

ON UNKNOTTING FIBERED POSITIVE KNOTS AND BRAIDS

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ABSTRACT. The unknotting number u and the genus g of braid positive knots are equal, as shown by Rudolph. We prove the stronger statement that any positive braid diagram of a genus g knot contains g crossings, such that changing them produces a diagram of the trivial knot. Then, we turn to unknotting the more general class of fibered positive knots, for which $u = g$ was conjectured by Stoimenow. We prove that the known ways to unknot braid positive knots do not generalize to fibered positive knots. Namely, we prove that there are fibered positive knots that cannot be unknotted optimally along fibered positive knots; there are fibered positive knots that do not arise as trefoil plumbings; and there are positive diagrams of fibered positive knots of genus g that do not contain g crossings, such that changing them produces a diagram of the trivial knot. In fact, we conjecture that one of our examples is a counterexample to Stoimenow's conjecture.

1. INTRODUCTION

This paper is inspired by the following conjecture posed by Stoimenow.

Conjecture 1.1 [Sto03] *If K is a positive and fibered knot, then its unknotting number $u(K)$ equals its 3-genus $g(K)$.*

Note that the inequality $u \geq g$ holds for all strongly quasipositive knots [Rud93], so the open part of the conjecture is to show $u \leq g$. While discussing the conjecture, we refer to Figure 2 for context. We will begin by examining the unknotting process of knots formed as the closure of positive braids, referred to as *braid positive knots*. These knots are both fibered [Sta78] and positive, and they satisfy $u = g$, i.e. the conjecture holds for them. An elegant proof of $u \leq g$ for braid positive knots is due to Rudolph [Rud83]. That proof is by induction, with the following induction step: every non-trivial braid positive knot K is related by a crossing change to another braid positive knot J such that $g(J) = g(K) - 1$. In other words, braid positive knots may be unknotted optimally along braid positive knots. Alternatively, the inequality $u(K) \leq g(K)$ for braid positive knots K may be

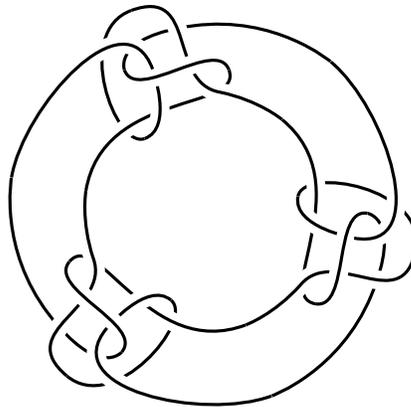


FIGURE 1. A potential counterexample \mathcal{K} to Conjecture 1.1.

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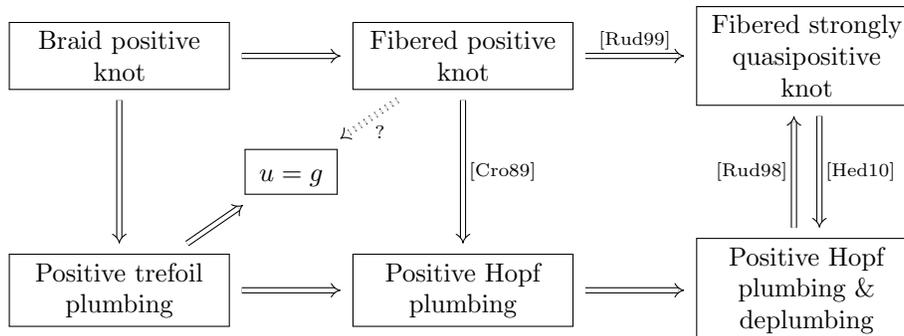


FIGURE 2. Implications between some properties of knots. No further implications hold among these properties, with the only potential exception of Conjecture 1.1, drawn dotted. For example, see [MM86] for a fibered strongly quasipositive knot that is not a positive Hopf plumbing.

deduced from the fact that such knots are *positive trefoil plumbings* [BD16], i.e. the fiber surface of K arises from a disk by finitely many successive plumbings of the fiber surface of the positive trefoil knot. The inequality $u(K) \leq g(K)$ holds for all positive trefoil plumbings K , since the genus is additive under knot plumbings and the effect of deplumbing a trefoil from a knot can be achieved by a crossing change (see Figure 6).

Now, we present a third proof of $u(K) \leq g(K)$ for braid positive knots K , as a corollary of the following theorem, which we prove in Section 2.

Theorem 1.2 *Let D be a knot diagram arising as the closure of a positive braid, and let g be the 3-genus of the represented knot. Then there exists a set of g crossings in D such that changing those crossings produces a diagram of the unknot.*

So far, we have discussed three different proofs that $u \leq g$ holds for braid positive knots. The following three theorems (proven respectively in Sections 3, 4 and 5) now demonstrate that each one of those proofs fails to generalize to fibered positive knots. Namely, fibered positive knots may not be unknotted optimally along fibered positive knots, they need not be positive trefoil plumbings, and they may admit positive diagrams that cannot be unknotted by performing g many crossing changes.

Theorem 1.3 *Let K be the fibered positive knot $K11n183$ shown in Figure 5, which satisfies $u(K) = g(K) = 3$. Then there is no crossing change relating K with a fibered positive knot J with $u(J) = 2$.*

Theorem 1.4 *The fiber surface of $K11n183$ is not a positive trefoil plumbing.*

Theorem 1.5 *Let D be the positive knot diagram shown in Figure 1. Then D represents a fibered positive knot \mathcal{K} of genus 7. However, the least number of crossings of D one needs to change to produce a diagram of the unknot equals 9.*

In fact, we make the following conjecture.

Conjecture 1.6 *The fibered positive knot \mathcal{K} shown in Figure 1 has unknotting number 9 and is thus a counterexample to Conjecture 1.1.*

In Section 5, we shall see that, however, no known lower bound for the unknotting number appears to be strong enough to settle Conjecture 1.6. We also present an infinite family of potential counterexamples for Conjecture 1.1, generalizing the knot \mathcal{K} .

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2. UNKNOTTING OF POSITIVE BRAIDS

This section is devoted to the proof of Theorem 1.2, stating that knots given by a positive braid diagram may be optimally unknotted within that diagram. An example is shown in Figure 4 from which the general argument can be deduced easily.

Proof of Theorem 1.2. Let K be a braid positive knot and D a positive braid diagram representing K ; see Figure 3(a) for an example. We may think of D as an oriented immersed self-transverse circle in \mathbb{R}^2 . The positivity of the crossings is then enough to determine the over-/undercrossing information at each double point. Note that the Seifert circles d_1, d_2, \dots, d_n that arise when Seifert's algorithm is applied

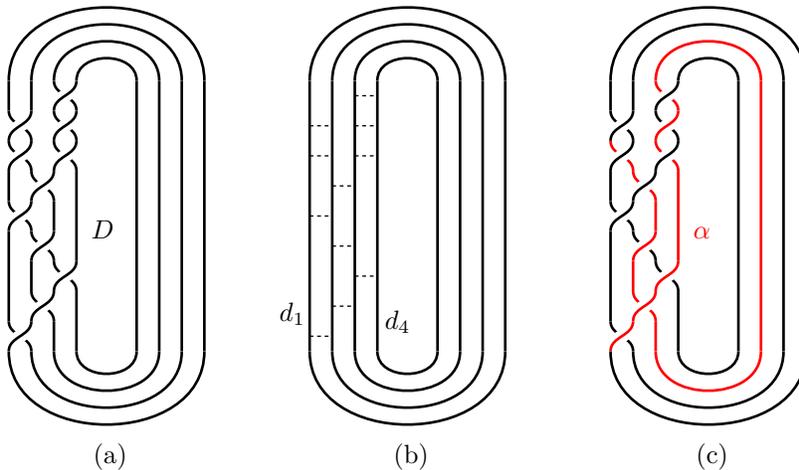


FIGURE 3. (a) Example of a diagram D representing a braid positive knot K . (b) Seifert circles obtained when applying Seifert's algorithm to D . (c) A sub-arc $\alpha \subset D$ with a single double point.

to D are nested, as in Figure 3(b). We choose the numbering in such a way that the disk bounded by d_i contains d_{i+1} , for all $i \in \{1, 2, \dots, n-1\}$. In particular, d_1 is the outermost and d_n is the innermost Seifert circle. Note that $d_i \cap D$ consists of finitely many sub-arcs of d_i , separated by the crossings that connect d_i with the adjacent Seifert circles.

We proceed by induction on the number of crossings of D . If D has no crossings, or, more generally, if K is the unknot, we are done. Hence, assume that K is non-trivial. Let $\alpha \subset D$ be an arc such that $\alpha \cap d_1$ has exactly two connected components, each containing one of the endpoints of α . In other words, α is the projection of an arc in K , starts at a point of d_1 , travels along D in the given orientation, and stops the first time it comes back to d_1 . An example of such an arc is shown in Figure 3(c). Since D is a positive braid diagram, every passage of α from a Seifert circle d_i to d_{i+1} through a double point is an overpass and every passage of α from d_{i+1} to d_i is an underpass. This implies that α contains the same number $\ell \geq 1$ of over- and underpasses. Note that there does indeed exist such a path α because K is non-trivial; in particular, there are at least two Seifert circles and at least one crossing connecting d_1 to d_2 .

First, consider the case in which every double point of D is traversed at most once by α . Changing the crossings that correspond to the ℓ underpasses on α , turns α into an arc that only contains overpasses. A planar isotopy can then be applied to remove all 2ℓ crossings encountered by α , preserving the rest of the diagram. We thus obtain a new positive braid diagram D' with 2ℓ fewer crossings than D . The knot K' represented by D' differs from K by ℓ crossing changes. When applying Seifert's algorithm to D' , we still obtain the same number n of Seifert circles; therefore $g(K') + \ell = g(K)$. By induction, we can find $g(K')$ crossing changes within D' which unknot K' . These $g(K')$ crossings did not move under the isotopy from D to D' , so we find them in the original diagram, D . Together with the ℓ crossings at the underpasses of α , we obtain $g(K') + \ell = g(K)$ crossings in D such that changing these turns D into a diagram of the unknot, as claimed.

Now consider the case in which α has at least one double point. We replace α by a sub-arc $\alpha_0 \subset \alpha$ that starts and ends at a double point c , that is, α_0 shall not self-intersect, whilst it may very well pass through crossings of D . Such a loop

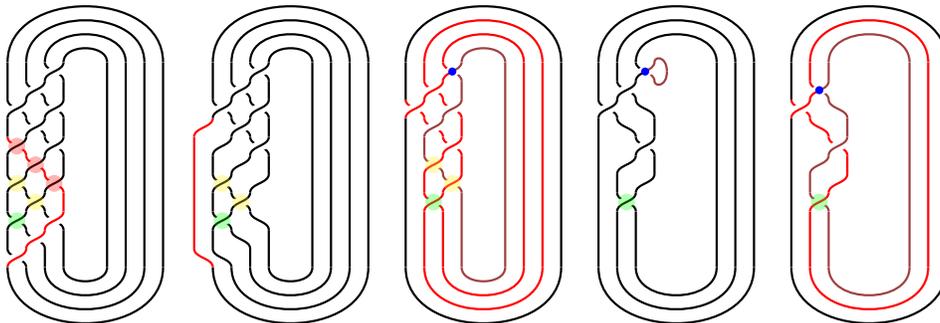


FIGURE 4. The algorithm from the proof of Theorem 1.2, applied to the torus knot $T(4, 5)$. Crossings to be changed are highlighted in red, yellow, and green (they are changed in this order); arcs α in red, loops α_0 in dark red, double points of loops α_0 marked with a blue dot.

can easily be found by walking along α until the first self-crossing of that walk occurs. Now we proceed as in the previous case: perform $\ell_0 \geq 0$ crossing changes in D , transforming the underpasses in α_0 (away from c , if there are any) into overpasses, apply a planar isotopy from α_0 to a small loop that does not contain any double points of D and finally perform a type (I) Reidemeister move removing the crossing c . This produces a positive braid diagram D' of a knot K' , having $2\ell_0 + 1$ fewer crossings than D , and K' is related to K by ℓ_0 crossing changes. The number of Seifert circles in D' is $n - 1$, and therefore $g(K') + \ell_0 = g(K)$. Again by induction, we can find $g(K')$ crossing changes in D' turning K' into the unknot; these $g(K')$ crossings did not move under the isotopy from D to D' , so we find them within the original diagram D . Together with the ℓ_0 crossings at the underpasses of α_0 , we found a set of $g(K') + \ell_0 = g(K)$ crossings in D such that changing them turns D into a diagram of the unknot. \square

3. OBSTRUCTING THE UNKNOTTING OF FIBERED POSITIVE KNOTS ALONG FIBERED POSITIVE KNOTS

This section contains the proof of Theorem 1.3, stating that there is no crossing change from the (positive, fibered) knot $K11n183$ with unknotting number 3 to any fibered positive knot with unknotting number 2. Figure 5(a) and (b) show a description of the protagonist of this section, the knot $K11n183$, as a plumbing of positive Hopf bands. Figure 5(b)–(d) moreover indicate how to unknot $K11n183$ with 3 crossing changes.

The Gordian distance of two knots is the minimum number of crossing changes required to transform one knot into the other. In the proof of Theorem 1.3 we make use of a new lower bound for Gordian distances in terms of the homology of branched coverings, which we give in Proposition 3.5. As preparation, let us make two definitions and prove one lemma in a rather general algebraic context.

Definition 3.1 Let R be a PID, $q \in R$ a prime element and $e \geq 1$ an integer. For a matrix A over R , denote by $n_{q,e}(A)$ the maximal integer such that the cokernel of A admits a summand $R/(q^{f_1}) \oplus \cdots \oplus R/(q^{f_{n_{q,e}(A)}})$ with $f_1, \dots, f_{n_{q,e}(A)} \geq e$.

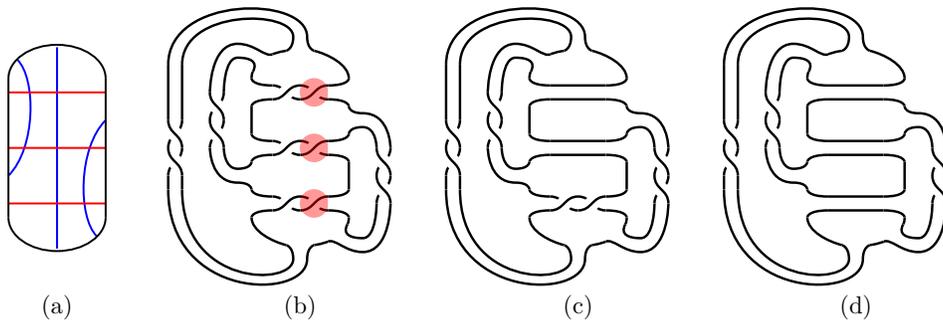


FIGURE 5. (a) Disc with prescribed arcs for plumbing positive Hopf bands from the top, in red, and from below, in blue. (b) Resulting positive diagram of $K11n183$. (c) The result of changing the top two of the three marked crossings is the twist knot $K6a3$, which is not fibered. (d) Diagram of the unknot obtained by changing all three marked crossings.

Definition 3.2 The *rank* $\text{rk}_S A$ of an $m \times m$ matrix A over a commutative ring S is the smallest number of generators for the submodule of S^m generated by the columns of A .

Lemma 3.3 *Let A be an $m \times m$ matrix over a PID R , $q \in R$ a prime element and $e \geq 1$ an integer. Let S be the ring $R/(q^e)$. Then $n_{q,e}(A) = m - \text{rk}_S A$.*

Proof. We can simplify the situation by applying the Smith normal form. Because R is a PID, there are invertible matrices U and V such that the product UAV is a diagonal matrix with entries d_1, \dots, d_m , where each d_i divides d_{i+1} . It is important to note that the rank and the isomorphism type of the cokernel of A remain unchanged when we multiply by U and V , i.e. $\text{rk}_S A = \text{rk}_S UAV$ and $n_{q,e}(A) = n_{q,e}(UAV)$. Hence $n_{q,e}(A)$ equals the number of d_i divisible by q^e . When we consider UAV as a matrix over the ring S , these entries d_i become 0. This proves $\text{rk}_S A \leq m - n_{q,e}(A)$.

Let us now show the reverse inequality. Let $\ell = m - n_{q,e}(A)$ and for $i \in \{1, \dots, \ell\}$, let f_i be maximal such that q^{f_i} divides d_i . By assumption, we have $0 \leq f_i < e$. Note that the S -module generated by the columns of UAV is isomorphic to $S/(q^{e-f_1}) \oplus \dots \oplus S/(q^{e-f_\ell})$. Let a system of a generators of this module be given, i.e. a surjective S -homomorphism

$$S^a \rightarrow S/(q^{e-f_1}) \oplus \dots \oplus S/(q^{e-f_\ell}).$$

Composing with multiplication by q^{e-f_i-1} in the i -th component, this yields a surjective homomorphism $S^a \rightarrow (S/(q))^\oplus \ell$. Since the target module is annihilated by q , this induces a surjective $S/(q)$ -homomorphism $(S/(q))^a \rightarrow (S/(q))^\oplus \ell$. But since $S/(q) \cong R_p/(q)$ is a field, we have $a \geq \ell$. Thus we have established that any system of generators of the column space of UAV over S has at least ℓ generators, and so the desired inequality $\text{rk}_S A \geq \ell = m - n_{q,e}(A)$ follows. \square

We shall now apply our newly found algebraic wisdom to the homology of branched coverings. Namely, for a knot K and an integer $n \geq 2$, denote by $\Sigma_n(K)$ the n -fold branched covering of S^3 along K (see e.g. [BZH14, Section 8.E] for an introduction). The action of the deck transformation group $\langle t | t^n \rangle$ on $\Sigma_n(K)$ makes the homology groups of $\Sigma_n(K)$ into modules over the ring $\mathbb{Z}[t]/(t^n - 1)$. Since $(1 + \dots + t^{n-1})$ acts trivially on homology, one may also consider homology as R_n -modules, for $R_n := \mathbb{Z}[t]/(1 + \dots + t^{n-1})$. To compute $H_1(\Sigma_n(K); R_n)$, we will use the following lemma.

Lemma 3.4 ([BZH14, Proposition 8.39]) *If B is a Seifert matrix of a knot K , then $tB - B^\top$ is a presentation matrix of $H_1(\Sigma_n(K); R_n)$ over R_n .*

In other words, $H_1(\Sigma_n(K); R_n)$ is isomorphic to the cokernel of $tB - B^\top$.

Proposition 3.5 *Let the integer $p \geq 2$ be a prime such that R_p is a PID.¹ Let $q \in R_p$ be a prime element and $e \geq 1$ an integer. For a knot K , denote by $n_{p,q,e}(K) \geq 0$ the maximal integer such that $H_1(\Sigma_p(K); R_p)$ admits a summand $R_p/(q^{f_1}) \oplus \dots \oplus R_p/(q^{f_{n_{p,q,e}(K)}})$ with $f_1, \dots, f_{n_{p,q,e}(K)} \geq e$. Then $|n_{p,q,e}(K) - n_{p,q,e}(J)|$ is a lower bound for the Gordian distance between the knots K and J .*

¹This is the case if and only if $p \leq 19$ (see e.g. [Was97, Theorem 11.1]). We expect that an adapted version of Proposition 3.5 holds for all primes p , using that R_p is always a Dedekind domain. But since we will only use the case $p = 3$ in this paper, we do not pursue this further.

By Lemma 3.4, $n_{p,q,e}(K)$ equals $n_{q,e}(tB - B^\top)$ (see Definition 3.1) for a Seifert matrix B of K . Let us first deduce Theorem 1.3 from Proposition 3.5, and prove the proposition afterwards.

Proof of Theorem 1.3. Let us assume towards a contradiction that J is a fibered positive knot with $u(J) = 2$, obtained from $K11n183$ by a single crossing change. It follows from [Cro89, Corollary 5.1] that the crossing number of a fibered positive knot is at most four times its genus. Thus $c(J) \leq 4 \cdot g(J) \leq 4 \cdot u(J) = 8$. A glance at the knot table [LM23] tells us that J must be $T(2, 3)^{\#2}$ or $T(2, 5)$. So it just remains to prove that $K11n183$ has Gordian distance at least 2 from both of those knots. Using the presentation matrices given by Lemma 3.4, and [Sag23] to compute their Smith normal forms, one finds:

$$\begin{aligned} H_1(\Sigma_3(K11n183); R_3) &\cong (R_3/(4))^{\oplus 2}, \\ H_1(\Sigma_3(T(2, 3)^{\#2}); R_3) &\cong (R_3/(2))^{\oplus 2}, \\ H_1(\Sigma_3(T(2, 5)); R_3) &\cong 0. \end{aligned}$$

Thus the values of $n_{3,2,2}$ on $K11n183$, $T(2, 3)^{\#2}$ and $T(2, 5)$ are 2, 0 and 0, respectively. By Proposition 3.5, this concludes the proof of Theorem 1.3. \square

Proof of Proposition 3.5. Assume that K and J are related by a single crossing change. It will suffice to show that $|n_{p,q,e}(K) - n_{p,q,e}(J)| \leq 1$. The knots K and J admit respective Seifert matrices B and C such that

$$C = \begin{pmatrix} B & v & 0 \\ w^\top & x & 1 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

for some integer column vectors v, w and for $x \in \mathbb{Z}$ [Sae99]. Consider the matrix E , which has the same size as matrix C but with all entries being zero except for the lower right corner, which is ∓ 1 . Now, let $\tilde{B} = C + E$. This matrix \tilde{B} is S-equivalent to the matrix B . Define $B' = t\tilde{B} - \tilde{B}^\top$, $C' = tC - C^\top$, and $E' = tE - E^\top$, all considered as matrices over the ring R_p . Note that by Lemma 3.4, B' and C' serve as presentation matrices for $H_1(\Sigma_p(K); R_p)$ and $H_1(\Sigma_p(J); R_p)$, respectively. Furthermore, we have the relationship $C' = B' - E'$. With $S = R_p/(q^e)$, the rank $\text{rk}_S E'$ of the matrix E' is equal to 1 (see Definition 3.2). We now have

$$n_{p,q,e}(J) = m - \text{rk}_S C' \geq m - \text{rk}_S B' - \text{rk}_S E' = n_{p,q,e}(K) - 1,$$

where the equalities are due to Lemma 3.3, and the inequality due to the subadditivity of rk_S , which follows quickly from its definition. Hence $n_{p,q,e}(K) - n_{p,q,e}(J) \leq 1$. Switching the roles of K and J gives us $|n_{p,q,e}(J) - n_{p,q,e}(K)| \leq 1$, as desired. \square

Remark 3.6 We are not aware of an alternative method to bound the Gordian distance of $K11n183$ and $T(2, 3)^{\#2}$. For the Gordian distance of $K11n183$ and $T(2, 5)$, on the other hand, one does not need the full strength of Proposition 3.5. Indeed, by the universal coefficient theorem, $H_1(\Sigma_3(K11n183); \mathbb{F}_2) \cong \mathbb{F}_2^4$ and $H_1(\Sigma_3(T(2, 5)); \mathbb{F}_2) = 0$. Now it suffices to use the following well-known lower bound for the Gordian distance of K and J , established by Wendt [Wen37] (in the very same paper in which the unknotting number was first introduced): namely, for all primes $p, q \geq 2$,

$$\frac{1}{p-1} \left| \dim_{\mathbb{F}_q} H_1(\Sigma_p(K); \mathbb{F}_q) - \dim_{\mathbb{F}_q} H_1(\Sigma_p(J); \mathbb{F}_q) \right|$$

is less than or equal to the Gordian distance of K and J .

Yet another way to show that $K11n183$ and $T(2, 5)$ have (algebraic) Gordian distance at least 2, is to use [Mur85, Corollary 2.8].

Remark 3.7 Since the lower bound in Proposition 3.5 (as well as Wendt’s bound) only depends on the S-equivalence class of the Seifert matrix, it follows that they are actually lower bounds for the *algebraic Gordian distance*. As a consequence, there is not even an algebraic unknotting move transforming any Seifert matrix of $K11n183$ into a Seifert matrix of a fibered positive knot J with $u(J) = 2$.

4. OBSTRUCTING TREFOIL PLUMBINGS

For context, we first remind the reader that every fibered link is related to the unknot via a sequence of Hopf plumbings and deplumbings [Har82, GG06, Gir02, Goo03]. Here, Hopf plumbing is the operation of gluing a (positive or negative) Hopf band onto a given Seifert surface S , where the gluing region is specified by a properly embedded interval on S (see Figure 6 (left arrow) for an illustration). The result of a Hopf plumbing operation is a new Seifert surface whose first Betti number is one more than that of S . Note that an isotopy of the plumbing arc (along properly embedded arcs in S) induces an isotopy of the resulting Seifert surface. Beware that the plumbing operation occurs locally in a neighbourhood of the plumbing interval; in particular, the plumbed Hopf band must not tangle with any part of the Seifert surface S .

It is known that a given fibered knot has, up to isotopy, a unique Seifert surface of minimal genus, namely its fiber surface (see e.g. [Rud05, Proposition 2.19] and [EL83, Lemma 5.1]). This allows us to make no distinction between fibered knots and their fiber surfaces. When we speak of a knot arising as a plumbing of Hopf bands (or trefoils), we mean that the knot is the boundary of a fiber surface obtained by plumbing Hopf bands (or trefoils).

In this section, we provide the proof of Theorem 1.4, which demonstrates that the knot $K11n183$ cannot be obtained by plumbing trefoils to an unknot. The trefoil plumbing operation is illustrated in Figure 6 (right arrow).

Proposition 4.1 *Let K be a knot arising as the plumbing of two positive trefoils. Then either K is one of the three knots $T(2, 3)^{\#2}$, $T(2, 5)$, $K10n14$, or the Levine–Tristram signature evaluated at $e^{0.11\pi i}$ equals -2 .*

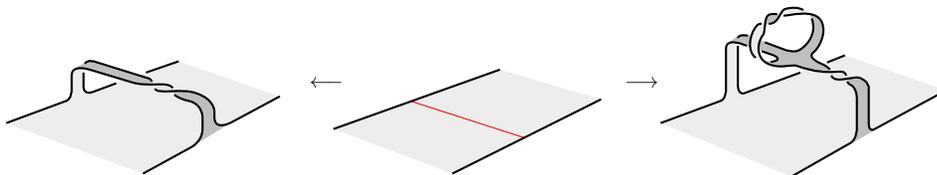


FIGURE 6. Plumbing a positive Hopf band (left) or a positive trefoil (right) to a Seifert surface (middle). The plumbing operation is specified by an interval (shown in red), properly embedded in the Seifert surface, and occurs in a neighbourhood of that interval.

Proof. Given a fiber surface obtained as plumbing of two positive trefoil fiber surfaces, it is possible to find an associated Seifert matrix of the form:

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ a & b & 0 & -1 \end{pmatrix}$$

for integers a, b . Note that a, b hold geometric significance: the second trefoil surface, attached to the first, consists of two Hopf bands (see Figure 6). One Hopf band is distant from the fiber surface associated to the first trefoil, while the other intersects it in a thickened interval. The pair (a, b) represents the algebraic intersection of this interval with the first trefoil surface's basis elements of the first homology group. Hence, either $(a, b) = (0, 0)$ or a and b are coprime. Since the fiber surface associated to a trefoil is a punctured torus, the homology class of the plumbing interval already determines its isotopy class. Thus the pair (a, b) determines the isotopy class of the resulting plumbing surface (and the knot spanned by it).

Setting $d := a^2 + ab + b^2 \geq 0$, one finds that the Alexander polynomial obtained from A equals

$$\Delta_d(t) := (t^{-1} - 1 + t)^2 + d \cdot (t^{-1} - 2 + t).$$

One verifies that either one of the following cases holds, or $d \geq 7$:

(a, b)	d	K
$(0, 0)$	0	$T(2, 3)^{\#2}$
$(1, 0), (0, 1), (-1, 1), (-1, 0), (0, -1), (1, -1)$	1	$T(2, 5)$
$(1, 1), (-1, 2), (-2, 1), (-1, -1), (1, -2), (2, -1)$	3	$K10n14$

So it just remains to show that if $d \geq 7$, then $\sigma_{e^{0.11\pi i}}(K) = -2$. Let us assume $d \geq 7$. One calculates

$$\Delta_d(t) = (\Delta_d(t) - \Delta_7(t)) + \Delta_7(t) = (d - 7) \cdot (t^{-1} - 2 + t) + \Delta_7(t).$$

Since the first summand is non-positive for $t = e^{0.11\pi i}$, we have

$$\Delta_d(e^{0.11\pi i}) \leq \Delta_7(e^{0.11\pi i}) < 0.$$

Recall that $\sigma_{e^{0.11\pi i}}(K)$ may be calculated as the signature of the Hermitian matrix $B := (1 - e^{0.11\pi i})A + (1 - e^{-0.11\pi i})A^\top$, i.e. $\sigma_{e^{0.11\pi i}}(K) = n_+(B) - n_-(B)$ for n_\pm the number of positive and negative eigenvalues of B , respectively. Note that the top left 3×3 submatrix of B does not depend on (a, b) , and one computes that it has one positive and two negative eigenvalues. It remains to prove that the fourth eigenvalue is also negative. Since $\det(B) = \Delta_d(e^{0.11\pi i})$ is negative, as we showed before, and $\text{sign}(\det B) = (-1)^{n_-(B)}$, it follows that $n_+(B) = 1$ and $n_-(B) = 3$, as desired. \square

Proof of Theorem 1.4. A plumbing of n trefoil fiber surfaces is a fiber surface, and thus genus-minimizing. Let us assume towards a contradiction that $K11n183$ is a plumbing of positive trefoils. Since the genus is additive under plumbing and $g(K11n183) = 3$, it follows that $K11n183$ is a plumbing of precisely three trefoils. By Proposition 4.1, the knot $K11n183$ is a plumbing of a (positive) trefoil to one of the following knots K : (1) $T(2, 3)^{\#2}$, (2) $T(2, 5)$, (3) $K10n14$, or (4) a knot K with $\sigma_{e^{0.11\pi i}}(K) = -2$. Let us rule out these cases one by one. Note that it is enough to

show that K cannot be obtained from $K11n183$ by a positive-to-negative crossing change.

In cases (1) and (2), Theorem 1.3 applies and gives the result. For case (3) we proceed similarly as in the proof of that theorem and compute

$$H_1(\Sigma_3(K10n14); R_3) \cong R_3/(5),$$

so Proposition 3.5 implies that the Gordian distance of $K11n183$ and $K10n14$ is at least 2. For case (4), recall that a positive-to-negative crossing change does not decrease the Levine–Tristram signature at a fixed point on the circle. But this contradicts $\sigma_{e^{0.11\pi i}}(K) = -2 < 0 = \sigma_{e^{0.11\pi i}}(K11n183)$. \square

Remark 4.2 There exists an obstruction for a knot K being the plumbing of a positive trefoil and another knot J , expressed in terms of the Alexander polynomial Δ_K of K , as demonstrated by Hironaka [Hir06]. In our context, Hironaka’s obstruction implies that if the knot K arises from the unknot by successive plumbings of positive trefoils, then the leading coefficient of Δ_K and the evaluation $\Delta_K(0)$, both of which are ± 1 , have the same sign. Note that this condition does not depend on the choice of normalization of Δ_K ; it is equivalent to Seifert matrices A associated to the fiber surface of K satisfying $\det(A) = 1$. However, one can see that this condition is in fact satisfied by all knots that arise from the unknot by the plumbing and deplumbing of positive Hopf bands, i.e. by all fibered strongly quasipositive knots. So Hironaka’s criterion cannot be used to obstruct a fibered positive knot from being a positive trefoil plumbing.

5. FIBERED POSITIVE KNOTS WITH POTENTIALLY LARGE UNKNOTTING NUMBER

Let us begin this section by proving Theorem 1.5, which claims that the knot \mathcal{K} is a fibered positive knot of genus 7, but one cannot unknot it by 7 crossing changes in its diagram D shown in Figure 1.

Proof of Theorem 1.5. Since D is a positive diagram, \mathcal{K} is a positive knot. To confirm that \mathcal{K} is fibered and compute its genus, observe that \mathcal{K} is a plumbing of 14 positive Hopf bands. These statements about \mathcal{K} are also the particular case $n = 1$ of Proposition 5.3(1) and (3) below, proven below in greater detail.

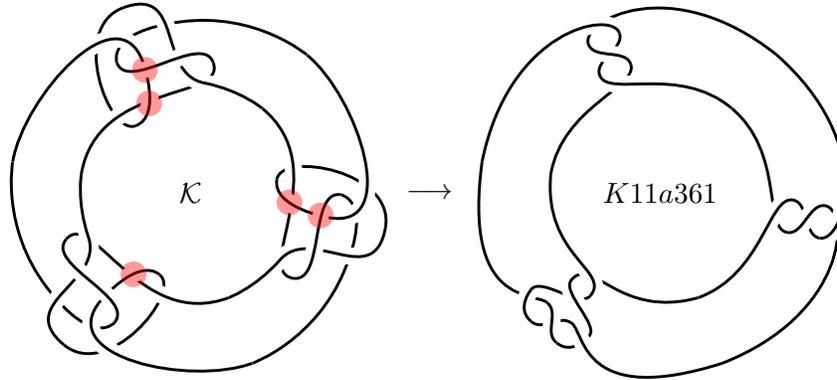


FIGURE 7. Applying five crossing changes to \mathcal{K} to obtain $K11a361$.

To see that the minimal number of crossing changes needed to transform D into a diagram of the unknot is 9, we have checked all possible combinations of at most 8 crossing changes in D and observed that the Alexander polynomials of the resulting knots are non-trivial. The computations can be accessed at [KLM⁺]. An unknotting sequence with 9 crossings is easily found in D (see also Proposition 5.3(4) below). \square

Remark 5.1 Since we cannot find 8 crossing changes that unknot \mathcal{K} , we conjectured (see Conjecture 1.6) that in fact $u(\mathcal{K}) = 9$, which would make \mathcal{K} a counterexample to Stoimenow’s Conjecture 1.1. Let us now discuss why $u(\mathcal{K}) > 7$ is hard to show. Although the literature is awash with lower bounds for the unknotting number, most of them fall into one of the following four types:

- (a) Bounds that are also lower bounds for the smooth 4-genus g_4 , for example, the bounds coming from the Levine–Tristram signatures [Lev69, Tri69], from the Υ -invariant [OSS17], or from slice-torus invariant such as the τ -invariant from knot Floer homology [OS03] or the s -invariant from Khovanov homology [Ras10].
- (b) Bounds that are also lower bounds for the *algebraic unknotting number* u_a , that is, the minimal number of crossing changes needed to transform a given knot into a knot with trivial Alexander polynomial. These are precisely the bounds determined by the S-equivalence class of the Seifert matrix, such as bounds coming from the Levine–Tristram signatures, or coming from homology of cyclic coverings (such as Proposition 3.5), and further bounds in [Nak81, Mur90, Fog93, Sae99, Sto04, BF14, BF15].
- (c) Bounds that are also bounds for the *proper rational unknotting number* u_q , that is, the minimal number of proper rational tangle replacements needed to transform a given knot into the unknot. See for example [Lin96, McC15, ILM21, Eft22, MZ23].
- (d) Bounds that are at most two [Sch85, Miy98, OS05, GL06, Gre14, McC17].

Note that the inequalities $g_4, u_a, u_q \leq u$ hold for all knots. As we shall see below in Proposition 5.2, $g_4, u_a, u_q \leq 7$ for \mathcal{K} , so none of the bounds of type (a), (b), (c) may be used to establish $u(\mathcal{K}) > 7$; nor may one use bounds of type (d), of course. One of the few lower bounds for u not subsumed by (a)–(d) is Owens’ obstruction [Owe08, Theorem 3] coming from Heegaard Floer homology, which can, however, only be used to show $u(\mathcal{K}) > |\sigma(\mathcal{K})/2| = 5$.

Proposition 5.2 *The knot \mathcal{K} shown in Figure 1 satisfies $g_4(\mathcal{K}) = 7$, $u_a(\mathcal{K}) \leq 7$ and $u_q(\mathcal{K}) \leq 5$.*

Proof. Since \mathcal{K} is positive, $g_4(\mathcal{K}) = g(\mathcal{K}) = 7$ follows. The statement $u_q(\mathcal{K}) \leq 5$ will be shown in greater generality in Proposition 5.3 below. Let us now prove $u_a(\mathcal{K}) \leq 7$. Figure 7 shows a set of five crossing changes from \mathcal{K} to the knot $K11a361$. So it suffices to show that $K11a361$ has algebraic unknotting number 2, which we shall do now (note that $u_a(K11a361) = 2$ is stated in the online tables [BF, LM23], but no certificate is provided).

Applying Seifert’s algorithm to the diagram of $K11a361$ shown in Figure 7 and appropriately choosing a basis of the first homology of the resulting surface yields

the following Seifert matrix of $K11a361$:

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

Consider the base change given by the matrix P defined below

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & -1 \\ -2 & -1 & -1 & 0 \end{pmatrix},$$

and apply an algebraic unknotting move by adding $w^\top w$ where $w = (1 \ 1 \ -1 \ -1)$ to obtain the matrix

$$B = PAP^\top + w^\top w = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & -3 \\ 0 & 1 & -2 & -4 \end{pmatrix}.$$

Applying a reduction to B to remove the first and the second rows/columns, and performing a new algebraic unknotting move by adding $v^\top v$ where $v = (2 \ 1)$ yields the matrix

$$\begin{pmatrix} -4 & -3 \\ -2 & -4 \end{pmatrix} + v^\top v = \begin{pmatrix} 0 & -1 \\ 0 & -3 \end{pmatrix},$$

which has trivial Alexander polynomial. We have thus algebraically unknotted $K11a361$ by two moves, showing $u_a(K11a361) \leq 2$. \square

Finally, let us consider a generalization of our example \mathcal{K} . Figure 8 shows an infinite family of knots \mathcal{K}_n , with $\mathcal{K}_1 = \mathcal{K}$.

Proposition 5.3 *For $n \geq 1$, let D_n be the positive knot diagram shown in Figure 8. Then, the knot \mathcal{K}_n represented by D_n has the following properties.*

- (1) \mathcal{K}_n is fibered and positive.
- (2) \mathcal{K}_n is not braid positive.
- (3) \mathcal{K}_n has 3- and 4-genus $g(\mathcal{K}_n) = g_4(\mathcal{K}_n) = 2 + 5n$.
- (4) There is a set of $2 + 7n$ crossings in the diagram D_n such that changing these crossings turns D_n into a diagram of the unknot. In particular,

$$2 + 5n \leq u(\mathcal{K}_n) \leq 2 + 7n.$$

- (5) The algebraic unknotting number $u_a(\mathcal{K}_n)$ of \mathcal{K}_n satisfies

$$1 + 2n \leq u_a(\mathcal{K}_n) \leq 7n.$$

- (6) The proper rational unknotting number $u_q(\mathcal{K}_n)$ of \mathcal{K}_n satisfies

$$1 + 2n \leq u_q(\mathcal{K}_n) \leq 1 + 4n.$$

The proposition motivates the following conjecture.

Conjecture 5.4 *The unknotting number of \mathcal{K}_n is $2 + 7n$.*

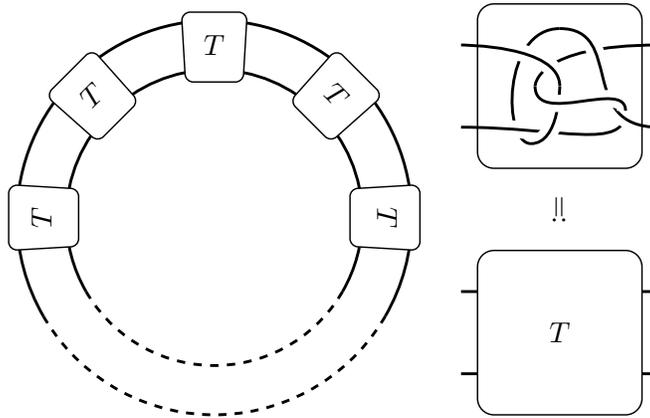


FIGURE 8. Diagram D_n representing \mathcal{K}_n is obtained by cyclically connecting $2n + 1$ copies of the tangle T , as indicated. D_1 is shown in Figure 1.

Note that this would imply that none of the \mathcal{K}_n can be obtained from the unknot by plumbing trefoils, and that all of them are counterexamples to Conjecture 1.1. It would even imply that the difference between the unknotting number and the 3-genus can be arbitrarily large for fibered positive knots (which is not even known for general knots). As shown in (3) and (6), $g_4(\mathcal{K}_n), u_q(\mathcal{K}_n) \leq g(\mathcal{K}_n)$, so no lower bound for u that is also a lower bound for g_4 or u_q (compare to Remark 5.1) can be used to establish Conjecture 5.4. On the other hand, the best upper bound we found for $u_a(\mathcal{K}_n)$ with $n > 1$ is $7n$, so a possible strategy to show that $u(\mathcal{K}_n) > g(\mathcal{K}_n)$ for $n > 1$ would be to bound $u_a(\mathcal{K}_n)$ from below.

Proof of Proposition 5.3. (1) All crossings in D_n are positive and thus \mathcal{K}_n is positive. To see that \mathcal{K}_n is fibered, observe that \mathcal{K}_n is obtained from the unknot by iterated Hopf plumbing. In particular, a Seifert surface for \mathcal{K}_n can be obtained from the fiber surface of the torus link $T(2, 4n + 2)$ by plumbing $6n + 3$ positive Hopf bands, three for each of the $2n + 1$ copies of T , as shown in Figure 9.

(2) To see that \mathcal{K}_1 is not braid positive, we compute in [KLM⁺] Ito's normalized version of the HOMFLYPT polynomial $\tilde{P}_{\mathcal{K}_1}(\alpha, z)$ and observe that some of its coefficients are negative, failing a condition for braid positive knots [Ito22, Theorem 1.1]. In particular, we observe that $\tilde{P}_{\mathcal{K}_1}(\alpha, 0) = -\alpha^4 + 7\alpha^3 + 9\alpha^2$. Applying the skein relation for the HOMFLYPT polynomial inductively to the $3n$ and $4n$ crossings involved in the crossing changes in Figure 10(b) and (c), we can inductively reduce the general case to the case $n = 1$. In particular, it follows that

$$\tilde{P}_{\mathcal{K}_n}(\alpha, 0) = -n\alpha^{2n+2} + (2 + 5n)\alpha^{2n+1} + (3 + 6n)\alpha^{2n},$$

which has a negative coefficient and thus \mathcal{K}_n is not braid positive.

(3) The genus of the surface obtained by applying Seifert's algorithm to D_n equals $2 + 5n$. Since D_n is positive, this surface realizes the 3- and 4-genus of \mathcal{K}_n .

(4) The lower bound for the unknotting number is given by the 4-genus of \mathcal{K}_n . To get the upper bound, we describe an explicit unknotting sequence in D_n . The tangle T can be transformed into a tangle consisting of a single positive (respectively negative) crossing by performing three (resp. four) crossing changes, as shown in

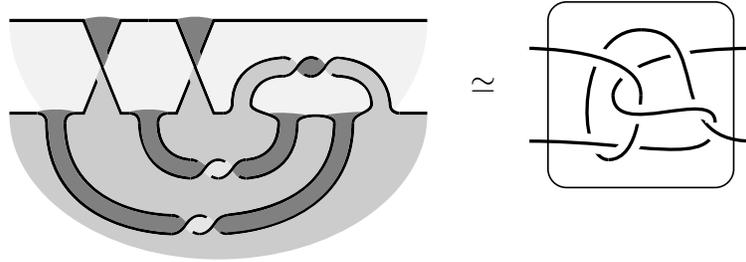


FIGURE 9. The fiber surface of \mathcal{K}_n is obtained from $T(2, 4n + 2)$ by plumbing 3 positive Hopf bands for each of the copies of T .

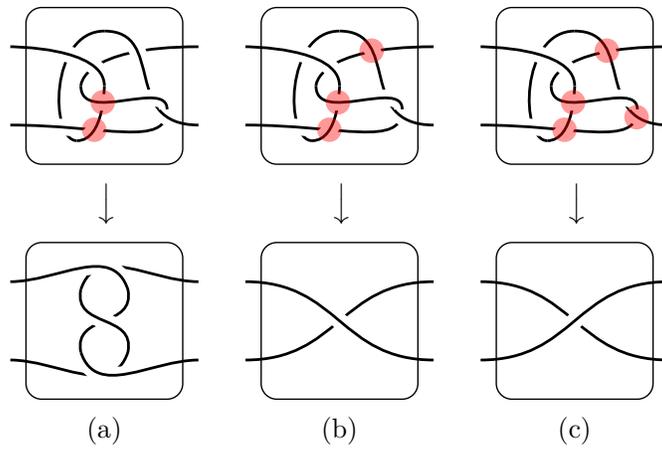


FIGURE 10. Changing the marked crossings of the tangle T in the upper row, yields tangles isotopic to the tangles shown in the lower row.

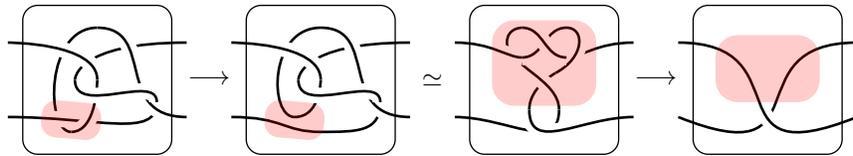


FIGURE 11. Two proper rational tangle replacements transform the tangle T into a single positive crossing.

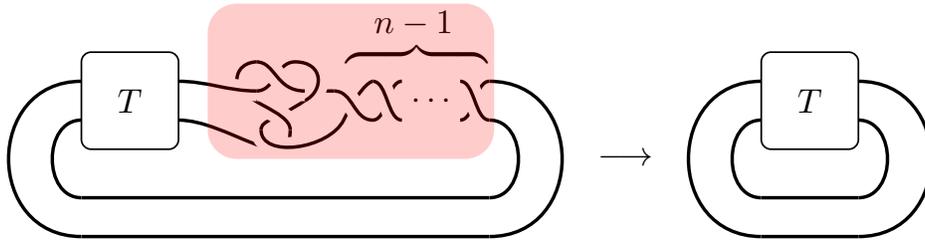


FIGURE 12. A proper rational tangle replacement.

Figure 10(b) (resp. in Figure 10(c)). We perform each of these two transformations in n of the $2n + 1$ copies of T in \mathcal{K}_n , thus replacing $2n$ copies of T by n positive and n negative crossings. In the remaining tangle, we perform two crossing changes, as shown in Figure 10(a). The resulting diagram represents the unknot and is related to D_n by a sequence of $2 + 3n + 4n$ crossing changes.

The lower bounds in (5) and (6) are given by the Montesinos trick: the double branched cover $\Sigma_2(K)$ of K can be obtained from S^3 by surgery on a link with $u_q(K)$ components [MZ23]. Moreover, $\Sigma_2(K)$ can be obtained from an integer homology sphere by surgery on a link with $u_a(K)$ components (since a knot with Alexander polynomial 1 has a ZHS^3 as double branched covering). In particular, the minimal numbers of required generators of $H_1(\Sigma_2(K); \mathbb{Z})$ is a lower bound for $u_a(K)$ and $u_q(K)$. To compute the first homology of the double branched cover we recall that the diagram D_n consists of the cyclic composition of $2n + 1$ copies of the tangle T . From that, it is not hard to see that the double branched cover of \mathcal{K}_n is obtained by taking $S^1 \times (S^2 \setminus \bigcup_{2n+1} \mathring{D}^2)$ and gluing $2n + 1$ copies of the double branched cover of T to it. On the other hand, the double branched cover of T can be seen to be the figure eight knot sister $m003$, a 1-cusped hyperbolic manifold realizing the minimum of volumes among such manifolds. (The other minimizer is the figure eight knot complement.) $m003$ can be seen as the complement of a nullhomologous knot in the lens space $L(5, 1)$ and thus has homology $\mathbb{Z} \oplus \mathbb{Z}_5$. In SnapPy [CDGW], `Manifold('L5a1(-5,1)(0,0)').identify()` shows that the (-5) -surgery on one of the components of the Whitehead link $L5a1$ is diffeomorphic to $m003$. (From that surgery description all the above statements are easy to verify.) Now a simple application of the Mayer–Vietoris sequence yields

$$H_1(\Sigma_2(\mathcal{K}_n); \mathbb{Z}) = \begin{cases} \mathbb{Z}_5^{2n+1} \oplus \mathbb{Z}_{2n+1} & \text{if } 2n + 1 \equiv 0 \pmod{5}, \\ \mathbb{Z}_5^{2n} \oplus \mathbb{Z}_{5(2n+1)} & \text{otherwise.} \end{cases}$$

In particular, the number of generators in $H_1(\Sigma_2(\mathcal{K}_n); \mathbb{Z})$ is either $2n + 1$ or $2n + 2$ depending on the divisibility of $2n + 1$.

The upper bound for $u_a(\mathcal{K}_n)$ and $u_q(\mathcal{K}_n)$ are obtained by explicit construction. Performing the crossing changes shown in Figure 10 (b) and (c) each in $n - 1$ copies of T , yields the knot \mathcal{K}_1 , for which $u_a(\mathcal{K}_1) \leq 7$ has already been established in Proposition 5.2. Overall, this gives $u_a(\mathcal{K}_n) \leq 3(n - 1) + 4(n - 1) + 7 = 7n$ as desired. Figure 11 shows a sequence of two proper rational tangle replacements transforming T into a single positive crossing. We apply these to $2n - 1$ of the tangles T in D_n , and to one of the remaining tangles T we just apply the first proper rational tangle replacement from Figure 11. This yields the knot shown in Figure 12 on the left. The proper rational unknotting replacement shown in that figure yields the closure of T , which is the torus knot $T(2, 5)$ with proper rational unknotting number $u_q(T(2, 5)) = 1$. In total, we can perform $4n + 1$ proper rational tangle replacements to get the unknot, getting the claimed upper bound. \square

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