Regularity theory for parabolic operators in the half-space with boundary degeneracy

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Abstract

We study elliptic and parabolic problems governed by the singular elliptic operators

$$\mathcal{L} = y^{\alpha_1}\text{Tr }\left(QD_x^2 u\right) + 2y^{\frac{\alpha_1+\alpha_2}{2}} q\cdot\nabla_x D_y + \gamma y^{\alpha_2} D_{yy} + c y^{\alpha_2-1} D_y \cancel{y^{\alpha_2-1} v\cdot\nabla}$$

under Neumann ~~or oblique derivative~~ boundary condition, in the half-space $\mathbb{R}^{N+1}_+ = \{(x,y) : x\in\mathbb{R}^N, y>0\}$. We prove elliptic and parabolic L^p-estimates and solvability for the associated problems. In the language of semigroup theory, we prove that $\mathcal{L}$ generates an analytic semigroup, characterize its domain as a weighted Sobolev space and show that it has maximal regularity.

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1 Introduction

In this paper we study solvability and regularity of elliptic and parabolic problems associated to the degenerate operators

$$\mathcal{L} = y^{\alpha_1}\text{Tr }\left(QD_x^2 u\right) + 2y^{\frac{\alpha_1+\alpha_2}{2}} q\cdot\nabla_x D_y + \gamma y^{\alpha_2} D_{yy} + c y^{\alpha_2-1} D_y \cancel{y^{\alpha_2-1} v\cdot\nabla} \tag{1}$$

and $D_t - \mathcal{L}$ in the half-space $\mathbb{R}^{N+1}_+ = \{(x,y) : x\in\mathbb{R}^N, y>0\}$ or in $(0,\infty)\times\mathbb{R}^{N+1}_+$ and under Neumann ~~or oblique derivative~~ boundary condition at $y=0$.

Here $c\in\mathbb{R}$ ~~$v=(b,c)\in\mathbb{R}^{N+1}$ with $b=0$ if $c=0$,~~ and $\begin{pmatrix} Q & q^t \\ q & \gamma \end{pmatrix}$ is a constant real elliptic matrix. The real numbers α_1, α_2 satisfy $\alpha_2 < 2$ and $\alpha_2 - \alpha_1 < 2$ but are not assumed to be nonnegative.

We write B_y to denote the 1-dimensional Bessel operator $D_{yy} + \frac{c}{y}D_y$. With this notation the special case where

$$\mathcal{L} = y^{\alpha_1}\Delta_x + y^{\alpha_2}B_y$$

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has been already studied in [17]. The main novelty here consists in the presence of the mixed derivatives $2y^{\frac{\alpha_1+\alpha_2}{2}} q \cdot \nabla_x D_y$ in the operator $\mathcal{L}$ which is a crucial step for treating degenerate operators in domains, through a localization procedure.

Our main result is the following, see Theorems 6.1, 6.3 and Appendix B for the definition of the weighted Sobolev spaces involved.

Theorem *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_2 < 2$, $\alpha_2 - \alpha_1 < 2$ and*

$$\alpha_1^- < \frac{m+1}{p} < \frac{c}{\gamma} + 1 - \alpha_2.$$

Then the operator

$$\mathcal{L} = y^{\alpha_1} Tr\ \left(QD_x^2 u\right) + 2y^{\frac{\alpha_1+\alpha_2}{2}} q \cdot \nabla_x D_y + \gamma y^{\alpha_2} D_{yy} + c y^{\alpha_2 - 1} D_y\ y^{\alpha_2-1} v \cdot \nabla$$

endowed with domain $W^{2,p}_{\mathcal{N}}(\alpha_1, \alpha_2, m)$ $W^{2,p}_{v}(\alpha_1, \alpha_2, m)$ when $c \neq 0$ and $W^{2,p}_{\mathcal{N}}(\alpha_1, \alpha_2, m)$ when $c = 0$, generates a bounded analytic semigroup in L^p_m which has maximal regularity.

Let us explain the meaning of the restrictions $\alpha_2 < 2$, $\alpha_2 - \alpha_1 < 2$ considering first the case where $\alpha_1 = \alpha_2 = \alpha$, so that the unique requirement is $\alpha < 2$.

It turns out that when $\alpha \geq 2$ the problem is easily treated in the strip $\mathbb{R}^N \times [0, 1]$ in the case of the Lebesgue measure, see [9], and all problems are due to the strong diffusion at infinity. The case $\alpha \geq 2$ in the strip $\mathbb{R}^N \times [1, \infty[$ requires therefore new investigation even though the 1-dimensional case is easily treated by the change of variables of Appendix B.

When $\alpha_1 \neq \alpha_2$, the further restriction $\alpha_2 - \alpha_1 < 2$ comes from the change of variables of Appendix B, see Section 6.

Let us briefly describe the previous literature on these operators. In [14, 16] we considered the simplest case of $\Delta_x + B_y$ making extensive use of the commutative structure of the operator. The non-commutative case of $y^{\alpha_1}\Delta_x + y^{\alpha_2} B_y$ have been later faced in [17]. Another source of non-commutativity comes from the presence of mixed derivatives. In [18], we treated the operator

$$\text{Tr}\ \left(QD_x^2 u\right) + 2q \cdot \nabla_x D_y + b \cdot \nabla_x + B_y$$

under Neumann boundary conditions. For related results with different methods and with VMO coefficients, we refer the reader also to [5, 3, 4, 6] for the case $\alpha_1 = \alpha_2 = 0$ (i.e. without the powers y^{α_1}, y^{α_2}) and to [7, 8] for the case $\alpha_1 = \alpha_2 \in (0, 2)$ with Dirichlet boundary conditions.

This paper is devoted to a final step in this direction, by adding (different) powers of y in front of the main terms of the operator. This is, by no means, an immediate generalization of the previous results or methods and many extra difficulties appear. Here we consider only constant matrices Q and constant q, γ, c. However a straightforward localization procedure allows to extend our results to the general case where Q, q, γ, c are bounded and uniformly continuous and allows to treat operators in smooth domains, whose degeneracy in the top order coefficients behaves like a power of the distance from the boundary. We shall treat these topics in a forthcoming paper. We refer also to [19] for the case of Dirichlet or oblique derivative boundary conditions.

As for the simpler operator $y^{\alpha_1}\Delta_x + y^{\alpha_2} B_y$, the case $\alpha_1 = \alpha_2$ implies all other cases by the change of variables described in Appendix B. However this modifies the underlying measure and the procedure works if one is able to deal with the complete scale of L^p_m spaces, where $L^p_m = L^p(\mathbb{R}^{N+1}_+; y^m dxdy)$.

The operators $\mathcal{L}$, $D_t-\mathcal{L}$, with $\alpha_1=\alpha_2=\alpha$, are studied through estimates like

$$\|y^\alpha\Delta_x u\|_{p,m}+\|y^\alpha\nabla_x D_y u\|_{p,m}+\|y^\alpha B_y u\|_{p,m}\le C\|\mathcal{L}u\|_{p,m}, \tag{2}$$

and

$$\|D_t u\|_{p,m}+\|\mathcal{L}u\|_{p,m}\le C\|(D_t-\mathcal{L})u\|_{p,m}, \tag{3}$$

where the L^p norms are taken over $\mathbb{R}^{N+1}_+$ and on $(0,\infty)\times\mathbb{R}^{N+1}_+$ respectively. This kind of estimates are quite natural in this context but not easy to prove. Of course they imply $\|y^\alpha D_{x_ix_j}u\|_{p,m}\le C\|\mathcal{L}u\|_{p,m}$, by the Calderón-Zygmund inequalities in the x-variables.

Let us explain how to obtain (2). Assuming that $y^\alpha(\Delta_x u+2a\cdot\nabla_x D_y u+B_y u)=f$ and taking the Fourier transform with respect to x (with covariable ξ) we obtain $-|\xi|^2\hat{u}(\xi,y)+2ia\cdot\xi D_y+B_y\hat{u}(\xi,y)=y^{-\alpha}\hat{f}(\xi,y)$. Denoting by $\mathcal{F}$ the Fourier transform with respect to x we get

$$\begin{aligned} y^\alpha\Delta_x\mathcal{L}^{-1}&=-\mathcal{F}^{-1}\left(y^\alpha|\xi|^2(|\xi|^2-2ia\xi D_y-B_y)^{-1}y^{-\alpha}\right)\mathcal{F}\\ y^\alpha\nabla_x D_y\mathcal{L}^{-1}&=-\mathcal{F}^{-1}\left(y^\alpha\xi D_y(|\xi|^2-2ia\xi D_y-B_y)^{-1}y^{-\alpha}\right)\mathcal{F}\end{aligned}$$

and the boundedness of $y^\alpha\Delta_x\mathcal{L}^{-1}$, $y^\alpha\nabla_x D_y\mathcal{L}^{-1}$ are equivalent to that of the multipliers $\xi\in\mathbb{R}^N\to y^\alpha|\xi|^2(|\xi|^2-2ia\cdot\xi D_y-B_y)^{-1}y^{-\alpha}$ and $\xi\in\mathbb{R}^N\to y^\alpha\xi D_y(|\xi|^2-2ia\cdot\xi D_y-B_y)^{-1}y^{-\alpha}$ in L^p_m.

The structure of these multipliers show the difficulties connected to the presence of the mixed operators. The parameter ξ not only appears as a spectral parameter but also in the operator $L_{2a\cdot\xi}:=B_y+2ia\cdot\xi D_y$. For this reason, we need a careful study of the 1- dimensional operator $L_{2a\cdot\xi}$ and variants like $y^\alpha(L_{2a\cdot\xi}-|\xi|^2)$, with estimates which depend explicitly on ξ.

Both the elliptic and parabolic estimates above share the name "maximal regularity" even though this term is often restricted to the parabolic case. We refer to [12] and the new books [10], [11] for the functional analytic approach to maximal regularity and to [2] for applications of these methods to uniformly parabolic operators.

The paper is organized as follows. In Section 2 we recall some results concerning a one-dimensional Bessel operator $y^\alpha B_y$ perturbed by a potential. In Section 3 we define and study a $1d$ auxiliary operator through a quadratic form. In Section 4 we investigate the boundedness of some multipliers related to the degenerate operator. In Section 5, which is the core of the paper, we prove generation results, maximal regularity and domain characterization for the operator $\mathcal{L}$, under Neumann boundary conditions. Finally, in Section 6, we extend our results to more general operators.

In the Appendices we briefly recall the harmonic analysis background needed in the paper, as square function estimates, $\mathcal{R}$-boundedness, a vector valued multiplier theorem and the changes of variables needed to reduce our operators to the simpler case where $\alpha_1=\alpha_2$.

Notation. For $N\ge 0$, $\mathbb{R}^{N+1}_+=\{(x,y):x\in\mathbb{R}^N,y>0\}$. For $m\in\mathbb{R}$ we consider the measure $y^m dxdy$ in $\mathbb{R}^{N+1}_+$ and we write $L^p_m(\mathbb{R}^{N+1}_+)$ for $L^p(\mathbb{R}^{N+1}_+;y^m dxdy)$ and often only L^p_m when $\mathbb{R}^{N+1}_+$ is understood. $\mathbb{C}^+=\{\lambda\in\mathbb{C}:Re\,\lambda>0\}$ and, for $|\theta|\le\pi$, we denote by Σ_θ the open sector $\{\lambda\in\mathbb{C}:\lambda\neq 0,\ |Arg(\lambda)|<\theta\}$. We denote by α^+ and α^- the positive and negative part of a real number, that is $\alpha^+=\max\{\alpha,0\}$, $\alpha^-=-\min\{\alpha,0\}$.

We use B for the one-dimensional Bessel operator $D_{yy}+\frac{c}{y}D_y$ and L_b for $B+ibD_y$. Here $c,b\in\mathbb{R}$ and both operators are defined on the half-line $(0,\infty)$.

2 The 1-dimensional operator $y^\alpha B - \mu y^\alpha$, $\mu \ge 0$

In this section we summarize the main results proved in [13] for the one dimensional operator $y^\alpha B - \mu y^\alpha = y^\alpha \left(D_{yy} + \frac{c}{y} D_y\right) - \mu y^\alpha$, $\mu \ge 0$, in L^p_m. To characterize the domain for $\mu > 0$, we note that the domain of the potential $V(y) = y^\alpha$ in L^p_m is

$$\{u \in L^p_m : \ y^\alpha u \in L^p_m\} = L^p_m \cap L^p_{m+\alpha p}.$$

We recall that the Sobolev spaces $W^{2,p}_{\mathcal{N}}(\alpha, m)$ are defined in Appendix B and the pedix $\mathcal{N}$ indicates a Neumann boundary condition at $y = 0$.

Theorem 2.1 *Let $\alpha < 2$, $c \in \mathbb{R}$ and $1 < p < \infty$.*

(i) If $0 < \frac{m+1}{p} < c + 1 - \alpha$, then the operator $y^\alpha B$ endowed with domain $W^{2,p}_{\mathcal{N}}(\alpha, m)$ generates a bounded analytic semigroup of angle $\pi/2$ on L^p_m which is positive for $z > 0$.

(ii) If $\mu > 0$ and $\alpha^- < \frac{m+1}{p} < c + 1 - \alpha$ then the operator $y^\alpha B - \mu y^\alpha$ endowed with domain $W^{2,p}_{\mathcal{N}}(\alpha, m) \cap L^p_{m+\alpha p}$ generates a bounded analytic semigroup in L^p_m which is positive for $z > 0$.

In both cases the set

$$\mathcal{D} = \{u \in C_c^\infty([0,\infty)),\ D_y u(y) = 0 \text{ for } y \le \delta \text{ and some } \delta > 0\} \tag{4}$$

is a core and the semigroups have maximal regularity.

Proof. See [13, Theorem 4.2] for (i), [13, Theorem 8.5] for (ii). The density of $\mathcal{D}$ is proved in proved in [13, Propositions 4.3, Theorem 8.5] and maximal regularity in [13, Theorem 7.2]. □

The following kernel estimates for the Bessel operator B (corresponding to $\alpha = \mu = 0$) will be used extensively.

Proposition 2.2 *Let $c > -1$. The semigroup $(e^{zB})_{z \in \mathbb{C}_+}$ consists of integral operators. Its heat kernel p_B, written with respect the measure $\rho^c d\rho$, satisfies for every $\varepsilon > 0$, $z \in \Sigma_{\frac{\pi}{2}-\varepsilon}$ and some $C_\varepsilon, \kappa_\varepsilon > 0$,*

$$|p_B(z, y, \rho)| \le C_\varepsilon |z|^{-\frac{1}{2}} \rho^{-c} \left(\frac{\rho}{|z|^{\frac{1}{2}}} \wedge 1\right)^c \exp\left(-\frac{|y-\rho|^2}{\kappa_\varepsilon |z|}\right),$$

$$|D_y p_B(z, y, \rho)| \le C_\varepsilon |z|^{-1} \rho^{-c} \left(\frac{y}{|z|^{\frac{1}{2}}} \wedge 1\right) \left(\frac{\rho}{|z|^{\frac{1}{2}}} \wedge 1\right)^c \exp\left(-\frac{|y-\rho|^2}{\kappa_\varepsilon |z|}\right).$$

Proof. See [14, Propositions 2.8 and 2.9]. □

3 The 1-dimensional operator $y^\alpha(B + 2ia \cdot \xi D_y - |\xi|^2)$

In this section we prove generation properties in L^p_m and heat kernel bounds for the operator

$$y^\alpha(B + 2ia \cdot \xi D_y - |\xi|^2) = y^\alpha L_{2a\cdot\xi} - y^\alpha |\xi|^2, \qquad |a| < 1, \alpha < 2,$$

where, according to our notation, $L_b = B + ibD_y$.

Having in mind the study of the operator $y^\alpha \Delta_x u + y^\alpha 2a \cdot \nabla_x D_y u + y^\alpha B_y u$ (see Section 5), from now on we assume the condition $|a| < 1$ which corresponds to the ellipticity of the top order coefficients and also $\alpha < 2$ as explained in the Introduction.

We start by the L^2 theory. We use the Sobolev spaces of Section 8 and also $H^1_c := \{u \in L^2_c : \nabla u \in L^2_c\}$ equipped with the inner product

$$\langle u, v\rangle_{H^1_c} := \langle u, v\rangle_{L^2_c} + \langle \nabla u, \nabla v\rangle_{L^2_c}.$$

and consider the form in $L^2_{c-\alpha}$ with $D(\mathfrak{a}) = H^1_c \cap L^2_{c-\alpha} \subset L^2_{c-\alpha}$ and

$$\begin{aligned}\mathfrak{a}(u, v) :&= \int_{\mathbb{R}_+} D_y u \, D_y \overline{v} \, y^c \, dy - 2ia \cdot \xi \int_{\mathbb{R}_+} (D_y u) \, \overline{v} \, y^c \, dy + \int_{\mathbb{R}_+} |\xi|^2 u \overline{v} y^c \, dy, \\ &= \int_{\mathbb{R}_+} y^\alpha D_y u D_y \overline{v} \, y^{c-\alpha} \, dy - 2ia \cdot \xi \int_{\mathbb{R}_+} y^\alpha D_y u \, \overline{v} \, y^{c-\alpha} \, dy + \int_{\mathbb{R}_+} |\xi|^2 y^\alpha u \overline{v} \, y^{c-\alpha} \, dy.\end{aligned}$$

The more pedantic second line above writes the form with respect to the reference measure $y^{c-\alpha} dy$, rather than $y^c dy$.

We define L in $L^2_{c-\alpha}$ as the operator associated to the form $\mathfrak{a}$, that is

$$\begin{aligned}&D(L) = \{u \in D(\mathfrak{a}) : \exists f \in L^2_{c-\alpha} \text{ such that } \mathfrak{a}(u, v) = \int_0^\infty f \overline{v} y^{c-\alpha} \, dy \text{ for every } v \in D(\mathfrak{a})\}, \\ &Lu = -f.\end{aligned}$$

If u, v are smooth functions with compact support, it is easy to see integrating by parts that

$$-\mathfrak{a}(u, v) = \langle y^\alpha (Bu + 2ia \cdot \xi D_y u - |\xi|^2 u), \overline{v}\rangle_{L^2_{c-\alpha}},$$

so that L is a realization of $y^\alpha (B + 2ia \cdot \xi D_y - |\xi|^2)$.

In the next lemmas we use two isometries which transform the operator into a simpler form.

The first one is $Tu(y) = \left(1 - \frac{\alpha}{2}\right)^{\frac{1}{2}} u(y^{1-\frac{\alpha}{2}})$ (this is the map $T_{-\frac{\alpha}{2}}$ of Appendix B) and allows to remove the power y^α in front of the Bessel operator B thus getting an equivalent operator $(\beta + 1)^{-2} \tilde{L}$ defined by

$$\tilde{L} = \tilde{B} + ib(\beta + 1) y^\beta D_y - (\beta + 1)^2 |\xi|^2 y^{2\beta}, \qquad \beta = \frac{\alpha}{2 - \alpha}$$

Here

$$\tilde{B} = D_{yy} + \frac{\tilde{c}}{y} D_y, \qquad \tilde{c} = \frac{2c - \alpha}{2 - \alpha}, \qquad b = 2a \cdot \xi$$

and, by the assumption $\alpha < 2$, one has $\beta + 1 > 0$ and moreover $\tilde{c} + 1 > 0$ if and only if $c + 1 - \alpha > 0$.

This is contained in Proposition 8.7 at level of operators but we state it below in the language of forms, omitting the elementary computations.

Lemma 3.1 *Let $c + 1 - \alpha > 0$. Setting $\tilde{c} = \frac{2c-\alpha}{2-\alpha} > -1$, let us consider the isometry*

$$T =: L^2_{\tilde{c}} \to L^2_{c-\alpha}, \quad Tu(y) = \left(1 - \frac{\alpha}{2}\right)^{\frac{1}{2}} u(y^{1-\frac{\alpha}{2}}).$$

Then, with $b = 2a \cdot \xi$, $\beta = \frac{\alpha}{2-\alpha}$, *one has* $T\left(H^1_{\tilde c} \cap L^2_{\tilde c+2\beta}\right) = H^1_c \cap L^2_{c-\alpha}$ *and*

$$\mathfrak{a}(u,v) = (\beta+1)^{-2}\, \tilde{\mathfrak{a}}(T^{-1}u, T^{-1}v), \qquad u,v \in H^1_c \cap L^2_{c-\alpha}$$

where $D(\tilde{\mathfrak{a}}) = H^1_{\tilde c} \cap L^2_{\tilde c+2\beta}$

$$\tilde{\mathfrak{a}}(u,v) = \int_{\mathbb{R}_+} D_y u\, D_y \overline{v}\, y^{\tilde c}\, dy - ib(\beta+1)\int_{\mathbb{R}_+} y^{\beta}(D_y u)\,\overline{v}\, y^{\tilde c}\, dy + |\xi|^2(\beta+1)^2 \int_{\mathbb{R}_+} y^{2\beta} u\overline{v} y^{\tilde c}\, dy.$$

We introduce now the quadratic form

$$Q_a(\xi) = |\xi|^2 - |a\cdot\xi|^2, \qquad (1-|a|^2)|\xi|^2 \le Q_a(\xi) \le |\xi|^2, \qquad \xi \in \mathbb{R}^N. \tag{5}$$

A second isometry S removes the term $ib(\beta+1)y^{\beta}D_y$ from the operator $\tilde L$ introducing a complex potential. This leads to the operator $(\beta+1)^{-2}\,\tilde A_{\tilde b,\beta}$ defined by

$$\tilde A_{\tilde b,\beta} = \tilde B - i\frac{\tilde b(\tilde c+\beta)}{2} y^{\beta-1} - (\beta+1)^2 Q_a(\xi) y^{2\beta} \tag{6}$$

where $\tilde c = \frac{c-\alpha}{2-\alpha}$, $\tilde b = b(\beta+1) = 2a\cdot\xi(\beta+1)$.

Lemma 3.2 *With the notation above let us consider the isometry*

$$S : L^2_{\tilde c} \to L^2_{\tilde c}, \qquad (Su)(y) = e^{-i\frac{\tilde b}{2(\beta+1)}y^{\beta+1}} u(y) = e^{-ia\cdot\xi y^{\frac{2}{2-\alpha}}} u(y).$$

If $D(\tilde{\mathfrak{a}}) = H^1_{\tilde c} \cap L^2_{\tilde c+2\beta}$, *then one has* $S(D(\tilde{\mathfrak{a}})) = D(\tilde{\mathfrak{a}})$ *and*

$$\tilde{\mathfrak{a}}(u,v) = \tilde{\mathfrak{a}}_{\tilde b}(S^{-1}u, S^{-1}v), \qquad u,\ v \in D(\tilde{\mathfrak{a}}).$$

$$\tilde{\mathfrak{a}}_{\tilde b}(u,v) = \int_{\mathbb{R}_+} D_y u D_y \overline{v} y^{\tilde c}\, dy - i\frac{\tilde b}{2}\int_{\mathbb{R}_+} D_y\left(u\overline{v}\right) y^{\tilde c+\beta}\, dy + (\beta+1)^2 Q_a(\xi)\int_{\mathbb{R}_+} u\overline{v} y^{\tilde c+2\beta}\, dy.$$

The above lemmas say that the operator $y^{\alpha}L_{2a\cdot\xi} - y^{\alpha}|\xi|^2$ and the associated form $\mathfrak{a}$ are equivalent, by mean of the isometry $T\circ S$, to $(\beta+1)^{-2}\tilde A_{\tilde b,\beta}$ and $(\beta+1)^{-2}\tilde{\mathfrak{a}}_{\tilde b}$ and motivates the next section.

Remark 3.3 Let us explain briefly the restrictions on the parameters c, α, β which appear in this section. The operator $B = D_{yy} + \frac{c}{y}D_y$ is considered here only for $c > -1$, so that the measure $y^c dy$ is finite in a neighborhood of 0, to impose Neumann boundary conditions. The case $c \le -1$ can be also considered under Dirichlet boundary conditions, see [14]. Note also that the set $\mathcal{D}$ defined in (4) is contained in the domain of the form associated with B, if and only if $c > -1$. All other restrictions on the parameters come from this choice. For example, the condition $c+1-\alpha > 0$ in Lemma 3.1 is equivalent to $\tilde c + 1 > 0$.

3.1 The auxiliary operator $A_{b,\beta} = B - i\frac{b(c+\beta)}{2}y^{\beta-1} - (\beta+1)^2 Q_a(\xi)y^{2\beta}$.

As explained in the above remark, we always assume that

$$|a| < 1, \quad c+1 > 0, \quad \beta+1 > 0.$$

Setting $b = 2a\cdot\xi(\beta+1)$, $Q_a(\xi) = |\xi|^2 - |a\cdot\xi|^2$, we consider the form $\tilde{\mathfrak{a}}_b$ defined on

$$D(\tilde{\mathfrak{a}}_b) = H^1_c \cap L^2_{c+2\beta} \subset L^2_c$$

by

$$\begin{aligned}\tilde{\mathfrak{a}}_b(u,v) &= \langle D_y u, D_y v\rangle_{L^2_c} - i\frac{b}{2}\langle u, D_y v\rangle_{L^2_{c+\beta}} - i\frac{b}{2}\langle D_y u, v\rangle_{L^2_{c+\beta}} + (\beta+1)^2 Q_a(\xi)\langle u, v\rangle_{L^2_{c+2\beta}} \\ &= \int_{\mathbb{R}_+} D_y u D_y \overline{v} y^c\, dy - i\frac{b}{2}\int_{\mathbb{R}_+} D_y\left(u\overline{v}\right) y^{c+\beta}\, dy + (\beta+1)^2 Q_a(\xi)\int_{\mathbb{R}_+} u\overline{v} y^{c+2\beta}\, dy \qquad (7)\end{aligned}$$

and its associated operator $A_{b,\beta}$ in L^2_c. Since for smooth functions with compact support away from the origin

$$(c+\beta)\int_0^\infty u\overline{v} y^{c+\beta-1}\, dy = \int_0^\infty u\big(D_y(y^{c+\beta}\overline{v}) - y^{c+\beta}D_y\overline{v}\big)\, dy = -\int_0^\infty D_y(u\overline{v}) y^{c+\beta}\, dy,$$

the operator $A_{b,\beta}$ is defined on smooth functions by

$$A_{b,\beta} := B - i\frac{b(c+\beta)}{2} y^{\beta-1} - (\beta+1)^2 Q_a(\xi) y^{2\beta}, \quad A_0 = B - (\beta+1)^2 Q_a(\xi) y^{2\beta}.$$

We collect in the following proposition the main properties satisfied by $\tilde{\mathfrak{a}}_b$.

Proposition 3.4 *The form $\tilde{\mathfrak{a}}_b$ is accretive and closed in L^2_c. Moreover*

(i) the adjoint form $\tilde{\mathfrak{a}}^*_b : (u,v) \mapsto \overline{\tilde{\mathfrak{a}}_b(v,u)}$ satisfies $\tilde{\mathfrak{a}}^*_b = \tilde{\mathfrak{a}}_{-b}$;

(ii) its real part is the positive form

$$\operatorname{Re}\tilde{\mathfrak{a}}_b(u,v) := \frac{\tilde{\mathfrak{a}}_b(u,v) + \tilde{\mathfrak{a}}^*_b(u,v)}{2} = \langle D_y u, D_y v\rangle_{L^2_c} + (\beta+1)^2 Q_a(\xi)\langle u, v\rangle_{L^2_{c+2\beta}};$$

(iii) for any $u \in H^1_c \cap L^2_{c+2\beta}$

$$|\operatorname{Im}\tilde{\mathfrak{a}}_b(u,u)| \le \frac{|b|}{(\beta+1)Q_a(\xi)^{\frac{1}{2}}}\operatorname{Re}\tilde{\mathfrak{a}}_b(u,u) = \frac{|a|}{\sqrt{1-|a|^2}}\operatorname{Re}\tilde{\mathfrak{a}}_b(u,u).$$

Proof. Properties (i) and (ii) are immediate consequences of the definition. Since $Re\,\tilde{\mathfrak{a}}_b(u,u) = \|D_y u\|^2_{L^2_c} + (\beta+1)^2 Q_a(\xi)\|u\|^2_{L^2_{c+2\beta}} \ge 0$, $\tilde{\mathfrak{a}}_b$ is accretive and, furthermore, the norm induced by the form $\tilde{\mathfrak{a}}_b$ coincides with the one of $H^1_c \cap L^2_{c+2\beta}$ and then $\tilde{\mathfrak{a}}_b$ is closed.

To prove (iii), we use Young's inequality and the elementary identity $D_y(|u|^2) = 2Re\,(\overline{u}D_y u)$ for $u \in H^1_c$. Then

$$\begin{aligned}|Im\,\tilde{\mathfrak{a}}_b(u,u)| &= \left|-\frac{b}{2}\int_0^\infty D_y\left(|u|^2\right) y^{c+\beta}\, dy\right| = \left|-b\int_0^\infty Re\,\left(\overline{u}D_y u\right) y^{c+\beta}\, dy\right| \\ &\le |b|\left(\|D_y u\|_{L^2_c}\|u\|_{L^2_{c+2\beta}}\right) = \frac{|b|}{(\beta+1)Q_a(\xi)^{\frac{1}{2}}}\left(\|D_y u\|_{L^2_c}\,(\beta+1)Q_a(\xi)^{\frac{1}{2}}\|u\|_{L^2_{c+2\beta}}\right) \\ &\le \frac{|b|}{2(\beta+1)Q_a(\xi)^{\frac{1}{2}}}\left(\|D_y u\|^2_{L^2_c} + (\beta+1)^2 Q_a(\xi)\|u\|^2_{L^2_{c+2\beta}}\right) \\ &= \frac{|b|}{2(\beta+1)Q_a(\xi)^{\frac{1}{2}}} Re\,\tilde{\mathfrak{a}}_b(u,u) = \frac{|a\cdot\xi|}{Q_a(\xi)^{\frac{1}{2}}} Re\,\tilde{\mathfrak{a}}_b(u,u) \le \frac{|a|}{\sqrt{1-|a|^2}} Re\,\tilde{\mathfrak{a}}_b(u,u).\end{aligned}$$

By standard theory on sesquilinear forms we have the following results.

Proposition 3.5 *The operator $A_{b,\beta}$ generates an analytic semigroup of angle $\frac{\pi}{2}-\arctan\frac{|a|}{\sqrt{1-|a|^2}}$ in L^2_c which satisfies*

$$\|e^{zA_{b,\beta}}f\|_{L^2_c}\le \|f\|_{L^2_c}, \qquad \forall z\in\Sigma_{\frac{\pi}{2}-\arctan\frac{|a|}{\sqrt{1-|a|^2}}}.$$

Moreover

(i) *The semigroup $\left(e^{tA_{b,\beta}}\right)_{t\ge 0}$ is L^∞-contractive and it is dominated by e^{tB}, that is*

$$|e^{tA_{b,\beta}}f|\le e^{tB}|f|, \qquad t>0, \quad f\in L^2_c.$$

(ii) *$\left(e^{tA_{b,\beta}}\right)_{t\ge 0}$ is a semigroup of integral operators and its heat kernel $\tilde p_{b,\beta}$, taken with respect to the measure $\rho^c d\rho$, satisfies for some constant C independent of b, β*

$$|\tilde p_{b,\beta}(t,y,\rho)|\le Ct^{-\frac12}\rho^{-c}\left(\frac{\rho}{t^{\frac12}}\wedge 1\right)^c\exp\left(-\frac{|y-\rho|^2}{\kappa t}\right), \qquad \textit{for a.e. } y,\rho>0.$$

(iii) *$A^*_{b,\beta}=A_{-b,\beta}$. and for any $s>0$ the operator satisfies the scaling property*

$$I_{\frac1s}\circ A_{b,\beta}\circ I_s=s^2\left(B-i\frac{b(c+\beta)}{2s^{1+\beta}}y^{\beta-1}-\frac{(\beta+1)^2}{s^{2\beta+2}}Q_a(\xi)y^{2\beta}\right), \qquad I_su(y):=u(sy).$$

Proof. The generation properties follows using Proposition 3.4 and [20, Teorems 1.52, 1.53]. To prove (i) we observe, preliminarily, that the operator B is associated with the form $\mathfrak{b}(u,v)=\langle D_yu,D_yv\rangle_{L^2_c}$ and its generated semigroup e^{tB} is sub-Markovian since $\mathfrak{b}$ satisfies the hypotheses of [20, Corollary 2.17]. The domination property for $e^{tA_{b,\beta}}$ then follows from [20, Theorem 2.21]. In particular $e^{tA_{b,\beta}}$ inherits the L^∞-contractivity of e^{tB}. (ii) is a consequence of [1, Proposition 1.9] since $e^{tA_{b,\beta}}$ is dominated by the positive integral operator e^{tB} whose kernel satisfies the stated estimate, see [14, Proposition 2.8] where, however, the kernel is written with respect to the Lebesgue measure. (iii) follows from (i) of Proposition 3.4 and by elementary computations. □

As in [18, Section 5], we can extend the above heat kernel estimates to complex times.

Theorem 3.6 *For every $0\le\nu<\frac{\pi}{2}-\arctan\frac{|a|}{\sqrt{1-|a|^2}}$ the heat kernel $\tilde p_{b,\beta}$, taken with respect to the measure $\rho^c d\rho$, satisfies for some constant C_ν independent of b, β*

$$|\tilde p_{b,\beta}(z,y,\rho)|\le C_\nu|z|^{-\frac12}\rho^{-c}\left(\frac{\rho}{|z|^{\frac12}}\wedge 1\right)^c\exp\left(-\frac{|y-\rho|^2}{\kappa|z|}\right),$$

for a.e. $y,\rho>0$, $\forall z\in\Sigma_\nu$.

3.2 Generation properties and domain characterization

Generation properties and kernel estimates for the original operator

$$y^\alpha L_{2a\cdot\xi}-y^\alpha|\xi|^2=y^\alpha(B+i2a\cdot\xi D_y-|\xi|^2)$$

can be deduced by the analogous properties of the auxiliary operator $A_{b,\beta}$ of Section 3.1. Indeed from Lemmas 3.1, 3.2 we have

$$\mathfrak{a}(u,v)=(\beta+1)^{-2}\,\tilde{\mathfrak{a}}_{\tilde b}(S^{-1}T^{-1}u,S^{-1}T^{-1}v), \qquad u,v\in D(\mathfrak{a}).$$

This implies that

$$y^\alpha L_{2a\cdot\xi} - y^\alpha|\xi|^2 = (T\circ S)\Big[(\beta+1)^{-2}\tilde{A}_{\tilde{b},\beta}\Big](T\circ S)^{-1}, \tag{8}$$

where $\beta = \frac{\alpha}{2-\alpha}$, $\tilde{b} = 2a\cdot\xi(\beta+1)$, $\tilde{c} = \frac{2c-\alpha}{2-\alpha}$ and

$$\tilde{A}_{\tilde{b},\beta} = \tilde{B} - i\frac{\tilde{b}(\tilde{c}+\beta)}{2}y^{\beta-1} - (\beta+1)^2 Q_a(\xi) y^{2\beta}.$$

Note that, by construction and by Proposition 8.7 we have

$$\tilde{B} = D_{yy} + \frac{\tilde{c}}{y}D_y, \qquad y^\alpha B = T\Big[(\beta+1)^{-2}\tilde{B}\Big]T^{-1}. \tag{9}$$

Theorem 3.7 *Let $c+1-\alpha > 0$. Then the operator $y^\alpha L_{2a\cdot\xi} - y^\alpha|\xi|^2$ generates a contractive analytic semigroup of angle $\frac{\pi}{2} - \arctan\frac{|a|}{\sqrt{1-|a|^2}}$ in $L^2_{c-\alpha}$ Moreover*

(i) The semigroup $\left(e^{ty^\alpha\left(L_{2a\cdot\xi}-|\xi|^2\right)}\right)_{t\geq 0}$ is dominated by $e^{ty^\alpha B}$, that is

$$|e^{ty^\alpha\left(L_{2a\cdot\xi}-|\xi|^2\right)}f| \leq e^{ty^\alpha B}|f|, \qquad t>0, \quad f\in L^2_{c-\alpha}.$$

(ii) $\left(e^{ty^\alpha\left(L_{2a\cdot\xi}-|\xi|^2\right)}\right)_{t\geq 0}$ is a semigroup of integral operators and its heat kernel p_α, taken with respect to the measure $\rho^{c-\alpha}d\rho$, satisfies for some constant C_ν

$$|p_\alpha(z,y,\rho)| \leq C_\nu |z|^{-\frac{1}{2}}\rho^{-\frac{\alpha}{2}}\left(\frac{\rho}{|z|^{\frac{1}{2-\alpha}}}\wedge 1\right)^{c+\frac{\alpha}{2}}\exp\left(-\frac{|y^{1-\frac{\alpha}{2}}-\rho^{1-\frac{\alpha}{2}}|^2}{\kappa|z|}\right),$$

for a.e. $y,\rho>0$, $\forall z\in\Sigma_\nu$, $0\leq\nu<\frac{\pi}{2}-\arctan\frac{|a|}{\sqrt{1-|a|^2}}$.

Proof. The proof is simply a translation of the results for $\tilde{A}_{\tilde{b},\beta}$ of Proposition 3.5 and of Theorem 3.6 by using the identity (8). For example, (i) follows since, by construction, we have for any g, $|TSg| = T|g|$ and therefore using (i) of Proposition 3.5 and (8), (9) we get for $t>0$, $f\in L^2_{c-\alpha}$

$$\begin{aligned}|e^{ty^\alpha\left(L_{2a\cdot\xi}-|\xi|^2\right)}f| &= |TSe^{t(\beta+1)^{-2}\tilde{A}_{\tilde{b},\beta}}S^{-1}T^{-1}f| = T|e^{t(\beta+1)^{-2}\tilde{A}_{\tilde{b},\beta}}S^{-1}T^{-1}f|\\ &\leq Te^{t(\beta+1)^{-2}\tilde{B}}|S^{-1}T^{-1}f| = Te^{t(\beta+1)^{-2}\tilde{B}}T^{-1}|f| = e^{ty^\alpha B}|f|.\end{aligned}$$

□

Now we prove that the semigroup $e^{z(y^\alpha(L_{2a\cdot\xi}-|\xi|^2))}$ extrapolates to the spaces L^p_m.

Proposition 3.8 *If $1<p<\infty$ and $0<\frac{m+1}{p}<c+1-\alpha$, then $(e^{z(y^\alpha(L_{2a\cdot\xi}-|\xi|^2))})$ is an analytic semigroup of angle $\frac{\pi}{2}-\arctan\frac{|a|}{\sqrt{1-|a|^2}}$ in L^p_m.*

Proof. All properties for $p=2$, $m=c-\alpha$ are contained in Theorem 3.7. The boundedness of $e^{z(y^\alpha(L_{2a\cdot\xi}-|\xi|^2))}$ in L^p_m then follows from [14, Proposition 12.2]. The semigroup law is inherited from the one of $L^2_{c-\alpha}$ via a density argument and we have only to prove the strong continuity at 0. Let $f, g \in C_c^\infty(\mathbb{R}_+)$. Then as $z \to 0$, $z \in \Sigma_{\frac{\pi}{2}-\arctan\frac{|a|}{\sqrt{1-|a|^2}}}$,

$$\int_0^\infty (e^{z(y^\alpha(L_{2a\cdot\xi}-|\xi|^2))}f)\, g\, y^m dy = \int_0^\infty (e^{z(y^\alpha(L_{2a\cdot\xi}-|\xi|^2))}f)\, g\, y^{m-c+\alpha}y^{c-\alpha}dy$$
$$\to \int_0^\infty fgy^{m-c+\alpha}y^{c-\alpha}dy = \int_0^\infty fgy^m dy,$$

by the strong continuity of $e^{z(y^\alpha(L_{2a\cdot\xi}-|\xi|^2))}$ in $L^2_{c-\alpha}$. Let us observe now that, using Theorem 3.7 and [14, Proposition 12.2], the family $\left\{e^{z(y^\alpha(L_{2a\cdot\xi}-|\xi|^2))}, z \in \Sigma_{\frac{\pi}{2}-\arctan\frac{|a|}{\sqrt{1-|a|^2}}}\right\}$ is uniformly bounded on $\mathcal{B}(L^p_m)$. By density, the previous limit holds for every $f \in L^p_m$, $g \in L^{p'}_m$. The semigroup is then weakly continuous, hence strongly continuous. □

Following the same lines of [13, Theorem 7.2], we get the following $\mathcal{R}$-boundedness result (see Appendix A for the relevant definitions).

Corollary 3.9 *Let $1 < p < \infty$ such that $0 < \frac{m+1}{p} < c+1-\alpha$. Then the following properties hold. For every $0 \le \nu < \pi - \arctan\frac{|a|}{\sqrt{1-|a|^2}}$ the families of operators*

$$\left\{e^{zy^\alpha(L_{2a\cdot\xi}-|\xi|^2)} :\ \xi \in \mathbb{R}^N \setminus \{0\},\ z \in \Sigma_\nu\right\},$$
$$\left\{\lambda\left(\lambda - y^\alpha(L_{2a\cdot\xi}-|\xi|^2)\right)^{-1} :\ \xi \in \mathbb{R}^N \setminus \{0\},\ \lambda \in \Sigma_\nu\right\}$$

are $\mathcal{R}$-bounded in L^p_m.

Proof. The proof follows almost identically to that of [13, Theorem 7.2] since from (ii) of Theorem 3.7 one has (using the notation of [13, Theorem 7.2]) for every $\le \nu < \frac{\pi}{2} - \arctan\frac{|a|}{\sqrt{1-|a|^2}}$ and for some positive constant C

$$\left|e^{zy^\alpha(L_{2a\cdot\xi}-|\xi|^2)}f\right| \le CS_\alpha^{-c}(|z|)|f|, \quad f \in L^p_m, \quad z \in \Sigma_\nu.$$

The $\mathcal{R}$-boundedness on $\xi \in \mathbb{R}^N \setminus \{0\}$ follows since the right hand side does not depend on ξ. □

In our investigations of degenerate N-d problems, we need, in the case $\alpha = 0$, to add a potential having non-negative real part to the operator of the latter proposition; this force to deal only with the semigroup on the real axis.

Let $V \in L^1_{loc}(\mathbb{R}^+, y^c\, dy)$ be a potential having non-negative real part and let $L_{2a\cdot\xi} - V$ be the operator in L^2_c associated with the form

$$\mathfrak{a}_V(u, v) = \int_{\mathbb{R}_+} (D_y u D_y \overline{v} - 2ia\cdot\xi D_y u\overline{v} + Vu\overline{v})\, y^c\, dy$$

defined on the domain

$$\mathcal{F} := H^1_c \cap L^2(\mathbb{R}_+, Vy^c dy).$$

Proposition 3.10 *Let $V \in L^1_{loc}(\mathbb{R}^+, y^c\,dy)$ be a potential having non-negative real part. Then for any $1 < p < \infty$ such that $0 < \frac{m+1}{p} < c+1$, $L_{2a\cdot\xi} - |\xi|^2 - V$ generates a C_0-semigroup on L^p_m. The generated semigroup consists of integral operators and the following estimate holds*

$$\left|e^{t(L_{2a\cdot\xi}-|\xi|^2-V)}f\right| \le e^{t(L_{2a\cdot\xi}-|\xi|^2)}|f| \le e^{tB}|f|, \qquad f \in L^p_m, \quad t \ge 0.$$

Moreover the families of operators

$$\left\{e^{t(L_{2a\cdot\xi}-|\xi|^2-V)} :\ t \ge 0,\ \xi \in \mathbb{R}^N \setminus \{0\},\ |a| < 1,\ V \in L^1_{loc}\left(\mathbb{R}^+, y^c\right),\ \mathrm{Re}\, V \ge 0\right\},$$

$$\left\{\lambda\left(\lambda - L_{2a\cdot\xi} + |\xi|^2 + V\right)^{-1} :\ \lambda > 0,\ \xi \in \mathbb{R}^N \setminus \{0\},\ |a| < 1,\ V \in L^1_{loc}\left(\mathbb{R}^+, y^c\right),\ \mathrm{Re}\, V \ge 0\right\} \tag{10}$$

are $\mathcal{R}$-bounded in L^p_m. In particular

$$\left\{|\xi|^2\left(|\xi|^2 - L_{2a\cdot\xi} + V\right)^{-1} :\ \xi \in \mathbb{R}^N \setminus \{0\},\ |a| < 1,\ V \in L^1_{loc}\left(\mathbb{R}^+, y^c\right),\ \mathrm{Re}\, V \ge 0\right\}$$

is $\mathcal{R}$-bounded in L^p_m.

Proof. The generation results can be proved as in Proposition 3.5. If $\mathfrak{a}$ is the form associated with $L_{2a\cdot\xi} - |\xi|^2$, then $L_{2a\cdot\xi} - |\xi|^2 - V$ is associated to $\mathfrak{a}_V := \mathfrak{a}(u,v) + \langle Vu, v\rangle_{L^2_c}$ and, by the standard theory on sesquilinear forms, $L_{2a\cdot\xi} - V$ generates a C_0-semigroup on L^2_c. The domination properties follow from [20, Theorem 2.21] again. The extrapolation on L^p_m follows as in Proposition 3.8. The $\mathcal{R}$-boundedness of the semigroup follows by domination from the $\mathcal{R}$-boundedness of $e^{t(L_{2a\cdot\xi}-|\xi|^2)}$ using Corollary 3.9 with $\alpha = 0$. The $\mathcal{R}$-boundedness of the resolvent family follows from Proposition 7.3 by writing the resolvent as the Laplace transform of the semigroup. The last claim follows by simply specializing (10) taking $\lambda = |\xi|^2$. □

We now prove that the domain of $y^\alpha L_{2a\xi} + |\xi|^2 y^\alpha$ is $W^{2,p}_{\mathcal{N}}(\alpha, m) \cap L^p_{m+\alpha p}$, under slightly more restrictive hypotheses than those of Proposition 3.8. Indeed, in what follows, we assume, besides $c + 1 - \alpha > 0$ also the condition $c + 1 > 0$.

Lemma 3.11 *Assume that $c+1 > 0$ and $c+1-\alpha > 0$. If $\lambda \in \mathbb{C}^+$ and $\xi \in \mathbb{R}^N \setminus \{0\}$, then*

$$\left(\lambda - y^\alpha L_{2a\xi} + |\xi|^2 y^\alpha\right)^{-1} f = \left(|\xi|^2 - L_{2a\cdot\xi} + \frac{\lambda}{y^\alpha}\right)^{-1}\left(\frac{f}{y^\alpha}\right), \qquad \forall f \in C_c^\infty((0,\infty)).$$

Proof. Under the assumptions $y^\alpha L_{2a\xi} - |\xi|^2 y^\alpha$ and $L_{2a\xi} - \lambda y^{-\alpha}$ generate a semigroup on $L^2_{c-\alpha}$ and L^2_c, respectively, see Theorem 3.7 and Proposition 3.10. Since $Re\,\lambda > 0$, both resolvents are well defined but act in different spaces.

Let $\mathfrak{a}$, $\mathfrak{a}_{\lambda y^{-\alpha}}$ be the forms associated to $y^\alpha L_{2a\xi} - |\xi|^2 y^\alpha$ in $L^2_{c-\alpha}$ and $L_{2a\xi} - \lambda y^{-\alpha}$ in L^2_c

$$\mathfrak{a}(u,v) = \int_{\mathbb{R}_+}\left(D_y u D_y \overline{v} + 2ia\cdot\xi D_y u\overline{v} + |\xi|^2 u\overline{v}\right) y^c\,dy,$$

$$\mathfrak{a}_{\lambda y^{-\alpha}}(u,v) = \int_{\mathbb{R}_+}\left(D_y u D_y \overline{v} + 2ia\cdot\xi D_y u\overline{v} + \lambda y^{-\alpha} u\overline{v}\right) y^c\,dy.$$

They are defined on the common domain

$$\mathcal{F} := \left\{u \in L^2_{c-\alpha} \cap L^2_c : D_y u \in L^2_c\right\}$$

Given $f \in C_c^\infty((0,\infty))$ let $u := \left(|\xi|^2 - L_{2a\cdot\xi} + \frac{\lambda}{y^\alpha}\right)^{-1}\left(\frac{f}{y^\alpha}\right)$. In order to prove that the equality $u = \left(\lambda - y^\alpha L_{2a\cdot\xi} + |\xi|^2 y^\alpha\right)^{-1} f$ holds, we have to show that $u \in \mathcal{F}$ and that for every $v \in \mathcal{F}$, u satisfies the weak equality

$$\int_0^\infty f\overline{v}y^{c-\alpha}\,dy = \int_0^\infty \lambda u\overline{v}y^{c-\alpha}\,dy + \mathfrak{a}_{\alpha,|\xi|^2y^\alpha}(u,v) \tag{11}$$

$$= \int_0^\infty (\lambda y^{-\alpha}u\overline{v} + D_y u D_y\overline{v} + 2ia\cdot\xi D_y u\overline{v} + |\xi|^2 u\overline{v})y^c\,dy. \tag{12}$$

By construction u is in the domain of $L_{2a\cdot\xi} - \lambda y^{-\alpha}$ which is contained in $\mathcal{F}$ and satisfies

$$\int_0^\infty \frac{f}{y^\alpha}\overline{v}y^c\,dy = \int_0^\infty |\xi|^2 u\overline{v}y^c\,dy + \mathfrak{a}_{\alpha,\lambda y^{-\alpha}}(u,v)$$

$$= \int_0^\infty (|\xi|^2 u\overline{v} + D_y u D_y\overline{v} + 2ia\cdot\xi D_y u\overline{v} + \lambda y^{-\alpha}u\overline{v})y^c\,dy,$$

which is the same as (11). □

Remark 3.12 In the next result we relate the resolvent of $y^\alpha L_{2a\cdot\xi} - y^\alpha$ with that of $L_{2a\cdot\xi} - \frac{1}{y^\alpha}$, in the sense specified below. We shall assume both the conditions $0 < \frac{m+1}{p} < c+1-\alpha$ and $-\alpha < \frac{m+1}{p} < c+1-\alpha$ (that is $\alpha^- < \frac{m+1}{p} < c+1-\alpha$). The first guarantees that $y^\alpha L_{2a\cdot\xi} - y^\alpha$ is a generator in L^p_m and the second that $L_{2a\cdot\xi} - \frac{1}{y^\alpha}$ is a generator in $L^p_{m+\alpha p}$.

Corollary 3.13 *Assume that $\alpha^- < \frac{m+1}{p} < c+1-\alpha$. If $\lambda \in \mathbb{C}^+$ and $\xi \in \mathbb{R}^N \setminus \{0\}$, then*

(i) for every $f \in L^p_m$

$$\left(\lambda - y^\alpha L_{2a\cdot\xi} + |\xi|^2 y^\alpha\right)^{-1} f = \left(|\xi|^2 - L_{2a\cdot\xi} + \frac{\lambda}{y^\alpha}\right)^{-1}\left(\frac{f}{y^\alpha}\right) \in L^p_{m+\alpha p} \cap L^p_m;$$

(ii) the operator $y^\alpha\left(\lambda - y^\alpha L_{2a\cdot\xi} + |\xi|^2 y^\alpha\right)^{-1}$ is bounded in L^p_m;

(iii) the operator $\frac{1}{y^\alpha}\left(|\xi|^2 - L_{2a\cdot\xi} + \frac{\lambda}{y^\alpha}\right)^{-1}$ is bounded in $L^p_{m+\alpha p}$.

Proof. Equality (i) is proved in Lemma 3.11 for any $f \in C_c^\infty((0,\infty))$. Since $\left(\lambda - y^\alpha L_{2a\cdot\xi} + |\xi|^2 y^\alpha\right)^{-1}$ is bounded form L^p_m into itself and $\left(|\xi|^2 - L_{2a\cdot\xi} + \frac{\lambda}{y^\alpha}\right)^{-1}\left(\frac{\cdot}{y^\alpha}\right)$ is bounded from L^p_m to $L^p_{m+\alpha p}$, by density, (i) holds for every $f \in L^p_m$. Parts (ii), (iii) are consequence of (i). □

To characterize the domain of $y^\alpha(L_{2a\cdot\xi} - |\xi|^2)$ we need the following lemmas.

Lemma 3.14 *Let $m \in \mathbb{R}$, $p > 1$. Then*

$$W^{2,p}_{\mathcal{N}}(\alpha, m) \cap L^p_{m+\alpha p} = W^{2,p}_{\mathcal{N}}(0, m+\alpha p) \cap L^p_m.$$

Proof. We observe preliminarily that from [15, Lemma 3.5] (which holds for $m \in \mathbb{R}$ and not only for $m < 2$), there exist $C > 0, \varepsilon_0 > 0$ such that for every $u \in W^{2,p}_{loc}(\mathbb{R}_+)$ one has

$$\|y^{\frac{\alpha}{2}} D_y u\|_{L^p_m((1,\infty))} \le C\left(\varepsilon\|y^\alpha D_{yy}u\|_{L^p_m((1,\infty))} + \frac{1}{\varepsilon}\|u\|_{L^p_m((1,\infty))}\right).$$

This and the elementary inequality $y^{\frac{\alpha}{2}} \leq y^{\alpha-1},\ y \leq 1$ grant that the term $y^{\frac{\alpha}{2}} D_y u$ can be discarded from the definition of the Sobolev space showing that

$$W^{2,p}_{\mathcal{N}}(\alpha, m) = \left\{u \in W^{2,p}_{loc}(\mathbb{R}_+) :\ u,\ y^\alpha D_{yy}u,\ y^{\alpha-1}D_y u \in L^p_m\right\}.$$

In view of the latter equality, the required identity becomes trivial. □

Lemma 3.15 *Let $1 < p < \infty$, $\alpha^- < \frac{m+1}{p} < c + 1 - \alpha$. Then there exists $C > 0$ such that for every $u \in W^{2,p}_{\mathcal{N}}(\alpha, m) \cap L^p_{m+\alpha p}$,*

$$\|y^\alpha D_y u\|_{L^p_m} \leq C\|y^\alpha Bu\|^{\frac{1}{2}}_{L^p_m} \|y^\alpha u\|^{\frac{1}{2}}_{L^p_m}.$$

It follows that for every $\varepsilon > 0$, $\xi \in \mathbb{R}^N$,

$$\||\xi| y^\alpha D_y u\|_{L^p_m} \leq \varepsilon\|y^\alpha Bu\|_{L^p_m} + \frac{C}{\varepsilon}\||\xi|^2 y^\alpha u\|_{L^p_m}.$$

Proof. We apply [18, Lemma 5.15] with $m + \alpha p$ in place of m thus obtaining

$$\|D_y u\|_{L^p_{m+\alpha p}} \leq C\|Bu\|^{\frac{1}{2}}_{L^p_{m+\alpha p}} \|u\|^{\frac{1}{2}}_{L^p_{m+\alpha p}}, \qquad u \in W^{2,p}_{\mathcal{N}}(0, m + \alpha p).$$

The first inequality then follows since by Lemma 3.14

$$W^{2,p}_{\mathcal{N}}(\alpha, m) \cap L^p_{m+\alpha p} = W^{2,p}_{\mathcal{N}}(0, m + \alpha p) \cap L^p_m.$$

The second one follows by Young inequality. □

Lemma 3.16 *Let $1 < p < \infty$, $\alpha^- < \frac{m+1}{p} < c + 1 - \alpha$. Then there exists $C > 0$, independent of $\xi \in \mathbb{R}^N$, $|a| < 1$, such that for every $u \in W^{2,p}_{\mathcal{N}}(\alpha, m) \cap L^p_{m+\alpha p}$, $\xi \in \mathbb{R}^N \setminus \{0\}$, $\mathrm{Re}\,\lambda > 0$,*

$$\||\xi|^2 y^\alpha u\|_{L^p_m} \leq C\|(\lambda - y^\alpha L_{2a\cdot\xi} + |\xi|^2 y^\alpha)u\|_{L^p_m}.$$

Proof. Let $u \in W^{2,p}_{\mathcal{N}}(\alpha, m) \cap L^p_{m+\alpha p}$ and set $f = (\lambda - y^\alpha L_{2a\cdot\xi} + |\xi|^2 y^\alpha)u$. Then, by Corollary 3.13 (i),

$$\||\xi|^2 y^\alpha u\|_{L^p_m} = \||\xi|^2 y^\alpha (\lambda - y^\alpha L_{2a\cdot\xi} + |\xi|^2 y^\alpha)^{-1} f\|_{L^p_m} = \left\||\xi|^2 y^\alpha \left(\frac{\lambda}{y^\alpha} - L_{2a\cdot\xi} + |\xi|^2\right)^{-1} \frac{f}{y^\alpha}\right\|_{L^p_m}.$$

By Proposition 3.10

$$\begin{aligned}\left\||\xi|^2 y^\alpha \left(\frac{\lambda}{y^\alpha} - L_{2a\cdot\xi} + |\xi|^2\right)^{-1} \frac{f}{y^\alpha}\right\|_{L^p_m} &= \left\||\xi|^2 \left(\frac{\lambda}{y^\alpha} - L_{2a\cdot\xi} + |\xi|^2\right)^{-1} \frac{f}{y^\alpha}\right\|_{L^p_{m+\alpha p}} \\ &\leq C\left\|\frac{f}{y^\alpha}\right\|_{L^p_{m+\alpha p}} = C\|f\|_{L^p_m}\end{aligned}$$

for some C independent of ξ. □

Theorem 3.17 *Let $1 < p < \infty$, $\alpha^- < \frac{m+1}{p} < c+1-\alpha$ and $\xi \in \mathbb{R}^N \setminus \{0\}$. Then the generator of $(e^{z(y^\alpha(L_{2a\cdot\xi}-|\xi|^2))})$ is the operator $y^\alpha(L_{2a\cdot\xi}-|\xi|^2)$ with domain $W^{2,p}_{\mathcal{N}}(\alpha,m)\cap L^p_{m+\alpha p}$. The set $\mathcal{D}$ defined in (4) is a core.*

Proof. We fix $0 \neq \xi$ and first prove that the equation $u - y^\alpha L_{2a\cdot\xi}u + |\xi|^2 y^\alpha u = f$, $f \in L^p_m$, is uniquely solvable in $W^{2,p}_{\mathcal{N}}(\alpha,m)\cap L^p_{m+\alpha p}$. Let $u \in W^{2,p}_{\mathcal{N}}(\alpha,m)\cap L^p_{m+\alpha p}$. By Theorem 2.1 (i) in the first inequality and then by Lemma 3.15 and Lemma 3.16, there exists a positive constant C such that for every $\varepsilon > 0$, $0 \le t \le 1$

$$
\begin{aligned}
&\|u\|_{W^{2,p}_{\mathcal{N}}(\alpha,m)} + \||y^\alpha u\|_{L^p_m} \le C\|u - y^\alpha Bu\| + \|y^\alpha u\|_{L^p_m} \\
&\le C\left(\|u - y^\alpha L_{2ta\cdot\xi}u + |\xi|^2 y^\alpha u\|_{L^p_m} + |2ta\cdot\xi|\|y^\alpha D_y u\|_{L^p_m} + \|(1+|\xi|^2)y^\alpha u\|_{L^p_m}\right) \\
&\le C\left(\|u - y^\alpha L_{2ta\cdot\xi}u + |\xi|^2 y^\alpha u\|_{L^p_m} + \varepsilon\|y^\alpha Bu\|_{L^p_m} + \frac{1}{\varepsilon}\|(1+|\xi|^2)y^\alpha u\|_{L^p_m}\right) \\
&\le C\left(\|u - y^\alpha L_{2ta\cdot\xi}u + |\xi|^2 y^\alpha u\|_{L^p_m} + \varepsilon\|u\|_{W^{2,p}_{\mathcal{N}}(\alpha,m)} + \frac{1}{\varepsilon}(1+|\xi|^{-2})\||\xi|^2 y^\alpha u\|_{L^p_m}\right) \\
&\le C\left(\|u - y^\alpha L_{2ta\cdot\xi}u + |\xi|^2 y^\alpha u\|_{L^p_m} + \varepsilon\|u\|_{W^{2,p}_{\mathcal{N}}(\alpha,m)} + \frac{1}{\varepsilon}(1+|\xi|^{-2})\|u - y^\alpha L_{2ta\cdot\xi}u + |\xi|^2 y^\alpha u\|_{L^p_m}\right).
\end{aligned}
$$

Note that for the last inequality we used the fact that the estimate in Lemma 3.16 is uniform in ξ and a. By choosing $\varepsilon = \frac{1}{2C}$ we deduce for some C depending on ξ but independent of t

$$\|u\|_{W^{2,p}_{\mathcal{N}}(\alpha,m)} + \|y^\alpha u\|_{L^p_m} \le C\|u - y^\alpha L_{2ta\cdot\xi}u + |\xi|^2 y^\alpha u\|_{L^p_m}.$$

Since, for $t = 0$, the operator $I - y^\alpha B + |\xi|^2 y^\alpha$ is invertible in $W^{2,p}_{\mathcal{N}}(\alpha,m)\cap L^p_{m+\alpha p}$ by Theorem 2.1(ii), the same holds for $I - y^\alpha L_{2a\cdot\xi} + |\xi|^2 y^\alpha$, by the method of continuity.

Let $(L_{m,p}, D_{m,p})$ be the generator of $(e^{t(y^\alpha L_{2a\cdot\xi}-|\xi|^2 y^\alpha)})$ in L^p_m and consider the set

$$\mathcal{D} = \{u \in C_c^\infty([0,\infty)) : u \text{ constant in a neighborhood of } 0\}$$

which is dense in $W^{2,p}_{\mathcal{N}}(\alpha,m)\cap L^p_{m+\alpha p}$, by Theorem 8.4. By using the definition of $y^\alpha L_{2a\cdot\xi} - |\xi|^2 y^\alpha$ through the form $\mathfrak{a}$ as in the beginning of Section 3, it is easy to see that $\mathcal{D} \subset D_{c-\alpha,2}$ and that $y^\alpha L_{2a\cdot\xi} - |\xi|^2 y^\alpha = L_{c-\alpha,2}$ on $\mathcal{D}$. Since $\mathcal{D}$ is dense in $W^{2,2}_{\mathcal{N}}(\alpha,c-\alpha)\cap L^2_{c+\alpha}$, $y^\alpha L_{2a\cdot\xi} - |\xi|^2 y^\alpha$ is closed on $W^{2,2}_{\mathcal{N}}(\alpha,c-\alpha)\cap L^2_{c+\alpha}$ and $L_{c-\alpha,2}$ is closed on $D_{c-\alpha,2}$, it follows that $W^{2,2}_{\mathcal{N}}(\alpha,c-\alpha)\cap L^2_{c+\alpha} \subset D_{c-\alpha,2}$ and then $W^{2,2}_{\mathcal{N}}(\alpha,c-\alpha)\cap L^2_{c+\alpha} = D_{c-\alpha,2}$, $y^\alpha L_{2a\cdot\xi} - |\xi|^2 y^\alpha = L_{c-\alpha,2}$, since both operators are invertible on their own domains and one is an extension of the other. This completes the proof in the special case $p = 2, m = c - \alpha$.

Take now $u \in \mathcal{D}$ and let $f = \lambda u - (y^\alpha L_{2a\cdot\xi} - |\xi|^2 y^\alpha)u \in L^p_m \cap L^2_{c-\alpha}$ for large λ. Let $v \in D_{m,p}$ solve $\lambda v - L_{m,p}v = f$. Since the semigroups are consistent, v coincides with the $L^2_{c-\alpha}$ solution which, by the previous step, is u. This gives $\mathcal{D} \subset D_{m,p}$ and that $y^\alpha L_{2a\cdot\xi} - |\xi|^2 y^\alpha = L_{m,p}$ on $\mathcal{D}$ and, as before, one concludes the proof for $p < \infty$. □

We remark that Proposition 3.8 assures that $y^\alpha(L_{2a\cdot\xi} - |\xi|^2)$ generates a semigroup on L^p_m under the milder assumption $0 < \frac{m+1}{p} < c+1-\alpha$. However, the hypothesis $(m+1)/p + \alpha > 0$ must be added when $\alpha < 0$ to have $\mathcal{D} \subset L^p_{m+\alpha p}$.

As consequence we deduce the domain of the operator $y^\alpha L_{2a\cdot\xi}$ in the special case $\alpha = 0$.

Corollary 3.18 *Let* $1 < p < \infty$, $0 < \frac{m+1}{p} < c+1$ *and* $b \in \mathbb{R}$. *Then the domain of the operator* L_b *is* $W^{2,p}_{\mathcal{N}}(0,m)$. *The set* $\mathcal{D}$ *defined in* (4) *is a core.*

Proof. By the arbitrariness of ξ, a we can write $b = 2a\cdot\xi$. The required claim then follows from Theorem 3.17 since the domains of $L_{2a\cdot\xi}$ coincides with the one of $L_{2a\cdot\xi} - |\xi|^2$ which for $\alpha = 0$ is is $W^{2,p}_{\mathcal{N}}(0,m) \cap L^p_m = W^{2,p}_{\mathcal{N}}(0,m)$. □

Using Corollary 3.13 (i) with m replaced by $m-\alpha p$, we can characterize the domain of $L_{2a\cdot\xi} - \frac{\lambda}{y^\alpha}$. In what follows we write $D_{m,p}(A)$ to denote the domain of an operator A on L^p_m.

Corollary 3.19 *Let* $1 < p < \infty$, $\alpha^+ < \frac{m+1}{p} < c+1$, $\xi \in \mathbb{R}^N \setminus \{0\}$ *and* $\lambda \in C^+$. *Then the generator of* $(e^{z(L_{2a\cdot\xi} - \frac{\lambda}{y^\alpha})})$ *is the operator* $L_{2a\cdot\xi} - \frac{\lambda}{y^\alpha}$ *with domain* $W^{2,p}_{\mathcal{N}}(0,m) \cap L^p_{m-\alpha p}$. *In particular the set* $\mathcal{D}$ *defined in* (4) *is a core.*

Proof. We use (i) of Corollary 3.13 and Theorem 3.17 with m replaced by $\tilde{m} := m - \alpha p$ (note that the condition $\alpha^+ < \frac{m+1}{p} < c+1$ and $\alpha^- < \frac{\tilde{m}+1}{p} < c+1-\alpha$ are equivalent) to obtain

$$D_{m,p}\left(L_{2a\cdot\xi} - \frac{\lambda}{y^\alpha}\right) = \left(|\xi|^2 - L_{2a\cdot\xi} + \frac{\lambda}{y^\alpha}\right)^{-1} (L^p_m) = \left(\lambda - y^\alpha L_{2a\cdot\xi} + |\xi|^2 y^\alpha\right)^{-1} (L^p_{m-\alpha p})$$

$$= D_{m-\alpha p,p}\left(y^\alpha L_{2a\cdot\xi} - |\xi|^2 y^\alpha\right) = W^{2,p}_{\mathcal{N}}(\alpha, m-\alpha p) \cap L^p_m.$$

Lemma 3.14 then implies

$$D_{m,p}\left(L_{2a\cdot\xi} - \frac{\lambda}{y^\alpha}\right) = W^{2,p}_{\mathcal{N}}(0,m) \cap L^p_{m-\alpha p}.$$

□

Remark 3.20 *Let* $1 < p < \infty$ *such that* $\alpha^- < \frac{m+1}{p} < c+1-\alpha$, $\xi \in \mathbb{R}^N \setminus \{0\}$ *and* $\lambda \in \mathbb{C}^+$. *Theorem 3.17, Corollary 3.19 and Lemma 3.14 show that the operators*

$$y^\alpha L_{2a\cdot\xi} - |\xi|^2 y^\alpha \quad \textit{in} \quad L^p_m, \qquad L_{2a\cdot\xi} - \frac{\lambda}{y^\alpha} \quad \textit{in} \quad L^p_{m+\alpha p}$$

endowed with the common domain

$$W^{2,p}_{\mathcal{N}}(\alpha,m) \cap L^p_{m+\alpha p} = W^{2,p}_{\mathcal{N}}(0, m+\alpha p) \cap L^p_m$$

generate a semigroup on L^p_m *and* $L^p_{m+\alpha p}$, *respectively. Their resolvents satisfy*

$$\left(\lambda - y^\alpha L_{2a\cdot\xi} + |\xi|^2 y^\alpha\right)^{-1} f = \left(|\xi|^2 - L_{2a\cdot\xi} + \frac{\lambda}{y^\alpha}\right)^{-1} \left(\frac{f}{y^\alpha}\right), \qquad f \in L^p_m$$

proving the equivalence between the two elliptic equations

$$\lambda u - y^\alpha L_{2a\cdot\xi} u + |\xi|^2 y^\alpha u = f, \qquad \frac{\lambda}{y^\alpha} u - L_{2a\cdot\xi} u + |\xi|^2 u = \frac{f}{y^\alpha}.$$

4 Multipliers

In this section we investigate the boundedness of some multipliers related to the degenerate operator

$$\mathcal{L} = y^\alpha \left(\Delta_x + 2\sum_{i=1}^{N} a_i D_{iy} + D_{yy} + \frac{c}{y} D_y\right), \qquad a \in \mathbb{R}^N,\ |a| < 1, \qquad \alpha < 2 \tag{13}$$

Assuming that

$$y^\alpha \left(\Delta_x u + 2a \cdot \nabla_x D_y u + B_y u\right) = f$$

and taking the Fourier transform (denoted by $\mathcal{F}$ or $\hat{\cdot}$) with respect to x (with covariable ξ) we obtain

$$-y^\alpha |\xi|^2\, \hat{u}(\xi, y) + y^\alpha i 2a \cdot \xi D_y \hat{u}(\xi, y) + y^\alpha B_y \hat{u}(\xi, y) = \hat{f}(\xi, y).$$

We consider the operator $L_{2a\cdot\xi} = B + 2ia \cdot \xi D_y$ of Section 3. The latter computation shows that formally

$$(\lambda - \mathcal{L})^{-1} = \mathcal{F}^{-1}\left(\lambda - y^\alpha L_{2a\cdot\xi} + y^\alpha |\xi|^2\right)^{-1} \mathcal{F}.$$

In order to prove that $\mathcal{L}$ generates an analytic semigroup and to prove regularity for the associated parabolic problem, we investigate the boundedness of the operator-valued multiplier

$$\xi \in \mathbb{R}^N \to R_\lambda(\xi) = \left(\lambda - y^\alpha L_{2a\cdot\xi} + y^\alpha |\xi|^2\right)^{-1}.$$

To characterize the domain of $\mathcal{L}$ we also consider the multipliers $|\xi|^2 y^\alpha R_\lambda$, $\xi y^\alpha D_y R_\lambda$ which are associated with the operators $y^\alpha \Delta_x (\lambda - \mathcal{L})^{-1}$, $y^\alpha D_{xy}(\lambda - \mathcal{L})^{-1}$, respectively. In the next results we prove that the above multipliers satisfy the hypotheses of Theorem 7.5.

We also need the operator-valued multiplier $\tilde{R}_\lambda(\xi)$ defined by

$$\xi \in \mathbb{R}^N \to \tilde{R}_\lambda(\xi) = \left(|\xi|^2 - L_{2a\cdot\xi} + \frac{\lambda}{y^\alpha}\right)^{-1} \in \mathcal{B}\left(L^p_{m+\alpha p}\right).$$

Note that the role of ξ and λ is interchanged between $R_\lambda(\xi)$ and $\tilde{R}_\lambda(\xi)$: λ is a spectral parameter in the first and plays a role of a complex potential in the second, where instead is ξ the spectral parameter. Nevertheless, we keep the same notation.

By Corollary 3.13 we have

$$y^\alpha R_\lambda(\xi) f = y^\alpha \tilde{R}_\lambda(\xi)\left(\frac{f}{y^\alpha}\right), \qquad y^\alpha D_y R_\lambda(\xi) f = y^\alpha D_y \tilde{R}_\lambda(\xi)\left(\frac{f}{y^\alpha}\right), \qquad f \in L^p_m. \tag{14}$$

Proposition 4.1 *Let $1 < p < \infty$, $\alpha < 2$ be such that $\alpha^- < \frac{m+1}{p} < c + 1 - \alpha$. Then the families*

$$\left\{\lambda R_\lambda(\xi),\ |\xi|^2 y^\alpha R_\lambda(\xi),\ \xi y^\alpha D_y R_\lambda(\xi) \in \mathcal{B}(L^p_m)\ :\ \lambda \in \mathbb{C}_+,\ \xi \in \mathbb{R}^N \setminus \{0\}\right\}$$

are $\mathcal{R}$-bounded.

Proof. The $\mathcal{R}$-boundedness of $\lambda R_\lambda(\xi)$ follows by Corollary 3.9.

The $\mathcal{R}$-boundedness of $|\xi|^2 y^\alpha R_\lambda(\xi)$ in $\mathcal{B}(L^p_m)$ follows by using formula (14). We write

$$|\xi|^2 y^\alpha R_\lambda(\xi) = y^\alpha \left(|\xi|^2 \tilde{R}_\lambda(\xi)\right)\left(\frac{\cdot}{y^\alpha}\right)$$

and use the $\mathcal{R}$-boundedness of $|\xi|^2\tilde{R}_\lambda(\xi)$ in $\mathcal{B}(L^p_{m+\alpha p})$ proved in Proposition 3.10 with $V(y)=\lambda y^{-\alpha}$.

Let us finally prove the $\mathcal{R}$-boundedness of $\xi y^\alpha D_y R_\lambda(\xi)$ in $\mathcal{B}(L^p_m)$. By formula (14) again we have

$$\xi y^\alpha D_y R_\lambda(\xi) = y^\alpha \left(\xi D_y \tilde{R}_\lambda(\xi)\right)\left(\frac{\cdot}{y^\alpha}\right).$$

Let us write

$$\begin{aligned}
\tilde{R}_\lambda(\xi) &= \left(|\xi|^2 - L_{2a\cdot\xi} + \frac{\lambda}{y^\alpha}\right)^{-1} \\
&= \left(|\xi|^2 - L_{2a\cdot\xi}\right)^{-1}\left(|\xi|^2 - L_{2a\cdot\xi}\right)\left(|\xi|^2 - L_{2a\cdot\xi} + \frac{\lambda}{y^\alpha}\right)^{-1} \\
&= \left(|\xi|^2 - L_{2a\cdot\xi}\right)^{-1}\left(Id - \frac{\lambda}{y^\alpha}\tilde{R}_\lambda(\xi)\right) \\
&= \left(|\xi|^2 - L_{2a\cdot\xi}\right)^{-1}\left(\frac{\cdot}{y^\alpha}\right)\left(y^\alpha - \lambda\tilde{R}_\lambda(\xi)\right)
\end{aligned}$$

(note that $\left(|\xi|^2 - L_{2a\cdot\xi}\right)\left(|\xi|^2 - L_{2a\cdot\xi} + \frac{\lambda}{y^\alpha}\right)^{-1}$ is well defined since $D\left(L_{2a\cdot\xi} - \frac{\lambda}{y^\alpha}\right) \subseteq D\left(L_{2a\cdot\xi}\right)$, by Corollary 3.19). The previous relations and (14) give

$$\begin{aligned}
\xi y^\alpha D_y R_\lambda(\xi) &= \xi y^\alpha D_y\left(|\xi|^2 - L_{2a\cdot\xi}\right)^{-1}\left(\frac{\cdot}{y^\alpha}\right)\left(y^\alpha - \lambda\tilde{R}_\lambda(\xi)\right)\left(\frac{\cdot}{y^\alpha}\right) \\
&= \xi y^\alpha D_y\left(|\xi|^2 - L_{2a\cdot\xi}\right)^{-1}\left(\frac{\cdot}{y^\alpha}\right)\left(I - \lambda\tilde{R}_\lambda(\xi)\left(\frac{\cdot}{y^\alpha}\right)\right) \\
&= \xi y^\alpha D_y\left(|\xi|^2 - L_{2a\cdot\xi}\right)^{-1}\left(\frac{\cdot}{y^\alpha}\right)\left(I - \lambda R_\lambda(\xi)\right).
\end{aligned}$$

The $\mathcal{R}$-boundedness of the family $\xi y^\alpha D_y R_\lambda(\xi)$ in $\mathcal{B}(L^p_m)$ then follows by composing the $\mathcal{R}$-boundedness of $\xi D_y\left(|\xi|^2 - L_{2a\cdot\xi}\right)^{-1}$ in $\mathcal{B}(L^p_{m+\alpha p})$ proved in [18, Corollary 6.4] and the $\mathcal{R}$-boundedness of the family $\lambda R_\lambda(\xi)$ in $\mathcal{B}(L^p_m)$ proved in the first step. □

To apply the Mikhlin multiplier theorem, we need a formula for the derivatives of the above functions with respect to ξ. In the following lemma $\mathcal{S}_n$ denotes the set of permutations of n elements.

Lemma 4.2 *Let $1<p<\infty$, $\alpha<2$ be such that $\alpha^- < \frac{m+1}{p} < c+1-\alpha$, and let us consider, for any fixed $\lambda\in\mathbb{C}_+$, the map*

$$\xi\in\mathbb{R}^N \to R_\lambda(\xi) = (\lambda - y^\alpha L_{2a\cdot\xi} + y^\alpha|\xi|^2)^{-1} \in B(L^p_m).$$

Then R_λ, $y^\alpha R_\lambda$, $y^\alpha D_y R_\lambda \in C^\infty\left(\mathbb{R}^N\setminus\{0\}; B(L^p_m)\right)$ and for any family of different indexes

$j_1, j_2, \ldots, j_n \in \{1, \ldots, N\}$ *one has*

$$D_{\xi_{j_1}} \cdots D_{\xi_{j_n}} R_\lambda(\xi) = \sum_{\sigma \in \mathcal{S}_n} R_\lambda(\xi) \prod_{k=1}^{n} \Big(2ia_{j_{\sigma(k)}} y^\alpha D_y R_\lambda(\xi) - 2\xi_{j_{\sigma(k)}} y^\alpha R_\lambda(\xi) \Big)$$

$$D_{\xi_{j_1}} \cdots D_{\xi_{j_n}} y^\alpha R_\lambda(\xi) = \sum_{\sigma \in \mathcal{S}_n} y^\alpha R_\lambda(\xi) \prod_{k=1}^{n} \Big(2ia_{j_{\sigma(k)}} y^\alpha D_y R_\lambda(\xi) - 2\xi_{j_{\sigma(k)}} y^\alpha R_\lambda(\xi) \Big) \tag{15}$$

$$D_{\xi_{j_1}} \cdots D_{\xi_{j_n}} y^\alpha D_y R_\lambda(\xi) = \sum_{\sigma \in \mathcal{S}_n} y^\alpha D_y R_\lambda(\xi) \prod_{k=1}^{n} \Big(2ia_{j_{\sigma(k)}} y^\alpha D_y R_\lambda(\xi) - 2\xi_{j_{\sigma(k)}} y^\alpha R_\lambda(\xi) \Big).$$

Proof. Let us fix $\lambda \in \mathbb{C}_+$. Let us prove the first equality in (15) for $n = 1$ that is, for $j = 1, \ldots, n$

$$\frac{\partial}{\partial \xi_j} (R_\lambda(\xi)) = R_\lambda(\xi) \Big(2ia_j y^\alpha D_y R_\lambda(\xi) - 2\xi_j y^\alpha R_\lambda(\xi) \Big), \qquad \xi \in \mathbb{R}^n \setminus \{0\}. \tag{16}$$

Indeed let us write for $|h| \leq 1$

$$\begin{aligned}
R_\lambda(\xi + he_j) - R_\lambda(\xi) &= \left(\lambda + y^\alpha |\xi + he_j|^2 - y^\alpha L_{2a\cdot(\xi + he_j)} \right)^{-1} - \left(\lambda + y^\alpha |\xi|^2 - y^\alpha L_{2a\cdot\xi} \right)^{-1} \\
&= R_\lambda(\xi) \Big[\left(\lambda + y^\alpha |\xi|^2 - y^\alpha L_{2a\cdot\xi} \right) \left(\lambda + y^\alpha |\xi + he_j|^2 - y^\alpha L_{2a\cdot(\xi+he_j)} \right)^{-1} - I \Big] \\
&= R_\lambda(\xi)\, y^\alpha \left(L_{2a\cdot(\xi+he_j)} - L_{2a\cdot\xi} + |\xi|^2 - |\xi + he_j|^2 \right) \left(\lambda + y^\alpha |\xi + he_j|^2 - y^\alpha L_{2a\cdot(\xi+he_j)} \right)^{-1} \\
&= R_\lambda(\xi)\, y^\alpha \left(2ia_j h D_y - 2\xi_j h - h^2 \right) R_\lambda(\xi + he_j) \\
&= 2ia_j h\, R_\lambda(\xi)\, y^\alpha D_y R_\lambda(\xi + he_j) - (2\xi_j h + h^2)\, R_\lambda(\xi)\, y^\alpha R_\lambda(\xi + he_j).
\end{aligned} \tag{17}$$

Using the $\mathcal{R}$-boundedness of $\lambda R_\lambda(\xi)$ $|\xi|^2 y^\alpha R_\lambda(\xi)$, $\xi y^\alpha D_y R_\lambda(\xi)$ of Proposition 4.1 (which implies uniform boundedness), the last equation implies in particular that

$$\begin{aligned}
&R_\lambda(\xi + he_j) \to R_\lambda(\xi), \qquad y^\alpha R_\lambda(\xi + he_j) \to y^\alpha R_\lambda(\xi), \\
&y^\alpha D_y R_\lambda(\xi + he_j) \to y^\alpha D_y R_\lambda(\xi) \qquad \text{in the norm of } B\left(L^p_m\right) \quad \text{as } h \to 0.
\end{aligned} \tag{18}$$

For example from (17) one has for some positive constant C

$$\begin{aligned}
\|R_\lambda(\xi + he_j) - R_\lambda(\xi)\|_{\mathcal{B}(L^p_m)} &\leq C \frac{\|R_\lambda(\xi)\|_{\mathcal{R}(\mathcal{B}(L^p_m))}}{|\lambda|} \\
&\times \left(|h| \frac{\|y^\alpha D_y R_\lambda(\xi)\|_{\mathcal{R}(\mathcal{B}(L^p_m))}}{|\xi + he_j|} + (|\xi| h + h^2) \frac{\|y^\alpha R_\lambda(\xi)\|_{\mathcal{R}(\mathcal{B}(L^p_m))}}{|\xi + he_j|^2} \right) \to 0 \qquad \text{as} \quad h \to 0.
\end{aligned}$$

The other limits in (18) follow similarly after applying to both sides of (17) y^α and $y^\alpha D_y$, respectively.

To end the proof we apply equality (17) again to get

$$\frac{R_\lambda(\xi + he_j) - R_\lambda(\xi)}{h} = R_\lambda(\xi) y^\alpha \left(2ia_j D_y - 2\xi_j \right) R_\lambda(\xi) - h\, R_\lambda(\xi)\, y^\alpha R_\lambda(\xi + he_j) \tag{19}$$

which tends to $R_\lambda(\xi)y^\alpha\left(2ia_jD_y-2\xi_j\right)R_\lambda(\xi)$ in the norm of $B\left(L^p_m\right)$ as $h\to 0$ since , by (18), the last term tends to 0. This proves (16).

The proof of the other equalities in (15) for $n=1$ that is, for $j=1,\dots,n$,

$$\frac{\partial}{\partial\xi_j}(y^\alpha R_\lambda(\xi))=y^\alpha R_\lambda(\xi)\left(2ia_jy^\alpha D_yR_\lambda(\xi)-2\xi_jy^\alpha R_\lambda(\xi)\right),\qquad \xi\in\mathbb{R}^n\setminus\{0\},$$

$$\frac{\partial}{\partial\xi_j}(y^\alpha D_yR_\lambda(\xi))=y^\alpha D_yR_\lambda(\xi)\left(2ia_jy^\alpha D_yR_\lambda(\xi)-2\xi_jy^\alpha R_\lambda(\xi)\right),\qquad \xi\in\mathbb{R}^n\setminus\{0\}$$

follow similarly by applying, respectively, the operators $y^\alpha Id$, $y^\alpha D_y$ to both sides of (19) and taking the limit for $h\to 0$. For example for the derivative $\frac{\partial}{\partial\xi_j}(y^\alpha D_yR_\lambda(\xi))$ we write, as in (17),

$$\begin{aligned}\frac{y^\alpha D_yR_\lambda(\xi+he_j)-y^\alpha D_yR_\lambda(\xi)}{h}=&y^\alpha D_yR_\lambda(\xi)y^\alpha\left(2ia_jD_y-2\xi_j\right)R_\lambda(\xi)\\&-h\,y^\alpha D_yR_\lambda(\xi)\,y^\alpha R_\lambda(\xi+he_j)\end{aligned}$$

which by (18) again tends to $y^\alpha D_yR_\lambda(\xi)y^\alpha\left(2ia_jD_y-2\xi_j\right)R_\lambda(\xi)$ in the norm of $B\left(L^p_m\right)$ as $h\to 0$.

Finally, (15) for $n>1$ follows by induction. For example if $n=2$ and $l\neq j$ one has

$$\begin{aligned}\frac{\partial^2}{\partial\xi_l\partial\xi_j}(R_\lambda(\xi))=&\frac{\partial}{\partial\xi_l}\Big[R_\lambda(\xi)\Big(2ia_jy^\alpha D_yR_\lambda(\xi)-2\xi_jy^\alpha R_\lambda(\xi)\Big)\Big]\\
=&\frac{\partial}{\partial\xi_l}(R_\lambda(\xi))\Big(2ia_jy^\alpha D_yR_\lambda(\xi)-2\xi_jy^\alpha R_\lambda(\xi)\Big)\\
&+R_\lambda(\xi)\left(2ia_j\frac{\partial}{\partial\xi_l}\left(y^\alpha D_yR_\lambda(\xi)\right)-2\xi_j\frac{\partial}{\partial\xi_l}\left(y^\alpha R_\lambda(\xi)\right)\right)\\
=&R_\lambda(\xi)\Big(2ia_ly^\alpha D_yR_\lambda(\xi)-2\xi_ly^\alpha R_\lambda(\xi)\Big)\Big(2ia_jy^\alpha D_yR_\lambda(\xi)-2\xi_jy^\alpha R_\lambda(\xi)\Big)\\
&+R_\lambda(\xi)\Big(2ia_jy^\alpha D_yR_\lambda(\xi)-2\xi_jy^\alpha R_\lambda(\xi)\Big)\Big(2ia_ly^\alpha D_yR_\lambda(\xi)-2\xi_ly^\alpha R_\lambda(\xi)\Big).\end{aligned}$$

□

Now we can finally prove that the multiplier $\lambda R_\lambda(\xi)$ associated with the operators $\lambda(\lambda-\mathcal{L})^{-1}$ satisfies the hypothesis of Theorem 7.5. This is crucial for proving that $\mathcal{L}$ generates an analytic semigroup in L^p_m.

Theorem 4.3 *Let $1<p<\infty$, $\alpha<2$ be such that $\alpha^-<\frac{m+1}{p}<c+1-\alpha$. Then the family*

$$\left\{\xi^\beta D^\beta_\xi(\lambda R_\lambda(\xi)):\xi\in\mathbb{R}^N\setminus\{0\},\ \beta\in\{0,1\}^N,\lambda\in\mathbb{C}_+\right\}$$

is $\mathcal{R}$-bounded in L^p_m.

Proof. Let $\beta=(\beta_1,\dots,\beta_N)\in\{0,1\}^N$, $|\beta|=n$. Let us suppose, without any loss of generality, $\beta_i=1$, for $i\leq n$ and $\beta_i=0$, for $i>n$. Then using (15) we get

$$\begin{aligned}\xi^\beta D^\beta_\xi\lambda R_\lambda(\xi)=&\xi_1\cdots\xi_n\ \left(D_{\xi_1}\cdots D_{\xi_n}\right)\lambda R_\lambda(\xi)\\
=&\sum_{\sigma\in\mathcal{S}_n}\lambda R_\lambda(\xi)\prod_{j=1}^n\Big(2ia_{\sigma(j)}\xi_{\sigma(j)}y^\alpha D_yR_\lambda(\xi)-2\xi^2_{\sigma(j)}y^\alpha R_\lambda(\xi)\Big).\end{aligned}$$

The $\mathcal{R}$-boundedness of $\xi^\beta D^\beta_\xi(\lambda R_\lambda)(\xi)$ then follows by composition and domination from the $\mathcal{R}$-boundedness of $\lambda R_\lambda(\xi)$, $|\xi|^2 y^\alpha R_\lambda(\xi)$, $\xi y^\alpha R_\lambda(\xi)$ using Proposition 4.1 and Corollary 7.2. □

The next two theorems show that the multipliers $|\xi|^2 y^\alpha R_\lambda$, $\xi y^\alpha D_y R_\lambda$, associated respectively with the operators $y^\alpha \Delta_x(\lambda-\mathcal{L})^{-1}$, $y^\alpha D_{xy}(\lambda-\mathcal{L})^{-1}$, satisfy the hypotheses of Theorem 7.5. This is essential for characterizing the domain of $\mathcal{L}$.

Theorem 4.4 *Let $1<p<\infty$, $\alpha<2$ be such that $\alpha^- < \frac{m+1}{p} < c+1-\alpha$. Then the family*

$$\left\{\xi^\beta D^\beta_\xi(|\xi|^2 y^\alpha R_\lambda(\xi)) : \xi\in\mathbb{R}^N\setminus\{0\},\ \beta\in\{0,1\}^N, \lambda\in\mathbb{C}_+\right\}$$

is $\mathcal{R}$-bounded in L^p_m.

Proof. Let us prove preliminarily that the family

$$\left\{|\xi|^2\xi^\beta D^\beta_\xi(y^\alpha R_\lambda(\xi)) : \xi\in\mathbb{R}^N\setminus\{0\},\ \beta\in\{0,1\}^N, \lambda\in\mathbb{C}_+\right\}$$

is $\mathcal{R}$-bounded in L^p_m. Let $\beta=(\beta_1,\dots,\beta_N)\in\{0,1\}^N$, $|\beta|=n$. Let us suppose, without any loss of generality, $\beta_i=1$, for $i\le n$ and $\beta_i=0$, for $i>n$. Then using (15) one has

$$\begin{aligned}|\xi|^2\xi^\beta D^\beta_\xi y^\alpha R_\lambda(\xi) &= |\xi|^2\xi_1\cdots\xi_n\ (D_{\xi_1}\cdots D_{\xi_n})\, y^\alpha R_\lambda(\xi)\\ &= \sum_{\sigma\in\mathcal{S}_n}|\xi|^2 y^\alpha R_\lambda(\xi)\prod_{j=1}^n\Big(2ia_{\sigma(j)}\xi_{\sigma(j)}y^\alpha D_y R_\lambda(\xi) - 2\xi^2_{\sigma(j)}y^\alpha R_\lambda(\xi)\Big)\end{aligned}$$

and the required $\mathcal{R}$-boundedness of $|\xi|^2\xi^\beta D^\beta_\xi y^\alpha R_\lambda(\xi)$ then follows as at the end of Theorem 4.3.

To prove the required claim let us observe that for any $\beta\in\{0,1\}^N$, $|\beta|=n$ there exist $\beta^j\in\{0,1\}^N$ satisfying

$$\beta^j_k=\beta_k,\quad k\neq j,\qquad \beta^j_j=0,\qquad |\beta^j|=n-1$$

and such that

$$D^\beta_\xi(|\xi|^2 y^\alpha R_\lambda(\xi)) = \sum_{j:\beta_j=1} 2\xi_j D^{\beta^j}_\xi y^\alpha R_\lambda(\xi) + |\xi|^2 D^\beta_\xi y^\alpha R_\lambda(\xi).$$

Then

$$\xi^\beta D^\beta_\xi(|\xi|^2 y^\alpha R_\lambda(\xi)) = \sum_{j:\beta_j=1} 2\xi_j^2\xi^{\beta^j} D^{\beta^j}_\xi y^\alpha R_\lambda(\xi) + |\xi|^2\xi^\beta D^\beta_\xi y^\alpha R_\lambda(\xi).$$

and the proof now follows by domination using the previous step. □

Theorem 4.5 *Let $1<p<\infty$, $\alpha<2$ be such that $\alpha^- < \frac{m+1}{p} < c+1-\alpha$. Then the family*

$$\left\{\xi^\beta D^\beta_\xi\,(\xi y^\alpha D_y R_\lambda(\xi)) : \xi\in\mathbb{R}^N\setminus\{0\},\ \beta\in\{0,1\}^N, \lambda\in\mathbb{C}_+\right\}$$

is $\mathcal{R}$-bounded in L^p_m.

Proof. As in the proof of the previous theorem we prove preliminarily that the family

$$\left\{\xi\, \xi^{\beta} D^{\beta}_{\xi}(y^{\alpha} D_y R_\lambda(\xi)) : \xi \in \mathbb{R}^N \setminus \{0\},\ \beta \in \{0,1\}^N, \lambda \in \mathbb{C}_+\right\}$$

is $\mathcal{R}$-bounded in L^p_m. Indeed let $\beta = (\beta_1, \dots, \beta_N) \in \{0,1\}^N$, $|\beta| = n$. Let us suppose, without any loss of generality, $\beta_i = 1$, for $i \le n$ and $\beta_i = 0$, for $i > n$. Then using (15) one has

$$\begin{aligned}\xi\, \xi^{\beta} D^{\beta}_{\xi} y^{\alpha} D_y R_\lambda(\xi) &= \xi\, \xi_1 \cdots \xi_n\ (D_{\xi_1} \cdots D_{\xi_n})\, y^{\alpha} D_y R_\lambda(\xi) \\ &= \sum_{\sigma \in \mathcal{S}_n} \xi\, y^{\alpha} D_y R_\lambda(\xi) \prod_{j=1}^{n} \Big(2i a_{\sigma(j)} \xi_{\sigma(j)} y^{\alpha} D_y R_\lambda(\xi) - 2\xi^2_{\sigma(j)} y^{\alpha} R_\lambda(\xi)\Big)\end{aligned}$$

and the required $\mathcal{R}$-boundedness of $\xi\, \xi^{\beta} D^{\beta}_{\xi} y^{\alpha} D_y R_\lambda(\xi)$ then follows as at the end of Theorem 4.3.

To prove the required claim let us fix $\beta \in \{0,1\}^N$, $|\beta| = n$ and let us observe that for any $j = 1, \dots N$ one has

$$D^{\beta}_{\xi}(\xi_j y^{\alpha} D_y R_\lambda)(\xi) = \xi_j D^{\beta}_{\xi}\left(y^{\alpha} D_y R_\lambda(\xi)\right) + \beta_j D^{\beta^j}_{\xi}\left(y^{\alpha} D_y R_\lambda(\xi)\right).$$

where $\beta^j \in \{0,1\}^N$ satisfies

$$\beta^j_k = \beta_k, \quad k \neq j, \qquad \beta^j_j = 0, \qquad |\beta^j| = n - 1.$$

The proof then follows by the previous step. □

5 The operator $\mathcal{L} = y^{\alpha}(\Delta_x + 2a \cdot \nabla_x D_y + B_y), \quad |a| < 1, \alpha < 2$

In this section we prove generation results, maximal regularity and domain characterization for the operator $\mathcal{L}$ defined in (13) in L^p_m. More general operators will be treated in the next section, based on this model case. We start with the L^2 theory.

As explained at the beginning of Section 4, we have formally for $\lambda \in \mathbb{C}_+$

$$\lambda(\lambda - \mathcal{L})^{-1} = \mathcal{F}^{-1}\left(\lambda R_\lambda(\xi)\right)\mathcal{F}, \qquad R_\lambda(\xi) = \left(\lambda - y^{\alpha} L_{2a\cdot\xi} + y^{\alpha}|\xi|^2\right)^{-1}$$

and consequently

$$y^{\alpha}\Delta_x(\lambda - \mathcal{L})^{-1} = -\mathcal{F}^{-1}\left(|\xi|^2 y^{\alpha} R_\lambda(\xi)\right)\mathcal{F}, \qquad i\nabla_x D_y(\lambda - \mathcal{L})^{-1} = \mathcal{F}^{-1}\left(\xi y^{\alpha} D_y R_\lambda(\xi)\right)\mathcal{F}.$$

All properties of $\mathcal{L}$ follow from the boundedness of the above multipliers, through Theorem 7.5.

5.1 The operator $\mathcal{L}$ in $L^2_{c-\alpha}$

We assume that $c + 1 - \alpha > 0$ so that the measure $y^{c-\alpha}\, dx\, dy$ is locally finite near $y = 0$ and use the Sobolev space $H^1_{\alpha,c} := \{u \in L^2_{c-\alpha} : y^{\frac{\alpha}{2}}\nabla u \in L^2_{c-\alpha}\}$ equipped with the inner product

$$\langle u, v\rangle_{H^1_{\alpha,c}} := \langle u, v\rangle_{L^2_{c-\alpha}} + \left\langle y^{\frac{\alpha}{2}}\nabla u, y^{\frac{\alpha}{2}}\nabla v\right\rangle_{L^2_{c-\alpha}}.$$

We consider the form in $L^2_{c-\alpha}$

$$\mathfrak{a}(u,v) := \int_{\mathbb{R}^{N+1}_+} \langle \nabla u, \nabla \overline{v}\rangle\, y^c dx\, dy + 2\int_{\mathbb{R}^{N+1}_+} D_y u\, a \cdot \nabla_x \overline{v}\, y^c dx\, dy, \quad D(\mathfrak{a}) = H^1_{\alpha,c}$$

and its adjoint $\mathfrak{a}^*(u,v)=\overline{\mathfrak{a}(v,u)}$

$$\mathfrak{a}^*(u,v)=\overline{\mathfrak{a}(v,u)}:=\int_{\mathbb{R}^{N+1}_+}\langle \nabla u,\nabla \overline{v}\rangle\, y^c dx\,dy+2\int_{\mathbb{R}^{N+1}_+} a\cdot\nabla_x u\, D_y\overline{v}\, y^c dx\,dy.$$

Proposition 5.1 *The forms $\mathfrak{a}$, $\mathfrak{a}^*$ are continuous, accretive and sectorial.*

Proof. We consider only the form $\mathfrak{a}$, the adjoint form can be handled similarly. If $u\in H^1_{\alpha,c}$

$$Re\,\mathfrak{a}(u,u)\geq \|\nabla_x u\|^2_{L^2_c}+\|D_y u\|^2_{L^2_c}-2|a|\|\nabla_x u\|_{L^2_c}\|D_y u\|_{L^2_c}\geq (1-|a|)(\|\nabla_x u\|^2_{L^2_c}+\|D_y u\|^2_{L^2_c}).$$

By the ellipticity assumption $|a|<1$, the accretivity follows. Moreover

$$|Im\,\mathfrak{a}(u,u)|\leq 2|a|\|\nabla_x u\|_{L^2_c}\|D_y u\|_{L^2_c}\leq |a|(\|\nabla_x u\|^2_{L^2_c}+\|D_y u\|^2_{L^2_c})\leq \frac{|a|}{(1-|a|)}Re\,\mathfrak{a}(u,u).$$

This proves the sectoriality and then the continuity of the form. □

We define the operators $\mathcal{L}$ and $\mathcal{L}^*$ associated respectively to the forms $\mathfrak{a}$ and $\mathfrak{a}^*$ by

$$\begin{aligned}&D(\mathcal{L})=\{u\in H^1_{\alpha,c}:\exists f\in L^2_{c-\alpha}\text{ such that }\mathfrak{a}(u,v)=\int_{\mathbb{R}^{N+1}_+} f\overline{v}y^c\,dz \text{ for every } v\in H^1_{\alpha,c}\},\\ &\mathcal{L}u=-f;\end{aligned}\tag{20}$$

$$\begin{aligned}&D(\mathcal{L}^*)=\{u\in H^1_{\alpha,c}:\exists f\in L^2_{c-\alpha}\text{ such that }\mathfrak{a}^*(u,v)=\int_{\mathbb{R}^{N+1}_+} f\overline{v}y^c\,dz \text{ for every } v\in H^1_{\alpha,c}\},\\ &\mathcal{L}^*u=-f.\end{aligned}\tag{21}$$

If u,v are smooth function with compact support in the closure of $\mathbb{R}^{N+1}_+$ (so that they do not need to vanish on the boundary), it is easy to see integrating by parts that

$$-\mathfrak{a}(u,v)=\langle y^\alpha(\Delta_x u+2a\cdot\nabla_x D_y u+B_y u),\overline{v}\rangle_{L^2_{c-\alpha}}$$

if $\lim_{y\to 0} y^c D_y u(x,y)=0$. This means that $\mathcal{L}$ is the operator $y^\alpha(\Delta_x+2a\cdot\nabla_x D_y+B_y)$ with Neumann boundary conditions at $y=0$. On the other hand

$$-\mathfrak{a}^*(u,v)=\left\langle y^\alpha\left(\Delta_x u+2a\cdot\nabla_x D_y u+2(c-\alpha)\frac{a\cdot\nabla_x u}{y}+B_y u\right),\overline{v}\right\rangle_{L^2_{c-\alpha}}$$

if $\lim_{y\to 0} y^c\left(D_y u(x,y)+2a\cdot\nabla_x u(x,y)\right)=0$ and therefore $\mathcal{L}^*$ is the operator

$$y^\alpha\left(\Delta_x+2a\cdot\nabla_x D_y+2c\frac{a\cdot\nabla_x u}{y}+B_y\right)$$

with the above oblique condition at $y=0$.

Proposition 5.2 *$\mathcal{L}$ and $\mathcal{L}^*$ generate contractive analytic semigroups $e^{z\mathcal{L}}$, $e^{z\mathcal{L}^*}$, $z\in\Sigma_{\frac{\pi}{2}-\arctan\frac{|a|}{1-|a|}}$, in $L^2_{c-\alpha}$. Moreover the semigroups $(e^{t\mathcal{L}})_{t\geq 0},(e^{t\mathcal{L}^*})_{t\geq 0}$ are positive and $L^p_{c-\alpha}$-contractive for $1\leq p\leq\infty$.*

PROOF. We argue only for $\mathcal{L}$. The generation result immediately follows from Proposition 5.1 and [20, Theorem 1.52]. The positivity follows by [20, Theorem 2.6] after observing that, if $u \in H^1_{\alpha,c}$, u real, then $u^+ \in H^1_{\alpha,c}$ and

$$\mathfrak{a}(u^+,u^-) := \int_{\mathbb{R}^{N+1}_+} \langle \nabla u^+, \nabla u^- \rangle\, y^c dx\, dy + 2\int_{\mathbb{R}^{N+1}_+} D_y u^+ a\cdot \nabla_x u^-\, y^c dx\, dy = 0.$$

Finally, the L^∞-contractivity follows by [20, Corollary 2.17] after observing that if $0 \le u \in H^1_{\alpha,c}$, then $1\wedge u, (u-1)^+ \in H^1_{\alpha,c}$ and, since $\nabla(1\wedge u) = \chi_{\{u<1\}}\nabla u$ and $\nabla(u-1)^+ = \chi_{\{u>1\}}\nabla u$, one has

$$\mathfrak{a}(1\wedge u, (u-1)^+) = 0.$$

□

The Stein interpolation theorem then shows that the above semigroups are analytic in $L^p_{c-\alpha}$ for $1<p<\infty$, see [20, Proposition 3.12] and a result by Lamberton yields maximal regularity in the same range, see [12, Theorem 5.6]. Since our results are more general, we do not state these consequences here.

Our aim is to characterize the domain of $\mathcal{L}$ in $L^2_{c-\alpha}$. As in [18, Section 7.1] and [17, Section 6.1], we can prove the following result.

Theorem 5.3 *If $c+1>|\alpha|$ then*

$$D(\mathcal{L}) = W^{2,2}_{\mathcal{N}}(\alpha,\alpha,c-\alpha).$$

In particular the set $C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}$, see (24), is a core for $\mathcal{L}$ in $L^2_{c-\alpha}(\mathbb{R}^{N+1}_+)$.

PROOF. The proof follows as in [18, Proposition 7.3, Theorem 7.4] using the boundedness of the multipliers $|\xi|^2 y^\alpha R_1(\xi)$, $\xi y^\alpha D_y R_1(\xi)$ in $L^2(\mathbb{R}^N; L^2_{c-\alpha}(\mathbb{R}_+)) = L^2_{c-\alpha}$, proved in Proposition 4.1. Note that the condition $c+1>|\alpha|$ is that of the quoted proposition with $p=2$ and $m=c-\alpha$.□

5.2 The operator $\mathcal{L}$ in L^p_m

In this section we prove domain characterization and maximal regularity for $\mathcal{L}$ in L^p_m. For clarity reasons we often write $\mathcal{L}_{m,p}$ to emphasize the underlying space on which the operator acts.

We shall use extensively the set (finite sums below)

$$C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D} = \left\{u(x,y) = \sum_i u_i(x)v_i(y),\ u_i\in C_c^\infty(\mathbb{R}^N),\ v_i\in\mathcal{D}\right\},$$

where $\mathcal{D}$ is defined in (4). We refer to Appendix 8 for further details and note that $\mathcal{L}$ is well defined on $C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}\subset L^p_m$ when $(m+1)/p>\alpha^-$.

The next results can be proved as in [18, Lemma 7.5, Lemma 7.6, Theorem 7.7, Corollary 7.8].

Lemma 5.4 *Let $\alpha^- < \frac{m+1}{p} < c+1-\alpha$. Then for any $\lambda\in\mathbb{C}^+$ the operators*

$$(\lambda-\mathcal{L}_{c-\alpha,2})^{-1},\quad y^\alpha\Delta_x(\lambda-\mathcal{L}_{c-\alpha,2})^{-1},\quad y^\alpha\nabla_x D_y(\lambda-\mathcal{L}_{c-\alpha,2})^{-1},\quad y^\alpha B^n_y(\lambda-\mathcal{L}_{c-\alpha,2})^{-1}$$

initially defined on $L^p_m\cap L^2_{c-\alpha}$ by Theorem 5.3, extend to bounded operators on L^p_m which we denote respectively by $\mathcal{R}(\lambda)$, $y^\alpha\Delta_x\mathcal{R}(\lambda)$, $y^\alpha\nabla_x D_y\mathcal{R}(\lambda)$, $y^\alpha B^n_y\mathcal{R}(\lambda)$. Moreover the family $\{\lambda\mathcal{R}(\lambda):\lambda\in\mathbb{C}^+\}$ is $\mathcal{R}$-bounded on L^p_m.

Proposition 5.5 *If $\alpha^- < \frac{m+1}{p} < c+1-\alpha$, an extension $\mathcal{L}_{m,p}$ of the operator $\mathcal{L}$, initially defined on $C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}$, generates a bounded analytic semigroup in $L^p_m(\mathbb{R}^{N+1}_+)$ which has maximal regularity and it is consistent with the semigroup generated by $\mathcal{L}_{c-\alpha,2}$ in $L^2_{c-\alpha}(\mathbb{R}^{N+1}_+)$.*

Finally we characterize the domain of $\mathcal{L}_{m,p}$.

Theorem 5.6 *If $\alpha^- < \frac{m+1}{p} < c+1-\alpha$, then*

$$D(\mathcal{L}_{m,p}) = W^{2,p}_{\mathcal{N}}(\alpha,\alpha,m)$$

and in particular $C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}$ is a core for $\mathcal{L}_{m,p}$.

Corollary 5.7 *Under the hypotheses of Theorem 5.6 we have for every $u \in W^{2,p}_{\mathcal{N}}(\alpha,\alpha,m)$*

$$\|y^\alpha D_{x_ix_j}u\|_{L^p_m} + \|y^\alpha D_{yy}u\|_{L^p_m} + \|y^\alpha D_{x_iy}u\|_{L^p_m} + \|y^{\alpha-1}D_yu\|_{L^p_m} \le C\|\mathcal{L}u\|_{L^p_m}.$$

6 Consequences for more general operators

The isometry introduced in Section 8 allows to deduce generation and domain properties in L^p_m for more general operators of the form

$$\mathcal{L} = y^{\alpha_1}\Delta_x + 2y^{\frac{\alpha_1+\alpha_2}{2}} a\cdot\nabla_x D_y + y^{\alpha_2}\left(D_{yy} + \frac{c}{y}D_y\right),$$

with $\alpha_1\alpha_2 \in \mathbb{R}$, $\alpha_2 < 2$, $\alpha_2 - \alpha_1 < 2$.

Theorem 6.1 *Let $\alpha_2 < 2$, $\alpha_2 - \alpha_1 < 2$ and*

$$\alpha_1^- < \frac{m+1}{p} < c+1-\alpha_2.$$

Then $\mathcal{L}$ with domain $W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m)$ generates a bounded analytic semigroup in L^p_m which has maximal regularity.

Proof. We use the isometry

$$T_{\frac{\alpha_1-\alpha_2}{2}} : L^p_{\tilde m} \to L^p_m, \qquad \tilde m = \frac{2m-\alpha_1+\alpha_2}{\alpha_1-\alpha_2+2}$$

which, according to Proposition 8.7, transforms $\mathcal{L}$ into

$$T^{-1}_{\frac{\alpha_1-\alpha_2}{2}}\,\mathcal{L}\,T_{\frac{\alpha_1-\alpha_2}{2}} = y^\alpha\Delta_x + y^\alpha 2\left(\frac{\alpha_1-\alpha_2+2}{2}\right) a\cdot\nabla_x D_y + \left(\frac{\alpha_1-\alpha_2+2}{2}\right)^2 y^\alpha \tilde B_y$$

where

$$\alpha = \frac{2\alpha_1}{\alpha_1-\alpha_2+2}, \quad \tilde B_y = D_{yy} + \frac{\tilde c}{y}D_y, \quad \tilde c = \frac{2c+\alpha_1-\alpha_2}{\alpha_1-\alpha_2+2}.$$

Observe the assumptions on the parameters translates into $\alpha < 2$ and $\alpha^- < \frac{\tilde m+1}{p} < \tilde c+1-\alpha$. The generation properties and maximal regularity for $\mathcal{L}$ in L^p_m are then immediate consequence of the same properties for the operator studied before. Concerning the domain, we have

$$D(\mathcal{L}) = T_{\frac{\alpha_1-\alpha_2}{2}}\left(W^{2,p}_{\mathcal{N}}(\alpha,\alpha,\tilde m)\right)$$

which, by Proposition 8.5, coincides with $W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m)$. □

Results for more general operators and boundary conditions follow by linear change of variables, as we explain below. We consider the operator in $\mathbb{R}^{N+1}_+$

$$\mathcal{L} = y^{\alpha_1}\mathrm{Tr}\,\left(QD_x^2 u\right) + 2y^{\frac{\alpha_1+\alpha_2}{2}} q\cdot\nabla_x D_y + y^{\alpha_2}\left(\gamma D_{yy} + \frac{c}{y}D_y\right)$$

$$= y^{\alpha_1}\sum_{i,j=1}^{N} q_{ij}D_{x_ix_j} + 2y^{\frac{\alpha_1+\alpha_2}{2}}\sum_{i=1}^{N} q_i D_{x_iy} + y^{\alpha_2}\left(\gamma D_{yy} + \frac{c}{y}D_y\right).$$

Here Q is the $N\times N$ matrix (q_{ij}), $q=(q_1,\dots,q_N)$ and we assume that the quadratic form $Q(\xi,\xi)+\gamma\eta^2+2q\cdot\xi\,\eta$ is positive definite. Through a linear change of variables in the x variables the term $\sum_{i,j=1}^N q_{ij}D_{x_ix_j}$ is transformed into $\gamma\Delta_x$ and all the results of Section 5 hold, replacing c with $\frac{c}{\gamma}$ in the statements (the condition $|a|<1$ of Section 5 is satisfied since the change of variables preserves the ellipticity). The case of variable coefficients can also be handled by freezing the coefficients and will be done in the future to deal with degenerate problems in bounded domains.

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A further change of variables allows to deal with the operator

$$\mathcal{L} = y^{\alpha_1}\mathrm{Tr}\,\left(QD_x^2 u\right) + 2y^{\frac{\alpha_1+\alpha_2}{2}} q\cdot\nabla_x D_y + y^{\alpha_2}\gamma D_{yy} + y^{\alpha_2-1}v\cdot\nabla$$

Here $v=(b,c)\in\mathbb{R}^{N+1}$, with $c\neq 0$, and we impose an oblique derivative boundary condition $y^{\alpha_2-1}v\cdot\nabla u(x,0)=0$ in the integral form

$$y^{\alpha_2-1}v\cdot\nabla u = y^{\alpha_2-1}\left(b\cdot\nabla_x u + cD_y u\right)\in L^p_m.$$

We define therefore

$$W^{2,p}_v(\alpha_1,\alpha_2,m) \overset{def}{=} \{u\in W^{2,p}(\alpha_1,\alpha_2,m):\ y^{\alpha_2-1}v\cdot\nabla u\in L^p_m\}.$$

We transform $\mathcal{L}$ into a similar operator with $b=0$ and Neumann boundary condition by mean of the following isometry of L^p_m

$$T\,u(x,y) := u\left(x-\frac{b}{c}y, y\right),\quad (x,y)\in\mathbb{R}^{N+1}_+. \tag{22}$$

Lemma 6.2 *Let $1<p<\infty$, $v=(b,c)\in\mathbb{R}^{N+1}$, $c\neq 0$. Then for $u\in W^{2,1}_{loc}\left(\mathbb{R}^{N+1}_+\right)$*

(i)

$$T^{-1}\left(y^{\alpha_1}\,Tr\,\left(QD_x^2u\right) + 2y^{\frac{\alpha_1+\alpha_2}{2}} q\cdot\nabla_x D_y + y^{\alpha_2}\gamma D_{yy} + y^{\alpha_2-1}v\cdot\nabla\right)T\,u$$

$$= y^{\alpha_1}\,Tr\,\left(\tilde{Q}D_x^2u\right) + 2y^{\frac{\alpha_1+\alpha_2}{2}}\tilde{q}\cdot\nabla_x D_y + y^{\alpha_2}\left(\gamma D_{yy}+\frac{c}{y}D_y\right)$$

where

$$\tilde{Q} = Q - \frac{2}{c}b\otimes q + \frac{\gamma}{c^2}b\otimes b,\qquad \tilde{q} = q - \frac{\gamma}{c}b$$

and the matrix $\left(\begin{array}{c|c}\tilde{Q} & \tilde{q}^t\\ \hline \tilde{q} & \gamma\end{array}\right)$ *is elliptic.*

(ii) $T\left(W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m)\right) = W^{2,p}_v(\alpha_1,\alpha_2,m)$.

PROOF. The proof follows by a straightforward computation. □

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We can therefore deduce results also for the last operator whose proofs follow directly from ~~the above lemma~~ Theorem 6.1.

Theorem 6.3 *Let* ~~$v=(b,c)\in\mathbb{R}^{N+1}$ *with* $b=0$ *if* $c=0$,~~ $c\in\mathbb{R}$ *and* $\begin{pmatrix} Q & q^t \\ q & \gamma \end{pmatrix}$ *an elliptic matrix and let* $\alpha_1,\alpha_2\in\mathbb{R}$ *such that* $\alpha_2<2$, $\alpha_2-\alpha_1<2$ *and*

$$\alpha_1^- < \frac{m+1}{p} < \frac{c}{\gamma}+1-\alpha_2.$$

Then the operator

$$\mathcal{L} = y^{\alpha_1}\, Tr\,\left(QD_x^2 u\right) + 2y^{\frac{\alpha_1+\alpha_2}{2}} q\cdot\nabla_x D_y + \gamma y^{\alpha_2} D_{yy} + cy^{\alpha_2-1}D_y \;\cancel{y^{\alpha_2-1} v\cdot\nabla}$$

endowed with domain $W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m)$ ~~$W^{2,p}_v(\alpha_1,\alpha_2,m)$ *when* $c\neq 0$ *and* $W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m)$ *when* $c=0$,~~ *generates a bounded analytic semigroup in* L^p_m *which has maximal regularity.*

We refer the reader to [19] for a further generalization of $\mathcal{L}$ involving Dirichlet or oblique derivative boundary conditions.

7 Appendix A: Vector-valued harmonic analysis

We review some results on vector-valued multiplier theorems referring the reader to [2], [21] or [12] for all proofs.

Let $\mathcal{S}$ be a subset of $B(X)$, the space of all bounded linear operators on a Banach space X. $\mathcal{S}$ is $\mathcal{R}$-bounded if there is a constant C such that

$$\|\sum_i \varepsilon_i S_i x_i\|_{L^p(\Omega;X)} \le C\|\sum_i \varepsilon_i x_i\|_{L^p(\Omega;X)}$$

for every finite sum as above, where $(x_i)\subset X, (S_i)\subset\mathcal{S}$ and $\varepsilon_i:\Omega\to\{-1,1\}$ are independent and symmetric random variables on a probability space Ω. In particular $\mathcal{S}$ is a bounded subset of $B(X)$. The smallest constant C for which the above definition holds is the $\mathcal{R}$-bound of $\mathcal{S}$, denoted by $\mathcal{R}(\mathcal{S})$. It is well-known that this definition does not depend on $1\le p<\infty$ (however, the constant $\mathcal{R}(\mathcal{S})$ does) and that $\mathcal{R}$-boundedness is equivalent to boundedness when X is an Hilbert space. When X is an $L^p(\Sigma)$ space (with respect to any σ-finite measure defined on a σ-algebra Σ), testing $\mathcal{R}$-boundedness is equivalent to proving square functions estimates, see [12, Remark 2.9].

Proposition 7.1 *Let* $\mathcal{S}\subset B(L^p(\Sigma))$, $1<p<\infty$. *Then* $\mathcal{S}$ *is* $\mathcal{R}$*-bounded if and only if there is a constant* $C>0$ *such that for every finite family* $(f_i)\in L^p(\Sigma), (S_i)\in\mathcal{S}$

$$\left\|\left(\sum_i |S_i f_i|^2\right)^{\frac12}\right\|_{L^p(\Sigma)} \le C\left\|\left(\sum_i |f_i|^2\right)^{\frac12}\right\|_{L^p(\Sigma)}.$$

The best constant C for which the above square functions estimates hold satisfies $\kappa^{-1}C \le \mathcal{R}(\mathcal{S}) \le \kappa C$ for a suitable $\kappa > 0$ (depending only on p). Using the proposition above, $\mathcal{R}$-boundedness follows from domination by a positive $\mathcal{R}$-bounded family.

Corollary 7.2 *Let $\mathcal{S}, \mathcal{T} \subset B(L^p(\Sigma))$, $1 < p < \infty$ and assume that $\mathcal{T}$ is an $\mathcal{R}$ bounded family of positive operators and that for every $S \in \mathcal{S}$ there exists $T \in \mathcal{T}$ such that $|Sf| \le T|f|$ pointwise, for every $f \in L^p(\Sigma)$. Then $\mathcal{S}$ is $\mathcal{R}$-bounded.*

We also need the following result about the integral mean of a $\mathcal{R}$-bounded family of operator which we state in the version we use.

Proposition 7.3 [12, Corollary 2.14] *Let X be a Banach space and let $\mathcal{F} \subset B(X)$ be an $\mathcal{R}$-bounded family of operator. For every strongly measurable $N : \Sigma \to B(X)$ on a σ-finite measure space (Σ, μ) with values in $\mathcal{F}$ and every $h \in L^1(\Sigma, \mu)$ we define the operator $T_{N,\mathcal{F}} \in B(X)$ by*

$$T_{N,\mathcal{F}}x = \int_\Sigma h(\omega)N(\omega)x\,d\mu(\omega), \qquad x \in X.$$

Then the family

$$\mathcal{C} = \{T_{N,\mathcal{F}} \ : \ \|h\|_{L^1} \le 1, N \text{ as above}\}$$

is $\mathcal{R}$ bounded and $\mathcal{R}(\mathcal{C}) \le 2\mathcal{R}(\mathcal{F})$.

Let $(A, D(A))$ be a sectorial operator in a Banach space X; this means that $\rho(-A) \supset \Sigma_{\pi-\phi}$ for some $\phi < \pi$ and that $\lambda(\lambda + A)^{-1}$ is bounded in $\Sigma_{\pi-\phi}$. The infimum of all such ϕ is called the spectral angle of A and denoted by ϕ_A. Note that $-A$ generates an analytic semigroup if and only if $\phi_A < \pi/2$. The definition of $\mathcal{R}$-sectorial operator is similar, substituting boundedness of $\lambda(\lambda + A)^{-1}$ with $\mathcal{R}$-boundedness in $\Sigma_{\pi-\phi}$. As above one denotes by ϕ_A^R the infimum of all ϕ for which this happens; since $\mathcal{R}$-boundedness implies boundedness, we have $\phi_A \le \phi_A^R$.

The $\mathcal{R}$-boundedness of the resolvent characterizes the regularity of the associated inhomogeneous parabolic problem, as we explain now.

An analytic semigroup $(e^{-tA})_{t\ge0}$ on a Banach space X with generator $-A$ has *maximal regularity of type L^q* $(1 < q < \infty)$ if for each $f \in L^q([0,T];X)$ the function $t \mapsto u(t) = \int_0^t e^{-(t-s)A)}f(s)\,ds$ belongs to $W^{1,q}([0,T];X) \cap L^q([0,T];D(A))$. This means that the mild solution of the evolution equation

$$u'(t) + Au(t) = f(t), \quad t > 0, \qquad u(0) = 0,$$

is in fact a strong solution and has the best regularity one can expect. It is known that this property does not depend on $1 < q < \infty$ and $T > 0$. A characterization of maximal regularity is available in UMD Banach spaces, through the $\mathcal{R}$-boundedness of the resolvent in a suitable sector $\omega + \Sigma_\phi$, with $\omega \in \mathbb{R}$ and $\phi > \pi/2$ or, equivalently, of the scaled semigroup $e^{-(A+\omega')t}$ in a sector around the positive axis. In the case of L^p spaces it can be restated in the following form, see [12, Theorem 1.11]

Theorem 7.4 *Let $(e^{-tA})_{t\ge0}$ be a bounded analytic semigroup in $L^p(\Sigma)$, $1 < p < \infty$, with generator $-A$. Then $T(\cdot)$ has maximal regularity of type L^q if and only if the set $\{\lambda(\lambda+A)^{-1}, \lambda \in \Sigma_{\pi/2+\phi}\}$ is $\mathcal{R}$- bounded for some $\phi > 0$. In an equivalent way, if and only if there are constants $0 < \phi < \pi/2$, $C > 0$ such that for every finite sequence $(\lambda_i) \subset \Sigma_{\pi/2+\phi}$, $(f_i) \subset L^p$*

$$\left\| \left(\sum_i |\lambda_i(\lambda_i + A)^{-1} f_i|^2 \right)^{\frac12} \right\|_{L^p(\Sigma)} \le C \left\| \left(\sum_i |f_i|^2 \right)^{\frac12} \right\|_{L^p(\Sigma)}$$

or, equivalently, there are constants $0 < \phi' < \pi/2$, $C' > 0$ such that for every finite sequence $(z_i) \subset \Sigma_{\phi'}$, $(f_i) \subset L^p$

$$\left\| \left(\sum_i |e^{-z_i A} f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Sigma)} \leq C' \left\| \left(\sum_i |f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Sigma)}.$$

Finally we state a version of the operator-valued Mikhlin multiplier theorem in the N-dimensional case, see e.g. [11, Corollary 8.3.22].

Theorem 7.5 *Let $1 < p < \infty$, $M \in C^N(\mathbb{R}^N \setminus \{0\}; B(L^p(\Sigma))$ be such that the set*

$$\left\{ \xi^\alpha D_\xi^\alpha M(\xi) : \xi \in \mathbb{R}^N \setminus \{0\},\ \alpha \in \{0,1\}^N \right\}$$

is $\mathcal{R}$-bounded. Then the operator $T_M = \mathcal{F}^{-1} M \mathcal{F}$ is bounded in $L^p(\mathbb{R}^N, L^p(\Sigma))$, where $\mathcal{F}$ denotes the Fourier transform.

8 Appendix B: Weighted spaces and similarity transformations

Let $p > 1$, $m, \alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\alpha_2 < 2, \qquad \alpha_2 - \alpha_1 < 2, \qquad \alpha_1^- < \frac{m+1}{p}.$$

In order to describe the domain of the operator

$$y^{\alpha_1} \Delta_x + 2y^{\frac{\alpha_1+\alpha_2}{2}} a \cdot \nabla_x D_y + y^{\alpha_2} \left(D_{yy} + \frac{c}{y} D_y \right),$$

we collect in this section the main results concerning anisotropic weighted Sobolev spaces, referring to [15] for further details and all the relative proofs. We define the Sobolev space

$$W^{2,p}(\alpha_1, \alpha_2, m) = \Big\{ u \in W^{2,p}_{loc}(\mathbb{R}^{N+1}_+) :\ u,\ y^{\alpha_1} D_{x_i x_j} u,\ y^{\frac{\alpha_1}{2}} D_{x_i} u, \\ y^{\alpha_2} D_{yy} u,\ y^{\frac{\alpha_2}{2}} D_y u,\ y^{\frac{\alpha_1+\alpha_2}{2}} D_y \nabla_x u \in L^p_m \Big\}$$

which is a Banach space equipped with the norm

$$\|u\|_{W^{2,p}(\alpha_1,\alpha_2,m)} = \|u\|_{L^p_m} + \sum_{i,j=1}^n \|y^{\alpha_1} D_{x_i x_j} u\|_{L^p_m} + \sum_{i=1}^n \|y^{\frac{\alpha_1}{2}} D_{x_i} u\|_{L^p_m} \\ + \|y^{\alpha_2} D_{yy} u\|_{L^p_m} + \|y^{\frac{\alpha_2}{2}} D_y u\|_{L^p_m} + \|y^{\frac{\alpha_1+\alpha_2}{2}} D_y \nabla_x u\|_{L^p_m}.$$

Next we add a Neumann boundary condition for $y = 0$ in the form $y^{\alpha_2 - 1} D_y u \in L^p_m$ and set

$$W^{2,p}_{\mathcal{N}}(\alpha_1, \alpha_2, m) = \{ u \in W^{2,p}(\alpha_1, \alpha_2, m) :\ y^{\alpha_2 - 1} D_y u\ \in L^p_m \}$$

with the norm

$$\|u\|_{W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m)} = \|u\|_{W^{2,p}(\alpha_1,\alpha_2,m)} + \|y^{\alpha_2-1} D_y u\|_{L^p_m}.$$

Remark 8.1 *With obvious changes we consider also the analogous Sobolev spaces $W^{2,p}(\alpha,m)$ and $W^{2,p}_{\mathcal{N}}(\alpha,m)$ on $\mathbb{R}_+$. For example we have*

$$W^{2,p}_{\mathcal{N}}(\alpha,m)=\left\{u\in W^{2,p}_{loc}(\mathbb{R}_+):\ u,\ y^\alpha D_{yy}u,\ y^{\frac{\alpha}{2}}D_y u,\ y^{\alpha-1}D_y u\in L^p_m\right\}.$$

All the results of this section will be valid also in $\mathbb{R}_+$ changing (when it appears) the condition $\alpha_1^-<\frac{m+1}{p}$ to $0<\frac{m+1}{p}$.

The next result clarifies in which sense the condition $y^{\alpha_2-1}D_y u\in L^p_m$ is a Neumann boundary condition.

Proposition 8.2 [15, Proposition 4.3] *The following assertions hold.*

(i) If $\frac{m+1}{p}>1-\alpha_2$, then $W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m)=W^{2,p}(\alpha_1,\alpha_2,m)$.

(ii) If $\frac{m+1}{p}<1-\alpha_2$, then

$$W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m)=\{u\in W^{2,p}(\alpha_1,\alpha_2,m):\lim_{y\to 0}D_y u(x,y)=0\text{ for a.e. x}\in\mathrm{\mathbb{R}^N}\}.$$

In both cases (i) and (ii), the norm of $W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m)$ is equivalent to that of $W^{2,p}(\alpha_1,\alpha_2,m)$.

The next results show the density of smooth functions in $W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m)$. Let

$$\mathcal{C}:=\left\{u\in C_c^\infty\left(\mathbb{R}^N\times[0,\infty)\right),\ D_y u(x,y)=0\text{ for }y\le\delta\text{ and some }\delta>0\right\},\tag{23}$$

its one dimensional version

$$\mathcal{D}=\{u\in C_c^\infty([0,\infty)),\ D_y u(y)=0\text{ for }y\le\delta\text{ and some }\delta>0\}\tag{24}$$

and finally (finite sums below)

$$C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}=\left\{u(x,y)=\sum_i u_i(x)v_i(y),\ u_i\in C_c^\infty(\mathbb{R}^N),\ v_i\in\mathcal{D}\right\}\subset\mathcal{C}.$$

Theorem 8.3 [15, Theorem 4.9] *$C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}$ is dense in $W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m)$.*

Note that the condition $(m+1)/p>\alpha_1^-$, or $m+1>0$ and $(m+1)/p+\alpha_1>0$, is necessary for the inclusion $C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}\subset W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m)$.

In the 1-dimensional case we need also the following density result proved in [13, Theorem 8.5].

Theorem 8.4 *Let $\alpha<2$, $\mu>0$, $c\in\mathbb{R}$. Then for any $1<p<\infty$ such that $\alpha^-<\frac{m+1}{p}<c+1-\alpha$, the set $\mathcal{D}$ is dense in $W^{2,p}_{\mathcal{N}}(\alpha,m)\cap L^p_{m+\alpha p}$.*

We consider now, for $\beta\in\mathbb{R}$, $\beta\neq-1$, the transformation

$$T_\beta u(x,y):=|\beta+1|^{\frac{1}{p}}u(x,y^{\beta+1}),\quad (x,y)\in\mathbb{R}^{N+1}_+.\tag{25}$$

Observe that

$$T_\beta^{-1}=T_{-\frac{\beta}{\beta+1}}.$$

Proposition 8.5 *Let $1 \le p \le \infty$, $\beta \in \mathbb{R}$, $\beta \neq -1$ and $m \in \mathbb{R}$. The following properties hold.*

(i) T_β maps isometrically $L^p_{\tilde m}$ onto L^p_m where $\tilde m = \frac{m-\beta}{\beta+1}$.

(ii) $W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m) = T_\beta\left(W^{2,p}_{\mathcal{N}}(\tilde\alpha_1,\tilde\alpha_2,\tilde m)\right)$, $\qquad \tilde\alpha_1 = \frac{\alpha_1}{\beta+1}$, $\quad \tilde\alpha_2 = \frac{\alpha_2+2\beta}{\beta+1}$.

In particular choosing $\beta = \frac{\alpha_1-\alpha_2}{2}$ and setting $\tilde\alpha = \frac{2\alpha_1}{\alpha_1-\alpha_2+2}$ one has

$$W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m) = T_{\frac{\alpha_1-\alpha_2}{2}}\left(W^{2,p}_{\mathcal{N}}(\tilde\alpha,\tilde\alpha,\tilde m)\right), \qquad \tilde\alpha = \frac{2\alpha_1}{\alpha_1-\alpha_2+2}, \quad \tilde m = \frac{2m-\alpha_1+\alpha_2}{\alpha_1-\alpha_2+2}.$$

PROOF. See [15, Lemma 2.1, Proposition 2.2] with $k=0$ and $T_\beta = T_{0,\beta}$. □

Remark 8.6 *It is essential to deal with $W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m)$: in general the map T_β does not transform $W^{2,p}(\tilde\alpha_1,\tilde\alpha_2,\tilde m)$ into $W^{2,p}(\alpha_1,\alpha_2,m)$.*

We consider now the operators

$$\mathcal{L} = y^{\alpha_1}\Delta_x + 2y^{\frac{\alpha_1+\alpha_2}{2}}(a,\nabla_x D_y) + y^{\alpha_2}B_y, \qquad a \in \mathbb{R}^N,\ |a|<1$$

in the space $L^p_m = L^p_m(\mathbb{R}^{N+1}_+)$. Here B is the Bessel operator

$$B = D_{yy} + \frac{c}{y}D_y, \qquad c > -1$$

on the half line $\mathbb{R}_+ =]0,\infty[$ (often we write B_y to indicate that it acts with respect to the y variable). The condition $|a|<1$ is equivalent to the ellipticity of the top order coefficients We investigate when these operators can be transformed one into the other by means of the transformation (25).

Proposition 8.7 *Let T_β be the isometry defined in (25). Then for every $u \in W^{2,1}_{loc}\left(\mathbb{R}^{N+1}_+\right)$ one has*

$$T_\beta^{-1}\Big(y^{\alpha_1}\Delta_x + 2y^{\frac{\alpha_1+\alpha_2}{2}}(a,\nabla_x D_y) + y^{\alpha_2}B_y\Big)T_\beta u$$
$$= \Big(y^{\frac{\alpha_1}{\beta+1}}\Delta_x + 2(\beta+1)y^{\frac{\alpha_1+\alpha_2+2\beta}{2(\beta+1)}}(a,\nabla_x D_y) + (\beta+1)^2 y^{\frac{\alpha_2+2\beta}{\beta+1}}\tilde B_y\Big)u$$

where

$$\tilde B = D_{yy} + \frac{\tilde c}{y}D_y, \qquad \tilde c = \frac{c+\beta}{\beta+1}. \tag{26}$$

In particular choosing $\beta = \frac{\alpha_1-\alpha_2}{2}$ and setting $\tilde\alpha = \frac{2\alpha_1}{\alpha_1-\alpha_2+2}$ one has

$$T_\beta^{-1}\Big(y^{\alpha_1}\Delta_x + 2y^{\frac{\alpha_1+\alpha_2}{2}}(a,\nabla_x D_y) + y^{\alpha_2}B_y\Big)T_\beta u$$
$$= y^{\tilde\alpha}\Big(\Delta_x + 2(\beta+1)(a,\nabla_x D_y) + (\beta+1)^2\tilde B_y\Big)u$$

PROOF. The proof follows using [17, Proposition 3.1, Proposition 3.2] with $k=0$ and the equalities

$$y^\alpha T_\beta u = T_\beta(y^{\frac{\alpha}{\beta+1}}u), \quad D_{x_i}(T_\beta u) = T_\beta(D_{x_i}u), \quad D_{xy}T_\beta u = T_\beta\left((\beta+1)y^{\frac{\beta}{\beta+1}}D_{xy}u\right).$$

□

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