CLASSIFICATION OF SOLUTIONS TO SOME LIOUVILLE TYPE SYSTEMS

ZHIJIE CHEN AND HANQING ZHAO

ABSTRACT. We give the classification of solutions for some Liouville type systems, which appear as limiting equations during the study of the bubbling phenomena of positive solutions for coupled nonlinear Schrödinger systems.

1. INTRODUCTION

The classification of solutions is a central problem in elliptic partial differential equations. In the seminal paper [5], Chen and Li used the moving planes method to classify all solutions of the Liouville equation

$$-\Delta u = e^u \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u dx < \infty. \tag{1.1}$$

They showed that all solutions of (1.1) are given by

$$u(x) = \log \frac{8\lambda^2}{(1+\lambda^2|x-x_0|^2)^2}, \quad \text{where } \lambda > 0,\ x_0 \in \mathbb{R}^2.$$

Among many important applications, this classification result plays a key role in the bubbling analysis of positive solutions for the Lane-Emden equation

$$-\Delta u = u^p, \quad u > 0, \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

as $p \to +\infty$, where Ω is a bounded domain in $\mathbb{R}^2$. More precisely, after a suitable scaling near a concerntration point, the Liouville equation (1.1) appears as a limiting equation of (1.2) as $p \to +\infty$; see [1, 8, 9, 14, 15].

Later, the classifications of solutions for the Liouville system

$$\begin{cases} -\Delta u_i = \sum\limits_{i=1}^{n} a_{ij} e^{u_j}, \quad \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{u_i} dx < \infty,\ i = 1, \cdots, n, \end{cases} \tag{1.3}$$

were studied by Chanillo and Kiessling [4], Chipot, Shafrir and Wolansky [6], and Lin and Zhang [12]. Chanillo and Kiessling [4] first proved that under some conditions on the matrix $A = (a_{ij})_{n\times n}$, all solutions of (1.3) are radially symmetric with respect to some points. Their result was improved by Chipot, Shafrir and Wolansky [6], who proved the following symmetry result for more general A.

Theorem A. [6] *Let $A = (a_{ij})_{n\times n}$ be*

$$\text{invertible, symmetric, non-negative and irreducible,} \tag{1.4}$$

and $(u_1, \dots, u_n)$ be a solution of (1.3). Then there exists $p \in \mathbb{R}^2$ such that all $u_1, \dots, u_n$ are radially symmetric and decreasing about p.

Here the matrix $A = (a_{ij})_{n\times n}$ is called non-negative if $a_{ij} \ge 0$ for all (i, j), irreducible if there is no partition of $\{1, \cdots, n\} = I_1 \cup I_2$ with $I_1 \cap I_2 = \emptyset$ such that $a_{ij} \equiv 0, \forall i \in I_1, j \in I_2$.

Comparing to the Liouville equation (1.1), the Liouville system (1.3) is more complicated and whether the explicit expressions of all solutions can be written down under the condition (1.4) still remains open. Lin and Zhang [12] studied a weaker problem that whether the solutions of (1.3) are unique up to translations and scalings, and proved the following result.

Theorem B. [12] *Let A satisfy (1.4), $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ be two radial solutions of (1.3) such that $\int_{\mathbb{R}^2} e^{u_i} dx = \int_{\mathbb{R}^2} e^{v_i} dx$ for all i. Then there exists $\delta > 0$ such that*

$$u_i(x) = v_i(\delta x) + 2\log\delta, \quad \forall i. \tag{1.5}$$

In this paper, motivated by the above results, we study the classification of solutions for the following Liouville type system

$$\begin{cases} -\Delta u_1 = \mu_1 e^{u_1} + \beta_0 e^{tu_1+(1-t)u_2} & \text{in } \mathbb{R}^2, \\ -\Delta u_2 = \mu_2 e^{u_2} + \beta_0 e^{tu_1+(1-t)u_2} & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{u_1} dx < \infty \ , \ \int_{\mathbb{R}^2} e^{u_2} dx < \infty, \end{cases} \tag{1.6}$$

where $\mu_1, \mu_2, \beta_0 \in (0, +\infty)$ and $t \in (0, 1)$ are all constants. Like (1.1)-(1.2), this system with $t = \frac{1}{2}$ appears as one of the limiting equations when we study the bubbling phenomena of the following coupled nonlinear Schrödinger system

$$\begin{cases} -\Delta u_1 = \mu_1 u_1^{p-1} + \beta_0 u_1^{\frac{p}{2}-1} u_2^{\frac{p}{2}} & \text{in } \Omega, \\ -\Delta u_2 = \mu_2 u_2^{p-1} + \beta_0 u_1^{\frac{p}{2}} u_2^{\frac{p}{2}-1} & \text{in } \Omega, \\ u_1, u_2 > 0 \text{ in } \Omega, \quad u_1 = u_2 = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.7}$$

as $p \to +\infty$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain. Therefore, it is very important for us to classify all solutions of (1.6). This is the first step for us to study the bubbling phenomena of (1.7) in future works. This Schrödinger system has received great interest in the past two decades, and many interesting results have been proved; see e.g. [2, 7, 13] and references therein.

Note that by letting $\beta = \beta_0/(\mu_1^t \mu_2^{1-t})$ and $v_k = u_k + \log \mu_k$ in (1.6), we only need to study the following equivalent system

$$(1.8) \qquad \begin{cases} -\Delta v_1 = e^{v_1} + \beta e^{tv_1+(1-t)v_2} & \text{in } \mathbb{R}^2, \\ -\Delta v_2 = e^{v_2} + \beta e^{tv_1+(1-t)v_2} & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{v_1} dx < \infty \ , \ \int_{\mathbb{R}^2} e^{v_2} dx < \infty. \end{cases}$$

Remark that, by letting $v_3 = tv_1 + (1-t)v_2$, we can rewrite (1.8) in the form of the Liouville system (1.3):

$$(1.9) \qquad \begin{cases} -\Delta v_1 = e^{v_1} + \beta e^{v_3} & \text{in } \mathbb{R}^2, \\ -\Delta v_2 = e^{v_2} + \beta e^{v_3} & \text{in } \mathbb{R}^2, \\ -\Delta v_3 = te^{v_1} + (1-t)e^{v_2} + \beta e^{v_3} & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{v_k} dx < \infty \ \text{ for } \ k = 1, 2, 3. \end{cases}$$

However, the corresponding matrix

$$\begin{pmatrix} 1 & 0 & \beta \\ 0 & 1 & \beta \\ t & 1-t & \beta \end{pmatrix}$$

is always degenerate, i.e. does not satisfy (1.4), so Theorems A and B can not apply. To the best of our knowledge, the classification of solutions for the system (1.9) is open. Our main result is

Theorem 1.1. *Let (v_1, v_2) with $v_1, v_2 \in L^1_{loc}(\mathbb{R}^2)$ be a solution of the system (1.8) (in the sense of distributions). Then there exist $\lambda \in (0, +\infty)$ and $x_0 \in \mathbb{R}^2$ such that*

$$(1.10) \qquad v_1(x) = v_2(x) = \log \frac{8\lambda^2}{(1 + \lambda^2 |x - x_0|^2)^2} - \ln(1 + \beta).$$

Our proof of Theorem 1.1 consists of two steps. The first step is to prove the radial symmetry of the solution via a Rellich-Pohozaev identity and the isoperimetric inequality; see Sections 2-3. This idea is borrowed from Chanillo-Kiessling [4]. The second step is to prove $v_1 = v_2$ via a new and interesting geometric inequality for radial functions which was introduced recently by Gui-Li [10]; see Section 4.

Remark 1.2. We are also interested in the classification of solutions to the following Liouville type system that seems more naturally in the point of symmetry

$$\begin{cases} -\Delta u_1 = \mu_1 e^{u_1} + \beta_0 e^{tu_1+(1-t)u_2} & \text{in } \mathbb{R}^2, \\ -\Delta u_2 = \mu_2 e^{u_2} + \beta_0 e^{tu_2+(1-t)u_1} & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{u_1} dx < \infty \ , \ \int_{\mathbb{R}^2} e^{u_2} dx < \infty. \end{cases}$$

Unfornaturely, it seems that the approach of this paper does not work for this system, because we can not obtain a useful Rellich-Pohozaev identity for this system. We will try to study this system elsewhere.

In the rest of this paper, we freely use the notation C to denote various constants. Denote $B_r(x) := \{y \in \mathbb{R}^2 : |x-y| < r\}$ and $B_r = B_r(0)$. Besides, for any function $f(x)$ that is radially symmetric with respect to 0, we also write $f(|x|) = f(x)$.

2. A Rellich-Pohozaev identity for entire solutions

In this section, we follow Chanillo-Kiessling's idea [4] to establish a Rellich-Pohozaev identity for entire solutions of (1.8). Let (v_1, v_2) with $v_1, v_2 \in L^1_{loc}(\mathbb{R}^2)$ be a solution of (1.8) and define

$$\alpha_1 := \int_{\mathbb{R}^2} e^{v_1} dx, \ \alpha_2 := \int_{\mathbb{R}^2} e^{v_2} dx \ \text{ and } \ \gamma := \int_{\mathbb{R}^2} e^{tv_1 + (1-t)v_2} dx. \tag{2.1}$$

The main result of this section is to prove the following Rellich-Pohozaev identity.

Proposition 2.1. *$t(\alpha_1 + \beta\gamma)^2 + (1-t)(\alpha_2 + \beta\gamma)^2 = 8\pi(t\alpha_1 + (1-t)\alpha_2 + \beta\gamma)$.*

Before proving Proposition 2.1, we need to study the asymptotic behavior of the solution (v_1, v_2) near infinity. First, we recall a result from Brezis and Merle [3].

Lemma 2.2. [3, Theorem 2] *Suppose $u \in L^1_{loc}(\mathbb{R}^2)$ satisfies*

$$-\Delta u = V(x) e^u \quad \text{in } \mathbb{R}^2 \tag{2.2}$$

with $V \in L^p(\mathbb{R}^2)$ and $e^u \in L^{p'}(\mathbb{R}^2)$ for some $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then $u^+ \in L^\infty(\mathbb{R}^2)$. Here $u^+ := \max\{u, 0\}$.

Lemma 2.3. *We have $v_1^+, v_2^+ \in L^\infty(\mathbb{R}^2)$ and $v_1, v_2 \in C^\infty(\mathbb{R}^2)$.*

Proof. Define $V(x) := te^{(1-t)v_1(x)} + t\beta e^{(1-t)v_2(x)}$, then

$$\|V\|_{L^{\frac{1}{1-t}}(\mathbb{R}^2)} \leq t\|e^{v_1}\|^{1-t}_{L^1(\mathbb{R}^2)} + t\beta \|e^{v_2}\|^{1-t}_{L^1(\mathbb{R}^2)} < \infty, \tag{2.3}$$

namely $V \in L^{\frac{1}{1-t}}(\mathbb{R}^2)$. By $-\Delta(tv_1) = Ve^{tv_1}$, $e^{tv_1} \in L^{\frac{1}{t}}(\mathbb{R}^2)$ and Lemma 2.2, it follows that $v_1^+ \in L^\infty(\mathbb{R}^2)$. Similarly, $v_2^+ \in L^\infty(\mathbb{R}^2)$. Thus $-\Delta v_1, -\Delta v_2 \in L^\infty(\mathbb{R}^2)$, so the standard elliptic regularity theory implies $v_1, v_2 \in C^{1,\alpha}_{loc}$, and then $v_1, v_2 \in C^\infty(\mathbb{R}^2)$ by a standard bootstrap argument. □

Lemma 2.4. *For $k = 1, 2$, we have*

$$v_k(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log\left(\frac{|x-y|}{|y|+1}\right) \left(e^{v_k(y)} + \beta e^{tv_1(y) + (1-t)v_2(y)}\right) dy + C_k, \tag{2.4}$$

where C_k is a constant. Moreover,

$$\frac{v_k(x)}{\log|x|} \to -\frac{1}{2\pi}(\alpha_k + \beta\gamma) \quad \text{uniformly as} \quad |x| \to \infty. \tag{2.5}$$

Proof. The proof is the same as that of [5, Lemma 1.2] with trivial modifications, and we omit it here. □

Note that the system (1.8) is conformally invariant, from which we have the following estimates.

Lemma 2.5. *$\alpha_k + \beta\gamma \geq 8\pi$ for $k = 1, 2$.*

Proof. We use the Kelvin transformation to define

$$\widetilde{v}_k(x) := v_k\left(\frac{x}{|x|^2}\right) - 4\log|x| = v_k(y) + 4\log|y|, \quad y = \frac{x}{|x|^2}. \tag{2.6}$$

By direct computations, we have

$$-\Delta_x \widetilde{v}_k(x) = |x|^{-4}\left(-\Delta_y v_k(y)\right) = e^{\widetilde{v}_k(x)} + \beta e^{t\widetilde{v}_1(x)+(1-t)\widetilde{v}_2(x)} \quad \text{in } \mathbb{R}^2 \setminus \{0\},$$

and

$$\int_{\mathbb{R}^2} e^{\widetilde{v}_k(x)} dx = \int_{\mathbb{R}^2\setminus\{0\}} |y|^4 e^{v_k(y)} \frac{dy}{|y|^4} = \int_{\mathbb{R}^2} e^{v_k(y)} dy = \alpha_k < \infty.$$

Clearly $\widetilde{v}_1, \widetilde{v}_2 \in C^\infty(\mathbb{R}^2\setminus\{0\})$, and Lemma 2.4 gives

$$\lim_{|x|\to 0} \frac{\widetilde{v}_k(x)}{\log|x|} = \lim_{|y|\to\infty} -\frac{v_k(y) + 4\log|y|}{\log|y|} = \frac{1}{2\pi}(\alpha_k + \beta\gamma) - 4, \tag{2.7}$$

which implies $\widetilde{v}_k \in L^1(B_r(0))$ and so $\widetilde{v}_k \in L^1_{loc}(\mathbb{R}^2)$. Therefore, $(\widetilde{v}_1, \widetilde{v}_2)$ is also a solution of the same system (1.8), so Lemma 2.3 yields $\widetilde{v}_k^+ \in L^\infty(\mathbb{R}^2)$ for $k = 1, 2$.

If $\alpha_k + \beta\gamma < 8\pi$ for some k, then it follows from (2.7) that

$$\lim_{|x|\to 0} \widetilde{v}_k(x) \geq \frac{1}{2}\left(\frac{1}{2\pi}(\alpha_k + \beta\gamma) - 4\right) \lim_{|x|\to 0} \log|x| = +\infty, \tag{2.8}$$

a contradiction with $\widetilde{v}_k^+ \in L^\infty(\mathbb{R}^2)$. This proves $\alpha_k + \beta\gamma \geq 8\pi$. □

A direct consquence of (2.5) and $\alpha_k + \beta\gamma \geq 8\pi$ is

Corollary 2.6. *There is $r > 2$ such that $e^{v_k(x)} \leq |x|^{-7/2}$ for any $|x| \geq r$ and $k = 1, 2$.*

Lemma 2.7. *For $k = 1, 2$ we have*

$$\lim_{|x|\to\infty} \langle x, \nabla v_k(x)\rangle = -\frac{1}{2\pi}(\alpha_k + \beta\gamma), \tag{2.9}$$

$$\lim_{|x|\to\infty} |x|\,|\nabla v_k(x)| = \frac{1}{2\pi}(\alpha_k + \beta\gamma), \tag{2.10}$$

uniformly in x.

Proof. **Step 1:** We prove

$$\limsup_{|x|\to\infty} |x|\,|\nabla v_k(x)| \leq \frac{1}{2\pi}(\alpha_k + \beta\gamma) \quad \text{uniformly in } x. \tag{2.11}$$

By (2.4) we have

$$\nabla v_k(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\left(e^{v_k(y)} + \beta e^{tv_1(y)+(1-t)v_2(y)}\right) dy. \tag{2.12}$$

It follows from $e^{tv_1+(1-t)v_2} \le te^{v_1} + (1-t)e^{v_2}$ that

$$\begin{aligned}
&|x|\,|\nabla v_k(x)| \\
\le &\frac{1}{2\pi}\left(\alpha_k+\beta\gamma\right) + \frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\frac{|x|}{|x-y|}-1\right)\left(e^{v_k(y)}+\beta e^{tv_1(y)+(1-t)v_2(y)}\right)dy \\
\le &\frac{1}{2\pi}\left(\alpha_k+\beta\gamma\right) + C\int_{\mathbb{R}^2}\frac{|y|}{|x-y|}\left(e^{v_1(y)}+e^{v_2(y)}\right)dy.
\end{aligned}$$

Therefore, in order to obtain (2.11), it is sufficient to show that for $k = 1, 2$,

$$\lim_{|x|\to\infty}\int_{\mathbb{R}^2}\frac{|y|}{|x-y|}e^{v_k(y)}dy = 0 \quad \text{uniformly in } x. \tag{2.13}$$

Fix $r > 2$ in Corollary 2.6 such that

$$e^{v_k(x)} \le |x|^{-7/2}, \quad \text{for any } |x| \ge r. \tag{2.14}$$

For any $|x| \ge r^2$, we split the integral domain $\mathbb{R}^2$ into three parts

$$\begin{aligned}
&\Omega_1 = \{y : |y| \le |x|/2\}, \quad \Omega_2 = \{y : |x|/2 < |y| \le 2|x|\}, \\
&\Omega_3 = \{y : |y| > 2|x|\}.
\end{aligned} \tag{2.15}$$

For $y \in \Omega_1$, we have $|x-y| \ge |x| - |y| \ge |x|/2$, so it follows from (2.14) that

$$\begin{aligned}
&\int_{\Omega_1}\frac{|y|}{|x-y|}e^{v_k(y)}dy \\
\le &\frac{2}{|x|}\left(\int_{\{|y|\le r\}}|y|e^{v_k(y)}dy + \int_{\{r\le|y|\le\frac{|x|}{2}\}}|y|^{-5/2}dy\right) \le \frac{C}{|x|},
\end{aligned} \tag{2.16}$$

where C depends on r but note that $r > 2$ is fixed.

For $y \in \Omega_2$, by (2.14) we have

$$\begin{aligned}
&\int_{\Omega_2}\frac{|y|}{|x-y|}e^{v_k(y)}dy \le \frac{C}{|x|^{5/2}}\int_{\{\frac{|x|}{2}<|y|\le 2|x|\}}\frac{1}{|x-y|}dy \\
\le &\frac{C}{|x|^{3/2}}\int_{\{\frac{1}{2}<|\mu|\le 2\}}\frac{1}{|\mu-\theta|}d\mu \le \frac{C}{|x|^{3/2}},
\end{aligned} \tag{2.17}$$

where $\theta = \frac{x}{|x|}$ and $\mu = \frac{y}{|x|}$.

For $y \in \Omega_3$, we have $|x-y| \ge |y| - |x| \ge |y|/2$, so

$$\int_{\Omega_3}\frac{|y|}{|x-y|}e^{v_k(y)}dy \le 2\int_{\{|y|>2|x|\}}e^{v_k(y)}dy. \tag{2.18}$$

By (2.16)-(2.18) we have

$$\int_{\mathbb{R}^2}\frac{|y|}{|x-y|}e^{v_k(y)}dy \le \frac{C}{|x|} + 2\int_{\{|y|>2|x|\}}e^{v_k(y)}dy, \quad \forall |x| \ge r^2.$$

Letting $|x| \to +\infty$ and using $e^{v_k} \in L^1(\mathbb{R}^2)$, we see that (2.13) holds, from which (2.11) follows.

Step 2: We prove (2.9).

By (2.12) we have

$$\langle x, \nabla v_k(x)\rangle = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\langle x, x-y\rangle}{|x-y|^2}\left(e^{v_k(y)} + \beta e^{tv_1(y)+(1-t)v_2(y)}\right) dy$$
$$= -\frac{1}{2\pi}(\alpha_k + \beta\gamma) - \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\langle y, x-y\rangle}{|x-y|^2}\left(e^{v_k(y)} + \beta e^{tv_1(y)+(1-t)v_2(y)}\right) dy.$$

Since (2.13) implies

$$\left|\int_{\mathbb{R}^2} \frac{\langle y, x-y\rangle}{|x-y|^2}\left(e^{v_k(y)} + \beta e^{tv_1(y)+(1-t)v_2(y)}\right) dy\right|$$
$$\leq C\int_{\mathbb{R}^2} \frac{|y|}{|x-y|}\left(e^{v_1(y)} + e^{v_2(y)}\right) dy \to 0 \quad \text{uniformly in } x \text{ as } |x| \to \infty,$$

we obtain (2.9).

Step 3: From (2.11) and (2.9), we have

$$\frac{1}{2\pi}(\alpha_k + \beta\gamma) = \lim_{|x|\to\infty} |\langle x, \nabla v_k(x)\rangle|$$
$$\leq \liminf_{|x|\to\infty} |x|\,|\nabla v_k(x)| \leq \limsup_{|x|\to\infty} |x|\,|\nabla v_k(x)| \leq \frac{1}{2\pi}(\alpha_k + \beta\gamma),$$

so (2.10) holds. This completes the proof. □

Lemma 2.8 (Pohozaev identity)**.** *For any solution (v_1, v_2) of (1.8), we have*

$$-2\int_{B_R}\left(te^{v_1} + (1-t)e^{v_2} + \beta e^{tv_1+(1-t)v_2}\right) dx$$
$$+R\int_{\partial B_R}\left(te^{v_1} + (1-t)e^{v_2} + \beta e^{tv_1+(1-t)v_2}\right) dS_x$$
$$= R\int_{\partial B_R} t\left(\frac{|\nabla v_1|^2}{2} - \left|\frac{\partial v_1}{\partial \vec{n}}\right|^2\right) + (1-t)\left(\frac{|\nabla v_2|^2}{2} - \left|\frac{\partial v_2}{\partial \vec{n}}\right|^2\right) dS_x, \tag{2.19}$$

where B_R is the open ball of radius R centered at 0 and $\vec{n}(x)$ denotes the outward normal vector of ∂B_R.

Proof. By direct computations, we have

$$\langle x, \nabla v_k(x)\rangle \Delta v_k(x) = \operatorname{div}\left(\langle x, \nabla v_k(x)\rangle \nabla v_k(x) - \frac{|\nabla v_k(x)|^2}{2}x\right),$$

$$\langle x, \nabla v_k(x)\rangle e^{v_k(x)} = \operatorname{div}\left(e^{v_k(x)}x\right) - 2e^{v_k(x)},$$

$$t\langle x, \nabla v_1(x)\rangle e^{tv_1(x)+(1-t)v_2(x)} + (1-t)\langle x, \nabla v_2(x)\rangle e^{tv_1(x)+(1-t)v_2(x)}$$
$$= \operatorname{div}\left(e^{tv_1(x)+(1-t)v_2(x)}x\right) - 2e^{tv_1(x)+(1-t)v_2(x)}.$$

Then multiplying $-\Delta v_1 = e^{v_1} + \beta e^{tv_1(x)+(1-t)v_2(x)}$ with $t\langle x, \nabla v_1(x)\rangle$, multiplying $-\Delta v_2 = e^{v_2} + \beta e^{tv_1(x)+(1-t)v_2(x)}$ with $(1-t)\langle x, \nabla v_2(x)\rangle$, and integrating over B_R, we easily obtain the desired identity. □

Proof of Proposition 2.1. Note that

(2.20)
$$\lim_{R\to\infty}\int_{B_R}\left(te^{v_1}+(1-t)e^{v_2}+\beta e^{tv_1+(1-t)v_2}\right)dx=t\alpha_1+(1-t)\alpha_2+\beta\gamma.$$

By Corollary 2.6, we have that for $R>r$,

$$\begin{aligned}&R\int_{\partial B_R}\left(te^{v_1}+(1-t)e^{v_2}+\beta e^{tv_1+(1-t)v_2}\right)dS_x\\ \le& CR\int_{\partial B_R}(e^{v_1}+e^{v_2})\,dS_x\le CR\int_{\partial B_R}|x|^{-7/2}dS_x\le CR^{-3/2},\end{aligned}$$

so

$$\lim_{R\to\infty}R\int_{\partial B_R}\left(te^{v_1}+(1-t)e^{v_2}+\beta e^{tv_1+(1-t)v_2}\right)dS_x=0. \tag{2.21}$$

Finally, (2.9)-(2.10) give

$$\begin{aligned}&\lim_{R\to\infty}R\int_{\partial B_R}\left(\frac{|\nabla v_k|^2}{2}-\left|\frac{\partial v_k}{\partial\vec{n}}\right|^2\right)dS_x\\ =&\lim_{R\to\infty}\frac{1}{2R}\int_{\partial B_R}(|x|\,|\nabla v_k|)^2-2\,|\langle x,\nabla v_k\rangle|^2\,dS_x=-\frac{(\alpha_k+\beta\gamma)^2}{4\pi}.\end{aligned} \tag{2.22}$$

Thus, inserting (2.20)-(2.22) into (2.19), we get

$$t(\alpha_1+\beta\gamma)^2+(1-t)(\alpha_2+\beta\gamma)^2=8\pi(t\alpha_1+(1-t)\alpha_2+\beta\gamma). \tag{2.23}$$

The proof is complete. □

Corollary 2.9. *$\alpha_k+\beta\gamma=8\pi$ for $k=1,2$. In particular, $\alpha_1=\alpha_2$.*

Proof. This follows from (2.23) and $\alpha_k+\beta\gamma\ge 8\pi$ that was proved in Lemma 2.5. □

3. Symmetry via the isoperimetric inequality

The main result of this section is to prove the radial symmetry of the solution (v_1,v_2) via the isoperimetric inequality.

Proposition 3.1. *v_1 and v_2 are radially symmetric and decreasing with respect to the same point.*

Similarly as [4], we will prove this result via the isoperimetric inequality. By Lemma 2.3, we let v_1^*,v_2^* and $(tv_1+(1-t)v_2)^*$ denote the equimeasurable, radially symmetric non-increasing rearrangement of v_1,v_2 and $tv_1+(1-t)v_2$ respectively, centered at 0. We also define

$$\begin{aligned}&\Lambda_c^k=\{x:v_k(x)\ge c\},\quad \Lambda_c^{k*}=\{x:v_k^*(x)\ge c\},\quad \text{for } k=1,2,\\ &\Xi_c=\{x:(tv_1+(1-t)v_2)(x)\ge c\},\\ &\Xi_c^*=\{x:(tv_1+(1-t)v_2)^*(x)\ge c\},\end{aligned}$$

with the radius R_c^k of the ball Λ_c^{k*} and the radius $\widetilde{R}_c$ of the ball Ξ_c^*. Since $\alpha_1, \alpha_2 < \infty$, both Λ_c^1 and Λ_c^2 are of finite measures. Furthermore, it follows from Sard's theorem that $\partial\Lambda_c^k \in C^2$ for almost all c, which implies that the unit outward normal vectors $\vec{n}(x)$ to $\partial\Lambda_c^k$ exist for almost all $\partial\Lambda_c^k$. In particular, $|\nabla v_k(x)| = -\langle \vec{n}(x), \nabla v_k(x)\rangle > 0$ on $\partial\Lambda_c^k$ for almost all $\partial\Lambda_c^k$.

First we recall some well-known results. The first lemma follows from the co-area formula and the isoperimetric inequality. See e.g. [16, Faber-Krahn Theorem].

Lemma 3.2. *For almost every c, we have*

$$\int_{\partial\Lambda_c^{k*}} |\nabla v_k^*| dS_x \le \int_{\partial\Lambda_c^k} |\nabla v_k| dS_x, \tag{3.1}$$

$$\int_{\partial\Xi_c^*} |\nabla (tv_1 + (1-t)v_2)^*| dS_x \le \int_{\partial\Xi_c} |\nabla (tv_1 + (1-t)v_2)| dS_x. \tag{3.2}$$

Moreover, if the equality in (3.1) (resp. (3.2)) holds for almost every c, then v_k (resp. $tv_1 + (1-t)v_2$) is radially symmetric with respect to some point.

The next lemma is a natural corollary of the simple rearrangement inequality [11, Theorem 3.4].

Lemma 3.3. *For almost every c, we have*

$$\int_{\Lambda_c^k} e^{v_k} dx = \int_{\Lambda_c^{k*}} e^{v_k^*} dx, \quad \int_{\Xi_c} e^{tv_1+(1-t)v_2} dx = \int_{\Xi_c^*} e^{(tv_1+(1-t)v_2)^*} dx$$

$$\int_{\Xi_c} e^{v_k} dx \le \int_{\Xi_c^*} e^{v_k^*} dx, \quad \int_{\Lambda_c^k} e^{tv_1+(1-t)v_2} dx \le \int_{\Lambda_c^{k*}} e^{(tv_1+(1-t)v_2)^*} dx.$$

Now we give the proof of Proposition 3.1.

Proof of Proposition 3.1. By Lemmas 3.2-3.3 and the divergence theorem, we have that for almost every c,

$$\begin{aligned}
-2\pi R_c^k (v_k^*)'(R_c^k) &= -\int_{\partial\Lambda_c^{k*}} \partial_r v_k^* dS_x = \int_{\partial\Lambda_c^{k*}} |\nabla v_k^*| dS_x \le \int_{\partial\Lambda_c^k} |\nabla v_k| dS_x \\
&= -\int_{\partial\Lambda_c^k} \langle \vec{n}(x), \nabla v_k\rangle dS_x = -\int_{\Lambda_c^k} \Delta v_k dx \\
&= \int_{\Lambda_c^k} \left(e^{v_k} + \beta e^{tv_1+(1-t)v_2} \right) dx \\
&\le \int_{\Lambda_c^{k*}} e^{v_k^*} dx + \beta \int_{\Lambda_c^{k*}} e^{(tv_1+(1-t)v_2)^*} dx.
\end{aligned} \tag{3.3}$$

Define

$$M_1(r) := \int_{B_r(0)} e^{v_1^*} dx, \quad M_2(r) := \int_{B_r(0)} e^{v_2^*} dx,$$
$$M_3(r) := \int_{B_r(0)} e^{(tv_1+(1-t)v_2)^*} dx.$$

By direct computations, we have that for almost every $r > 0$,

$$\begin{aligned} M_k'(r) =& 2\pi r e^{v_1^*(r)}, \;\; M_k''(r) = 2\pi(1 + r(v_k^*)'(r))e^{v_1^*(r)}, \quad k = 1, 2, \\ M_3'(r) =& 2\pi r e^{(tv_1+(1-t)v_2)^*(r)}, \\ M_3''(r) =& 2\pi(1 + r((tv_1 + (1-t)v_2)^*)'(r))e^{(tv_1+(1-t)v_2)^*(r)}, \end{aligned}$$

and so

$$r(v_k^*)'(r) = \frac{rM_k''(r)}{M_k'(r)} - 1, \quad k = 1, 2, \tag{3.4}$$

$$r((tv_1 + (1-t)v_2)^*)'(r) = \frac{rM_3''(r)}{M_3'(r)} - 1.$$

Since (3.3) says that for almost every $r > 0$,

$$-2\pi r(v_k^*)'(r) \le M_k(r) + \beta M_3(r), \quad k = 1, 2,$$

it follows from (3.4) that for almost every $r > 0$,

$$\begin{aligned} 2\pi(rM_k'(r))' &= 2\pi r M_k''(r) + 2\pi M_k'(r) \\ &\ge 4\pi M_k'(r) - (M_k(r) + \beta M_3(r))M_k'(r), \quad k = 1, 2. \end{aligned} \tag{3.5}$$

Note from $\int_{\mathbb{R}^2} e^{v_k^*} = \int_{\mathbb{R}^2} e^{v_k} = \alpha_k < +\infty$ that there exists $r_n \to +\infty$ such that

$$\lim_{n\to\infty} r_n M_k'(r_n) = \lim_{n\to\infty} 2\pi r_n^2 e^{v_k^*(r_n)} = 0.$$

Then by integrating (3.5) from 0 to r_n and letting $n \to \infty$, we obtain

$$0 \ge 4\pi\alpha_k - \frac{\alpha_k^2}{2} - \beta \int_0^\infty M_3(r)M_k'(r)dr, \quad k = 1, 2. \tag{3.6}$$

Since

$$-\Delta(tv_1 + (1-t)v_2) = te^{v_1} + (1-t)e^{v_2} + \beta e^{tv_1+(1-t)v_2},$$

a similar argument as (3.3)-(3.6) gives

$$\begin{aligned} -2\pi\widetilde{R}_c((tv_1 + (1-t)v_2)^*)'(\widetilde{R}_c) &= \int_{\partial\Xi_c^*} |\nabla(tv_1 + (1-t)v_2)^*| dS_x \\ &\le \int_{\partial\Xi_c} |\nabla(tv_1 + (1-t)v_2)| dS_x \\ &\le \int_{\Xi_c^*} te^{v_1^*} + (1-t)e^{v_2^*} + \beta e^{(tv_1+(1-t)v_2)^*} dx, \end{aligned}$$

$$2\pi(rM_3'(r))' \ge 4\pi M_3'(r) - (tM_1(r) + (1-t)M_2(r) + \beta M_3(r))M_3'(r),$$

and so

$$0 \ge 4\pi\gamma - \frac{\beta}{2}\gamma^2 - \int_0^\infty M_3'(r)(tM_1(r) + (1-t)M_2(r))dr. \tag{3.7}$$

By (3.6) and (3.7), we easily obtain

$$t(\alpha_1 + \beta\gamma)^2 + (1-t)(\alpha_2 + \beta\gamma)^2 \ge 8\pi\left(t\alpha_1 + (1-t)\alpha_2 + \beta\gamma\right). \tag{3.8}$$

Since Proposition 2.1 proves that (3.8) is actually an identity, we see from the above arguments that the inequalities of (3.3) are also identities for almost every c, so it follows from Lemma 3.2 that v_k is radially symmetric with respect to some point $x_k \in \mathbb{R}^2$ for $k = 1, 2$. Then

$$-e^{-tv_1}\Delta v_1 - e^{(1-t)v_1} = \beta e^{(1-t)v_2}$$

implies $x_1 = x_2$. Up to a translation we may assume $x_1 = x_2 = 0$. Then

$$rv_k'(r) = -\int_0^r s(e^{v_k(s)} + \beta e^{tv_1(s)+(1-t)v_2(s)})ds < 0$$

for any $r > 0$, so v_k is radially symmetric and decreasing. The proof is complete. □

4. Uniqueness via geometric inequalities

In this section, we complete the proof of Theorem 1.1 by applying the following interesting geometric inequality from Gui-Li [10].

Theorem 4.1. [10, Theorem 1.10] *Let w_1 be a radially symmetric function such that*

$$\Delta w_1 + e^{w_1} \le 0 \quad in \ \mathbb{R}^2. \tag{4.1}$$

Let w_2 be another radially symmetric function defined in $\mathbb{R}^2$. If for some disk B_r we have

$$\Delta w_2 + e^{w_2} \le \Delta w_1 + e^{w_1}, \ w_2 < w_1 \ \ in \ \ B_r \quad and \ w_2 = w_1 \ \ on \ \ \partial B_r, \tag{4.2}$$

then

$$\int_{B_r} e^{w_1} + e^{w_2} dy \ge \int_{\mathbb{R}^2} e^{w_1} dy. \tag{4.3}$$

Furthermore, if the equality in (4.3) holds, then

$$\Delta w_2 + e^{w_2} = \Delta w_1 + e^{w_1} = 0 \quad in \ \ B_r. \tag{4.4}$$

Proof of Theorem 1.1. By Proposition 3.1 and up to a translation, we can assume that v_1 and v_2 are radially symmetric and decreasing with respect to 0. Thus, we can consider the following ODE problem

$$\begin{cases} (rv_1'(r))' + re^{v_1} + \beta re^{tv_1+(1-t)v_2} = 0, \\ (rv_2'(r))' + re^{v_2} + \beta re^{tv_1+(1-t)v_2} = 0, \\ \int_0^{+\infty} re^{v_1} dr < \infty, \ \int_0^{+\infty} re^{v_2} dr < \infty, \end{cases} \tag{4.5}$$

Step 1: We prove $v_1 \equiv v_2$.

Assume by contradiction that $v_1 \not\equiv v_2$. Then $v_1(0) \neq v_2(0)$ (otherwise by $v_1'(0) = v_2'(0) = 0$ and the uniqueness for ODEs, we obtain $v_1 \equiv v_2$). Without loss of generality, we may assume $v_1(0) > v_2(0)$. If $v_1 > v_2$ in $\mathbb{R}^2$, we have $\alpha_1 = \int_{\mathbb{R}^2} e^{v_1} dx > \alpha_2 = \int_{\mathbb{R}^2} e^{v_2} dx$, which contradicts with $\alpha_1 = \alpha_2$

in Proposition 2.9. Thus, there exists $R_0 > 0$ such that $v_1 > v_2$ in B_{R_0} and $v_1 = v_2$ on ∂B_{R_0}. Since

$$\Delta v_1 + e^{v_1} = \Delta v_2 + e^{v_2} = -\beta e^{tv_1+(1-t)v_2} < 0 \quad \text{in } \mathbb{R}^2,$$

we can apply Theorem 4.1 to obtain

$$\int_{B_{R_0}} e^{v_1} dx + \int_{B_{R_0}} e^{v_2} dx > \int_{\mathbb{R}^2} e^{v_1} dx = \alpha_1 = \alpha_2. \tag{4.6}$$

On the other hand, similarly as Lemma 2.5, we use the Kelvin transformation to define

$$\widetilde{v}_k(r) := v_k(\frac{1}{r}) - 4\log(r), \quad k = 1, 2.$$

Then $(\widetilde{v}_1, \widetilde{v}_2)$ satisfies

$$\begin{cases} (r\widetilde{v}_1'(r))' + re^{\widetilde{v}_1} + \beta re^{t\widetilde{v}_1+(1-t)\widetilde{v}_2} = 0, \\ (r\widetilde{v}_2'(r))' + re^{\widetilde{v}_2} + \beta re^{t\widetilde{v}_1+(1-t)\widetilde{v}_2} = 0, \\ \int_0^{+\infty} re^{\widetilde{v}_1} dr < \infty, \ \int_0^{+\infty} re^{\widetilde{v}_2} dr < \infty. \end{cases}$$

Again we have $\widetilde{v}_1(0) \neq \widetilde{v}_2(0)$, so there exists $R_1 \in (0, 1/R_0]$ such that $\widetilde{v}_1 > \widetilde{v}_2$ in B_{R_1} (or $\widetilde{v}_1 < \widetilde{v}_2$ in B_{R_1}) and $\widetilde{v}_1 = \widetilde{v}_2$ on ∂B_{R_1}. Since we also have

$$\Delta \widetilde{v}_1 + e^{\widetilde{v}_1} = \Delta \widetilde{v}_2 + e^{\widetilde{v}_2} = -\beta e^{t\widetilde{v}_1+(1-t)\widetilde{v}_2} < 0 \quad \text{in } \mathbb{R}^2,$$

we can apply Theorem 4.1 to obtain

$$\int_{B_{R_1}} e^{\widetilde{v}_1} dx + \int_{B_{R_1}} e^{\widetilde{v}_2} dx > \int_{\mathbb{R}^2} e^{\widetilde{v}_1} dx \ \left(\text{or } \int_{\mathbb{R}^2} e^{\widetilde{v}_2} dx \right),$$

or equivalently,

$$\int_{\mathbb{R}^2 \setminus B_{\frac{1}{R_1}}} e^{v_1} dx + \int_{\mathbb{R}^2 \setminus B_{\frac{1}{R_1}}} e^{v_2} dx > \int_{\mathbb{R}^2} e^{v_1} dx \ \left(\text{or } \int_{\mathbb{R}^2} e^{v_2} dx \right) = \alpha_1 = \alpha_2. \tag{4.7}$$

Since $R_0 \leq \frac{1}{R_1}$, we conclude from (4.6) and (4.7) that

$$\alpha_1 + \alpha_2 = \int_{\mathbb{R}^2} e^{v_1} dx + \int_{\mathbb{R}^2} e^{v_2} dx > \alpha_1 + \alpha_2,$$

a contradiction. This proves $v_1 \equiv v_2$.

Step 2: We prove (1.10).

Since $v_1 \equiv v_2$, we have

$$\begin{cases} -\Delta v_1 = (1+\beta) e^{v_1} \quad \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{v_1} dx < \infty. \end{cases} \tag{4.8}$$

Then by Chen-Li's classification result [5, Theorem 1], there exist $\lambda \in (0, +\infty)$ and $x_0 \in \mathbb{R}^2$ such that (1.10) holds. This completes the proof. □

Acknowledgements. The authors thank the referee very much for valuable comments. Z.Chen is supported by National Key R&D Program of China (Grant 2023YFA1010002) and NSFC (No. 12222109).

REFERENCES

[1] ADIMURTHI AND M. GROSSI, *Asymptotic estimates for a two-dimensional problem with polynomial nonlinearity*, Proc. Amer. Math. Soc. **132** (2004), 1013–1019.

[2] T. BARTSCH, N. DANCER AND Z.-Q. WANG, *A Liouville theorem, a priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system*, Calc. Var. Partial Differential Equations **37** (2010), 345–361.

[3] H. BREZIS AND F. MERLE, *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. Partial Differential Equations **16** (1991), 1223–1253.

[4] S. CHANILLO AND M. KIESSLING, *Conformally invariant systems of nonlinear PDE of Liouville type*, Geom. Funct. Anal. **5** (1995), 924–947.

[5] W. CHEN AND C. LI, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. **63** (1991), 615–622.

[6] M. CHIPOT, I. SHAFRIR AND G. WOLANSKY, *On the solutions of Liouville systems*, J. Differential Equations **140** (1997), 59–105.

[7] N. DANCER, J. WEI AND T. WETH, *A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), 953–969.

[8] F. DE MARCHIS, M. GROSSI, I. IANNI AND PACELLA, F, *L^∞-norm and energy quantization for the planar Lane-Emden problem with large exponent*, Arch. Math. **111** (2018), 421–429.

[9] F. DE MARCHIS, I. IANNI AND F. PACELLA, *Asymptotic profile of positive solutions of Lane-Emden problems in dimension two*, J. Fixed Point Theory Appl. **19** (2017), 889–916.

[10] C. GUI AND Q. LI, *Some geometric inequalities related to Liouville equation*, Math. Z. **305** (2023), Paper No. 40, 30 pp.

[11] E.-H. LIEB AND M. LOSS, *Analysis*, vol. 14 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second ed., 2001.

[12] C.-S. LIN AND L. ZHANG, *Profile of bubbling solutions to a Liouville system*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), 117–143.

[13] T. LIN AND J. WEI, *Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \le 3$*, Comm. Math. Phys. **255** (2005), 629–653.

[14] X. REN AND J. WEI, *On a two-dimensional elliptic problem with large exponent in nonlinearity*, Trans. Amer. Math. Soc. **343** (1994), 749–763.

[15] X. REN AND J. WEI, *Single-point condensation and least-energy solutions*, Proc. Amer. Math. Soc. **124** (1996), 111–120.

[16] R. SCHOEN AND S.-T. YAU, *Lectures on differential geometry*, vol. I of Conference Proceedings and Lecture Notes in Geometry and Topology, International Press, Cambridge, MA, 1994.

DEPARTMENT OF MATHEMATICAL SCIENCES, YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, 100084, CHINA
Email address: zjchen2016@tsinghua.edu.cn

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING, 100084, CHINA
Email address: zhq20@mails.tsinghua.edu.cn