

# Proof of a conjecture by H. Dullin and R. Montgomery\*

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## Abstract

In the framework of the planar Euler problem in the quasi-periodic regime, the formulae of the periods available in the literature are simple only on one side of their singularity. In this paper, we complement such formulae with others, which result simpler on the other side. The derivation of such new formulae uses the Keplerian limit and complex analysis tools. As an application, we prove a conjecture by H. Dullin and R. Montgomery, which states that such periods, as well as their ratio, the *rotation number*, are monotone functions of their non-trivial first integral, at any fixed energy level.

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# 1 Introduction

**1.1 Overview on the Euler problem** The *two fixed centers* (or *Euler-*) problem is the Hamiltonian system – firstly considered by Euler and next proven to be integrable by Jacobi [9, 13] – of the motion of a free particle undergoing gravitational attraction by two masses in fixed positions in the Euclidean space. As an integrable problem, the Euler problem has attracted the attention of many mathematicians, maybe because of the peculiarity of the process of integration, intimately connected to a spectacular regularization of collisions with *both* the attracting centers, which now we briefly recall (see [12, 1, 8, 25] and references therein for a complete discussion). The Hamiltonian of the Euler problem is

$$J = \frac{\|\mathbf{y}\|^2}{2} - \frac{M}{\|\mathbf{x} + \mathbf{v}_0\|} - \frac{m}{\|\mathbf{x} - \mathbf{v}_0\|} \quad (1)$$

where  $M, m$  are the masses of the attracting bodies,  $\mathbf{v}_0$  provides the direction of the mass  $m$ , which we also fix as the first axis  $\mathbf{i}$  of a reference frame:  $\mathbf{v}_0 = v_0 \mathbf{i}$ , with  $v_0 > 0$ ;  $\|\cdot\|$  is the Euclidean norm and the gravity constant has been put equal to 1. Without loss of generality, we assume  $M \geq m \geq 0$ , but  $(M, m) \neq (0, 0)$ . Even though the result of this paper will concern the planar case (where “planar” means that we fix  $\mathbf{y}, \mathbf{x}$  to take values on a fixed plane through  $\mathbf{i}$ ), some formulae that we shall write hold for the more general spatial problem, so we regard  $\mathbf{y}, \mathbf{x}$  as objects of  $\mathbb{R}^3$ . And precisely in the spatial version one observes that the Hamiltonian (1) remains unvaried applying a rotation about the  $\mathbf{i}$  axis, due to existence of the following first integral

$$\Theta(\mathbf{y}, \mathbf{x}) := x_2 y_3 - x_3 y_2 \quad (2)$$

consisting of the projection along the  $\mathbf{i}$ -axis of the angular momentum

$$\mathbf{M}(\mathbf{y}, \mathbf{x}) := \mathbf{x} \times \mathbf{y} \quad (3)$$

of the particle. Note that  $\Theta$  vanishes in the planar case.

The solution found out by Jacobi relies on switching to new position coordinates  $\alpha, \beta, \vartheta$ , with

$$\alpha \geq 1, \quad |\beta| \leq 1, \quad \vartheta \in \mathbb{T}$$

defined as

$$\begin{cases} \alpha(\mathbf{x}) := \frac{\|\mathbf{x} + \mathbf{v}_0\| + \|\mathbf{x} - \mathbf{v}_0\|}{2v_0} \\ \beta(\mathbf{x}) := \frac{\|\mathbf{x} + \mathbf{v}_0\| - \|\mathbf{x} - \mathbf{v}_0\|}{2v_0} \\ \vartheta(\mathbf{x}) := \arg(x_3, -x_2) \end{cases}$$

We denote as  $A(\mathbf{y}, \mathbf{x}), B(\mathbf{y}, \mathbf{x}), \Theta(\mathbf{y}, \mathbf{x})$  are the generalized impulses conjugated to  $\alpha, \beta, \vartheta$  (with  $\Theta$  being precisely the quantity in (2)). Using such coordinates,  $J$  is carried to the nice form (see [1] for the details)

$$J = \frac{1}{v_0^2(\alpha^2 - \beta^2)} \left( \frac{A^2(\alpha^2 - 1)}{2} + \frac{B^2(1 - \beta^2)}{2} + \frac{\Theta^2}{2(\alpha^2 - 1)} + \frac{\Theta^2}{2(1 - \beta^2)} - M_+ v_0 \alpha + M_- v_0 \beta \right) \quad (4)$$

with

$$M_- = M - m, \quad M_+ = M + m. \quad (5)$$

Given the lucky expression in (4), Jacobi proposed to switch to the new, completely separated, Hamiltonian

$$J_{\text{reg}} = J_+(A, \alpha, J_0, \Theta_0, M_+) + J_-(B, \beta, J_0, \Theta_0, M_-) \quad (6)$$

where

$$\begin{aligned} J_+(A, \alpha, J_0, \Theta_0, M) &= \frac{A^2(\alpha^2 - 1)}{2} + \frac{\Theta_0^2}{2(\alpha^2 - 1)} - Mv_0\alpha - J_0v_0^2(\alpha^2 - 1) \\ J_-(B, \beta, J_0, \Theta_0, M) &= \frac{B^2(1 - \beta^2)}{2} + \frac{\Theta_0^2}{2(1 - \beta^2)} + Mv_0\beta - J_0v_0^2(1 - \beta^2) \end{aligned} \quad (7)$$

The motions of Hamiltonian  $J_{\text{reg}}$  in (6) are related to the ones of  $J$  in (4) as follows: any orbit of  $J_{\text{reg}}$  on the zero energy level in the time  $\tau$  provides an orbit of  $J$  on the  $J_0$  energy level in the time  $t$ , related to  $\tau$  via

$$\frac{d\tau}{dt} = \frac{1}{v_0^2(\alpha(t)^2 - \beta(t)^2)}. \quad (8)$$

In addition, by the complete separability of  $J_{\text{reg}}$ , the zero energy level of  $J_{\text{reg}}$  enforces a further constant of motion, which we call  $F_0$ , defined via

$$\begin{cases} J_+(A, \alpha, J_0, \Theta_0, M_+) = -\frac{F_0}{2} \\ J_-(B, \beta, J_0, \Theta_0, M_-) = \frac{F_0}{2} \end{cases} \quad (9)$$

**1.2 Problem and result** In this paper, we are interested to the periods (hereafter called *Jacobi periods*, and denoted as  $\tau_+$ ,  $\tau_-$ ) of the Euler problem, in the quasi-periodic regime. The classical analysis that we have reviewed in the previous section is fit to completely describe the set  $\mathbb{P}$  of parameters  $(J_0, F_0, \Theta_0)$  corresponding to periodic orbits, and provide an expression of their periods, in form of elliptic integrals. In this paper, we aim to complement the formulae for  $\tau_+$ ,  $\tau_-$  existing in the literature with others, so as to write down the simplest expressions possible. For simplicity, we focus on the planar case ( $\Theta_0 = 0$ ) (even though we do not see obstructions to treat the general case with the same techniques here) and switch from  $J_0$  and  $F_0$  to the quantities

$$d := -4v_0J_0, \quad f := -2J_0F_0 \quad (10)$$

Observe that, by the assumptions on  $m$ ,  $M$ , the mass parameters  $M_-$ ,  $M_+$  in (5) verify

$$0 \leq M_- \leq M_+, \quad (M_-, M_+) \neq (0, 0) \quad (11)$$

with  $M_- = 0$  corresponding to  $m = M$ , while  $M_- = M_+$  to  $m = 0$ . From now on, we shall implicitly assume (11), without further mention. Then the set  $\mathbb{P}$  is a convex set, twice disconnected, having the form (see Fig. 1)

$$\mathbb{P} = \left\{ (d, f) \in \mathbb{R}^2 : d > 0, \quad f_{M_-}^-(d) < f < f_{M_+}^+(d), \quad f \neq f_{M_-}^s(d), \quad f \neq f_{M_+}^s(d) \right\} \quad (12)$$

with suitable  $f_M^-(d)$ ,  $f_M^s(d)$ ,  $f_M^+(d)$  precisely defined in the course of the paper (see Equation (22) below) and verifying

$$f_{M_+}^-(d) \leq f_{M_-}^-(d) \leq f_{M_-}^s(d) \leq f_{M_+}^s(d) \leq f_{M_+}^+(d) \quad \forall d > 0. \quad (13)$$

We now define, for each  $M \in \{M_-, M_+\}$ , the set

$$\mathbb{Q}_M := \left\{ (d, f) \in \mathbb{R}^2 : d > 0, \quad f > f_M^-(d), \quad f \neq f_M^s(d) \right\} \quad (14)$$

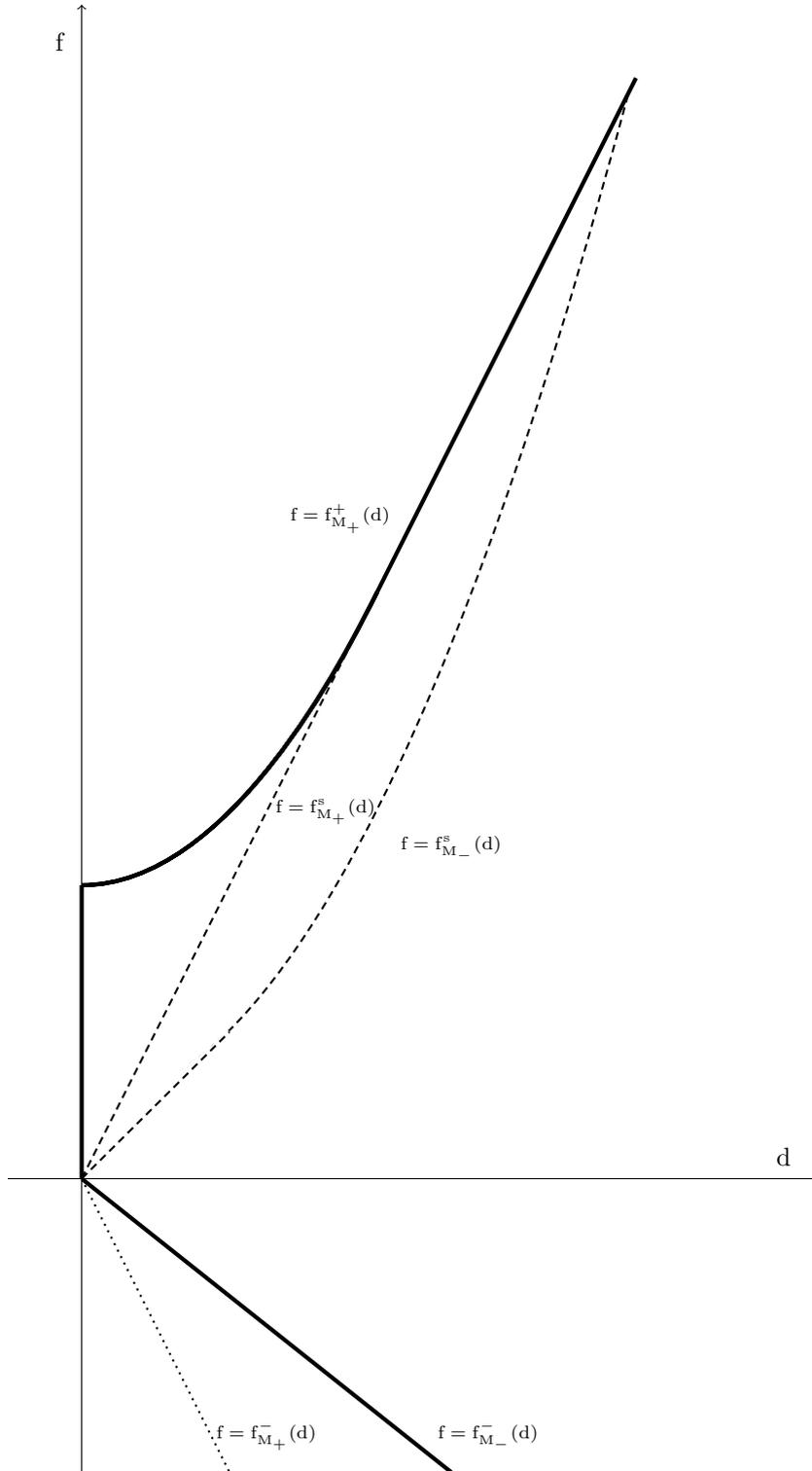


Figure 1: Graphical representation of the domain  $\mathbb{P}$ , in the plane  $(d, f)$ . The singular lines  $f = f_{M_{\pm}}^s(d)$  (dashed) and the boundary lines of  $\mathbb{P}$  (thick) are reported. The lower boundary  $f = f_{M-}^-(d)$  of  $\mathbb{P}$  is also a boundary line of  $\mathbb{Q}_{M-}$ , while the line  $f = f_{M+}^-(d)$  (dotted), lower boundary line of  $\mathbb{Q}_{M+}$ , is external to  $\mathbb{P}$ . It has been reported for comparison with Fig. 2.

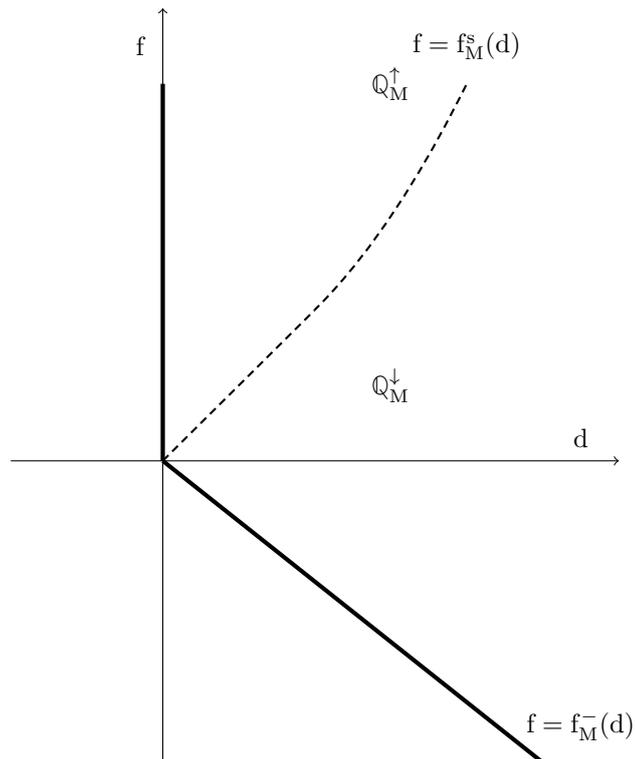


Figure 2: Graphical representation of the domain  $Q_M$ , and of its sub-domains  $Q_M^\downarrow$ ,  $Q_M^\uparrow$ , in the plane  $(d, f)$ . The “singular line”  $f = f_M^s(d)$  (dashed) and the “minimum line”  $f = f_M^-(d)$  are reported. The domain  $\mathbb{P}$  in Fig. 1 is a subset of  $Q_{M_+}^\uparrow \cap Q_{M_+}^\downarrow$ .

By (12) and (13), we have:  $\mathbb{Q}_{M_+}, \mathbb{Q}_{M_-} \supset \mathbb{P}$ . We split each  $\mathbb{Q}_M$  as  $\mathbb{Q}_M = \mathbb{Q}_M^\uparrow \cup \mathbb{Q}_M^\downarrow$ , where  $\mathbb{Q}_M^\uparrow$  is the part of  $\mathbb{Q}_M$  above the line  $\{f = f_M^s(d)\}$ , while  $\mathbb{Q}_M^\downarrow$  is the one below (see Fig. 2). On each part of  $\mathbb{Q}_M$ , we define the function

$$\theta_M(d, f) := \begin{cases} \theta_M^\uparrow := 2\sqrt{\frac{2d}{v_0}} \int_{-1}^1 \frac{dx}{\sqrt{(1-x^2)(d^2x^2 - 4dMx + 4f - d^2)}} & \text{if } (d, f) \in \mathbb{Q}_M^\uparrow \\ \theta_M^\downarrow := \sqrt{\frac{2d}{v_0}} \int_0^2 \frac{dz}{\sqrt{z(2-z)(M^2z^2 - 2fz + d^2)}} & \text{if } (d, f) \in \mathbb{Q}_M^\downarrow \end{cases} \quad (15)$$

In this paper we prove that

**Theorem 1.1** *The function  $\theta_M$  is well-defined on  $\mathbb{Q}_M$  and, moreover, the following identity holds*

$$\tau_\pm = \theta_{M_\pm} |_{\mathbb{P}} \quad \forall (d, f) \in \mathbb{P}.$$

We remark that the well posedness of  $\theta_M^\uparrow$  on  $\mathbb{Q}_M^\uparrow$  and the validity of

$$\tau_\pm = \theta_{M_\pm}^\uparrow |_{\mathbb{P}} \quad \forall (d, f) \in \mathbb{P} \cap \mathbb{Q}_{M_\pm}^\uparrow$$

is a mere consequence of the classical analysis, as shown in Section 2. The novelty of the paper relies on the well posedness of  $\theta_M^\downarrow$  on  $\mathbb{Q}_M^\downarrow$  and the validity of

$$\tau_\pm = \theta_{M_\pm}^\downarrow |_{\mathbb{P}} \quad \forall (d, f) \in \mathbb{P} \cap \mathbb{Q}_{M_\pm}^\downarrow.$$

This is discussed in Sections 3 and 4.

**1.3 An application** As an application of Theorem 1.1, we shall prove a conjecture posed<sup>1</sup> by H. Dullin and R. Montgomery in [8]. Besides the functions  $\tau_-(d, f)$ ,  $\tau_+(d, f)$ , consider also their ratio, the *rotation number*:

$$\omega(d, f) := \frac{\tau_-(d, f)}{\tau_+(d, f)}.$$

We focus on the dependence of the functions  $\tau_-(d, f)$ ,  $\tau_+(d, f)$  and  $\omega(d, f)$  on  $f$ . To this end, given  $\mathbb{A} \subset \mathbb{R}^2$ , we denote as  $\mathbb{A}_d$  the set of  $f$  such that  $(d, f) \in \mathbb{A}$ , for a fixed  $d > 0$ , and observe that  $\mathbb{P}_d$  is twice disconnected along  $f = f_{M_-}^s(d)$ ,  $f = f_{M_+}^s(d)$  when  $0 \leq M_- < M_+$ ; once when  $M_- = M_+$ . We prove that

**Theorem 1.2** *The functions  $\tau_-(d, \cdot)$ ,  $\tau_+(d, \cdot)$  and  $\omega(d, \cdot)$  are differentiable and monotone on each connected component of  $\mathbb{P}_d$ .*

Besides proving a problem left open in [8], Theorem 1.2 has been recently applied in the framework of Rabinowitz Floer homology [10, 24]. In addition, we foresee applications to close-to-be integrable systems having the two-center problem as limiting case, like the problem of three or more bodies. Various results in this directions have been obtained by the author and collaborators: see [20] for the description of a possible setting; [21, 5] for rigorous results; [6, 7] for numerical studies. Moreover, as the monotonicity of the periods is somewhat a convexity assertion, we foresee connections to Nekhoroshev theory [17, 18, 22, 19, 11], while the monotonicity of

<sup>1</sup>Precisely, the conjecture regards the monotonicity of  $\tau_+$  on the  $f$ -fibers of  $\mathbb{P} \cap \mathbb{Q}_{M_+}^\uparrow$  and of  $\omega_S, \omega_P$  on the  $f$ -fibers of  $\mathbb{D}_S, \mathbb{D}_P$ , respectively, as the monotonicity of  $\tau_+$  on the  $f$ -fibers of  $\mathbb{P} \cap \mathbb{Q}_{M_+}^\downarrow$ , of  $\tau_-$  on the  $f$ -fibers of  $\mathbb{P}$  and (hence of)  $\omega_L$  on the  $f$ -fibers of  $\mathbb{D}_L$  has been proven in [8], using the theory of elliptic functions.

the rotation number should allow interactions with weak KAM theory: see [14, 15, 16, 3, 23] for general information; [4] for an application of Aubry–Mather theory to the Euler problem. All such possible directions of research are here mentioned just as possible hints, being definitely far from the purposes of the paper. Here we limit to show how Theorem 1.2 follows from Theorem 1.1.

**Proof of Theorem 1.2** The differentiability of  $\tau_-(d, \cdot)$ ,  $\tau_+(d, \cdot)$  and  $\omega(d, \cdot)$  and monotonicity of  $\tau_-(d, \cdot)$  and  $\tau_+(d, \cdot)$  on  $\mathbb{P}_d$  are an immediate consequence<sup>2</sup> of Theorem 1.1 and of the formula in (15). Concerning  $\omega(d, \cdot)$  for  $f \in \mathbb{P}_d$ , it is sufficient consider the case  $0 \leq M_- < M_+$ , as when  $M_- = M_+$ , we are in the case of the Kepler problem, so  $\tau_+ = \tau_-$ , whence  $\omega(d, f) \equiv 1$  is trivially monotone. As  $\mathbb{Q}_{M_-}^\downarrow \subset \mathbb{Q}_{M_+}^\downarrow$ ,  $\mathbb{Q}_{M_+}^\uparrow \subset \mathbb{Q}_{M_-}^\uparrow$  for all  $0 \leq M_- < M_+$ , then  $\omega(d, f)$  is given by

$$\omega(d, f) = \begin{cases} \omega_S(d, f) := \frac{\theta_{M_-}^\downarrow(d, f)}{\theta_{M_+}^\downarrow(d, f)} & \text{if } (d, f) \in \mathbb{D}_S := \mathbb{Q}_{M_-}^\downarrow \\ \omega_L(d, f) := \frac{\theta_{M_-}^\uparrow(d, f)}{\theta_{M_+}^\uparrow(d, f)} & \text{if } (d, f) \in \mathbb{D}_L := \mathbb{Q}_{M_-}^\uparrow \cap \mathbb{Q}_{M_+}^\downarrow \\ \omega_P(d, f) := \frac{\theta_{M_-}^\uparrow(d, f)}{\theta_{M_+}^\uparrow(d, f)} & \text{if } (d, f) \in \mathbb{D}_P := \mathbb{Q}_{M_+}^\uparrow \end{cases}$$

where, for ease of comparison, we have used the same notations as in [8], for what concerns the subscripts  $S, L, P$ . We shall prove the following assertions which, incidentally, correspond to the second figure of the ones numbered as 12 in [8], but obtained numerically therein.

- (S) The function  $\omega_S(d, \cdot)$  increases from a positive finite value to  $+\infty$  (as  $f \rightarrow f_{M_-}^s(d)^-$ ) while  $f \in \mathbb{D}_{S,d}$ .
- (L) The function  $\omega_L(d, \cdot)$  decreases from  $+\infty$  (as  $f \rightarrow f_{M_-}^s(d)^+$ ) to 0 (as  $f \rightarrow f_{M_+}^s(d)^-$ ) while  $f \in \mathbb{D}_{S,d}$ .
- (P) The function  $\omega_P(d, f)$  increases from 0 (as  $f \rightarrow f_{M_+}^s(d)^+$ ) to a positive finite value while  $f \in \mathbb{D}_{P,d}$ .

The only assertions among the one listed above which deserve a discussion concern the increasing monotonicity of  $\omega_S(d, \cdot)$  and of  $\omega_P(d, \cdot)$ , as the limiting values of  $\omega_S(d, f)$ ,  $\omega_L(d, f)$  and  $\omega_P(d, f)$  as  $f \rightarrow f_{M_\pm}^s(d)^\pm$  as well as the decreasing monotonicity of  $\omega_L(d, \cdot)$  are a trivial consequence of the formula (15) and Theorem 1.1. In turn, the proofs of the increasing monotonicity of  $\omega_S(d, \cdot)$  and of  $\omega_P(d, \cdot)$  are similar, so we only discuss the increasing monotonicity of  $\omega_S(d, \cdot)$ . But the formula of the  $f$ -derivative of  $\omega_S$

$$\partial_f \omega_S(d, f) = \partial_f \left( \frac{\theta_{M_-}^\downarrow(d, f)}{\theta_{M_+}^\downarrow(d, f)} \right) = \omega_S(d, f) \left( \frac{\partial_f \theta_{M_-}^\downarrow}{\theta_{M_-}^\downarrow} - \frac{\partial_f \theta_{M_+}^\downarrow}{\theta_{M_+}^\downarrow} \right)$$

shows that this is equivalent to check that the function  $\frac{\partial_f \theta_M^\downarrow}{\theta_M^\downarrow}$  is decreasing with  $M$ . Taking then the  $M$ -derivative of  $\frac{\partial_f \theta_M^\downarrow}{\theta_M^\downarrow}$ , this finally reduces to verify that

$$\theta_M^\downarrow \partial_{fM}^2 \theta_M^\downarrow - \partial_M \theta_M^\downarrow \partial_f \theta_M^\downarrow < 0.$$

<sup>2</sup>As a matter of fact, when  $M_- \neq M_+$ , Theorem 1.1 and the formula in (15) show a little more, namely, that  $\tau_-(d, \cdot)$  and  $\tau_+(d, \cdot)$  are respectively differentiable and monotone for  $f \in \mathbb{P}_d \cup \{f = f_{M_+}^s(d)\}$ ,  $f \in \mathbb{P}_d \cup \{f = f_{M_-}^s(d)\}$ , respectively.

Apart for a negligible positive factor, the left hand side of this inequality can be written as

$$-3 \int_0^2 f(x)^2 p(x) dx \int_0^2 p(y) dy + \int_0^2 f(x) p(x) dx \int_0^2 f(y) p(y) dy$$

with

$$f(x) := \frac{1}{(M^2 x^2 - 2fx + d^2)}, \quad p(x) := \frac{1}{\sqrt{x(2-x)(M^2 x^2 - 2fx + d^2)}}.$$

An application of the Chebychev Integral Inequality (Proposition A.1) with  $f = g$  concludes the proof.  $\square$

**1.4 On the proof of Theorem 1.1** We shall prove a more general result (Proposition 3.1 below) which, in combination with the classical representation of the periods (Proposition 2.1) implies, in particular, Theorem 1.1. In turn, the proof of Proposition 3.1 will be developed in two steps, corresponding to Sections 3 and 4, respectively. Such two steps correspond to the proof of Proposition 3.1 for values of the parameters  $(d, f)$  in (10) belonging to two different subsets of  $\mathbb{Q}_M$  (subsets whose union gives  $\mathbb{Q}_M$ , possibly deprived of a finite number of curves). Such two proofs are completely different one from the other, as the proof in Sections 3 uses dynamical arguments, while the one in Section 4 is purely analytic, and uses the theory of elliptic functions. Here we provide a brief account of the former only. All starts with observing that, due to the separability of  $J_{\text{reg}}$  in (6),  $\tau_+$  and  $\tau_-$  depend on the masses  $m$  and  $M$  only via the combinations  $M_+$ ,  $M_-$  in (5). We write  $\tau_+ = \tau_+(M_+)$ ,  $\tau_- = \tau_-(M_-)$ . Then it is quite natural to try to reconstruct  $\tau_+$ ,  $\tau_-$  from the expressions of the same quantities of the case when  $m = 0$  and  $M \in \{M_+, M_-\}$ . But (provided that  $M_- \neq 0$ ) this is nothing else than looking at the orbits of the Kepler Hamiltonian  $K$  (given in (45) below) with sun mass equal to  $M_-$ ,  $M_+$ , respectively, on the level set defined by  $F_0$  and  $K = J_0$ , along the time  $\tau$  in (8). As quasi-periodic orbits of the Kepler problem are actually periodic, the values of  $\tau_+(M_+)$  and  $\tau_-(M_-)$  for such case coincide, and, calling  $\tau_M^{\text{Kep}}$  such common value, it turns out that the domains of  $\tau_{M_+}^{\text{Kep}}$ ,  $\tau_{M_-}^{\text{Kep}}$  are strictly smaller than the original domain of  $\tau_+(M_+)$ ,  $\tau_-(M_-)$ . Were it not enough, we shall be able to derive a really simple formula for  $\tau_M^{\text{Kep}}$  (corresponding to the second line in (15)) only on a smaller subset of the domain of  $\tau_M^{\text{Kep}}$ . To extend the formulae of  $\tau_{M_+}^{\text{Kep}}$ ,  $\tau_{M_-}^{\text{Kep}}$  on the original domains of  $\tau_+(M_+)$ ,  $\tau_-(M_-)$ , and show that such extended formulae coincide with the correct formulae of  $\tau_+(M_+)$ ,  $\tau_-(M_-)$ , we shall need an analytic discussion, which we provide in Section 4.

## 2 Classical formulae for Jacobi periods

Define the function

$$\tau_M := \begin{cases} \tau_M^\uparrow := 2\sqrt{\frac{2d}{v_0}} \int_{-1}^1 \frac{dx}{\sqrt{(1-x^2)(d^2x^2 - 4dMx + 4f - d^2)}} & \text{if } (d, f) \in \mathbb{Q}_M^\uparrow \\ \tau_M^\downarrow := 2\sqrt{\frac{2d}{v_0}} \int_{-1}^{x_M(d, f)} \frac{dx}{\sqrt{(1-x^2)(d^2x^2 - 4dMx + 4f - d^2)}} & \text{if } (d, f) \in \mathbb{Q}_M^\downarrow \end{cases} \quad (16)$$

where  $x_M(d, f)$  is the minimum root of  $d^2x^2 - 4dMx + 4f - d^2$  and  $\mathbb{Q}_M^\downarrow$ ,  $\mathbb{Q}_M^\uparrow$  are defined as in the introduction. The purpose of this paper is to show that the classical analysis reviewed in Section 1 leads to the following

**Proposition 2.1** *The function  $\tau_M$  is well-defined on  $\mathbb{Q}_M$ , tends to  $+\infty$  as  $f \rightarrow f_M^s(d)^\pm$  and, moreover, the following identity holds*

$$\tau_\pm = \tau_{M_\pm}|_{\mathbb{P}} \quad \forall (d, f) \in \mathbb{P}$$

**Remark 2.1** Remark that  $\tau_M^\uparrow = \theta_M^\uparrow$ , while  $\tau_M^\downarrow$  is much more complicated compared to  $\theta_M^\downarrow$  (compare (15) and (16)). In particular, the proof of Theorem 1.2 based on the formulae in (16) does not seem simple.

The separability of the Hamiltonian  $J_{\text{reg}}$  in (6) allows to consider the motions of the variables  $\alpha$  and  $\beta$  independently one of the other. In particular, one can first define the two respective ‘‘Hill sets’’

$$\mathbb{H}_+ := \left\{ (J_0, F_0, \Theta_0) \in \mathbb{R}^3 : \mathbb{A}(J_0, F_0, \Theta_0) \neq \emptyset \right\}, \quad \mathbb{H}_- := \left\{ (J_0, F_0, \Theta_0) \in \mathbb{R}^3 : \mathbb{B}(J_0, F_0, \Theta_0) \neq \emptyset \right\} \quad (17)$$

where

$$\begin{aligned} \mathbb{A}(J_0, F_0, \Theta_0) &:= \left\{ \alpha \in (1, +\infty) : (\alpha^2 - 1)(2J_0 v_0^2 \alpha^2 + 2M_+ v_0 \alpha - (2J_0 v_0^2 + F_0)) - \Theta_0^2 \geq 0 \right\} \\ \mathbb{B}(J_0, F_0, \Theta_0) &:= \left\{ \beta \in (-1, 1) : (1 - \beta^2)(-2J_0 v_0^2 \beta^2 - 2M_- v_0 \beta + (2J_0 v_0^2 + F_0)) - \Theta_0^2 \geq 0 \right\} \end{aligned}$$

and next consider the ‘‘Hill set of  $J_{\text{reg}}$ ’’, defined as the intersection

$$\mathbb{H} := \mathbb{H}_+ \cap \mathbb{H}_- .$$

Likewise, one can depict the set of parameters giving rise to periodic orbits for  $J_{\text{reg}}$  as the intersection of analogous sets for the two planes, separately:

$$\mathbb{P}(M_+, M_-) := \mathbb{P}_+(M_+) \cap \mathbb{P}_-(M_-) \quad (18)$$

where

$$\begin{aligned} \mathbb{P}_+(M_+) &:= \left\{ (J_0, F_0, \Theta_0) \in \mathbb{H}_+ : \text{the } \alpha\text{-level curves are smooth, closed and connected} \right\} \\ \mathbb{P}_-(M_-) &:= \left\{ (J_0, F_0, \Theta_0) \in \mathbb{H}_- : \text{the } \beta\text{-level curves are smooth, closed and connected} \right\} \end{aligned}$$

where ‘‘ $\alpha, \beta$  level curves’’ are defined as the curves in the planes  $(\alpha, \alpha')$ ,  $(\beta, \beta')$  defined<sup>3</sup> via

$$\alpha' = \pm \sqrt{(\alpha^2 - 1)(2J_0 v_0^2 \alpha^2 + 2M_+ v_0 \alpha - (2J_0 v_0^2 + F_0)) - \Theta_0^2} \quad (19)$$

$$\beta' = \pm \sqrt{(1 - \beta^2)(-2J_0 v_0^2 \beta^2 - 2M_- v_0 \beta + (2J_0 v_0^2 + F_0)) - \Theta_0^2} \quad (20)$$

for  $(J_0, F_0, \Theta_0) \in \mathbb{H}_+, \mathbb{H}_-$ , respectively. The classical analysis reviewed in Section 1 leads to the

<sup>3</sup>Combining the evolution equation for  $\alpha'$  obtained from the Hamilton equations of  $J_+$  and fixing the level (9) one obtains

$$\alpha' = A(\alpha^2 - 1) = \pm \sqrt{(\alpha^2 - 1)(2J_0 v_0^2 \alpha^2 + 2M_+ v_0 \alpha - (2J_0 v_0^2 + F_0)) - \Theta_0^2}$$

The equation for  $\beta'$  is obtained similarly.

following expressions for the periods

$$\begin{aligned}\tau_+(J_0, F_0, \Theta_0) &= 2 \int_{\alpha_{\min}(J_0, F_0, \Theta_0)}^{\alpha_{\max}(J_0, F_0, \Theta_0)} \frac{d\alpha}{\sqrt{(\alpha^2 - 1)(2J_0 v_0^2 \alpha^2 + 2M_+ v_0 \alpha - (F_0 + 2J_0 v_0^2)) - \Theta_0^2}} \\ \tau_-(J_0, F_0, \Theta_0) &= 2 \int_{\beta_{\min}(J_0, F_0, \Theta_0)}^{\beta_{\max}(J_0, F_0, \Theta_0)} \frac{d\beta}{\sqrt{(1 - \beta^2)(-2J_0 v_0^2 \beta^2 - 2M_- v_0 \beta + (F_0 + 2J_0 v_0^2)) - \Theta_0^2}}\end{aligned}\quad (21)$$

where  $\alpha_{\min}$ ,  $\alpha_{\max}$  ( $\beta_{\min}$ ,  $\beta_{\max}$ ) are the intersections of the  $\alpha$  ( $\beta$ ) smooth level curves with the axis  $\alpha' = 0$  ( $\beta' = 0$ ). From now on, we focus on the case  $\Theta_0 = 0$  and use the parameters  $(d, f)$  in (10), instead of  $J_0, F_0$ . Let  $f_M^-(d)$ ,  $f_M^+(d)$  and  $f_M^s(d)$  be defined as

$$\begin{aligned}f_M^-(d) &:= -Md \\ f_M^s(d) &:= \begin{cases} Md & \text{if } 0 < d \leq 2M \\ M^2 + \frac{d^2}{4} & \text{if } d > 2M \end{cases} \quad f_M^+(d) := \begin{cases} Md & \text{if } d > 2M \\ M^2 + \frac{d^2}{4} & \text{if } 0 < d \leq 2M \end{cases}\end{aligned}\quad (22)$$

Here, we have assigned the superscripts “+”, “-” and “s” to recall the words “maximum”, “minimum” and “singular”, as in the Keplerian case ( $M_+ = M_- = M$ ),  $f_M^+(d)$ ,  $f_M^-(d)$  are, respectively, the maximum and the minimum possible values of  $f$ , while  $f_M^s(d)$  is the unique value of  $f$  such that the Jacobi period is not defined (compare Proposition 3.3 below).

**Proposition 2.2** *The sets  $\mathbb{P}_+(M_+)$ ,  $\mathbb{P}_-(M_-)$  are given by*

$$\begin{aligned}\mathbb{P}_+(M_+) &= \{(d, f) \in \mathbb{R}^2 : d \geq 0, \quad f \leq f_{M_+}^+(d), \quad f \neq f_{M_+}^s(d)\} \\ \mathbb{P}_-(M_-) &= \{(d, f) \in \mathbb{R}^2 : d \geq 0, \quad f \geq -M_- d, \quad f \neq f_{M_-}^s(d)\}\end{aligned}\quad (23)$$

The periods  $\tau_+(M_+)$ ,  $\tau_-(M_-)$  in (21) are given by

$$\tau_-(M_-) = 2\sqrt{\frac{2d}{v_0}} \int_{-1}^{\min\{1, \beta_-\}} \frac{d\beta}{\sqrt{(1 - \beta^2)(d^2 \beta^2 - 4dM_- \beta + 4f - d^2)}} \quad \text{if } (d, f) \in \mathbb{P}_-(M_-) \quad (24)$$

$$\tau_+(M_+) = 2\sqrt{\frac{2d}{v_0}} \int_{\max\{\alpha_-, 1\}}^{\alpha_+} \frac{d\alpha}{\sqrt{(\alpha^2 - 1)(-d^2 \alpha^2 + 4dM_+ \alpha - 4f + d^2)}} \quad \text{if } (d, f) \in \mathbb{P}_+(M_+) \quad (25)$$

where

$$\alpha_{\pm} := \frac{2}{d} \left( M_{\pm} \pm \sqrt{M_{\pm}^2 + \frac{d^2}{4} - f} \right), \quad \beta_-(d, f) = \begin{cases} \frac{2}{d} \left( M_- - \sqrt{M_-^2 + \frac{d^2}{4} - f} \right) & \text{if } f \leq M_-^2 + \frac{d^2}{4} \\ 1 & \text{otherwise.} \end{cases}\quad (26)$$

Furthermore, the following identity holds:

$$\tau_+(M) = \tau_-(M) \quad \text{if } (d, f) \in \mathbb{P}(M, M). \quad (27)$$

**Proof** Let  $\Theta_0 = 0$ . It is immediate to see that values of  $J_0 \geq 0$  do not give rise to periodic motions, so we restrict to  $J_0 < 0$ , and denote as  $\bar{\mathbb{P}}_{\pm} := \mathbb{H}_{\pm} \cap \{J_0 < 0\}$ . Using the parameters  $(d, f)$  in (10), we show that

$$\bar{\mathbb{P}}_+ = \{(d, f) \in \mathbb{R}^2 : d \geq 0, \quad f \leq f_{M_+}^+(d)\}, \quad \bar{\mathbb{P}}_- = \{(d, f) \in \mathbb{R}^2 : d \geq 0, \quad f \geq -M_- d\}. \quad (28)$$

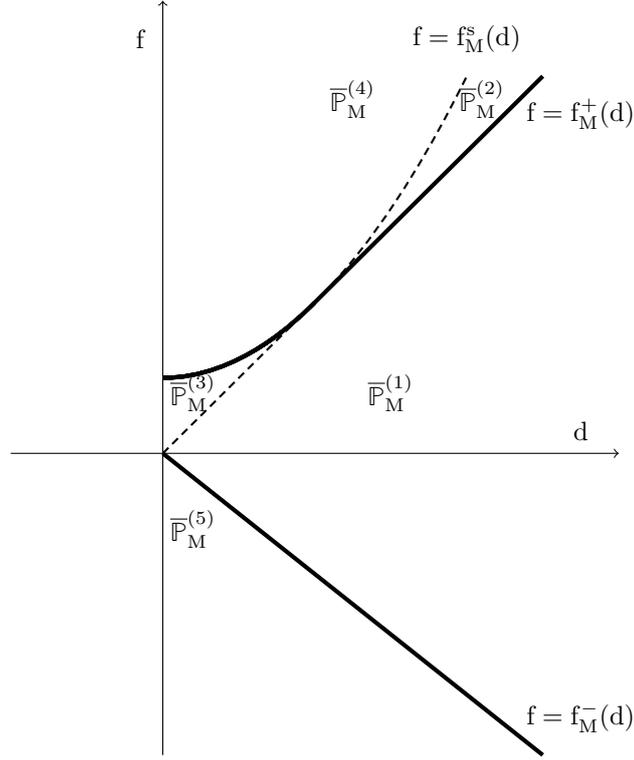


Figure 3: Graphical representation of the lines  $f = f_M^-(d)$  (thick),  $f = f_M^+(d)$  (thick),  $f = f_M^s(d)$  (dashed) and of the domains  $\bar{\mathbb{P}}_M^{(i)}$ , in the plane  $(d, f)$ .

Conditions in (17) are that the systems

$$\begin{cases} d^2\alpha^2 - 4M_+d\alpha + 4f - d^2 \leq 0 \\ \alpha \geq 1 \end{cases} \quad \begin{cases} d^2\beta^2 - 4M_-d\beta + 4f - d^2 \geq 0 \\ -1 \leq \beta \leq 1 \end{cases}$$

have non-empty solutions for  $\alpha, \beta$ , respectively. Introducing the notations

$$\begin{aligned} \bar{\mathbb{P}}_M^{(1)} &:= \left\{ (d, f) \in \mathbb{R}^2 : d \geq 0, -Md \leq f \leq Md \right\} \\ \bar{\mathbb{P}}_M^{(2)} &:= \left\{ (d, f) \in \mathbb{R}^2 : d \geq 2M, Md \leq f \leq M^2 + \frac{d^2}{4} \right\} \\ \bar{\mathbb{P}}_M^{(3)} &:= \left\{ (d, f) \in \mathbb{R}^2 : 0 \leq d \leq 2M, Md \leq f \leq M^2 + \frac{d^2}{4} \right\} \\ \bar{\mathbb{P}}_M^{(4)} &:= \left\{ (d, f) \in \mathbb{R}^2 : d \geq 0, f \geq M^2 + \frac{d^2}{4} \right\} \\ \bar{\mathbb{P}}_M^{(5)} &:= \left\{ (d, f) \in \mathbb{R}^2 : d \geq 0, f \leq -Md \right\} \end{aligned} \tag{29}$$

(see Fig. 3) one sees that the solutions of such systems are

$$\begin{cases} \alpha_- \leq \alpha \leq \alpha_+ & \text{if } (d, f) \in \overline{\mathbb{P}}_{M_+}^{(3)} \\ 1 \leq \alpha \leq \alpha_+ & \text{if } (d, f) \in \overline{\mathbb{P}}_{M_+}^{(1)} \cup \overline{\mathbb{P}}_{M_+}^{(5)} \\ \emptyset & \text{otherwise} \end{cases}
\begin{cases} -1 \leq \beta \leq 1 & \text{if } (d, f) \in \overline{\mathbb{P}}_{M_-}^{(3)} \cup \overline{\mathbb{P}}_{M_-}^{(4)} \\ -1 \leq \beta \leq \beta_- & \text{if } (d, f) \in \overline{\mathbb{P}}_{M_-}^{(1)} \\ -1 \leq \beta \leq \beta_- \ \& \ \beta_+ \leq \beta \leq 1 & \text{if } (d, f) \in \overline{\mathbb{P}}_{M_-}^{(2)} \\ \emptyset & \text{otherwise} \end{cases} \quad (30)$$

where

$$\alpha_{\pm} := \frac{2}{d} \left( M_{\pm} \pm \sqrt{M_{\pm}^2 + \frac{d^2}{4} - f} \right), \quad \beta_{\pm} = \frac{2}{d} \left( M_{\pm} \pm \sqrt{M_{\pm}^2 + \frac{d^2}{4} - f} \right).$$

Taking the union of the values of  $(d, f)$  which provide non-empty solutions separately for  $\alpha$  and  $\beta$ , we obtain the sets

$$\overline{\mathbb{P}}_+ = \overline{\mathbb{P}}_{M_+}^{(1)} \cup \overline{\mathbb{P}}_{M_+}^{(3)} \cup \overline{\mathbb{P}}_{M_+}^{(5)}, \quad \overline{\mathbb{P}}_- = \overline{\mathbb{P}}_{M_-}^{(1)} \cup \overline{\mathbb{P}}_{M_-}^{(2)} \cup \overline{\mathbb{P}}_{M_-}^{(3)} \cup \overline{\mathbb{P}}_{M_-}^{(4)}.$$

These are just the sets in (28). The level curves in (19), (20) for  $(d, f) \in \overline{\mathbb{P}}_+, \overline{\mathbb{P}}_-$ , respectively, are

$$\alpha' = \pm \sqrt{\frac{v_0}{2d}} \sqrt{(\alpha^2 - 1)(-d^2\alpha^2 + 4dM_+\alpha - 4f + d^2)} \quad (31)$$

$$\beta' = \pm \sqrt{\frac{v_0}{2d}} \sqrt{(1 - \beta^2)(d^2\beta^2 - 4M_-d\beta + 4f - d^2)}. \quad (32)$$

They intersect the horizontal axes, in both  $(\alpha, \alpha')$  and  $(\beta, \beta')$  planes at the end-points of the non-empty solutions for  $\alpha, \beta$  (respectively) in (30). The sets  $\mathbb{P}_{\pm}$  is then the subset of  $\overline{\mathbb{P}}_{\pm}$  such that the polynomials under the square roots in (31), (32) have not double roots at such end-points. But such double roots occur if and only if  $f = f_{M_+}^s(d)$  (for the  $(\alpha, \alpha')$  level curve), or  $f = f_{M_-}^s(d)$  (for the  $(\beta, \beta')$  level curve). Then  $\mathbb{P}_{\pm} = \overline{\mathbb{P}}_{\pm} \setminus \{f = f_{M_{\pm}}^s(d)\}$ . These are precisely the sets in (23). At this point, the formulae in (24), (25) immediately follow, with the observation that if  $(d, f) \in \overline{\mathbb{P}}_{M_-}^{(2)} \cap \mathbb{P}$ , two periodic orbits for  $\beta$  arise (one with  $-1 \leq \beta \leq \beta_-$ , another with  $\beta_+ \leq \beta \leq 1$ ), but they have the same period, by an easy consequence of Cauchy Theorem. Also the equality in (27) follows from the Cauchy Theorem. It is crucial to observe at this respect that, while  $(d, f) \in \mathbb{P}(M, M)$ , the roots of the polynomial  $d^2x^2 - 4dMx + 4f - d^2$  are real-valued, and never fall in  $(-\infty, -1)$ .  $\square$

The sets  $\mathbb{P}_+, \mathbb{P}_+$  in (23) are strictly larger than  $\mathbb{P}(M_+, M_-)$  in (18). We show that if  $(d, f) \in \mathbb{P}(M_+, M_-)$  then the formulae for  $\tau_-, \tau_+$  obtained through (24), (25) coincide with the ones obtained through Equation (16) and Proposition 2.1.

**Proof of Proposition 2.1** We first check the formula for  $\tau_+$ . At this purpose, observe that, as  $M_+ \geq M_-$ , then  $\mathbb{P}(M_+, M_-) \subset \mathbb{P}(M_+, M_+)$ . Then we can use (27) with  $M = M_+$  and (24) with  $M_-$  replaced by  $M_+$ . We obtain

$$\tau_+(M_+) = \tau_-(M_+) = 2\sqrt{\frac{2d}{v_0}} \int_{-1}^{\min\{1, \beta_+^+\}} \frac{d\beta}{\sqrt{(1 - \beta^2)(d^2\beta^2 - 4dM_+\beta + 4f - d^2)}}$$

for all  $(d, f) \in \mathbb{P}(M_+, M_-)$ , with (observing that the occurrence  $f > M_+^2 + \frac{d^2}{4}$  never happens for  $(d, f) \in \mathbb{P}(M_+, M_-)$ )

$$\beta_{\pm}^{\pm} = \frac{2}{d} \left( M_+ - \sqrt{M_+^2 + \frac{d^2}{4} - f} \right)$$

being the minimal root of the polynomial  $d^2\beta^2 - 4dM_+\beta + 4f - d^2$ , hence real for all  $(d, f) \in \mathbb{P}(M_+, M_-)$ . It is immediate to check that  $\beta_{\pm}^{\pm} > 1$  if and only if  $(d, f) \in \mathbb{P}(M_+, M_-) \cap \mathbb{Q}_{M_+}^{\uparrow}$ , whence we have nothing else to prove for  $\tau_+$ . Now we check the formula for  $\tau_-$ . The definition of  $\beta_-$  in (26) implies that the supremum in the integral (24) is strictly less than 1, and coincides with the minimum root of  $d^2x^2 - 4dMx + 4f - d^2$  if and only if  $(d, f) \in \mathbb{P}(M_+, M_-) \cap \mathbb{Q}_{M_-}^{\downarrow}$ .  $\square$

### 3 Dynamical part

The purpose of this and the next section is to provide a new formula for the function  $\tau_M$  in (16). Especially, we aim at simplifying its expression on  $\mathbb{Q}_M^{\downarrow}$ .

Define the function

$$t_M = \begin{cases} t_M^{\uparrow} := \sqrt{\frac{2d}{v_0}} \int_0^{z_M(d,f)} \frac{dz}{\sqrt{z(2-z)(M^2z^2 - 2fz + d^2)}} & \text{if } (d, f) \in \mathbb{Q}_M^{\uparrow} \\ t_M^{\downarrow} := \sqrt{\frac{2d}{v_0}} \int_0^2 \frac{dz}{\sqrt{z(2-z)(M^2z^2 - 2fz + d^2)}} & \text{if } (d, f) \in \mathbb{Q}_M^{\downarrow} \end{cases}$$

where  $z_M(d, f)$  is the minimum roots of  $M^2z^2 - 2fz + d^2$ , turning out to be real while  $(d, f) \in \mathbb{Q}_M^{\uparrow}$ . We shall prove that

**Proposition 3.1** *The function  $t_M$  is well-defined on  $\mathbb{Q}_M$  and verifies*

$$t_M = \tau_M \quad \forall M \geq 0, \quad \forall (d, f) \in \mathbb{Q}_M$$

Proposition 3.1 will be proved in two steps, the former developed in this section, the latter in the next one. Similarly as in the proof of Proposition 2.2, we consider the following subsets of  $\mathbb{Q}_M$

$$\begin{aligned} \mathbb{P}_M^{(1)} &:= \left\{ (d, f) \in \mathbb{R}^2 : d \geq 0, -Md < f < Md \right\} \\ \mathbb{P}_M^{(2)} &:= \left\{ (d, f) \in \mathbb{R}^2 : d \geq 2M, Md < f < M^2 + \frac{d^2}{4} \right\} \\ \mathbb{P}_M^{(3)} &:= \left\{ (d, f) \in \mathbb{R}^2 : 0 < d < 2M, Md < f < M^2 + \frac{d^2}{4} \right\} \\ \mathbb{P}_M^{(4)} &:= \left\{ (d, f) \in \mathbb{R}^2 : d \geq 0, f > M^2 + \frac{d^2}{4} \right\} \end{aligned} \quad (33)$$

which are the inner parts of the ones in (29), hence are open. The union the  $\mathbb{P}_M^{(j)}$ 's above coincides with  $\mathbb{Q}_M$ , apart for the boundaries of the  $\mathbb{P}_M^{(j)}$  belonging to  $\mathbb{Q}_M$ , which may be neglected for the purpose of proving Proposition 3.1. Note that, in the case  $M = 0$ ,  $\mathbb{P}_0^{(1)} = \mathbb{P}_0^{(3)} = \emptyset$ . In this section, we prove that

$$\tau_M = \sqrt{\frac{2d}{v_0}} \int_0^2 \frac{dz}{\sqrt{z(2-z)(M^2z^2 - 2fz + d^2)}} \quad \forall M > 0, \quad \forall (d, f) \in \mathbb{P}_M^{(1)} \quad (34)$$

In the next Section 4, we shall prove

$$\tau_M = \sqrt{\frac{2d}{v_0}} \int_0^{\min\{2, z_M(d,f)\}} \frac{dz}{\sqrt{z(2-z)(M^2z^2 - 2fz + d^2)}} \quad \forall M \geq 0, \quad \forall (d, f) \in \mathbb{P}_M^{(2)} \cup \mathbb{P}_M^{(3)} \cup \mathbb{P}_M^{(4)} \quad (35)$$

(this equality includes the case  $M = 0$ , where one has to keep in mind  $\mathbb{P}_0^{(3)} = \emptyset$ ). Using

$$\overline{\mathbb{P}^{(1)} \cup \mathbb{P}^{(2)}} = \overline{\mathbb{Q}_M^{\downarrow}}, \quad \overline{\mathbb{P}^{(3)} \cup \mathbb{P}^{(4)}} = \overline{\mathbb{Q}_M^{\uparrow}}, \quad z_M(d, f) \geq 2, \quad \forall (d, f) \in \mathbb{P}_M^{(2)}$$

The ‘‘bar’’ denotes set closure, we have that (34) and (35) imply Proposition 3.1.

### 3.1 Set up

We fix the following terms.

- The coordinates

$$(A(\mathbf{y}, \mathbf{x}), B(\mathbf{y}, \mathbf{x}), \Theta(\mathbf{y}, \mathbf{x}), \alpha(\mathbf{x}), \beta(\mathbf{x}), \vartheta(\mathbf{x}))$$

described in the introduction will be referred to as *co-focal coordinates associated to*  $(\mathbf{y}, \mathbf{x})$ .

- Time  $\tau$ -solutions<sup>4</sup>

$$\begin{cases} \tau \rightarrow (A(\tau, J_0, F_0, \Theta_0, \alpha_0), \alpha(\tau, J_0, F_0, \Theta_0, \alpha_0)) \\ \tau \rightarrow (B(\tau, J_0, F_0, \Theta_0, \beta_0), \beta(\tau, J_0, F_0, \Theta_0, \beta_0)) \end{cases} \quad (36)$$

of the Hamiltonians  $J_+$ ,  $J_-$  in (7) satisfying (9) and, moreover,

$$\begin{cases} \alpha(0, J_0, F_0, \Theta_0, \alpha_0) = \alpha_0 \\ \beta(0, J_0, F_0, \Theta_0, \beta_0) = \beta_0 \end{cases}$$

are called *Jacobi solutions determined by*  $(J_0, F_0, \Theta_0, \alpha_0, \beta_0)$ .

- The periods of the Jacobi solutions (36), namely, the functions  $\tau_+(J_0, F_0, \Theta_0)$ ,  $\tau_-(J_0, F_0, \Theta_0)$  in (21), will be called *Jacobi periods*.
- Time  $t$ -solutions

$$\begin{cases} t \rightarrow (\bar{A}(t, J_0, F_0, \Theta_0, \alpha_0, \beta_0), \bar{\alpha}(t, J_0, F_0, \Theta_0, \alpha_0, \beta_0)) \\ t \rightarrow (\bar{B}(t, J_0, F_0, \Theta_0, \alpha_0, \beta_0), \bar{\beta}(t, J_0, F_0, \Theta_0, \alpha_0, \beta_0)) \end{cases} \quad (37)$$

of the Hamiltonian  $J$  in (4) verifying the second equality in (4), the equalities in (9) and, moreover,

$$\begin{cases} \bar{\alpha}(0, J_0, F_0, \Theta_0, \alpha_0, \beta_0) = \alpha_0 \\ \bar{\beta}(0, J_0, F_0, \Theta_0, \alpha_0, \beta_0) = \beta_0 \end{cases}$$

will be called *natural solutions determined by*  $(J_0, F_0, \Theta_0, \alpha_0, \beta_0)$ .

- The periods of the natural solutions (37) will be called *natural periods*, and denoted as  $\bar{\tau}_+(J_0, F_0, \Theta_0)$ ,  $\bar{\tau}_-(J_0, F_0, \Theta_0)$ .

### 3.2 Jacobi periods

In this section we construct the natural solutions (37) starting from the solutions of the Hamiltonian  $J$  in (1), regarding the parameters  $(J_0, F_0, \Theta_0)$  as the fixed values of their corresponding first integrals.

As we shall exploit the “Keplerian limit” (namely, the limit of (1) when  $m \rightarrow 0$ ), we preliminarily shift the initial coordinates  $(\mathbf{y}, \mathbf{x})$  in (1) as

$$(\mathbf{y}, \mathbf{x}) \rightarrow (\mathbf{y}, \mathbf{x} - \mathbf{v}_0). \quad (38)$$

---

<sup>4</sup>As a rule, when the discussion refers to the general problem, we keep the initial parameters  $(J_0, F_0, \Theta_0)$ . The parameters  $(d, f)$  in (10) will be used only in the planar case  $(\Theta_0 = 0)$ .

This carries  $J$  to

$$J(\mathbf{y}, \mathbf{x}) = \frac{\|\mathbf{y}\|^2}{2} - \frac{M}{\|\mathbf{x}\|} - \frac{m}{\|\mathbf{x} - 2\mathbf{v}_0\|}. \quad (39)$$

The function  $\Theta$  in (2) is left unvaried by the transformation (38). We let

$$F(\mathbf{y}, \mathbf{x}, M, m) = \|\mathbf{G}(\mathbf{y}, \mathbf{x})\|^2 - 2\mathbf{v}_0 \cdot \left( \mathbf{y} \times \mathbf{G}(\mathbf{y}, \mathbf{x}) - M \frac{\mathbf{x}}{\|\mathbf{x}\|} + m \frac{\mathbf{x} - 2\mathbf{v}_0}{\|\mathbf{x} - 2\mathbf{v}_0\|} \right) \quad (40)$$

where

$$\mathbf{G}(\mathbf{y}, \mathbf{x}) := \mathbf{x} \times \mathbf{y} \quad (41)$$

the angular<sup>5</sup> momentum. The function  $F(\mathbf{y}, \mathbf{x}, M, m)$  in (40) will be referred to as *Euler Integral*.

**Proposition 3.2** *The couple of functions (37) is a natural solution determined by  $(J_0, F_0, \Theta_0, \alpha_0, \beta_0)$  if and only if there exists a solution*

$$t \rightarrow (\bar{\mathbf{y}}(t, J_0, F_0, \Theta_0, \alpha_0, \beta_0), \bar{\mathbf{x}}(t, J_0, F_0, \Theta_0, \alpha_0, \beta_0))$$

of the Hamilton equations of  $J$  in (39), with initial datum  $(\mathbf{y}_0, \mathbf{x}_0)$  such that

$$\left\{ \begin{array}{l} J(\mathbf{y}_0, \mathbf{x}_0, M, m) = J_0 \\ F(\mathbf{y}_0, \mathbf{x}_0, M, m) = F_0 \\ \Theta(\mathbf{y}_0, \mathbf{x}_0) = \Theta_0 \\ \alpha(\mathbf{x}_0) = \alpha_0 \\ \beta(\mathbf{x}_0) = \beta_0. \end{array} \right. \quad (42)$$

such that (37) are the co-focal coordinates associated to

$$\left( \bar{\mathbf{y}}(t, J_0, F_0, \Theta_0, \alpha_0, \beta_0), \bar{\mathbf{x}}(t, J_0, F_0, \Theta_0, \alpha_0, \beta_0) - \mathbf{v}_0 \right).$$

The proof of Proposition 3.2 is standard, hence is deferred to Section 3.5.

In view of the definitions above, we rewrite the relation in (8) between the times  $t$  and  $\tau$  as

$$\tau = \int_0^t \frac{dt'}{\|\bar{\mathbf{x}}(t', J_0, F_0, \Theta_0, \alpha_0, \beta_0)\| \|\bar{\mathbf{x}}(t', J_0, F_0, \Theta_0, \alpha_0, \beta_0) - 2\mathbf{v}_0\|} \quad (43)$$

The formula (43) allows to establish the following relation between natural and Jacobi periods:

$$\tau_{\pm}(J_0, F_0, \Theta_0) = \int_0^{\bar{t}_{\pm}(J_0, F_0, \Theta_0)} \frac{dt'}{\|\bar{\mathbf{x}}(t', J_0, F_0, \Theta_0, \alpha_0, \beta_0)\| \|\bar{\mathbf{x}}(t', J_0, F_0, \Theta_0, \alpha_0, \beta_0) - 2\mathbf{v}_0\|} \quad (44)$$

for any  $\bar{\mathbf{x}}(t, J_0, F_0, \Theta_0, \alpha_0, \beta_0)$  as in Proposition 3.2.

<sup>5</sup>Note that, because of the shift (38),  $\mathbf{G}(\mathbf{y}, \mathbf{x})$  in (41) is related to  $\mathbf{M}(\mathbf{y}, \mathbf{x})$  in (3) via

$$\mathbf{G} = \mathbf{M} + \mathbf{M}_0 \quad \text{with} \quad \mathbf{M}_0 := \mathbf{v}_0 \times \mathbf{y}$$

### 3.3 Jacobi Kepler period

In this section we assume  $M > 0$ . We consider the Kepler's Hamiltonian

$$K = \frac{\|\mathbf{y}\|^2}{2} - \frac{M}{\|\mathbf{x}\|} \quad (45)$$

which we regard as the limiting value of the Hamiltonian (39) when  $m = 0$ . We shall denote as

$$(\bar{\mathbf{y}}^{\text{Kep}}(t, J_0, F_0, \Theta_0, \alpha_0, \beta_0), \bar{\mathbf{x}}^{\text{Kep}}(t, J_0, F_0, \Theta_0, \alpha_0, \beta_0))$$

any solution of (45) verifying

$$\begin{cases} K(\bar{\mathbf{y}}^{\text{Kep}}, \bar{\mathbf{x}}^{\text{Kep}}) = \frac{\|\bar{\mathbf{y}}^{\text{Kep}}\|^2}{2} - \frac{M}{\|\bar{\mathbf{x}}^{\text{Kep}}\|} = J_0 \\ E(\bar{\mathbf{y}}^{\text{Kep}}, \bar{\mathbf{x}}^{\text{Kep}}) = \|\mathbf{G}(\bar{\mathbf{y}}^{\text{Kep}}, \bar{\mathbf{x}}^{\text{Kep}})\|^2 - 2\mathbf{v}_0 \cdot \left( \bar{\mathbf{y}}^{\text{Kep}} \times \mathbf{G}(\bar{\mathbf{y}}^{\text{Kep}}, \bar{\mathbf{x}}^{\text{Kep}}) - M \frac{\bar{\mathbf{x}}^{\text{Kep}}}{\|\bar{\mathbf{x}}^{\text{Kep}}\|} \right) = F_0 \\ \Theta(\bar{\mathbf{y}}^{\text{Kep}}, \bar{\mathbf{x}}^{\text{Kep}}) = \mathbf{G}(\bar{\mathbf{y}}^{\text{Kep}}, \bar{\mathbf{x}}^{\text{Kep}}) \cdot \mathbf{i} = \Theta_0 \end{cases} \quad (46)$$

As well-known, the Keplerian orbit  $t \rightarrow \bar{\mathbf{x}}^{\text{Kep}}(t, J_0, F_0, \Theta_0, \alpha_0, \beta_0)$  describes an ellipse, the semi-major axis, eccentricity, anomaly of perihelion of which will be denoted as  $a$ ,  $e$ ,  $\omega$ , respectively. The periods of the Keplerian orbits, here denoted as  $t_M^{\text{Kep}}$ , are well-known to depend only on the semi-major axis  $a$ , or, equivalently, on the value  $J_0$  of the energy:

$$t_M^{\text{Kep}}(J_0) = 2\pi \sqrt{\frac{a^3}{M}} = 2\pi M \sqrt{\frac{1}{8|J_0|^3}}$$

as  $a = \frac{M}{2|J_0|}$ . With the terminology of the previous section, we have that the natural periods  $\bar{t}_+$ ,  $\bar{t}_-$  for the Keplerian case coincide, and their common value is precisely  $t_M^{\text{Kep}}(J_0)$ . But then, also the Jacobi periods (*Jacobi Kepler periods*, hereafter) coincide. They will be denoted as  $\tau_M^{\text{Kep}}$  and, through (44), are given by

$$\tau_M^{\text{Kep}}(J_0, F_0, \Theta_0) = \int_0^{t_M^{\text{Kep}}(J_0)} \frac{dt'}{\|\bar{\mathbf{x}}^{\text{Kep}}(t', J_0, F_0, \Theta_0, \alpha_0, \beta_0)\| \|\bar{\mathbf{x}}^{\text{Kep}}(t', J_0, F_0, \Theta_0, \alpha_0, \beta_0) - 2\mathbf{v}_0\|} \quad (47)$$

It is convenient to use the eccentric anomaly  $\xi$  in the integral (47), as it is related to  $t'$  via:

$$\frac{dt'}{d\xi} = \sqrt{\frac{a}{M}} \|\bar{\mathbf{x}}^{\text{Kep}}\|$$

This way,  $\|\bar{\mathbf{x}}^{\text{Kep}}\|$  is cancelled, and (47) becomes

$$\tau_M^{\text{Kep}}(J_0, F_0, \Theta_0) = \sqrt{\frac{a}{M}} \int_0^{2\pi} \frac{d\xi}{\|\bar{\mathbf{x}}^{\text{Kep}}(\xi, J_0, F_0, \Theta_0, \alpha_0, \beta_0) - 2\mathbf{v}_0\|}$$

We now focus on the case  $\Theta_0 = 0$ . Using the orbital elements  $a$ ,  $e$ ,  $\omega$ , we obtain

$$\tau_M^{\text{Kep}}(J_0, F_0) = \sqrt{\frac{a}{M}} \int_0^{2\pi} \frac{d\xi}{\sqrt{a^2(1 - e \cos \xi)^2 - 4av_0((\cos \xi - e) \cos \omega - \sqrt{1 - e^2} \sin \xi \sin \omega) + 4v_0^2}} \quad (48)$$

where  $a$ ,  $e$  and  $\omega$  are to be regarded as functions of  $(J_0, F_0)$ . To this end, we rewrite two first equations in (46) using  $a$ ,  $e$  and  $\omega$ :

$$\begin{cases} -\frac{M}{2a} = J_0 \\ M(a(1-e^2) - 2v_0e \cos \omega) = F_0 \end{cases}$$

Switching, via (10), to  $(d, f)$ , they become

$$\begin{cases} a = \frac{2v_0M}{d} \\ e^2 + \frac{d}{M}e \cos \omega = 1 - \frac{f}{M^2} \end{cases} \quad (49)$$

Observe that we have only two parameters,  $(d, f)$ , for the three quantities  $a$ ,  $e$  and  $\omega$ . This will enable us to fix one among  $(e, \omega)$  arbitrarily. At this respect, the second equation above has non-empty solutions for  $(e, \omega) \in [0, 1) \times \mathbb{T}$  for any choice of  $(d, f)$  in the set

$$\overline{\mathbb{P}}_M^{\text{Kep}} := \{(d, f) : d \geq 0, \quad f_M^-(d) \leq f \leq f_M^+(d)\} \quad (50)$$

with  $f_M^-(d)$ ,  $f_M^+(d)$  the functions we have already encountered in (22). With

$$\mathbb{P}_M^{\text{Kep}} := \overline{\mathbb{P}}_M^{\text{Kep}} \setminus \{f = f_M^s(d)\} \quad (51)$$

we have

**Proposition 3.3**  $\tau_M^{\text{Kep}}(d, f)$  is bounded if and only if  $(d, f) \in \mathbb{P}_M^{\text{Kep}}$ . Moreover, it is given by

$$\begin{aligned} & \tau_M^{\text{Kep}}(d, f) \\ &= \sqrt{\frac{d}{2v_0}} \int_0^{2\pi} \frac{d\xi}{\sqrt{M^2(1-e \cos \xi)^2 - 2Md((\cos \xi - e) \cos \omega - \sqrt{1-e^2} \sin \xi \sin \omega) + d^2}} \end{aligned} \quad (52)$$

for all  $(d, f) \in \mathbb{P}_M^{\text{Kep}}$ . Here,  $(e, \omega)$  is any couple in  $[0, 1) \times \mathbb{T}$  solving

$$e^2 + \frac{d}{M}e \cos \omega = 1 - \frac{f}{M^2} \quad (53)$$

**Proof** The formula in (52) is the same as in (48), after the substituting  $a$  via (49). For any  $(d, f) \in \overline{\mathbb{P}}_M^{\text{Kep}}$ , the couple

$$(e_*, \omega_*) = \left( \frac{\widehat{e}}{M}, \cos^{-1} \sigma \right)$$

with

$$\widehat{e} := \sigma \left( -\frac{d}{2} + \sqrt{M^2 + \frac{d^2}{4} - f} \right) \quad \sigma := \begin{cases} +1 & \text{if } -Md \leq f < M^2 \\ -1 & \text{if } M^2 \leq f < f_M^+(d) \end{cases}$$

verifies  $(e_*, \omega_*) \in [0, 1) \times \mathbb{T}$  and solves (53). Inserting such values in (52), the term with  $\sin \xi$  is cancelled, and one obtains the formula

$$\tau_M^{\text{Kep}}(d, f) = \sqrt{\frac{d}{2v_0}} \int_0^{2\pi} \frac{d\xi}{\sqrt{\widehat{e}^2 \cos^2 \xi^2 - 2M(\widehat{e} + \sigma d) \cos \xi + M^2 + 2\sigma \widehat{e}d + d^2}} \quad (54)$$

Now, the integral in (54) is (real-valued and) finite provided that the polynomial

$$P(z) := \widehat{e}^2 z^2 - 2M(\widehat{e} + \sigma d)z + M^2 + 2\sigma \widehat{e}d + d^2$$

has no roots in  $\pm 1$  and no double roots in  $(-1, 1)$ . This defines the domain (51).  $\square$

The domains  $\mathbb{P}_{M_-}^{\text{Kep}}$ ,  $\mathbb{P}_{M_+}^{\text{Kep}}$  defined via (50), (51) are strictly smaller<sup>6</sup> than the domains  $\mathbb{P}_-$ ,  $\mathbb{P}_+$  in (23). By construction, the periods  $\tau_-$ ,  $\tau_+$  in (24), (25) coincide with  $\tau_{M_+}^{\text{Kep}}$ ,  $\tau_{M_-}^{\text{Kep}}$  limited to the values of  $(d, f) \in \mathbb{P}_{M_+}^{\text{Kep}}$ ,  $\mathbb{P}_{M_-}^{\text{Kep}}$ , respectively. Notice however that the formula (54) exhibited inside the proof of Proposition 3.3 is involved, and seems to be useless in order to prove Theorem 1.1. However, restricting  $(d, f)$  to the domains  $\mathbb{P}_{M_-}^{(1)}$ ,  $\mathbb{P}_{M_+}^{(1)}$  defined in (33), and moreover using  $-$  as in the proof of Proposition 3.3 – the remarked freedom of choice of one among  $(e, \omega)$  in equation (53), we obtain a simpler formula, as we show in the next section.

### 3.4 Proof of (34)

When  $(d, f)$  takes values in the set  $\mathbb{P}_M^{(1)}$  in (29), the limiting couple

$$(e^*, \omega^*) = \left( 1, -\cos^{-1} \frac{f}{Md} \right)$$

solves Equation (53), and provides (through Proposition 3.3) the expression

$$\begin{aligned} t_M^*(d, f) &= \sqrt{\frac{d}{2v_0}} \int_0^{2\pi} \frac{d\xi}{\sqrt{M^2(1 - \cos \xi)^2 - 2f(1 - \cos \xi) + d^2}} \\ &= \sqrt{\frac{2d}{v_0}} \int_0^2 \frac{dz}{\sqrt{z(2-z)(M^2 z^2 - 2fz + d^2)}} \end{aligned}$$

having let  $z := 1 - \cos \xi$  in the half-period. This proves (34).

### 3.5 Proof of Proposition 3.2

The proof of the following lemma is omitted:

**Lemma 3.1** *Using the coordinates  $(A, B, \Theta, \alpha, \beta, \vartheta)$  described in Section 1, the functions  $J$  and  $F$  in (39), (40) have the expressions*

$$\begin{cases} J(A, B, \Theta, \alpha, \beta, M, m) = \frac{1}{v_0^2(\alpha^2 - \beta^2)} (J_-(B, \beta, 0, \Theta, M_-) + J_+(A, \alpha, 0, \Theta, M_+)) \\ F(A, B, \Theta, \alpha, \beta, M, m) = 2 \frac{\alpha^2 - 1}{\alpha^2 - \beta^2} J_-(B, \beta, 0, \Theta, M_-) - 2 \frac{1 - \beta^2}{\alpha^2 - \beta^2} J_+(A, \alpha, 0, \Theta, M_+) \end{cases} \quad (55)$$

where  $J_+(A, \alpha, J_0, \Theta, M)$ ,  $J_-(B, \beta, \Theta, J_0, M)$  are as in (7),  $M_+$ ,  $M_-$  as in (5).

**Proof of Proposition 3.2** We look at the system (42) using the co-focal coordinates. Namely, we look at motions of  $J(A, B, \Theta, \alpha, \beta, M, m)$  in (4) with an initial datum  $(A(0), B(0), \Theta(0), \alpha(0), \beta(0))$

<sup>6</sup>Observe however that  $\mathbb{P}_{M_+}^{\text{Kep}}$  is larger than  $\mathbb{P}(M_+, M_-)$  in (18). This already allows us to identify  $\tau_+ = \tau_{M_+}^{\text{Kep}}$  for all  $(d, f) \in \mathbb{P}(M_+, M_-)$ , even though a similar formula for  $\tau_-$  is still missing at this stage.

such that

$$\begin{cases} J(A(0), B(0), \Theta(0), \alpha(0), \beta(0), M, m) = \overline{J_0} \\ F(A(0), B(0), \Theta(0), \alpha(0), \beta(0), M, m) = F_0 \\ \Theta(0) = \Theta_0 \\ \alpha(0) = \alpha_0 \\ \beta(0) = \beta_0. \end{cases} \quad (56)$$

Using (55), we see that, if  $(\alpha, \beta) \neq (1, \pm 1)$ , the system (56) is equivalent to

$$\begin{cases} J_+(A, \alpha, 0, \Theta, M_+) = v_0^2(\alpha^2 - 1)J_0 - \frac{F_0}{2} \\ J_-(B, \beta, 0, \Theta, M_-) = v_0^2(1 - \beta^2)J_0 + \frac{F_0}{2} \end{cases}$$

This is the same as (9), as  $J_+(A, \alpha, 0, \Theta, M_+) - v_0^2(\alpha^2 - 1)J_0 = J_+(A, \alpha, J_0, \Theta, M_+)$ , and  $J_-(B, \beta, 0, \Theta, M_-) - v_0^2(1 - \beta^2)J_0 = J_-(B, \beta, J_0, \Theta, M_-)$ .  $\square$

## 4 Analytic part

In this section, we prove a result on elliptic integrals. For notices on elliptic integrals, we refer to [2].

### 4.1 Equivalences of complex integrals

**Lemma 4.1** *Given  $\varrho \in \mathbb{C} \setminus \{\pm 1\}$ , the unique circle in  $\mathbb{C}$  through  $z = 1$ ,  $z = -1$  and  $z = \varrho$  has equation*

$$|z|^2 + \left( \frac{1 - |\varrho|^2}{\Im \varrho} \right) \Im z = 1 \quad (57)$$

**Definition 4.1** We denote as:

- $\mathcal{C}(\varrho)$  the circle in (57);
- $\mathcal{S}(\varrho)$  the segment of  $\mathcal{C}(\varrho)$  where  $\varrho$  does not belong to;
- $\mathcal{S}^*(\varrho) := \mathcal{C}(\varrho) \setminus \mathcal{S}(\varrho)$ ;
- $\mathcal{A}(\varrho)$  the closed bounded region delimited by  $\mathcal{S}(\varrho)$  and the interval  $[-1, 1]$  in the real axis;
- $\mathcal{A}^*(\varrho)$  the closed bounded region delimited by  $\mathcal{S}^*(\varrho)$  and the interval  $[-1, 1]$  in the real axis.

**Lemma 4.2** *Let  $\varrho \in \mathbb{C} \setminus \{\pm 1\}$ . The invertible transformation*

$$\Gamma_\varrho : \quad z \in \mathbb{C} \cup \{\infty\} \rightarrow \tau = \frac{1 - z\varrho}{z - \varrho} \in \mathbb{C} \cup \{\infty\}$$

*sends  $1 \rightarrow 1$ ,  $(-1) \rightarrow (-1)$ ,  $\infty \rightarrow \varrho$  and, moreover, the interval  $[-1, 1]$  in the  $z$ -plane to the path  $s_\varrho$  in the  $\tau$ -plane, given by:*

- (i) *if  $\varrho \in \mathbb{R}$  and  $|\varrho| > 1$  : the interval  $[-1, 1]^+$  in the  $z$  to itself in the  $\tau$ -plane; the half-plane  $\{\Im z > 0\}$ , ( $\{\Im z < 0\}$ ) to itself in the  $\tau$ -plane;*

(ii) if  $\varrho \in \mathbb{R}$ ,  $|\varrho| < 1$  : the interval  $[-1, 1]^+$  in the  $z$ -plane to  $(-\infty, -1]^- \cup [1, +\infty)^-$  in the  $\tau$ -plane; the half-plane  $\{\Im z > 0\}$ , ( $\{\Im z < 0\}$ ) to  $\{\Im \tau < 0\}$ , ( $\{\Im \tau > 0\}$ );

(iii) the circular segment  $\mathcal{S}(-\varrho)$ , run from  $\tau = -1$  to  $\tau = 1$ , if  $\varrho \in \mathbb{C} \setminus \mathbb{R}$ .

**Proof** We prove only (iii), as (i) and (ii) are trivial. Let  $t \in [-1, 1]$ . We prove that  $\tau = \Gamma_\varrho(t) \in \mathcal{C}(-\varrho)$ . Writing  $\varrho = a + ib$  with  $a, b \in \mathbb{R}$ , we have

$$\begin{aligned} |\tau|^2 - \frac{1 - |\varrho|^2}{\Im(\varrho)} \Im \tau &= \left| \frac{1 - t(a + ib)}{t - (a + ib)} \right|^2 - \frac{1 - a^2 - b^2}{b} \Im \left( \frac{1 - t(a + ib)}{t - (a + ib)} \right) \\ &= \frac{(1 - ta)^2 + t^2 b^2}{(t - a)^2 + b^2} - \frac{1 - a^2 - b^2}{b} \frac{(1 - ta)b - tb(t - a)}{(t - a)^2 + b^2} \\ &= \frac{(1 - ta)^2 + t^2 b^2}{(t - a)^2 + b^2} - \frac{(1 - a^2 - b^2)(1 - t^2)}{(t - a)^2 + b^2} \\ &= 1. \end{aligned}$$

On the other hand, the image of  $[-1, 1]$  through  $\Gamma_\varrho$  must be the circular segment of  $\mathcal{C}(-\varrho)$  with extremes  $\tau = -1$ ,  $\tau = 1$  and lying on the opposite side with respect to  $-\varrho$ , as  $-\varrho$  is the image of  $\infty$ . This is precisely  $\mathcal{S}(-\varrho)$ .  $\square$

The main results of this section is the following one. Notice that we do not assume that  $P$  and  $R$  have real-valued coefficients.

**Proposition 4.1** *Let*

$$P(z) = az^2 - 2bz + c, \quad R(z) = \alpha z^2 - 2\beta z + \gamma$$

where  $a, b, c, \alpha, \beta, \gamma \in \mathbb{C}$  verify

$$a \pm 2b + c \neq 0 \tag{58}$$

$$\text{if } a \neq 0 \text{ and } b^2 - ac = 0 \text{ then } \sqrt{\frac{c}{a}} \notin (-1, 1) \tag{59}$$

$$a - c = \alpha - \gamma \tag{60}$$

$$b^2 - ac = \beta^2 - \alpha\gamma. \tag{61}$$

Choose one root  $\varrho \in \mathbb{C}$  of the polynomial

$$Q(z) := (a - \alpha)z^2 - 2bz + c + \alpha. \tag{62}$$

Then  $\varrho \notin \{\pm 1\}$  and

a) the equality

$$\int_{[-1, 1]} \frac{dz}{\sqrt{(1 - z^2)P(z)}} = \int_{[-1, 1]} \frac{dz}{\sqrt{(1 - z^2)R(z)}} \tag{63}$$

hold if

a<sub>1</sub>)  $\varrho \in \mathbb{R}$ , with  $|\varrho| > 1$ ;

a<sub>2</sub>)  $\varrho \in \mathbb{R}$ , with  $|\varrho| < 1$  and at least one of the half-planes  $\{\Im z > 0\}$ ,  $\{\Im z < 0\}$  is free from roots of  $P$  (of  $R$ );

a<sub>3</sub>)  $\varrho \in \mathbb{C} \setminus \mathbb{R}$  and no root of  $P$  lies in  $\mathcal{A}(\varrho)$  (no root of  $R$  lies in  $\mathcal{A}(-\varrho)$ );

b) the equality

$$\int_{[-1,1]} \frac{dz}{\sqrt{(1-z^2)P(z)}} = - \int_{[-\infty,-1] \cup [1,+\infty)} \frac{dz}{\sqrt{(1-z^2)R(z)}}$$

hold if

- b<sub>1</sub>)  $\varrho \in \mathbb{R}$ , with  $|\varrho| > 1$  and at least one of the half-planes  $\{\Im z > 0\}$ ,  $\{\Im z < 0\}$  is free from roots of  $P$  (of  $R$ );
- b<sub>2</sub>)  $\varrho \in \mathbb{R}$ , with  $|\varrho| < 1$ ;
- b<sub>3</sub>)  $\varrho \in \mathbb{C} \setminus \mathbb{R}$  and no root of  $P$  lies in  $\mathcal{A}^*(\varrho)$  (equivalently, no root of  $R$  lies in  $\mathcal{A}^*(-\varrho)$ ).

c) the equality

$$\int_{[-\infty,-1] \cup [1,+\infty)} \frac{dz}{\sqrt{(1-z^2)P(z)}} = \int_{[-\infty,-1] \cup [1,+\infty)} \frac{dz}{\sqrt{(1-z^2)R(z)}}$$

hold if

- c<sub>1</sub>)  $\varrho \in \mathbb{R}$ , with  $|\varrho| > 1$ ;
- c<sub>2</sub>)  $\varrho \in \mathbb{R}$ , with  $|\varrho| < 1$  and at least one of the half-planes  $\{\Im z > 0\}$ ,  $\{\Im z < 0\}$  is free from roots of  $P$  (of  $R$ );
- c<sub>3</sub>)  $\varrho \in \mathbb{C} \setminus \mathbb{R}$  and no root of  $P$  lies in  $\mathcal{A}^*(1/\varrho)$  (no root of  $R$  lies in  $\mathcal{A}^*(-1/\varrho)$ );

d) the equality

$$\int_{[-\infty,-1] \cup [1,+\infty)} \frac{dz}{\sqrt{(1-z^2)P(z)}} = - \int_{[-1,1]} \frac{dz}{\sqrt{(1-z^2)R(z)}}$$

holds if

- d<sub>1</sub>)  $\varrho \in \mathbb{R}$ , with  $|\varrho| > 1$  and at least one of the half-planes  $\{\Im z > 0\}$ ,  $\{\Im z < 0\}$  is free from roots of  $P$  (of  $R$ );
- d<sub>2</sub>)  $\varrho \in \mathbb{R}$ , with  $|\varrho| < 1$ ;
- d<sub>3</sub>)  $\varrho \in \mathbb{C} \setminus \mathbb{R}$  and no root of  $P$  lies in  $\mathcal{A}(1/\varrho)$  (equivalently, no root of  $R$  lies in  $\mathcal{A}(-1/\varrho)$ ).

e) If  $\varrho \in \mathbb{R}$ , the equality

$$\int_{[x_0,1]} \frac{dz}{\sqrt{(1-z^2)P(z)}} = \int_{[y_0(\varrho),1]} \frac{dz}{\sqrt{(1-z^2)R(z)}}$$

also holds, for any  $x_0 \in \mathbb{R}$ , and with  $y_0(\varrho) = \Gamma_\varrho(x_0)$ .

**Remark 4.1** The reason why the sets  $\mathcal{A}(\varrho)$ ,  $\mathcal{A}^*(\varrho)$ ,  $\mathcal{A}^*(1/\varrho)$ ,  $\mathcal{A}(1/\varrho)$  appearing in assumptions a<sub>3</sub>), b<sub>3</sub>), c<sub>3</sub>), d<sub>3</sub>) are chosen to be closed is to avoid annoying questions of definition of the complex square root (as the proof uses arguments of holomorphic functions).

**Proof of Proposition 4.1** We prove only the statements in a), as the proof of the statements in b) is similar, b), c) are obtained from a), b) respectively, changing coordinate  $z = \zeta^{-1}$ . Finally, statement in e) a generalization of a<sub>1</sub>), a<sub>2</sub>).

Let  $\varrho \in \mathbb{C}$  solve (62). Due to condition (58), we have  $\varrho \in \mathbb{C} \setminus \{\pm 1\}$ . We check that  $\alpha$ ,  $\beta$  and  $\gamma$  verify

$$\alpha = \frac{a\varrho^2 - 2b\varrho + c}{\varrho^2 - 1}, \quad \beta = \frac{b\varrho^2 + (a+c)\varrho + b}{\varrho^2 - 1}, \quad \gamma = \frac{c\varrho^2 - 2b\varrho + a}{\varrho^2 - 1}. \quad (64)$$

Indeed, the first equality in (64) is equivalent to (62) combined with  $\varrho \in \mathbb{C} \setminus \{\pm 1\}$ . The second and the third equality follow from it and (60) and (61). Let now  $\mathcal{I}_1$  denote the integral at left hand side in (63). We change variable, letting

$$z = \Gamma_{-\varrho}(\tau) = \frac{1 + \tau\varrho}{\tau + \varrho}$$

where the coordinate  $\tau$  runs the path  $s_\varrho := \Gamma_\varrho([-1, 1])$ , as in the thesis of Lemma 4.2. The integral becomes

$$\mathcal{I}_1 = \int_{s_\varrho} \frac{w(\tau)d\tau}{\sqrt{(1-\tau^2)(\alpha\tau^2 - 2\beta\tau + \gamma)w(\tau)^2}} = \int_{s_\varrho} \frac{d\tau}{\sqrt{(1-\tau^2)(\alpha\tau^2 - 2\beta\tau + \gamma)}}$$

with

$$w(\tau) := \frac{\varrho^2 - 1}{(\tau + \varrho)^2}$$

and  $\alpha$ ,  $\beta$  and  $\gamma$  as in (64).

$a_1$ ) Let  $\varrho \in \mathbb{R}$ , with  $|\varrho| > 1$ . As, in such case,  $s_\varrho = [-1, 1]$ , we immediately have the thesis.

$a_2$ ) Let  $\varrho \in \mathbb{R}$ , with  $|\varrho| < 1$ . We show that  $\alpha$  and  $\beta$  do not simultaneously vanish. By (59),  $P(z)$  does not have double roots in  $(-1, 1)$ . If  $\alpha = \beta = 0$ , then  $b^2 - ac = \beta^2 - \alpha\gamma = 0$  and  $\varrho \in (-1, 1)$  satisfies

$$a\varrho^2 - 2b\varrho + c = \alpha(\varrho^2 - 1) = 0$$

which contradicts that  $P$  does not have double roots in  $(-1, 1)$ . As  $\alpha$  and  $\beta$  do not simultaneously vanish, and  $s_\varrho = (-\infty, -1]^- \cup [1, +\infty)^-$ , we get

$$\mathcal{I}_1 = \int_{(-\infty, -1]^- \cup [1, +\infty)^-} \frac{d\tau}{\sqrt{(1-\tau^2)(\alpha\tau^2 - 2\beta\tau + \gamma)}}$$

As the transformation  $\tau = \Gamma_\varrho(z)$  sends roots of  $P$  to roots of  $R$  and (due to  $\varrho \in (-1, 1)$ ) the half-planes  $\{\Im z > 0\}$ ,  $\{\Im z < 0\}$  to  $\{\Im \tau < 0\}$ ,  $\{\Im \tau > 0\}$ , respectively, then at least one of the half-planes  $\{\Im \tau > 0\}$ ,  $\{\Im \tau < 0\}$  is free from roots of  $R$ . Then we can use Cauchy theorem, choosing a path integral made of the real axis and half-circle centered at  $\tau = 0$  with radius  $r \rightarrow \infty$  on the side of the complex plane where no roots of  $R$  occur. Then

$$\int_{(-\infty, -1]^- \cup [1, +\infty)^-} \frac{d\tau}{\sqrt{(1-\tau^2)(\alpha\tau^2 - 2\beta\tau + \gamma)}} = \int_{[-1, 1]^+} \frac{d\tau}{\sqrt{(1-\tau^2)(\alpha\tau^2 - 2\beta\tau + \gamma)}}$$

which concludes.

$a_3$ ) Let  $\varrho \in \mathbb{C} \setminus \mathbb{R}$  and assume that no root of  $P$  lies in  $\mathcal{A}(\varrho)$ . Then no root of  $R$  lies in  $\mathcal{A}(-\varrho)$ . As  $s_\varrho = \mathcal{S}(-\varrho)$  has as end-points  $-1$  and  $1$ , by the absence of roots of  $R$  in  $\mathcal{A}(-\varrho)$ , we can deform the path integral, from  $\mathcal{S}(-\varrho)^+$  to  $[-1, 1]^+$ .  $\square$

## 4.2 Proof of (35)

In this section we assume  $M \geq 0$ . When  $M = 0$ , the part of proof below involving the set  $\mathbb{P}_0^{(3)} = \emptyset$  is to be disregarded. The proof is organized in two steps.

*Step 1* We prove that

$$\tau_M = C \begin{cases} \int_{-1}^{t_0} \frac{dt}{\sqrt{a_2(1-t^2)(t_0-t)}} & \text{if } (d, f) \in \mathbb{P}_M^{(3)} \cup \mathbb{P}_M^{(4)} \\ \frac{1}{2} \int_{-1}^1 \frac{dt}{\sqrt{a_2(1-t^2)(t_0-t)}} & \text{if } (d, f) \in \mathbb{P}_M^{(2)} \end{cases} \quad (65)$$

with

$$C := 2\sqrt{\frac{2d}{v_0}}, \quad a_2 := 2\sqrt{f^2 - M^2d^2}, \quad t_0 := \frac{d^2 - 2f}{2\sqrt{f^2 - M^2d^2}}.$$

We denote as

$$a_1 := 2\sqrt{f^2 - M^2d^2} + d^2 - 2f, \quad c_1 := 2\sqrt{f^2 - M^2d^2} - d^2 + 2f.$$

Consider the triple of polynomials

$$\begin{aligned} P_1(z) &= a_1z^2 + c_1 \\ Q_1(z) &= 2\left(\sqrt{f^2 - M^2d^2} - f\right)z^2 + 2\left(\sqrt{f^2 - M^2d^2} + f\right) \\ R_1(z) &= d^2z^2 - 4dMz + 4f - d^2 \end{aligned} \quad (66)$$

which verify (58) to (61). The roots  $\varrho_{1\pm}$  of  $Q_1$  are real, opposite and verify  $-\varrho_{1-} = \varrho_{1+} > 1$  for all  $(d, f) \in \mathbb{P}_M^{(2)} \cup \mathbb{P}_M^{(3)} \cup \mathbb{P}_M^{(4)}$ . Then, for all  $x_0 \in \mathbb{R}$  and for all  $(d, f) \in \mathbb{P}_M^{(2)} \cup \mathbb{P}_M^{(3)} \cup \mathbb{P}_M^{(4)}$ ,

$$\int_{x_0}^1 \frac{dz}{\sqrt{(1-z^2)(a_1z^2 + c_1)}} = \int_{y_0}^1 \frac{dz}{\sqrt{(1-z^2)(d^2z^2 - 4dMz + 4f - d^2)}}$$

where  $y_0 = \Gamma_{\varrho_{1+}(x_0)}$ . We next distinguish three cases. In all cases, we denote as  $-z_{1-} = z_{1+} = \sqrt{-\frac{c_1}{a_1}}$  the roots of  $P_1$ , and we use the change of coordinates (between real variables)

$$x = \sqrt{\frac{t-1}{1+t}} : [1, +\infty) \rightarrow [0, 1). \quad (67)$$

- $(d, f) \in \mathbb{P}_M^{(2)}$ . In this case,

$$a_1 > 0, \quad c_1 < 0, \quad t_0 \in (1, +\infty), \quad -z_- = z_+ \in (0, 1).$$

We take  $x_0 = z_{1+}$  the maximum root of  $P_1$ , so  $y_0 = y_{1+} = \frac{2}{d} \left( M + \sqrt{M^2 + \frac{d^2}{4} - f} \right)$  is the maximum root of  $R_1$ . We have the equality

$$\begin{aligned} \tau_M &= C \int_{y_{1+}}^1 \frac{dz}{\sqrt{(1-z^2)(d^2z^2 - 4dMz + 4f - d^2)}} \\ &= C \int_{z_{1+}}^1 \frac{dz}{\sqrt{(1-z^2)(a_1z^2 + c_1)}} \end{aligned}$$

where the last integral is meant in real sense. The change (67) transforms the right hand side into

$$\tau_M = \frac{C}{2} \int_{t_0}^{+\infty} \frac{dt}{\sqrt{a_2(t^2 - 1)(t - t_0)}} = \frac{C}{2} \int_{-1}^1 \frac{dt}{\sqrt{a_2(1-t^2)(t_0-t)}}$$

where we have used a modification of the integral path, after using that  $t_0 \in (1, +\infty)$  while  $(d, f) \in \mathbb{P}_M^{(2)}$  and that the function under the integral has only real branching points and  $(-1, 1), (t_0, +\infty)$  are the only intervals in the real axis where the function under the integral is real-valued.

- $(d, f) \in \mathbb{P}_M^{(3)}$ . In this case,

$$a_1 < 0, \quad c_1 > 0, \quad t_0 \in (-1, 0], \quad -z_{1-} = z_{1+} \in (1, +\infty).$$

We take  $x_0 = -1$ , so  $y_0 = -1$ . We have the equality

$$\begin{aligned} \tau_M &= C \int_{-1}^1 \frac{dz}{\sqrt{(1-z^2)(d^2z^2 - 4dMz + 4f - d^2)}} \\ &= C \int_{-1}^1 \frac{dz}{\sqrt{(1-z^2)(a_1z^2 + c_1)}} \\ &= 2C \int_0^1 \frac{dx}{\sqrt{(1-x^2)(a_1x^2 + c_1)}} \end{aligned}$$

where the last integral is meant in real sense. In the last step, we have used  $-z_{1-} = z_{1+} \in (1, +\infty)$ . With the change (67) the right hand side becomes

$$C \int_1^{+\infty} \frac{dt}{\sqrt{a_2(t^2 - 1)(t - t_0)}} = C \int_{-1}^{t_0} \frac{dt}{\sqrt{a_2(1 - t^2)(t_0 - t)}}$$

Here, we have: simplified the factor 2; interpreted the integral as a complex one and modified the integral path, after using that  $t_0 \in (-1, 0]$  while  $(d, f) \in \mathbb{P}_M^{(3)}$  and that the function under the integral has only real branching points and  $[1, \infty), [-1, t_0]$  are the only intervals in the real axis where the function under the integral is real-valued.

- $(d, f) \in \mathbb{P}_M^{(4)}$ . In this case,

$$a_1 > 0, \quad c_1 > 0, \quad t_0 \in (-1, 1), \quad -z_- = z_+ \in i\mathbb{R}$$

We take  $x_0 = -1$ , so  $y_0 = -1$ . We have the equality

$$\begin{aligned} \tau_M &= C \int_{-1}^1 \frac{dz}{\sqrt{(1-z^2)(d^2z^2 - 4dMz + 4f - d^2)}} = C \int_{-1}^1 \frac{dz}{\sqrt{(1-z^2)(a_1z^2 + c_1)}} \\ &= 2C \int_0^1 \frac{dz}{\sqrt{(1-z^2)(a_1z^2 + c_1)}} \end{aligned}$$

where the last integral is meant in real sense. In the last step, we have used  $-z_{1-} = z_{1+} \in i\mathbb{R}$ . The change (67) transforms the right hand side into

$$\tau_M = C \int_1^{+\infty} \frac{dt}{\sqrt{a_2(t^2 - 1)(t - t_0)}} = C \int_{-1}^{t_0} \frac{dt}{\sqrt{a_2(1 - t^2)(t_0 - t)}}$$

again by a modification of the integral path, after using  $t_0 \in (-1, 1)$  while  $(d, f) \in \mathbb{P}_M^{(4)}$  and that the function under the integral has only real branching points and  $(-1, t_0), (1, +\infty)$  are the only intervals in the real axis where the function under the integral is real-valued.

Then (65) is completely proved.

*Step 2* We now consider the polynomials

$$\begin{aligned} P_2(t) &= Q_2(t) = M^2 t^2 - 2(M^2 - f)t + M^2 - 2f + d^2 \\ R_2(t) &= a_2(t_0 - t) = -2t\sqrt{f^2 - M^2 d^2} - 2f + d^2 \end{aligned}$$

which verify (58) to (61). As  $P_2 = Q_2$  has real-valued roots while  $(d, f) \in \mathbb{P}_M^{(2)} \cup \mathbb{P}_M^{(3)} \cup \mathbb{P}_M^{(4)}$ , we have

$$\begin{aligned} \int_{-1}^{t_0} \frac{dt}{\sqrt{(1-t^2)(-2t\sqrt{f^2 - M^2 d^2} - 2f + d^2)}} &= \int_{-1}^{z_0} \frac{dz}{\sqrt{(1-z^2)(M^2 z^2 - 2(M^2 - f)z + M^2 - 2f + d^2)}} \\ &= \int_0^{w_0} \frac{dw}{\sqrt{w(2-w)(M^2 w^2 - 2fw + d^2)}} \end{aligned}$$

having let  $w := 1 - z$ , and having named and  $z_0$  the minimum root of  $M^2 z^2 - 2(M^2 - f)z + M^2 - 2f + d^2$  and  $w_0$  the minimum root of  $M^2 w^2 - 2fw + d^2$ .  $\square$

**Remark 4.2** One might wonder if the method of proof of (35) would work also in the case  $(d, f) \in \mathbb{P}_M^{(1)}$ , which has been treated separately in Section 3, via dynamical arguments. As a matter of fact, when  $(d, f) \in \mathbb{P}_M^{(1)}$ ,  $\mathcal{C}_{\varrho_{1\pm}}$  is the unit circle centered at 0 and the roots of  $P_1$  are complex and both belong to  $\mathcal{C}_{\varrho_{1\pm}}$ . In that case, there is no room to apply Proposition 4.1, with the choice of  $P_1, Q_1, R_1$  in (66).

## A Chebyshev Integral Inequality

**Proposition A.1** *Let  $f, g, p : \mathbb{R} \rightarrow \mathbb{R}$ , with  $p \geq 0$ , and  $p, fp, gp, fgp$  integrable on  $\mathbb{R}$ ;  $\int_{\mathbb{R}} p(x)dx = 1$  and  $f, g$  decreasing or increasing on the support of  $p$ . Then*

$$\left( \int_{\mathbb{R}} f(x)p(x)dx \right) \left( \int_{\mathbb{R}} g(y)p(y)dy \right) \leq \left( \int_{\mathbb{R}} f(x)g(x)p(x)dx \right) \left( \int_{\mathbb{R}} p(y)dy \right).$$

*If  $f = g$ , the monotonicity assumption can be released.*

**Proof** As  $f, g$  are decreasing or increasing on the support of  $p$ , for any choice of  $x, y$  on such support, we have

$$0 \leq (f(x) - f(y))(g(x) - g(y)) = f(x)g(x) - f(x)g(y) - f(y)g(x) + f(y)g(y). \quad (68)$$

Multiplying by  $p(x)p(y)$  and taking the integral on  $\mathbb{R}^2$  we get the result. If  $f = g$ , the inequality (68) holds without the monotonicity assumption.  $\square$

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