\kappa\sqrt{C_\alpha}m^{-\frac{\eta-2b}{12}} < r_0,$$

which is our contradiction. This will follow immediately once we show that

$$\begin{aligned} d(\mathbf{x}_q\mathbf{a}^T, \mathbf{x}_p\,\mathbf{u}^{\bar{s}_p-s_p}\mathbf{a}^T) + d(\mathbf{x}_p\,\mathbf{u}^{\bar{s}_p-s_p}\mathbf{a}^T, \mathbf{y}_p\,\mathbf{u}^{\bar{t}_p-t_p}\mathbf{a}^T) \\ + d(\mathbf{y}_p\,\mathbf{u}^{\bar{t}_p-t_p}\mathbf{a}^T, \mathbf{y}_p\,\mathbf{u}^{t_q-t_p}\mathbf{a}^T) + d(\mathbf{y}_q\mathbf{a}^T, \mathbf{x}_q\mathbf{a}^T) \leq 20\kappa\sqrt{C_\alpha}m^{-\frac{\eta-2b}{12}}, \end{aligned} \tag{35}$$

since $d(\mathbf{y}_p\,\mathbf{u}^{t_q-t_p}\mathbf{a}^T, \gamma^{-1}\mathbf{x}_q\mathbf{a}^T) = d(\mathbf{y}_q\mathbf{a}^T, \mathbf{x}_q\mathbf{a}^T)$. Let us consider each of the four summands above separately.

For the first summand, we have

$$\begin{aligned} d(\mathbf{x}_q\mathbf{a}^T, \mathbf{x}_p\,\mathbf{u}^{\bar{s}_p-s_p}\mathbf{a}^T) &= d(\mathbf{e}, \mathbf{a}^{-T}\mathbf{u}^{s_q-\bar{s}_p}\mathbf{a}^T) = d(\mathbf{e}, \mathbf{u}^{e^{-T}d(J_p,J_q)}) \\ &= e^{-T}d(J_p,J_q) \leq L^{-\left(1+\frac{1}{6}\eta+\frac{2}{3}b\right)+(1+b)} \leq L^{-\frac{\eta-2b}{6}}, \end{aligned}$$

which is smaller than $\sqrt{C_\alpha}m^{-\frac{\eta-2b}{12}}$. In a similar way, we can bound the third summand by

$$d(\mathbf{y}_p\,\mathbf{u}^{\bar{t}_p-t_p}\mathbf{a}^T, \mathbf{y}_p\,\mathbf{u}^{t_q-t_p}\mathbf{a}^T) = d(\mathbf{u}^{e^{-2T}(t_q-\bar{t}_p)}, \mathbf{e}),$$

which, in turn, is less than $5\sqrt{C_\alpha}m^{-\frac{\eta-2b}{12}}$, since either $s_q - \bar{s}_p > m$ or $t_q - \bar{t}_p > m$, and hence $t_q - \bar{t}_p \leq s_q - \bar{s}_p + 4(s_q - \bar{s}_p)^{1-\eta} \leq 5(s_q - \bar{s}_p)$.

It remains to bound the second and fourth term. Let us start from the latter. By Lemma 4.3.2, we are precisely in the setting of Corollary 3.2.2. Hence, we can apply Lemma 3.2.3: since $T \leq \left(1 + \frac{1}{3}\eta + \frac{1}{3}b\right)\log L$, we get

$$d(\mathbf{y}_q\mathbf{a}^T, \mathbf{x}_q\mathbf{a}^T) \leq 3\kappa\left(L^{-\frac{1}{2}+\frac{5}{6}\eta} + L^{-\frac{\eta}{2}+\frac{1}{3}\eta+\frac{1}{3}b}\right) = 6\kappa L^{-\frac{\eta-2b}{6}} \leq 6\kappa\sqrt{C_\alpha}L^{-\frac{\eta-2b}{12}}.$$

The proof of the desired inequality

$$d(\mathbf{x}_p\,\mathbf{u}^{\bar{s}_p-s_p}\mathbf{a}^T, \mathbf{y}_p\,\mathbf{u}^{\bar{t}_p-t_p}\mathbf{a}^T) \leq 6\kappa\sqrt{C_\alpha}L^{-\frac{\eta-2b}{12}}$$

for the remaining summand is completely analogous: it follows again from Lemma 3.2.3, by considering the negative parametrizations $s' = -s$ and $t'(s) = -t(-s)$, $s \in [-L, 0]$. This completes the proof of (35) and hence yields the result. □

4.3.9. We can finally complete the proof of Ratner's Basic Lemma.

Proof of Lemma 4.2.1. We start by recalling that all the choice of the relevant constants were done in §4.1.1. In particular, we have chosen b. Let $\theta = \theta(b)$ given by Solovay's Lemma 4.3.1.

Let $x, y \in M_2$, and let λ and A, with

$$\frac{|A \cap [0, \lambda]|}{\lambda} > 1 - \frac{\theta}{8},$$

be as in the statement. We consider $\mathbf{x}_1$ and $\mathbf{y}_1$ lifts of x and y such that $d(\mathbf{x}_1, \mathbf{y}_1) < \rho$. With these data we build blocks $\{J_1, \dots, J_n\}$ as in §4.3. By applying the superblocks algorithm to this collection of blocks we obtain a collection of superblocks as in Lemma 4.3.2.

The collection of superblocks we have obtained thus satisfies the assumptions of Lemma 4.3.1. This implies that there exists a superblock $J = [s_J, \bar{s_J}]$ of length $|J| \geq 3/4\lambda$. Lemma 4.3.2 guarantees that we can apply Corollary 3.2.2 to the points $\mathbf{x}_J$ and $\mathbf{y}_J$ which are lifts to G of the points $x\mathbf{u}^{s_J}$ and $y\mathbf{u}^{\tau(s_J)}$ respectively. We obtain that

$$\mathbf{g} = \mathbf{x}_J^{-1} \mathbf{y}_J = \exp\Big(\sum_{\iota, j} c_{j,\iota} E_{j,\iota} \Big) \exp(aA) \exp(\bar{u}\bar{U})\, \mathbf{u}^u$$

satisfies

$$|u| \leq \varepsilon, \quad |a| \leq \kappa \lambda^{-\eta/2}, \quad |\bar{u}| \leq \kappa \lambda^{-1-\eta/2}, \quad |c_{\iota,j}| \leq \kappa\varepsilon \lambda^{-j+\eta/2}.$$

which proves the first part

Let us now show the second claim. Since, by assumption, there exists $\lambda_0 \geq m$ such that we have that $|A \cap [0, \lambda]| \geq (1 - \theta/8)\lambda$ for all $\lambda \geq \lambda_0$, we can find a sequence $\lambda_n \to \infty$ such that $\lambda_n < \lambda_{n+1} < 8\lambda_n/7$. For all such λ_n we can repeat the above construction and obtain an interval $J_n = [s_n, \bar{s_n}]$ having length $|J_n| \geq 3/4\lambda_n$. By our choice of λ_n we have

$$J_n \cap J_{n+1} \neq \varnothing,$$

and hence $J_n \subset J_{n+1}$ for all n; more precisely, $J_n = [s_0, \bar{s_n}]$ where $s_0 \leq \lambda_0$. Applying the same bounds given by Corollary 3.2.2 we have, for $\mathbf{g} = (\mathbf{x}\mathbf{u}^{s_0})^{-1}\mathbf{y}\mathbf{u}^{\tau(s_0)}$

$$|u| \leq \varepsilon, \quad |a| \leq \kappa \lambda_n^{-\eta/2} \quad |\bar{u}| \leq \kappa \lambda_n^{-1-\eta/2}, \quad \text{and} \quad |c_{j,\iota}| \leq \kappa\varepsilon \lambda_n^{-j+\eta/2}.$$

for all n. This implies that $|a|$, $|\bar{u}|$ and $|c_{j,\iota}|$ are 0, if $j \neq 0$. Thus, there exists $\mathbf{g}$, commuting with $\mathbf{u}$ such that $\mathbf{y} = \mathbf{x}\mathbf{g}$ as we wanted. □

5. Normaliser

5.0.1. In this section we will apply the Basic Lemma 4.2.1. Before we state formally the main result of this section, let us recall all the constants we have defined so far.

The two functions α_1 and α_2 defining the time-changes $\widetilde{\phi}_1^t$ and $\widetilde{\phi}_2^t$, define the constant $C_\alpha > 0$ in Assumption 1. The strong spectral gap yields $\eta \in (0, 1/4)$ in Corollary 2.1.5. From equation (25) we obtain the constants b and $c(b)$. Finally $\theta = \theta(b) \in (0, 1)$ is the one given by Solovay's Lemma 4.3.1 and used in the Basic

Lemma 4.2.1. Having recalled all the important constants, we pick an $\omega > 0$ satisfying

$$\omega < \frac{\theta}{50C_\alpha^4}.$$

Using Proposition 4.1.1 we obtain a set $\mathcal{G}$ and constants m_0, ρ. Let $\delta > 0$ be the uniform continuity constant of ψ on $\mathcal{G}$ so that, if $x, y \in \mathcal{G}$, then

$$d(x,y) < \delta \implies d(\psi(x), \psi(y)) < \rho.$$

Finally, we choose $\delta' < \rho$ and we recall that $m = 2C_\alpha^4 m_0 > m_0$.

For any $n \in \mathbb{N}$, there exist $s_n > 0$ and a set $K_{\mathbf{u}_1}(n) \subset M_1$ of measure greater than $1 - 2^{-n}$ such that for all $x \in K_{\mathbf{u}_1}(n)$ and all $s \geq s_n$

$$|\{t \in [0,s] : x\mathbf{u}_1^t \in \mathcal{G}\}| \geq \left(1 - \frac{\theta}{40C_\alpha^4}\right) s.$$

We can take $s_n > m\omega_0^{-1}$. This section, §5, is devoted to the proof of the following result.

Proposition 5.0.1. *Let $\mathbf{g}_1 \in G_1$, $\mathbf{g}_2 \in G_2$, with $d(\mathbf{e}_1, \mathbf{g}_1) < \delta$ and $d(\mathbf{e}_2, \mathbf{g}_2) < \delta'$, satisfy the commuting relations $\mathbf{g}_1\mathbf{u}_1^t\mathbf{g}_1^{-1} = \mathbf{u}_1^{c^{-1}t}$ and $\mathbf{g}_2\mathbf{u}_2^t\mathbf{g}_2^{-1} = \mathbf{u}_2^{c^{-1}t}$ for all $t \in \mathbb{R}$. Fix $n \in \mathbb{N}$, and let $x \in K_{\mathbf{u}_1}(n)$, $y = x\mathbf{g} \in K_{\mathbf{u}_1}(n)$. Then, for every $s \geq s_n$ there exist*

- *points $x_0 = x\mathbf{u}_1^{t_0}$, $y_0 = y\mathbf{u}_1^{ct_0} = x_0\mathbf{g}$, with $0 \leq t_0 \leq 4\omega s$,*
- *an interval $[0, \lambda_s] \subset [0, s]$, with $\lambda_s \geq C_\alpha^{-2}\left(1 - \frac{\theta}{16C_\alpha^4}\right)s$,*
- *a compact subset $A_s \subset [0, \lambda_s]$,*
- *a strictly increasing Lipschitz function $\tau = \tau_{x_0,y_0} : [0, \lambda_s] \to \mathbb{R}_{>0}$ with $\tau(0) = 0$,*

such that, if we set $x_0' = \psi(x_0)$, $y_0' = \psi(y_0)$, and $\overline{x}_0' = \psi(y_0)\mathbf{g}_2^{-1}$, then

- *$0, \lambda_s \in A_s$,*
- *$|A_s| > \left(1 - \frac{\theta}{8}\right)\lambda_s$,*
- *$x_0'\mathbf{u}_2^r$ and $\overline{x}_0'\mathbf{u}_2^{\tau(r)}$ belong to $\psi(\mathcal{G})$ for all $r \in A_s$ and satisfy*

$$d\left(x_0'\mathbf{u}_2^r, \overline{x}_0'\mathbf{u}_2^{\tau(r)}\right) < \rho,$$

- *if $r', r \in A_s$ and $r' - r > m$ or $\tau(r') - \tau(r) > m$, then*

$$|\tau(r') - \tau(r) - (r' - r)| \leq 4(r' - r)^{1-\eta}.$$

5.0.2. Let $\mathbf{g}_1$, x, and y be as in the assumptions of Proposition 5.0.1, in particular $d(x,y) < \delta$. We can assume that $\frac{1}{2} \leq c \leq 2$.

Clearly,

$$y\mathbf{u}_1^{cs} = x\mathbf{g}_1\mathbf{u}_1^{cs} = x\mathbf{u}_1^s\mathbf{g}_1,$$

and

$$\psi(x\mathbf{u}_1^s) = \psi(x)\mathbf{u}_2^{z(x,s)}, \qquad \psi(y\mathbf{u}_1^{cs}) = \psi(y)\mathbf{u}_2^{z(y,cs)}.$$

Define

$$\forall s \geq 0: \quad B_s := \{r \in [0,s] : x\mathbf{u}_1^r \in \mathcal{G}, y\mathbf{u}_1^{cr} \in \mathcal{G}\},$$

which is a compact subset of $[0, s]$. By definition of $K_{\mathbf{u}_1}(n)$, the estimate on the measure of $\mathcal{G}$ in Proposition 4.1.1, and our choice of ω, since $c \leq 2$, we have

$$\forall s > 2s_n: \quad |B_s| \geq \left(1 - \frac{\theta}{16C_\alpha^4}\right) s. \tag{36}$$

Let us assume now that $d(\mathbf{e},\mathbf{g}) = d(x,y) < \delta$, and consequently $d(x\mathbf{u}_1^s, y\mathbf{u}_1^{cs}) < \delta$ for all $s \in \mathbb{R}$. Then, by Proposition 4.1.1, we have

$$d\left(\psi(x)\mathbf{u}_2^{z(x,s)}, \psi(y)\mathbf{u}_2^{z(y,cs)}\right) < \rho, \quad \text{for all } s \in B_s. \tag{37}$$

Lemma 5.0.2. *Let $t \in B_s$. Assume that there exists $t < t' \leq s$ such that*

$$z(x,t') - z(x,t) > cm \quad \text{or} \quad z(y,t') - z(y,t) > cm.$$

Then,

$$\left| c^{-1}\big(z(y,ct') - z(y,ct)\big) - (z(x,t') - z(x,t)) \right| \leq 4\,(z(x,t') - z(x,t))^{1-\eta}.$$

Proof. Using Lemma 2.2.2 and the definition of $m = 2C_\alpha^4 m_0$, under our assumption it is easy to see that

$$z(x,t') - z(x,t) > m_0 \quad \text{and} \quad z(y,t') - z(y,t) > m_0 \quad \text{and} \quad t' - t > m_0.$$

Recall the definition $z(x,t) := w_2(\psi(x), \xi_1(x,t))$, of w_2 and of ξ_1. By Lemma 2.2.2,

$$\begin{aligned}
&z(x,t') - z(x,t) - (\xi_1(x,t') - \xi_1(x,t)) \\
&\quad = z(x,t') - z(x,t) - \int_{w_2(\psi(x),\xi_1(x,t))}^{w_2(\psi(x),\xi_1(x,t'))} \alpha_2(\psi(x)\mathbf{u}_2^s)\,\mathrm{d}s \\
&\quad = \int_{z(x,t)}^{z(x,t')} (1-\alpha_2)(\psi(x)\mathbf{u}_2^s)\,\mathrm{d}s = \int_0^{z(x\mathbf{u}_1^t, t'-t)} (1-\alpha_2)(\psi(x\mathbf{u}_1^t)\mathbf{u}_2^s)\,\mathrm{d}s.
\end{aligned}$$

Since $x\mathbf{u}_1^t \in \mathcal{G}$ and $z(x,t') - z(x,t) > m_0$, by Corollary 2.1.5

$$|z(x,t') - z(x,t) - (\xi_1(x,t') - \xi_1(x,t))| \leq \frac{1}{C_\alpha^4}(z(x\mathbf{u}_1^t, t'-t))^{1-\eta} \leq \frac{1}{C_\alpha^2}(t'-t)^{1-\eta}.$$

Similarly,

$$\begin{aligned}
&|\xi_1(x,t') - \xi_1(x,t) - (t'-t)| \\
&\quad = |\xi_1(x,t') - \xi_1(x,t) - (w_1(x,\xi_1(x,t')) - w_1(x,\xi_1(x,t)))| \\
&\quad = \left| \int_{w_1(x,\xi_1(x,t))}^{w_1(x,\xi_1(x,t'))} (1-\alpha_1)(x\mathbf{u}_1^s)\,\mathrm{d}s \right| = \left| \int_t^{t'} (1-\alpha_1)(x\mathbf{u}_1^s)\,\mathrm{d}s \right| \leq \frac{1}{C_\alpha^4}(t'-t)^{1-\eta}.
\end{aligned}$$

Therefore,

$$|z(x,t') - z(x,t) - (t'-t)| \leq \frac{2}{C_\alpha^2}(t'-t)^{1-\eta}.$$

We can apply the same reasoning for y instead of x, yielding

$$\left| c^{-1}(z(y,ct') - z(y,ct)) - (t'-t) \right| \leq \frac{2}{C_\alpha^2}(t'-t)^{1-\eta}.$$

We deduce that

$$\left| c^{-1}(z(y,ct') - z(y,ct)) - (z(x,t') - z(x,t)) \right| \leq \frac{4}{C_\alpha^2}(t'-t)^{1-\eta}.$$

The rough bounds on z and the cocycle relation in Lemma 2.2.2 conclude the proof. □

5.0.3. *Proof of Proposition 5.0.1.* By the compactness of B_s, the points

$$t_0 = \inf B_s \quad \text{and} \quad t_1 = \sup B_s$$

are in B_s. Let

$$t_0' = z(x, t_0), \quad t_1' = z(x, t_1) \quad \text{and} \quad \lambda_s = t_1' - t_0'.$$

Set

$$B_s' = z(x, B_s) \subset [t_0', t_1']$$

From (36) we have

$$t_0 < \frac{\theta}{16C_\alpha^4} s, \qquad \left(1 - \frac{\theta}{16C_\alpha^4}\right) s \leq t_1 - t_0 \leq s.$$

By the uniform Lipschitz estimate on $z(x, \cdot)$, this yields

$$C_\alpha^{-2}\left(1 - \frac{\theta}{16C_\alpha^4}\right) s \leq \lambda_s = t_1' - t_0' \leq C_\alpha^2 s,$$

and

$$\frac{|B_s'|}{\lambda_s} \geq 1 - \frac{\theta}{16}.$$

We let

$$A_s' = \{t_0'\} \cup (B_s' \cap [t_0' + m, t_1']), \quad A_s = A_s' - t_0'.$$

Then, since $s \geq s_n$, if n is sufficiently large, we have

$$\frac{|A_s'|}{\lambda_s} \geq 1 - \frac{\theta}{8}.$$

Let

$$x_0 = x\mathbf{u}_1^{t_0}, \quad y_0 = y\mathbf{u}_1^{ct_0} = x_0\mathbf{g}$$

$$x_0' = \psi(x_0) = \psi(x)\mathbf{u}_2^{z(x,t_0)} = \psi(x)\mathbf{u}_2^{s_0}, \quad y_0' = \psi(y_0) = \psi(y)\mathbf{u}_2^{z(y,ct_0)}.$$

By Lemma 2.2.2, we have

$$\psi(x_0\mathbf{u}_1^t) = \psi(x\mathbf{u}_1^{t+t_0}) = \psi(x)\mathbf{u}_2^{z(x,t+t_0)} = \psi(x_0)\mathbf{u}_2^{z(x,t+t_0)-z(x,t_0)} = \psi(x_0)\mathbf{u}_2^{z(x_0,t)},$$

and similarly

$$\psi(y_0\mathbf{u}_1^{ct}) = \psi(y_0)\mathbf{u}_2^{z(y_0,ct)}.$$

We know from (37) that the U_2-orbit of y_0' shadows the U_2-orbit of x_0' at times rescaled by c. We now define a point $\overline{x}_0'$ whose U_2-orbit shadows the U_2-orbit of x_0' at synchronous times.

Let $\widetilde{\mathbf{g}}$, with $d(\mathbf{e}, \widetilde{\mathbf{g}}) < \delta'$, be as in the assumptions: an element in the normaliser of U_2 such that $\widetilde{\mathbf{g}}\mathbf{u}_2^s\widetilde{\mathbf{g}}^{-1} = \mathbf{u}_2^{c^{-1}s}$. We set

$$\bar{x}_0' = \psi(y_0)\widetilde{\mathbf{g}}^{-1}.$$

Then

$$\bar{x}_0'\mathbf{u}_2^t = \left(\psi(y_0)\mathbf{u}_2^{ct}\right)\widetilde{\mathbf{g}}^{-1}, \quad \forall t \in \mathbb{R},$$

and therefore,

$$d(\bar{x}_0'\mathbf{u}_2^t, \psi(y_0)\mathbf{u}_2^{ct}) < \delta', \quad \forall t \in \mathbb{R}. \tag{38}$$

Define

$$\tau(r) = \tau_{x_0,y_0}(r) = c^{-1}z(y_0, c\eta(x_0, r)),$$

where $\eta(x_0,\cdot)$ is the inverse of $z(x_0,\cdot)$. It follows from the definition of A_s and Proposition 4.1.1 that for $r \in A_s$ we have

$$x_0'\mathbf{u}_2^r \in \psi(\mathcal{G}), \quad y_0'\mathbf{u}_2^{\tau_{x_0,y_0}(r)} \in \psi(\mathcal{G}),$$
$$\text{and} \quad d(x_0'\mathbf{u}_2^r, \bar{x}_0'\mathbf{u}_2^{\tau_{x_0,y_0}(r)}) < \rho + \delta' < 2\rho,$$

where we used (37) and (38). In particular

$$d(x_0', \bar{x}_0') < \rho.$$

since $0 \in A_s$. Suppose

$$r' - r > m, \qquad \text{or} \qquad \tau_{x_0,y_0}(r') - \tau_{x_0,y_0}(r) > m.$$

These conditions are equivalent to $z(x,t') - z(x,t) > m$ or $z(y,ct') - z(y,ct) > m$. If furthermore $r \in A_s$, then Lemma 5.0.2 applies and we have

$$\Big|\big(\tau_{x_0,y_0}(r') - \tau_{x_0,y_0}(r)\big) - \big(r' - r\big)\Big| \leq 4\big(r' - r\big)^{1-\eta}. \tag{39}$$

The proof is then complete. □

5.1. **Action of the conjugacy on the normaliser.** Thanks to the Basic Lemma, we can now show that the isomorphism ψ maps the foliation with leaves tangent to the normaliser of U_1 to the one with leaves tangent to the normaliser of U_2. More precisely, we prove the following proposition.

Proposition 5.1.1. *Let $\mathbf{g} \in G_1$ with $d(\mathbf{e},\mathbf{g}) < \delta$ be such that $\mathbf{g}\mathbf{u}_1^{ct} = \mathbf{u}_1^t\mathbf{g}$ for some $c \in (1/2, 2)$. For almost every $x \in M_1$ there exists $\Phi(x,\mathbf{g}) \in G_2$ satisfying $\Phi(x,\mathbf{g})\mathbf{u}_2^{ct} = \mathbf{u}_2^t\Phi(x,\mathbf{g})$ and such that*

$$\psi(x\mathbf{g}) = \psi(x)\Phi(x,\mathbf{g}).$$

Moreover, for every $y \in \mathbb{R}$, we have

$$\Phi(x\mathbf{u}_1^t,\mathbf{g})\,\Phi(x,\mathbf{g})^{-1} = \mathbf{u}_2^{z(x\mathbf{g},ct) - cz(x,t)}.$$

Finally, $\Phi(x,\mathbf{g})\exp(\mathbb{R}\mathbf{u}_2) \in N_{G_2}(\mathbf{u}_2)/\exp(\mathbb{R}\mathbf{u}_2)$ is constant almost everywhere and for almost all $x \in M_1$, the map

$$\Phi(x,\cdot)\colon N_{G_1}(\mathbf{u}_1)/\exp(\mathbb{R}\mathbf{u}_1) \to N_{G_2}(\mathbf{u}_2)/\exp(\mathbb{R}\mathbf{u}_2)$$

is a group homomorphism.

Proof. From §5.0.1, we have that for every $n \in \mathbb{N}$, the measure of the set $K_{\mathbf{u}_1}(n) \cap K_{\mathbf{u}_1}(n)\mathbf{g}^{-1}$ is at least $1 - 2^{-n+1}$. Therefore, for almost every $x \in M_1$, there exists $n = n(x) \in \mathbb{N}$ such that $x \in K_{\mathbf{u}_1}(n)$ and $y = x\mathbf{g} \in K_{\mathbf{u}_1}(n)$. Choose $\widetilde{\mathbf{g}} \in G$ such that $d(\mathbf{e},\widetilde{\mathbf{g}}) < \delta'$ and $\widetilde{\mathbf{g}}\mathbf{u}_2^{ct} = \mathbf{u}_2^t\widetilde{\mathbf{g}}$. Combining Proposition 5.0.1 with Lemma 4.2.1, we deduce that there exists $\mathbf{g}_0 = \mathbf{g}_0(x) \in G_2$, depending on $\mathbf{g}$ and possibly on x, which commutes with $\mathbf{u}_2^t$ and with $d(\mathbf{e},\mathbf{g}_0) \leq \varepsilon$ such that

$$\psi(y\mathbf{u}_1^{ct_0})\widetilde{\mathbf{g}}^{-1} = \psi(x\mathbf{u}_1^{t_0})\mathbf{g}_0, \tag{40}$$

where $0 \leq t_0 \leq s_{n(x)}$.

We can rewrite (40) as

$$\begin{aligned} \psi(x\mathbf{g}) &= \psi(x)\mathbf{u}_2^{z(x,t_0)}\mathbf{g}_0\widetilde{\mathbf{g}}\mathbf{u}_2^{-z(y,ct_0)} \\ &= \psi(x)\mathbf{g}_0\widetilde{\mathbf{g}}\mathbf{u}_2^{cz(x,t_0) - z(y,ct_0)}, \end{aligned} \tag{41}$$

where the element $\mathbf{g}_0\widetilde{\mathbf{g}}$ satisfies $d(\mathbf{e}, \mathbf{g}_0\widetilde{\mathbf{g}}) \leq \delta' + \rho < 2\rho$ and belongs to the normaliser $N_{G_2}(\mathbf{u}_2)$ of $\mathbf{u}_2^t$, namely $(\mathbf{g}_0\widetilde{\mathbf{g}})\mathbf{u}_2^{ct} = \mathbf{u}_2^t(\mathbf{g}_0\widetilde{\mathbf{g}})$. Equality (41) proves the existence of $\Phi(x, \mathbf{g}) \in N_{G_2}(\mathbf{u}_2)$ satisfying $\psi(x\mathbf{g}) = \psi(x)\Phi(x, \mathbf{g})$.

We have a well-defined measurable map

$$\begin{aligned} \Phi_{\mathbf{g}} \colon M_1 &\to N_G(\mathbf{u}_2)/\exp(\mathbb{R}U_2) \\ x &\mapsto \mathbf{g}_0(x)\widetilde{\mathbf{g}}\exp(\mathbb{R}U_2). \end{aligned}$$

By construction, it is easy to see that $\Phi_{\mathbf{g}}(x\mathbf{u}_1^t) = \Phi_{\mathbf{g}}(x)$ for all $t \in \mathbb{R}$. Since $\Phi_{\mathbf{g}}$ is bounded, by ergodicity of the unipotent flow $\mathbf{u}_1^t$, the map $\Phi_{\mathbf{g}}$ must be constant. This proves that $\Phi(x, \mathbf{g})\exp(\mathbb{R}\mathbf{u}_2)$ is constant almost everywhere. Note that the map Φ satisfies

$$\Phi(x, \mathbf{g}_1\mathbf{g}_2) = \Phi(x, \mathbf{g}_1)\Phi(x\mathbf{g}_1, \mathbf{g}_2)$$

for all $\mathbf{g}_1, \mathbf{g}_2$ in the normaliser of $\mathbf{u}_1$. In particular, $\Phi(x, \cdot)$ descends to a group homomorphism between $N_{G_1}(\mathbf{u}_1)/\exp(\mathbb{R}\mathbf{u}_1)$ and $N_{G_2}(\mathbf{u}_2)/\exp(\mathbb{R}\mathbf{u}_2)$.

Finally, let $t \in \mathbb{R}$ be fixed. Then,

$$\begin{aligned} \psi(x\mathbf{u}_1^t\mathbf{g}) &= \psi(x\mathbf{g})\mathbf{u}_2^{z(x\mathbf{g},ct)} = \psi(x)\Phi(x, \mathbf{g})\mathbf{u}_2^{z(x\mathbf{g},ct)} \\ &= \psi(x\mathbf{u}_1^t)\mathbf{u}_2^{-z(x,t)}\Phi(x, \mathbf{g})\mathbf{u}_2^{z(x\mathbf{g},ct)} \\ &= \psi(x\mathbf{u}_1^t)\Phi(x, \mathbf{g})\mathbf{u}_2^{-cz(x,t)+z(x\mathbf{g},ct)}. \end{aligned}$$

Since the expression above also equals $\psi(x\mathbf{u}_1^t)\Phi(x\mathbf{u}_1^t, \mathbf{g})$, the proof of the cocycle relation for Φ is complete. □

Let us fix $x_0 \in M_1$, and denote $\Phi(\mathbf{g}) = \Phi(x_0, \mathbf{g})$. Since $\Phi(x, \mathbf{g})\exp(\mathbb{R}\mathbf{u}_2) = \Phi(\mathbf{g})\exp(\mathbb{R}\mathbf{u}_2)$ almost everywhere, for almost every $x \in M_1$ there exists $\beta(x, \mathbf{g}) \in \mathbb{R}$ such that

$$\Phi(x, \mathbf{g}) = \Phi(\mathbf{g})\mathbf{u}_2^{\beta(x,\mathbf{g})}.$$

The proof of Proposition 5.1.1 implies that

$$d(\mathbf{e}, \Phi(\mathbf{g})) \leq \varepsilon, \qquad \text{and} \qquad \text{if } x \in K_{\mathbf{u}_1}(n), \text{ then } |\beta(x, \mathbf{g})| \leq s_n.$$

With this notation, from Proposition 5.1.1, we now obtain the following result about the cohomology of the time-change functions α_1 and α_2, which is a more general version of Theorem C.

Corollary 5.1.2. *Let $\mathbf{g} \in G$ with $d(\mathbf{e}, \mathbf{g}) < \delta$ be such that $\mathbf{g}\mathbf{u}_1^{ct} = \mathbf{u}_1^t\mathbf{g}$ for some $c \in (1/2, 2)$. Then we have*

$$\int_0^t \alpha_1(x\mathbf{u}_1^s\mathbf{g}) \, \mathrm{d}s - \int_0^{z(x,t)} \alpha_2(\psi(x)\mathbf{u}_2^s\Phi(\mathbf{g})) \, \mathrm{d}s = f(x\mathbf{u}_1^t) - f(x),$$

where

$$f(x) = \frac{1}{c}\int_0^{\beta(x,\mathbf{g})} \alpha_2(\psi(x)\Phi(\mathbf{g})\mathbf{u}_2^s) \, \mathrm{d}s.$$

In particular, if $\alpha_2 = 1$, then $\alpha_1(x)$ is cohomologous with $\alpha_1(x\mathbf{c})$, where $\mathbf{c}$ belongs to the centraliser of $\mathbf{u}_1$, and the same holds for α_2 if α_1 is trivial.

Proof. By a direct computation, recalling the definition of $z(x,t)$, we have

$$\begin{aligned}\int_0^t \alpha_1(x\mathbf{u}_1^s\mathbf{g})\ \mathrm{d}s &= \frac{1}{c}\int_0^{ct} \alpha_1(x\mathbf{g}\mathbf{u}_1^s)\ \mathrm{d}s \\ &= \frac{1}{c}\int_0^{z(x\mathbf{g},ct)} \alpha_2(\psi(x\mathbf{g})\mathbf{u}_2^s)\ \mathrm{d}s \\ &= \frac{1}{c}\int_0^{z(x\mathbf{g},ct)} \alpha_2(\psi(x)\Phi(\mathbf{g})\mathbf{u}_2^{\beta(x,\mathbf{g})+s})\ \mathrm{d}s,\end{aligned}$$

which we can write as

$$\frac{1}{c}\int_0^{z(x\mathbf{g},ct)+\beta(x,\mathbf{g})} \alpha_2(\psi(x)\Phi(\mathbf{g})\mathbf{u}_2^s)\ \mathrm{d}s - \frac{1}{c}\int_0^{\beta(x,\mathbf{g})} \alpha_2(\psi(x)\Phi(\mathbf{g})\mathbf{u}_2^s)\ \mathrm{d}s,$$

using $\beta(x\mathbf{u}_1^t,\mathbf{g}) - \beta(x,\mathbf{g}) = z(x\mathbf{g},ct) - cz(x,t)$, we have

$$\begin{aligned}&\int_0^t \alpha_1(x\mathbf{u}_1^s\mathbf{g})\ \mathrm{d}s = \\ &\qquad \frac{1}{c}\int_0^{cz(x,t)+\beta(x\mathbf{u}_1^t,\mathbf{g})} \alpha_2(\psi(x)\Phi(\mathbf{g})\mathbf{u}_2^s)\ \mathrm{d}s - \frac{1}{c}\int_0^{\beta(x,\mathbf{g})} \alpha_2(\psi(x)\Phi(\mathbf{g})\mathbf{u}_2^s)\ \mathrm{d}s.\end{aligned}$$

Since

$$\int_0^{z(x,t)} \alpha_2(\psi(x)\mathbf{u}_2^s\Phi(\mathbf{g}))\ \mathrm{d}s = \frac{1}{c}\int_0^{cz(x,t)} \alpha_2(\psi(x)\Phi(\mathbf{g})\mathbf{u}_2^s)\ \mathrm{d}s,$$

joining these two pieces we obtain the Corollary. □

6. Proof of the Main Result

In this section we prove our main result, namely Theorem A.

6.1. **Joinings and a map ζ.** We recall that a *joining* of the flows $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$ is a measure μ on $M_1 \times M_2$ which is invariant under the action of $\widetilde{\phi}_1 \times \widetilde{\phi}_2$ and such that, when projected to any of the two factors M_i, coincides with $\alpha_i\mu_i$. The measure obtaine by projections to the factors are called *marginals*. For more details, see, e.g., [10].

Since ψ is a measurable conjugacy between the two flows, we can construct a joining, supported on the graph $\mathrm{Gr}_\psi = \{(x,\psi(x)), x \in M_1\}$, such that

$$\mu_\psi(B) = \widetilde{\mu_1}(\{x \in M_1 : (x,\psi(x)) \in B\}) = \widetilde{\mu_2}(\{y \in M_2 : (\psi^{-1}(y), y) \in B\}).$$

Let $\phi_A^t = \phi_{A_1\times A_2}^t \colon M_1 \times M_2$ the diagonal action given by $(x,y) \mapsto (x\mathbf{a}_1^t, y\mathbf{a}_2^t)$. If we push forward the joining μ_ψ by the action of A^t we obtain a new measure whose support is given by

$$\phi_A^t \,\mathrm{Gr}_\psi = \{(x\mathbf{a}_1^t, \psi(x)\mathbf{a}_2^t), x \in M_1\} = \{(x, \psi_t(x)), x \in M_1\} = \mathrm{Gr}_{\psi_t},$$

where $\psi_t(x) = \psi(x\mathbf{a}_1^{-t})\mathbf{a}_2^t$. The marginals of this measure are given by

$$(p_1 \circ \phi_A^t)_*\mu_\psi = (\tau_1 \circ \phi_{A_1}^{-t})\mu_1 \qquad \text{and} \qquad (p_2 \circ \phi_A^t)_*\mu_\psi = (\tau_2 \circ \phi_{A_2}^{-t})\mu_2,$$

where $p_i \colon M_1 \times M_2 \to M_i$ is the projection on the i^{th} coordinate. Hence ψ_t is a measurable conjugacy between the (ergodic) flows generated by $(\tau_1 \circ \phi_{A_1}^{-t})^{-1}U_1$ and $(\tau_2 \circ \phi_{A_2}^{-t})^{-1}U_2$.

In this section, to ease the notation, let us call $G = G_1 \times G_2$ and $M = M_1 \times M_2$.

Lemma 6.1.1. *The set $\{(\phi_A^t)_*\mu_\psi : t > 0\}$ is relatively sequentially compact in the space of probability measures on M with the topology of weak convergence.*

Proof. It is sufficient to prove that $\{(\phi_A^t)_*\mu_\psi : t > 0\}$ is tight. For any given $\varepsilon > 0$, there exist compact sets $K_i \subset M_i$ of measure $\mu_i(K_i) \geq 1 - \varepsilon/(2C_\alpha)$. Let $K = K_1 \times K_2 \subset M$. Then,

$$\begin{aligned}(\phi_A^t)_*\mu_\psi(K) &= \widetilde{\mu_1}\{x \in M_1 : x \in K_1\mathbf{a}_1^{-t},\ \psi(x) \in K_2\mathbf{a}_2^{-t}\}\\ &\geq 1 - \widetilde{\mu_1}(M_1 \setminus K_1\mathbf{a}_1^{-t}) - \widetilde{\mu_2}(M_2 \setminus K_2\mathbf{a}_2^{-t})\\ &\geq 1 - C_\alpha\mu_1(M_1 \setminus K_1) - C_\alpha\mu_2(M_2 \setminus K_2) \geq 1 - \varepsilon.\end{aligned}$$

This proves tightness and hence the compactness claim. □

By Lemma 6.1.1, there exist weak-* limit points of the family $\{(\phi_A^t)_*\mu_\psi : t > 0\}$. The following result shows that any such limit point ν must be invariant under diagonal action of the unipotent flows $(x, y) \mapsto (x\mathbf{u}_1^t, y\mathbf{u}_2^t)$.

Lemma 6.1.2. *Let ν a weak-* limit point of $\{(\phi_A^t)_*\mu_\psi : t > 0\}$. Then, $(\phi_U^r)_*\nu = \nu$, where $\phi_U^r(x, y) = (x\mathbf{u}_1^r, y\mathbf{u}_2^r)$.*

Proof. Let $t_n \to \infty$ be an increasing sequence such that $\mu_n := (\phi_A^{t_n})_*\mu_\psi$ converges weakly to ν. It is enough to show that

$$\lim_{n\to\infty} |((\phi_U^r)_*\mu_n)(f_1 \otimes f_2) - \mu_n(f_1 \otimes f_2)| = 0, \tag{42}$$

for every pair of bounded Lipschitz functions $f_i\colon M_i \to \mathbb{C}$.

Let us call $r_n = e^{t_n}r$. We compute

$$\begin{aligned}((\phi_U^r)_*\mu_n)(f_1 \otimes f_2) &= \mu_n(f_1 \circ \phi_1^r \otimes f_2 \circ \phi_2^r) = \mu_\psi(f_1 \circ \phi_1^r \circ \phi_{A_1}^{t_n} \otimes f_2 \circ \phi_2^r \circ \phi_{A_2}^{t_n})\\ &= \mu_\psi(f_1 \circ \phi_{A_1}^{t_n} \circ \phi_1^{r_n} \otimes f_2 \circ \phi_{A_2}^{t_n} \circ \phi_2^{r_n})\\ &= \mu_\psi(f_1 \circ \phi_{A_1}^{t_n} \circ \widetilde{\phi_1}^{\xi_1(\cdot, r_n)} \otimes f_2 \circ \phi_{A_2}^{t_n} \circ \widetilde{\phi_2}^{\xi_2(\cdot, r_n)})\\ &= \mu_\psi(f_1 \circ \phi_{A_1}^{t_n} \circ \widetilde{\phi_1}^{\xi_1(\cdot, r_n) - r_n} \otimes f_2 \circ \phi_{A_2}^{t_n} \circ \widetilde{\phi_2}^{\xi_2(\cdot, r_n) - r_n}),\end{aligned}$$

where, in the last line, we used the invariance of μ_ψ under the product flow $\widetilde{\phi_1^t} \times \widetilde{\phi_2^t}$. We will now call $e_i(x, n) := \xi_i(x, r_n) - r_n$. Denoting by C_i the Lipschitz constant of f_i, we obtain

$$\begin{aligned}|f_i \circ \phi_{A_i}^{t_n} \circ \widetilde{\phi}_i^{e_i(x,n)}(x) - f_i \circ \phi_{A_i}^{t_n}(x)| &= |f_i(x\mathbf{u}_i^{w_i(x, e_i(x,n))}\mathbf{a}_i^{t_n}) - f_i(x\mathbf{a}_i^{t_n})|\\ &= |f_i(x\mathbf{a}_i^{t_n}\mathbf{u}_i^{e^{-t_n}w_i(x, e_i(x,n))}) - f_i(x\mathbf{a}_i^{t_n})| \leq C_i e^{-t_n}|w_i(x, e_i(x, n))|\end{aligned}$$

for all $x \in M_i$.

Fix $\varepsilon > 0$. By Corollary 2.1.5, there exist sets $Y_i \subset M_i$ of measure $\widetilde{\mu_i}(Y_i) \geq 1 - \varepsilon$ and $N > 1$ such that for all $x \in Y_i$ and $n \geq N$ we have

$$\begin{aligned}|f_i \circ \phi_{A_i}^{t_n} \circ \widetilde{\phi}_i^{e_i(x,n)}(x) - f_i \circ \phi_{A_i}^{t_n}(x)| &\leq C_i e^{-t_n}|w_i(x, e_i(x, n))|\\ &\leq C_i\, C_\alpha e^{-t_n}|e_i(x, n)| \leq C_i C_\alpha^{-3} e^{-\eta t_n} r^{1-\eta} < \varepsilon.\end{aligned}$$

The claim (42) follows combining the bounds

$$\begin{aligned}&|((\phi_U^r)_*\mu_n)(f_1 \otimes f_2) - \mu_\psi(\mathbb{1}_{Y_1} \cdot (f_1 \circ \phi_{A_1}^{t_n} \circ \widetilde{\phi_1}^{e_1(\cdot, n)}) \otimes \mathbb{1}_{Y_2} \cdot (f_2 \circ \phi_{A_2}^{t_n} \circ \widetilde{\phi_2}^{e_2(\cdot, n)}))|\\ &\qquad \leq \|f_1\|_\infty \cdot \|f_2\|_\infty[\widetilde{\mu_1}(M_1 \setminus Y_1) + \widetilde{\mu_1}(\psi^{-1}(M_2 \setminus Y_2))] \leq 2\|f_1\|_\infty \cdot \|f_2\|_\infty\varepsilon\end{aligned}$$

and

$$
\begin{aligned}
&|\mu_n(f_1 \otimes f_2) - \mu_\psi(\mathbb{1}_{Y_1} \cdot (f_1 \circ \phi_{A_1}^{t_n} \circ \widetilde{\phi_1}^{e_1(\cdot,n)}) \otimes \mathbb{1}_{Y_2} \cdot (f_2 \circ \phi_{A_2}^{t_n} \circ \widetilde{\phi_2}^{e_2(\cdot,n)}))| \\
&\quad \leq \Big| \int_{Y_1 \cap \psi^{-1}(Y_2)} f_1 \circ \phi_{A_1}^{t_n} \circ \widetilde{\phi_1}^{e_1(x,n)}(x) f_2 \circ \phi_{A_2}^{t_n} \circ \widetilde{\phi_2}^{e_2(\psi(x),n)}(\psi(x)) \\
&\quad - f_1 \circ \phi_{A_1}^{t_n}(x) f_2 \circ \phi_{A_2}^{t_n}(\psi(x)) \,\mathrm{d}\widetilde{\mu_1}(x) \Big| + \|f_1\|_\infty \cdot \|f_2\|_\infty \cdot \widetilde{\mu_1}[M_1 \setminus (Y_1 \cap \psi^{-1}(Y_2))] \\
&\quad \leq (\|f_1\|_\infty + \|f_2\|_\infty + 2\|f_1\|_\infty \cdot \|f_2\|_\infty)\varepsilon,
\end{aligned}
$$

thus the proof is complete. □

By Lemma 6.1.2, the projections $(p_i)_*\nu$ of any limit point ν as above are invariant under the action of $\mathbf{u}_i^t$. Thanks to Ratner's Measure Classification Theorem [29–31], any of its ergodic component is an algebraic measure supported on a closed subgroup $\exp(\mathbb{R}\mathbf{u}_i) < L_i < G_i$. Since the measures $(p_i)_*[(\phi_A^t)_*\mu_\psi]$ are all absolutely continuous with respect to the Haar measure on G_i with density uniformly bounded away from 0, it follows that $L_i = G_i$. Therefore, the projections $(p_i)_*\nu$ are the Haar measures μ_i on G_i. In summary, we have shown that any weak-* limit ν is in turn an ergodic joining of $\mathbf{u}_1^t$ and $\mathbf{u}_2^t$. To this we can apply the Joinings Theorem of [29], as we will now explain.

We want to use the limiting joining ν to construct a measurable map $\zeta\colon M_1 \to M_2$. By construction this map will be the limit of the sequence ψ_{t_n}.

By algebraicity of ν there exists a point $x(\nu) \in M$ such that the measure ν is supported on the orbit $x(\nu)\Lambda(\nu)$, where $\Lambda(\nu) = \mathrm{Stab}_G(\nu)$ are the elements of G which leave invariant the measure ν. Let us define the groups

$$
\Lambda_1(\nu) = \{\mathbf{g} \in G_1 : (\mathbf{g}, \mathbf{e}) \in \Lambda(\nu)\}, \qquad \Lambda_2(\nu) = \{\mathbf{g} \in G_2 : (\mathbf{e}, \mathbf{g}) \in \Lambda(\nu)\}.
$$

In [29], it is shown that these groups are closed normal subgroups of G_1 and G_2 respectively and, moreover, $\Gamma_i \cap \Lambda_i$ is a lattice in Λ_i. In particular, this means that $\Gamma_i\Lambda_i$ is a closed orbit in $M_i = \Gamma_i \backslash G_i$. Therefore, $\Gamma_i\Lambda_i$ is a discrete subgroup of the factor G_i/Λ_i. Since the lattices Γ_i are irreducible, this implies that either $\Lambda_i = G_i$, or Λ_i is *finite*. We remark that the first possibility occurs if and only if ν is the trivial joining $\mu_1 \otimes \mu_2$, thus we now assume that ν is not $\mu_1 \otimes \mu_2$.

For $x \in M_1$ let

$$
\zeta(x) = \zeta_\nu(x) = \{y \in M_2 : (x, y) \in x(\nu)\Lambda(\nu)\}
$$

the fiber of ν at x. The finiteness of the Λ_i's implies that this fiber is finite. In fact, it follows from [29, Theorem 2] that there exist an element $\overline{\mathbf{g}} \in G_2$ and a continuous surjective homomorphism $\varpi\colon G_1 \to G_2/\Lambda_2$, with kernel Λ_1 such that $\varpi(\mathbf{u}_1) = \mathbf{u}_2\Lambda_2$ and

$$
\zeta(\Gamma_1\mathbf{x}) = \{\Gamma_2\overline{\mathbf{g}}\beta_i\varpi(\mathbf{x}), i = 1, \ldots, n\},
$$

where $\varpi(\Gamma_1) = \{\overline{\Gamma}\beta_i : i = 1, \ldots, n\}$, with $\overline{\Gamma} = \varpi(\Gamma_1) \cap \overline{\mathbf{g}}^{-1}\Gamma_2\overline{\mathbf{g}}\Lambda_2$ of finite index, equal to n, inside both $\varpi(\Gamma_1)$ and $\overline{\mathbf{g}}^{-1}\Gamma_2\overline{\mathbf{g}}\Lambda_2$. Up to passing to a finite quotient, we can assume that both Λ_i are trivial and that $\varpi\colon G_1 \to G_2$ is an isomorphism, with $\varpi(\Gamma_1) \subset \overline{\mathbf{g}}^{-1}\Gamma_2\overline{\mathbf{g}}$. Hence, we obtain a map $\zeta\colon M_1 \to M_2$, given by $\zeta(\Gamma_1\mathbf{x}) = \Gamma_2\overline{\mathbf{g}}\varpi(\mathbf{x})$, for μ_1-a.e. $\Gamma_1\mathbf{x}$, which, by construction satisfies $\zeta(x\mathbf{u}_1^s) = \zeta(x)\mathbf{u}_2^s$.

6.1.1. We are now ready to prove our main result.

Proof of Theorem A. We want to show that the conjugacy ψ is given by $\psi(x) = \zeta(x)\mathbf{c}(x)\mathbf{u}_2^{t(x)}$, where $\mathbf{c}(x)$ belongs to the centraliser of $\mathbf{u}_2$, and $t(x) \in \mathbb{R}$.

By compactness of the set $\mathcal{G}$, we can find an n_0 such that if $n \geq n_0$ and $x \in \mathcal{G}$ then $d(\psi_{t_n}(x), \zeta(x)) < \rho$. Let $\mathcal{G}_n = \phi_{A_1}^{t_n}(\mathcal{G}) \cap \mathcal{G}$, whose measure is $\mu_1(\mathcal{G}_n) > 1 - 2\omega$. Denote Q the generic set of $\mathcal{G}_n$ for $\mathbf{u}_1^t$ and $Q_n = \mathcal{G}_n \cap Q$. Then for $x \in Q_n$ we introduce the set

$$A = A(x) = \{s \in \mathbb{R}_+ : x\mathbf{u}_1^s \in \mathcal{G}_n\}.$$

By genericity we have that the relative length of A is large. In fact, as $T \to \infty$, we have

$$\frac{|A \cap [0, T]|}{T} \to 1 - 2\omega > 1 - \frac{\theta}{8},$$

where the last inequality follows by our choice of ω in §5.0.1.

Let $x' = \psi_{t_n}(x) \in M_2$. By Proposition 5.1.1, we can define a function σ such that

$$x'\mathbf{u}_2^{\sigma(s)} = \psi_{t_n}(x\mathbf{u}_1^s).$$

We want to apply Ratner's Basic Lemma 4.2.1 to the pair of points x' and $\zeta(x)$ and the set A. By construction, if $s \in A$, then

$$d(x'\mathbf{u}_2^{\sigma(s)}, \zeta(x)\mathbf{u}_2^s) = d(\psi_{t_n}(x\mathbf{u}_1^s), \zeta(x\mathbf{u}_1^s)) < \rho.$$

Since $0 \in A$, it only remains to check the last condition in the Lemma. Let us assume that $s'-s > m$ and show that $|(\sigma(s')-\sigma(s))-(s'-s)| \leq 4(s'-s)^{1-\eta}$. The case when $\sigma(s') - \sigma(s) > m$ is similar. We have that $x\mathbf{u}_1^s\mathbf{a}_1^{-t_n} \in \mathcal{G}$ and $\psi(x\mathbf{u}_1^s\mathbf{a}_1^{-t_n}) \in \psi(\mathcal{G})$. Moreover

$$x\mathbf{u}_1^{s'}\mathbf{a}_1^{-t_n} = x\mathbf{u}_1^s\mathbf{a}_1^{-t_n}\mathbf{u}_1^{r_n(s'-s)},$$

and

$$\psi(x\mathbf{u}_1^{s'}\mathbf{a}_1^{-t_n}) = \psi(x\mathbf{u}_1^s\mathbf{a}_1^{-t_n})\mathbf{u}_2^{z(x\mathbf{u}_1^s\mathbf{a}_1^{-t_n}, r_n(s'-s))}.$$

Hence, by definition of $\sigma(s)$ we have

$$\begin{aligned} x'\mathbf{u}_2^{\sigma(s')} &= \psi_{t_n}(x\mathbf{u}_1^{s'}) \\ &= \psi_{t_n}(x\mathbf{u}_1^s)\mathbf{u}_2^{z(x\mathbf{u}_1^s\mathbf{a}_1^{-t_n}, r_n(s'-s))/r_n} \\ &= x'\mathbf{u}_2^{z(x\mathbf{u}_1^s\mathbf{a}_1^{-t_n}, r_n(s'-s))/r_n}\mathbf{u}_2^{\sigma(s)}, \end{aligned}$$

which implies

$$\sigma(s') - \sigma(s) = \frac{z(x\mathbf{u}_1^s\mathbf{a}_1^{-t_n}, r_n(s'-s))}{r_n}.$$

As in Lemma 5.0.2, we have

$$|z(x\mathbf{u}_1^s\mathbf{a}_1^{-t_n}, r_n(s'-s)) - r_n(s'-s)| \leq \frac{2}{C_\alpha} r_n(s'-s)^{1-\eta},$$

and, finally,

$$|(\sigma(s') - \sigma(s)) - (s'-s)| \leq 4(s'-s)^{1-\eta}.$$

By the Basic Lemma 4.2.1, we have that $x' = \zeta(x)\mathbf{c}\mathbf{u}_2^t$, for some time $t = t(x) \in \mathbb{R}$ and where $\mathbf{c} = \mathbf{c}(x)$ commutes with $\mathbf{u}_2$.

By construction, we have that $\zeta \circ \phi_{A_1}^t = \phi_{A_2}^t \circ \zeta$. Then, for every point $x \in \phi_{A_1}^{-t_n} Q_n$ we have $\psi(x) = \zeta(x)\mathbf{c}\mathbf{u}_2^t$, as above. The set

$$\{x \in M_1 : \psi(x) = \zeta(x)\mathbf{c}\mathbf{u}_2^t, \text{ for some } t \in \mathbb{R} \text{ and } \mathbf{c} \text{ commuting with } \mathbf{u}_2\}$$

is $\mathbf{u}_1^t$-invariant. Since it contains the set $\phi_{A_1}^{-t_n} Q_n$, which has positive measure, we conclude by ergodicity of $\mathbf{u}_1^t$, that $\psi(x) = \zeta(x)\mathbf{c}\mathbf{u}_2^t$, for μ_1-a.e. x, as we wanted. □

7. Proof of Theorem B

In this section we prove Theorem B, which is a direct generalisation of the main result of [27]. We will specialise to the case $G_i = \mathrm{SL}_2(\mathbb{R}) \times G_i'$, where, for $i = 1, 2$, G_i' is a connected semisimple linear group with algebra $\mathfrak{g}_i'$, and $\Gamma_i \leq G_i$ is an irreducible lattice. We consider the unipotent 1-parameter subgroup

$$\mathbf{u}_i^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \times \mathbf{e},$$

where $\mathbf{e} = \mathbf{e}_i \in G_i'$ is the identity. We remark that $\mathbf{u}_i^t$ commutes with multiplication by elements in G_i'. We let

$$\mathbf{a}_i^t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \times \mathbf{e}, \qquad \text{and} \qquad \overline{\mathbf{u}}_i^t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \times \mathbf{e},$$

and denote by U_i, A_i and $\overline{U}_i$ the corresponding elements in the Lie algebras $\mathfrak{g}_i$. Note that any element $\mathbf{g}_i \in N_{G_i}(\mathbf{u}_i)$ of the normaliser $N_{G_i}(\mathbf{u}_i)$ of $\mathbf{u}_i$ can be written as $\mathbf{g}_i = \mathbf{a}_i^a \overline{\mathbf{u}}_i^{\overline{u}} \mathbf{g}_i'$, where $\mathbf{g}_i' \in G_i'$ and $a, \overline{u} \in \mathbb{R}$.

7.1. Convergence of ψ_{t_n}.

In the previous section, in order to ensure the convergence, up to a subsequence, of ψ_{t_n} we had to pass to joinings, and then exploit the fact that sequences of (tight) measures have limit points. In the more restricted setting of this section, we can prove directly, following Ratner's original strategy in [27], that ψ_{t_n} converges. The key difference is that now we know that the *only* direction which is expanded, when we push by the Cartan elements $\mathbf{a}_i$, is the one of the unipotent flow.

7.1.1. We recall that, in §5.0.1 we have constructed, for any $n \in \mathbb{N}$ a set $K_{\mathbf{u}_1}(n) \subset M_1$ of measure greater than $1 - 2^{-n}$ such that, if $x \in K_{\mathbf{u}_1}(n)$ and $s \geq s_n$, then the $\mathbf{u}_1$-orbit of x enters $\mathcal{G}$ before time s_n, and actually belongs to $\mathcal{G}$ for a large frequency of times.

Fix a constant $0 < \gamma < \frac{\eta}{2}$ and let $t_n = \log s_n^{1+\gamma}$. As in §6 we will consider the sequence of isomorphisms given by $\psi_{t_n}(x) = \psi(x\mathbf{a}_1^{-t_n})\mathbf{a}_2^{t_n}$. Let

$$V = \bigcap_{n \in \mathbb{N}} \phi_{A_1}^{t_n}(K_{\mathbf{u}_1}(n)). \tag{43}$$

Since $\mu_1(K_{\mathbf{u}_1}(n)) > 1 - 2^{-n}$, then $\mu_1(V) > \frac{1}{2}$.

We now show that the restriction to V of ψ_{t_n} "commutes in the limit" with the multiplication by elements in the normaliser of $\mathbf{u}_1$.

Lemma 7.1.1. *Let $\mathbf{g}_1 \in N_{G_1}(\mathbf{u}_1)$ with $d(\mathbf{a}, g_1) \leq \delta$. There exists $\Phi(\mathbf{g}_1) \in N_{G_2}(\mathbf{u}_2)$ with $d(\mathbf{e}, \Phi(\mathbf{g}_1)) \leq \varepsilon$ such that for any $x \in V$, and any $y = x\mathbf{g}_1 \in V$ we have*

$$d(\psi_{t_n}(x)\Phi(\mathbf{g}), \psi_{t_n}(y)) \to 0,$$

as $n \to \infty$.

Proof. We can write $\mathbf{g}_1 = \mathbf{g}_1' \mathbf{a}_1^a \mathbf{u}_1^u$ for some $|a|, |u| \leq \delta$ and $\mathbf{g}_1' \in G_1'$ with $d(\mathbf{e}, \mathbf{g}_1') \leq \delta$. Define $z = x\mathbf{g}_1'\mathbf{a}_1^a$.

Let $x_n = x\mathbf{a}_1^{-t_n}$, $z_n = z\mathbf{a}_1^{-t_n} = x_n\mathbf{g}_1'\mathbf{a}_1^a$, and $y_n = y\mathbf{a}_1^{-t_n} = z_n\mathbf{u}_1^{s_n^{1+\gamma}u}$. Since $x_n \in K_{\mathbf{u}_1}(n)$, Proposition 5.1.1 and the discussion after it imply that we can write

$$\psi(z_n) = \psi(x_n)\Phi(\mathbf{g}_1'\mathbf{a}_1^a)\mathbf{u}_2^{\beta_n},$$

where $|\beta_n| \leq s_n$ and $\Phi(\mathbf{g}_1'\mathbf{a}_1^a) \in \exp(\mathbb{R}A_2 \oplus \mathfrak{g}_2')$. Therefore, since $\psi(y_n) = \psi(z_n)\mathbf{u}_2^{z(z_n,s_n^{1+\gamma}u)}$, we deduce that

$$\psi(y_n)\,\mathbf{a}_2^{t_n} = \psi(x_n)\mathbf{a}_2^{t_n}\Phi(\mathbf{g}_1'\mathbf{a}_1^a)\mathbf{u}_2^{s_n^{-(1+\gamma)}(\beta_n+z(z_n,s_n^{1+\gamma}u))}.$$

Note that $|s_n^{-(1+\gamma)}\beta_n| \to 0$ as $n \to \infty$, hence we can focus on the term $s_n^{-(1+\gamma)}z(z_n, s_n^{1+\gamma}u)$.

By definition, since $z_n \in K_{\mathbf{u}_1}(n)$, there exists $0 \leq r_n \leq s_n$ such that $z_n\mathbf{u}_1^{r_n} \in \mathcal{G}$. Using the cocycle relation of Lemma 2.2.2 and the properties of the set $\mathcal{G}$ in Proposition 4.1.1, we deduce that

$$\begin{aligned}|z(z_n, s_n^{1+\gamma}u) - s_n^{1+\gamma}u| &\leq |z(z_n, r_n)| + |z(z_n\mathbf{u}_1^{r_n}, s_n^{1+\gamma}u - r_n) - s_n^{1+\gamma}u|\\ &\leq C_\alpha^2 s_n + r_n + C_\alpha^{-4}(s_n^{1+\gamma}u - r_n)^{1-\eta}.\end{aligned}$$

Thus, it follows that $s_n^{-(1+\gamma)}z(z_n, s_n^{1+\gamma}u) \to u$. We conclude that

$$d(\psi_{t_n}(y), \psi_{t_n}(x)\Phi(\mathbf{g}_1'\mathbf{a}_1^a)\mathbf{u}_2^u) \leq |s_n^{-(1+\gamma)}z(z_n, s_n^{1+\gamma}u) - u| \to 0,$$

which proves the lemma. □

Lemma 7.1.2. *For any $x \in V$ and $y = x\overline{\mathbf{u}}_1^{\delta_0} \in V$, with $|\delta_0| < \delta$, we have*

$$d(\psi_{t_n}(x)\overline{\mathbf{u}}_2^{\delta_0}, \psi_{t_n}(y)) \to 0,$$

as $n \to \infty$.

The proof of this result follows the same strategy and actually follows quite closely the proof of Proposition 5.0.1. We will use the parametrisation $q(t)$ we defined in §3.1.4.

Proof. Let $x_n = x\mathbf{a}_1^{-t_n} \in K_{\mathbf{u}_1}(n)$ and $y_n = y\mathbf{a}_1^{-t_n} = x_n\overline{\mathbf{u}}_1^{\delta_n} \in K_{\mathbf{u}_1}(n)$, where $\delta_n = \delta_0 s_n^{-1-\gamma}$. Let $t^* \in [0, s_n]$ the first time that both the $\mathbf{u}_1$-orbit of x_n and y_n belong to $\mathcal{G}$. Call $v_n = x_n\mathbf{u}_1^{t^*}$ and $w_n = y_n\mathbf{u}_1^{t^*}$. Then, we have that $w_n = v_n\mathbf{a}_1^a\overline{\mathbf{u}}_1^{\bar{u}}$ for some very small a and $\bar{u}$. More precisely, using (19), we have that

$$a = 2\log(1 - \delta_n t^*), \qquad \bar{u} = \delta_n(1 - \delta_n t^*), \tag{44}$$

in particular, $\bar{u} \leq \delta_n$. Using the notation of §3.1.4, the parametrization $q(t)$ is defined by

$$v_n\mathbf{u}_1^t\exp(a(t)A_1)\exp(\bar{u}(t)\overline{U}_1) = w_n\mathbf{u}_1^{q(t)}.$$

where, by (20),

$$q(t) = t - \frac{\bar{u}t^2 + t(1 - e^a)}{(e^a - \bar{u}t)}.$$

Call

$$B_n = \{t \in [0, s_n] : v_n\mathbf{u}_1^t \in \mathcal{G} \text{ and } w_n\mathbf{u}_1^{q(t)} \in \mathcal{G}\},$$

which is a compact subset of $[0, s_n]$. We remark that $0 \in B_n$.

The fact that $x_n \in K_{\mathbf{u}_1}(n)$ implies that its $\mathbf{u}_1$-orbit is outside of $\mathcal{G}$ for at most $\frac{1}{40C_\alpha^4}s_n$ time, and the same holds for y_n. Since we are assuming that t^* is the first time both their orbits are in $\mathcal{G}$, then the orbits of v_n and w_n, under $\mathbf{u}_1$, spend at least $(1-\frac{1}{20C_\alpha^4})ss_n$ of their time inside $\mathcal{G}$. Finally, as $|q'(t)-1| \leq \delta_0 s_n^{-\gamma}$, which is arbitrarily small if n is sufficiently large, we deduce that

$$\frac{|B_n|}{s_n} \geq \left(1 - \frac{\theta}{18C_\alpha^4}\right). \tag{45}$$

By definition, using again (19), we have that

$$d(v_n \mathbf{u}_1^t, w_n \mathbf{u}_1^{q(t)}) \leq |a(t)| + |\bar{u}(t)| \leq 4\delta_n s_n \leq \delta$$

for all $t \in B_n$. Therefore the points $v_n' = \psi(v_n)$ and $w_n' = \psi(w_n)$ satisfy

$$v_n' \mathbf{u}_2^{z(v_n,t)} \in \psi(\mathcal{G}), \qquad w_n' \mathbf{u}_2^{z(w_n,q(t))} \in \psi(\mathcal{G}),$$

and

$$d(v_n' \mathbf{u}_2^{z(v_n,t)}, w_n' \mathbf{u}_2^{z(w_n,q(t))}) \leq \rho, \qquad \text{for all } t \in B_n.$$

Let us consider the point $\overline{v}_n' = v_n' \mathbf{a}_2^a \overline{\mathbf{u}}_2^{\bar{u}}$. As before, we have that

$$d(\overline{v}_n' \mathbf{u}_2^{q(t)}, v_n' \mathbf{u}_2^t) \leq \rho,$$

for all $t \in B_n$, where the function $q(t)$ is the same as above.

Let $t = t(r)$ be the Lipschitz function with $t(0) = 0$ uniquely defined by $r = z(w_n, q(t))$, and let

$$\chi(r) = q(z(v_n, t(r))).$$

It is easy to see that also χ is Lipschitz and satisfies $\chi(0) = 0$. Then, by the triangle inequality,

$$d(\overline{v}_n' \mathbf{u}_2^{\chi(r)}, w_n' \mathbf{u}_2^r) \leq \rho, \qquad \text{for all } r \in B_n' := z(w_n, q(B_n)).$$

We remark that B_n' is compact and that $0 \in B_n'$.

Let $t_1 = \sup B_n$ and $\lambda_n = z(w_n, q(t_1))$. From (45) we have

$$\left(1 - \frac{\theta}{18C_\alpha^4}\right) s_n \leq t_1 \leq s_n.$$

By the uniform Lipschitz estimate on z, this yields

$$C_\alpha^{-2}\left(1 - \frac{\theta}{18C_\alpha^4}\right) s_n \leq \lambda_n \leq C_\alpha^2 s_n,$$

and

$$\frac{|B_n'|}{\lambda_n} \geq 1 - \frac{\theta}{16}.$$

We let

$$A_n = \{0\} \cup (B_n' \cap [m, s_n]).$$

Then, if n is sufficiently large, we have

$$\frac{|A_n|}{\lambda_n} \geq 1 - \frac{\theta}{8}.$$

Since $r \in B_n'$ if and only if $t(r) \in B_n$, then $w_n' \mathbf{u}_2^r = w_n' \mathbf{u}_2^{z(w_n,q(t))} \in \psi(\mathcal{G})$ and hence statisfies the Injectivity Condition IC(ρ, m) and the Frequently Bounded Radius Condition FBR$(T_0, c(b), r_0)$.

We want to apply the Basic Lemma, to the parametrization χ and the points w_n' and $\overline{v}_n'$, with the exponent γ instead of η in Lemma 4.2.1. It only remains to verify Condition 4. To do so, let $r', r \in B_n'$ and assume that either $r' - r > m$ or that

$\chi(r') - \chi(r) > m$. Let $t' = t(r')$ and $t = t(r)$ be the corresponding times in B_n. We claim that we have that

$$q(t') - q(t) > m_0, \qquad \text{and} \qquad t' - t > m_0. \tag{46}$$

Assuming the claim for a moment, let us estimate $|(\chi(r') - \chi(r)) - (r' - r)|$.

By the cocycle relation, we have that

$$r' - r = z(w_n, q(t')) - z(w_n, q(t)) = z(w_n \mathbf{u}_1^{q(t)}, q(t') - q(t)), \tag{47}$$

with $w_n \mathbf{u}_1^{q(t)} \in \mathcal{G}$. Then, by Corollary 2.1.5 applied to z (see also the proof of Lemma 5.0.2), and using our claim, we obtain

$$\begin{aligned} z(w_n \mathbf{u}_1^{q(t)}, q(t') - q(t)) &\leq (q(t') - q(t)) + \frac{1}{C_\alpha^4}(q(t') - q(t))^{1-\eta} \\ &\leq q'(\xi)\left((t' - t) + \frac{1}{C_\alpha^4}(t' - t)^{1-\eta}\right), \end{aligned}$$

for some $\xi \in [0, s_n]$.

Similarly we have, for some $\tilde{\xi} \in [0, s_n]$, that

$$\begin{aligned} \chi(r') - \chi(r) &= q(z(v_n, t')) - q(z(v_n, t)) \\ &= q'(\tilde{\xi})(z(v_n, t') - z((v_n, t)) \\ &= q'(\tilde{\xi}) z(v_n \mathbf{u}_1^t, t' - t) \\ &\leq q'(\tilde{\xi})\left((t' - t) + \frac{1}{C_\alpha^4}(t' - t)^{1-\eta}\right), \end{aligned} \tag{48}$$

since $v_n \mathbf{u}_1^t \in \mathcal{G}$.

Putting the previous inequalities together, we obtain

$$|(\chi(r') - \chi(r)) - (r' - r)| \leq (t' - t) \cdot |q'(\xi) - q'(\tilde{\xi})| + \frac{4}{C_\alpha^4}(t' - t)^{1-\eta}. \tag{49}$$

Using again the mean value theorem, we obtain a point ξ^* such that $|q'(\xi) - q'(\tilde{\xi})| = |\tilde{\xi} - \xi| \cdot |q''(\xi^*)|$. By a direct computation,

$$|q''(\xi^*)| = \frac{2\bar{u}e^a}{(e^a - \bar{u}\xi^*)^3} \leq 2\delta_n, \tag{50}$$

where we used the estimates (44) and the fact that $\xi^* \in [0, s_n]$ to deduce that $\bar{u}\xi^* \leq \delta_0 s_n^{-\gamma}$ and hence the denominator is between 0 and 1.

Moreover, since $\tilde{\xi}$ and ξ belong to the interval $[t, t']$ and both are lesser than s_n, we have that

$$\delta_n = \delta_0 s_n^{-1-\gamma} \leq \delta_0 \cdot |\tilde{\xi} - \xi|^{-1} \cdot (t' - t)^{-\gamma}. \tag{51}$$

Plugging (50) and (51) into (49), we get

$$|(\chi(r') - \chi(r)) - (r' - r)| \leq 2\delta_0 (t' - t)^{1-\gamma} + \frac{4}{C_\alpha^4}(t' - t)^{1-\eta}.$$

We now go back from $(t' - t)$ to $(r' - r)$. Using (47) and the Lipschitz estimate on z we have

$$\frac{1}{2C_\alpha^2}(t' - t) \leq \frac{1}{C_\alpha^2}(q(t') - q(t)) \leq r' - r \leq C_\alpha^2(q(t') - q(t)) \leq 2C_\alpha^2(t' - t). \tag{52}$$

Thus, we have shown that

$$|(\chi(r') - \chi(r)) - (r' - r)| \leq 4(r' - r)^{1-\gamma},$$

which proves Condition 4 of the Basic Lemma 4.2.1, as we wanted.

It remains to prove that, if $r' - r > m$ or $\chi(r') - \chi(r) > m$ then (46) holds. In the former case, from (52) we have

$$t' - t \geq \frac{r' - r}{2C_\alpha^2} > \frac{m}{2C_\alpha^2} > m_0.$$

Similarly, since $\frac{1}{2} \leq q'(\tilde{\xi}) \leq 2$,

$$q(t') - q(t) = q'(\xi)(t' - t) \geq \frac{t' - t}{2} \geq \frac{r' - r}{4C_\alpha^2} > \frac{m}{4C_\alpha^2} > m_0.$$

The latter case, when $\chi(r') - \chi(r) > m$, can be dealt similarly using (48) to obtain

$$\frac{1}{2C_\alpha^2}(t' - t) \leq \chi(r') - \chi(r) \leq 2C_\alpha^2(t' - t).$$

Having verified all the assumptions of the Basic Lemma 4.2.1, we conclude that there exists a $T \in [0, s_n]$ such that

$$\overline{v}'_n \mathbf{u}_2^T = w'_n \mathbf{u}_2^{\chi(T)} \mathbf{g}_c \exp(a_T A_2) \exp(\bar{u}_T \overline{U}_2) \mathbf{u}_2^{\varepsilon_T},$$

with $\mathbf{g}_c \in G'_2$ such that $d(\mathbf{e}, \mathbf{g}_c) < \varepsilon$, and

$$|a_T| \leq C s_n^{-\frac{\eta}{2}}, \qquad |\bar{u}_T| \leq C s_n^{-1-\frac{\eta}{2}}, \qquad |\varepsilon_T| \leq \varepsilon. \tag{53}$$

By the definition of $\overline{v}'_n$, we have

$$v'_n \mathbf{a}_2^a \overline{\mathbf{u}}_2^{\bar{u}} = w'_n \mathbf{u}_2^{\chi(T)} \mathbf{g}_c \mathbf{a}_2^{a_T} \overline{\mathbf{u}}_2^{\bar{u}_T} \mathbf{u}_2^{\varepsilon_T - T},$$

applying $\mathbf{a}_2^{t_n}$ to both side, this yields

$$v'_n \mathbf{a}_2^{a+t_n} \overline{\mathbf{u}}_2^{\bar{u} s_n^{1+\gamma}} = w'_n \mathbf{a}_2^{t_n} \mathbf{u}_2^{\chi(T) s_n^{-(1+\gamma)}} \mathbf{g}_c \mathbf{a}_2^{a_T} \overline{\mathbf{u}}_2^{\bar{u}_T s_n^{1+\gamma}} \mathbf{u}_2^{(\varepsilon_T - T) s_n^{-(1+\gamma)}}.$$

Using the definition of v'_n, the left hand side is equal to

$$\psi_{t_n}\left(x \mathbf{u}_1^{t^* e^{-t_n}}\right) \mathbf{a}_2^a \overline{\mathbf{u}}_2^{\bar{u} s_n^{1+\gamma}}.$$

Since $t^* \leq s_n$, using the definition of t_n and (44), we have

$$d\left(\psi_{t_n}\left(x \mathbf{u}_1^{t^* e^{-t_n}}\right) \mathbf{a}_2^a \overline{\mathbf{u}}_2^{\bar{u} s_n^{1+\gamma}}, \psi_{t_n}(x) \overline{\mathbf{u}}_2^{\delta_0}\right) \to 0$$

as $n \to \infty$. In a similar fashion,

$$w'_n \mathbf{a}_2^{t_n} = \psi_{t_n}\left(y \mathbf{u}_1^{t^* e^{-t_n}}\right),$$

which approaches $\psi_{t_n}(y)$ for large n. Finally, by the estimates (53), since $\gamma < \frac{\eta}{2}$,

$$d\left(\psi_{t_n}(y), \mathbf{u}_2^{\chi(T) s_n^{-(1+\gamma)}} \mathbf{g}_c \mathbf{a}_2^{a_T} \overline{\mathbf{u}}_2^{\bar{u}_T s_n^{1+\gamma}} \mathbf{u}_2^{(\varepsilon_T - T) s_n^{-(1+\gamma)}}\right) \to 0,$$

as $n \to \infty$. In summary, we have obtained

$$d(\psi_{t_n}(x) \overline{\mathbf{u}}_2^{\delta_0}, \psi_{t_n}(y)) \to 0,$$

as we wanted. $\square$

We want to show that the sequence of isomorphisms ψ_{t_n} converges to a function ζ, which we will later show to be a measurable conjugacy of the unipotent flows $\phi^t_{U_1}$ and $\phi^t_{U_2}$. Using elementary arguments, given a point $x \in M_1$, we could find a subsequence, depending on the point, such that $\psi_{t_n}(x)$ converges along that subsequence. However, thanks to the previous Lemmas, we can obtain a subsequence which is *independent* of the point, and ensure convergence of ψ_{t_n} on a full measure set, as we now show.

Corollary 7.1.3. *There exist a $\phi^t_{U_1}$-invariant set $\Omega \subset M_1$ with $\mu_1(\Omega) = 1$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ such that, if $x \in \Omega$ then $\lim_{k\to\infty} \psi_{t_{n_k}}(x) =: \zeta(x) \in M_2$ exists and $\zeta(x\mathbf{u}^t_1) = \zeta(x)\mathbf{u}^t_2$, for all $x \in \Omega$ and $t \in \mathbb{R}$.*

Proof. We begin by showing that, on a set of full measure, the sequence of isomorphisms ψ_n has a converging subsequence.

Let K_n be an exhaustion of compact sets of M_2 such that $\mu_2(M_2 \setminus K_n) < 2^{-n}$, for all $n \in \mathbb{N}$. Denote $K^{\complement}_n$ the complement of K_n inside M_2 and

$$F_n = \phi^{t_n}_{A_1} \circ \psi^{-1} \circ \phi^{-t_n}_{A_2}(K^{\complement}_n).$$

Since $\sum_{n=1}^{\infty} \mu_1(F_n) < \infty$, by the Borel-Cantelli lemma the set

$$F = \{x \in M_1 : x \text{ belongs to finitely many } F_n\}$$

has full μ_1-measure. If $x \in F$, then $\psi_{t_n}(x)$ belongs to finitely many sets $K^{\complement}_n$, so there exists a subsequence $n_k = n_k(x)$ such that $\psi_{t_{n_k}}(x)$ converges in M_2.

Since the set V, defined in (43), has positive measure, and F has full measure, we can assume $V \subset F$. Moreover, there exists a point $x^* \in V$ such that, if $W(x^*, \delta)$ is the local stable leaf of x^*, then $V \cap W(x^*, \delta)$ has positive measure, with respect to the natural Riemannian volume on the leaf itself. As $x^* \in F$, there exists a subsequence $n_k = n_k(x^*)$ such that $\psi_{t_{n_k}}(x^*)$ converges in M_2. Let

$$\Omega = \{x\mathbf{u}^t_1, t \in \mathbb{R}, x \in V \cap W(x^*, \delta)\}.$$

By construction, the set Ω is invariant under the flow defined by $\mathbf{u}^t_1$, and has positive μ_1-measure. By ergodicity, $\mu_1(\Omega) = 1$.

Thanks to Lemma 7.1.1 and Lemma 7.1.2, we can define $\zeta(x) = \lim_{k\to\infty} \psi_{t_{n_k}}(x)$, for every $x \in \Omega$, which satisfies $\zeta(x\mathbf{u}^t_1) = \zeta(x)\mathbf{u}^t_2$ for all $t \in \mathbb{R}$ and $x \in \Omega$. □

We can now complete the proof of Theorem B.

Proof of Theorem B. Let Ω and n_k be given by Corollary 7.1.3. We assume, for simplicity, that $\Omega = M_1$ and $n_k = n$. Then, for every $x \in M_1$ we can define the function

$$\zeta(x) := \lim_{n\to\infty} \psi_{t_n}(x),$$

which satisfies

$$\zeta(x\mathbf{u}^t_1) = \zeta(x)\mathbf{u}^t_2,$$

for all $x \in M_1$ and all times $t \in \mathbb{R}$. In other words, $\zeta \colon M_1 \to M_2$ is a measurable conjugacy between the two unipotent flows $\phi^t_{U_1}$ and $\phi^t_{U_2}$. By the Rigidity Theorem in [29], there exists an isomorphism $\varpi \colon G_1 \to G_2$, with $\varpi(\Gamma_1) \subset \overline{\mathbf{g}}^{-1}\Gamma_2\overline{\mathbf{g}}$, such that $\zeta(\Gamma_1\mathbf{x}) = \Gamma_2\overline{\mathbf{g}}\varpi(\mathbf{x})$, for μ_1-a.e. $\Gamma_1\mathbf{x}$. From here, the proof follows verbatim the one of Theorem A. □

Appendix A. A consequence of Chevalley's Lemma

In this short appendix we prove a more precise version of Lemma 3.1.1. We begin by recalling the setup.

Let $H < G$ be a Zariski closed subgroup of G isomorphic to $\mathrm{SL}_2(\mathbb{R})$, and let $\mathfrak{s} < \mathfrak{g}$ be its Lie algebra. Let us write $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{s}$, where $\mathfrak{m}$ is a $\mathfrak{s}$-submodule of $\mathfrak{g}$. Then, any $\mathbf{g} \in G$ with $d(\mathbf{e}, \mathbf{g}) < 1$ can be written in a unique way as $\mathbf{g} = \exp(Y)\mathbf{h} \in G$, with $\mathbf{h} \in H$ and $Y \in \mathfrak{m}$ such that $d(\mathbf{e}, \mathbf{h}) < 1$ and $d(\mathbf{e}, \exp(Y)) < 1$. In the following lemma, we prove the converse result, namely we will show that if an element $\mathbf{g} \in G$ sufficiently close to the identity can be written as $\mathbf{g} = \exp(Y)\mathbf{h}$ with $\mathbf{h} \in H$ and $Y \in \mathfrak{m}$, then automatically we have that $\exp(Y)$ and $\mathbf{h}$ are close to the identity in G.

Lemma A.0.1. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathbf{g} = \exp(Y)\mathbf{h} \in G$, with $\mathbf{h} \in H$ and $Y \in \mathfrak{m}$, is such that $d(\mathbf{e}, \mathbf{g}) < \delta$, then $d(\mathbf{e}, \mathbf{h}) < \varepsilon$ and $d(\mathbf{e}, \exp(Y)) < \varepsilon$.*

In the proof of this result, we will use Chevalley's Lemma, whose proof can be found in, e.g., [4, 7.9] or [43, 3.1.4].

Lemma A.0.2 (Chevalley's Lemma)**.** *Let G be a linear algebraic group over $\mathbb{C}$ and let $H < G$ be a Zariski closed subgroup. Assume that H is semisimple. There exist a rational representation $\rho \colon G \to \mathrm{GL}(V)$, where V is a finite-dimensional vector space, and a vector $v \in V$ such that $H = \mathrm{Stab}_G(v)$.*

A.0.1. *Proof of Lemma A.0.1.* Let V, v, and ρ be given by Chevalley's Lemma. In the following, we will identify the tangent space T_vV at v with V, so that we can simply write the exponential map $\exp_{T_vV} \colon T_vV \simeq V \to V$ as $\exp_{T_vV}(w) = v + w$. Let us also fix a norm $\| \cdot \|$ on V, and for simplicity assume $\|v\| = 1$.

Call $e_v \colon \mathrm{GL}(V) \to V$ the evaluation map $e_v(L) = Lv$. Note that we can write its differential $\mathrm{d}e_v \colon \mathfrak{gl}(V) \to V$ as $\mathrm{d}e_v(A) = \exp_{\mathfrak{gl}(V)}(A)v - v$. Indeed, if $A \in \mathfrak{gl}(V)$, then we have

$$\mathrm{d}e_v(A) + v = (\exp_{T_vV} \circ \mathrm{d}e_v)(A) = (e_v \circ \exp_{\mathfrak{gl}(V)})(A) = \exp_{\mathfrak{gl}(V)}(A)v,$$

which proves the claim.

Define $\Phi = e_v \circ \rho \colon G \to V$. Then, by the chain rule, we can write its differential $\mathrm{d}\Phi \colon \mathfrak{g} \to T_vV \simeq V$ as

$$\mathrm{d}\Phi \colon X \mapsto \exp_{\mathfrak{gl}(V)}(\mathrm{d}\rho(X))v - v = \rho(\exp(X)).v - v.$$

It is immediate to see that $\mathrm{d}\Phi(X) = 0$ if and only if $\rho(\exp(X)).v = v$; that is, if and only if $\exp(X) \in \mathrm{Stab}_G(v) = H$. This shows that $\ker \mathrm{d}\Phi = \mathfrak{s}$. Identifying $\mathfrak{m}$ with $\mathfrak{g}/\mathfrak{s}$, we obtain a linear isomorphism (which, by a little abuse of notation, we still denote by $\mathrm{d}\Phi$) between $\mathfrak{m}$ and the image $\mathrm{d}\Phi(\mathfrak{g})$.

Let now $\varepsilon > 0$ be fixed. There exists $\delta_0 > 0$ such that if $\|X\|_{\mathfrak{g}} < \delta_0$ then $d(\mathbf{e}, \exp(X)) < \varepsilon/2$. Since the inverse $(\mathrm{d}\Phi)^{-1}$ is a linear, and hence continuous, map, there exists $\delta_1 > 0$ such that for all $w \in \mathrm{d}\Phi(\mathfrak{g})$ with $\|w\| \leq \delta_1$, we have $\|(\mathrm{d}\Phi)^{-1}w\|_{\mathfrak{g}} \leq \delta_0$. Finally, by continuity of ρ, let $\delta > 0$ be such that if $\mathbf{g} \in G$ is such that $d(\mathbf{e}, \mathbf{g}) < \delta$, then $\|\rho(\mathbf{g}) - \mathrm{Id}\| < \delta_1$. Without loss of generality, we can assume that $\delta < \varepsilon/2$.

Let $\mathbf{g} = \exp(Y)\mathbf{h} \in G$ be such that $d(\mathbf{e}, \mathbf{g}) < \delta$. Then, we have

$$\|\rho(\exp(Y)).v - v\| = \|\rho(\exp(Y))\rho(\mathbf{h}).v - v\| = \|\rho(\mathbf{g}).v - v\| \leq \|\rho(\mathbf{g}) - \mathrm{Id}\| < \delta_1,$$

and, clearly, $\rho(\exp(Y)).v - v = \mathrm{d}\Phi(Y) \in \mathrm{d}\Phi(\mathfrak{g})$. Since $Y \in \mathfrak{m}$, we deduce that $Y = (\mathrm{d}\Phi)^{-1}(\mathrm{d}\Phi(Y))$, and therefore

$$\|Y\|_{\mathfrak{g}} = \|(\mathrm{d}\Phi)^{-1}(\rho(\exp(Y)).v - v)\| < \delta_0.$$

We conclude that $d(\mathbf{e}, \exp(Y)) < \varepsilon/2$ and $d(\mathbf{e}, \mathbf{h}) = d(\mathbf{e}, \exp(-Y)\mathbf{g}) \leq \delta + \varepsilon/2 < \varepsilon$, hence the proof is complete.

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